Approximation of the Hilbert Transform on the Real Line Using Freud Weights

Incoronata Notarangelo

Dedicated to Professor Gradimir V. Milovanović on his 60th birthday

1 Introduction

Let us consider the Cauchy principal value integral

$$\mathcal{H}(G,y) := \int_{\mathbb{R}} \frac{G(x)}{x-y}\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{G(x)}{x-y}\,\mathrm{d}x,$$

where $y \in \mathbb{R}$. If the limit exists, we call $\mathcal{H}(G)$ the Hilbert transform of the function G. It is well known that $\mathcal{H}$ is a bounded operator in $L^p(\mathbb{R})$ for $1 < p < \infty$, while it is, in general, unbounded in the space of continuous functions. Nevertheless, if the function G satisfies the Dini-type condition

$$\int_0^1 \frac{\omega(G,t)}{t}\,\mathrm{d}t < \infty,$$

where ω is its usual modulus of smoothness with step t, then its Hilbert transform $\mathcal{H}(G)$ is continuous on $\mathbb{R}$ (see [12, p. 218]).

The numerical approximation of the Hilbert transform on $\mathbb{R}$ has interested several authors (see, for instance, [1, 3–5, 13, 14, 23–25]). To be more precise, to this end the zeros of Hermite and Markov–Sonin polynomials have been used in [4] and [5], respectively.

Incoronata Notarangelo
PhD student "International Doctoral Seminar entitled J. Bolyai",
Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata,
Via dell'Ateneo Lucano 10, I-85100 Potenza, Italy,
incoronata.notarangelo@unibas.it

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In this paper, we want to compute integrals of the form

$$\mathcal{H}(fw,y) = \int_{\mathbb{R}} \frac{f(x)w(x)}{x-y}\,\mathrm{d}x\,,$$

where $w(x) = \mathrm{e}^{-|x|^\alpha}$, $\alpha > 1$, is a Freud weight. Concerning the study of the smoothness of the function $\mathcal{H}(fw)$, we refer the reader to [19]. We remark that integrals of this form appear in Cauchy singular integral equations, and the numerical treatment of these equations usually requires the approximation of Hilbert transforms (see [19]).

For this purpose, we suggest some simple quadrature rules obtained from a Gauss-type formula based on the zeros of Freud polynomials. These rules, in some aspects, are different from those used in [4, 5]. We will consider two different cases: the first when y is sufficiently small, and the second otherwise. The main effort in this paper will be to prove the stability and the convergence of the proposed rules.

The paper is organized as follows. In Sect. 2, we recall some basic facts. In Sect. 3, we introduce the quadrature rules and state our main results. In Sect. 4, some numerical examples are described. In Sect. 5, we prove our main results. Finally, the Appendix deals with the computation of the Hilbert transform of a Freud weight.

2 Preliminary Results and Notations

In the sequel, C will stand for a positive constant that can assume different values in each formula, and we shall write $\mathrm{C} \neq \mathrm{C}(a,b,\ldots)$ when C is independent of $a,b,\ldots$. Furthermore, $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant C independent of these parameters such that $(A/B)^{\pm 1} \leq \mathrm{C}$.

2.1 Function Spaces

Let us consider the weight function u defined by

$$u(x) = (1+|x|)^\beta\,\mathrm{e}^{-|x|^\alpha}, \quad \beta \geq 0,\ \ \alpha > 1\,,$$

for $x \in \mathbb{R}$.

We denote by C_u the following set of continuous functions,

$$C_u = \left\{ f \in C^0(\mathbb{R}) : \lim_{x\to\pm\infty} f(x)u(x) = 0 \right\},$$

equipped with the norm

$$\|fu\| := \|fu\|_\infty = \sup_{x\in\mathbb{R}} |f(x)u(x)|\,.$$

In the sequel, we will write

$$\|fu\|_E = \sup_{x\in E} |f(x)u(x)|$$

for any $E \subset \mathbb{R}$. We note that the Weierstrass theorem implies the limit conditions in the definition of C_u.

Subspaces of C_u are the Sobolev spaces, defined by

$$W_r(u) = \left\{ f \in C_u : \ f^{(r-1)} \in AC(\mathbb{R}), \ \|f^{(r)}u\| < \infty \right\}, \quad r \in \mathbb{Z}^+,$$

where $AC(\mathbb{R})$ denotes the set of all functions which are absolutely continuous on every closed subset of $\mathbb{R}$. We equip these spaces with the norm

$$\|f\|_{W_r(u)} = \|fu\| + \|f^{(r)}u\|\,.$$

For any $f \in C_u$, we consider the following main part of the r-th modulus of smoothness (see [6])

$$\Omega^r(f,t)_u = \sup_{0<h\le t} \|\Delta_h^r(f)\,u\|_{\mathcal{I}_{r,h}}\,, \quad r \in \mathbb{Z}^+,$$

where $\mathcal{I}_{r,h} = \left[-\mathrm{A}rh^{-1/(\alpha-1)}, \mathrm{A}rh^{-1/(\alpha-1)}\right]$, $\mathrm{A} > 0$ is a constant, and

$$\Delta_h f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right), \quad \Delta^r = \Delta\left(\Delta^{r-1}\right).$$

The r-th modulus of smoothness is given by

$$\omega^r(f,t)_u = \Omega^r(f,t)_u + \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)\,u\|_{(-\infty,-\mathrm{A}rt^*)} + \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)\,u\|_{(\mathrm{A}rt^*,+\infty)}\,,$$

with step $t < t_0$ (t_0 sufficiently small) and $t^* := t^{-1/(\alpha-1)}$.

This modulus of smoothness is equivalent to the following K-functional

$$K(f,t^r)_u = \inf_{g\in W_r(u)} \left\{ \|(f-g)\,u\| + t^r \|g^{(r)}u\| \right\},$$

namely $\omega^r(f,t)_u \sim K(f,t^r)_u$. It follows that

$$\omega^r(f,t)_u \le \mathrm{C}t^r \|f^{(r)}u\| \tag{1}$$

for any $f \in W_r(u)$, with $\mathrm{C} \ne \mathrm{C}(f,t)$.

Let us denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m and by $E_m(f)_u = \inf_{P\in\mathbb{P}_m} \|(f-P)\,u\|$ the error of best polynomial approximation of $f \in C_u$. The following Jackson and Stechkin-type inequalities hold true (see [6]):

$$E_m(f)_u \le \mathrm{C}\,\omega^r\left(f, \frac{a_m}{m}\right)_u, \quad r < m, \tag{2}$$

$$\omega^r\left(f,\frac{a_m}{m}\right)_u \le \mathrm{C}\left(\frac{a_m}{m}\right)^r \sum_{k=1}^{m}\left(\frac{k}{a_k}\right)^r \frac{E_k(f)_u}{k}, \tag{3}$$

where $f \in C_u$, $a_m := a_m(u) \sim m^{1/\alpha}$ is the Mhaskar–Rahmanov–Saff (M–R–S) number related to the weight u; in both cases, C is a positive constant independent of f and m, and $\lfloor a \rfloor$ stands for the largest integer smaller than or equal to $a \in \mathbb{R}^+$.

An estimate, weaker than (2), for the error of best polynomial approximation is given by

$$E_m(f)_u \le \mathrm{C}\int_0^{a_m/m} \frac{\Omega^r(f,t)_u}{t}\,\mathrm{d}t \tag{4}$$

with $\mathrm{C} \neq \mathrm{C}(m,f)$.

By means of the modulus of smoothness, we can define the Zygmund spaces

$$Z_s(u) = \left\{ f \in C_u :\ \sup_{t>0}\frac{\omega^r(f,t)_u}{t^s} < \infty,\ r > s \right\}, \quad s \in \mathbb{R}^+,$$

with the norm

$$\|f\|_{Z_s(u)} = \|fu\| + \sup_{t>0}\frac{\omega^r(f,t)_u}{t^s}.$$

We remark that, by inequalities (4) and (3), $\sup_{t>0}\Omega^r(f,t)_u t^{-s} < \infty$ implies $\sup_{t>0}\omega^r(f,t)_u t^{-s} < \infty$. Therefore, in the definition of the Zygmund space, $\omega^r(f,t)_u$ can be replaced by $\Omega^r(f,t)_u$.

Finally, in the sequel we will write $\omega(f,t)_u$ and $\Omega(f,t)_u$ in place of $\omega^1(f,t)_u$ and $\Omega^1(f,t)_u$.

2.2 *Orthonormal Polynomials and Gaussian Rule*

Let us consider the Freud weight $w(x) = \mathrm{e}^{-|x|^\alpha}$, $\alpha > 1$, $x \in \mathbb{R}$, and its related M-R-S number a_m, given by (see for instance [15])

$$a_m = a_m(w) = \left[\frac{2\pi}{\alpha B((\alpha+1)/2, 1/2)}\right]^{1/\alpha} m^{1/\alpha},$$

where B is the beta function.

Let $\{p_m(w)\}_{m\in\mathbb{N}}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficient and degree m. We denote by $x_k := x_{m,k}$, $1 \le k \le \lfloor m/2 \rfloor$, the positive zeros of $p_m(w)$ and by $x_{-k} := x_{m,-k} = -x_{m,k}$ the negative ones, both ordered increasingly. If m is odd then $x_{m,0} = 0$ is a zero of $p_m(w)$. These zeros satisfy (see for instance [16])

$$-a_m(w) < x_{m,-\lfloor m/2 \rfloor} < \cdots < x_{m,1} < x_{m,2} < \cdots < x_{m,\lfloor m/2 \rfloor} < a_m(w).$$

For a fixed $\theta \in (0,1)$, we define an index $j = j(m)$ such that

$$x_j = \min_{1\le k\le \lfloor m/2 \rfloor} \{x_k : x_k \ge \theta a_m(w)\}. \tag{5}$$

Hence, letting $\Delta x_k := \Delta x_{m,k} = x_{m,k+1} - x_{m,k}$, $k \in \{-\lfloor m/2 \rfloor, \dots, \lfloor m/2 \rfloor - 1\}$, we have (see [18])

$$\Delta x_{m,k} \sim \frac{a_m}{m}, \quad |k| \le j. \tag{6}$$

The following proposition will be useful in the sequel.

Proposition 1. *Let $x_{m+1,k}$, $|k| \le \lfloor (m+1)/2 \rfloor$, be the zeros of $p_{m+1}(w)$. If $x_{m+1,k+1}$, $x_{m,k} \in (-\theta a_m, \theta a_m)$, with a fixed $\theta \in (0,1)$, we have*

$$x_{m+1,k+1} - x_{m,k} \sim \frac{a_m}{m},$$

where the constants in "$\sim$" are independent of m and k.

Finally, we will need the following "truncated" Gaussian quadrature rule,

$$\int_{\mathbb{R}} f(x)w(x)\,\mathrm{d}x = \sum_{|k|\le j} \lambda_{m,k}(w) f(x_{m,k}) + \rho_m(f), \tag{7}$$

where $\rho_m(f)$ is the remainder term, $\lambda_k := \lambda_{m,k}(w)$ are the coefficients of the usual Gaussian rule and $x_{m,k}$ are the zeros of $p_m(w)$. An estimate of the remainder term is given by the following proposition, proved in [18].

Proposition 2. *Let $f \in C_u$, where $u(x) = (1+|x|)^\beta \mathrm{e}^{-|x|^\alpha}$, $\alpha > 1$. If $\beta > 1$, then there holds*

$$|\rho_m(f)| \le \mathrm{C}\left\{E_M(f)_u + \mathrm{e}^{-\mathrm{A}m}\|fu\|\right\},$$

where $M = \lfloor (\theta/(\theta+1))^\alpha m/2 \rfloor \sim m$, C and A are positive constants independent of m and f.

3 Main Results

To compute the Hilbert transform

$$\mathcal{H}(fw,y) = \int_{\mathbb{R}} \frac{f(x)w(x)}{x-y}\,\mathrm{d}x,$$

where the integral is understood in the Cauchy principal value sense, we use the well-known decomposition

$$\mathcal{H}(fw,y) = \int_{\mathbb{R}} \frac{f(x)-f(y)}{x-y} w(x)\,\mathrm{d}x + f(y)\int_{\mathbb{R}} \frac{w(x)}{x-y}\,\mathrm{d}x. \tag{8}$$

We assume that the second integral can be calculated with the required precision (in the Appendix, we will give some examples of how this can be done). The other is an ordinary improper integral, and so we apply a quadrature rule to compute it. The presence of the weight w in the first integral in (8) leads us to use the Gauss-type rule (7). Hence, we get

$$\begin{aligned} \mathcal{H}(fw,y) &= \sum_{|k|\le j} \lambda_{m,k}(w)\frac{f(x_{m,k})-f(y)}{x_{m,k}-y} + f(y)\int_{\mathbb{R}} \frac{w(x)}{x-y}\mathrm{d}x + e_m(f,y) \\ &= \sum_{|k|\le j} \lambda_{m,k}(w)\frac{f(x_{m,k})}{x_{m,k}-y} + \\ &\qquad + f(y)\left[\int_{\mathbb{R}} \frac{w(x)}{x-y}\mathrm{d}x - \sum_{|k|\le j}\frac{\lambda_{m,k}(w)}{x_{m,k}-y}\right] + e_m(f,y) \qquad (9) \\ &=: \sum_{|k|\le j} \lambda_{m,k}(w)\frac{f(x_{m,k})}{x_{m,k}-y} + f(y)A_m(y) + e_m(f,y) \qquad (10) \\ &=: H_m(f,y) + e_m(f,y), \end{aligned}$$

$e_m(f,y)$ being the remainder term, and assuming $x_{m,k} \neq y$, $|k| \le j$.

Since the quantities $x_{m,k} - y$ could be "too small", the rule $H_m(f,y)$ is essentially unstable. Nevertheless, it can be productively used, making some "careful choices".

First of all, we observe that if, for fixed y and m, there holds $|y| \ge x_{m,j} + 1$, then (9) can be replaced by the simpler formula

$$\int_{\mathbb{R}} \frac{f(x)w(x)}{x-y}\mathrm{d}x = \sum_{|k|\le j} \lambda_{m,k}(w)\frac{f(x_{m,k})}{x_{m,k}-y} + \rho_m(f,y), \qquad (11)$$

where the remainder term $\rho_m(f,y)$ can be estimated using the following theorem.

Theorem 1. *Assume $|y| \ge x_{m,j}+1$. For any $f \in W_r(u)$, with $\beta > 1$, and for a sufficiently large m (say, $m \ge m_0$), we have*

$$|\rho_m(f,y)| \le \mathrm{C}\left(\frac{a_M}{M}\right)^r \|f\|_{W_r(u)}, \qquad (12)$$

where $M = \lfloor (\theta/(\theta+1))^\alpha m/2 \rfloor \sim m$ and C is a positive constant independent of m and f.

Now we observe that, for every fixed y and for $m \ge m_0$ ($m_0 = m_0(y,\theta)$), we have $|y| \le x_{m,j}$. Therefore, let us consider the rule $H_m(f,y)$ under the assumption $|y| \le x_{m,j}$. We remark that the term in (9) producing instability is the one related to the knot $x_{m,d}$ closest to y, i.e., $\lambda_{m,d}(w)/(x_{m,d}-y)$. But for $y \in [-x_{m,j}, x_{m,j}]$ there holds

$$\frac{\lambda_{m,d}(w)}{x_{m,d}-y} \sim \frac{a_m}{m}\frac{w(x_{m,d})}{x_{m,d}-y}.$$

Thus, for a fixed $y \in [-x_{m,j}, x_{m,j}]$, we are going to construct a sequence of integers $\{m^*\} \subset \mathbb{N}$, and then a sequence $\{H_{m^*}(f,y)\}_{m^* \in \mathbb{N}}$, such that

$$|x_{m^*,d} - y| \sim \frac{a_m}{m}.$$

This is possible by virtue of Proposition 1, and our choice is determined as follows.

Assume that $x_{m,d} \le y < x_{m,d+1}$ for some $d \in \{-j, \dots, j-1\}$. Because of the interlacing properties of the zeros of $p_{m+1}(w)$ with those of $p_m(w)$, two cases are possible:

(a) $x_{m,d} \le y < x_{m+1,d+1}$; (b) $x_{m+1,d+1} \le y < x_{m,d+1}$.

In the case (a), if $y < (x_{m,d} + x_{m+1,d+1})/2$, then we choose $m^* = m+1$ and we use the quadrature rule $H_{m+1}(f,y)$, otherwise we choose $m^* = m$ and we use $H_m(f,y)$. We make a similar choice in the case (b).

Thus, for every fixed $y \in [-x_{m,j}, x_{m,j}]$ we can define a sequence $\{H_{m^*}(f,y)\}$, $m^* \in \{m, m+1\}$. The next theorem proves the stability and the convergence of this sequence.

Theorem 2. *Let $y \in [-x_{m^*,j}, x_{m^*,j}]$ be fixed, with j defined by (5). For any $f \in C_u$, with $\beta > 1$, we have*

$$|H_{m^*}(f,y)| \le \mathrm{C}\, \|fu\| \log m. \tag{13}$$

Moreover, if f is such that

$$\int_0^1 \frac{\omega^r(f,t)_u}{t}\, \mathrm{d}t < \infty, \quad r \in \mathbb{Z}^+,$$

there holds

$$|e_{m^*}(f,y)| \le \mathrm{C} \left\{ \log m \int_0^{\frac{a_M}{M}} \frac{\omega^r(f,t)_u}{t}\, \mathrm{d}t + \mathrm{e}^{-\mathrm{A}m} \|fu\| \right\}. \tag{14}$$

In both inequalities, C *and* A *are positive constants independent of m, y and f, while $M = \lfloor (\theta/(\theta+1))^\alpha m/2 \rfloor \sim m$.*

In particular, by Theorem 2, for $m \ge m_0$ and for $\beta > 1$, if $f \in Z_s(u)$, $s > 0$, we have

$$|e_{m^*}(f,y)| \le \mathrm{C} \log m \left(\frac{a_m}{m} \right)^s \|f\|_{Z_s(u)},$$

while if $f \in W_r(u)$ we get

$$|e_{m^*}(f,y)| \le \mathrm{C} \log m \left(\frac{a_m}{m} \right)^r \|f\|_{W_r(u)}, \tag{15}$$

using inequality (1).

Thus, Theorem 2 shows that the rule $H_{m^*}(f,y)$ is stable and its remainder term converges with the same order of the best polynomial approximation, apart from an extra factor $\log m$.

From a numerical point of view, we remark that we have to compute only $2j$ (or $2j+1$ if m^* is odd) Christoffel numbers, zeros of $p_{m^*}(w)$, and values of the function f. Moreover, if $\alpha=2$, then $\{p_{m^*}(w)\}_{m\in\mathbb{N}}$ is the sequence of Hermite polynomials, that is the simplest case, and one can use the routine "gaussq" (see [10, 11]) or the routines "recur" and "gauss" (see [9]). In the case $\alpha\neq 2$, we can use the Mathematica Package "`OrthogonalPolynomials`" (see [2]).

Remark 1. We can use the same method to evaluate

$$\mathcal{H}(fw_\delta,y)=\int_{\mathbb{R}}\frac{f(x)w_\delta(x)}{x-y}\,\mathrm{d}x,$$

where $w_\delta(x)=\mathrm{e}^{-\delta|x|^\alpha}$, $\delta>0$.

In fact, since $\{p_m(w_\delta,x)\}_{m\in\mathbb{N}}=\{\delta^{1/(2\alpha)}p_m(w,\delta^{1/\alpha}x)\}_{m\in\mathbb{N}}$, we have

$$p_m(w_\delta,z_k)=0\Leftrightarrow z_k=\frac{x_{m,k}(w)}{\delta^{1/\alpha}},\quad |k|\leq\left\lfloor\frac{m}{2}\right\rfloor,$$

and the same relation holds for the Christoffel numbers.

4 Numerical Examples

In this section, we show some approximate values for the integral $\mathcal{H}(fw,y)$, $y\in\mathbb{R}$, obtained by using the algorithms described in Sect. 3.

Since the exact value of the integral is not known, in all the tables we report the digits which are correct according to the results obtained for $m=400$.

Moreover, $J=J(m)$ will denote the number of the points that we use in the quadrature rules (9) and (11). To be more precise, J will be equal to $2j$, with j given by (5), if m^* is even, and it will be equal to $2j+1$ otherwise.

All the computations have been done in double-precision arithmetic (ca. 16 decimal digits).

Example 1. We want to evaluate the following integral

$$\int_{\mathbb{R}}\frac{|x-1|^4\mathrm{e}^{-|x|^3}}{x-y}\,\mathrm{d}x.$$

Since the function $f(x)=|x-1|^4\in W_4(u)$, with $u(x)=(1+|x|)^\beta\mathrm{e}^{-|x|^3}$ and $\beta>1$, the theoretical error behaves like $m^{-8/3}\log m$ by using (15) and $a_m\sim m^{1/3}$. In Table 1, we approximate the above integral by using quadrature rules (9) and (11), choosing $\theta=3/5$ in (5). We can see that the numerical results agree with the theoretical ones.

Table 1 Approximate values obtained for $\theta = 3/5$

m	J	$y = 1$	J	$y = 10$
16	10	−3.7	10	−0.5
32	21	−3.7859	20	−0.592
64	42	−3.785959023	42	−0.592611694
128	83	−3.78595902313849	82	−0.592611694730986

Example 2. Let us consider the integral

$$\int_{\mathbb{R}} \frac{\cosh x \, \mathrm{e}^{-x^4}}{x-y} \, \mathrm{d}x .$$

Since the function $f(x) = \cosh x = (\mathrm{e}^x + \mathrm{e}^{-x})/2$ is an analytic function, we obtain very accurate results. In Table 2, we report the results obtained for $\theta = 3/5$.

We remark that one obtains the same approximations for $m = 384$ and $\theta = 3/5$ and for $m = 32$ and $\theta = 0.95$. Therefore, the parameter θ influences the numerical results, notably for very smooth functions and small values of m, but the appropriate choice of this parameter is not yet totally clear.

Table 2 Approximate values obtained for $\theta = 3/5$

m	J	$y = 0.5$	J	$y = -6$
16	11	−1.16	10	0.3
32	18	−1.167	18	0.360
64	39	−1.167487	38	0.36052
128	75	−1.16748708017	76	0.36052114077
256	150	−1.16748708017153	150	0.360521140776152
384	225	−1.167487080171531	224	0.360521140776152

Example 3. Now we want to evaluate the integral

$$\int_{\mathbb{R}} \frac{\mathrm{e}^{-|x|^3}}{(1+x^2)^7 (x-y)} \, \mathrm{d}x .$$

As in Example 2, the function $f(x) = (1+x^2)^{-7}$ is very smooth and one could observe the influence of the parameter θ in the numerical results. In Table 3, we have chosen $\theta = 1/2$.

Example 4. Finally, we consider the integral

$$\int_{\mathbb{R}} \frac{\mathrm{d}x}{(1+x^2)^7 (x-y)} = \int_{\mathbb{R}} \frac{f(x)\, \mathrm{e}^{-|x|^3}}{x-y} \, \mathrm{d}x .$$

Table 3 Approximate values obtained for $\theta = 1/2$

m	J	$y = 0.2$	J	$y = 7$
16	9	−1.553	8	$-9.72\,E-2$
32	17	−1.553	16	$-9.7383\,E-2$
64	35	−1.5536242	34	$-9.738383637\,E-2$
128	68	−1.5536242711590	68	$-9.7383836377130\,E-2$
256	135	−1.55362427115905	134	$-9.7383836377130\,E-2$
384	201	−1.553624271159052	202	$-9.738383637713081\,E-2$

Since $f(x) = (1+x^2)^{-7}e^{|x|^3} \in W_6(u)$, with $u(x) = (1+|x|)^2 e^{-|x|^3}$, the theoretical error behaves like $m^{-4}\log m$. Note that $\|f^{(6)}u\| = \mathcal{O}(10^5)$, and this influences in a negative way the numerical results in Table 4, obtained for $\theta = 1/2$.

Table 4 Approximate values obtained for $\theta = 1/2$

m	J	$y = 0.2$	J	$y = 7$
16	9	−1.5	8	−0.101
32	17	−1.519	16	−0.1014
64	35	−1.5192	34	−0.10143
128	68	−1.5192	68	−0.101431
256	135	−1.51924	134	−0.1014318
384	201	−1.5192406	202	−0.1014318

5 Proofs

First of all, we recall a well-known inequality (see [16]). Let $1 \le p \le \infty$ and let $u(x) = (1+|x|)^\beta e^{-|x|^\alpha}$, with $\alpha > 1$, $\beta \ge 0$. For every $P \in \mathbb{P}_m$ and $\theta > 0$, there holds

$$\|Pu\|_{L^p(J_m)} \le \mathrm{C}e^{-\mathrm{A}m}\|Pu\|_p, \tag{16}$$

where $J_m = \{x \in \mathbb{R} : \ |x| > (1+\theta)a_m(u)\}$, $a_m(u) \sim m^{1/\alpha} \sim a_{m/2}(w)$, C and A are positive constants independent of m and P.

By (16), it easily follows that (see for instance [21]) the inequality

$$\|fu\|_{J_m} \le \mathrm{C}\left\{E_m(f)_u + e^{-\mathrm{A}m}\|fu\|\right\} \tag{17}$$

holds for any $f \in C_u$.

Proof of Proposition 1. An analog of this proposition was proved in [4] (Lemma 2.1) for Hermite zeros. Similar arguments apply in the case of Freud zeros, taking into account (6). Therefore, we omit the proof. □

We recall the Posse–Markov–Stieltjes inequalities (see [8], p. 33 and [17]). If a function g is such that $g^{(k)}(x) \geq 0$, $k = 0,1,\dots,2m-1$, $m > 1$, for $x \in (-\infty, x_d]$, $d = -\lfloor m/2 \rfloor + 1, \dots, \lfloor m/2 \rfloor$, then we have

$$\sum_{k=-\lfloor m/2 \rfloor}^{d-1} \lambda_k g(x_k) \leq \int_{-\infty}^{x_d} g(x) w(x)\, dx \leq \sum_{k=-\lfloor m/2 \rfloor}^{d} \lambda_k g(x_k). \tag{18}$$

On the other hand, if $(-1)^k g^{(k)}(x) \geq 0$, $k = 0,1,\dots,2m-1$, $m > 1$, for $x \in [x_d, +\infty)$, $d = -\lfloor m/2 \rfloor, \dots, \lfloor m/2 \rfloor - 1$, then we have

$$\sum_{k=d+1}^{\lfloor m/2 \rfloor} \lambda_k g(x_k) \leq \int_{x_d}^{+\infty} g(x) w(x)\, dx \leq \sum_{k=d}^{\lfloor m/2 \rfloor} \lambda_k g(x_k). \tag{19}$$

Proposition 3. *Let $A_m(y)$ and j be given by (10) and (5), respectively. Let us assume that $|y| \leq x_j$ and $\min_k |x_k - y| \geq \mathrm{c}\, a_m/m$, $\mathrm{c} \neq \mathrm{c}(m)$. Then we have*

$$|A_m(y)| \leq \mathrm{C}\, w(y) \log m, \tag{20}$$

where C *is a positive constant independent of m and y.*

Proof. Let $x_{d-1} < x_d \leq y < x_{d+1}$, $|d| \leq j$. We can write

$$|A_m(y)| \leq \left| \mathcal{H}(w,y) - \sum_{|k| \leq \lfloor m/2 \rfloor} \frac{\lambda_k}{x_k - y} \right| + \left| \sum_{|k| > j} \frac{\lambda_k}{x_k - y} \right| =: |B_1| + |B_2|. \tag{21}$$

We can decompose the integral in B_1 as

$$\int_{\mathbb{R}} \frac{w(x)}{x-y}\, dx = \left\{ \int_{-\infty}^{x_{d-1}} + \int_{x_{d-1}}^{x_{d+1}} + \int_{x_{d+1}}^{+\infty} \right\} \frac{w(x)}{x-y}\, dx.$$

Using the Posse–Markov–Stieltjes inequalities (18) for the first integral, with $g(x) = 1/(y-x)$, and (19) for the last integral, with $g(x) = 1/(x-y)$, we get

$$\int_{x_{d-1}}^{x_{d+1}} \frac{w(x)}{x-y}\, dx - \frac{\lambda_{d+1}}{x_{d+1} - y} - \frac{\lambda_d}{x_d - y} \leq B_1 \leq \int_{x_{d-1}}^{x_{d+1}} \frac{w(x)}{x-y}\, dx - \frac{\lambda_{d-1}}{x_{d-1} - y} - \frac{\lambda_d}{x_d - y}.$$

It follows that

$$|B_1| \leq \left| \int_{x_{d-1}}^{x_{d+1}} \frac{w(x)}{x-y}\, dx \right| + \left| \frac{\lambda_d}{x_d - y} \right| + \max \left\{ \left| \frac{\lambda_{d-1}}{x_{d-1} - y} \right|, \left| \frac{\lambda_{d+1}}{x_{d+1} - y} \right| \right\}. \tag{22}$$

Since $y - x_d \sim a_m/m \sim x_{d+1} - x_d$, $|d| \leq j$, we have (see for instance [18])

$$\frac{\lambda_d}{|x_d - y|} \sim \frac{\Delta x_d\, w(x_d)}{|x_d - y|} \leq \mathrm{C}\, w(x_d) \sim w(y), \tag{23}$$

using

$$|x-y| \le C\frac{a_m}{m} \quad \Rightarrow \quad w(x) \sim w(y). \tag{24}$$

Analogously, we obtain

$$\frac{\lambda_{d\pm1}}{|x_{d\pm1}-y|} \le C\,w(y). \tag{25}$$

On the other hand, by the mean value theorem, we have

$$\begin{aligned}\left|\int_{x_{d-1}}^{x_{d+1}} \frac{w(x)}{x-y}\,\mathrm{d}x\right| &\le \left|\int_{x_{d-1}}^{x_{d+1}} \frac{w(x)-w(y)}{x-y}\,\mathrm{d}x\right| + w(y)\left|\int_{x_{d-1}}^{x_{d+1}} \frac{\mathrm{d}x}{x-y}\right| \\ &\le \alpha \int_{x_{d-1}}^{x_{d+1}} |\xi_x|^{\alpha-1} w(\xi_x)\,\mathrm{d}x + w(y)\left|\log\frac{x_{d+1}-y}{y-x_{d-1}}\right|, \end{aligned} \tag{26}$$

for some $\xi_x \in (x,y)$. Since $x_{d+1}-y \sim y-x_{d-1}$, by (24) and (6), the right-hand side of (26) is equivalent to

$$\int_{x_{d-1}}^{x_{d+1}} \mathrm{d}w(x) + w(y) \sim w(y) \tag{27}$$

Combining (23), (25)–(27) in (22), we get

$$|B_1| \le C\,w(y)\,. \tag{28}$$

Now let us consider the term B_2. Since $w(x_k) \le w(y)$ for $|k| > j$, we have

$$|B_2| \le C \sum_{|k|>j} \frac{\Delta x_k w(x_k)}{|x_k-y|} \le C\,w(y) \sum_{|k|>j} \frac{\Delta x_k}{|x_k-y|} \le C\,w(y)\log m\,. \tag{29}$$

Combining (28) and (29) in (21), we obtain (20). □

Proof of Theorem 1. We will consider only the case $f \in W_1(u)$, since the case $r \ge 2$ can be proved by iteration.

Given $\Psi \in C^\infty(\mathbb{R})$ an arbitrary nondecreasing function such that

$$\Psi(x) = \begin{cases} 0, & x \le 0\,, \\ 1, & x \ge 1\,, \end{cases}$$

and x_j defined by (5), we set

$$\Psi_j(x) = \Psi\left(\frac{|x|-x_j}{\Delta x_j}\right) = \begin{cases} 1, & |x| \ge x_{j+1}\,, \\ 0, & |x| \le x_j\,. \end{cases}$$

Then we can write

$$\frac{f(x)}{x-y} = \frac{[1-\Psi_j(x)]\,f(x)}{x-y} + \frac{\Psi_j(x)f(x)}{x-y} =: F_1(x) + F_2(x),$$

and so

$$\int_{\mathbb{R}} \frac{f(x)}{x-y}\,\mathrm{d}x = \int_{\mathbb{R}} F_1(x)\,\mathrm{d}x + \int_{\mathbb{R}} F_2(x)\,\mathrm{d}x.$$

Using the truncated Gaussian rule (7), we obtain

$$\int_{\mathbb{R}} F_1(x)w(x)\,\mathrm{d}x = \sum_{|k|\le j} \frac{\lambda_k f(x_k)}{x_k - y} + \rho_m(F_1)\,.$$

Setting

$$\rho_m(f,y) = \rho_m(F_1) + \int_{\mathbb{R}} F_2(x)\,\mathrm{d}x, \tag{30}$$

we have

$$\int_{\mathbb{R}} \frac{f(x)}{x-y}\,\mathrm{d}x = \sum_{|k|\le j} \frac{\lambda_k f(x_k)}{x_k - y} + \rho_m(f,y)\,.$$

Let us consider the first term on the right of (30). By Proposition 2, since $\beta > 1$, we get

$$\begin{aligned} |\rho_m(F_1)| &\le \mathrm{C}\left\{E_M(F_1)_u + \mathrm{e}^{-\mathrm{A}m}\|F_1 u\|\right\} \\ &\le \mathrm{C}\left\{\frac{a_M}{M}\|F_1' u\| + \mathrm{e}^{-\mathrm{A}m}\|fu\|\right\} \le \mathrm{C}\,\frac{a_M}{M}\|f\|_{W_1(u)}, \end{aligned} \tag{31}$$

by Jackson's inequality (2) and (1). In fact for $m \ge m_0$ and $|x| \le x_{j+1}$, we have $|x-y| \ge 1 - \Delta x_j \ge 1/2$ and then $\|F_1 u\| \le 2\,\|fu\|$. Moreover, we can write

$$\begin{aligned} F_1'(x) &= -\frac{\Psi_j'(x)f(x)}{x-y} + \frac{[1-\Psi_j(x)]\,f'(x)}{x-y} - \frac{[1-\Psi_j(x)]\,f(x)}{(x-y)^2} \\ &=: G_1(x) + G_2(x) + G_3(x)\,. \end{aligned} \tag{32}$$

There holds

$$\|G_3 u\| \le \sup_{|x|\le x_{j+1}} \frac{|fu|(x)}{(x-y)^2} \le 4\,\|fu\| \tag{33}$$

and

$$\|G_2 u\| \le \sup_{|x|\le x_{j+1}} \frac{|f'u|(x)}{|x-y|} \le 2\,\|f'u\|\,, \tag{34}$$

since $|y| \ge x_j + 1$ and $|x-y| \ge 1 - \Delta x_j \ge 1/2$. Furthermore, using inequalities (17), (2) and (1), we get

$$\begin{aligned} \|G_1 u\| &= \sup_{x_j \le |x| \le x_{j+1}} \frac{|fu|(x)}{\Delta x_j |x-y|} \le \mathrm{C}\,\frac{m}{a_m} \sup_{x_j \le |x| \le x_{j+1}} |fu|(x) \\ &\le \mathrm{C}\left\{\frac{m}{a_m} E_M(f)_u + \mathrm{e}^{-\mathrm{A}m}\|fu\|\right\} \le \mathrm{C}\,\|f\|_{W_1(u)}, \end{aligned} \tag{35}$$

since $M \sim m$. Therefore, by (33)–(35) and (32), we obtain (31).

To handle the term in (30) containing F_2, we assume for simplicity $y>0$, since the other case is similar. We have

$$\int_{\mathbb{R}} F_2(x)w(x)\,\mathrm{d}x = \left\{\int_{x_j}^{+\infty} + \int_{-\infty}^{-x_j}\right\} \frac{\Psi_j(x)f(x)w(x)}{x-y}\,\mathrm{d}x =: I_1 + I_2\,. \tag{36}$$

Using (17) we get

$$\begin{aligned} |I_2| &\le \int_{-\infty}^{-x_j} \left|\frac{f(x)w(x)}{x-y}\right| \mathrm{d}x \le \|fu\|_{(-\infty,-x_j]} \int_{-\infty}^{-x_j} \frac{\mathrm{d}x}{(y-x)(1+|x|)^\beta} \\ &\le \mathrm{C}\left\{E_M(f)_u + \mathrm{e}^{-\mathrm{A}m}\|fu\|\right\} \le \mathrm{C}\,\frac{a_M}{M}\|f\|_{W_1(u)}\,, \end{aligned} \tag{37}$$

since $\beta>1$ and $M = \lfloor(\theta/(\theta+1))^\alpha m/2\rfloor$.

However, setting $\varepsilon = a_m/m$, we have

$$I_1 = \left\{\int_{x_j}^{y-\varepsilon} + \int_{y-\varepsilon}^{y-\varepsilon} + \int_{y+\varepsilon}^{+\infty}\right\} \frac{\Psi_j(x)f(x)w(x)}{x-y}\,\mathrm{d}x =: J_1 + J_2 + J_3. \tag{38}$$

The integrals J_1 and J_3 are nonsingular and using the same arguments as in the estimate of I_2, we obtain

$$|J_1| + |J_3| \le \mathrm{C}\,\frac{a_M}{M}\|f\|_{W_1(u)}. \tag{39}$$

To estimate J_2, we can write

$$\begin{aligned} |J_2| &= \left|\int_{y-\varepsilon}^{y+\varepsilon} \frac{f(x)w(x)}{x-y}\,\mathrm{d}x\right| \\ &\le w(y)\left|\int_{y-\varepsilon}^{y+\varepsilon} \frac{f(x)-f(y)}{x-y}\,\mathrm{d}x\right| + \left|\int_{y-\varepsilon}^{y+\varepsilon} f(x)\frac{w(x)-w(y)}{x-y}\,\mathrm{d}x\right| \\ &=: A_1 + A_2. \end{aligned} \tag{40}$$

Using the mean value theorem, with $\xi, \xi_x \in (y-\varepsilon, y+\varepsilon)$, and by (24), since $\beta>1$, we have

$$A_1 \le 2\varepsilon\, w(y)|f'(\xi)| \le \mathrm{C}\,\frac{a_m}{m}\|f'u\| \tag{41}$$

and

$$\begin{aligned} A_2 &\le |f(\xi)| \int_{y-\varepsilon}^{y+\varepsilon} |w'(\xi_x)|\,\mathrm{d}x \\ &\sim |f(\xi)| \int_{y-\varepsilon}^{y+\varepsilon} -\mathrm{d}w(x) = |f(\xi)|\,[w(y-\varepsilon) - w(y+\varepsilon)] \\ &\le \mathrm{C}\|fu\|_{[x_j,+\infty)} \le \mathrm{C}\left\{E_M(f)_u + \mathrm{e}^{-\mathrm{A}m}\|fu\|\right\} \\ &\le \mathrm{C}\,\frac{a_M}{M}\|f\|_{W_1(u)}, \end{aligned} \tag{42}$$

by (17). Combining (41) and (42) in (40), we get

$$|J_2| \le \mathrm{C}\,\frac{a_M}{M}\|f\|_{W_1(u)}.$$

It follows that recalling (39) and (38),

$$|I_1| \le \mathrm{C}\,\frac{a_M}{M}\|f\|_{W_1(u)}. \tag{43}$$

By (43), (37), and (36) we obtain

$$\left|\int_{\mathbb{R}} F_2(x)w(x)\,\mathrm{d}x\right| \le \mathrm{C}\,\frac{a_M}{M}\|f\|_{W_1(u)}. \tag{44}$$

Finally, combining (31) and (44) in (30), we get (12). □

To prove Theorem 2 we need the following lemmas.

Lemma 1. *Let* $w(x) = \mathrm{e}^{-|x|^\alpha}$*,* $\alpha > 1$*, and* $u(x) = (1+|x|)^\beta w(x)$*,* $\beta > 0$*. For* $\theta \in (0,1)$ *and* $|y| \le \theta a_m$ *we have*

$$\left|\int_{\mathbb{R}} \frac{f(x)w(x)}{x-y}\,\mathrm{d}x\right| \le \mathrm{C}\left\{\|fu\|\log m + \int_0^1 \frac{\Omega(f,t)_u}{t}\,\mathrm{d}t\right\}, \tag{45}$$

with C *a constant independent of* f*,* m *and* y*.*

Proof. We assume $y > 0$, since the other case can be treated in a similar way.

Letting $\varepsilon = a_m/m$, we can write

$$\begin{aligned}\int_{\mathbb{R}} \frac{f(x)w(x)}{x-y}\,\mathrm{d}x &= \left\{\int_{-\infty}^{-2a_m} + \int_{-2a_m}^{y-\varepsilon} + \int_{y-\varepsilon}^{y+\varepsilon} + \int_{y+\varepsilon}^{2a_m} + \int_{2a_m}^{+\infty}\right\} \frac{f(x)w(x)}{x-y}\,\mathrm{d}x \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{46}$$

Since $\beta > 0$, we get

$$|I_5| \le \|fu\| \int_{2a_m}^{+\infty} \frac{\mathrm{d}x}{(x-y)(1+|x|)^\beta} \le \mathrm{C}\|fu\| \int_{2a_m}^{+\infty} \frac{\mathrm{d}x}{x(1+|x|)^\beta} \le \mathrm{C}\,\|fu\| \tag{47}$$

and also

$$|I_1| \le \mathrm{C}\,\|fu\|. \tag{48}$$

Moreover, we have

$$|I_4| \le \|fu\| \int_{y+\varepsilon}^{2a_m} \frac{\mathrm{d}x}{x-y} \le \mathrm{C}\|fu\| \log\frac{2a_m - y}{\varepsilon} \le \mathrm{C}\,\|fu\|\log m. \tag{49}$$

Analogously, we obtain

$$|I_2| \le \mathrm{C}\,\|fu\|\log m. \tag{50}$$

Now let us consider the term I_3. We can write

$$|I_3| = \left| \int_{y-\varepsilon}^{y+\varepsilon} \frac{f(x)w(x) - f(y)w(y)}{x-y}\,\mathrm{d}x \right| = \left| \int_0^{2\varepsilon} \frac{[fw](y+t/2) - [fw](y-t/2)}{t}\,\mathrm{d}t \right|.$$

Since $w(y+t/2) < w(y) < u(y)$, we obtain

$$\begin{aligned} |I_3| &\le \int_0^{2\varepsilon} \frac{|\Delta_t f(y)|\,u(y)}{t}\,\mathrm{d}t + \|fu\| \int_0^{2\varepsilon} \frac{|\Delta_t w(y)|}{t\,w(y-t/2)}\,\mathrm{d}t \\ &\le \int_0^1 \frac{\Omega(f,t)_u}{t}\,\mathrm{d}t + \|fu\| \int_0^{2\varepsilon} \frac{|\Delta_t w(y)|}{t\,w(y-t/2)}\,\mathrm{d}t. \end{aligned} \tag{51}$$

As in the proof of Proposition 3, using the mean value theorem, (24) and (6), we have

$$\int_0^{2\varepsilon} \frac{|\Delta_t w(y)|}{t\,w(y-t/2)}\,\mathrm{d}t \le \int_0^{2\varepsilon} \frac{|\xi_t|^{\alpha-1} w(\xi_t)}{w(y-t/2)}\,\mathrm{d}t \le \mathrm{C} \int_0^{2\varepsilon} |\xi_t|^{\alpha-1}\,\mathrm{d}t \le \mathrm{C}\,\frac{a_m}{m} a_m^{\alpha-1} \le \mathrm{C}. \tag{52}$$

Combining (47)–(52) in (46) we get (45). □

Lemma 2. *For any function f such that*

$$\int_0^1 \frac{\omega^r(f,t)_u}{t}\,\mathrm{d}t < \infty, \quad r \in \mathbb{Z}^+,$$

and with $P \in \mathbb{P}_m$ the polynomial of best approximation of f, we have

$$\int_0^1 \frac{\Omega(f-P,t)_u}{t}\,\mathrm{d}t \le \mathrm{C} \log m \int_0^{a_m/m} \frac{\omega^r(f,t)_u}{t}\,\mathrm{d}t, \tag{53}$$

with C a positive constant independent of m, f and P.

Proof. By inequality (4), we have

$$\begin{aligned} \int_0^1 \frac{\Omega(f-P,t)_u}{t}\,\mathrm{d}t &= \int_0^{a_m/m} \frac{\Omega(f-P,t)_u}{t}\,\mathrm{d}t + \int_{a_m/m}^1 \frac{\Omega(f-P,t)_u}{t}\,\mathrm{d}t \\ &\le \int_0^{a_m/m} \frac{\Omega(f-P,t)_u}{t}\,\mathrm{d}t + E_m(f)_u \log m \\ &\le \int_0^{a_m/m} \frac{\Omega(f-P,t)_u}{t}\,\mathrm{d}t + \mathrm{C} \log m \int_0^{a_m/m} \frac{\Omega^r(f,t)_u}{t}\,\mathrm{d}t. \end{aligned}$$

By using the Jackson and Stechkin-type inequalities (2) and (3), and proceeding as in the proof of Proposition 4.2 in [20], we obtain

$$\int_0^{a_m/m} \frac{\Omega(f-P,t)_u}{t}\,\mathrm{d}t \le \mathrm{C} \int_0^{a_m/m} \frac{\omega^r(f,t)_u}{t}\,\mathrm{d}t,$$

from which we get (53). □

Proof of Theorem 2. Let us first prove (13). By Proposition 3, with $\beta > 1$, we have

$$\begin{aligned} |H_{m^*}(f,y)| &\le |f(y)|\,|A_{m^*}(y)| + \left| \sum_{|k|\le j} \lambda_{m^*,k}(w) \frac{f(x_{m^*,k})}{x_{m^*,k}-y} \right| \\ &\le \mathrm{C}\|fu\| \log m + \lambda_{m^*,d}(w) \left| \frac{f(x_{m^*,d})}{x_{m^*,d}-y} \right| \\ &\quad + \left| \sum_{|k|\le j, k\ne d} \lambda_{m^*,k}(w) \frac{f(x_{m^*,k})}{x_{m^*,k}-y} \right| \\ &=: \mathrm{C}\|fu\| \log m + |B_1| + |B_2|, \end{aligned} \tag{54}$$

$x_{m^*,d}$ being a zero closest to y.

Since $\beta > 1$, by (23), we get

$$|B_1| \le \mathrm{C}\|fu\|. \tag{55}$$

Moreover, we have

$$|B_2| \le \mathrm{C}\|fu\| \sum_{|k|\le j, k\ne d} \frac{\Delta x_{m^*,k}}{|x_{m^*,k}-y|} \le \mathrm{C}\|fu\| \log m. \tag{56}$$

Combining (55) and (56) in (54), we obtain (13).

Now, we prove (14). Letting $P \in \mathbb{P}_M$ be the polynomial of best approximation of f of degree $M = \lfloor (\theta/(\theta+1))^\alpha m/2 \rfloor \sim m$, we have

$$|e_{m^*}(f,y)| \le |e_{m^*}(f-P,y)| + |e_{m^*}(P,y)|. \tag{57}$$

Since the ordinary Gaussian rule is exact for polynomials of degree at most $2m-1$, we get

$$\begin{aligned} |e_{m^*}(P,y)| &= \left| \sum_{|k|>j} \lambda_{m^*,k}(w) \frac{P(x_{m^*,k}) - P(y)}{x_{m^*,k}-y} \right| \\ &\le \left| \sum_{|k|>j} \lambda_{m^*,k}(w) \frac{P(x_{m^*,k})}{x_{m^*,k}-y} \right| + |P(y)| \sum_{|k|>j} \frac{\lambda_{m^*,k}(w)}{|x_{m^*,k}-y|} \\ &=: |S_1| + |S_2|. \end{aligned} \tag{58}$$

By (29), since $\beta > 1$, and by (16), we have

$$|S_1| \le \mathrm{C} \sum_{|k|>j} \Delta x_{m^*,k} \frac{|P(x_{m^*,k})|\, w(x_{m^*,k})}{|x_{m^*,k}-y|} \le \mathrm{C}\, \mathrm{e}^{-\mathrm{A}m} \|Pu\|. \tag{59}$$

Moreover, we have

$$|S_2| \le \mathrm{C}\, \|Pu\|_{\{x\in\mathbb{R}:|x|\ge|y|\}} \sum_{|k|>j} \frac{\Delta x_{m^*,k}\, \mathrm{e}^{|y|^\alpha - |x_{m^*,k}|^\alpha}}{|x_{m^*,k}-y|}.$$

Two cases are possible. If $|y| \le x_{m^*,j+1}/2^{1/\alpha}$, then

$$|y|^\alpha - |x_{m^*,k}|^\alpha \le -\frac{x^\alpha_{m^*,j+1}}{2} \le -\frac{(\theta a_m)^\alpha}{2} \le -\frac{m}{2}.$$

Otherwise, by (16), we get

$$\|Pu\|_{\{x\in\mathbb{R}:|x|\ge|y|\}} \le \|Pu\|_{\{x\in\mathbb{R}:|x|\ge x_{m^*,j+1}/2\}} \le \mathrm{C}\,\mathrm{e}^{-\mathrm{A}m}\|Pu\|.$$

In both cases, we obtain

$$|S_2| \le \mathrm{C}\,\mathrm{e}^{-\mathrm{A}m}\|Pu\|. \tag{60}$$

Combining (59) and (60) in (58), we have

$$|e_{m^*}(P,y)| \le \mathrm{C}\,\mathrm{e}^{-\mathrm{A}m}\|Pu\| \le \mathrm{C}\left\{E_M(f)_u + \mathrm{e}^{-\mathrm{A}m}\|fu\|\right\}. \tag{61}$$

On the other hand, using Proposition 3 and proceeding as in the proof of inequality (13), we have

$$\begin{aligned}|e_{m^*}(f-P,y)| &\le |f(y)-P(y)|\,|A_{m^*}(y)| \\ &\quad + \left|\sum_{|k|\le j}\lambda_{m^*,k}(w)\frac{[f-P](x_{m^*,k})}{x_{m^*,k}-y}\right| + \left|\int_{\mathbb{R}}\frac{[f-P](x)w(x)}{x-y}\,\mathrm{d}x\right| \\ &\le \mathrm{C}E_M(f)_u\log m + \left|\int_{\mathbb{R}}\frac{[f-P](x)w(x)}{x-y}\,\mathrm{d}x\right|.\end{aligned}$$

By using Lemmas 1 and 2, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}}\frac{[f-P](x)w(x)}{x-y}\,\mathrm{d}x\right| &\le \mathrm{C}\left\{E_M(f)_u\log m + \int_0^1\frac{\Omega(f-P,t)_u}{t}\,\mathrm{d}t\right\} \\ &\le \mathrm{C}\left\{E_M(f)_u\log m + \log m\int_0^{a_M/M}\frac{\omega^r(f,t)_u}{t}\,\mathrm{d}t\right\}.\end{aligned}$$

Therefore, we have

$$|e_{m^*}(f-P,y)| \le \mathrm{C}\log m\left\{E_M(f)_u + \int_0^{a_M/M}\frac{\omega^r(f,t)_u}{t}\,\mathrm{d}t\right\}. \tag{62}$$

By inequalities (61), (62) and (57), using (4), we get (14). □

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Appendix

We want to show a simple case for computing

$$\mathcal{H}(w,y) = \int_{\mathbb{R}} \frac{e^{-|x|^\alpha}}{x-y}\,dx, \quad \alpha > 1,$$

that is if $\alpha \in \mathbb{N}$. We recall that for the case $\alpha = 2$ one can use the error function (see for instance [22]).

Let us assume $y > 0$, since $\mathcal{H}(w)$ is an odd function. We can write

$$\int_{-\infty}^{+\infty} \frac{e^{-|x|^\alpha}}{x-y}\,dx = \int_0^{+\infty} \frac{e^{-x^\alpha}}{x-y}\,dx - \int_0^{+\infty} \frac{e^{-x^\alpha}}{x+y}\,dx$$
$$= \frac{1}{\alpha}\left[\int_0^{+\infty} \frac{x^{1/\alpha-1}e^{-x}}{x^{1/\alpha}-y}\,dx - \int_0^{+\infty} \frac{x^{1/\alpha-1}e^{-x}}{x^{1/\alpha}+y}\,dx\right]. \tag{63}$$

By (63), if α is even, we get

$$\int_{\mathbb{R}} \frac{e^{-x^\alpha}}{x-y}\,dx = \frac{2}{\alpha}\sum_{k=1}^{\alpha/2} y^{2k-1}\int_0^{+\infty} \frac{x^{(1-2k)/\alpha}e^{-x}}{x-y^\alpha}\,dx, \tag{64}$$

while if α is odd we have

$$\int_{\mathbb{R}} \frac{e^{-|x|^\alpha}}{x-y}\,dx = \frac{1}{\alpha}\sum_{k=1}^{\alpha} y^{k-1}\left[\int_0^{+\infty} \frac{x^{(1-k)/\alpha}e^{-x}}{x-y^\alpha}\,dx\right.$$
$$\left. +(-1)^{k-1}\int_0^{+\infty} \frac{x^{(1-k)/\alpha}e^{-x}}{x+y^\alpha}\,dx\right]. \tag{65}$$

The Cauchy principal value integrals in (64) and (65) are confluent hypergeometric functions, while, for the Stieltjes transforms in (65), for any $k = 1,\ldots,\alpha$, there holds (see for instance [7])

$$y^{k-1}\int_0^{+\infty} \frac{x^{(1-k)/\alpha}e^{-x}}{x+y^\alpha}\,dx = e^{y^\alpha}\Gamma\left(1+\frac{1-k}{\alpha}\right)\Gamma\left(\frac{k-1}{\alpha}, y^\alpha\right),$$

where Γ stands for the Gamma and the incomplete Gamma functions.

Finally, we remark that a similar method can be applied also if α is rational.

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