

Pseudo-differential equations, and the Bethe ansatz for the classical Lie algebras

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Abstract

The correspondence between ordinary differential equations and Bethe ansatz equations for integrable lattice models in their continuum limits is generalised to vertex models related to classical simple Lie algebras. New families of pseudo-differential equations are proposed, and a link between specific generalised eigenvalue problems for these equations and the Bethe ansatz is deduced. The pseudo-differential operators resemble in form the Miura-transformed Lax operators studied in work on generalised KdV equations, classical W-algebras and, more recently, in the context of the geometric Langlands correspondence. Negative-dimension and boundary-condition dualities are also observed.

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1. Introduction

A recent observation [1] has established an unexpected link between two-dimensional conformal field theory (CFT) and the theory of ordinary differential equations. This rests on a correspondence between the transfer matrix eigenvalues of certain integrable models (IMs), in their conformal limits [2,3], and the spectral determinants [4,5] of ordinary differential equations.

The initial results [1,6–8] connected conformal field theories with Virasoro central charge $c \leq 1$ with Schrödinger problems for one-dimensional anharmonic oscillators. These conformal field theories are naturally associated to the Lie algebra A_1 , but a generalisation to models related to A_{n-1} , with additional extended W-algebra symmetries, was soon established [9–12]. However, contrary to initial expectations, a simple Lie-algebraic structure did not emerge immediately, and the extension of the correspondence to the theories associated with other simple Lie algebras \mathfrak{g} has proved surprisingly elusive.

The purpose of this paper is to begin to fill this gap, by establishing a link between CFTs related to the classical simple Lie algebras and spectral problems associated with a set of ordinary (pseudo-) differential equations.

We shall also prove for $\mathfrak{g} = A_{n-1}$, and conjecture for the other simple Lie algebras, the existence of closed systems of functional equations (ψ -systems) among uniquely-defined solutions $\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(\text{rank}(\mathfrak{g}))}$ of a set of $\text{rank}(\mathfrak{g})$ pseudo-differential operators, with each pair $\psi^{(a)}/(\text{operator})_a$ being naturally associated to a node of the Dynkin diagram. These ψ -systems are very similar to the systems of functional relations introduced by Mukhin and Varchenko in the framework of the Bethe ansatz (BA) method for \mathfrak{g} -XXX quantum spin chains [13–16], and in the context of the so-called Miura-opers related to the geometric Langlands correspondence (see, for example, [17,18]). This similarity is related to the fact that the homogeneous ‘differential’ parts of the operators studied here resemble, in form, the Miura-transformed Lax operators introduced by Drinfel’d and Sokolov in their studies of generalised KdV equations and classical W-algebras [19].

The rest of the paper is organised as follows. Section 2 gathers together some known, or easily deduced, properties of the Bethe equations for \mathfrak{g} -type quantum spin chains in their continuum limits. Our main results are summarised in Section 3, while extra details and numerical support for the specific A , D , B and C proposals are given in Sections 4–7 respectively, and Section 8 contains our general conclusions. There are two appendices: Appendix A deals with the semiclassical analysis for A_1 -related ODEs in the presence of string solutions, and Appendix B describes a simple algorithm useful for the numerical solutions of the differential equations. The algorithm is a generalisation of Cheng’s method from Regge pole theory [20], and relies on an elegant dual formulation of the relevant boundary problems.

2. The Bethe ansatz equations and their string solutions

For any simple Lie algebra \mathfrak{g} of type A_{n-1} to G_2 , a set of Bethe ansatz equations (BAEs), depending on a set of $\text{rank}(\mathfrak{g})$ twist parameters $\gamma = \{\gamma_a\}$, can be written in a universal form as

[21–26]¹

$$\prod_{b=1}^{\text{rank}(\mathfrak{g})} \Omega^{B_{ab}\gamma_b} \frac{Q_{B_{ab}}^{(b)}(E_i^{(a)}, \gamma)}{Q_{-B_{ab}}^{(b)}(E_i^{(a)}, \gamma)} = -1, \quad i = 0, 1, 2, \dots, \tag{2.1}$$

where

$$Q_k^{(a)}(E, \gamma) = Q^{(a)}(\Omega^k E, \gamma), \tag{2.2}$$

and the numbers $E_i^{(a)}$ are the—in general complex—zeros of the functions $Q^{(a)}$:

$$Q^{(a)}(E_i^{(a)}, \gamma) = 0. \tag{2.3}$$

In (2.1) and (2.3) the indices a and b label the simple roots of the Lie algebra, the matrix B_{ab} is defined by

$$B_{ab} = \frac{(\alpha_a, \alpha_b)}{|\text{long roots}|^2}, \quad a, b = 1, 2, \dots, \text{rank}(\mathfrak{g}) \tag{2.4}$$

and the α 's are the simple roots of \mathfrak{g} . The constant Ω is a pure phase, parameterised in terms of a real number $\mu > 0$ as

$$\Omega = \exp\left(i \frac{2\pi}{h^\vee \mu}\right) \tag{2.5}$$

where h^\vee is the dual Coxeter number of \mathfrak{g} . Strictly speaking, the BAE (2.1) arise from taking a suitable continuum or field theory limit of the lattice model BAE, in the fashion explained in, for example, [27].

The functions $Q^{(a)}$ appearing in (2.1) have a characteristic asymptotic behaviour at large values of $-E$

$$\ln Q^{(a)}(-E, \gamma) = m_a \frac{\sin(\frac{\pi}{h^\vee})}{\sin(\frac{\pi}{h^\vee} B_{aa})} (-E)^\mu + \dots \tag{2.6}$$

For the A_{n-1} , B_n , C_n and D_n models the sets $\{m_a\}$ are given in Table 1.² The only free parameter—the overall constant m in Table 1—depends on the way the conformal field theory limit is reached.

The negative real E axis is also the direction of *maximal* growth for $\ln Q^{(a)}(E)$ as $|E| \rightarrow \infty$. From (2.6), the Hadamard order of $Q^{(a)}$ is therefore μ and, in the so-called ‘semiclassical regime’ $0 < \mu < 1$, $Q^{(a)}$ can be written as a convergent infinite product over its zeros as

$$Q^{(a)}(E, \gamma) = Q^{(a)}(0, \gamma) \prod_{i=0}^{\infty} \left(1 - \frac{E}{E_i^{(a)}}\right). \tag{2.7}$$

¹ For finite lattice models, the explicit diagonalisation of the A_{n-1} cases has been performed through the algebraic Bethe ansatz by Schulz [21] and also by Babelon et al. [22]. For C_n and D_n models, it has been done by Reshetikhin [23,24]. There is a shortcut to reach the same conclusions via the so-called analytic Bethe ansatz of Reshetikhin [25], and Wiegmann and Reshetikhin [26].

² The constants $\{m_a\}$ are related to a particular matrix K_{ab} emerging from the analysis of the Bethe ansatz. For simply-laced algebras, K_{ab} is proportional to the Cartan matrix and $\vec{v} = (m_1, m_2, \dots, m_r)$ is its Perron–Frobenius eigenvector.

Table 1

Dual Coxeter numbers and coefficients $\{m_a\}$ for models based on classical simple Lie algebras

Model	h^\vee	m_a
A_{n-1}	n	$m_a = 2m \sin(a \frac{\pi}{h^\vee})$ ($a = 1, \dots, n-1$)
D_n	$2n-2$	$m_{n-1} = m_n = m, m_a = 2m \sin(a \frac{\pi}{h^\vee})$ ($a = 1, \dots, n-2$)
B_n	$2n-1$	$m_n = m, m_a = 2m \sin(a \frac{\pi}{h^\vee})$ ($a = 1, \dots, n-1$)
C_n	$n+1$	$m_a = 2m \sin(a \frac{\pi}{2h^\vee})$ ($a = 1, \dots, n$)

It turns out that the Bethe ansatz roots generally split into multiplets with approximately equal modulus $|E_i^{(a)}|$, and that the ground state of the quantum spin chain corresponds to a ‘pure’ configuration of roots, containing only multiplets with a common dimension

$$d_a = \frac{K}{B_{aa}}. \quad (2.8)$$

The integer K in (2.8) depends on the particular spin chain under discussion, and corresponds to the degree of fusion [28–30]. For $\mathfrak{g} = A_1$, the spin- j A_1 quantum chains, $K = d_1 = 2j$.

It is generally expected [31] that for large values of i the zeros asymptotically tend to the *perfect* string configurations:

$$\arg E_i^{(a)} \sim (d_a + 1 - 2l) \frac{B_{aa}\pi}{h^\vee \mu}, \quad l = 1, 2, \dots, d_a. \quad (2.9)$$

Appendix A contains some further discussion of the asymptotic behaviour of these string solutions.

3. Summary of the main results

This paper is about the correspondence between the Bethe ansatz equations (2.1) for $\mathfrak{g} = A_{n-1}, B_n, C_n, D_n$ and spectral problems associated to solutions $\psi(x, E, \mathfrak{g})$ of particular pseudo-differential equations with vanishing boundary conditions

$$\psi(x, E, \mathfrak{g}) = o(e^{-\frac{x^{M+1}}{M+1}}) \quad (M > K/(h^\vee - K)) \quad (3.1)$$

imposed at large x on the positive real axis. To specify these equations we introduce the n th-order differential operator [11]

$$D_n(\mathfrak{g}) = D(g_{n-1} - (n-1))D(g_{n-2} - (n-2)) \cdots D(g_1 - 1)D(g_0), \quad (3.2)$$

$$D(g) = \left(\frac{d}{dx} - \frac{g}{x} \right), \quad (3.3)$$

with

$$\mathfrak{g} = \{g_{n-1}, \dots, g_1, g_0\}, \quad \mathfrak{g}^\dagger = \{n-1-g_0, n-1-g_1, \dots, n-1-g_{n-1}\}. \quad (3.4)$$

We also use the inverse differential operator $(d/dx)^{-1}$, generally defined through its formal action

$$\left(\frac{d}{dx} \right)^{-1} x^s = \frac{x^{s+1}}{s+1}. \quad (3.5)$$

The following properties hold

$$\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)^{-1}x^s = x^s, \quad \left(\frac{d}{dx}\right)^{-1}\left(\frac{d}{dx}\right)x^s = x^s \quad (s \neq 0), \tag{3.6}$$

and the integration by parts property

$$\left(\frac{d}{dx}\right)^{-1}\left[f(x)\frac{d}{dx}g(x)\right] = f(x)g(x) - \left(\frac{d}{dx}\right)^{-1}\left[g(x)\frac{d}{dx}f(x)\right] \tag{3.7}$$

is satisfied, provided in the x -expansion of $f(x)g(x)$ about the origin the constant term is absent. In the following we shall assume the validity of (3.7) by working implicitly with non-integer values for the parameters g_i introduced in (3.4), and by invoking continuity of the final results in these parameters.

Finally, we define a basic ‘potential’

$$P_K(E, x) = (x^{h \vee M/K} - E)^K. \tag{3.8}$$

With this notation in place, the following pseudo-differential equations are the main concern of this article:

A_{n-1} (su(n)):

$$((-1)^n D_n(\mathbf{g}) - P_K(x, E))\psi(x, E, \mathbf{g}) = 0, \tag{3.9}$$

with the constraint $\sum_{i=0}^{n-1} g_i = \frac{n(n-1)}{2}$.

D_n (so($2n$)):

$$\left(D_n(\mathbf{g}^\dagger)\left(\frac{d}{dx}\right)^{-1}D_n(\mathbf{g}) - \sqrt{P_K(x, E)}\left(\frac{d}{dx}\right)\sqrt{P_K(x, E)}\right)\psi(x, E, \mathbf{g}) = 0. \tag{3.10}$$

B_n (so($2n + 1$)):

$$\left(D_n(\mathbf{g}^\dagger)D_n(\mathbf{g}) + \sqrt{P_K(x, E)}\left(\frac{d}{dx}\right)\sqrt{P_K(x, E)}\right)\psi(x, E, \mathbf{g}) = 0. \tag{3.11}$$

C_n (sp($2n$)):

$$\left(D_n(\mathbf{g}^\dagger)\left(\frac{d}{dx}\right)D_n(\mathbf{g}) - P_K(x, E)\left(\frac{d}{dx}\right)^{-1}P_K(x, E)\right)\psi(x, E, \mathbf{g}) = 0. \tag{3.12}$$

The correspondence we propose links the ground-state $Q^{(1)}$ ’s of (2.1) to particular solutions ψ (3.1) of Eqs. (3.18)–(3.21).

In order to clarify this statement we introduce an alternative basis of solutions $\{\chi_i(x, E, \mathbf{g})\}$ to (3.18)–(3.21), characterised by their behaviour near the origin

$$\chi_i(x, E, \mathbf{g}) \sim x^{\lambda_i} + \dots, \quad x \rightarrow 0, \tag{3.13}$$

where the λ ’s are the ordered ($\lambda_0 < \lambda_1 < \dots$) roots of the appropriate indicial equation (see Table 2).

Writing ψ as a linear combination of the χ ’s, we have in general

$$\psi(x, E, \mathbf{g}) = Q_{[0]}^{(1)}(E, \mathbf{g})\chi_0(x, E, \mathbf{g}) + Q_{[1]}^{(1)}(E, \mathbf{g})\chi_1(x, E, \mathbf{g}) + \dots. \tag{3.14}$$

Table 2

Indicial equations

Model	Indicial equation
A_{n-1}	$\prod_{i=0}^{n-1} (\lambda - g_i) = 0$
D_n	$(\lambda - h^\vee/2)^{-1} \prod_{i=0}^{n-1} (\lambda - g_i)(\lambda - h^\vee + g_i) = 0$
B_n	$\prod_{i=0}^{n-1} (\lambda - g_i)(\lambda - h^\vee + g_i) = 0$
C_n	$(\lambda - n) \prod_{i=0}^{n-1} (\lambda - g_i)(\lambda - 2n + g_i) = 0$

If the zeros of $Q_{[0]}^{(1)}(E, \mathbf{g})$ are $\{E_i^{(1)}\}$, then for $E \in \{E_i^{(1)}\}$ the function $x^{-\lambda_0} \psi(x, E, \mathbf{g})$ vanishes exceptionally both at $x = \infty$ and at $x = 0$. This allows the coefficient $Q_{[0]}^{(1)}(E, \mathbf{g})$ to be identified with the spectral determinant for a boundary problem defined on the positive real axis (see, for example, [4,5]). An alternative (dual) definition of the spectral functions $Q_{[0]}^{(1)}(E, \mathbf{g})$ in terms of the adjoint equations to (3.18)–(3.21) is briefly discussed in Appendix B.

We claim that for classical Lie algebras with arbitrary degree of fusion K , the ground-state $Q^{(1)}(E, \gamma)$'s in (2.1) and the functions $Q_{[0]}^{(1)}(E, \mathbf{g})$ in (3.14) coincide up to a trivial normalisation, so that

$$\frac{Q^{(1)}(E, \gamma)}{Q^{(1)}(0, \gamma)} = \frac{Q_{[0]}^{(1)}(E, \mathbf{g})}{Q_{[0]}^{(1)}(0, \mathbf{g})}. \tag{3.15}$$

Moreover, from the WKB approximation

$$\ln Q_{[0]}^{(1)}(-E, \mathbf{g}) = \kappa(-E)^{\hat{\mu}} + \dots \quad (E \gg 0) \tag{3.16}$$

with

$$\hat{\mu} = \frac{K(M+1)}{h^\vee M}, \quad \kappa = \kappa\left(\frac{h^\vee M}{K}, \frac{h^\vee}{K}\right) \tag{3.17}$$

and

$$\kappa(a, b) = \int_0^\infty dx \left((x^a + 1)^b - x^{ab} \right) = \frac{\Gamma(1 + 1/a)\Gamma(1 + 1/b) \sin(\pi/b)}{\Gamma(1 + 1/a + 1/b) \sin(\pi/b + \pi/a)}. \tag{3.18}$$

Therefore, in order to have a match between (3.16) and (2.6), we must set

$$\mu = \hat{\mu}, \quad m_1 = \kappa \frac{\sin(\frac{\pi}{h^\vee} B_{11})}{\sin(\frac{\pi}{h^\vee})}, \quad \Omega = \exp\left(i \frac{2\pi M}{K(M+1)}\right). \tag{3.19}$$

Given a particular ordering convention, the relationship between the twist parameters $\{\gamma_a\}$ and the constants $\{g_a\}$ is given in Table 3.

Various consistency checks, including the WKB approach and numerical work, support the correspondence both qualitatively and quantitatively.

Finally, starting from Eqs. (3.18) to (3.21), the Bethe ansatz equations and Table 3 were obtained with the help of a system of functional relations involving $\psi^{(1)}(x, E, \mathbf{g}) = \psi(x, E, \mathbf{g})$ together with other auxiliary functions $\psi^{(a)}(x, E, \mathbf{g})$, ($a = 2, \dots, \text{rank}(\mathbf{g})$) (see Sections 4–7 and

Table 3

The relationship between the set of parameters $\{g_a\} \leftrightarrow \{\gamma_a\}$ with $\alpha = 2K/(h^\vee M)$

Model	Ordering $\forall i < j$	$\{g_a\} \leftrightarrow \{\gamma_a\}$
A_{n-1}	$g_i < g_j$	$\gamma_a = \alpha(\sum_{i=0}^{a-1} g_i - \frac{a(h^\vee-1)}{2})$ $\gamma_a = \alpha(\sum_{i=0}^{a-1} g_i - \frac{a}{2}h^\vee)$ ($a = 1, \dots, n-2$)
D_n	$g_i < g_j < h^\vee/2$	$\gamma_{n-1} = \frac{\alpha}{2}(\sum_{i=0}^{n-1} g_i - \frac{n}{2}h^\vee)$ $\gamma_n = \frac{\alpha}{2}(\sum_{i=0}^{n-2} g_i - g_{n-1} - \frac{n-2}{2}h^\vee)$
B_n	$g_i < g_j < h^\vee/2$	$\gamma_a = \alpha(\sum_{i=0}^{a-1} g_i - \frac{a}{2}h^\vee)$
C_n	$g_i < g_j < n$	$\gamma_a = \alpha(\sum_{i=0}^{a-1} g_i - an)$ $\gamma_n = \frac{\alpha}{2}(\sum_{i=0}^{n-1} g_i - n^2)$

[9,11]). We set³

$$\psi_k^{(a)} = \psi^{(a)}(\omega^k x, \Omega^k E, \mathbf{g}) \tag{3.20}$$

where

$$\Omega = \exp\left(i \frac{2\pi M}{K(M+1)}\right) \tag{3.21}$$

as in (3.19), and

$$\omega = \Omega^{K/(h^\vee M)} = \exp\left(i \frac{2\pi}{h^\vee(M+1)}\right).$$

For the simply-laced algebras the ψ -systems can then be written in the compact form

$$W[\psi_{-\frac{1}{2}}^{(a)}, \psi_{\frac{1}{2}}^{(a)}] = \prod_{b=1}^{\text{rank}(\mathfrak{g})} (\psi^{(b)})^{A_{ab}}, \tag{3.22}$$

where $A_{ab} = 2\delta_{ab} - 2B_{ab}$ is the incidence matrix of the corresponding Dynkin diagram and W the Wronskian:

$$W[f, g] = f(x) \frac{d}{dx} g(x) - g(x) \frac{d}{dx} f(x). \tag{3.23}$$

Eq. (3.22) is proven in Section 4.2 for $\mathfrak{g} = A_{n-1}$, and our numerical results indirectly support the validity of (3.22) for $\mathfrak{g} = D_n$. Currently, we have no analogous pseudo-differential equations for the exceptional Lie algebras but the similarity between (3.22), the relations proposed in [13–15] and the other functional equations (Y-systems and T-systems) discovered in the framework of integrable models [32–37] suggests the validity of (3.22) in its more general form. For the non-simply-laced algebras our conjectures are

$$\begin{aligned} B_n: \quad W[\psi_{-\frac{1}{2}}^{(a)}, \psi_{\frac{1}{2}}^{(a)}] &= \psi^{(a-1)} \psi^{(a+1)}, \quad a = 1, \dots, n-1, \\ W[\psi_{-\frac{1}{4}}^{(n)}, \psi_{\frac{1}{4}}^{(n)}] &= \psi_{-\frac{1}{4}}^{(n-1)} \psi_{\frac{1}{4}}^{(n-1)}. \end{aligned} \tag{3.24}$$

³ In the A_{n-1} models $\psi_k^{(1)}(x, E, \mathbf{g}) = \omega^{(n-1)k/2} y_{-k}(x, E, \mathbf{g})$, where y_k is the function defined in Section 3 of [11].

(Our root convention is $(\alpha_i|\alpha_i) = 2$ for $i = 1, 2, \dots, n-1$ and $(\alpha_n|\alpha_n) = 1$.)

$$\begin{aligned}
 C_n: \quad W\left[\psi_{-\frac{1}{4}}^{(a)}, \psi_{\frac{1}{4}}^{(a)}\right] &= \psi^{(a-1)}\psi^{(a+1)}, \quad a = 1, \dots, n-2, \\
 W\left[\psi_{-\frac{1}{4}}^{(n-1)}, \psi_{\frac{1}{4}}^{(n-1)}\right] &= \psi^{(n-2)}\psi_{-\frac{1}{4}}^{(n)}\psi_{\frac{1}{4}}^{(n)}, \\
 W\left[\psi_{-\frac{1}{2}}^{(n)}, \psi_{\frac{1}{2}}^{(n)}\right] &= \psi^{(n-1)}.
 \end{aligned} \tag{3.25}$$

(Here, $(\alpha_i|\alpha_i) = 1$ for $i = 1, 2, \dots, n-1$ and $(\alpha_n|\alpha_n) = 2$.)

$$\begin{aligned}
 F_4: \quad W\left[\psi_{-\frac{1}{4}}^{(1)}, \psi_{\frac{1}{4}}^{(1)}\right] &= \psi^{(2)}, \\
 W\left[\psi_{-\frac{1}{4}}^{(2)}, \psi_{\frac{1}{4}}^{(2)}\right] &= \psi^{(1)}\psi_{-\frac{1}{4}}^{(3)}\psi_{\frac{1}{4}}^{(3)}, \\
 W\left[\psi_{-\frac{1}{2}}^{(3)}, \psi_{\frac{1}{2}}^{(3)}\right] &= \psi^{(2)}\psi^{(4)}, \\
 W\left[\psi_{-\frac{1}{2}}^{(4)}, \psi_{\frac{1}{2}}^{(4)}\right] &= \psi^{(3)}.
 \end{aligned} \tag{3.26}$$

(Here, $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 1$ and $(\alpha_3, \alpha_3) = (\alpha_4, \alpha_4) = 2$.)

$$\begin{aligned}
 G_2: \quad W\left[\psi_{-\frac{1}{2}}^{(1)}, \psi_{\frac{1}{2}}^{(1)}\right] &= \psi^{(2)}, \\
 W\left[\psi_{-\frac{1}{6}}^{(2)}, \psi_{\frac{1}{6}}^{(2)}\right] &= \psi^{(1)}\psi_{-\frac{2}{6}}^{(1)}\psi_{\frac{2}{6}}^{(1)}.
 \end{aligned} \tag{3.27}$$

(Here, $(\alpha_1|\alpha_1) = 3$ and $(\alpha_2|\alpha_2) = 1$.)

Again, these relations are not proven but we have indirect numerical evidence for $\mathfrak{g} = B_n, C_n$. Further details and numerical support for the above conjectures are provided in the following sections, which examine the A, D, B and C cases in turn.

4. The A_{n-1} models

The ODE for the A_{n-1} models is

$$\left((-1)^{n+1}D_n(\mathfrak{g}) + P_K(x, E)\right)\psi(x, E, \mathfrak{g}) = 0, \tag{4.1}$$

where the operator $D_n(\mathfrak{g})$ and the generalised potential $P_K(x, E)$ were defined in Section 3, and the additional constraint

$$\sum_{i=0}^{n-1} g_i = \frac{n(n-1)}{2} \tag{4.2}$$

ensures that the term $x^{-1} \frac{d^{n-1}}{dx^{n-1}}$ is absent.

The function $\psi(x, E, \mathfrak{g})$ is defined to be the most subdominant solution on the positive real axis, with asymptotic behaviour, for $M > K/(h^\vee - K)$, given by

$$\psi(x, E, \mathfrak{g}) \sim \mathcal{N}x^{(1-n)M/2} \exp(-x^{M+1}/(M+1)) \tag{4.3}$$

as $x \rightarrow \infty$. The coefficient \mathcal{N} represents an E - and \mathfrak{g} -independent normalisation constant.

The $K = 1$ cases have been extensively discussed in [9–12]; they are related to the WA_{n-1} conformal field theories with integer Virasoro central charge $c = n - 1$. Alternatively, at particular values of the parameters \mathfrak{g} and M they can also be put in correspondence with the minimal coset

conformal field theories

$$\frac{(\hat{A}_{n-1})_1 \times (\hat{A}_{n-1})_L}{(\hat{A}_{n-1})_{L+1}}, \tag{4.4}$$

with a simple relationship between L in (4.4) and the parameter M in (3.8).

The generalisation to integer $K > 1$ comes from an observation by Sergei Lukyanov [38] for the A_1 case, for which numerical and analytic support was later provided in [39] and in [40]. It is reasonable to conjecture that this generalisation works both for A_{n-1} with $n > 2$ and, up to minor modifications, for the other models to be discussed in this paper. In analogy to (4.4), at particular values of \mathfrak{g} and M , the integer $K > 1$ cases should correspond to the cosets

$$\frac{\hat{\mathfrak{g}}_K \times \hat{\mathfrak{g}}_L}{\hat{\mathfrak{g}}_{K+L}}, \tag{4.5}$$

which describe conformal field theories with $\mathfrak{g} = A_{n-1}, B_n, C_n, D_n$. In Appendix A we shall explain, in the simplest case, why potentials of such forms naturally lead to string patterns of roots of the sort mentioned in the introduction.

4.1. Negative-dimension dualities

It is interesting to note that there are formal duality relations among our pseudo-differential equations involving negative values of n and K . Consider the A_{n-1} ODEs with the twists $\mathfrak{g} = \{n - 1, n - 2, \dots, 1, 0\}$:

$$\left((-1)^{n+1} \frac{d^n}{dx^n} + P_K(x, E) \right) \psi(x, E) = 0. \tag{4.6}$$

Setting $\tilde{\psi}(x, E) = P_K(x, E)\psi(x, E)$ and multiplying from the left by $(\frac{d}{dx})^{-n}$, the result is

$$\left((-1)^{n+1} \left(\frac{d}{dx} \right)^{-n} + P_{-K}(x, E) \right) \tilde{\psi}(x, E) = 0. \tag{4.7}$$

Comparing (4.7) with (4.6) and taking into account the boundary conditions, we see that there is a formal duality and a spectral equivalence between the initial n th-order ODE and the pseudo-differential equation (4.7):

$$\{n, M, K\} \leftrightarrow \{-n, M, -K\}. \tag{4.8}$$

Though the above manipulation might look purely formal, it strongly resembles previously-observed W-algebra dualities [41]:

$$\frac{(\hat{A}_{-n})_K \times (\hat{A}_{-n})_L}{(\hat{A}_{-n})_{K+L}} \sim \frac{(\hat{A}_n)_{-K} \times (\hat{A}_n)_{-L}}{(\hat{A}_n)_{-K-L}}. \tag{4.9}$$

A discussion of the precise relation between L and the ODE parameters $\{n, M, K, \mathfrak{g}\}$ is not important for the current naive considerations and we shall postpone it to the future.

Whilst the duality (4.8) remains at the moment a purely formal observation, the second duality discussed in [41]

$$\frac{(\hat{D}_{-n})_K \times (\hat{D}_{-n})_L}{(\hat{D}_{-n})_{K+L}} \sim \frac{(\hat{C}_n)_{-K/2} \times (\hat{C}_n)_{-L/2}}{(\hat{C}_n)_{-K/2-L/2}}, \tag{4.10}$$

will lead us to the proposal (3.12) for the C_n -related equations. (For the quantisation of the classical D_n W-algebras in relation to the Miura-operators see, for example, [42].)

Negative-dimension dualities resembling those described here are also well-known to group theorists; many more details can be found in [43].

4.2. Auxiliary functions and the ψ -system

The solution $\psi(x, E, \mathbf{g})$ of (4.1) with the asymptotic behavior (4.3) is not the only function associated to the A_{n-1} Bethe ansatz equations. In order to derive the Bethe ansatz itself a total of $n - 1$ functions $\psi^{(1)}, \dots, \psi^{(n-1)}$, one for each node of the A_{n-1} Dynkin diagram, should be introduced. These auxiliary functions are solutions of generally more-complicated ordinary differential equations.

Following [11], all the functions $\psi^{(a)}(x, E, \mathbf{g})$ decaying at large x on the positive real axis can be constructed from $\psi \equiv \psi^{(1)}$ as

$$\psi^{(a)} = W^{(a)}\left[\frac{1-a}{2}, \frac{3-a}{2}, \dots, \frac{a-1}{2}\right] \equiv W^{(a)}\left[\psi_{\frac{1-a}{2}}, \psi_{\frac{3-a}{2}}, \dots, \psi_{\frac{a-1}{2}}\right], \tag{4.11}$$

where $a = 1, 2, \dots, n - 1$, $W^{(a)}[f_1, \dots, f_a]$ denotes the generalised Wronskian of the set of functions $\{f_a\}$

$$W^{(a)}[f_1, \dots, f_a] = \det\left[\left(\vec{f}, \frac{d}{dx}\vec{f}, \dots, \frac{d^{a-1}}{dx^{a-1}}\vec{f}\right)\right] \tag{4.12}$$

with $\vec{f} = (f_1, f_2, \dots, f_a)$ (so that $W^{(2)}[f, g] \equiv W[f, g]$, cf. Eq. (3.23)), and ψ_k denotes the ‘rotated’ solution (3.20).

Finally, normalising $\psi^{(1)}$ by

$$\mathcal{N} = \frac{i^{(n-1)/2}}{\sqrt{n}} \implies \psi^{(n)}(x, E, \mathbf{g}) = 1. \tag{4.13}$$

Since ψ is a solution of an n th-order ODE, a naive counting of degrees of freedom shows that the order of the ODE satisfied by $\psi^{(a)}$ should be

$$\frac{n!}{(n-a)!a!}. \tag{4.14}$$

The $(n - a + 1)$ and the (a) equations are related by a $(\mathbf{g} \leftrightarrow \mathbf{g}^\dagger)$ -conjugation [9,11], arising from the \mathbb{Z}_2 symmetry of the Dynkin diagram. Notice that the result (4.14) exactly matches the dimensions of the basic representations of A_{n-1} ; these are again in one-to-one correspondence with the nodes of the Dynkin diagram.

Fortunately, in order to derive the Bethe ansatz equations an explicit knowledge of the remaining $n - 2$ ODEs is unnecessary: the derivation of [11] was instead based on the Stokes relation associated to (4.1)

$$\sum_{k=0}^n (-1)^k C^{(k)}(E, \mathbf{g}) y_k(x, E, \mathbf{g}) = 0, \tag{4.15}$$

where, according to [11],

$$y_k(x, E, \mathbf{g}) = \omega^{(n-1)k/2} \psi(\omega^{-k}x, \Omega^{-k}E, \mathbf{g}), \tag{4.16}$$

$C^{(0)}(E, \mathbf{g}) = 1$ and the Stokes multipliers $C^{(k)}(E, \mathbf{g})$ with $k > 0$ are analytic functions of E and \mathbf{g} . Stokes relations for the D_n , B_n and C_n Eqs. (3.10), (3.11) and (3.12) also exist, but we have encountered some subtle complications⁴ in generalising the approach of [11] to these cases.

Instead, the strategy here is based on the conjectured validity of a simple ‘universal’ system of functional equations among the $\psi^{(a)}$ functions, which leads immediately to the Bethe ansatz equations and bypasses the analysis of Stokes relations.

We shall now prove the A_{n-1} ψ -system (3.22):

$$\psi^{(a-1)}\psi^{(a+1)} = W\left[\psi_{-\frac{1}{2}}^{(a)}, \psi_{\frac{1}{2}}^{(a)}\right], \quad \psi^{(0)} = \psi^{(n)} = 1. \tag{4.17}$$

The proof is based on the observation that determinants satisfy functional equations, in particular the so-called Jacobi identity

$$\Delta\Delta[p, q|p, q] = \Delta[p|p]\Delta[q|q] - \Delta[p|q]\Delta[q|p]. \tag{4.18}$$

Here, Δ is the determinant of an $(a + 1) \times (a + 1)$ matrix and $\Delta[p_1, p_2|q_1, q_2]$ denotes the determinant of the same matrix with the $p_{1,2}$ th rows and $q_{1,2}$ th columns removed.

In order to prove (4.17) we have to consider three different cases: $a = 1$, $1 < a < n - 1$ and $a = n - 1$. The $a = 1$ case follows from the definition of $\psi^{(2)}$ given in (4.11). Eq. (4.17) for $1 < a < n - 1$ follows from the following chain of identities

$$\begin{aligned} \prod_b (\psi^{(b)})^{A_{ab}} &= W^{(a+1)}\left[-\frac{a}{2}, -\frac{a-2}{2}, \dots, \frac{a-2}{2}, \frac{a}{2}\right] W^{(a-1)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}\right] \\ &= (-1)^{(a-1)} W^{(a+1)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}, -\frac{a}{2}, \frac{a}{2}\right] \\ &\quad \times W^{(a-1)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}\right] \\ &= (-1)^{(a-1)} \Delta\Delta[a, a+1|a, a+1], \end{aligned} \tag{4.19}$$

where we have identified

$$\Delta \equiv W^{(a+1)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}, -\frac{a}{2}, \frac{a}{2}\right] \tag{4.20}$$

and

$$\Delta[a, a+1|a, a+1] = W^{(a-1)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}\right]. \tag{4.21}$$

This is nothing but the LHS of the Jacobi identity (4.18). Then an application of the Jacobi identity naturally proves (4.17) in the following way:

$$\begin{aligned} \prod_b (\psi^{(b)})^{A_{ab}} &= (-1)^{(a-1)} (\Delta[a|a]\Delta[a+1|a+1] - \Delta[a|a+1]\Delta[a+1|a]) \\ &= (-1)^{(a-1)} \left(W^{(a)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}, \frac{a}{2}\right] \right) \end{aligned}$$

⁴ In the D_n case these complications are probably a consequence of the fact that the ODEs associated to the \mathbb{Z}_2 -conjugate nodes in the Dynkin diagram are somehow more fundamental than Eq. (3.10). This latter equation is more naturally associated to the first node on the ‘tail’ of the diagram.

$$\begin{aligned}
 & \times W^{(a)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}, -\frac{a}{2}\right] \\
 & - W^{(a)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}, -\frac{a}{2}\right] W^{(a)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}, \frac{a}{2}\right] \\
 & = W^{(a)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}, \frac{a}{2}\right] W^{(a)}\left[-\frac{a}{2}, -\frac{a-2}{2}, \dots, \frac{a-2}{2}\right] \\
 & - W^{(a)}\left[-\frac{a}{2}, -\frac{a-2}{2}, \dots, \frac{a-2}{2}\right] W^{(a)}\left[-\frac{a-2}{2}, \dots, \frac{a-2}{2}, \frac{a}{2}\right] \\
 & = \psi_{-\frac{1}{2}}^{(a)} \psi_{\frac{1}{2}}^{\prime(a)} - \psi_{-\frac{1}{2}}^{\prime(a)} \psi_{\frac{1}{2}}^{(a)} = W\left[\psi_{-\frac{1}{2}}^{(a)}, \psi_{\frac{1}{2}}^{(a)}\right].
 \end{aligned} \tag{4.22}$$

Finally, for $a = n - 1$ the previous calculation shows that

$$\psi^{(n-2)} \psi^{(n)} = W\left[\psi_{-\frac{1}{2}}^{(n-1)}, \psi_{\frac{1}{2}}^{(n-1)}\right]. \tag{4.23}$$

Choosing $\mathcal{N} = \frac{i^{(n-1)/2}}{\sqrt{n}}$ gives $\psi^{(n)} = 1$ and (4.17) is proved.

4.3. The A_{n-1} Bethe ansatz equations

In this section we shall show that the BAEs are a simple consequence of the ψ -system. First, recall the alternative χ -basis of solutions (3.13) and the formal ordering of Table 3

$$g_i < g_j, \quad \forall i < j. \tag{4.24}$$

These solutions are defined by their behaviour as $x \rightarrow 0$

$$\chi_i(x, E, \mathbf{g}) \sim x^{\lambda_i} + O(x^{\lambda_i+n}), \quad \lambda_i = g_i. \tag{4.25}$$

Next, expand $\psi(x, E, \mathbf{g})$ in the χ -basis

$$\psi(x, E, \mathbf{g}) = \sum_{i=0}^{n-1} Q_{[i]}^{(1)}(E, \mathbf{g}) \chi_i(x, E, \mathbf{g}), \tag{4.26}$$

and use the property

$$\chi_i(\omega^k x, \Omega^k E, \mathbf{g}) = \omega^{k\lambda_i} \chi_i(x, E, \mathbf{g}), \tag{4.27}$$

to obtain

$$\psi_k(x, E, \mathbf{g}) = \sum_{i=0}^{n-1} Q_{[i]}^{(1)}(\Omega^k E, \mathbf{g}) \omega^{k\lambda_i} \chi_i(x, E, \mathbf{g}). \tag{4.28}$$

Now expanding the determinants in the determinants for this new basis leads to

$$\begin{aligned}
 \psi^{(a)}(x, E, \mathbf{g}) & = \sum_{\mathbf{i}} \left(\prod_j Q_{[i_j]}^{(1)}(\Omega^j E, \mathbf{g}) \omega^{j\lambda_{i_j}} \right) W^{(a)}\left[\chi_{i_{\frac{1-a}{2}}}, \dots, \chi_{i_{\frac{a-1}{2}}}\right] \\
 & = \sum_{\mathbf{i}} Q_{[i_{\frac{1-a}{2}}, \dots, i_{\frac{a-1}{2}}]}^{(a)}(E, \mathbf{g}) W^{(a)}\left[\chi_{i_{\frac{1-a}{2}}}, \dots, \chi_{i_{\frac{a-1}{2}}}\right]
 \end{aligned} \tag{4.29}$$

where $j = \frac{1-a}{2}, \frac{3-a}{2}, \dots, \frac{a-1}{2}$, $\sum_{\mathbf{i}}$ denotes the sum from 0 to $n - 1$ of the set $\{i_j\}$ while in $\sum_{\mathbf{i}}$ there is the additional constraint $0 \leq i_{\frac{1-a}{2}} \leq i_{\frac{3-a}{2}} \leq \dots \leq i_{\frac{a-1}{2}}$.

A family of x -independent equations is obtained by identifying from the LHS and RHS of (4.17) the terms corresponding to the same power. It is possible a-priori to identify from every determinant only the highest, second highest, lowest and second lowest orders. We shall extract the leading orders, though similar results can be obtained from the subdominant ones.

Setting

$$Q_k^{(a)}(E, \mathbf{g}) = Q_{[0, \dots, a-1]}^{(a)}(\Omega^k E, \mathbf{g}), \quad \bar{Q}_k^{(a)}(E, \mathbf{g}) = Q_{[0, \dots, a-2, a]}^{(a)}(\Omega^k E, \mathbf{g}) \tag{4.30}$$

we have

$$\begin{aligned} \psi_k^{(a)}(x, E, \mathbf{g}) &= \omega^{k\alpha_a} Q_k^{(a)}(E, \mathbf{g}) W^{(a)}[\chi_0, \dots, \chi_{a-1}] \\ &\quad + \omega^{k(\alpha_{a+1} - \lambda_{a-1})} \bar{Q}_k^{(a)}(E, \mathbf{g}) W^{(a)}[\chi_0, \dots, \chi_{a-2}, \chi_a] + \dots \end{aligned} \tag{4.31}$$

The orders of the first and the second terms in (4.31) are $\alpha_a - a(a - 1)/2$ and $\alpha_{a+1} - \lambda_{a-1} - a(a - 1)/2$ respectively, where

$$\alpha_a = \sum_{i=0}^{a-1} \lambda_i. \tag{4.32}$$

Substituting (4.31) into (4.17), comparing the leading terms of both sides for small x and using the relation

$$\begin{aligned} W^{(a+1)} W^{(a-1)} &= W[W^{(a)}, \hat{W}^{(a)}], \\ W^{(a)} &:= W^{(a)}[\chi_0, \dots, \chi_{a-1}], \quad \hat{W}^{(a)} := W^{(a)}[\chi_0, \dots, \chi_{a-2}, \chi_a] \end{aligned}$$

(also proved through the Jacobi identity), we find

$$Q^{(a+1)} Q^{(a-1)} = \omega^{\frac{1}{2}(\lambda_a - \lambda_{a-1})} Q_{-\frac{1}{2}}^{(a)} \bar{Q}_{\frac{1}{2}}^{(a)} - \omega^{\frac{1}{2}(\lambda_{a-1} - \lambda_a)} Q_{\frac{1}{2}}^{(a)} \bar{Q}_{-\frac{1}{2}}^{(a)}, \tag{4.33}$$

$Q^{(0)} = 1$. Finally, let $E_i^{(a)}$ be a zero of $Q^{(a)}(E, \mathbf{g})$. Evaluating the above equation at $E = \Omega^{1/2} E_i^{(a)}$ and at $E = \Omega^{-1/2} E_i^{(a)}$, and by dividing the two equations thus obtained, we find the A_{n-1} Bethe ansatz equations

$$\prod_{b=1}^{n-1} \Omega^{B_{ab}\gamma_b} \frac{Q_{B_{ab}}^{(b)}(E_i^{(a)})}{Q_{-B_{ab}}^{(b)}(E_i^{(a)})} = -1, \quad i = 0, 1, 2, \dots, \tag{4.34}$$

where

$$\gamma_a = \frac{2K}{Mh^\vee} \left(\sum_{i=0}^{a-1} g_i - a \frac{(n-1)}{2} \right). \tag{4.35}$$

Notice that in writing relation (4.35) we have used the identity

$$\frac{K}{Mh^\vee} (\lambda_a - \lambda_{a-1}) = - \sum_{b=1}^{n-1} B_{ab}\gamma_b, \tag{4.36}$$

the ordering (4.24) and imposed the constraint $\gamma_0 = \gamma_n = 0$.

As shown in Table 4, for $K = 1$ there is very good agreement between the IM results obtained from the solution of a suitable nonlinear integral equation (NLIE) (see Section 6 in [11]) and the direct numerical solution of the ODE.

Table 5 also shows the good agreement between the IM results obtained using the spin-1 NLIEs [44,45] and the exact solution of the ODE with $K = 2$ (see Section 6.2).

Table 4

Comparison of IM results (A_4 NLIE) with the direct numerical solution of the A_4 ODE with $K = 1$, $M = 10/21$ and $\mathbf{g} = \{3.421, 2.3, 1.02, 0.2\}$

Level	A_4 NLIE	ODE numerics
$E_0^{(1)}$	14.0495626907	14.0495626922
$E_1^{(1)}$	47.7146839354	47.7146839363
$E_2^{(1)}$	95.1785845453	95.1785845456
$E_3^{(1)}$	154.202021469	154.202021470
$E_4^{(1)}$	223.483044292	223.483044292

Table 5

Comparison of IM results (A_1 NLIE) with the exact solution of the A_1 ODE with $K = 2$, $M = 1$ and $g_0 = 0$. The set $\{\sqrt{2}E_i^{(1)}\}$ are the exact eigenvalues of the B_1 linear ODE of Section 6.2

Level	A_1 NLIE with $K = 2$	ODE numerics
$E_0^{(1)}$	$1.49259741085 \pm 1.60304589242i$	$1.49259741085 \pm 1.60304589242i$
$E_1^{(1)}$	$2.31180377628 \pm 2.38537059826i$	$2.31180377628 \pm 2.38537059826i$
$E_2^{(1)}$	$2.91183770898 \pm 2.97068128676i$	$2.91183770898 \pm 2.97068128676i$
$E_3^{(1)}$	$3.40837129214 \pm 3.45880577384i$	$3.40837129216 \pm 3.45880577388i$
$E_4^{(1)}$	$3.84143464742 \pm 3.88626414305i$	$3.84143464640 \pm 3.88626414641i$

5. The D_n models

The D_n pseudo-differential equation (3.10) is

$$\left(D_n(\mathbf{g}^\dagger) \left(\frac{d}{dx} \right)^{-1} D_n(\mathbf{g}) - \sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)} \right) \psi(x, E, \mathbf{g}) = 0. \tag{5.1}$$

Following the A_{n-1} example, we start from the solution $\psi(x, E, \mathbf{g})$ of (5.1) with asymptotic behaviour

$$\psi(x, E, \mathbf{g}) \sim \mathcal{N} x^{-h^\vee M/2} \exp\left(-\frac{x^{M+1}}{M+1}\right) \quad (M > K/(h^\vee - K)) \tag{5.2}$$

as $x \rightarrow \infty$ on the positive real axis, and introduce the alternative basis of solutions (3.13)

$$\chi_i(x, E, \mathbf{g}), \quad i = 0, 1, \dots, 2n - 1 \tag{5.3}$$

characterised by their behaviour near the origin

$$\chi_i(x, E, \mathbf{g}) \sim x^{\lambda_i} + O(x^{\lambda_i+2n-1}), \quad \begin{cases} \lambda_i = g_i, & \text{for } i \leq n - 1, \\ \lambda_i = h^\vee - g_{2n-1-i}, & \text{for } i > n - 1. \end{cases} \tag{5.4}$$

In (5.4) the parameters λ_i represent the $2n$ solutions of the indicial equation (see Table 2) with the ordering

$$g_i < g_j \leq \frac{h^\vee}{2}, \quad \lambda_i < \lambda_j, \quad \forall i < j. \tag{5.5}$$

5.1. *The ψ -system and the D_n Bethe ansatz equations*

To extract the D_n BAE, we start from

$$\psi_k^{(1)} = \psi_k(x, E, \mathbf{g}) = \psi(\omega^k x, \Omega^k E, \mathbf{g}), \tag{5.6}$$

where Ω and ω are as defined in (3.19) and (3.20), and assume the validity, for a suitable value of the normalisation constant \mathcal{N} , of the ψ -system (3.22)

$$\begin{aligned} W[\psi_{-\frac{1}{2}}^{(a)}, \psi_{\frac{1}{2}}^{(a)}] &= \psi^{(a-1)}\psi^{(a+1)}, \quad a = 1, \dots, n-3, \\ W[\psi_{-\frac{1}{2}}^{(n-2)}, \psi_{\frac{1}{2}}^{(n-2)}] &= \psi^{(n-3)}\psi^{(n-1)}\psi^{(n)}, \\ W[\psi_{-\frac{1}{2}}^{(n-1)}, \psi_{\frac{1}{2}}^{(n-1)}] &= W[\psi_{-\frac{1}{2}}^{(n)}, \psi_{\frac{1}{2}}^{(n)}] = \psi^{(n-2)}. \end{aligned} \tag{5.7}$$

Eqs. (5.7) and Jacobi identity (4.18) used in reverse imply the following relations linking the remaining functions $\psi^{(a)}(x, E, \mathbf{g})$ to $\psi^{(1)}(x, E, \mathbf{g})$:

$$\phi^{(a)} \equiv \psi^{(a)} = W^{(a)}[\psi_{\frac{1-2a}{2}}, \psi_{\frac{3-2a}{2}}, \dots, \psi_{\frac{a-1}{2}}], \quad a = 1, \dots, n-2, \tag{5.8}$$

$$\phi^{(n-1)} \equiv \psi^{(n-1)}\psi^{(n)} = W^{(n-1)}[\psi_{\frac{2-n}{2}}, \psi_{\frac{4-n}{2}}, \dots, \psi_{\frac{n-2}{2}}], \tag{5.9}$$

and

$$\phi^{(n)} \equiv \psi_{-\frac{1}{2}}^{(n-1)}\psi_{\frac{1}{2}}^{(n-1)} + \psi_{-\frac{1}{2}}^{(n)}\psi_{\frac{1}{2}}^{(n)} = W^{(n)}[\psi_{\frac{1-n}{2}}, \psi_{\frac{3-n}{2}}, \dots, \psi_{\frac{n-1}{2}}]. \tag{5.10}$$

Now notice that the auxiliary functions $\phi^{(a)}(x, E, \mathbf{g})$ defined in (5.8), (5.9) and (5.10) satisfy an A-type ψ -system

$$\phi^{(a-1)}\phi^{(a+1)} = W[\phi_{-\frac{1}{2}}^{(a)}, \phi_{\frac{1}{2}}^{(a)}], \quad a = 1, \dots, n-1. \tag{5.11}$$

Therefore, the arguments applied in Section 4.3 go through in the same way:

$$\begin{aligned} \phi_k^{(a)}(x, E, \mathbf{g}) &= \omega^{k\alpha_a} \hat{Q}_k^{(a)}(E, \mathbf{g}) W^{(a)}[\chi_0, \dots, \chi_{a-1}] \\ &\quad + \omega^{k(\alpha_{a+1} - \lambda_{a-1})} \bar{Q}_k^{(a)}(E, \mathbf{g}) W^{(a)}[\chi_0, \dots, \chi_{a-2}, \chi_a] + \dots \end{aligned} \tag{5.12}$$

($a = 1, 2, \dots, n-1$). The orders of the first and the second terms in (5.12) are given by $\alpha_a - a(a-1)/2$ and $\alpha_{a+1} - \lambda_{a-1} - a(a-1)/2$ respectively, with $\alpha_a = \sum_{i=0}^{a-1} \lambda_i$ and

$$\hat{Q}_k^{(a)}(E, \mathbf{g}) = \hat{Q}^{(a)}(\Omega^k E, \mathbf{g}), \quad \bar{Q}_k^{(a)}(E, \mathbf{g}) = \bar{Q}^{(a)}(\Omega^k E, \mathbf{g}) \tag{5.13}$$

are ϕ -related spectral determinants.

Using Eq. (5.11) we establish the following identity among ϕ -related spectral determinants

$$\begin{aligned} \hat{Q}^{(a+1)}(E)\hat{Q}^{(a-1)}(E) &= \omega^{\frac{1}{2}(\lambda_a - \lambda_{a-1})} \hat{Q}_{-\frac{1}{2}}^{(a)}(E)\bar{Q}_{\frac{1}{2}}^{(a)}(E) - \omega^{\frac{1}{2}(\lambda_{a-1} - \lambda_a)} \hat{Q}_{\frac{1}{2}}^{(a)}(E)\bar{Q}_{-\frac{1}{2}}^{(a)}(E); \end{aligned} \tag{5.14}$$

which leads to

$$\frac{\hat{Q}_{-\frac{1}{2}}^{(a-1)}(E_i^{(a)})}{\hat{Q}_{\frac{1}{2}}^{(a-1)}(E_i^{(a)})} \frac{\hat{Q}_1^{(a)}(E_i^{(a)})}{\hat{Q}_{-1}^{(a)}(E_i^{(a)})} \frac{\hat{Q}_{-\frac{1}{2}}^{(a+1)}(E_i^{(a)})}{\hat{Q}_{\frac{1}{2}}^{(a+1)}(E_i^{(a)})} = -\Omega^{\frac{\alpha}{2}(\lambda_a - \lambda_{a-1})} \tag{5.15}$$

with $a = 1, 2, \dots, n - 1$ and $\alpha = \frac{2K}{Mh^\vee}$. We then make the following identifications

$$\hat{Q}^{(0)}(E, \mathbf{g}) = Q^{(0)}(E, \mathbf{g}) = 1, \tag{5.16}$$

$$\hat{Q}^{(a)}(E, \mathbf{g}) = Q^{(a)}(E, \mathbf{g}) \quad (a = 1, \dots, n - 1), \tag{5.17}$$

$$\hat{Q}^{(n-1)}(E, \mathbf{g}) = Q^{(n-1)}(E, \mathbf{g})Q^{(n)}(E, \mathbf{g}), \tag{5.18}$$

$$\hat{Q}^{(n)}(E, \mathbf{g}) = Q^{(n-1)}_{-\frac{1}{2}}(E, \mathbf{g})Q^{\frac{1}{2}}_{-\frac{1}{2}}(E, \mathbf{g}), \tag{5.19}$$

which reflect the relations among the ϕ 's and the ψ 's in the above. In (5.19) we have implicitly assumed

$$\psi^{(n)}(x, E, \mathbf{g}) = o(\psi^{(n-1)}(x, E, \mathbf{g})) \tag{5.20}$$

as $x \rightarrow 0$ and also that (see the discussion in Section 5.2)

$$Q^{(n-1)}(E, \{g_{n-1}, g_{n-2}, \dots, g_0\}) \equiv Q^{(n)}(E, \{h^\vee - g_{n-1}, g_{n-2}, \dots, g_0\}). \tag{5.21}$$

Plugging relations (5.17), (5.18) and (5.19) into (5.15) and using (5.21) is it easy to check that (5.15) can be recast in the universal form (2.1)

$$\prod_{b=1}^n \Omega^{B_{ab}\gamma_b} \frac{Q^{(b)}_{B_{ab}}(E_i^{(a)}, \gamma)}{Q^{(b)}_{-B_{ab}}(E_i^{(a)}, \gamma)} = -1, \quad i = 0, 1, 2, \dots \tag{5.22}$$

where B_{ab} is the D_n -related matrix defined according to (2.4) and we have imposed the extra condition

$$\gamma_n - \gamma_{n-1} = n - 1 - g_{n-1} \tag{5.23}$$

to fix the exact $\{g_a\} \leftrightarrow \{\gamma_a\}$ relation as given in Table 3.

Condition (5.23) guarantees that when $g_{n-1} - n + 1 = 0$ the operator $(d/dx)^{-1}$ in (5.1) acts directly on a d/dx and the relevant equation reduces to an $(2n - 1)$ -order ODE. When this occurs $\gamma_n = \gamma_{n-1}$ and so the pair of \mathbb{Z}_2 -conjugate nodes of the Dynkin diagram are ‘twisted’ in exactly the same way. Further, a change of sign in the RHS of (5.23) swaps γ_n and γ_{n-1} , a property that naturally reflects the presence of the \mathbb{Z}_2 -symmetry in the D_n Dynkin diagram. All these properties are confirmed by the analysis of Sections 5.2 and 5.3 and by 12-digits of numerical agreement at $K = 1$ with $\mathbf{g} = \{n - 1, n - 2, \dots, 1, 0\}$,

$$\gamma_a = \alpha \left(\frac{(a - 1)a}{2} - a \frac{h^\vee}{2} \right), \quad \gamma_n = \gamma_{n-1} = -\frac{nh^\vee}{4}, \tag{5.24}$$

between NLIE and ODE results.

Table 6 shows the (still-excellent) agreement at $K = 1$ away from the $\gamma_n = \gamma_{n-1}$ surface. Appropriate $K > 1$ NLIEs are unknown but numerical results qualitatively reproduce the expected IM scenario of Section 2. Further analytic support to the proposed ODE/IM correspondence for D_n is given in Sections 5.2, 5.3 and 5.4.

5.2. Example 1: $D_2 \sim A_1 \oplus A_1$

The D_2 algebra can be decomposed into a pair of independent A_1 algebras, mirroring an analogous factorisation in the BAE. In this section we shall prove that the solution $\psi(x, E, \mathbf{g})$ to (5.1) with $n = 2$ is the product of two solutions of A_1 -related ODEs.

Table 6

Comparison of IM results (D_4 NLIE) with the direct numerical solution of the D_4 pseudo-differential equation with $K = 1$, $M = 1/3$ and $\mathbf{g} = \{2.95, 2.3, 1.1, 0.2\}$

Level	D_4 NLIE	ODE numerics
$E_0^{(1)}$	17.8625636061	17.8625636061
$E_1^{(1)}$	50.2942213433	50.2942213430
$E_2^{(1)}$	92.8267466445	92.8267466442
$E_3^{(1)}$	143.348705065	143.348705065
$E_4^{(1)}$	200.738324171	200.738324172

We start from the general D_2 equation:

$$\left(\left(\frac{d}{dx} + \frac{g_0}{x} \right) \left(\frac{d}{dx} + \frac{g_1 - 1}{x} \right) \left(\frac{d}{dx} \right)^{-1} \left(\frac{d}{dx} - \frac{g_1 - 1}{x} \right) \left(\frac{d}{dx} - \frac{g_0}{x} \right) - \sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)} \right) \psi(x, E, \mathbf{g}) = 0. \tag{5.25}$$

Expanding and integrating by parts, we obtain an equivalent equation:

$$\left(-\frac{d^3}{dx^3} + 4P(x, E, g') \frac{d}{dx} + 2\frac{dP}{dx}(x, E, g') + \frac{(1 - g_0)^2(1 - g_1)^2}{x^2} \left(\frac{d}{dx} \right)^{-1} \frac{1}{x^2} \right) \psi(x, E, \mathbf{g}) = 0 \tag{5.26}$$

where for this subsection it is convenient to define $P(x, E, k) = \frac{1}{4}[(x^{2M/K} - E)^K + \frac{k}{x^2}]$, and $g' = g_0^2 - 2g_0 + g_1^2 - 2g_1 + 1$.

We now set $Z(x) = \chi_1(x)\chi_2(x)$, a product of the solutions of two A_1 (spin- $\frac{1}{2}$) equations, which for general ρ and σ satisfy

$$\frac{d^2}{dx^2} \chi_1(x, E, \rho) = P(x, E, \rho) \chi_1(x, E, \rho), \tag{5.27}$$

$$\frac{d^2}{dx^2} \chi_2(x, E, \sigma) = P(x, E, \sigma) \chi_2(x, E, \sigma). \tag{5.28}$$

To show that $Z(x)$ satisfies equation (5.26), we differentiate and repeatedly use the A_1 equations to find

$$\frac{d^3 Z}{dx^3} = 2\frac{dP}{dx} \left(x, E, \frac{\rho + \sigma}{2} \right) Z + 2P \left(x, E, \frac{\rho + \sigma}{2} \right) \frac{dZ}{dx} + 2P(x, E, \rho) \chi_1 \frac{d\chi_2}{dx} + 2P(x, E, \sigma) \frac{d\chi_1}{dx} \chi_2. \tag{5.29}$$

If we now define the Wronskian

$$W = \chi_1 \frac{d\chi_2}{dx} - \frac{d\chi_1}{dx} \chi_2 \tag{5.30}$$

and use

$$\chi_1 \frac{d\chi_2}{dx} = \frac{1}{2} \left(\frac{dZ}{dx} + W \right), \quad \frac{d\chi_1}{dx} \chi_2 = \frac{1}{2} \left(\frac{dZ}{dx} - W \right), \tag{5.31}$$

then (5.29) can be written as

$$\frac{d^3 Z}{dx^3} = 2 \frac{dP}{dx} \left(x, E, \frac{\rho + \sigma}{2} \right) Z + 4P \left(x, E, \frac{\rho + \sigma}{2} \right) \frac{dZ}{dx} + \frac{\rho - \sigma}{4x^2} W. \tag{5.32}$$

In order to express W in terms of Z we differentiate, apply (5.27) and (5.28), and then integrate:

$$\frac{dW}{dx} = \chi_1 \frac{d^2 \chi_2}{dx^2} - \frac{d^2 \chi_1}{dx^2} \chi_2 = \frac{\sigma - \rho}{4x^2} Z \quad \rightarrow \quad W = (\sigma - \rho) \left(\frac{d}{dx} \right)^{-1} \frac{Z}{4x^2}. \tag{5.33}$$

The resulting equation for Z ,

$$\begin{aligned} & \left(-\frac{d^3}{dx^3} + 4P \left(x, E, \frac{\rho + \sigma}{2} \right) \frac{d}{dx} + 2 \frac{dP}{dx} \left(x, E, \frac{\rho + \sigma}{2} \right) \right. \\ & \left. - \frac{(\rho - \sigma)^2}{16x^2} \left(\frac{d}{dx} \right)^{-1} \frac{1}{x^2} \right) Z(x, E, \rho, \sigma) = 0, \end{aligned} \tag{5.34}$$

exactly matches equation (5.26) provided the following relations between g_0 and g_1 , and ρ and σ hold:

$$\rho + \sigma = 2(g_0^2 - 2g_0 + g_1^2 - 2g_1 + 1), \tag{5.35}$$

$$\frac{\rho - \sigma}{4} = (g_0 - 1)(g_1 - 1). \tag{5.36}$$

If $\rho = \sigma$ then either g_0 or g_1 has to be zero and the integral operator in the D_2 equation acts on a total derivative. This observation agrees with the discussion in Section 5.1 about the relation between $(d/dx)^{-1}$ and an asymmetric choice of the twists γ_n and γ_{n-1} .

5.3. Example 2: $D_3 \sim A_3$

The BAE for A_3 and D_3 are the same under identification of Bethe roots. It is therefore interesting to discuss the exact correspondence between the two models at the level of the pseudo-differential equations. Actually, it was the study of this case that lead us to the general D_n -related equations.

We start from the observation that the solution ψ of the A_3 -related ODE

$$(D_4(\mathbf{g}) - P_K(x, E))\psi(x, E, \mathbf{g}) = 0, \tag{5.37}$$

is associated to the first node of the A_3 Dynkin diagram, while the solution of the D_3 equation

$$\left(D_3(\bar{\mathbf{g}}^\dagger) \left(\frac{d}{dx} \right)^{-1} D_3(\bar{\mathbf{g}}) - \tau \sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)} \right) \phi(x, E, \bar{\mathbf{g}}) = 0 \tag{5.38}$$

is more naturally associated to the central node of the $D_3 = A_3$ Dynkin diagram. (A constant factor τ has been included in Eq. (5.38) to take into account the possibly-different normalisations for the E parameters.)

Therefore we are looking for a relationship between $\phi(x, E, \bar{\mathbf{g}})$ and

$$\psi^{(2)}(x, E, \mathbf{g}) = W[\psi_{-\frac{1}{2}}^{(1)}, \psi_{\frac{1}{2}}^{(1)}]. \tag{5.39}$$

For simplicity, we perform the calculation at $\mathbf{g} = \{2, 1, 0\}$. We set

$$\psi^{(2)}(x, E) = [0, 1] \tag{5.40}$$

where we have introduced the shorthand notation

$$[i, j] = \left(\frac{d^i}{dx^i} \psi_{-\frac{1}{2}}^{(1)} \right) \left(\frac{d^j}{dx^j} \psi_{\frac{1}{2}}^{(1)} \right) - \left(\frac{d^j}{dx^j} \psi_{-\frac{1}{2}}^{(1)} \right) \left(\frac{d^i}{dx^i} \psi_{\frac{1}{2}}^{(1)} \right) \tag{5.41}$$

so that

$$\frac{d}{dx}[i, j] = [i + 1, j] + [i, j + 1], \quad [i, i] = 0, \tag{5.42}$$

and

$$\frac{d^4}{dx^4} \psi_{\pm\frac{1}{2}}^{(1)}(x, E) = -P_K(x, E) \psi_{\pm\frac{1}{2}}^{(1)}(x, E). \tag{5.43}$$

Taking derivatives five times and using the above equation we have

$$\begin{aligned} \psi^{(2)} &= [0, 1], & \frac{d}{dx} \psi^{(2)} &= [0, 2], & \frac{d^2}{dx^2} \psi^{(2)} &= [1, 2] + [0, 3], \\ \frac{d^3}{dx^3} \psi^{(2)} &= 2[1, 3] + [0, 4] = 2[1, 3] - P_K[0, 0] = 2[1, 3], \\ \frac{d^4}{dx^4} \psi^{(2)} &= 2[2, 3] + 2[1, 4] = 2[2, 3] + 2P_K \psi^{(2)}, \\ \frac{d^5}{dx^5} \psi^{(2)} &= 2[2, 4] + 2 \frac{d}{dx} (P_K \psi^{(2)}) = 2P_K \frac{d}{dx} \psi^{(2)} + 2 \frac{d}{dx} (P_K \psi^{(2)}), \end{aligned} \tag{5.44}$$

and therefore obtain the desired ODE

$$\left(\frac{d^5}{dx^5} - 2\sqrt{P_K(x, E)} \frac{d}{dx} \sqrt{P_K(x, E)} \right) \psi^{(2)}(x, E) = 0, \tag{5.45}$$

which matches (5.38) at $\tau = 2$.

We have also checked that the solution $\psi^{(1)}(x, E, \mathbf{g})$ of the more general A_3 -related differential equation (5.37) leads to a function $\psi^{(2)}(x, E, \mathbf{g}) = W[\psi_{-\frac{1}{2}}^{(1)}, \psi_{\frac{1}{2}}^{(1)}]$, which is the solution of

$$\left(D_3(\bar{\mathbf{g}}^\dagger) \left(\frac{d}{dx} \right)^{-1} D_3(\bar{\mathbf{g}}) - 2\sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)} \right) \psi^{(2)}(x, E, \mathbf{g}) = 0. \tag{5.46}$$

As already seen in Section 5.2, in order to recast the resulting equation in the factorised form (5.46) one has perform a number of integrations by parts.

The exact relation between the A_3 and D_3 sets of parameters is

$$\begin{aligned} 2g_0 &= 1 + \bar{g}_0 + \bar{g}_1 - \bar{g}_2, \\ 2g_1 &= 1 + \bar{g}_0 - \bar{g}_1 + \bar{g}_2, \\ 2g_2 &= 1 - \bar{g}_0 + \bar{g}_1 + \bar{g}_2. \end{aligned} \tag{5.47}$$

5.4. Relationship with the sine-Gordon model

The reader may have noticed that the sets of numbers $\{m_a\}$ for the D_n and B_n models summarised in Table 1 match the mass spectra of the sine-Gordon model at particular values of the coupling constant. From the sine-Gordon point of view the D_n -related mass spectrum emerges

at the reflectionless points where the scattering between the solitons becomes purely diagonal. This link between the sine-Gordon model, affine Toda field theories and perturbed coset CFTs has been discussed in various places [46–49].

In this section we would like to point out that there is a simple connection between equation (5.1) taken at

$$K = 1, \quad M = 1/(n - 1), \quad \mathbf{g} = \{n - 1, n - 2, \dots, 1, 0\} \tag{5.48}$$

and the Schrödinger problem associated, through the first instance of the ODE/IM correspondence [1,6], to the CFT limit of the sine-Gordon model. We start from (5.1) with parameters (5.48):

$$\left(-\frac{d^{2n-1}}{dx^{2n-1}} + (x^2 - E)\frac{d}{dx} + x \right) \chi(x, E) = 0 \tag{5.49}$$

and require $\chi(x, E)$ to be absolutely integrable on the full real line; this restricts the possible values taken by E to a discrete set. Fourier transforming (5.49) yields

$$\left(-\frac{d^2}{dk^2} - \frac{1}{k} \frac{d}{dk} + ((-1)^n k^{2n-2} - E) \right) \tilde{\chi}(k, E) = 0, \tag{5.50}$$

and replacing

$$k \rightarrow ik, \quad E \rightarrow -E, \quad \tilde{\chi}(k, E) \rightarrow k^{-1/2} \tilde{\chi}(k, E) \tag{5.51}$$

we finally find

$$\left(-\frac{d^2}{dk^2} + k^{2n-2} - \frac{1}{4k^2} - E \right) \tilde{\chi}(k, E) = 0. \tag{5.52}$$

Eq. (5.52) exactly matches the ODE associated in [1,6] to the reflectionless points of the untwisted sine-Gordon model at its $c = c_{\text{eff}} = 1$ conformal point.

This simple observation gives extra support to the correctness of the D_n proposal (5.1), and it leads naturally to the B_n proposals discussed in the next section.

6. The B_n models

The discussion in Section 5.4 of the link between the sine-Gordon and D_n scattering theories at specific values of the parameters can be extended to the B_n models [48,49]. This and further considerations led us to the ODE (3.11), which we repeat here:

$$\left(D_n(\mathbf{g}^\dagger) D_n(\mathbf{g}) + \sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)} \right) \psi(x, E, \mathbf{g}) = 0. \tag{6.1}$$

The results of Sections 5.1 and 5.2 suggest a link between the presence of the integral operator $(d/dx)^{-1}$ and the possibility of breaking the symmetry between the n and $n - 1$ nodes of the D_n Dynkin diagram by choosing $\gamma_n \neq \gamma_{n-1}$ in the BAE. Therefore, in writing (6.1) we have omitted the integral operator $(d/dx)^{-1}$ of (5.1) because, in contrast to the D_n Dynkin diagrams, the B_n diagrams have no \mathbb{Z}_2 symmetry.

The relevant solution $\psi(x, E, \mathbf{g})$ to (6.1) has the asymptotic $x \rightarrow \infty$ behaviour

$$\psi(x, E, \mathbf{g}) \sim \mathcal{N} x^{-h^\vee M/2} \exp\left(-\frac{x^{M+1}}{M+1}\right) \quad (M > K/(h^\vee - K)) \tag{6.2}$$

on the positive real axis. The solutions (3.13)

$$\chi_i(x, E, \mathbf{g}), \quad i = 0, 1, \dots, 2n - 1 \tag{6.3}$$

are instead characterised by the $x \rightarrow 0$ behaviour

$$\chi_i(x, E, \mathbf{g}) \sim x^{\lambda_i} + O(x^{\lambda_i+2n}), \quad \begin{cases} \lambda_i = g_i, & \text{for } i \leq n - 1, \\ \lambda_i = h^\vee - g_{2n-1-i}, & \text{for } i > n - 1. \end{cases} \tag{6.4}$$

In (6.4) the λ 's represent the $2n$ solutions of the indicial equation in Table 2 with the ordering

$$g_i < g_j < \frac{h^\vee}{2}, \quad \lambda_i < \lambda_j, \quad \forall i < j. \tag{6.5}$$

6.1. The ψ -system and the B_n Bethe ansatz equations

The B_n ψ -system is

$$\psi^{(a-1)}\psi^{(a+1)} = W[\psi_{-\frac{1}{2}}^{(a)}, \psi_{\frac{1}{2}}^{(a)}], \quad a = 1, \dots, n - 1, \tag{6.6}$$

$$\psi_{-\frac{1}{4}}^{(n-1)}\psi_{\frac{1}{4}}^{(n-1)} = W[\psi_{-\frac{1}{4}}^{(n)}, \psi_{\frac{1}{4}}^{(n)}]. \tag{6.7}$$

Using the identity (4.18) we can express $\psi^{(a)}$ with $a > 1$ in terms of $\psi^{(1)} \equiv \psi$. The result is

$$\psi^{(a)} = W[\psi_{\frac{1-2}{2}a}, \dots, \psi_{\frac{1-2}{2}a-1}], \quad a = 1, 2, \dots, n. \tag{6.8}$$

Following the derivation in Section 4.3, define $\alpha_a = \sum_{i=0}^{a-1} \lambda_i$,

$$Q_k^{(a)}(E, \mathbf{g}) = Q_{[0, \dots, a-1]}^{(a)}(\Omega^k E, \mathbf{g}), \quad \bar{Q}_k^{(a)}(E, \mathbf{g}) = Q_{[0, \dots, a-2, a]}^{(a)}(\Omega^k E, \mathbf{g}) \tag{6.9}$$

where $Q_{[0, \dots, a-1]}^{(a)}$ and $Q_{[0, \dots, a-2, a]}^{(a)}$ are defined as in (4.29) and

$$\begin{aligned} \psi_k^{(a)}(x, E, \mathbf{g}) &= \omega^{k\alpha_a} Q_k^{(a)}(E, \mathbf{g}) W^a[\chi_0, \dots, \chi_{a-1}] \\ &+ \omega^{k(\alpha_{a+1} - \lambda_{a-1})} \bar{Q}_k^{(a)}(E, \mathbf{g}) W^a[\chi_0, \dots, \chi_{a-2}, \chi_a] + \dots \end{aligned} \tag{6.10}$$

Using (6.10) in (6.6) and identifying the leading contributions about $x = 0$ gives

$$\begin{aligned} Q^{(a+1)}(E)Q^{(a-1)}(E) \\ = \omega^{\frac{1}{2}(\lambda_a - \lambda_{a-1})} Q_{-\frac{1}{2}}^{(a)}(E)\bar{Q}_{\frac{1}{2}}^{(a)}(E) - \omega^{\frac{1}{2}(\lambda_{a-1} - \lambda_a)} Q_{\frac{1}{2}}^{(a)}(E)\bar{Q}_{-\frac{1}{2}}^{(a)}(E) \end{aligned} \tag{6.11}$$

with $Q^{(0)}(E) = 1$ and

$$\frac{Q_{-\frac{1}{2}}^{(a-1)}(E_i^{(a)})}{Q_{\frac{1}{2}}^{(a-1)}(E_i^{(a)})} \frac{Q_1^{(a)}(E_i^{(a)})}{Q_{-1}^{(a)}(E_i^{(a)})} \frac{Q_{-\frac{1}{2}}^{(a+1)}(E_i^{(a)})}{Q_{\frac{1}{2}}^{(a+1)}(E_i^{(a)})} = -\Omega^{-2\gamma_a + \gamma_{a-1} + \gamma_{a+1}}. \tag{6.12}$$

In (6.12), $a = 1, \dots, n - 1$ and

$$\gamma_a = \alpha \left(\sum_{i=0}^{a-1} \lambda_i + av \right), \quad a = 1, 2, \dots, n, \tag{6.13}$$

where $\alpha = \frac{2K}{Mh^\vee}$ and v is still to be fixed. Plugging (6.10) into (6.7) leads to

$$\begin{aligned} Q_{-\frac{1}{4}}^{(n-1)}(E) Q_{\frac{1}{4}}^{(n-1)}(E) \\ = \omega^{\frac{1}{4}(\lambda_n - \lambda_{n-1})} Q_{-\frac{1}{4}}^{(n)}(E) \bar{Q}_{\frac{1}{4}}^{(n)}(E) - \omega^{\frac{1}{4}(\lambda_{n-1} - \lambda_n)} Q_{\frac{1}{4}}^{(n)}(E) \bar{Q}_{-\frac{1}{4}}^{(n)}(E) \end{aligned} \tag{6.14}$$

and

$$\frac{Q_{-\frac{1}{2}}^{(n-1)}(E_i^{(n)}) Q_{\frac{1}{2}}^{(n)}(E_i^{(n)})}{Q_{\frac{1}{2}}^{(n-1)}(E_i^{(n)}) Q_{-\frac{1}{2}}^{(n)}(E_i^{(n)})} = -\Omega^{-\gamma_n + \frac{1}{2}(\gamma_{n-1} + \gamma_{n+1})}. \tag{6.15}$$

The boundary condition $\gamma_{n+1} = \gamma_{n-1}$ fixes $v = -h^\vee/2$, the $\{g_a\} \leftrightarrow \{\gamma_a\}$ relation in Table 3 and allows (6.12) and (6.15) to be recast in the form (2.1).

We have checked the consistency of the $n = 2$ case both numerically and analytically. The relation with the sine-Gordon model briefly mentioned at the beginning of Section 6 and the analysis of Sections 6.2 and 7.1.2 lend extra analytic support to the proposal.

6.2. Example 3: B_1

This is a singular limit for the analytic BAE study in [37]. It however suggests the equivalence of the $K = 1$ case of B_1 to the $K = 2$ case of A_1 in the integrable system. Indeed, the differential equation is second order in the former case and it can be written in the more-standard form

$$\left(\frac{d^2}{dx^2} + P_K(x, E) \frac{d}{dx} + \frac{1}{2} \left(\frac{d}{dx} P_K(x, E) \right) - \frac{g_0(g_0 - 1)}{x^2} \right) \psi(x, E, g_0) = 0. \tag{6.16}$$

Performing a Liouville transformation

$$\psi(x, E, g_0) \rightarrow \psi(x, E, g_0) \exp\left(-\frac{1}{2} \int^x P_K(\xi, E) d\xi\right) \tag{6.17}$$

we find

$$\left(-\frac{d^2}{dx^2} + \frac{1}{4}(x^{M/K} - E)^{2K} + \frac{g_0(g_0 - 1)}{x^2} \right) \psi(x, E, g_0) = 0. \tag{6.18}$$

Eq. (6.18) coincides with the equations studied by Sergei Lukyanov [38] which are related to the A_1 lattice models with integer spin

$$j = K, \quad K = 1, 2, 3, \dots \tag{6.19}$$

The cases $g_0 = 0$ with $M = K$ can be solved in closed form. After a shift $x \rightarrow x + E$, the simplest case $K = 1$ becomes

$$\left(\frac{d^2}{dx^2} + x \frac{d}{dx} + \frac{1}{2} \right) \psi(x) = 0 \tag{6.20}$$

which has general solution

$$y(x) = c_1 e^{-\frac{x^2}{2}} H_{-\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right) + c_2 e^{-\frac{x^2}{4}} \sqrt{x} I_{-\frac{1}{4}}\left(\frac{x^2}{4}\right) \tag{6.21}$$

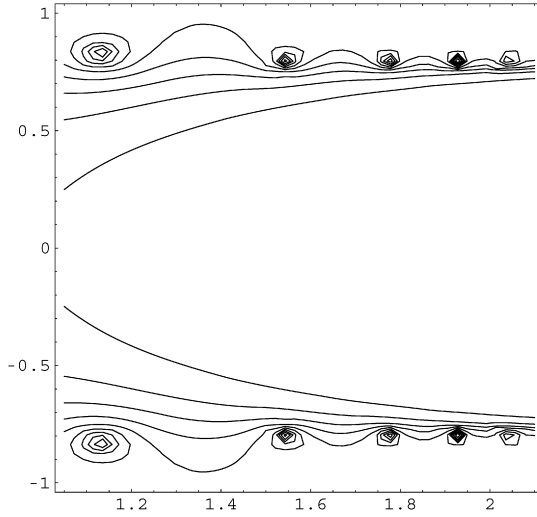


Fig. 1. 2-strings in the B_1 model at $g_0 = 0, M = K = 1$. Contour plot of $1/(1+|H_{-1/2}(-\frac{1}{\sqrt{2}}e^\theta)|)$ in the complex θ -plane.

where H and I are respectively the Hermite and the Bessel functions. Since

$$H_\nu(x) \sim 2^\nu x^\nu (1 + O(1/x)) \tag{6.22}$$

as $\text{Re}(x) > 0, |x| \rightarrow \infty$, while

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} (1 + O(1/x)), \tag{6.23}$$

the most subdominant solution at large x on the positive real axis is

$$\psi(x) = e^{-\frac{x^2}{2}} H_{-\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right). \tag{6.24}$$

Fig. 1 shows the first five complex-conjugate pairs of zeros of $\psi(-e^\theta)$. This 2-string pattern is typical of A_1 -related spin-1 integrable models. The exact eigenvalues are reported in Table 5 above. There is also good agreement between the position of the first pairs of zeros shown in Fig. 1 and the WKB asymptotic prediction of Appendix A.

7. The C_n models

The analytic and numerical results of Section 5 support the conjectured link between the D_n BAE (2.1) and Eq. (5.1). At $\mathbf{g} = \{n - 1, n - 2, \dots, 1, 0\}$, the D_n ODE is

$$\left(\frac{d^{2n-1}}{dx^{2n-1}} - \sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)} \right) \psi(x, E) = 0. \tag{7.1}$$

In this section, we start from (7.1) and consider a second duality relation

$$\frac{(\hat{D}_{-n})_K \times (\hat{D}_{-n})_L}{(\hat{D}_{-n})_{K+L}} \sim \frac{(\hat{C}_n)_{-K/2} \times (\hat{C}_n)_{-L/2}}{(\hat{C}_n)_{-K/2-L/2}} \tag{7.2}$$

discussed by Hornfeck [41]. Motivated by the results of Section 4.1 on the analogy between the $A_n \leftrightarrow A_{-n}$ spectral duality and the A_n -related duality in [41], we change $n \rightarrow -n$ and $K \rightarrow -2K$ in (7.1):

$$\left((-1)^{-2n-1} \frac{d^{-2n-1}}{dx^{-2n-1}} + \sqrt{P_{-2K}(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_{-2K}(x, E)} \right) \psi(x, E) = 0, \tag{7.3}$$

where

$$P_{-2K}(x, E) = \left(x^{\frac{M(2n+2)}{-2K}} - E \right)^{-2K}. \tag{7.4}$$

Replacing

$$\psi(x, E) \rightarrow \left((P_{-2K}(x, E))^{-\frac{1}{2}} \left(\frac{d}{dx} \right)^{-1} (P_{-2K}(x, E))^{-\frac{1}{2}} \right) \psi(x, E), \tag{7.5}$$

multiplying by $\frac{d^{2n+1}}{dx^{2n+1}}$ and noticing that

$$(P_{-2K}(x, E))^{-\frac{1}{2}} \equiv P_K(x, E) = (x^{h^\vee M/K} - E)^K, \tag{7.6}$$

where $h^\vee = n + 1$ is the dual Coxeter number of C_n , we can write the resulting equation as

$$\left(\frac{d^{2n+1}}{dx^{2n+1}} - P_K(x, E) \left(\frac{d}{dx} \right)^{-1} P_K(x, E) \right) \psi(x, E) = 0. \tag{7.7}$$

Eq. (7.7) is our C_n candidate at $\mathbf{g} = \mathbf{g}_0 = \{n - 1, n - 2, \dots, 1, 0\}$. Further, adapting the discussion of Section 5 that led to the full B_n equation, and noting the similarity between the pseudo-differential operators (3.9), (3.10) and (3.11), we replace

$$\frac{d^{2n+1}}{dx^{2n+1}} \equiv D_n(\mathbf{g}_0^\dagger) \left(\frac{d}{dx} \right) D_n(\mathbf{g}_0) \implies D_n(\mathbf{g}^\dagger) \left(\frac{d}{dx} \right) D_n(\mathbf{g}), \tag{7.8}$$

and the final C_n proposal becomes (3.12):

$$\left(D_n(\mathbf{g}^\dagger) \left(\frac{d}{dx} \right) D_n(\mathbf{g}) - P_K(x, E) \left(\frac{d}{dx} \right)^{-1} P_K(x, E) \right) \psi(x, E, \mathbf{g}) = 0. \tag{7.9}$$

The relevant solution of (7.9) has the asymptotic $x \rightarrow \infty$ behaviour

$$\psi(x, E, \mathbf{g}) \sim \mathcal{N} x^{-nM} \exp\left(-\frac{x^{M+1}}{M+1}\right) \quad (M > K/(h^\vee - K)) \tag{7.10}$$

on the positive real axis. The solutions (3.13)

$$\chi_i(x, E, \mathbf{g}), \quad i = 0, 1, \dots, 2n + 1 \tag{7.11}$$

are characterised by the $x \rightarrow 0$ behaviour

$$\chi_i(x, E, \mathbf{g}) \sim x^{\lambda_i} + O(x^{\lambda_i+2n+1}), \quad \begin{cases} \lambda_i = g_i, & \text{for } i \leq n - 1, \\ \lambda_n = n, \\ \lambda_i = 2n - g_{2n-i}, & \text{for } i > n. \end{cases} \tag{7.12}$$

In (7.12) the λ 's represent the $2n + 1$ roots of the indicial equation in Table 2 with the ordering

$$g_i < g_j < n, \quad \lambda_i < \lambda_j, \quad \forall i < j. \tag{7.13}$$

7.1. The ψ -system and the C_n Bethe ansatz equations

We now deduce the C_n Bethe ansatz equations from the proposed C_n ψ -system:

$$\begin{aligned} W[\psi_{-\frac{1}{4}}^{(a)}, \psi_{\frac{1}{4}}^{(a)}] &= \psi^{(a-1)}\psi^{(a+1)}, \quad a = 1, 2, \dots, n - 2, \\ W[\psi_{-\frac{1}{4}}^{(n-1)}, \psi_{\frac{1}{4}}^{(n-1)}] &= \psi^{(n-2)}\psi_{-\frac{1}{4}}^{(n)}\psi_{\frac{1}{4}}^{(n)}, \\ W[\psi_{-\frac{1}{2}}^{(n)}, \psi_{\frac{1}{2}}^{(n)}] &= \psi^{(n-1)}. \end{aligned} \tag{7.14}$$

Using the Jacobi identity we find

$$\phi^{(a)} \equiv \psi^{(a)} = W^{(a)}[\psi_{\frac{1}{4}-a}, \psi_{\frac{3}{4}-a}, \dots, \psi_{\frac{a-1}{4}}], \quad a = 1, 2, \dots, n - 1, \tag{7.15}$$

$$\phi^{(n)} \equiv \psi_{-\frac{1}{4}}^{(n)}\psi_{\frac{1}{4}}^{(n)} = W^{(n)}[\psi_{\frac{1}{4}-n}, \psi_{\frac{3}{4}-n}, \dots, \psi_{\frac{n-1}{4}}], \tag{7.16}$$

where the functions $\phi^{(a)}(x, E, \mathbf{g})$ satisfy the following system of functional relations

$$W[\phi_{-\frac{1}{4}}^{(a)}, \phi_{\frac{1}{4}}^{(a)}] = \phi^{(a-1)}\phi^{(a+1)}, \quad a = 1, \dots, n - 1, \tag{7.17}$$

and

$$W[\phi_{-\frac{1}{4}}^{(n)}, \phi_{\frac{1}{4}}^{(n)}] = \phi^{(n-1)}(\psi^{(n)})^2. \tag{7.18}$$

From (7.18) we see that

$$\psi^{(n)}(x, E, \mathbf{g}) = \sqrt{W^{(n+1)}[\psi_{-\frac{n}{4}}, \dots, \psi_{\frac{n}{4}}]}. \tag{7.19}$$

Now set

$$\begin{aligned} \phi_k^{(a)}(x, E, \mathbf{g}) &= \omega^{k\alpha_a} \hat{Q}_k^{(a)}(E, \mathbf{g}) W^{(a)}[\chi_0, \dots, \chi_{a-1}] \\ &\quad + \omega^{k(\alpha_{a+1}-\lambda_{a-1})} \bar{Q}_k^{(a)}(E, \mathbf{g}) W^{(a)}[\chi_0, \dots, \chi_{a-2}, \chi_a] + \dots \end{aligned} \tag{7.20}$$

with $a = 1, 2, \dots, n$. The orders of the first and the second term in (7.20) are given respectively by $\alpha_a - a(a - 1)/2$ and $\alpha_{a+1} - \lambda_{a-1} - a(a - 1)/2$, where $\alpha_a = \sum_{i=0}^{a-1} \lambda_i$. Assuming

$$\psi_k^{(n)}(x, E, \mathbf{g}) = \omega^{k\beta} x^\beta Q_k^{(n)}(E, \mathbf{g}) + \omega^{k\sigma} x^\sigma \tilde{Q}_k^{(n)}(E, \mathbf{g}) + \dots \tag{7.21}$$

where $\beta < \sigma$, we can make the following identifications

$$\beta = \alpha_n/2 - n(n - 1)/4, \quad \sigma = \beta + (\lambda_n - \lambda_{n-1}) \tag{7.22}$$

and

$$\hat{Q}^{(0)}(E) = Q^{(0)}(E) = 1, \tag{7.23}$$

$$\hat{Q}^{(a)}(E) = Q^{(a)}(E), \quad a = 1, 2, \dots, n - 1, \tag{7.24}$$

$$\hat{Q}^{(n)}(E) \propto Q_{-\frac{1}{4}}^{(n)}(E) Q_{\frac{1}{4}}^{(n)}(E), \tag{7.25}$$

$$\bar{Q}^{(n)}(E) \propto \omega^{\frac{1}{4}(\lambda_n - \lambda_{n-1})} Q_{-\frac{1}{4}}^{(n)}(E) \tilde{Q}_{\frac{1}{4}}^{(n)}(E) + \omega^{\frac{1}{4}(\lambda_{n-1} - \lambda_n)} Q_{\frac{1}{4}}^{(n)}(E) \tilde{Q}_{-\frac{1}{4}}^{(n)}(E). \tag{7.26}$$

Using (7.20) in (7.17) and selecting the leading terms we find

$$\begin{aligned} & \hat{Q}^{(a+1)}(E)\hat{Q}^{(a-1)}(E) \\ &= \omega^{\frac{1}{4}(\lambda_a-\lambda_{a-1})}\hat{Q}_{-\frac{1}{4}}^{(a)}(E)\bar{Q}_{\frac{1}{4}}^{(a)}(E) - \omega^{\frac{1}{4}(\lambda_{a-1}-\lambda_a)}\hat{Q}_{\frac{1}{4}}^{(a)}(E)\bar{Q}_{-\frac{1}{4}}^{(a)}(E). \end{aligned} \tag{7.27}$$

With the identifications (7.24) and (7.25) this leads to

$$\prod_{b=1}^n \frac{Q_{B_{ab}}^{(b)}(E_i^{(a)})}{Q_{-B_{ab}}^{(b)}(E_i^{(a)})} = -\Omega^{\frac{\alpha}{4}(\lambda_a-\lambda_{a-1})}, \tag{7.28}$$

where $a = 1, 2, \dots, n - 1$ and $\alpha = \frac{2K}{Mh^\nabla}$. Using (7.20) with $a = n$ in (7.18), (7.26) and (7.25) gives instead

$$Q^{(n-1)}(E) = \omega^{\frac{1}{2}(\lambda_n-\lambda_{n-1})}Q_{-\frac{1}{2}}^{(n)}(E)\tilde{Q}_{\frac{1}{2}}^{(n)}(E) - \omega^{\frac{1}{2}(\lambda_{n-1}-\lambda_n)}Q_{\frac{1}{2}}^{(n)}(E)\tilde{Q}_{-\frac{1}{2}}^{(n)}(E), \tag{7.29}$$

which leads to

$$\frac{Q_{-\frac{1}{2}}^{(n-1)}(E_i^{(n)})}{Q_{\frac{1}{2}}^{(n-1)}(E_i^{(n)})} \frac{Q_1^{(n)}(E_i^{(n)})}{Q_{-1}^{(n)}(E_i^{(n)})} = -\Omega^{\frac{\alpha}{2}(\lambda_n-\lambda_{n-1})}. \tag{7.30}$$

Finally, with the identification (7.12) and the choice in Table 3 for the $\{g_a\} \leftrightarrow \{\gamma_a\}$ relation, Eqs. (7.28) and (7.30) can be assembled into the universal form (2.1).

7.1.1. Example 4: C_1

The $n = 1$ case is again a singular limit of the analytic BAE, but it also suggests the similarity of this case to the A_1 models [37]. The pseudo-differential equation is, however, not second order but instead third order

$$\left(\frac{d^3}{dx^3} - \frac{\mathcal{L}}{x^2} \frac{d}{dx} + \frac{\mathcal{L}}{x^3} - P_K(x, E) \left(\frac{d}{dx} \right)^{-1} P_K(x, E) \right) \psi(x, E, g_0) = 0, \tag{7.31}$$

where $\mathcal{L} = g_0(g_0 - 2)$. It is nevertheless easy to check that (7.31) is solved by the product of two functions satisfying second order ODEs:

$$\psi(x, E, g_0) = \chi_-(x, E, g_0)\chi_+(x, E, g_0) \tag{7.32}$$

where χ_{\pm} originates from a single function χ as follows

$$\chi_{\pm}(x, E, g_0) = \chi(\omega^{\pm 1/4}x, \Omega^{\pm 1/4}E, g_0). \tag{7.33}$$

Since we assume that χ satisfies the standard ODE associated with A_1 ,

$$\left(\frac{d^2}{dx^2} - \frac{1}{2}P_K(x, E) - \frac{\mathcal{L}}{4x^2} \right) \chi(x, E, g_0) = 0, \tag{7.34}$$

the functions χ_{\pm} satisfy the following:

$$\frac{d^2}{dx^2} \chi_{\pm}(x, E, g_0) = \left(\pm \frac{i}{2}P_K(x, E) + \frac{\mathcal{L}}{4x^2} \right) \chi_{\pm}(x, E, g_0). \tag{7.35}$$

Starting from (7.33) and differentiating $\psi = \psi(x, E, g_0)$ three times we find

$$\begin{aligned} \frac{d\psi}{dx} &= \frac{d\chi_+}{dx}\chi_- + \chi_+ \frac{d\chi_-}{dx}, & \frac{d^2\psi}{dx^2} &= 2\frac{d\chi_-}{dx}\frac{d\chi_+}{dx} + \frac{\mathcal{L}}{2x^2}\psi, \\ \frac{d^3\psi}{dx^3} &= iP_K\left(\chi_+ \frac{d\chi_-}{dx} - \frac{d\chi_+}{dx}\chi_-\right) + \frac{\mathcal{L}}{x^2}\frac{d\psi}{dx} - \frac{\mathcal{L}}{x^3}\psi. \end{aligned} \tag{7.36}$$

We notice that

$$\frac{d}{dx}\left(\chi_+ \frac{d\chi_-}{dx} - \frac{d\chi_+}{dx}\chi_-\right) = -iP_K\psi \tag{7.37}$$

and therefore

$$\chi_+ \frac{d\chi_-}{dx} - \frac{d\chi_+}{dx}\chi_- = -i\left(\frac{d}{dx}\right)^{-1}P_K\psi. \tag{7.38}$$

Inserting (7.38) into (7.36) we finally arrive at Eq. (7.31).

7.1.2. Example 5: $C_2 \sim B_2$

The ODEs for B_2 and for C_2 are both deduced from the D_n -type ODE, but through completely different routes. It is thus a good test to check the equivalence of these two cases. We start with the B_2 -related ODE at $\mathbf{g} = \{1, 0\}$

$$\frac{d^4\psi}{dx^4} + P_K\frac{d\psi}{dx} + \frac{1}{2}\frac{dP_K}{dx}\psi = 0. \tag{7.39}$$

The ODE associated with the second node of B_2 , which is nothing but the first node of C_2 , would be satisfied by the following function

$$\psi^{(2)} = W[\psi_{-\frac{1}{2}}, \psi_{\frac{1}{2}}] = [0, 1], \tag{7.40}$$

where

$$\frac{d^4\psi_{\pm\frac{1}{2}}}{dx^4} = P_K\frac{d\psi_{\pm\frac{1}{2}}}{dx} + \frac{1}{2}\frac{dP_K}{dx}\psi_{\pm\frac{1}{2}}. \tag{7.41}$$

We then easily evaluate the derivatives of $\psi^{(2)}$:

$$\begin{aligned} \frac{d\psi^{(2)}}{dx} &= [0, 2], & \frac{d^2\psi^{(2)}}{dx^2} &= [1, 2] + [0, 3], & \frac{d^3\psi^{(2)}}{dx^3} &= 2[1, 3] + P_K\psi^{(2)}, \\ \frac{d^4\psi^{(2)}}{dx^4} &= 2[2, 3] + P_K\frac{d\psi^{(2)}}{dx}, & \frac{d^5\psi^{(2)}}{dx^5} &= P_K\frac{d^2\psi^{(2)}}{dx^2} - 2P_K[1, 2]. \end{aligned} \tag{7.42}$$

Finally, noticing that

$$2[1, 2] = 2\left(\frac{d}{dx}\right)^{-1}[1, 3] = \frac{d^2\psi^{(2)}}{dx^2} - \left(\frac{d}{dx}\right)^{-1}P_K\psi^{(2)} \tag{7.43}$$

we have obtained a closed form ODE for $\psi^{(2)}$:

$$\left(\frac{d^5\psi^{(2)}}{dx^5} - P_K(x, E)\left(\frac{d}{dx}\right)^{-1}P_K(x, E)\right)\psi^{(2)}(x, E) = 0. \tag{7.44}$$

This equation is exactly the $\mathbf{g} = \{1, 0\}$ equation associated to the first node of C_2 . This verifies the consistency of our proposal.

Table 7

$C_2 = B_2$: comparison of Bethe ansatz results with the numerical solution of the B_2 and C_2 equations using the algorithm described in Appendix B. (B_2 node convention with $\mathbf{g} = \{1, 0\}$, $K = 1$ and $M = 2/3$)

BA numerics		ODE numerics	
1st node	2nd node	1st node	2nd node
6.28405	$6.8368 \pm 5.8640i$	6.28390	$6.8365 \pm 5.8637i$
13.2379	$18.216 \pm 14.265i$	13.2376	$18.214 \pm 14.264i$
21.6307	$30.996 \pm 23.645i$	21.6303	$30.992 \pm 23.642i$
30.5039	$44.747 \pm 33.707i$	30.5034	$44.739 \pm 33.700i$
39.8617	$59.254 \pm 44.304i$	39.8613	$59.240 \pm 44.292i$

More generally, the correspondence between the B_2 parameters $\mathbf{g} = \{g_1, g_0\}$ and the C_2 parameters $\bar{\mathbf{g}} = \{\bar{g}_1, \bar{g}_0\}$ is established by

$$\begin{aligned} 2g_0 &= \bar{g}_0 + \bar{g}_1 - 1, \\ 2g_1 &= \bar{g}_0 - \bar{g}_1 + 3. \end{aligned} \tag{7.45}$$

Finally we have directly checked the $C_2 = B_2$ ψ -system at $\mathbf{g} = \{1, 0\}$ by verifying that both sides of

$$W[\psi_{-\frac{1}{4}}^{(1)}, \psi_{\frac{1}{4}}^{(1)}] = \psi_{-\frac{1}{4}}^{(2)} \psi_{\frac{1}{4}}^{(2)}, \tag{7.46}$$

satisfy the same 15th order ODE.

Unfortunately we do not currently have NLIEs for the B_n/C_n models and the numerical checks presented below are instead based on an approximate solution similar to that used by Voros in [5]. For the B_2 BAEs for the case $\mathbf{g} = \{1, 0\}$, $K = 1$ and $M = 2/3$, we started from a perfect-string estimate for the first 1000 roots as follows:

$$E_j = \left(4\sqrt{\frac{\pi}{3}} \frac{\Gamma(\frac{11}{6})}{\Gamma(\frac{4}{3})} j \right)^{\frac{6}{5}} \quad \text{1st node,} \tag{7.47}$$

$$E_j = e^{\pm i\frac{\pi}{5}} \left(4\sqrt{\pi} \frac{\Gamma(\frac{11}{6})}{\Gamma(\frac{4}{3})} \left(j - \frac{1}{6} \right) \right)^{\frac{6}{5}} \quad \text{2nd node.} \tag{7.48}$$

We then solved the BAEs recursively using the Newton–Raphson method on the first 20 roots, keeping the remaining roots fixed. Table 7 compares the lowest roots thus obtained with the results from the solution of the (pseudo-)differential equations. The relatively low accuracy in comparison with Tables 4, 5 and 6 is most likely to be a consequence of the slow convergence rate of the algorithm used to solve the Bethe ansatz equations, and in particular the systematic errors introduced by fixing the higher levels ($E_j, j > 20$) to their perfect-string values.

8. Conclusions

There are many aspects about the correspondence between integrable models and the spectral theory of ordinary differential equations that we would like to explore and understand at a deeper level. Given the applications of the Bethe ansatz to the study of QCD in its leading-logarithm approximation [50] and to the study of anomalous dimensions of composite operators in Yang–Mills theories [51,52], the extension of the correspondence to lattice models is certainly desirable from a physical point of view. On the other hand the mathematical structures arising from the

generalisation of the correspondence to other conformal field theories with extended symmetries has the potential to link areas of modern and classical mathematics in an elegant way. In this paper our results were obtained very much on a case-by-case basis, but we already saw the emergence of interesting mathematical objects: the ψ -systems, negative-dimension dualities, and the formal similarity with the Miura-operators studied both in the classical work [19] (see also [53–55]) and more recently in [13–15,17,18].

It would be very interesting to generalise equations (3.18)–(3.21) to encompass the excited states of the integrable models; to date this has been completed for the $K = 1$ case of A_1 [56, 57]. More challenging, but also extremely interesting, would be to extend the correspondence to perturbed conformal field theories defined on a cylinder, both for the ground state [58] and for excited states [59,60].

Finally, even remaining inside the current setup, the ODE/IM correspondence has already had an impact on condensed matter physics: it has been applied to interacting Bose liquids [61], the single electron box [62] and quantum dots [63].

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Appendix A. Root strings from the complex WKB method

As mentioned in Section 2, a characteristic feature of the ‘fused’ models with $K > 1$ is that the asymptotic roots are not necessarily real, but rather are grouped into so-called ‘strings’ of complex roots. Here we show how this behaviour can be recovered from a treatment of the ODE using the complex WKB method. We shall restrict the analysis to the A_1 case and for simplicity set $\mathbf{g} = \{1, 0\}$. The ODE is the $n = 2$ case of (4.1):

$$\left(-\frac{d^2}{dz^2} + P_K(z)\right)\psi(z, E) = 0, \quad P_K(z) = (z^{2M/K} - E)^K \tag{A.1}$$

and the boundary condition determining the eigenvalues is that there should be a solution which decays as $z \rightarrow \infty$ on the positive real axis, and is simultaneously zero at $z = 0$. (In this section we use z instead of x to emphasise that it must be considered in the complex plane.)

Before giving the WKB treatment, we summarise the expectations from integrable models. A useful discussion of the $K = 2$ case, for generic twist parameter, is in Appendix 2 of [64]. For a finite lattice of size N , the integral equation derived in that paper leads to the following asymptotic condition on the Bethe ansatz roots θ_j^\pm

$$N \ln \left(\tanh \left(\frac{\pi \theta_j^\pm}{2\gamma} \right) \right) = \pm N \pi i \mp \left(2j - 1 + \frac{\phi}{\pi - 2\gamma} \right) \pi i + \frac{1}{2} \ln 2, \tag{A.2}$$

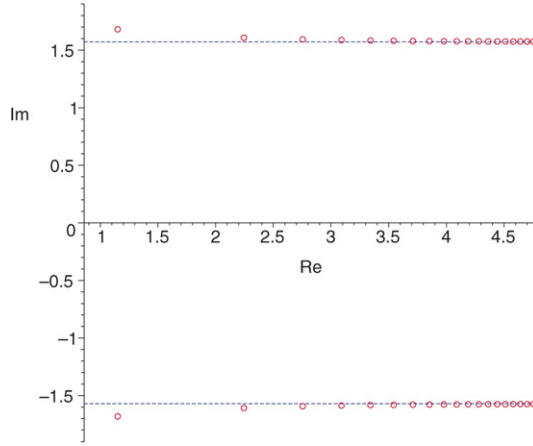


Fig. 2. Two-strings from the asymptotics of the NLIE [64].

where $1 \leq j < N/2 + 1$. This equation describes the approximate position of complex-conjugate pairs of roots $\{\theta_j^-, \theta_j^+\}$, with θ_j^- and θ_j^+ lying in the upper and lower halves of the complex plane. The relationship between γ, ϕ in (A.2) and M and $\mathbf{g} = \{1 - g_0, g_0\}$ is

$$\mu = \frac{M + 1}{M} = \frac{\pi}{2\gamma}, \quad \frac{\phi}{\pi - 2\gamma} = \frac{1}{2} - g_0. \tag{A.3}$$

In the continuum limit, $N \rightarrow \infty$ and the numbers $e^{\theta_j^\pm}$ with $1 \leq j \ll N/2$ tend to zero. Shifting θ_j^\pm appropriately, Eq. (A.2) becomes

$$e^{2\mu\theta_j^\pm} = \pm \left(2j - \frac{1}{2} - g_0\right) \pi i - \frac{1}{2} \ln 2, \quad j = 1, 2, \dots \tag{A.4}$$

The second term causes deviations from the string pattern, even within this asymptotic approximation. Set $\theta_j^\pm = \pm i\pi/(4\mu) + \beta_j^\pm$; then

$$e^{2\mu\beta_j^\pm} = \left(2j - \frac{1}{2} - g_0\right) \pi \pm \frac{i}{2} \ln 2 \tag{A.5}$$

or, taking logs and expanding for large j ,

$$2\mu\beta_j^\pm = \ln \left(2j - \frac{1}{2} - g_0\right) \pi \pm i \frac{\ln 2}{2(2j - \frac{1}{2} - g_0)\pi} + \dots \tag{A.6}$$

Eq. (A.6) exhibits the asymptotic deviations from the perfect string configurations, which are only recovered in the limit $j \rightarrow \infty$. This qualitative pattern matches the results of Table 5 and Fig. 1, and is illustrated in Fig. 2. To compare with the $A_1, K = 2$ results in Section 4.3 one has to identify

$$\{e^{2\mu\theta_j^+}, e^{2\mu\theta_j^-}\} = \{\mu^{-1}(E_{2j-1})^\mu, \mu^{-1}(E_{2j})^\mu\} \tag{A.7}$$

in (A.4).

These features (and their generalisations to higher K) can all be recovered from a complex WKB treatment of (A.1). The practicalities of this method have been very clearly explained by

Heading [65,66], and we shall, more-or-less, follow his notations, which we now summarise. The behaviour of (A.1) is controlled by $P_K(z) \equiv (z^{2M/K} - E)^K$, and the possibly-complex zeros of this function are called turning (or transition) points. For each turning point z_0 , a branch cut is inserted joining z_0 to infinity; and for $2M/K \notin \mathbb{Z}$, another is added starting from the origin along the negative-imaginary axis. After this, taking $\arg(P_K(z))$ to tend to zero along the positive real z axis renders $(P_K(z))^{1/2}$ and $(P_K(z))^{1/4}$ uniquely defined. From each turning point z_0 there is a set of lines

$$\Im \int_{z_0}^z dt \sqrt{P_K(t)} = 0 \tag{A.8}$$

called Stokes lines, and another set

$$\Re \int_{z_0}^z dt \sqrt{P_K(t)} = 0 \tag{A.9}$$

called anti-Stokes lines.⁵ A K th order turning point has $K + 2$ Stokes lines, and $K + 2$ anti-Stokes lines, emanating from it. Leading-order WKB solutions are written as

$$(z_0, z) \equiv (P_K(z))^{-1/4} \exp\left(\int_{z_0}^z dt \sqrt{P_K(t)}\right). \tag{A.10}$$

If $\Re \int_{z_0}^z dt \sqrt{P_K(t)} > 0$, a subscript d is added, signifying that $(z_0, z)_d$ is a dominant solution; (z, z_0) is then subdominant, and is written $(z, z_0)_s$. On anti-Stokes lines, dominant and subdominant solutions swap roles.⁶

Formal WKB solutions $\alpha(z, z_0) + \beta(z_0, z)$ are only valid in restricted domains: the Stokes phenomenon means that the coefficient of the subdominant term changes by an amount proportional to the dominant term when a Stokes line is crossed, so that $\alpha(z, z_0)_d + \beta(z_0, z)_s$ is replaced by $\alpha(z, z_0)_d + (\beta + T\alpha)(z_0, z)_s$, where the Stokes multiplier T is a constant characteristic of the given Stokes line and crossing direction. (Note, since the discontinuity always occurs in the coefficient of a subdominant term, there is no contradiction with the fact that the WKB solution provides an asymptotic approximation to the exact—continuous—solutions of the original equation.) For a Stokes line emanating from a K th order turning point, traversed in a positive (anticlockwise) sense, the Stokes multiplier is [67]

$$T = 2i \cos(\pi/(K + 2)). \tag{A.11}$$

This can alternatively be recovered via the $E = 0, l = 0, M = K/2$ value of the function $C(E, l)$ discussed in [8], Eq. (3.10). For a clockwise traverse, T is replaced by $-T$.

The key feature of (A.1), leading to the formation of strings, is that the turning points which control the WKB eigenvalues all have order K . If a solution which decays along the positive

⁵ Here we follow the conventions of Heading and, for example, Berry, but beware that other’s conventions are exactly the reverse. Note also that our $P_K(z)$ is *minus* the function $q(z)$ used by Heading.

⁶ Note, in contrast to elsewhere in this paper, the terms dominant and subdominant are used here with respect to the WKB expansion parameter, which for brevity we have set equal to 1. The parameter, a factor of ε^2 in front of the derivative term in (A.1), can easily be restored if desired. In Heading’s notations, ε corresponds to k^{-1} .

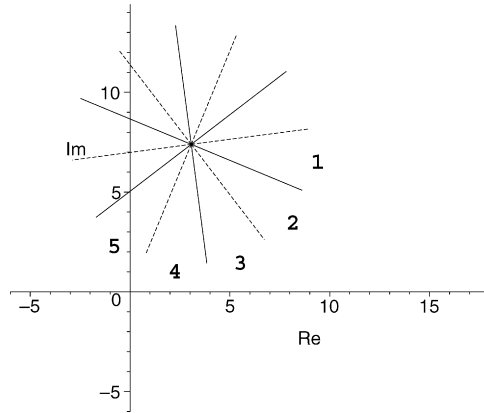


Fig. 3. Initial directions of Stokes lines (continuous) and anti-Stokes lines (dotted) from a 4th order turning point.

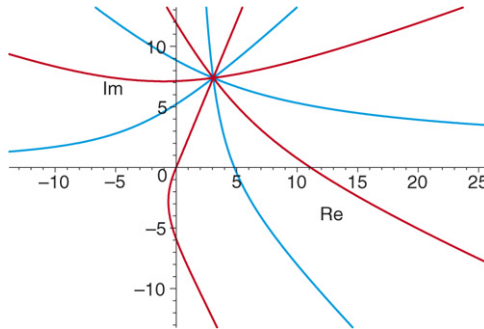


Fig. 4. The full Stokes and anti-Stokes lines for the situation shown in Fig. 3.

real axis is traced in to a turning point, there are K different directions away from that turning point along which the two WKB solutions (z_0, z) and (z, z_0) might come into balance and have a chance to cancel; if the continuation of any of these directions can be arranged to hit the origin, then a (WKB) eigenvalue is possible. When the Stokes multipliers have unit modulus, these directions are precisely the anti-Stokes lines; departures from unit modulus cause small shifts, and lead to deviations from the string pattern.

An example should make this clear. Fig. 3 shows the set of Stokes and anti-Stokes directions away from a turning point of (A.1) with $M = 3$, $K = 4$. Between each pair of anti-Stokes lines there is a Stokes sector, which is further divided into two segments by the Stokes line which lies exactly in the middle of the Stokes sector. The first few of these segments are labelled 1 . . . 5 in the figure; note that segments 1 and 2 make up one Stokes sector, and segments 3 and 4 another. For the figure, $\arg(E)$ was chosen so that the origin lies in the union of segments 4 and 5, and the positive real axis in the union of the continuations of segments 1 and 2. Across the (dotted) anti-Stokes lines, dominant and subdominant solutions swap over, while across the (continuous) Stokes lines, the coefficients of subdominant terms may change by the Stokes phenomenon. Fig. 4 shows the continuation of the Stokes directions shown in Fig. 3 into the full complex plane, justifying the claim that the real axis lies in the continuation of segments 1 and 2.

Our aim is to find an approximate solution which decays as $z \rightarrow \infty$ on the positive real axis, and is also zero at the origin. Since the positive real axis lies in the continuations of segments **1** and **2**, we start with a solution subdominant in segments **1** and **2**, and continue it clockwise around the turning point to find the behaviour in the region of the origin. There is no Stokes phenomenon passing between **1** to **2** since the dominant term is absent in both of these segments, but from then on the phenomenon must be taken into account each time a Stokes line is crossed. With z_0 the location of the turning point, and (z, z_0) the approximate subdominant solution in segments **1** and **2**, we have:

- 1, 2:** $(z, z_0)_s$,
- 3:** $(z, z_0)_d$,
- 4:** $(z, z_0)_d - 2i \cos(\pi/6)(z_0, z)_s$,
- 5:** $(z, z_0)_s - 2i \cos(\pi/6)(z_0, z)_d$.

Near to the anti-Stokes line between segments **4** and **5**, the WKB solutions (z, z_0) and $2i \cos(\pi/6)(z_0, z)$ are of similar magnitude, and the (WKB) condition yielding the eigenvalues is found by demanding that they cancel exactly at $z = 0$:

$$(z_0, 0) - 2i \cos(\pi/6)(0, z_0) = 0. \tag{A.12}$$

That is

$$\exp\left(2 \int_0^{z_0} dt \sqrt{P_K(t)} + i\pi/2\right) = \frac{1}{2 \cos(\pi/6)} \tag{A.13}$$

or

$$2 \int_0^{z_0} dt \sqrt{P_K(t)} = \left(2j - \frac{1}{2}\right)\pi i - \ln(2 \cos(\pi/6)). \tag{A.14}$$

Remembering that $z_0 = E^{K/(2M)}$, the integral can be evaluated to

$$\begin{aligned} \int_0^{E^{K/(2M)}} (t^{2M/K} - E)^{K/2} dt &= (-1)^{K/2} E^\mu \int_0^1 (1 - u^{2M/K})^{K/2} du \\ &= (-1)^{K/2} \tilde{\kappa}(2M/K, 2/K) E^\mu, \end{aligned} \tag{A.15}$$

where $\mu = K(M + 1)/(2M)$ and

$$\tilde{\kappa}(a, b) = \frac{\Gamma(1 + 1/a)\Gamma(1 + 1/b)}{\Gamma(1 + 1/a + 1/b)}. \tag{A.16}$$

Eqs. (A.14)–(A.15) give the WKB prediction at $K = 4$. Had we instead performed the calculation for $K = 2$, the logarithm on the RHS of (A.14) would have been replaced by $-\ln(2 \cos(\pi/4)) = -\frac{1}{2} \ln 2$, and (A.4) would have been reproduced. More generally, the Stokes multipliers solve a stationary T-system, and the constant determining the deviation of the j th root in a K -string is found to be

$$\ln(\sin(\pi j/(K + 2))/\sin(\pi(j + 1)/(K + 2))), \quad j = 1, 2, \dots, K. \tag{A.17}$$

The reason why only K out of the $K + 2$ anti-Stokes directions allow for zeros of the wavefunction is that on the two anti-Stokes lines next to the sector where the initial subdominant solution

is defined (the anti-Stokes lines bordering the union of segments **1** and **2** in the example) there is only a single ‘pure’ WKB solution, and thus there is no chance to get the cancellation between two WKB solutions which leads to the wavefunction zeros near the other anti-Stokes directions.

It would be interesting to reproduce these results from a study of the relevant nonlinear integral equations in the integrable models, but we will leave a more detailed study of such issues for future work.

Appendix B. The numerical algorithm and the dual formulation of the boundary problem

In this appendix we describe the numerical methods used to test our conjectures. We solved the pseudo-differential equations using an iterative power-series method, which we describe below. On the integrable model side, there is a well-known method to rewrite an infinite set of Bethe ansatz equations such as (2.1) into a finite set of nonlinear integral equations (NLIEs) [68]. We applied this to the $K = 1$ simply-laced cases A_{n-1} and D_n .

B.1. The χ -functions and the generalised Cheng algorithm

The special solution $\psi(x, E, \mathbf{g})$ and its derivatives can be written in terms of the χ -functions (3.13) and the spectral determinants as

$$\frac{d^m}{dx^m} \psi(x, E, \mathbf{g}) = Q_{[0]}^{(1)}(E, \mathbf{g}) \frac{d^m}{dx^m} \chi_0(x, E, \mathbf{g}) + Q_{[1]}^{(1)}(E, \mathbf{g}) \frac{d^m}{dx^m} \chi_1(x, E, \mathbf{g}) + \dots \tag{B.1}$$

By solving for $Q_{[0]}^{(1)}(E, \mathbf{g})$:

$$Q_{[0]}^{(1)}(E, \mathbf{g}) = \frac{W[\psi, \chi_1, \chi_2, \dots]}{W[\chi_0, \chi_1, \chi_2, \dots]} \tag{B.2}$$

In order to calculate the spectral determinant from (B.2) we must first find the χ -solutions by numerically solving the pseudo-differential equations. This can be done, with very high precision, using a generalisation of the iteration method introduced by Cheng many years ago [20] and more recently used in [69]. In the following we shall define the iterative solution for each Lie algebra on a case-by-case basis.

A_{n-1} : We begin by defining a solution for (3.9)

$$D_n(\mathbf{g}) \chi_a(x, E, \mathbf{g}) = (-1)^n P_K(x, E) \chi_a(x, E, \mathbf{g}), \tag{B.3}$$

and its adjoint equation

$$D_n(\mathbf{g}^\dagger) \chi_a^\dagger(x, E, \mathbf{g}^\dagger) = P_K(x, E) \chi_a^\dagger(x, E, \mathbf{g}^\dagger), \tag{B.4}$$

where $\mathbf{g} = \{g_{n-1}, g_{n-2}, \dots, g_1, g_0\}$ and $\mathbf{g}^\dagger = \{n-1-g_0, n-1-g_1, \dots, n-1-g_{n-1}\}$.

Eq. (B.4) was defined, as usual, by applying the rule $(d^p/dx^p)^\dagger = (-1)^p (d^p/dx^p)$. Notice also that (B.4) is not the $\psi^{(n-1)}$ -related ODE, which is instead obtained by simply replacing \mathbf{g} with \mathbf{g}^\dagger in (B.3).

Consider equation (B.3). The first step is to define a linear operator

$$L_{\mathbf{g}}^A(x^p) = \frac{x^{p+n}}{\prod_{b=0}^{n-1} (p+n-g_b)} \tag{B.5}$$

such that for any polynomial $\mathcal{P}(x)$ of x

$$D_n(\mathbf{g})L_{\mathbf{g}}^A(\mathcal{P}(x)) = \mathcal{P}(x). \tag{B.6}$$

Now it is easy to check that

$$\chi_a(x, E, \mathbf{g}) = x^{g_a} + (-1)^n L_{\mathbf{g}}^A(P_K(x, E)\chi_a(x, E, \mathbf{g})) \tag{B.7}$$

defines a solution to (B.3) solvable by iteration. Setting $g_a^\dagger = n - 1 - g_{n-1-a}$,

$$\chi_a^\dagger(x, E, \mathbf{g}^\dagger) = x^{g_a^\dagger} + L_{\mathbf{g}^\dagger}^A(P_K(x, E)\chi_a^\dagger(x, E, \mathbf{g}^\dagger)), \tag{B.8}$$

instead defines an iterative solution to the adjoint equation (B.4).

D_n : The appropriate linear operator for the self-adjoint equation (3.10)

$$D_n(\mathbf{g}^\dagger) \left(\frac{d}{dx} \right)^{-1} D_n(\mathbf{g})\chi_a(x, E, \mathbf{g}) = \sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)}\chi_a(x, E, \mathbf{g}) \tag{B.9}$$

is

$$L_{\mathbf{g}}^D(x^p) = \frac{(p + \frac{h^\vee}{2} + 1)x^{p+h^\vee+1}}{\prod_{b=0}^{2n-1} (p + 1 + \lambda_b)}, \tag{B.10}$$

and the corresponding iterative solution is

$$\chi_a(x, E, \mathbf{g}) = x^{\lambda_a} + L_{\mathbf{g}}^D \left(\sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)}\chi_a(x, E, \mathbf{g}) \right). \tag{B.11}$$

The relation between the λ 's and the g 's is given in (5.4).

B_n : The iterative solution of the B_n Eq. (3.11)

$$D_n(\mathbf{g}^\dagger)D_n(\mathbf{g})\chi_a(x, E, \mathbf{g}) = -\sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)}\chi_a(x, E, \mathbf{g}) \tag{B.12}$$

and its adjoint

$$D_n(\mathbf{g}^\dagger)D_n(\mathbf{g})\chi_a^\dagger(x, E, \mathbf{g}) = \sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)}\chi_a^\dagger(x, E, \mathbf{g}) \tag{B.13}$$

are, respectively,

$$\chi_a(x, E, \mathbf{g}) = x^{\lambda_a} - L_{\mathbf{g}}^B \left(\sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)}\chi_a(x, E, \mathbf{g}) \right) \tag{B.14}$$

and

$$\chi_a^\dagger(x, E, \mathbf{g}) = x^{\lambda_a} + L_{\mathbf{g}}^B \left(\sqrt{P_K(x, E)} \left(\frac{d}{dx} \right) \sqrt{P_K(x, E)}\chi_a^\dagger(x, E, \mathbf{g}) \right), \tag{B.15}$$

with

$$L_{\mathbf{g}}^B(x^p) = \frac{x^{p+h^\vee+1}}{\prod_{b=0}^{2n-1} (p + 1 + \lambda_b)}. \tag{B.16}$$

The relation between the λ 's and the g 's is given in (6.4).

C_n : The χ -solutions to the self-adjoint equation (3.12)

$$D_n(\mathbf{g}^\dagger) \left(\frac{d}{dx} \right) D_n(\mathbf{g}) \chi_a(x, E, \mathbf{g}) = P_K(x, E) \left(\frac{d}{dx} \right)^{-1} P_K(x, E) \chi_a(x, E, \mathbf{g}) \tag{B.17}$$

satisfy

$$\chi_a(x, E, \mathbf{g}) = x^{\lambda_a} + L_{\mathbf{g}}^C \left(P_K(x, E) \left(\frac{d}{dx} \right)^{-1} P_K(x, E) \chi_a(x, E, \mathbf{g}) \right) \tag{B.18}$$

with

$$L_{\mathbf{g}}^C(x^p) = \frac{x^{p+2n+1}}{\prod_{b=0}^{2n-1} (p+1+\lambda_b)}, \tag{B.19}$$

and the relation between the λ 's and the \mathbf{g} 's given in (7.12).

B.2. Dual formulation of the boundary problem

From (B.2), with the numerical estimates for the χ -functions and their derivatives at large values of x , replacing $\psi(x, E, \mathbf{g})$ by its asymptotic behaviour and by varying E one has in principle access to $Q_{[0]}^{(1)}(E, \mathbf{g})$ and in particular to its zeros. This process is certainly possible but it is tedious and the CPU time increases at least quadratically with the order of the equation. Surprisingly, there is a short-cut that makes the algorithm essentially order-independent.

To see this, consider the A_{n-1} case

$$C Q_{[0]}^{(1)}(E, \mathbf{g}) = W[\psi, \chi_1, \chi_2, \dots, \chi_{n-1}], \tag{B.20}$$

where

$$C = W[\chi_0, \chi_1, \chi_2, \dots, \chi_{n-1}] = (-1)^{\text{int}[n/2]} \prod_{i=0, j=i+1}^{n-1} (\lambda_i - \lambda_j). \tag{B.21}$$

Expand (B.20) with respect to the first column:

$$C Q_{[0]}^{(1)}(E, \mathbf{g}) = \sum_{p=0}^{n-1} (-1)^p w_p \frac{d^p \psi}{dx^p}, \tag{B.22}$$

where, borrowing the short-hand notation of Section 5.3,

$$w_p(E, \mathbf{g}) = [0, 1, \dots, p-1, p+1, \dots, n-1] \\ = \det \left[\left(\vec{\chi}, \frac{d}{dx} \vec{\chi}, \dots, \frac{d^{p-1}}{dx^{p-1}} \vec{\chi}, \frac{d^{p+1}}{dx^{p+1}} \vec{\chi}, \dots, \frac{d^{n-1}}{dx^{n-1}} \vec{\chi} \right) \right] \tag{B.23}$$

with $\vec{\chi} = (\chi_1, \chi_2, \dots, \chi_{n-1})$. At large x on the positive real axis

$$\frac{d^p}{dx^p} \psi(x) \sim (-1)^p x^{(1-n+2p)M/2} \exp(-x^{M+1}/(M+1)) \quad (M > K/(h^\vee - K)). \tag{B.24}$$

Therefore, if for all functions $w_p(E, \mathbf{g})$ we have

$$w_p(e_i, \mathbf{g}) = o(x^{(1-n)M/2} \exp(x^{M+1}/(M+1))), \quad p = 0, 1, \dots, n-1 \tag{B.25}$$

for some $E = e_i$, then from (B.22)

$$\mathcal{Q}_{[0]}^{(1)}(e_i, \mathbf{g}) = 0. \tag{B.26}$$

This shows that $\{e_i\} \subset \{E_i^{(1)}\}$. We shall now argue that the set $\{e_i\}$ is not empty and further prove that

$$\{e_i\} = \{E_i^{(1)}\}. \tag{B.27}$$

To do this consider $w_{n-1}(E, \mathbf{g}) = [0, 1, 3, \dots, n - 2]$. By repeated differentiation and substitution of the ODE, one can prove as in [11] that

$$w_{n-1}(E, \mathbf{g}) = \left((-1)^{\text{int}[(n-1)/2]} \prod_{i=1, j=i+1}^{n-1} (g_i - g_j) \right) \chi_{n-1}^\dagger(x, E, \mathbf{g}^\dagger) \tag{B.28}$$

i.e. w_{n-1} satisfies the adjoint A_{n-1} -equation (B.4) and thus can be determined using (B.8) with $g_{n-1}^\dagger = n - 1 - g_0$.

At large x ,

$$w_{n-1}(x) = S(E, \mathbf{g}) \left(x^{(1-n)M/2} + o(x^{(1-n)M/2}) \right) \exp\left(\frac{x^{M+1}}{M+1}\right) + \dots \tag{B.29}$$

where the dots in (B.29) indicate terms that grow exponentially at most as

$$\exp\left(\cos\left(\frac{2\pi}{n}\right) \frac{x^{M+1}}{M+1}\right). \tag{B.30}$$

Therefore, the condition (B.25) for $p = n - 1$ is fulfilled for any value of $E = e_i$ such that $S(e_i, \mathbf{g}) = 0$. Since this a single condition imposed on a nontrivial function of E , we expect an infinite but countable set of solutions.

For the next step, we want to argue that if the condition (B.25) is satisfied for $p = n - 1$ it is simultaneously satisfied for $p = 0, 1, \dots, n - 2$.

From (B.29) we see that at $S(e_i, \mathbf{g}) = 0$ also

$$\frac{d^p}{dx^p} w_{n-1}(e_i, \mathbf{g}) = o(x^b \exp(x^{M+1}/(M+1))) \tag{B.31}$$

for any finite complex number b . We now claim that all of the w_a can be written as a linear combination of w_{n-1} and its derivatives:

$$w_p(x, E, \mathbf{g}) = \sum_{a=0}^{n-1-p} \frac{K_a^{(p)}}{x^{n-1-p-a}} \frac{d^a}{dx^a} w_{n-1}(x, E, \mathbf{g}), \tag{B.32}$$

where the constants $K_a^{(p)}$ depend only on \mathbf{g} . It is very simple to see that (B.32) is true in general. Write the adjoint A_{n-1} ODE in the ‘expanded’ form

$$\frac{d^n}{dx^n} w_{n-1} = \left(\frac{A_2}{x^2} \frac{d^{n-2}}{dx^{n-2}} + \frac{A_3}{x^3} \frac{d^{n-3}}{dx^{n-3}} + \dots + P_K(x, E) \right) w_{n-1}. \tag{B.33}$$

Differentiating once

$$\frac{d}{dx} w_{n-1} = [0, 1, 2, \dots, n - 3, n - 1] = w_{n-2}, \tag{B.34}$$

taking further derivatives and using (B.33) the general result is

$$w_{p-1} = \frac{d}{dx} w_p - \frac{A_{n-p}}{x^{n-p}} w_{n-1}. \tag{B.35}$$

Solving (B.35) recursively proves (B.32).

Eqs. (B.32) and (B.31) together are equivalent to (B.25). To prove (B.27), substitute the asymptotic behaviours (B.24) and

$$w_{n-1-i} \sim \frac{d^i}{dx^i} w_{n-1}(x) \sim S(E, \mathbf{g}) x^{(1-n+2i)M/2} \exp\left(\frac{x^{M+1}}{M+1}\right) \tag{B.36}$$

into (B.22) to see that

$$C Q_{[0]}^{(1)}(E, \mathbf{g}) = nS(E, \mathbf{g}). \tag{B.37}$$

The eigenvalues are obtained by solving $\chi_{n-1}^\dagger(x, E, \mathbf{g}^\dagger) = 0$ for fixed x , chosen large enough to capture the asymptotic behaviour of the solution while also ensuring the power series is reliable. This condition is different from the precise requirement (B.31), but considering (B.29) and (B.30) it is clear that, provided x is very large, the error can be minimised and it selects the points on the complex E -plane where the function $S(E, \mathbf{g})$ is approximately zero.

Similarly, for B_n the eigenvalues of the spectral determinant $Q_{[0]}^{(1)}$ are found by solving (B.15) for $\chi_{2n-1}^\dagger(x, E, \mathbf{g})$. The situation is more complicated in the presence of the integral operators that appear in the D_n and the C_n models, and we have not yet fully completed the analysis. However, guided by the above and the results when the integral operator is absent, the obvious prescription to compute $\chi_{2n-1}(x, E, \mathbf{g})$ from (B.11) and $\chi_{2n+1}(x, E, \mathbf{g})$ from (B.18) for D_n and C_n respectively works very well.

B.3. NLIEs

The Bethe ansatz equations of type A_{n-1} and D_n for $K = 1$ can be rephrased as a set of r nonlinear integral equations where r is the rank of the algebra [68] (see also [3,9,11,64,70–73]). They are compactly written as

$$f^{(a)}(\theta) = -\frac{2i\pi}{h^\vee} \frac{\gamma_a}{\alpha} - 2ib_0 e^\theta + \sum_{b=1}^r \left(\int_{\mathcal{C}_1} d\theta' \varphi_{ab}(\theta - \theta') \ln(1 + e^{f^{(b)}(\theta')}) - \int_{\mathcal{C}_2} d\theta' \varphi_{ab}(\theta - \theta') \ln(1 + e^{-f^{(b)}(\theta')}) \right), \tag{B.38}$$

with $b_0 = m_1 \sin \mu$ and

$$\varphi_{ab}(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\theta} \left(\delta_{ab} - \frac{\sinh(\pi\mu k)}{\sinh(\frac{\pi k}{h^\vee}(h^\vee\mu - 1)k) \cosh(\frac{\pi k}{h^\vee})} C_{ab}^{-1}(k) \right). \tag{B.39}$$

The integration contours \mathcal{C}_1 and \mathcal{C}_2 run from $-\infty$ to ∞ just below and just above the real axis respectively. The algebra dependence is encoded in the constant b_0 , the ‘twists’ $\{\gamma_a/\alpha\}$, and the deformed Cartan matrix

$$C_{ab}(k) = \begin{cases} 2 & a = b, \\ \frac{-1}{\cosh(\frac{\pi k}{h^\vee})} & \langle ab \rangle. \end{cases} \tag{B.40}$$

Above, (ab) implies the nodes a and b of the Dynkin diagram are connected. The explicit expressions for the inverse Cartan matrices can be found in [68].

By construction, the spectrum of the ordinary differential equation is encoded in the zeros θ_j of $1 + \exp(f^{(a)}(\theta))$ that lie on the real axis of the complex- θ plane [1]. The eigenvalues of the relevant ODE are obtained via the relation

$$E_j = e^{\theta_j/\mu}. \quad (\text{B.41})$$

The large- E WKB-like behaviour can be obtained from the large- θ limit of these equations. In this limit the convolution terms in (B.38) can be dropped, and imposing $f^{(1)}(\theta_j) = i\pi(2j-1)$ for integer j yields the semiclassical prediction

$$2 \sin(\mu\pi) m_a (E_j^{(a)})^\mu \sim \left(2j + 1 - \frac{2}{\alpha h^\vee} \gamma_a \right) \pi, \quad j = 0, 1, 2, \dots \quad (\text{B.42})$$

with $\alpha = 2(\mu - 1/h^\vee)$.

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