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Evolution of teachers' and researchers' praxeologies for designing inquiry mathematics tasks: the role of teachers' beliefs

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Abstract
This paper is focused on the collaborative work of two communities, one of teachers and one of researchers, during a teacher professional development program on the inquiry-based learning approach in mathematics, addressed to lower secondary school in-service mathematics teachers. We conceptualize the design of inquiry mathematics task as the boundary object on which the two communities work collaboratively. We aim to study the evolution of teachers' and researchers' meta-didactical praxeologies for designing inquiry mathematics tasks, with the Meta-Didactical Transposition framework, to understand if and how their collaboration favors their convergence toward shared components of final meta-didactical praxeologies. In the results, we show that this convergence is reached thanks to internalization processes of praxeological components for designing inquiry mathematics tasks, as a consequence of the learning mechanisms activated by the common work on the boundary object. In this paper, we address also the issue of understanding the complex relationship between teachers' beliefs on inquiry mathematics tasks and the meta-didactical praxeologies of teachers and researchers. As a result, we propose a model in which the evolution of teachers' beliefs is taken into account both as an agent and a consequence of the evolution of the meta-didactical praxeologies of the two communities involved in the teacher professional development program.

Keywords Boundary object · Meta-didactical praxeologies · Teacher beliefs · Teacher professional development

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Introduction

The different forms, context and outcomes of mathematics teacher collaboration have been the focus of several studies in the last years (e.g., Robutti et al., 2016). Many of these studies take into account the interactions between mathematics teachers and researchers in mathematics education as a key factor for teachers' learning (Borko & Potari, 2020). However, in the study of teacher professional development (PD), it is important to also address the issues that the collaborative work of these two communities could raise. Indeed, teachers' and researchers' roles and perspectives while participating in a PD program are different, causing sometimes a distance between them, especially when they have to collaboratively design instructional resources for students (Wake et al., 2016).

In this paper, we study the interactions between teachers and researchers having the role of teacher educators, involved in a collaborative work during a PD program organized by the University of Turin, focused on the inquiry-based learning approach in mathematics (Artigue & Blomhøj, 2013; Maaß & Artigue, 2013). The PD program is aimed at involving teachers in designing inquiry mathematics tasks for their students. The long-term objective of their collaboration is to reach a convergence on some practical and theoretical aspects related to their task design activity, despite an initial distance due to different perspectives on the design of inquiry mathematics tasks. To understand the causes of this distance between the two communities, we focus on what is recognized as a crucial aspect in teacher PD: teachers' beliefs about students' needs in terms of teaching approach (Sztajn, 2003). For example, Leikin et al. (2006) found that the majority of teachers involved in PD are reluctant to propose challenging tasks—such as inquiry mathematics tasks—to their students because of their concern on whether an approach based on this kind of tasks could meet the learning needs of low-achieving students. Taking into account this issue, we aim to study the evolution of teachers' practices and discourses justifying their practices, occurring as a result of the collaboration with the researchers, in connection with the evolution of teachers' beliefs about inquiry mathematics tasks and about students' needs.

We also take into account that, in order to reach a convergence, not only teachers', but also researchers' practices and justifying discourses need to evolve, according to teachers' needs and feedback. So, our study also aims at understanding if and how, at the end of the PD program, a convergence is reached on some practices and justifying discourses, shared among the two communities, for designing inquiry mathematics tasks for students. To deepen the insight into our research problem, we conceptualize the design of inquiry mathematics tasks as the *Boundary Object* (BO) (Star, 2010) on which the two communities involved in the PD program work collaboratively. In fact, it is a common working terrain for them, and it is flexible enough to allow both teachers and researchers to bring their peculiar perspectives, favoring the evolution of both communities working on it.

To study the evolution processes of both teachers and researchers during the PD program under scrutiny, we adopt the *Meta-Didactical Transposition* (MDT) framework (Arzarello et al., 2014; Robutti, 2020), which has been specifically introduced to analyze this kind of context in a dynamic way, assuming that both communities—teachers and researchers—should undergo profound changes. We also rely on an evolution of MDT, named MDT.2, which focuses in particular on the evolution of the practices and the discourses justifying the practices of the two communities, modeling them through *internalization processes* (Cusi et al., 2023). In the study of the *internalization processes* at the basis of the evolution of their practices and justifying discourses, an important role is played by the various agents (Prodromou et al., 2019) favoring and supporting this evolution, especially the motivational agents. In the MDT.2 framework (Cusi et al., 2023), for example, teachers' beliefs are taken into account as motivational agents for teachers' evolution.

With respect to MDT.2, we aim to do something more: to develop that frame, positioning the evolution of teachers' beliefs in relation to the evolution of their practices and discourses justifying their practices. In literature, indeed, several studies (e.g., Guskey, 1986; Lloyd, 1999; Wilson & Cooney, 2002) have already highlighted that not only a change in teachers' beliefs influences their practices, but also the other way around: experimenting with new practices allows teachers to reconsider their beliefs. Guskey (2000, 2002), for example, suggested that the relationship between classroom practices and teachers' beliefs is highly complex, because they influence each other in both directions. In addition, Swan (2007) describes a double direction relationship between the implementation of some specific tasks in the classroom and a change in teachers' beliefs. With respect to this literature, we would like to investigate something more: the relationship between teachers' beliefs and not only their practices but also their discourses justifying those specific practices.

Therefore, taking into account both the communities involved in the PD program, we aim to study the interconnections between the evolution of their collaborative design of inquiry mathematics tasks and the evolution of teachers' beliefs on this kind of tasks and on their students' needs.

Theoretical frameworks

Meta-Didactical Transposition Framework

MDT framework (Arzarello et al., 2014; Robutti, 2020) is aimed to describe and interpret the interactions between a community of researchers, acting as teacher educators, and a community of teachers participating in a PD program in an institutional context (e.g., organized by a University). It is based on the Anthropological Theory of Didactics (ATD) (Chevallard, 1985, 2019; Chevallard & Bosch, 2020), whose fundamental concept is that of praxeology. A praxeology consists of two main blocks: *praxis* (practical knowledge, or "know-how") and *logos* (or "know-why", explanations and justifications of the *praxis*, accompanied by theoretical discourses). The *praxis* block consists of a type of tasks and a technique suitable to solve it; the *logos* block consists of technology (discourses about the techniques) and theory (discourses which clarify, unify or justify the technology) (Chevallard, 2019).

In the ATD framework, praxeologies serve to model human knowledge in general. The specific case of mathematical praxeologies involves mathematical tasks and techniques, with the *logos* block constituted by "explanations of techniques, definitions of terms, rules, theorems, proofs, and so on" (Miyakawa & Winsløw, 2019, p. 284). Teaching mathematics could be modeled by didactic praxeologies, in which *praxis* and *logos* blocks are related to teach some specific mathematical praxeology. Miyakawa and Winsløw (2019) also introduced the concept of *paradidactic praxeologies* to study teachers' practices related to teaching, which are not directly teaching in their classrooms (e.g., preparation of teaching materials, meeting with colleagues). In the MDT framework, the praxeologies under scrutiny are meta-didactical ones, in the sense that they are referred to teacher PD contexts, in which *praxis* and *logos* are related to the collaborative building of didactic praxeologies among teachers and researchers. Meta-didactical praxeologies entail reflections on didactical praxeologies, and transformations and evolutions of them through the interactions occurring between the two communities during a PD program.

One of the fundamental hypotheses of the MDT framework is that, at the beginning of a PD program, both communities—the teachers participating in the PD program and the researchers having the role of teacher educators—have initial meta-didactical praxeologies. These praxeologies evolve thanks to *internalization processes*, thus converging toward shared components of final meta-didactical praxeologies (Arzarello et al., 2014; Robutti, 2020). *Internalization*, indeed, is the process through which a community start to habitually adopt praxeologies never adopted, or even not known, before. The *internalization processes* have been studied from different perspectives in the work of Cusi et. al. (2023), through the integration in the MDT model of theoretical elements coming from other frameworks, thus conducting to an evolution (MDT.2) of the original model. In particular, to describe how an *internalization process* can occur during the collaborative work of teachers and researchers in a PD program, Cusi et al. (2023) interpret the object of the work in terms of the BO framework. The actions that the members of both communities carry out on the BO are interpreted with Carlile's (2004) model: *transfer* (at syntactic level), *translation* (at semantic level), and *transformation* (at pragmatic level). To describe why an *internalization process* can occur, instead, Cusi et al. (2023) rely on different types of agents (methodological, institutional, material and technological, motivational) determining teacher praxeologies' evolution (Prodromou et al., 2018). In the end of their work, as venues for further research, they indicate the need to deepen the study of motivational agents, in particular "the role played by beliefs, emotions attitudes and values in influencing the elaboration of the justifying discourses within the teachers' and researchers' meta-didactical praxeologies" (ibid., p. 18).

In our study, we focus on the interactions occurring between teachers and researchers during a PD program, deeply influenced by teachers' beliefs on inquiry mathematics tasks and on students' needs. So, we aim to take up the need highlighted by Cusi et al. (2023) as a challenge and to study how the evolution of teachers' beliefs influences their meta-didactical praxeologies and their *internalization processes*. Furthermore, we aim to analyze the other direction of the relationship that is how the evolution of the meta-didactical praxeologies of both communities, due to *internalization processes*, influence the evolution of teachers' beliefs.

Boundary object

A BO, as conceptualized by Star (2010), is a shared action space for different communities, allowing them to work and evolve together without preliminary consensus on all aspects due to its flexibility. "Its mediational qualities seemed to be that it sat in the middle between different groups, very ill-structured or sketchy in the common usage" (Star, 2010, p. 608).

The BO construct has been applied several times in studies on teacher PD. For example, Nolen et al., (2011) study assessment tools and artifacts as BOs between the social worlds in which novice teachers learn to teach, crossing boundaries between university and school. Kynigos and Kalogeria (2012) address mathematics teacher PD as a form of boundary crossing between teachers' habitual communities and the community of teacher educators. Scenarios and half-baked microworlds are two kinds of artefacts that play the role of BOs in their work. In a similar way, Sztajn et al. (2014) examine researchers and teachers exchanging knowledge during a mathematics teacher PD, as a boundary encounter. They recognize that this kind of encounter impacts both researchers' and teachers' practices. Sinclair et al. (2020) apply the BO construct to study the collaboration between researchers and elementary school mathematics teachers, involved in the design of tasks focused on multiplication, with a gesture-based application named TouchTimes (Jackiw & Sinclair, 2019). They see the application as a BO in their "joint effort to understand multiplication in its various instantiations in the classroom, the research literature, the teachers' experience, the textbooks, other teaching and learning resources" (Sinclair et al., 2020, p. 1471).

More generally, when professionals of different disciplines and institutions collaborate for a common goal, there is a need to coordinate different types of expertise. This challenge is often conceptualized as a *boundary crossing* (Akkerman & Bruining, 2016). On the basis of a review of 181 studies, Akkerman and Bakker (2011) describe what learning can result from *boundary crossing*, identifying four types of learning mechanisms which could take place:

- *identification*, whereby practices at the boundary are re-conceptualized in light of one another, in order to legitimate their coexistence;
- *coordination*, whereby people of different communities cooperate efficiently, even without consensus (Star, 2010), to maintain the flow of work, creating routines to rely on;
- *reflection*, that is a mutual definition of different perspectives, with the effort to accept others' perspectives to look at one's own practice;
- *transformation*, whereby changes occur in existing practices, leading to new in-between practices. It is characterized by the confrontation with a lack or problem, the recognition of a shared problem space, the hybridization of perspectives and practices and the crystallization of new practices and discourses (Akkerman & Bruining, 2016).

Robutti and colleagues (2020) adopt the BO framework to explain the evolution of praxeologies from external to internal to the community of the researchers and/or the teachers, during a PD program. They connect this evolution to the changes occurred in the BO, that is a dynamic object, according to Star (2010), constituted by different components. In this way, the BO can be used as a means to explain the praxeologies of the different communities acting on it. Robutti and colleagues.(2020) focus, in particular, on the learning mechanism of *transformation* (Akkerman & Bakker, 2011), which could lead to the creation of a new, in-between practice, which is what MDT defines a "shared praxeology." "This evolution of praxeologies is thus simultaneously a process of professional development and a product of a learning activity made possible by the joint work of subjects (teachers, researchers) on a boundary object" (Robutti et al., 2020, p. 216).

In our study, we interpret the design of inquiry mathematics tasks during the PD program "Medie 2.0" as the BO on which teachers and researchers work together (Fig. 1). The two communities, indeed, have different perspectives on it and can participate with their different background and expertise to the work during their *boundary encounters*. Researcher' and teachers' role in the context of a PD program are of course different, with an unequal distribution of power, but both communities can make a fundamental and original contribution. The BO itself can evolve according to the evolution of the meta-didactical praxeologies of researchers and teachers for the design of inquiry mathematics tasks.

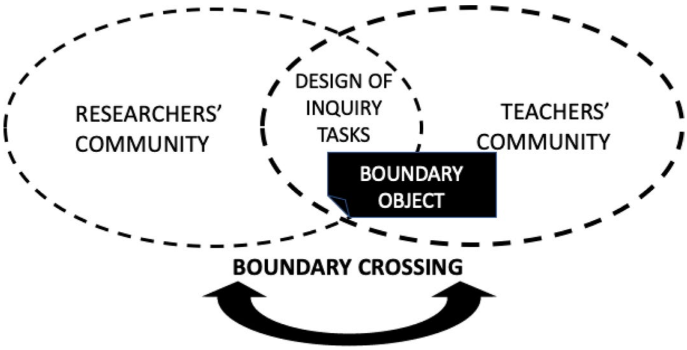


Fig. 1 Representation of the *boundary crossing* process involving researchers and teachers

Teachers' beliefs and practice

The relationship between teachers' beliefs and practices has been a main trend of research in recent decades and nowadays it seems clear that there is no linear relationship in a specific direction. More than thirty years ago, Cobb et al. (1990) claimed that beliefs and practices develop together and are dialectically related. However, both consistencies and inconsistencies between teachers' beliefs and practices have been found in several studies (Cross, 2009; Wilkins, 2008). To explain some apparent inconsistencies, for example, Leatham (2006) invites researchers to deeper investigate the possible unexpected implications of some beliefs or the fact that other beliefs, different from those under scrutiny, "took precedence in that particular situation" (p.95). Skott (2009) emphasizes the role of the context in shaping teachers' practices, explaining apparent inconsistencies between teachers' beliefs and practices as a result of teachers' acting within a local context, in actual or virtual communities that deeply influence their practice. Along the same lines is the work of Wong et al. (2016), who describe teachers' beliefs as situational, manifesting in instructional practices only in relation to the complexities of the classroom context.

Many studies have been conducted on how to promote change in teachers' beliefs during a PD program. For example, Liljedahl (2010) shows how it is possible to cause a profound change in teachers' practices and beliefs in a short time, thus pointing out that beliefs may be less stable than is usually thought (Liljedahl, 2012). To favor a change in teachers' beliefs, Cross (2009) emphasizes the importance for teachers to be engaged in mathematics activities which can foster a disposition toward mathematics as inquiry, in order for them to appreciate the social and constructive aspects of the discipline. Also, in Wilkins' (2008) study, teachers' beliefs about effective mathematics teaching are found to be positively related to the frequency with which teachers used inquiry mathematics tasks in their classrooms.

According to Guskey (2002), a significant change in teachers' beliefs primarily occurs after they experiment changes in their classroom practices and gain evidence of improvements in students' learning or attitude. The crucial point of Guskey's argumentation is that the experience of successful implementation of new teaching practices is at the basis of any changes in teachers' beliefs. However, he acknowledges that this model in some ways "overly simplifies a highly complex process" (Guskey, 2002, p. 385), opening up to the possibility that beliefs and practices influence each other in both directions.

In this paper, we aim to explore this complexity, to fully capture the intertwined evolutions of teachers' beliefs, practices (*praxis* component of their meta-didactical praxeologies) and justifying discourses (*logos* component of their meta-didactical praxeologies). Our hypothesis is that all these evolutions are triggered by teachers' collaboration with the researchers on the design of inquiry mathematics tasks, considered as a BO cross the two communities.

Research questions

In light of the theoretical frameworks introduced in the previous sections, the research questions we aim to answer in this paper are:

RQ1) How teachers' and researchers' meta-didactical praxeologies for designing inquiry mathematics tasks evolve during their collaborative work on the BO, possibly converging toward shared components of final meta-didactical praxeologies?

RQ2) What are the relationships between the evolution of teachers' beliefs on inquiry mathematics tasks and the evolution of the meta-didactical praxeologies of both teachers and researchers involved in the PD program?

Methodology

The context

The context of this study is a teacher PD program, part of the SSPM (Scuole Secondarie Potenziate in Matematica, i.e., Secondary Schools Enhanced in Mathematics) project (Pocalana & Robutti, 2022; 2023; Pocalana et al., 2023) of the University of Turin. The SSPM project is part of the national Liceo Matematico[1] project, which includes 27 Italian universities and is aimed at fostering the teaching/learning of mathematics in the Italian schools. The different universities may choose whether to direct the programs to both upper and lower secondary schools or only to upper ones (Licei), and the University of Turin made the choice to address it to both secondary levels (and also to primary schools). The SSPM project implemented in Turin provides:

- To the students, 33 mathematics classes per year, additional to the curricular ones
- To the mathematics teachers, a PD program of 10 meetings per year, led by mathematics education researchers.

The in-service teachers participating in the SSPM project have a double role:

- As learners, they attend 20 h (2 h for each meeting) of professional development per year of synchronous online or face-to-face meetings with the researchers (in the academic years 2020/21 and 2021/22 only online, due the Covid-19 pandemic)

[1] www.liceomatematico.it

- As teachers, they experiment in their classrooms with the activities they worked on in the meetings, during the 33 h of additional mathematics classes connected with the SSPM project, or during their usual everyday classes.

The PD program addressed to lower secondary school teachers is divided into two levels. The three-year first-level PD program is aimed to introduce the teachers to inquiry in mathematics and to the inquiry-based approach principles (Artigue & Blomhøj, 2013; Laursen & Rasmussen, 2019; Maaß & Artigue, 2013), as well as to the "mathematics laboratory" (Anichini et al., 2004; Arzarello & Robutti, 2008) approach, based on group work, sharing and comparison of ideas, classroom discussions led by the teacher, problem posing and problem solving, use of artefacts and materials. The researchers provide teachers with mathematics tasks, ready to be presented in their classrooms, with all the details of the scenario for their implementation. The teachers provide reports of their classroom implementations, with comments, reflections and suggestions for future improvements in the task design. Their implementations are mainly conducted during the additional mathematics classes connected with the SSPM project. In the case of lower secondary schools (that are compulsory in the Italian school system), these additional classes are devoted mostly to high-achieving and highly motivated students, selected by the mathematics teacher.

The second level PD program, named "Medie 2.0", is held by the first author, a PhD student in mathematics education, who is former secondary school teacher, under the supervision of the second author, a full professor in mathematics education, both working at the University of Turin. It has among its goals to extend teachers' implementations of inquiry mathematics tasks to their whole classes, according to the inclusive spirit of the inquiry-based approach (Laursen & Rasmussen, 2019). To achieve this goal, the researchers actively involve teachers in designing inquiry mathematics tasks for their students, so that they could introduce the adaptations and differentiations they deemed necessary for different types of students. The researchers provide teachers with ideas ad hints for inquiry mathematics tasks, but not with the complete scenario and the task formulation for students, that have to be designed cooperatively.

The sample of our study is constituted by 17 lower secondary school (grades 6–8) mathematics teachers, who have already successfully completed the first-level program and are attending the second level PD program "Medie 2.0" (Fig. 2).

Data collection

For this study, part of a bigger PhD project (Pocalana et al., 2023; Pocalana & Robutti, 2023), we collected data from two years of the second level PD program "Medie 2.0." In particular:

- the transcripts of the collective discussions, occurred during the 20 meetings (10 per year);
- the materials created by the researchers for the work with the teachers during the PD meetings (slides with theoretical references, teacher-sheets, etc.);
- the written protocols handed in by the teachers, with proposals for the design of inquiry mathematics tasks, based on the researchers' hints;
- the teachers' written answers to the reflection questions on their design proposals, included in the teacher-sheets provided by the researchers;

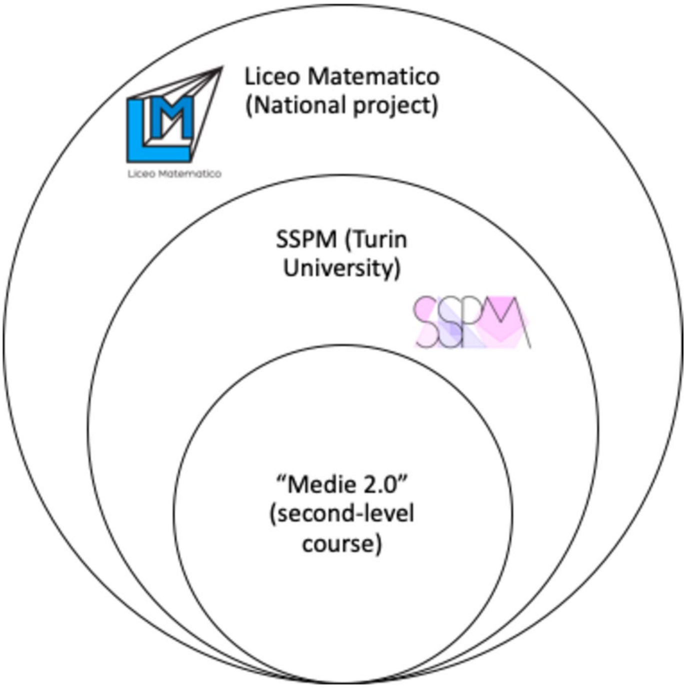


Fig. 2 Institutional context of the PD program

- the teachers' reports of classroom implementations of the designed tasks;
- the field notes taken by the researchers during their discussions aimed at planning the PD program meetings.

In the following example of teacher-sheet (Figs. 3 and 4), provided by the researchers during a meeting with the teachers, it is possible to retrieve all the usual sections in which the proposals for the teachers were divided: Ideas for the inquiry mathematics task—Task design—Justification of task design—Questionnaire for reflection.

Data analysis

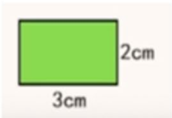
To address RQ1, among the collected data, we looked for evidences of teachers' meta-didactical praxeologies, related to the type of tasks: *designing inquiry mathematics tasks for students during the PD program "Medie 2.0."* Specifically, we looked for techniques to address this type of tasks (*praxis* block), and technologies and theories, that is discourses justifying their techniques (*logos* block). We derived them from the teachers' answers to the reflection questions on their task design proposals and from their reports of classroom implementations, accompanied by discourses justifying their choices, especially for the choice of which students to involve in the solution of the inquiry tasks. Due to the characteristics of the justifying discourses in the kind of praxeologies under scrutiny, we chose to analyze technologies and theories as a unique block, referring to them simply as *logos*. For example, when teachers reported that they designed inquiry mathematics

IDEA/HINT FOR ACTIVITY
LCM and GCD ... squared!

1) Starting from a rectangle (supplied on cardboard, drawn in the notebook or made with GeoGebra), ask to build a square made up of many rectangles identical to the one given.

2) Ask what is the size of the side of the smallest square that can be constructed in this way.

Example (NOT to be provided to students):

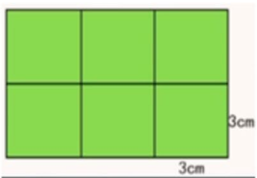


→ 6 is the least common multiple for 3 and 2

3) Starting from a given rectangle, ask to divide it into identical squares.

4) Ask how long the side of the biggest squares is with which the starting rectangle can be divided.

Example (NOT to be provided to students):



→ 3 is the greatest common divisor for 9 and 6.

5) Ask to carry out all the previous points, starting from a rectangle having the two dimensions which measure, respectively, 15 cm and 5 cm.

6) Start a discussion about what happens when one dimension of the starting rectangle is multiple of the other.

7) Propose increasingly complex examples, for example rectangles with dimensions of 10 cm and 15 cm, respectively etc.

8) Start a discussion by asking the students to explain their solution strategies.

Fig. 3 Example of teacher-sheet—first part, with ideas for the mathematics task

tasks only for high-achieving students and guided tasks for all the others, this has been recognized as the technique component of a teachers' meta-didactical praxeology. The justifications provided by the teachers for their choices are recognized as the *logos* block of that teachers' meta-didactical praxeology.

From the materials (slides with theoretical references, teacher-sheets, etc.) created by the researchers for the meetings with the teachers and from the field notes of their discussions, we traced the evolution of researchers' techniques to address the type of tasks *designing inquiry mathematics tasks for students during the PD program "Medie 2.0"* and the *logos* justifying them. We analyzed the *logos* block of researchers' meta-didactical praxeologies

TASK DESIGN

Working in groups, design the activity sheet for students.

JUSTIFICATION OF TASK DESIGN

Justify your task design choices.

QUESTIONNAIRE FOR REFLECTION

1) In which class and in what period would you propose this activity?
2) How would you describe the teachers' role during the activity?
3) Which interventions do you imagine the teacher should carry out during the activity and why?

Fig. 4 Example of teacher-sheet—second part, with requests for the teachers

with a specific focus on the theoretical frameworks constituting the theory on which the PD program is based, which increase and expand over the two years under scrutiny. This evolution is influenced by their common work with the teachers on the BO, that coincides with the type of tasks of the meta-didactical praxeologies of both communities. In particular, we analyzed researchers' choices of theoretical references to be presented to the teachers, aimed at triggering new reflections and an evolution in teachers' practices and justifying discourses. We are aware that the analysis of the researchers' *logos* should take into account elements connected with their understanding of what inquiry in mathematics means and, more generally, with their beliefs about how it would be implemented in schools. However, we acknowledge as a limitation of this study the difficulties of studying one's own personal beliefs with an external view point, due to the peculiar position of the researchers, who are both the teacher educators and the authors of the study.

To describe the intertwining evolution processes of the meta-didactical praxeologies of both communities, in teachers' productions, we sought elements recalling the theoretical references proposed by the researchers during the meetings, which increased and changed as a consequence of the evolution of researchers' meta-didactical praxeologies for the work with the teachers. Studying the evolutions occurred for both communities, we identified the *internalization* of new components of meta-didactical praxeologies, which were not present at the beginning of the PD program. We also identified the components of final meta-didactical praxeologies shared among the two communities, in the case of accordance between the researchers' proposals and what reported by the teachers in terms of reflections on their task design. We analyzed the achievement of shared components of final meta-didactical praxeologies in terms of the learning mechanisms of *transformation* (Akkerman & Bruining, 2016), entailing the creation of new, in-between *praxis* and *logos* components between the two communities, through the steps of confrontation with a lack or problem, recognition of a shared problem space, hybridization of perspectives and practices and crystallization of new practices and justifying discourses.

To address RQ2, in all the teachers' oral and written productions, we sought connections between their meta-didactical praxeologies and their beliefs about the appropriateness of inquiry mathematics tasks for all the students and about students' needs. The conceptualization of the *logos* block of teachers' praxeologies adopted in our analysis includes the discourses justifying the techniques related to a specific type of tasks. So, it

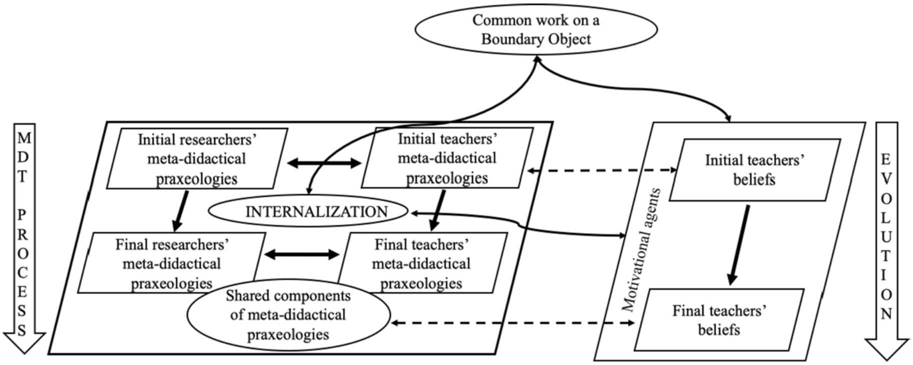


Fig. 5 Adaptation of MDT.2 model, which takes into account the double role of teachers' beliefs

is contextualized in the institutional constraints of the PD program under scrutiny, and it can include theoretical references coming from the literature presented to them by the researchers. According to Chevallard (2019), "the logos block corresponds to what most people have in mind when they use the term 'knowledge'— although one can reasonably argue that knowledge is the dialectical union of logos and praxis" (p.92). Teachers' beliefs are intended as more general lenses through which teachers see mathematics, teaching strategies, and their students' needs (Pocalana et al., 2023; Cross, 2009). Teachers could be more or less aware of their beliefs (Furinghetti & Pehkonen, 2002), which are not only related to specific tasks and corresponding practices, but general views influencing several aspects of their teaching. Our hypothesis is that some instances of teachers' beliefs could be connected with the *logos* block of their meta-didactical praxeologies. We presume that researchers' beliefs also influence the *logos* of their meta-didactical praxeologies and that they can evolve. This aspect, not addressed in the present paper, will be the object of future studies.

Toward the end of the second year of the PD program "Medie 2.0", the teachers have been explicitly asked by the researchers for which of their students the inquiry mathematics tasks they had designed were appropriate. Their answers to this question, as well as all the other teachers' productions collected during the PD program, have been coded according to the themes "appropriateness of inquiry tasks for all the students"/ "appropriateness of inquiry tasks for only some students." We selected these themes because they represent the core of the *meta-didactical conflict* between teachers and researchers, emerged during the PD program. Teachers' beliefs on these themes have been particularly relevant for us, because of their connection with the teachers' discourses (*logos* component of their meta-didactical praxeologies) justifying their choice to propose inquiry mathematics tasks to all their students or only to the high-achieving ones.

In light of the analysis described above, we felt the need of the integration in the MDT.2 framework (Cusi et al., 2023) of the double direction relationship between the evolution of teachers' beliefs and the evolution of the meta-didactical praxeologies of both the communities working on the BO. In the model that we propose as working hypothesis (Fig. 5); teachers' beliefs are considered as motivational agents which trigger *internalization processes* of new meta-didactical praxeologies for both teachers and researchers. At the same time, however, they are influenced by the *internalization processes,* and they undergo an evolution, which, in a cyclical way, can trigger new learning mechanisms and *internalization processes*. The collaborative work on the BO influence both the *praxis* and the *logos* block of the meta-didactical praxeologies of teachers and researchers, activating learning mechanisms. In our model, it is highlighted that it influences also the evolution of teachers' beliefs, which in turn favor, the path toward shared components of final meta-didactical praxeologies among the two communities.

The case study that we will present in the following section can contribute to test the validity of the model that we propose.

Results

To address RQ1, in the following sections, we will describe *internalization processes* both for teachers and for researchers, due to the learning mechanism of *transformation*, activated by the common work on the BO, during the PD program "Medie 2.0." To address RQ2, we will show how the evolution of teachers' beliefs is connected with the evolution of the meta-didactical praxeologies of both the communities of teachers and researchers.

Initial meta-didactical praxeologies of the communities of teachers and researchers

The teachers and the researchers involved in the PD program had a common type of tasks that is *designing inquiry mathematics tasks for students during the PD program "Medie 2.0."* This is considered, in our analysis, as the BO on which they worked collaboratively in a common effort, due to its interpretative flexibility (Star, 2010). To accomplish this type of tasks, the two communities both had their own techniques and *logos*, that, at the beginning of the PD program, highlighted a *meta-didactical conflict* (Arzarello & Ferretti, 2022), especially on a specific point, the target in terms of students of the inquiry mathematics tasks:

- On the one side, the researchers thought that the inquiry mathematics tasks designed in the "Medie 2.0" PD program should be appropriate for all the students. For this reason, they asked teachers, as part of the PD, to design them for their whole classes;
- On the other side, the teachers thought that the inquiry mathematics tasks designed in the "Medie 2.0" PD program should be proposed only to the groups of high-achieving students participating in the SSPM project.

At the basis of the discourses justifying teachers' choices there were shared, rooted beliefs about the needs of low-achieving students, who, in teachers' opinion, should be guided in procedural tasks, to avoid frustration (Pocalana & Robutti, 2023; 2022; Pocalana et al., 2023). For this reason, the teachers, more or less consciously, behave as if they had to address two different subtasks of the general type of tasks, for which they adopt different techniques: designing inquiry mathematics tasks for the SSPM additional classes with restricted groups of high-achieving students and designing procedural mathematics tasks for their everyday classes with all the students.

Coherently with what has been said, the initial meta-didactical praxeologies of researchers (Table 1) and teachers (Table 2) present substantial differences.

From the analyzed data (§ 3.2), we found that:

Table 1 Researchers' initial meta-didactical praxeologies

Praxis	Logos
Type of task: Designing inquiry mathematics tasks for students during the PD program "Medie 2.0" Techniques: Providing teachers with sheets containing hints and ideas for inquiry mathematics tasks intended for all the students and discussing with teachers the task design	The inquiry mathematics tasks designed during the PD program "Medie 2.0" should be suitable to be proposed to the whole classes This idea is deeply rooted in researchers' experience and knowledge and in the whole literature on inquiry-based learning in mathematics. For example, one of the "pillars" of inquiry-based mathematics education, according to Laursen & Rasmussen (2029) is "Instructors foster equity in their design and facilitation choices" (p. 138)

- On the one side, in the coding of the teachers' oral and written productions at the beginning of the PD program "Medie 2.0" according to the themes "appropriateness of inquiry tasks for all the students"/"appropriateness of inquiry tasks for only some students", only the second theme emerged. The first theme did not emerge from any teacher participating in the PD program.
- On the other side, the proposal made by the researchers to the teachers was about involving all the students in inquiry mathematics tasks. This choice was justified by the literature in the field of mathematics education, which report that inquiry-based learning is an inclusive approach, appropriate for all students (e.g., Artigue & Blomhøj, 2013; Laursen & Rasmussen, 2019; Maaß & Artigue, 2013).

We interpret the facing of this *meta-didactical conflict* between the two communities as the first step of the learning mechanism of *transformation* (§2.2), because we recognize it as the confrontation with a lack or problem and the recognition of a shared problem space (Akkerman & Bruining, 2016) among teachers and researchers. In fact, the two communities had different perspectives on the appropriateness of inquiry mathematics tasks for all the students, due to their different roles in the PD program and different discourses justifying their choices.

In the following sections, we will show that the learning mechanisms of *transformation* triggered *internalization processes* of components of meta-didactical praxeologies for both communities. This led, over the two years of PD program, toward shared practices and justifying discourses, allowing teachers and researchers to accomplish their common task with a shared perspective. At the same time, we will trace an intertwined evolution of teachers' beliefs about the appropriateness of inquiry mathematics tasks for all the students. This evolution in terms of beliefs has been both an agent and a consequence of the convergence toward shared components of final meta-didactical praxeologies among the two communities of teachers and researchers. As far as researchers are concerned, there have been several integrations to their initial theoretical references for the PD program, due to the interactions with the teachers during the two years, in response to their needs and feedback. For example, they integrated theoretical elements from the European project FAsMEd (Cusi et al., 2017) and from Peter Liljedahl's (2018, 2023) model of *flow*, based on Csíkszentmihályi's (1996) idea of *optimal experience*, as we will describe in the following sections.

Table 2 Teachers' initial meta-didactical praxeologies and beliefs

Praxis		Logos
Teachers' initial meta-didactical praxeologies		
Type of task: Designing inquiry mathematics tasks for students during the PD program "Medie 2.0"		The inquiry mathematics tasks designed during the PD program "Medie 2.0" of the SSPM project are meant to be proposed only to the students participating in the project itself Teachers' theoretical basis for the design of inquiry mathematics tasks are mostly rooted in the slides with "theory pills" on inquiry-based learning provided by the researchers. Teachers' idea of inclusion is deeply rooted also in their broad previous experience and knowledge, partially conflicting with researchers' logos
Subtask 1: Designing inquiry mathematics tasks for the SSPM additional mathematics classes Techniques 1: Using the hints and ideas provided by the researchers to design inquiry mathematics tasks for the restricted groups of high-achieving students participating in the SSPM project	Subtask 2: Designing tasks for their everyday classes with all the students Techniques 2: Adapting the hints and ideas provided by the researchers to design procedural tasks, with step-by-step instructions, for their everyday classes with all the students	
Initial teachers' beliefs		
Inquiry mathematics tasks are only appropriate for high-achieving students Low-achieving students need to apply known procedures, with teacher's step-by-step guidance		
Excerpts showing examples of teachers' justifying discourses		
Antonella	"It [*an inquiry activity proposed during the PD program*] is certainly appropriate for motivated and skilled students. The weakest ones would soon reach the state of frustration […]. It is not appropriate for all those students who, for many reasons, […] do not proceed unless they are guided within known processes. That is, they do not know how to move independently and are unable to maintain attention except for short periods."	
Paola	"To involve all the students, we have to bring them all to at least a basic level, guiding them a lot. We cannot let them free to explore."	
Carla	"At the secondary school, we have to build problem solving ability, the horizon is limited. We have to work slowly, with guided problems. Only sometimes, someone in the class is able to treat a situation with something a little different from the acquired method."	

Internalization processes triggered by the learning mechanism of *transformation*

Helping worksheets from the FAsMEd project

During the first year of the PD program “Medie 2.0”, the researchers recognized the teachers’ need to adapt the inquiry mathematics tasks designed during the PD program to the different kinds of students in their classes. For this reason, they presented the teachers with a theoretical reference which could constitute a source of inspiration for the design of tasks for their whole classes. It came from the work of Cusi et al. (2017), as part of the European project FAsMEd on formative assessment strategies. In the context of FAsMEd project, they designed three types of (digital) worksheets for each activity to be proposed in the classroom: *problem worksheets*, introducing the problem and asking one or more questions; *helping worksheets*, “aimed at supporting students who face difficulties with the problem worksheets by making specific suggestions (e.g., guiding questions)” (Cusi et al., 2017, p. 758); *poll worksheets*, prompting a poll within students. The researchers, when presenting the teachers with this reference, focused their attention especially on the *helping worksheets*, thought for those students who experienced some difficulties with the task. They consist of gradual questions, different kinds of representations for the problem data, or reminders of previous tasks already solved by the students. The most important idea is that the provided help must not be a ready-made solution, but it had to respect the active role of the students in the search for the solution.

The teachers re-elaborated the example of the *helping worksheets*, proposing a new element, as a modification of that presented by the researchers: the *help cards* (Pocalana & Robutti, 2022). *Help cards* are small, flexible hints, questions, different representations of the same problem etc., to be given according to the different needs of different groups of students. They are not collected in a single worksheet, but are many, different prompts, to be given one at a time, on the spot, according to the situation of the students. The introduction *of help cards* can be recognized as an example of hybridization of practices that is an integral part of the learning mechanism of *transformation* (Akkerman & Bruining, 2016). The discourse justifying the new, hybrid practices rely on the literature references (Cusi et al., 2017) presented by the researchers during the PD program.

> For example, the activity for students proposed by Franca during a meeting of the second year, based on a researchers’ hint (Fig. 3), was:

Divide a rectangle, having dimensions 6 cm and 9 cm, into many squares, equal to each other and which have the maximum possible size.

A *help card* proposed by Franca consisted in a table, with five examples of rectangles, in which the students had to fill two columns, one with the two dimensions of the proposed rectangles and the other with the dimension of the chosen squares (Fig. 6).

A second *help card*, to be given, if needed, after the first one, consisted in a question aimed at focusing the students’ attention on the fact that the side length of each square should be the greatest common divisor of the measurements of the sides’ lengths of the corresponding rectangle:

> Does the side of each square you have obtained represent a "special divisor" for the dimensions of the rectangle?

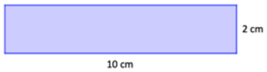
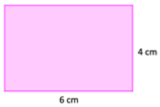
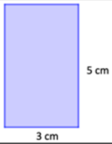
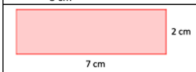
	Sides of the rectangle	Side of the square
		
		
		
		
		

Fig. 6 *Help card* proposed by Franca

To take into account the needs of the high-achieving students, Franca proposed a further step of the task, with a further question, intended only for the students who successfully completed the first steps. She called this further request *enhancement card*.

> Franca: "I propose to investigate multiples and divisors of the numbers 12, 24, 60, to prompt reflections on the different divisibility of these numbers and on their use. For example, in the case of 60, for the measurement of time or in the subdivision of the year into months and seasons."

This idea of the *enhancement cards* will be resumed by the researchers during the second year of the "Medie 2.0" PD program, with the aim of addressing the different needs of the students in the class.

The *flow* model

During the second year of the PD program "Medie 2.0", the researchers presented the teachers with new theoretical references, as a means to deepen the reflection on the didactical proposals elaborated during the previous year of the PD program and to provide new interpretative lens for the work carried on collaboratively. The source of the new theoretical proposal was Peter Liljedahl's (2018, 2023) elaboration of the Csíkszentmihályi's (1996) model of *flow*, based on the idea of *optimal experience*. According to Csíkszentmihályi, the balance between the challenge represented by an activity and the ability of the doer is fundamental for *optimal experience* that is a state in which the doer is deeply involved in the activity, experiences enjoyment and loses all track of time. The *flow* represents the balance between challenge and ability, while frustration (or anxiety) is the state obtained when the challenge of the activity exceeds the doer's ability.

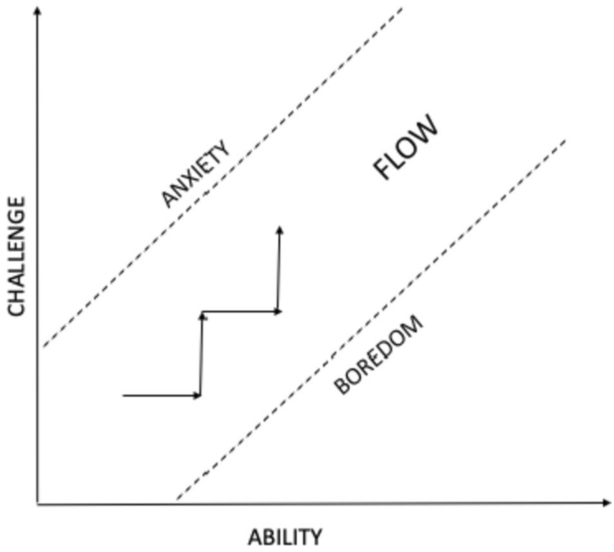


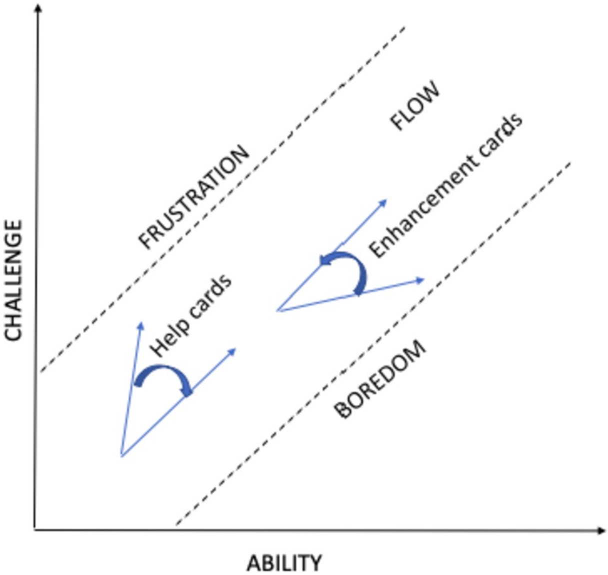
Fig. 7 *Flow* model (adapted by Liljedahl, 2018), where the vectors represent the dynamic state of a student (or a group of students)

Vice versa, boredom is obtained when the ability of the doer is greater than the challenge represented by the activity. Liljedahl (2018, 2023) applied these ideas to the mathematics classroom tasks, interpreting *flow* as the ideal state in which students should be maintained by teacher's actions. In order to obtain this effect, the teacher should gradually increase the challenge of the activity, according to the increasing students' ability, in a dynamic way.

In Fig. 7, which represents the model of *flow* according to Peter Liljedahl (2018), the ideal direction in which a student is considered in *flow* is that of the bisector of the quadrant formed by the "Ability" axis (horizontal) and the "Challenge" axis (vertical). In order to avoid student's frustration (anxiety), the teacher can give some help, if needed, and to avoid student's boredom, the teacher can make further requests, when the student completed the previous ones. The graph is totally qualitative, and it is meant to provide a metaphoric image of a dynamic process occurring during a classroom activity.

The *flow* model was deemed by the researchers as particularly in line with the inclusive teaching/learning proposal, entailing the introduction of *help cards* and *enhancement cards*, obtained as a result of the collaborative work between teachers and researchers during the PD program. For this reason, the researchers presented the teachers with a new version of Liljedahl's (2018, 2023) model, that they created including the key elements elaborated during the previous PD program meetings: *help cards* and *enhancement cards*. In this new version of the model (Fig. 8), the state of a group of students is represented by a blue arrow, indicating its evolution in time, during the classroom activity. When the direction of the blue arrow tends toward the vertical, it means that the group is at risk of experiencing anxiety, because the increase in the challenge of the activity is faster than the increase in their ability. *Help cards* give support in the form of hints, prompts, or questions at disposal of those groups who need some help to maintain the state of *flow*. The curved blue arrow, directed clockwise, represents exactly the students' returning to the *flow* direction thanks to the *help cards*. When the direction of the blue arrow tends toward the horizontal, it means that the group is at risk of experiencing boredom, because the increase in their ability is faster than the increase in the challenge of the activity. *Enhancement cards* are further requests, or new questions for the groups of students who have already accomplished the

Fig. 8 *Flow* model created by the authors, inspired by Liljedhal's (2018; 2023) model, with the integration of *help cards* and *enhancement cards*



assigned tasks and are at risk of experiencing boredom. The curved blue arrow, directed counterclockwise, represents the students' returning to the *flow* direction thanks to the *enhancement cards*. The slopes of the arrows have not to be intended in any way as a measure of the phenomenon represented, being the graph totally qualitative.

Elements of the model presented above soon became part of the *logos* block of the teachers' meta-didactical praxeologies. Indeed, the teachers started to make (explicit or implicit) reference to it during the design of their mathematics tasks, thus showing *internalization* processes of *logos* components, justifying their new *praxis*, as exemplified in the following excerpt:

> Paola: "Our idea is that of designing an activity that starts with the whole class, structured on subsequent steps, the last of these steps intended only for the group of high-achieving students. We prepared *help cards* for the students who struggle with the activity and *enhancement cards* for the students of the enhancement project."

It is noteworthy that the *enhancement cards*, introduced by the researchers in the *flow* model, came from a teacher's idea (§4.2.1), and they soon became shared among all the teachers, thus exemplifying how the evolution of the meta-didactical praxeologies of both communities are intertwined and influence each other.

Internalization processes for both communities

Internalization processes can be traced in the *logos* block of the meta-didactical praxeologies of both communities, related to the integration, in the theoretical references for the PD program, of elements coming from the FAsMEd project (Cusi et al., 2017) and the model of *flow* (Liljedahl, 2018, 2023). The researchers were already aware of these approaches, but they had never adopted them for the design of inquiry mathematics tasks in the context of the PD program "Medie 2.0", while the teachers did not know them at all. These approaches are in line with the expressed teachers' needs and feedback and complement the

Table 3 Internalized components of meta-didactical praxeologies for teachers and researchers

Type of task: designing inquiry mathematics tasks for students during the PD program "Medie 2.0"	
Praxis	*Logos*
Researchers	
Techniques Proposing teachers to design *help cards* (Pocalana & Robutti, 2022) to be given to the groups of students needing some help Proposing teachers to design *enhancement cards* for the groups of students needing further stimuli	New theoretical references useful to be proposed to teachers for the design of inclusive tasks are *Helping worksheets* of the FAsMEd project (Cusi et al., 2017) Model of *flow* (Liljedahl, 2018, 2023)
Teachers	
Techniques Using researchers' hints and ideas to design the same inquiry mathematics task, developed in gradual steps, intended for all students, who work in groups Designing *help cards* (Pocalana & Robutti, 2022) corresponding to that task, for the groups needing some help; Designing *enhancement cards* corresponding to that task, for the groups of students needing further stimuli	The inquiry mathematics tasks designed during the PD program "Medie 2.0" can be proposed to the whole classes New theoretical references for the task design: *Helping worksheets* of the FAsMEd project (Cusi et al., 2017); Model of *flow* (Liljedahl, 2018, 2023)

original theoretical framework of inquiry-based approach (e.g., Artigue & Blomhøj, 2013; Laursen & Rasmussen, 2019; Maaß & Artigue, 2013) on which the PD program is based. Due to these internalized components, the teachers also internalized a new awareness that became part of their final *logos* block: the inquiry mathematics tasks designed during the PD program "Medie 2.0" should be conceived to be proposed to the whole classes.

On the other hand, correspondent *internalization processes* can be traced in the *praxis* block of both communities. More specifically, for teachers, *internalization* concerns: 1) the design of inquiry mathematics tasks, intended for all students working in groups of three (or in pairs) (Anichini et al., 2004; Arzarello & Robutti, 2008; Cusi et al., 2017); 2) the design of inquiry mathematics tasks structured in gradual steps of increasing difficulty; 3) the design of corresponding *help cards* (Pocalana & Robutti, 2023) and *enhancement cards*. For the researchers, the first two techniques components were already usually adopted and proposed in several PD programs, while the third component has been introduced and internalized in the context of the PD program "Medie 2.0." So, we can summarize the internalized components of meta-didactical praxeologies for both communities, in Table 3.

Figure 9 shows an example of a task designed collaboratively by researchers and teachers toward the end of the second year of PD program, thought to be proposed to all the students in the classroom, divided in groups of three (or pairs). The task takes the form of a game, based on the concepts of factors and divisibility. The question proposed at the end of the task is conceived as the first step of the activity, accessible to all the groups.

Besides the task, the teachers, following the researchers' requests, also designed different types of *help cards* (Fig. 10) and *enhancement cards* (Fig. 11), to be given to the different groups, according to their needs.

The first *help card* (at the top of Fig. 10) is intended for students with difficulties in recognizing the factors of the numbers. The second *help card* (at the bottom of Fig. 10)

Game: Thief of factors

- Start with a collection of 12 tickets, with the numbers from 1 to 12 (one number for each ticket).
- You choose one ticket to keep.
- Once you choose, the thief gets all tickets remaining with numbers that are factors of the number you chose.
- The thief must get at least one ticket after every move. If you have no moves that give the thief a ticket, then the game is over and the thief gets all the remaining tickets.

- Get at least 22!
- Can you describe your strategy?

Fig. 9 Example of task designed at the end of the second year of the PD program "Medie 2.0"

THE FACTORS OF THE NUMBERS FROM 1 TO 12

1 → 1	7 → 1; 7
2 → 1; 2	8 → 1; 2; 4; 8
3 → 1; 3	9 → 1; 3; 9
4 → 1; 2; 4	10 → 1; 2; 5; 10
5 → 1; 5	11 → 1; 11
6 → 1; 2; 3; 6	12 → 1; 2; 3; 6; 12

HOW CAN YOU USE THE PRIME NUMBERS 7 AND 11?

Fig. 10 Two examples of *help cards*, designed for groups with different needs

is a reflection question aimed at focusing the students' attention on the peculiar role of prime numbers in the strategy to address the task.

Figure 11 shows examples of *enhancements cards*, designed by the teachers for the groups who need further stimuli, because they successfully find a strategy to address the original task. They represent subsequent steps of increasing difficulty in which the task is subdivided.

In the first two examples of *enhancement cards* (at the top of Fig. 11), there are extensions of the original task in terms of the final sum to be reached. In the last two

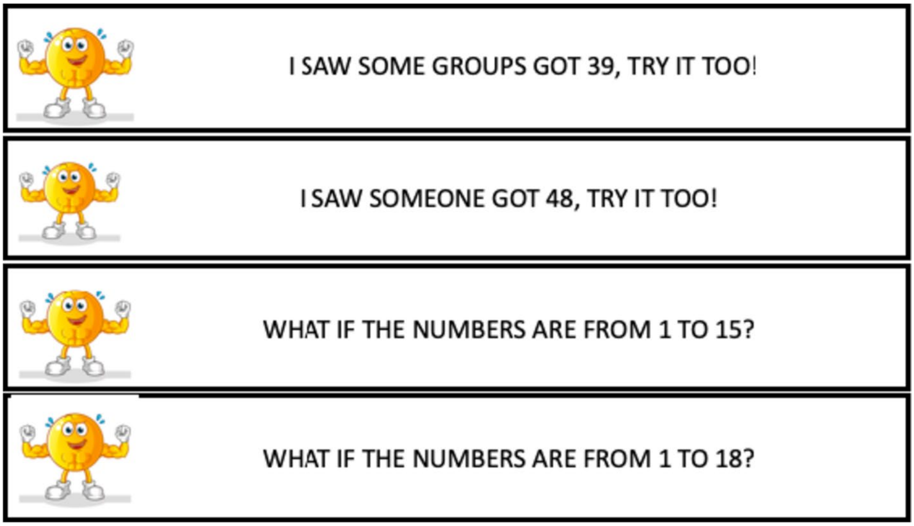


Fig. 11 Four examples of *enhancement cards*, representing subsequent steps of the task

examples (at the bottom of Fig. 11), there are extensions of the original task in terms of the amount of numbers to consider.

As a consequence of the *internalization processes* described in Table 3 and exemplified in Figs. 9, 10 and 11, it can be revealed a hybridization, as well as a crystallization of new practices and justifying discourses (Akkerman & Bruining, 2016) for the two communities, at the end of the second year of the PD program. Parallel to this, we can trace an evolution of teachers' beliefs, which are intrinsically connected to the discourses justifying their practices, as we will show in the following section.

Evolution of teacher beliefs

Toward the end of the second year of PD program "Medie 2.0," the researchers asked the teachers to answer two questions, in writing, about the inquiry mathematics task they had been working on during the last meetings, described in §4.2 (Figs. 9, 10 and 11). The questions were meant to understand if there had been an evolution in their beliefs about the appropriateness of this kind of tasks for all their students. The questions were:

1) *For how many students in your class, and for which ones, do you think this task would be appropriate?*
2) *For whom do you think this task may not be appropriate?*

Out of 17 respondents, only one declared that the inquiry mathematics task was not appropriate for all the students. Another one said that it would not be appropriate only for a student with a disability certification, who would have needed some adjustments. In the coding of the teachers' oral and written productions, at the end of the PD program "Medie 2.0," the theme "appropriateness of inquiry tasks for all the students" was predominant, with only one teacher expressing beliefs coded with the theme "appropriateness of

Table 4 Teachers' beliefs at the end of the PD program

Inquiry mathematics tasks can be appropriate for all students, with specific adaptations for different groups of students
Excerpts showing examples of teachers' justifying discourses
Beatrice: "We have imagined different levels, so the task can involve the whole class."
Paola: "[*This inquiry task is appropriate*] for all students, with increasingly complex questions. The activity starts in the whole class, with the involvement of everyone in working groups. The initial stage is appropriate for everyone. The last steps are intended only for the group of high-achieving students."
Carolina: "[*This inquiry task is appropriate*] for all students, using *help cards* and dividing them into working groups. If I think about my current classes, even the most fragile students, using *help cards*, would be able to carry out the activity."
Bianca: "In my 7th grade class, for a student severely autistic the task would be scaled down a lot but it might not be impossible […]. For the rest of the class I think it is a task that everyone can face, as well as for my 8th grade class."
Lucia: "In my opinion, there are no restrictions on who can face the task."
Paola: "This is an inclusive task, for everyone. So, we thought at different levels, to involve all the students of our 7th grade classes."
Franca: "I'm thinking about questions that we would give to the students one at a time. Maybe they get by themselves, they understand perfectly, they pinpoint the task. If they don't, we can go with the question that should address them in the right way."
Eleonora and Bianca: "In our opinion, proposing tasks of increasing difficulty allows greater inclusion. The weakest students would stop earlier than the others, but they would still have been able to work with their classmates. Eventually, we can think of providing *help cards* to guide the reasoning of the most fragile students, thus allowing them to move on to the next steps."

inquiry tasks for only some students." So, a remarkable evolution can be traced in teachers' beliefs about this aspect, which represented an important cause of the *meta-didactical conflict* between teachers and researchers at the beginning of the PD program "Medie 2.0." As exemplified in the excerpts in Table 4, this evolution goes hand in hand with the evolution of teachers' practices and justifying discourses (meta-didactical praxeologies).

The evolution toward shared components of final meta-didactical praxeologies among the members of the two communities, triggered by the learning mechanism of *transformation* (§4.2), is concomitant and intertwined with the evolution of teachers' beliefs on inquiry mathematics tasks. These findings confirmed our hypothesis of adaptation of the MDT.2 model (Fig. 5).

Discussion and conclusions

In this paper, we explored the interactions between mathematics teachers and researchers in mathematics education, involved in a PD program, working together collaboratively on a BO, that is constituted by the design of inquiry mathematics tasks for students. To address our first research question, we described the evolution of teachers' and researchers' meta-didactical praxeologies for designing inquiry mathematics tasks, toward components of final meta-didactical praxeologies shared among the members of both communities. These evolution processes toward shared practices and justifying discourses occurred thanks to *internalization processes* for the members of both communities. In turn, the *internalization processes* have been interpreted as consequences of the learning mechanism of *transformation* (Akkerman & Bakker, 2011), activated by the common work on the BO. A pivotal role in triggering the *transformation* mechanism and, consequently, fostering the convergence of both communities toward shared components of final meta-didactical praxeologies has been fulfilled by the sharing of new theoretical references during the PD program meetings. These references, coming from distinct frameworks, specifically the helping worksheets from the FAsMEd project (Cusi et al., 2017) and the model of *flow* (Liljedahl, 2018, 2023), constituted the basis of the internalization (Cusi et al., 2023) of *logos* components and correspondent *praxis* components for both communities. The new *logos* components introduced by the researchers allowed overcoming the meta-didactical conflict (Arzarello & Ferretti, 2022) with the teachers.

To address our second research question, we tested the model (Fig. 5) that we introduced as a hypothesis, in which the evolution of teachers' beliefs during the PD program is considered both as a motivational agent and a consequence of the *internalization processes* and, consequently, of the *transformation* mechanism occurred for both communities. Through our analysis, we found that there is a richer relationship between teachers' beliefs and their meta-didactical praxeologies than that described in Cusi et al. (2023): indeed, the evolution of teachers' beliefs on inquiry mathematics tasks during the PD program proved to be both an agent and a consequence of the evolution of their meta-didactical praxeologies, which, in turn, is intertwined with the evolution of researchers' meta-didactical praxeologies. This is coherent with Guskey's (2000, 2002) and Swan's (2007) conceptualization of the double direction relationship between teachers' practices and beliefs, but it goes further, also taking into account the *logos* component of their praxeologies and their relationship with researchers' praxeologies. We are aware that it is not always possible to make a strict distinction between teachers' *logos* and beliefs because they are intricately woven and interconnected. However, in this study, we have considered *logos* to be closely tied to the type of tasks and the techniques of the praxeologies under examination, strongly linked to the institutional context under consideration. Conversely, we have regarded beliefs as more general views, lenses through which teachers see mathematics, teaching strategies, and their students' needs, which influence their praxeologies and are in turn influenced by them. So, the analysis of the case study reported in this paper confirmed the validity of our model, obtained as an adaptation of MDT.2 model (Cusi et al., 2023).

A limitation of this study is that the teachers' classroom practices have been inferred from their discussions during the PD program meetings and from their reports and written productions, without a direct observation by the researchers, due to the Covid-19 restrictions. However, we think that this circumstance does not affect the validity of our results, because our analysis is focused on teachers' meta-didactical praxeologies for designing inquiry mathematics tasks. Therefore, our interest has been focused on the teachers' practices and justifying discourses for the work during the PD program, in the task design activity and in the reflection on the task design. Further research is needed in order to deepen the analysis of the intertwined evolution of researchers' meta-didactical praxeologies and beliefs. Since it is not easy for researchers to study their own beliefs, it may be productive to study the correlation between meta-didactical praxeologies and beliefs of different researchers, conducting PD programs in different contexts. It may also be fruitful to conceptualize the experience of the convergence among teachers and researchers as a PD path for both of them, analyzing the evolution of the respective meta-didactical praxeologies and beliefs.

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Data availability Not applicable.

Code availability Not applicable.

Declarations

Conflict of interest The authors declare that they do not have any conflict of interest.

References

Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132–169. https://doi.org/10.3102/0034654311404435

Akkerman, S., & Bruining, T. (2016). Multilevel Boundary Crossing in a Professional Development School Partnership. *Journal of the Learning Sciences, 25*(2), 240–284. https://doi.org/10.1080/10508406.2016.1147448

Anichini, G., Arzarello, F., Ciarrapico, L. and Robutti, O. (Eds.). (2004). *Matematica 2003. Attività didattiche e prove di verifica per un nuovo curricolo di matematica (ciclo secondario)*. Matteoni Stampatore.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM - Mathematics Education, 45*, 797–810. https://doi.org/10.1007/s11858-013-0506-6

Arzarello, F., & Robutti, O. (2008). Framing the embodied mind approach within a multimodal paradigm. In: English, L.D., & Kirshner, D. (Eds.). *Handbook of International Research in Mathematics Education*. https://doi.org/10.4324/9780203930236

Arzarello, F., & Ferretti, F. (2022). Links between the INVALSI Mathematics test and teaching practices: An exploratory study. In P. Falzetti (Ed.), *INVALSI data to investigate the characteristics of students, schools and society* (pp. 96–109). Franco Angeli.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., & Martignone, F. (2014). Meta-didactical transposition: A theoretical model for teacher education programmes. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 347–372). Springer Science+Business Media.

Borko, H., & Potari, D. (Eds). (2020). *Teachers of Mathematics Working and Learning in Collaborative Groups. The 25th ICMI Study*. Springer.

Carlile, P. (2004). Transferring, translating, and transforming: An integrative framework for managing knowledge across boundaries. *Organization Science, 15*(5), 555–568.

Chevallard, Y. (1985). *La transposition didactique du savoir savant au savoir enseigné*. La Pensée Sauvage.

Chevallard, Y., & Bosch, M. (2020). Anthropological theory of the didactic (ATD). In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 53–61). Springer. https://doi.org/10.1007/978-3-030-15789-0_100034

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114. https://doi.org/10.24529/hjme.1205

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 125–146). National Council of Teachers of Mathematics.

Cross, D. I. (2009). Alignment, cohesion, and change: Examining mathematics teachers' belief structures and their influence on instructional practices. *Journal of Mathematics Teacher Education, 12*, 325–346. https://doi.org/10.1007/s10857-009-9120-5

Csíkszentmihályi, M. (1996). *Creativity: Flow and the psychology of discovery and invention*. Harper Perennial.

Cusi, A., Robutti, O., Panero, M., Taranto, E., & Aldon, G. (2023). Meta-Didactical Transposition.2: the evolution of a framework to analyse teachers' collaborative work with researchers in technological settings. In A. Clark-Wilson, O. Robutti & N. Sinclair (Eds.), *Mathematics Teacher in the Digital Era – 2nd Edition*. Springer. https://doi.org/10.1007/978-3-031-05254-5_14

Cusi, A., Morselli, F., & Sabena, C. (2017). Promoting formative assessment in a connected classroom environment: Design and implementation of digital resources. *ZDM - Mathematics Education, 49*(5), 755–767. https://doi.org/10.1007/s11858-017-0878-0

Furinghetti, F., & Pehkonen, E. (2002). Rethinking characterizations of beliefs. In G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 39–57). Springer. https://doi.org/10.1007/0-306-47958-3_3

Guskey, T. R. (1986). Staff development and the process of teacher change. *Educational Researcher, 15*(5), 5–12. https://doi.org/10.3102/0013189X015005005

Guskey, T. R. (2000). *Evaluating professional development*. Corwin Press.

Guskey, T. R. (2002). Professional development and teacher change. *Teachers and Teaching: Theory and Practice, 8*(3/4), 381–391. https://doi.org/10.1080/135406002100000512

Jackiw, N. & Sinclair, N. (2019). TouchTimes [iPad application software]. Burnaby, BC: Tangible Mathematics Group. https://apps.apple.com/ca/app/touchtimes/id1469862750

Kynigos, C., & Kalogeria, E. (2012). Boundary crossing through in-service online mathematics teacher education: The case of scenarios and half-baked microworlds. *ZDM - Mathematics Education, 44*(6), 733–745. https://doi.org/10.1007/s11858-012-0455-5

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education, 5*, 129–146. https://doi.org/10.1007/s40753-019-00085-6

Leatham, K. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9*, 91–102. https://doi.org/10.1007/s10857-006-9006-8

Leikin, D. (2006). Motion, technology, gestures in interpreting graphs. *International Journal for Technology in Mathematics Education, 13*(3), 117–125.

Liljedahl, P. (2010). Noticing rapid and profound mathematics teacher change. *Journal of Mathematics Teacher Education, 13*(5), 411–423. https://doi.org/10.1007/s10857-010-9151-y

Liljedahl, P. (2018). On the edges of flow: Student problem solving behavior. In S. Carreira, N. Amado, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect* (pp. 505–524). Springer.

Liljedahl, P. (2023). Flow and variation theory: powerful allies in creating and maintaining thinking in the classroom. In R. Leikin (Ed.), *Mathematical Challenges for All* (pp. 539–563). Springer.

Liljedahl, P., Oesterle, S., & Bernèche, C. (2012). Stability of beliefs in mathematics education: A critical analysis. *Nordic Studies in Mathematics Education, 17*(3–4), 101–118.

Lloyd, G. M. (1999). Two teachers' conceptions of a reform-oriented curriculum: Implications for mathematics teacher development. *Journal of Mathematics Teacher Education, 2*, 227–252.

Maaß, K., & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: A synthesis. *ZDM - Mathematics Education, 45*(6), 779–795. https://doi.org/10.1007/s11858-013-0528-0

Minisola, R., Robutti, O., & Miyakawa, T. (2024). Didacticians introducing lesson study for the professional development of prospective mathematics teachers. *Asian Journal for Mathematics Education*. https://doi.org/10.1177/27527263241228324

Miyakawa, T., & Winsløw, C. (2019). Paradidactic infrastructure for sharing and documenting mathematics teacher knowledge: A case study of "practice research" in Japan. *Journal of Mathematics Teacher Education, 22*, 281–303. https://doi.org/10.1007/s10857-017-9394-y

Nolen, S. B., Horn, I. S., Ward, C. J., & Childers, S. A. (2011). Novice teacher learning and motivation across contexts: Assessment tools as boundary objects. *Cognition and Instruction, 29*(1), 88–122. https://doi.org/10.1080/07370008.2010.533221

Prodromou, T., Robutti, O., & Panero, M. (2018). Making sense out of the emerging complexity inherent in professional development. *Mathematics Education Research Journal, 30*(4), 445–473. https://doi.org/10.1007/s13394-017-0229-z

Pocalana, G., & Robutti, O. (2022). Mathematics teacher educators' work to foster an inquiry community. In Fernández, C., Llinares, S., Gutiérrez, A., & Planas, N. (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3). Alicante, Spain: PME.

Pocalana, G., Robutti, O., & Liljedahl, P. (2023). Inquiry activities are not for everyone: teachers' beliefs and professional development. *International Journal of Mathematical Education in Science and Technology, 54*(8), 1557–1580. https://doi.org/10.1080/0020739X.2023.2176795

Pocalana, G., & Robutti, O. (2023). Evolution of didacticians' meta-didactical praxeologies and documentation work. *International Journal of Science and Mathematics Education*, 22, 211–233. https://doi.org/10.1007/s10763-023-10367-w

Robutti, O., Cusi, A., Clark-Wilson, A., Jaworski, B., Chapman, O., Esteley, C., Goos, M., Isoda, M., & Joubert, M. (2016). ICME international survey on teachers working and learning through collaboration. *ZDM - Mathematics Education*, *48*, 651–690. https://doi.org/10.1007/s11858-016-0797-5

Robutti, O. (2020). Meta-didactical Transposition. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 611 – 619). Springer. https://doi.org/10.1007/978-3-319-77487-9_100012-1

Robutti, O., Aldon, G., Cusi, A., Olsher, S., Panero, M., Cooper, J., Carante, P. & Prodromou, T. (2020). Boundary Objects in Mathematics Education and Their Role across Communities of Teachers and Researchers in Interaction. In G. M. Liloyd & O. Chapman (Eds.), *International Handbook of Mathematics Teacher, 2nd Edition. Volume 3: Participants in Mathematics Teacher Education* (pp. 211–240). Brill-Sense. https://doi.org/10.1163/9789004419230_009

Sinclair, N., Chorney, S., Gunes, C., & Bakos, S. (2020). Disruptions in meanings: Teachers' experiences of multiplication in TouchTimes. *ZDM - Mathematics Education*. https://doi.org/10.1007/s11858-020-01163-9

Skott, J. (2009). Contextualising the notion of 'belief enactment.' *Journal of Mathematics Teacher Education, 12*, 27–46. https://doi.org/10.1007/s10857-008-9093-9

Star, S. L. (2010). This is not a boundary object: Reflections on the origin of a concept. *Science, Technology, & Human Values, 35*(5), 601–617. https://doi.org/10.1177/0162243910377624

Swan, M. (2007). The impact of task-based professional development on teachers' practices and beliefs: A design research study. *Journal of Mathematics Teachers Education, 10*, 217–237. https://doi.org/10.1007/s10857-007-9038-8

Sztajn, P. (2003). Adapting reform ideas in different mathematics classrooms: Beliefs beyond mathematics. *Journal of Mathematics Teacher Education, 6*, 53–75. https://doi.org/10.1023/A:1022171531285

Sztajn, P., Wilson, P. H., Edgington, C., & Myers, M. (2014). Mathematics professional development as design for boundary encounters. *ZDM - Mathematics Education, 46*(2), 201–212. https://doi.org/10.1007/s11858-013-0560-0

Wake, G., Swan, M., & Foster, C. (2016). Professional learning through the collaborative design of problem-solving lessons. *Journal of Mathematics Teacher Education, 19*(2), 1–18. https://doi.org/10.13140/RG.2.1.3219.1849

Wilkins, J. L. M. (2008). The relationship among elementary teachers' content knowledge, attitudes, beliefs, and practices. *Journal of Mathematics Teacher Education, 11*(2), 139–164. https://doi.org/10.1007/s10857-007-9068-2

Wilson, M. S., & Cooney, T. (2002). Mathematics teacher change and development. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education* (pp. 127–147). Kluwer. https://doi.org/10.1007/0-306-47958-3_8

Wong, N. Y., Ding, R., & Zhang, Q. P. (2016). From classroom environment to conception of mathematics. In R. B. King & A. B. I. Bernardo (Eds.), *The psychology of Asian learners* (pp. 541–557). Springer.