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INVARIANTS OF VANISHING BRAUER CLASSES

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Abstract. A specialization of a $K3$ surface with Picard rank one to a $K3$ with rank two defines a vanishing class of order two in the Brauer group of the general $K3$ surface. We give the B-field invariants of this class. We apply this to the $K3$ double plane defined by a cubic fourfold with a plane. The specialization of such a cubic fourfold whose group of codimension two cycles has rank two to one which has rank three induces such a specialization of the double planes. We determine the Picard lattice of the specialized double plane as well as the vanishing Brauer class and its relation to the natural 'Clifford' Brauer class. This provides more insight in the specializations. It allows us to explicitly determine the $K3$ surfaces associated to infinitely many of the conjecturally rational cubic fourfolds obtained as such specializations.

Introduction

In this paper, S will be a complex projective $K3$ surface. An element α in the Brauer group $\mathrm{Br}(S)$ defines α-twisted sheaves on S which generate a twisted derived category ([HuySt05]). A locally free α-twisted sheaf of rank n defines a projective space bundle over S. Conversely, a projective bundle over S is the projectivization of an α-twisted locally free sheaf. A class $\alpha \in \mathrm{Br}(S)$ also defines a Hodge substructure $T_\alpha(S)$ of the transcendental lattice $T(S)$.

The two-torsion subgroup $\mathrm{Br}(S)_2$ allows one to describe the conic bundles over S that are the exceptional divisors in $K3^{[2]}$-type hyperkähler manifolds ([vGK23]). In the case that S is a general $K3$ surface of degree two and $\alpha \in \mathrm{Br}(S)_2$ is a non-trivial class, the Hodge substructure $T_\alpha(S)$ is Hodge isometric to either the transcendental lattice of a general cubic fourfold with a plane or a general $K3$ surface of degree eight or to neither of these.

In this paper we will be particularly concerned with specializations of a $K3$ surface S and their impact on $\mathrm{Br}(S)_2$. That is, we consider a family of $K3$ surfaces over a disc with general fiber S and special fiber S_{2d}. We focus on the case where the rank of the Picard groups are one and two respectively, see [C02], [MT23] for more general cases. Here the index $2d$ refers to the degree of the generator of $\mathrm{Pic}(S)$. There are then natural identifications of the second cohomology groups of the general and the special fiber. This easily implies that there is a restriction map of Brauer groups from $\mathrm{Br}(S)_2$ to $\mathrm{Br}(S_{2d})_2$ which has a kernel of order two. The generator of this kernel is called the vanishing Brauer class (of the specialization) and we denote it by

$\alpha_{van} \in \mathrm{Br}(S)_2$,

$$\langle \alpha_{van} \rangle \;=\; \ker \big(\mathrm{Br}(S)_2 \;\longrightarrow\; \mathrm{Br}(S_{2d})_2 \big) \;.$$

We work out the invariants of this Brauer class. In the case of a $K3$ of degree two these invariants determine whether the Brauer class corresponds to a point of order two in $J(C)$, where C is the ramification curve defined by S, or to an even or odd theta characteristic on C. These results were in in a sense anticipated in the papers [IOOV17], [Sk17] where the restriction map $\mathrm{Pic}(S) \to \mathrm{Pic}(C)$ is studied in relation to $\mathrm{Br}(S)_2$.

In the remainder of the introduction we discuss an application of vanishing Brauer classes to cubic fourfolds. A well-known conjecture states that a (complex) cubic fourfold X is rational if and only if it has an associated $K3$ surface S, that is the transcendental lattice $T(X)$ of X is Hodge isometric to $T(S)(-1)$, the transcendental lattice of S with the opposite intersection form ([Has00],[Ku10],[AdTh14]). If it exists, S is called a $K3$ surface associated to X. The general cubic fourfold X' does not have an associated $K3$ surface since $T(X')$ then has rank 22 whereas $T(S)$ has rank at most 21.

A much studied case is the one of a cubic fourfold X containing a plane P. In that case X defines a $K3$ double plane $S = S_X$ with an odd theta characteristic, corresponding to a Brauer class $\alpha_X \in \mathrm{Br}(S_X)_2$. In case (X,P) is general, with group of codimension two cycles generated by the square of the hyperplane class and P, the $K3$ surface is *not* an associated $K3$ surface since $T(X) \not\cong T(S_X)(-1)$. Instead there is a Hodge isometry $T(X) \cong T_{\alpha_X}(S_X)(-1)$, where $T_{\alpha_X}(S_X)$ is the index two sublattice defined by the class α_X ([Vo86]).

We consider now a specialization of a general (X,P) to a fourfold where the group of algebraic codimension two cycles $N^2(X)$ has rank three. These rank three lattices have been classified and they are isomorphic to lattices $M_{\tau,n}$ for a pair (τ,n) of integers, with $\tau \in \{0,\ldots,4\}$, $n \geq 2$, with a few cases that actually do not occur ([YY23]). We denote the specialization of X by $X_{\tau,n}$ and the double plane it defines by $S_{\tau,n}$. The (τ,n) such that $X_{\tau,n}$ has an associated $K3$ surface are given in ([YY23, Cor. 8.14]).

The $K3$ surface $S_{\tau,n}$ is a specializiation of S_X with Picard rank of $S_{\tau,n}$ equal to two, hence this specialization defines a vanishing Brauer class $\alpha_{van} \in \mathrm{Br}(S_X)_2$. In $\mathrm{Br}(S_X)_2$ we now have two Brauer classes, α_X and α_{van}. A complete description of the specialization from S_X to $S_{\tau,n}$ requires taking into account not only the Picard lattice of $S_{\tau,n}$ and invariants of α_{van} but also the relation between α_X and α_{van}. Our main application of vanishing Brauer classes, Theorem 4.4.1, gives all this information. The rather long proof consists of explicit computations with lattices.

A well-known case that we recover is the case that $\alpha_X = \alpha_{van}$ which was studied by Hassett ([Has99]). Then $\tau = 1,3$ and $S_{\tau,n}$ *is* a $K3$ surface associated to $X_{\tau,n}$, and these are the only cases in which $S_{\tau,n}$ is associated to $X_{\tau,n}$. The quadratic surface bundle over $\mathbb{P}^2$ defined by $(X_{\tau,n},P)$ then

has a rational section and this implies that it is a rational fourfold. Since this fourfold is birational to $X_{\tau,n}$, also $X_{\tau,n}$ is rational, thus verifying the conjecture.
In case $\tau = 0, 4$, the vanishing Brauer class corresponds to a point of order two on the ramification curve C of the double plane S_X. The sum $\beta_X := \alpha_X + \alpha_{van}$ corresponds to a theta characteristic on C, which is even if and only if n is odd. A cubic fourfold $X_{\tau,n}$ has an associated $K3$ surface if and only if n is odd. So it is of some interest to have a concrete description of this associated $K3$ surface. Let $S_{\tau,n}$ be the $K3$ double plane defined by $X_{\tau,n}$ and let $C_{\tau,n}$ be the branch curve of the double cover $S_{\tau,n} \to \mathbb{P}^2$. Let β be the even theta characteristic on the branch curve $C_{\tau,n}$ which is the specialization of β_X on C. Then β defines a $K3$ surface S_β which has a natural degree eight polarization and there is a Hodge isometry $T(S_\beta) \cong T_\beta(S_{\tau,n})$. In Proposition 5.1.4 we show that S_β is a $K3$ surface associated to $X_{\tau,n}$. We also discuss the example in [ABBV14], which has $(\tau, n) = (4, 5)$, in our context in the Section 5.2.

1. Brauer Groups of $K3$ surfaces

1.2. **Brauer classes and B-fields.** Let S be a $K3$ surface. The *Brauer group* of S is (cf. [Huy16, 18.1])

$$\mathrm{Br}(S) = H^2(S, \mathcal{O}_S^*)_{tors} \ .$$

The exponential sequence in this case gives

$$0 \longrightarrow H^2(S,\mathbb{Z})/\mathrm{Pic}(S) \longrightarrow H^2(S,\mathcal{O}_S) \longrightarrow H^2(S,\mathcal{O}_S^*) \longrightarrow 0 \ .$$

A two-torsion class $\alpha \in \mathrm{Br}(S)_2$ has a lift $\tilde{\alpha}$ to the one dimensional complex vector space $H^2(S,\mathcal{O}_S)$ with $2\tilde{\alpha} \in H^2(S,\mathbb{Z})/\mathrm{Pic}(S)$. Any class $B = B_\alpha \in \frac{1}{2}H^2(S,\mathbb{Z}) \subset H^2(S,\mathbb{Q})$ mapping to $\tilde{\alpha}$ is called a B-field representative of α (see [HuySt05]). A B-field B_α is unique up to $(1/2)\mathrm{Pic}(S) + H^2(S,\mathbb{Z})$:

$$B'_\alpha = B_\alpha + \frac{1}{2}p + c, \qquad p \in \mathrm{Pic}(S), \quad c \in H^2(S,\mathbb{Z}) \ ,$$

(see [Huy05, §4], see [Ku10, §6]). Assume now that S is a general polarized $K3$ surface. There is the following

Lemma 1.2.1. *(*[Ku10, Lemma 6.1], [vGK23, Lemma 2.1]*) Let S be a $K3$ surface such that $\mathrm{Pic}(S) = \mathbb{Z}h$, $h^2 = 2d > 0$. Let $\alpha \in \mathrm{Br}(S)_2$ and $B_\alpha \in \frac{1}{2}H^2(S,\mathbb{Z}) \subset H^2(S,\mathbb{Q})$ a B-field representing α. The intersection numbers*

(1) $B_\alpha h \mod \mathbb{Z}$,
(2) $B_\alpha^2 \mod \mathbb{Z}$, *only in the case that* $4B_\alpha h + h^2 \equiv 0 \mod 4$,

are invariants of α.

1.2. **Brauer Groups and $K3$ Lattices.** There is an isomorphism, with $\rho(S)$ the Picard number of S,

$$\mathrm{Br}(S) \cong \big(H^2(S,\mathbb{Z})/\mathrm{Pic}(S)\big) \otimes \mathbb{Q}/\mathbb{Z} \;\cong\; (\mathbb{Q}/\mathbb{Z})^{22-\rho(S)} \;.$$

The lattice $H^2(S,\mathbb{Z})$ is selfdual since it has a unimodular intersection form. A class $\alpha \in \mathrm{Br}(S)_2$ can thus be identified with a homomorphism

$$\alpha : T(S) \longrightarrow \mathbb{Z}/2\mathbb{Z} \;.$$

If B_α represents α, this homomorphism is given by

$$\alpha : x \longmapsto x \cdot B_\alpha \mod \mathbb{Z} \quad (\text{in } \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}) \;.$$

The class α defines a sublattice $T_\alpha(S) := \ker(\alpha)$ in $T(S)$. For a $K3$ surface with Picard rank one the invariants of α given in Lemma 1.2.1 are invariants of the lattice $T_\alpha(S)$. In fact, index two sublattices T_α, T_β of $T(S)$ are isometric if and only if α and β have the same invariants ([vGK23, Thm.2.3]).

2. Vanishing Brauer Classes and Invariants

Definition 2.1.1. Let (S,h) be a general polarized $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z}h$, $h^2 = 2d > 0$. Consider a specialization S_{2d} of S where the Picard rank of S_{2d} is two, so $\mathrm{Pic}(S_{2d}) = \mathbb{Z}h \oplus \mathbb{Z}k$, for some divisor class k which is primitive in $H^2(S_{2d},\mathbb{Z})$. (By a specialization of S we mean a family of quasi-polarized $K3$'s over a complex disc Δ such that the fiber over $0 \in \Delta$ is S_{2d} and the fiber over some non-zero $a \in \Delta$ is S). We may then identify

$$H^2(S,\mathbb{Z}) \;=\; H^2(S_{2d},\mathbb{Z}) \;,$$

and we have inclusions

$$\mathrm{Pic}(S) \subset \mathrm{Pic}(S_{2d}), \qquad T(S) \supset T(S_{2d}).$$

Thus there is a restriction map $\mathrm{Br}(S) \to \mathrm{Br}(S_{2d})$ given by restriction of the homomorphism α to $T(S_{2d})$. Since $\mathrm{Br}(S)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{21}$ and $\mathrm{Br}(S_{2d})_2 \cong (\mathbb{Z}/2\mathbb{Z})^{20}$, there is a unique order two Brauer class that becomes trivial in $\mathrm{Br}(S_{2d})$, that is, α generates the kernel of the restriction map. This class α_{van} is the *vanishing Brauer class* (in this specialization).

Proposition 2.1.2. *A B-field representative of α_{van} is provided by $B = k/2$ $(\in \frac{1}{2}H^2(S,\mathbb{Z}))$.*

Proof. Since $k/2 \notin (1/2)\mathrm{Pic}(S) + H^2(S,\mathbb{Z})$, it defines a non-trivial class in $\mathrm{Br}(S)_2$. On the other hand, obviously $k/2 \in (1/2)\mathrm{Pic}(S_{2d})$, hence $k/2$ defines the trivial class in $\mathrm{Br}(S_{2d})_2$. Therefore $B = k/2$ is a B-field representative of α_{van}. $\square$

This allows us to read off the invariants of α_{van} from the intersection matrix of $\mathrm{Pic}(S_{2d}) = \mathbb{Z}h \oplus \mathbb{Z}k$ which can be written as

$$\begin{pmatrix} h^2 & hk \\ hk & k^2 \end{pmatrix} = \begin{pmatrix} 2d & b \\ b & 2c \end{pmatrix} \quad \text{for some } \; b, c \in \mathbb{Z} \;.$$

Using the B-field representative $k/2$ of α_{van} given in Proposition 2.1.2 one finds the corollary below.

Corollary 2.1.3. *The invariants of $\alpha_{van} \in \mathrm{Br}(S)$ are*

$$B_{van}h \equiv (1/2)b \mod \mathbb{Z}, \quad B_{van}^2 \equiv (1/2)c \mod \mathbb{Z},$$

where B_{van} is any B-field representing α_{van}. In particular, $2B_{van}h \equiv \mathrm{disc}(\mathrm{Pic}(S_{2d}))$ mod 2.

2.2. **Vanishing Brauer Classes of Double Planes.** We now consider the case where we specialize a double plane S, that is a double cover of the plane branched over a smooth sextic curve C_6 with Picard rank one, to a $K3$ surface S_2. In this case

$$\mathrm{Pic}(S_2) = \left(\mathbb{Z}h \oplus \mathbb{Z}k, \begin{pmatrix} 2 & b \\ b & 2c \end{pmatrix}\right) . \tag{1}$$

The change of basis which fixes h and maps $k \mapsto k - mh$ where $b = 2m, 2m+1$, shows that we may assume $b = 0, 1$.
The two-torsion Brauer classes on a double plane with Picard rank one correspond to the points of order two and the theta characteristics on the genus 10 branch curve C_6. We find the following characterization of the class α_{van} in terms of these line bundles on C_6.

Proposition 2.2.1. *Let (S, h) be a general double plane specializing to S_2 with $\mathrm{Pic}(S_2) = \mathbb{Z}h \oplus \mathbb{Z}k$ and intersection matrix of the form (1). Then,*

(1) *If $B_{\alpha_{van}}h \equiv 0$ (b is even), α_{van} corresponds to a point of order two $p \in Jac(C_6)$.*
(2) *If $B_{\alpha_{van}}h \equiv \frac{1}{2}$ and $B_{\alpha_{van}}^2 \equiv \frac{1}{2}$, ($b, c$ odd), α_{van} corresponds to an odd theta characteristic in C_6.*
(3) *If $B_{\alpha_{van}}h \equiv \frac{1}{2}$ and $B_{\alpha_{van}}^2 \equiv 0$ (b odd and c even), α_{van} corresponds to an even theta characteristic in C_6.*

Proof. This follows from Corollary 2.1.3 and [vG05], [IOOV17]. Notice that [IOOV17, Theorem 1.1] shows that the vanishing Brauer class is obtained from the restriction of a line bundle on S to the ramification curve $C_6 \subset \mathbb{P}^2$. $\square$

3. Cubic fourfolds containing a plane

3.1. **Cubics with a plane and $K3$ double planes.** Let X be a smooth cubic hypersurface in $\mathbb{P}^5(\mathbb{C})$ containing a plane P. Consider the projection from the plane P onto a plane in $\mathbb{P}^5$ disjoint from P. Blowing up X along P, one obtains a quadric surface bundle $\pi : Y \longrightarrow \mathbb{P}^2$. The rulings of the quadrics define a double cover $S = S_X$ of $\mathbb{P}^2$ branched over a degree six curve C_6, the discriminant sextic. If X does not contain a second plane intersecting P, the curve C_6 smooth and S is a $K3$ surface (see [Vo86, §1 Lemme 2]).

$$\begin{tikzcd} Y = Bl_P(X) \arrow[r] \arrow[dr, "q"] & X \supset P \arrow[d, dotted] \\ C_6 \arrow[r, hook, "i"] & \mathbb{P}^2 \end{tikzcd} \tag{2}$$

The rulings of the quadrics of the bundle also define a $\mathbb{P}^1$-bundle F over S which gives a Brauer class $\alpha_X \in \mathrm{Br}(S)_2$, also known as the Clifford class. This class α_X corresponds to an odd theta characteristic L on C_6 with $h^0(L) = 1$ (see [Vo86, §2]).
Conversely, a smooth plane sextic with such an odd theta characteristic defines a cubic fourfold with a plane which is obtained as in loc. cit. and also from the minimal resolution of the push-forward of L to $\mathbb{P}^2$ as in [Be00].

3.2. **Lattices.** The cohomology group $H^4(X, \mathbb{Z})$ with the intersection form is a rank 23 odd, unimodular, lattice of signature $(2+, 21-)$. It is also a Hodge structure with Hodge numbers $h^{3,1} = 1$, $h^{2,2} = 21$. Let $h_3 \in H^2(X, \mathbb{Z})$ be the class of a hyperplane section and let $h_3^2 \in H^4(X, \mathbb{Z})$ be its square. Denote with $N^2(X) \subset H^4(X, \mathbb{Z})$ the odd, positive definite, lattice of classes of codimension two algebraic cycles. The transcendental lattice of X is the even lattice defined as

$$T(X) := N^2(X)^\perp \subset H^4(X, \mathbb{Z}) .$$

The following proposition follows from [Vo86].

Proposition 3.2.1. *Let X be a smooth cubic fourfold with a plane and let S be the $K3$ double plane defined by X. Then there is a Hodge isometry:*

$$T(X) \cong T_{\alpha_X}(S)(-1)$$

with $T_{\alpha_X}(S) := \ker \alpha_X : T(S) \to \mathbb{Z}/2\mathbb{Z}$ (it is a sublattice of index two in $T(S)$ if α_X is non-trivial). It follows that

$$rank(N^2(X)) = rank(Pic(S)) + 1.$$

The general cubic fourfold X with a plane P has

$$N^2(X) = \left(\mathbb{Z}h_3^2 \oplus \mathbb{Z}P,\ K_8 := \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \right) .$$

4. Noether-Lefschetz divisors in $\mathcal{C}_8$ and double planes

4.1. **The divisors $\mathcal{C}_d$ in $\mathcal{C}$.** Hassett determined all Noether-Lefschetz divisors in the moduli space $\mathcal{C}$ of cubic fourfolds. These (irreducible) divisors are denoted by $\mathcal{C}_d$ and $d > 6$, $d \equiv 0, 2 \mod 6$. The divisor $\mathcal{C}_d$ parametrizes fourfolds X with a certain rank two sublattice, containing h_3^2, denoted by $K_d \subset N^2(X)$ where $d = \mathrm{disc}(K_d)$. Thus $\mathcal{C}_8$ parametrizes the cubic fourfolds with a plane since then $K_8 \subset N^2(X)$.

4.2. **The divisors $\mathcal{C}_M$ in $\mathcal{C}_8$.** Yang and Yu give a classification of all Noether-Lefschetz divisors in $\mathcal{C}_8$, that is the divisors that parametrize cubics X with a plane and $\mathrm{rank} N^2(X) > 2$. They correspond to positive definite saturated sublattices of rank three $M \subset H^4(X, \mathbb{Z})$ with $K_8 \subset M$, up to isometry. Denote by $\mathcal{C}_M \subset \mathcal{C}_8$ the divisor of smooth cubic fourfolds X with such an isometry class of embeddings $M \hookrightarrow H^4(X, \mathbb{Z})$.

Proposition 4.2.1. *(*[YY23, Corollary 8.14]*) Consider the pairs of integers (τ, n) such that $\tau = 0, ..., 4$, $n \geq 2$ and $(\tau, n) \neq (3, 2), (4, 2), (4, 3)$. Let $M_{\tau,n}$ be the rank three positive definite lattice with intersection matrix given by*

$$A_{\tau,n} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & \tau \\ 0 & \tau & 2n \end{pmatrix}, \qquad M_{\tau,n} := (\mathbb{Z}^3, A_{\tau,n}) .$$

(1) *If M is a positive definite rank 3 lattice such that $K_8 \subset M$ and such that M has a saturated embedding in $H^4(X, \mathbb{Z})$ then $M \cong M_{\tau,n}$ with (τ, n) as above.*
(2) *Up to isometry, there is a unique embedding $M_{\tau,n} \hookrightarrow L \cong H^4(X, \mathbb{Z})$ such that the first basis vector maps to h_3^2.*
(3) *The divisor $\mathcal{C}_{M_{\tau,n}}$ in $\mathcal{C}_8$ is non-empty and irreducible. Moreover $\mathcal{C}_{M_{\tau,n}} = \mathcal{C}_{M_{\tau',n'}}$ if and only if $(\tau, n) = (\tau', n')$.*

4.3. **The group $(\mathrm{Pic}(C)/\langle K_C \rangle)_2$.** Let C be a smooth curve of genus g. Recall that $x \in \mathrm{Pic}(C)$ is a two-torsion point if $2x = 0$ and it is a theta characteristic if $2x = K_C$, the canonical class of C. A theta characteristic L is called even/odd if $h^0(L)$ is even/odd, there are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd theta characteristics. The parity of a theta characteristic does not change under a deformation of C.
The two-torsion points are a group, denoted by $J(C)_2$ ($= \mathrm{Pic}(C)_2$); the sum $p + L$ of a point of order two p with a theta characteristic L is again a theta characteristic, but the parity may change; the sum $L + M$ of two theta characteristics can be written as $K_C + p$ for a unique two-torsion point p. The union of the sets of two torsion points and theta characteristics thus has a group structure, this group can be identified with the two-torsion group $(\mathrm{Pic}(C)/\langle K_C \rangle)_2$, which has order 2^{2g+1}. For any double plane S with smooth branch curve C, there is a surjective map $\mathcal{A} : (\mathrm{Pic}(C)/\langle K_C \rangle)_2 \to \mathrm{Br}(S)_2$ which is thus an isomorphism if $\mathrm{rank}(\mathrm{Pic}(S)) = 1$ ([IOOV17, Thm 1.1]). The kernel of this map is given by restrictions of certain line bundles on S to C.

Let $\mathcal{U} \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(6)) \cong \mathbb{P}^{27}$ be the open subset whose points define smooth sextic curves, such a curve has genus 10. There is a finite unramified covering $\tilde{\mathcal{U}} \to \mathcal{U}$ of degree 2^{21} whose fiber over C is the group $(\mathrm{Pic}(C)/\langle K_C \rangle)_2$. It follows from [Be86] that this covering has four connected components defined by the subsets: $\{0\}$, $J(C)_2$ and the sets of even theta and odd theta characteristics respectively. In particular, the monodromy group of this covering, which is $Sp(20,\mathbb{F}_2)$, acts transitively on the odd theta characteristics.

Lemma 4.3.1. *The stabilizer of an odd theta characteristic α is an orthogonal subgroup $O^-(20,\mathbb{F}_2)\subset Sp(20,\mathbb{F}_2)$. The orbits of this stabilizer on a fiber are well known:*

(1) $\{0\}$,
(2) $\{p\in J(C)_2-\{0\}:\ p+\alpha \textit{ is odd}\}$,
(3) $\{p\in J(C)_2-\{0\}:\ p+\alpha \textit{ is even}\}$,
(4) $\{\alpha\}$,
(5) $\{$*odd theta characteristics distinct from* $\alpha\}$,
(6) $\{$*the even theta characteristics*$\}$.

Proof. This can be deduced for example from the results of Igusa [Ig72, V.6]. The group $J(C)_2$, with the Weil pairing, can be identified with his P and the bilinear alternating map $e:P\times P\to\{\pm1\}$. For a theta characteristic L on C define the quadratic form $q_L:J(C_2)\to\mathbb{F}_2$ by $q_L(p):=h^0(L+p)-h^0(L)$ mod 2 ([Har82, Theorem 1.13]). The theta characteristics are then identified with the set T of maps $c:P\to\mu$ such that $c(r+s)=c(r)c(s)e(r,s)$ where $c(r)=(-1)^{q_L(r)}$.
Then [Ig72, Corollary p. 213] states that the symplectic group defined by e on P is doubly transitive on both the even and on the odd theta characteristics. Thus the stabilizer of an odd theta characteristic is transitive on the set of the remaining odd theta characteristics. From [Ig72, Proposition 2] one deduces that the stabilizer of an odd theta characteristic is transitive on the even theta characteristics. Since the second and third orbits (in $J(C)_2$) are in bijection with the fifth and sixth orbits (in T) respectively, the Lemma follows. □

4.4. **The specializations of X and S.** Let X be a cubic fourfold with a plane with $N^2(X)=K_8$, which has rank two. Let S be the $K3$ double plane defined by (X,P) with branch locus C and Brauer class α_X which we identify with an odd theta characteristic on C.
We specialize X to $X_{\tau,n}$ when $N^2(X_{\tau,n})=M_{\tau,n}$, which has rank three. Let $S_2=S_{\tau,n}$ be the $K3$ double plane defined by $X_{\tau,n}$, it has Picard rank two. The specialization of cubic fourfolds defines a specialization of $K3$ surface S to $S_{\tau,n}$. Hence it defines a vanishing Brauer class $\alpha_{van}\in\mathrm{Br}(S)$. This vanishing Brauer class is non-trivial and thus lies in exactly one of the orbits $(2)\dots(6)$ of the stabilizer of α_X.
In the following theorem we determine the Picard lattice of the specialization $S_{\tau,n}=S_2$ of S and we also determine the orbit of α_{van}.

Theorem 4.4.1. *Let X be a general cubic fourfold with a plane, so with $\mathrm{rank}\,N^2(X)=2$. Let S be the $K3$ double plane defined by X, let C be* *the branch curve and let* $\alpha_X \in \mathrm{Br}(S)_2$ *be the Clifford class. Let* $X_{\tau,n}$ *be a specialization of* X *such that*

$$N^2(X) \cong M_{\tau,n} .$$

Let $S_{\tau,n}$ *be the* $K3$ *double plane defined by* $X_{\tau,n}$*. Then the specialization of* $K3$ *double planes from* S *to* $S_2 = S_{\tau,n}$ *has the following properties.*

(1) $\tau = 0$ *: The Picard lattice of* $S_{0,n}$ *is*

$$\mathrm{Pic}(S_{0,n}) \cong \begin{pmatrix} 2 & 0 \\ 0 & -2n \end{pmatrix} .$$

The Brauer class α_{van} *corresponds to a point of order two* $p \in Jac(C_6)$*. Moreover, the theta characteristic* $p + \alpha_X$ *is even/odd exactly when* n *is odd/even,*

(2) $\tau = 1$ *: The Picard lattice of* $S_{1,n}$ *is*

$$\mathrm{Pic}(S_{1,n}) \cong \begin{pmatrix} 2 & 1 \\ 1 & 2-8n \end{pmatrix} .$$

Moreover $\alpha_X = \alpha_{van}$*, so these two classes coincide.*

(3) $\tau = 2$ *: The Picard lattice of* $S_{2,n}$ *is*

$$\mathrm{Pic}(S_{2,n}) \cong \begin{pmatrix} 2 & 1 \\ 1 & 2-2n \end{pmatrix} .$$

Moreover, $\alpha_X \neq \alpha_{van}$ *and* α_{van} *corresponds to a theta characteristic which is even/odd when* n *is odd/even.*

(4) $\tau = 3$*: The Picard lattice of* $S_{3,n}$ *is*

$$\mathrm{Pic}(S_{3,n}) \cong \begin{pmatrix} 2 & 1 \\ 1 & 14-8n \end{pmatrix} .$$

Moreover, $\alpha_X = \alpha_{van}$*, so the two classes coincide.*

(5) $\tau = 4$ *:The Picard lattice of* $S_{4,n}$ *is*

$$\mathrm{Pic}(S_{4,n}) \cong \begin{pmatrix} 2 & 0 \\ 0 & 6-2n \end{pmatrix} .$$

The Brauer class α_{van} *corresponds to a point of order two* $p \in Jac(C_6)$*. The theta characteristic* $p + \alpha_X$ *is even/odd when* n *is odd/even.*

4.5. **Remark.** Consider a $K3$ double plane with a Picard lattice $diag(2, 2c)$ for some $c < 0$. Choose an odd theta characteristic with $h^0 = 1$ on the branch curve and let N^2 be the rank three lattice of algebraic codimension two cycles on the associated cubic fourfold. Then the theorem above shows that N^2 is isometric to either $M_{0,c}$ or $M_{4,3-c}$. One needs information on the orbit of the vanishing Brauer class of the specialization of a general double plane with Picard rank one to the $K3$ under consideration to determine which of the two is the correct one.

4.6. **The proof of the main result.** The remainder of this section is devoted to the proof of Theorem 4.4.1. First of all, for an $\alpha \in \mathrm{Br}(S)_2$ corresponding to an odd theta characteristic, we work out an explicit inclusion $T_\alpha(S) \subset \Lambda$ where Λ is a lattice isometric to $H^2(S,\mathbb{Z})$ in 4.7. Let X be the cubic fourfold with a plane such that $\alpha_X = \alpha$. There is an isometry $T(X)(-1) \cong T_\alpha(S)$. Therefore $K_8 \oplus T_\alpha(S)(-1)$ has an overlattice $L \cong H^4(X,\mathbb{Z})$, which is in fact unique. We determine L explicitly in 4.8. Next, for τ, n as in Proposition 4.2.1, we choose an explicit primitive embedding $M_{\tau,n} \subset L$, compatible with $K_8 \hookrightarrow L$. Then the perpendicular $M_{\tau,n}^\perp$ in L is isometric to $T(X_{\tau,n}) = T_\alpha(S_{\tau,n})(-1)$. Since this lattice is contained in $T(X) = T_\alpha(S)(-1) \subset \Lambda(-1)$, we have found the sublattice $T_\alpha(S_{\tau,n}) \subset T_\alpha(S) \subset \Lambda$. In the diagram below, the lattices in the first row are in $\Lambda \cong H^2(S,\mathbb{Z})$, those in the second row are in $L \cong H^4(X,\mathbb{Z})$.

$$\begin{array}{ccccc} T(S) \supset T(S_\alpha) & & T_\alpha(S_{\tau,n}) \subseteq T(S_{\tau,n}) & & \subset \Lambda \\ \cong & & \cong & & \\ T(X)(-1) & \supset M_{\tau,n}(-1)^\perp = & T(X_{\tau,n})(-1) & & \subset L \end{array}$$

The perpendicular of $T_\alpha(S_{\tau,n})$, and also of $T(S_{\tau,n})$, in Λ is then $\mathrm{Pic}(S_{\tau,n})$. Finally we determine the vanishing Brauer class and the orbit it lies in.

4.7. **The lattice $T_\alpha(S) \subset \Lambda$.** Let (S,h) be a $K3$ surface of degree 2 with $\mathrm{Pic}(S) = \mathbb{Z}h$ and let $\alpha \in \mathrm{Br}(S)_2$ be a Brauer class defined by an odd theta characteristic on C_6, the branch curve of $\phi_h : S \to \mathbb{P}^2$.

There is an isomorphism

$$H^2(S,\mathbb{Z}) \xrightarrow{\cong} \Lambda := U^3 \oplus E_8(-1)^2 = U \oplus U \oplus \Lambda'$$

where $U = \left(\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$. Under the isomorphism, we may assume that

$$h = \left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0\right), \qquad B = B_\alpha = (1/2)\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), 0\right),$$

here we use that any two odd theta characteristics are in the same orbit of the monodromy group and that for the B-field representative B of an odd theta characteristic one has $Bh = B^2 = 1/2$.

The transcendental lattice of S is then

$$T(S) = h^\perp = \left\langle \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \right\rangle \oplus U \oplus \Lambda'.$$

The Brauer class α corresponds to the homomorphism, again denoted by α:

$$\alpha : T(S) \longrightarrow \mathbb{Z}/2\mathbb{Z}, \qquad t \longmapsto (t, 2B) \mod 2.$$

Since the image of $((p,-p),(q,r),v) \in T(S)$ is $p+q+r \mod 2$, the index two (non-primitive) sublattice $T_\alpha(S) = \ker(\alpha)$ of $T(S)$ is

$$T_\alpha := T_\alpha(S) := \ker(\alpha) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle \oplus \Lambda',$$

where

$$\gamma_1 := \left(\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), 0\right), \qquad \gamma_2 := \left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), 0\right), \qquad \gamma_3 := \left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right), 0\right).$$

Hence T_α is the lattice

$$T_\alpha(S) \;=\; \left(\oplus_{i=1}^{3}\mathbb{Z}\gamma_i,\ \begin{pmatrix}-2 & 1 & 0\\ 1 & 2 & 2\\ 0 & 2 & 0\end{pmatrix}\right) \oplus \Lambda' \;.$$

To glue $T_\alpha(S)$ to K_8 we need to know its discriminant group. Let

$$\gamma_\alpha^* \;:=\; \tfrac{1}{8}(2\gamma_1 + 4\gamma_2 - 5\gamma_3) \;=\; \tfrac{1}{8}\left(\left(\begin{smallmatrix}2\\-2\end{smallmatrix}\right), \left(\begin{smallmatrix}4\\-4\end{smallmatrix}\right), 0\right) \;.$$

Since $\det(T_\alpha) = 8$, the discriminant group of T_α has order eight. Notice that $(\gamma_\alpha^*, \sum a_i\gamma_i) = a_3$ and thus $\gamma_\alpha^* \in T_\alpha^*$, the dual lattice. Since γ_α^* has order eight in the discriminant group T_α^*/T_α, we conclude that it is a generator. So the discriminant group is cyclic of order 8 and $(\gamma_\alpha^*, \gamma_\alpha^*) = -5/8$.

4.8. **The overlattice L of $K_8 \oplus T_\alpha(S)(-1)$.** A general cubic fourfold X with a plane has $N^2(X) \cong K_8$ where

$$K_8 \;:=\; \left(\mathbb{Z}h_3^2 \oplus \mathbb{Z}P,\ \begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\right), \qquad \text{let} \quad \gamma_8^* \;=\; (1/8)(3h_3^2 - P) \;.$$

Notice that the intersection form on K_8 has determinant 8 and, since $(\gamma_8^*, ah_3 + bP) = b$, the discriminant group is generated by γ_8^*. As $(\gamma_8^*, \gamma_8^*) = 3/8$ and $(\gamma_\alpha^*, \gamma_\alpha^*) = 5/8$ in $T_\alpha(-1)_{\mathbb{Q}}$ we can glue the lattices K_8 and $T_\alpha(-1)$ by adding $\gamma_8^* + \gamma_\alpha^*$ to their direct sum. Let

$$L \;:=\; \mathbb{Z}(\gamma_8^* + \gamma_\alpha^*) \;+\; \big(K_8 \oplus T_\alpha(S)(-1)\big) \;,$$

it is a unimodular overlattice of $K_8 \oplus T_\alpha(S)(-1)$ and $(\gamma_8^* + \gamma_\alpha^*)^2 = 3/8 + 5/8 = 1$. The lattice L is well-known to be unique, with unique sublattice K_8, all up to isometry. The lattice L is odd (because K_8 is an odd sublattice) and has signature $(2+19, 2)$. As it is unimodular, it must be isometric to $<1>^{21} \oplus <-1>^2 \cong H^4(X, \mathbb{Z})$.

For our computations it is convenient to write L as

$$L \;=\; \mathbb{Z}h_3 \oplus \mathbb{Z}P \oplus \mathbb{Z}(\gamma_8^* + \gamma_\alpha^*) \oplus \mathbb{Z}\gamma_1 \oplus \mathbb{Z}(\gamma_2 - \gamma_3) \oplus \Lambda'(-1) \;.$$

Since we work in L, the sublattice $T(X) = T_\alpha(S)(-1)$ which has opposite intersection form. The Gram matrix of the first five summands is

$$M \;:=\; \begin{pmatrix}3 & 1 & 1 & 0 & 0\\ 1 & 3 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 2 & -1\\ 0 & 0 & 1 & -1 & 2\end{pmatrix}, \qquad \text{let} \quad S \;:=\; \begin{pmatrix}1 & 0 & 0 & 0 & 2\\ -1 & 0 & 0 & 0 & -1\\ 3 & -1 & -1 & 1 & -6\\ -1 & 1 & 0 & 0 & 2\\ -2 & 1 & 1 & 0 & 4\end{pmatrix} \;.$$

Then we have

$${}^tSMS \;=\; \mathrm{diag}(1,1,1,1,-1), \quad \text{hence} \quad L \cong <1>^{\oplus 4} \;\oplus\; <-1> \oplus \Lambda'(-1) \;.$$

Finally we observe that $h_3^2 = 3$ and $(h_3^2)^\perp$ is an even lattice, which is all that is required of the class that corresponds to the square of the hyperplane section. The embedding $K_8 \hookrightarrow H^2(X, \mathbb{Z})$ is unique up to isometry and thus we may assume that the second generator corresponds to the class of a plane P in X.

4.9. **An embedding $M_{\tau,n} \hookrightarrow L$.** For each $\tau = 0,1,2,3,4$ we explicitly find an $m_{\tau,n} \in L$ such that

$$h_3^2 m_{\tau,n} \,=\, 0, \quad P m_{\tau,n} \,=\tau, \quad m_{\tau,n}^2 \,=\, 2n,$$

and such that, with the lattice $M_{\tau,n}$ defined in Proposition 4.2.1,

$$M_{\tau,n} \xrightarrow{\cong} \langle\, h_3, P, m_{\tau,n}\,\rangle \subset\, (\mathbb{Z}^5, M) \,\oplus\, \Lambda'(-1) \,=\, L$$

is a saturated embedding.
We use the basis of $\mathbb{Z}^5$ as above so that

$$(a_1,\ldots,a_5) \,=\, a_1 h_3 + a_2 P + a_3(\gamma_8^* + \gamma_\alpha^*) + a_4\gamma_1 + a_5(\gamma_2 - \gamma_3).$$

In the list below $v_{\tau,n} \in \Lambda'(-1)$ is such that $m_{\tau,n}^2 = 2n$, such a $v_{\tau,n}$ exists for any n,τ as one can choose a suitable $v_{\tau,n}$ in the summand $U \subset \Lambda'$.

$$\begin{array}{lcllcccl}
m_{0,n} & = & (0,0,0,1,0) + v_{0,n} & 2n & = & m_{0,n}^2 & = & 2 + v_{0,n}^2\ , \\
m_{1,n} & = & (1,0,-3,1,1) + v_{1,n} & 2n & = & m_{1,n}^2 & = & 2 + v_{1,n}^2\ , \\
m_{2,n} & = & (2,0,-6,1,3) + v_{2,n} & 2n & = & m_{2,n}^2 & = & 2 + v_{2,n}^2\ , \\
m_{3,n} & = & (0,1,-1,1,1) + v_{3,n} & 2n & = & m_{3,n}^2 & = & 4 + v_{3,n}^2\ , \\
m_{4,n} & = & (1,1,-4,1,2) + v_{4,n} & 2n & = & m_{4,n}^2 & = & 6 + v_{4,n}^2\ .
\end{array}$$

Since $h_3 = (1,0,0,0,0)$, $P = (0,1,0,0,0)$ and the coefficient of γ_1 in each of these $m_{\tau,n}$ is equal to one, the sublattice generated by $h_3, P, m_{\tau,n}$ is primitive.
Next we determine a primitive vector $t_{\tau,n} \in L$ (unique up to sign) such that

$$A_{\tau,n} \cap K_8^\perp \,=\, A_{\tau,n} \cap \langle h_3, P\rangle^\perp \,=\, \mathbb{Z} t_{\tau,n}\ .$$

We found:

$$\begin{array}{lclcr}
t_{0,n} & = & 0\cdot h_3 + 0\cdot P + 1\cdot m_{0,n} & = & (0,0,0,1,0) + v_{0,n}\ , \\
t_{1,n} & = & 1\cdot h_3 - 3\cdot P + 8\cdot m_{1,n} & = & (9,-3,-24,8,8) + 8v_{1,n}\ , \\
t_{2,n} & = & 1\cdot h_3 - 3\cdot P + 4\cdot m_{2,n} & = & (9,-3,-24,4,12) + 4v_{2,n}\ , \\
t_{3,n} & = & 3\cdot h_3 - 9\cdot P + 8\cdot m_{3,n} & = & (3,-1,-8,8,8) + 8v_{3,n}\ , \\
t_{4,n} & = & 1\cdot h_3 - 3\cdot P + 2\cdot m_{4,n} & = & (3,-1,-8,2,4) + 2v_{4,n}\ .
\end{array}$$

Since $t_{\tau,n} \in K_8^\perp = T_\alpha(-1)$, it is an integral linear combination of $\gamma_1,\ldots,\gamma_3$ and an element in Λ'. As a first step we use that $8\gamma_8^* = 3h_3^2 - P$, from this we see that

$$t_{\tau,n} \,=\, (a_1,\ldots,a_5) \quad\Longrightarrow\quad t_{\tau,n} \,=\, a_3\gamma_8^* + a_4\gamma_1 + a_5(\gamma_2 - \gamma_3)\ .$$

Next we use the definition of γ^*_α, to avoid unnecessary fractions we let $a_3 = 8a'_3$:

$$t_{\tau,n} = (2a'_3 + a_4)\gamma_1 + (4a'_3 + a_5)\gamma_2 + (-5a'_3 - a_5)\gamma_3, \qquad (a_3 = 8a'_3) .$$

Finally we use the definition of the $\gamma_i \in U^2 \oplus \Lambda'$ and notice that the sign of the bilinear form changes again since we are back in $T_\alpha \subset \Lambda$.

$$\begin{array}{rclrcl} t_{0,n} &=& \left(\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), v_{0,n} \right), & v_{0,n}^2 &=& -2n+2, \\ t_{1,n} &=& \left(\left(\begin{smallmatrix} 2 \\ -2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -4 \\ 12 \end{smallmatrix}\right), 8v_{1,n} \right), & v_{1,n}^2 &=& -2n+2, \\ t_{2,n} &=& \left(\left(\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right), 4v_{2,n} \right), & v_{2,n}^2 &=& -2n+2, \\ t_{3,n} &=& \left(\left(\begin{smallmatrix} 6 \\ -6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 \\ 4 \end{smallmatrix}\right), 8v_{3,n} \right), & v_{3,n}^2 &=& -2n+4, \\ t_{4,n} &=& \left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right), 2v_{4,n} \right), & v_{4,n}^2 &=& -2n+6. \end{array}$$

The class $t_{\tau,n}$ is transcendental in $H^2(S,\mathbb{Z})$, but in the specialization under consideration it becomes algebraic, so $t_{\tau,n} \in \mathrm{Pic}(S_{\tau,n})$.

4.10. **The Picard lattice** $\mathrm{Pic}(S_{\tau,n})$ **and the Brauer class** α_{van}**.** We compute the Picard group of $S_{\tau,n}$, the $K3$ surface associated to a cubic fourfold $X_{\tau,n}$ with $N^2(X) = M_{\tau,n}$ as:

$$\mathrm{Pic}(S_{\tau,n}) = \langle h,\, t_{\tau,n}\rangle_{sat},$$

We will do so for each of the five cases for τ in the next sections. We determine the vanishing Brauer class $\alpha_{van} \in \mathrm{Br}(S)$ for the specialization of S to $S_{\tau,n}$ induced by the one of a general cubic fourfold with a plane to one with $N^2(X) = M_{\tau,n}$. The finer classification of α_{van} in terms of the orbits of the stabilizer of the Brauer class (the Clifford invariant) $\alpha = \alpha_X \in \mathrm{Br}(S)$ defined by the cubic fourfold X is also given. Recall from §4.7 that α has B-field representative

$$B_\alpha = \tfrac{1}{2}\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), 0 \right) ,$$

and that $\mathrm{Pic}(S) = \mathbb{Z}h$, with

$$h = \tfrac{1}{2}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0 \right) .$$

4.11. **The case** $\tau = 0$**.** In this case the sublattice generated by $h, t_{0,n}$ is primitive, hence

$$\mathrm{Pic}(S_{0,n}) = \langle h,\, t_{0,n} = \left(\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), v_{0,n} \right) \rangle_{sat} = \langle h, t_{0,n}\rangle = \begin{pmatrix} 2 & 0 \\ 0 & -2n \end{pmatrix} .$$

Notice that $\det(\mathrm{Pic}(S_{0,n})) = -4n$ whereas $\det(A_{0,n}) = 16 \cdot n - 3 \cdot 0^2 = 16n$. The invariants of the vanishing Brauer class are determined from Corollary 2.1.3. The Gram matrix of $\mathrm{Pic}(S_{0,n})$ has $b = 0$, $2c = -2n$. Hence the vanishing Brauer class has invariants $B_{van}h = 0$ and $B_{van}^2 = 0$. By Proposition 2.2.1 it corresponds to a point of order two (as b is even).

A B-field representing the vanishing Brauer class is obtained from the second basis vector of $\mathrm{Pic}(S_{0,n})$:

$$B_{van} := \tfrac{1}{2}t_{0,n} = \tfrac{1}{2}\left(\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), v_{0,n}\right) .$$

Now we use the addition in the Brauer group. The sum of the Clifford class and the vanishing Brauer class has a B-field representative given by

$$B_\alpha + B_{van} = \tfrac{1}{2}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right), v_{0,n}\right) .$$

This B-field has invariant $h \cdot (B_\alpha + B_{van}) = \frac{1}{2}$, so the sum of the Brauer classes corresponds to a theta characteristic. Using $v_{0,n}^2 = -2n+2$ one finds

$$(B_\alpha + B_{van})^2 = \tfrac{1}{4}(0 + 4 + v_{0,n}^2) = \tfrac{1}{2}(-n+3) .$$

Using Proposition 2.2.1 again, we find that the theta characteristic is even when $n \equiv 1 \mod 2$ and it is odd otherwise.

4.12. **The case $\tau = 1$.** The Picard lattice is (notice that $2h+t_{1,n}$ is divisible by 4):

$$\mathrm{Pic}(S_{1,n}) = \langle h,\, t_{1,n} = \left(\left(\begin{smallmatrix}2\\-2\end{smallmatrix}\right), \left(\begin{smallmatrix}-4\\12\end{smallmatrix}\right), 8v_{1,\tau}\right)\rangle_{sat} = \langle h,\, \left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\3\end{smallmatrix}\right), 2v_{1,\tau}\right)\rangle .$$

The last generator has norm $-6 + 4v_{1,\tau}^2 = 2 - 8n$ and the Gram matrix of the Picard lattice w.r.t. this basis is

$$\mathrm{Pic}(S_{1,n}) = \begin{pmatrix} 2 & 1 \\ 1 & 2-8n \end{pmatrix} .$$

Notice that $\det(\mathrm{Pic}(S_{1,n})) = 3-16n$ which is the opposite of $\det(A_{1,n})$. The Gram matrix has $b = 1$, $2c = 2 - 8n$. Hence the vanishing Brauer class is defined by a theta characteristic (as b is odd) which is odd since $c = 1 - 4n$ is odd.

A B-field representing the vanishing Brauer class is obtained from the second basis vector of $\mathrm{Pic}(S_{1,n})$:

$$B_{van} := \tfrac{1}{2}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\3\end{smallmatrix}\right), 2v\right) \equiv \tfrac{1}{2}\left(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), 0\right) ,$$

where the congruence is modulo $\frac{1}{2}\mathrm{Pic}(S_{1,n}) + H^2(S_{1,n}, \mathbb{Z})$, hence the vanishing Brauer class coincides with the one defined by B_α, the Clifford class α_X.

4.13. **The case $\tau = 2$.** The Picard lattice is ($2h + t_{2,n}$ is divisible by 4), hence:

$$\mathrm{Pic}(S_{2,n}) = \left\langle h,\, t_{2,n} = \left(\left(\begin{smallmatrix}-2\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\4\end{smallmatrix}\right), 4v\right)\right\rangle_{sat} = \langle h,\, \left(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), v\right)\rangle$$

so the Gram matrix is:

$$\mathrm{Pic}(S_{2,n}) = \begin{pmatrix} 2 & 1 \\ 1 & 2-2n \end{pmatrix} .$$

Notice that $\det(\mathrm{Pic}(S_{2,n})) = 3 - 4n$ whereas $\det(A_{2,n}) = 16 \cdot n - 3 \cdot 2^2 = 4(4n-3)$. The Gram matrix of $\mathrm{Pic}(S_{2,n})$ has $b = 1$, $2c = 2 - 2n$. Hence

the vanishing Brauer class is defined by a theta characteristic (as b is odd) which is even/odd iff $c = 1 - n$ is even/odd iff n is odd/even.
A B-field representing the vanishing Brauer class is obtained from the second basis vector of $\mathrm{Pic}(S_{2,n})$:

$$B_{van} := \tfrac{1}{2}\big(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), v\big) .$$

The transcendental lattice contains the vector

$$t := \big(\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), 0\big) \in T(S_{2,n}) = \mathrm{Pic}(S_{2,n})^{\perp} .$$

The restriction of the Brauer class α_X to $T(S_{2,n})$ is not trivial since $B_\alpha \cdot t = \frac{1}{2}(-1+2) \not\equiv 0 \mod \mathbb{Z}$. Hence $\alpha_X \neq \alpha_{van}$.

4.14. **The case $\tau = 3$.** The Picard lattice is (notice that $2h + t_{3,n}$ is divisible by 4):

$$\mathrm{Pic}(S_{3,n}) = \langle h,\, t_{3,n} = \big(\left(\begin{smallmatrix}6\\-6\end{smallmatrix}\right), \left(\begin{smallmatrix}4\\4\end{smallmatrix}\right), 8v\big) \rangle_{sat} = \langle \big(h,\, \left(\begin{smallmatrix}2\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), 2v\big) \rangle ,$$

as the last vector has length $-4 + 2 - 4v_{3,n}^2 = 14 - 8n$ we find the Gram matrix:

$$\mathrm{Pic}(S_{3,n}) = \begin{pmatrix} 2 & 1 \\ 1 & 14 - 8n \end{pmatrix} .$$

Notice that $\det(\mathrm{Pic}(S_{3,n})) = 27 - 16n$ which is the opposite of $\det(A_{3,n})$. The Gram matrix of $\mathrm{Pic}(S_{3,n})$ has $b = 1, 2c = 14 - 8n$. Hence the vanishing Brauer class is defined by a theta characteristic (as b is odd) which is odd since $c = 7 - 4n$ is odd.
A B-field representing the vanishing Brauer class is obtained from the second basis vector of $\mathrm{Pic}(S_{3,n})$:

$$B_{van} := \tfrac{1}{2}\big(\left(\begin{smallmatrix}2\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), 2v\big) \equiv \tfrac{1}{2}\big(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), 0\big) \equiv B_\alpha \mod H^2(S_{3,n}, \mathbb{Z}) ,$$

hence the vanishing Brauer class coincides, as in the case $\tau = 1$, with α_X.

4.15. **The case $\tau = 4$.** The Picard lattice is (notice that $t_{4,n}$ is divisible by 2 in $H^2(S_{4,n}, \mathbb{Z})$):

$$\mathrm{Pic}(S_2) = \langle h,\, t_{4,n} = \big(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\2\end{smallmatrix}\right), 2v_{4,n}\big) \rangle_{sat} = \langle h,\, \big(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), v_{4,n}\big) \rangle .$$

Hence

$$\mathrm{Pic}(S_{4,n}) = \begin{pmatrix} 2 & 0 \\ 0 & 6 - 2n \end{pmatrix} .$$

Notice that $\det(\mathrm{Pic}(S_{4,n})) = 12 - 4n$ whereas $\det(A_{4,n}) = 16 \cdot n - 3 \cdot 4^2 = 4 \cdot (4n - 12)$. Therefore the vanishing Brauer class is defined by a point of order two (as b is even).
A B-field representing the vanishing Brauer class is obtained from the second basis vector of $\mathrm{Pic}(S_{4,n})$:

$$B_{van} := \tfrac{1}{2}\big(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), v_{4,n}\big) .$$

Notice that

$$B_\alpha + B_{van} = \tfrac{1}{2}\big(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), 0\big) + \tfrac{1}{2}\big(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), v_{4,n}\big) = \tfrac{1}{2}\big(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right), v_{4,n}\big)$$

is a B-field with invariants $h\cdot(B_\alpha+B_{van})=\frac{1}{2}$, so it corresponds to a theta characteristic, and

$$(B_\alpha+B_{van})^2 \;=\; \tfrac{1}{4}(0+4+v^2)\equiv\tfrac{1}{4}(10-2n)\equiv\tfrac{1}{2}(1-n)\quad \mathrm{mod}\ \mathbb{Z}\ .$$

The theta characteristic corresponding to $\alpha_X+\alpha_{van}$ is thus even iff n is odd. □

4.16. **Remark.** In 4.11,. . . ,4.15 of the proof of Theorem 4.4.1 we observed that $-4\det(Pic(S_{\tau,n}))=\det(M_{\tau,n})$ for $\tau=0,2,4$ whereas $-\det(Pic(S_{\tau,n}))=\det(M_{\tau,n})$ if $\tau=1,3$. This relation was already observed in [ABBV14, Proposition 1]. Notice also that $\det(Pic(S_{\tau,n}))$ is even iff $\tau=0,4$ and that in these cases $\alpha_X\neq\alpha_{van}$, so α_X restricts to a non-trivial Brauer class on $S_{\tau,n}$. This was already shown in [ABBV14, Proposition 2]. See also [Gal17, Theorem 4.8 and Proposition 4.10].

5. Associated $K3$ surfaces and the divisors $\mathcal{C}_M$

5.1. **Classification of admissible Lattices in $\mathcal{C}_8$.** In [YY23] there is also a lattice-theoretic characterization of the cubic fourfolds in $\mathcal{C}_8$ with an associated $K3$ surface, that is, of those that are conjecturally rational.

Definition 5.1.1. ([YY23, Definition 8.1]) A lattice $M_{\tau,n}$ is admissible if $T(X_{\tau,n})(-1)$ is Hodge isometric to the transcendental lattice of a $K3$ surface.

Remark 5.1.2. The definition in [YY23] is different, but it is equivalent.

Proposition 5.1.3. *(*[YY23, Corollary 8.14]*) The lattice $M_{\tau,n}$ is admissible if and only if one of the following conditions is true*

(1) *(a) $\tau=1,3$;*
(2) *(b) $\tau=0,2,4$ and n is odd.*

We already discussed the cases $\tau=1,3$ in the introduction. In these cases $\alpha_{van}=\alpha_X$, where X is a cubic fourfold with $N^2(X)\cong K_8$ and $S_{\tau,n}$, the $K3$ double plane defined by $X_{\tau,n}$, is a $K3$ surface associated to $X_{\tau,n}$. Moreover, Hassett proved the rationality of these cubic fourfolds in [Has99].

The case $\tau=2$, n odd, is still under investigation. In the remaining cases we have identified an associated $K3$ surface, see the proposition below. We are investigating its geometry in relation to the cubic fourfolds.

Proposition 5.1.4. *Let $X_{\tau,n}$ be a cubic fourfold with $N^2(X_{\tau,n})\cong M_{\tau,n}$. Let $S_{\tau,n}$ be the $K3$ double plane defined by $X_{\tau,n}$ and let $C_{\tau,n}$ be the branch curve of the double cover $S_{\tau,n}\to\mathbb{P}^2$. Let $\tau=0,4$ and n odd, and let β be the even theta characteristic β on $C_{\tau,n}$ which is the specialization of the even theta characteristic $\beta_X:=\alpha_{van}+\alpha_X$ on the branch curve of S_X. Then the $K3$ surface S_β, a degree 8 surface in $\mathbb{P}^5$, defined by the even theta characteristic β is a $K3$ surface associated to $X_{\tau,n}$.*

Proof. Let X be a general cubic fourfold with a plane and let C_6 be the branch curve of $S_X \to \mathbb{P}^2$. We proved that for $\tau = 0, 4$ the vanishing Brauer class α_{van} corresponds to a point of order two in $J(C_6)$ and that $\beta_X := \alpha_{van} + \alpha_X$ corresponds to an even theta characteristic on C_6. Specializing C_6 to $C_{\tau,n}$, we obtain an even theta characteristic β on $C_{\tau,n}$. Then (the push forward to $\mathbb{P}^2$ of) β admits a minimal resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^6 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^6 \longrightarrow \beta \longrightarrow 0$$

where M is a 6×6 matrix of linear forms on $\mathbb{P}^2$ and $\det M = F$ where $F = 0$ is an equation defining $C_{\tau,n}$ ([Be00, Proposition 4.2]). The base locus of these quadrics is a $K3$ surface S_β of degree 8 in $\mathbb{P}^5$.
The transcendental lattice of S_β is $T_\beta(S_{\tau,n})$, see [vG05], [IOOV17] for the case of a $K3$ with Picard rank one, and by specialization it also holds for $S_{\tau,n}$. Since α_{van} is trivial on $T(S_{\tau,n})$, the homomorphisms β and α_X have the same restriction to $T(S_{\tau,n})$, hence

$$T(S_\beta) \cong \ker(\beta : T(S_{\tau,n}) \to \mathbb{Z}/2\mathbb{Z}) = \ker(\alpha_X : T(S_{\tau,n}) \to \mathbb{Z}/2\mathbb{Z}) = T(X_{\tau,n})(-1)\,.$$

Therefore the $K3$ surface S_β is associated to X. $\square$

5.2. **Pfaffian cubic fourfolds with a plane.** An interesting example of a pfaffian, hence rational, cubic fourfold X with a plane and $\mathrm{rank}(N^2(X)) = 3$, but with $\alpha_X \neq 0$, is given in [ABBV14, §4]. They determine τ, n explicitly (but notice that they use a different convention for writing the lattices $M_{\tau,n}$). We verify this here, using Theorem 4.4.1.
The double plane $S = S_X$ is branched along a smooth sextic $C = C_6 \subset \mathbb{P}^2$ with a tangent conic. Since the inverse image of this conic consists of two smooth rational curves n, n' in S, the Picard lattice of S is

$$\mathrm{Pic}(S) = \left(\mathbb{Z}h \oplus \mathbb{Z}n, \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}\right) \cong \left(\mathbb{Z}h \oplus \mathbb{Z}(h-n), \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}\right)\,.$$

So we are in case $\tau = 0, n = 2$ or in the case $\tau = 4$, $n = 5$. The vanishing Brauer class is thus a point of order two in $J(C)_2$ and $\alpha_X + \alpha_{van}$ is an odd/even theta characteristic in the first/second second case respectively.
The tangent conic cuts out a divisor $2D$ on $C \subset \mathbb{P}^2$ and the rational curves n, n' each cut out D on $C \subset S$. The intersection of h with $C \subset S$ is a divisor class D_l such that $2D_l \equiv 2D \in \mathrm{Pic}(C)$. The image of $h - n \in \mathrm{Pic}(S)$ is then $p := D_l - D \in \mathrm{Pic}(C)$, which a point of order two, and p must correspond to the vanishing Brauer class by [IOOV17, Theorem 1.1].
The Clifford class α_X corresponds to a theta characteristic L on C with $h^0(L) = 1$ and $2L \in |K_C = 3D_l|$ is cut out by a cubic curve C_3 tangent to C. Following [Vo86] and using the description of X from [ABBV14, §4], one finds that

$$C_3 : \quad 14x^3 + 15x^2y + 4x^2z + 9xyz + 14xz^2 + 16y^3 + 11y^2z + 8yz^2 + z^3 = 0$$

is the determinant of the 3×3 submatrix of linear forms in the 4×4 matrix (obtained from the minimal resolution of L) defining the quadric bundle on the cubic fourfold ([Be00, Proposition 4.2b]). So if (with $x = x_0, \ldots, w = u_2$)

$$F(x_0, x_1, x_2, u_0, u_1, u_2) = \\ (x_0 - 4x_1 - x_2)u_0^2 + \ldots - x_2^3 = \sum_{0 \le i,j \le 2} F_{ij}(x_0, x_1, x_2)u_i u_j + \ldots$$

is the defining (pfaffian) cubic polynomial for X, where the remaining terms are of degree at most one in the u_i, then C_3 is defined by $\det(F_{ij})_{i,j=0,1,2} = 0$. The theta characteristic $L + p = L + D_l - D = L + D - D_l$ has, using Serre duality on C:

$$h^0(L + p) = h^0(K_C - (L + D - D_l)) = h^0(4D_l - (L + D)) \ .$$

Since $|4D_l|$ is cut out by degree four curves on C and an explicit (Magma) computation shows that there are no such curves passing through the support of $L + D$, we conclude that $h^0(L + p) = 0$, so $L + p$ is an even theta characteristic.

Comparing with Theorem 4.4.1 we see that we are indeed in the case that $(\tau, n) = (4, 5)$, that is $N^2(X) \cong M_{4,5}$.

In particular, there is a $K3$ surface associated to $X = X_{4,5}$ which is identified in [ABBV14] with the pfaffian $K3$ surface associated by Beauville and Donagi in [BD85] to the pfaffian fourfold X. This $K3$ surface has a natural embedding of degree 14 in $G(2, 6)$. Instead, we associated the $K3$ surface S_β to X in Proposition 5.1.4. In a sequel to this paper we intend to investigate S_β in this particular case as well as for $\tau = 4$ and any odd n.

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