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A HANDY-type model with non renewable resources

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ABSTRACT

In this paper we study a modified HANDY model, describing interactions between nature resources and human exploitation. It is a system of four ODEs, whose vector field is non differentiable at certain points. The main novelty of our approach is the introduction of a variable describing non renewable resources, whose equation contains both a consumption and a replenishment term. We first establish the existence and positivity of solution for any time $t \geq 0$, and we get some results on the asymptotic behavior of x (population) and w (wealth). Then we compute all the equilibrium points of the system, and we study their stability. We find several stability and instability results, depending on the parameters of the system. Some numerical simulations confirm the theoretical results, and give suggestions for future research.

1. Introduction

The HANDY model (Human And Nature DYnamics) was introduced in 2014 (see [1]) as a theoretical model describing how human population interacts with the environment in ways that can lead to societal collapse. It is a system of four differential equations relating population, wealth and natural resources. Building on previous studies (see the bibliography in the quoted paper), the authors develop an original approach that includes wealth as a variable, and the partition of the population into two groups ("Commoners" and "Elites"). The model has led to concerning conclusions about the future of human society, suggesting that overexploitation of natural resources, combined with strong inequalities in wealth distribution, could lead to societal collapse. However, the HANDY model also suggests that such collapses could be avoided through sustainable policies and equitable wealth distribution. The HANDY model has been then further developed in several papers (see [2–8]). Of course, there are many other types of models dedicated to the study of interactions between nature and society: see for example [9] and the references therein.

In this paper we introduce a modified HANDY model, by distinguishing renewable and non renewable resource. Indeed, in the HANDY model the natural resources are lead by a modified logistic equation, so, in absence of human exploitation, they are pushed to the carrying capacity of the environment. In the present paper we use for the renewable resources the same equation as in HANDY model, but we introduce a new variable, which we call z, for the non renewable resources. The equation for z' is the following

$$z' = -\delta xz + k\,\frac{xw}{xw+1}\,z \tag{1}$$

where x is the population and w is the wealth. The right-hand side in (1) is the sum of two terms, the first of which is a depletion term due to human exploitation. The second term tries to describe the replenishment of these resources due to investment, science and technology. In the study of population dynamics, a function of the form $H(t) = t/(1+t)$ is called a Holling II function (see [10], p.25 and passim) and it is used to model saturation effects. In our case, we use such a functional dependance in view of the long standing debate between different views, that we can summarize as "optimistic" and "pessimistic". For the pessimists, under human

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exploitation, the non renewable resources must necessarily vanish, in the long run. For the optimists, science and technology will always succeed in substituting depleted resources with new ones, in an unpredictable way. The second term in Eq. (1) expresses an intermediate position between optimism and pessimism, because it says that the non renewable resources are indeed replenished thanks to human ingenuity, but there is a saturation effect in the investment of human energy and wealth (which is given by xw), so in any case the rate of replenishment cannot be greater of a fixed factor k.

We introduce this equation in the HANDY model and, for the sake of simplicity, in this paper we drop the partition of population in Commoners and Elites. We intend re-introduce this partition in future work.

To write down our model, we took particular inspiration from the work of A.Tonnelier [5], which is a careful and deep analysis of several aspects of HANDY model. In particular, our model is linked to the analysis of Section 3 of Tonnelier's paper, which is devoted to the "Egalitarian society", that is, a society with no Elites. In the paper by Tonnelier the equations of the original HANDY paper are simplified, allowing a better understanding of the model. Using the same simplifications, our model can be written as

$$\begin{cases} x'(t) = (\beta - \alpha_m)x - \dfrac{1}{\rho}(\alpha_M - \alpha_m)(\rho x - w)^+ \\ y'(t) = \gamma y(\lambda - y) - \delta_1 xy \\ z'(t) = -\delta_2 xz + k_1 \dfrac{\delta_3 xw}{\delta_3 xw + k_2} z \\ w'(t) = \delta_1 xy + \delta_2 xz - \delta_3 xw - s\left[x - \dfrac{1}{\rho}(\rho x - w)^+\right]. \end{cases} \tag{2}$$

Here β is the birth rate of the population assumed to be constant, while the death rate depend on wealth and is described by the function

$$f(x,w) = \alpha_m + (\alpha_M - \alpha_m)\frac{(w_{th} - w)^+}{w_{th}}.$$

x^+ is the usual positive part function, $\alpha_M > \alpha_m > 0$ are two constant parameters giving respectively the maximum and minimum death rate. w_{th} is a threshold below which famine and deprivation begin and death rate grows, and is defined as $w_{th} = \rho x$ with $\rho > 0$ representing a minimum rate of consumption for individuals. The variable y represents the renewable resources. The regeneration of these resources is described by a logistic model with carrying capacity $\lambda > 0$ and a depletion term that is proportional to both population and resources and is modulated by a parameter $\delta_1 > 0$.
We have discussed above the equation for z, the non-renewable resources.

Accumulated wealth w depends on the population's work on natural resources (renewable and not) and such wealth decreases due to consumption and investment for the replenishment of non renewable resources. The consumption is weighted by the function

$$s - s\frac{(w_{th} - w)^+}{w_{th}}$$

where the parameter $s > 0$ represents the subsistence wage per individual. The investment is given by $\delta_3 xw$.

We use some standard rescaling to set $\rho = \delta_3 = k_2 = 1$. As to the parameters δ_1 and δ_2, they represent the rate of exploitation of renewable and non-renewable resources, respectively. In this paper we want to pursue study of some relevant mathematical aspects of this model, and we have no knowledge about the effective values of these parameters. It seems reasonable to guess that they are not so different, at least for contemporary societies, for which both kinds of resources are relevant. So it seems acceptable to start our study with a simplification and to assume $\delta_1 = \delta_2$. In Section 6 we have done simulations for some cases in which $\delta_1 \neq \delta_2$. The general case will be treated in the future steps of our research.

The system is then reduced to the following form

$$\begin{cases} x'(t) = (\beta - \alpha_m)x - (\alpha_M - \alpha_m)(x - w)^+ \\ y'(t) = \gamma y(\lambda - y) - \delta xy \\ z'(t) = -\delta xz + k\dfrac{xw}{xw + 1} z \\ w'(t) = \delta xy + \delta xz - xw - s\left[x - (x - w)^+\right], \end{cases} \tag{3}$$

This is the ODE system that will be studied in the following of the present paper.

The paper is organized as follows: after the introduction, in Section 2 we obtain general results of existence and positivity of the solutions, and also some results on their asymptotic behavior. In particular, Propositions 3 and 4 states the it cannot happen $x \to +\infty$ or $w \to +\infty$ as $t \to +\infty$.

In Sections 3 and 4 we compute the equilibrium points of system (3), and we study their stability properties. As it is clear from the equations of the system, it will be convenient to distinguish the cases $x < w$, $x = w$, $x > w$. In the first and third case we use standard linearization techniques, while this is impossible when $x = w$, because there the vector field is not differentiable and the jacobian matrix is not defined. We will obtain several results of stability and instability, also in the non standard case $x = w$ (see in particular Propositions 15 and 16), however we are still far from a complete description of the situation. In Section 5 we give some numerical simulation that may help to understand the evolution of the solutions, and also can spread light on the situations for which we do not have theoretical results. In Section 6 we give a list of open problems and topics of future research.

We end this introduction by indicating, for the reader's sake, some of the results obtained in the present paper.

- In Section 2 we obtain that it cannot be $x(t) \to +\infty$ or $w(t) \to +\infty$ as $t \to +\infty$. This does not necessarily mean that population and wealth are bounded, but if they grow above a critical level, then they will start to oscillate. Of course, much more work is needed to determine this critical level.
- The critical points with $x = 0$ are all unstable, except a particular case described in Proposition 17.
- For some ranges of the parameters, there are asymptotically stable equilibrium points with positive values for populations and wealth (see Section 4). For all these equilibria it holds $x > w$.
- Numerical simulations seem to suggest that nearby some unstable equilibrium it is possible to find periodic solutions (see Section 5).

2. Properties of the solutions

We are interested in non negative solutions, so we work the cone

$$C = \left\{(x, y, z, w) \in \mathbb{R}^4 \mid x > 0,\ y > 0,\ z > 0,\ w > 0\right\}$$

or possibly in its closure

$$\overline{C} = \left\{(x, y, z, w) \in \mathbb{R}^4 \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ w \geq 0\right\}.$$

In all this paper it is implied that the initial conditions are in C or in $\overline{C}$. In this sections we will prove some results assuring that also the trajectories stay there.

The system (3) can be written in the form $X' = F(X)$ with $X = X(t) = (x(t), y(t), z(t), w(t))$ and $F(X) = (f_1(x, w), f_2(x, y), f_3(x, w, z), f_4(x, y, z, w))$ with

$$\begin{cases} f_1(x, w) = (\beta - \alpha_m)x - (\alpha_M - \alpha_m)(x - w)^+ \\ f_2(x, y) = \gamma y(\lambda - y) - \delta xy \\ f_3(x, z, w) = -\delta xz + k\dfrac{xw}{xw + 1}z \\ f_4(x, y, z, w) = \delta xy + \delta xz - xw - s\left[x - (x - w)^+\right], \end{cases}$$

Setting $D = \left\{(x, y, z, w) \in \mathbb{R}^4 \mid xw \neq -1\right\}$, it is easy to verify that $F : D \to \mathbb{R}^4$ is locally Lipschitz because all of its components are, and D is an open set in $\mathbb{R}^4$ containing $\overline{C}$. Consequently

Proposition 1. *The problem*

$$\begin{cases} X'(t) = F(X) \\ X'(t_0) = X_0 \end{cases} \tag{4}$$

admits a unique local solution for any $(t_0, X_0) \in D$.

We are interested in the case in which $t_0 = 0$ and $X_0 = (x_0, y_0, z_0, w_0)$ is such that $x_0 > 0$, $y_0 > 0$, $z_0 > 0$, $w_0 \geq 0$. For any such X_0 we consider the maximal solutions of (4), with maximal interval (a, b) and of course $0 = t_0 \in (a, b)$. We will prove the following proposition

Proposition 2. *Let $X(t) = (x(t), y(y), z(t), w(t))$ be a solution to problem (4) with $t_0 = 0$ and $X_0 = (x_0, y_0, z_0, w_0)$, where $x_0 > 0, y_0 > 0, z_0 > 0, w_0 \geq 0$. Let (a, b) be the maximal interval. Then we have $b = +\infty$, so that the solution is defined in $[0, +\infty)$. Also we have $X(t) \in C$ for all $t \in (0, +\infty)$.*

We will get the proof of Proposition 2 by several lemmas. First, in Lemmas 3, 4, 5, 6 we prove that $X(t) \in C$ for all $t \in (0, b)$. Then in Lemmas 7, 8 we prove that $b = +\infty$. In all these lemmas we assume the hypotheses of Proposition 2.

Lemma 3. *Let $X(t)$ be a solution of the system with the hypotheses of Proposition 2. Then $y(t) > 0$, $\forall t \in [0, b)$.*

Proof. The function $y(t)$ satisfies the equation

$$y'(t) = \gamma y(\lambda - y) - \delta xy.$$

Considering $x(t)$ as a given C^1 function, this is a one dimensional equation for which we have local uniqueness. Of course, it admits the constant $\bar{y}(t) = 0$ as a solution. Therefore, no solution different from $\bar{y}$ can intersect $\bar{y}$, thus if $y_0 > 0$ it results $y(t) > 0$, $\forall t \in [0, b)$. $\square$

Lemma 4. *Let $X(t)$ be a solution of the system with the hypotheses of Proposition 2. Then $z(t) > 0$, $\forall t \in [0, b)$.*

Proof. As in the previous lemma, $z(t)$ is a solution of equation

$$z'(t) = -\delta xz + k\frac{xw}{xw + 1}z,$$

which is a one dimensional equation if we consider $x(t), w(t)$ as given C^1 functions. But the constant $\bar{z}(t) = 0$ is also a solution of this equation, hence the solution $z(t)$ cannot intersect the constant $\bar{z}(t) = 0$, so $z(t) > 0$ for all $t \in [0, b)$. $\square$

Lemma 5. *Let $X(t)$ be a solution of the system with the hypotheses of Proposition 2. Then $x(t) > 0$ for all $t \in [0, b)$.*

Proof. Let us define $t_1 = \sup\{t \in (0, b) \mid x(s) > 0, \forall s \in [0, t)\}$. Due to continuity and because $x_0 > 0$, we have that $t_1 \in (0, b]$ and $x(t) > 0$, $\forall t \in [0, t_1)$. If $t_1 = b$ then $x(t) > 0$ in $[0, b)$, and the lemma holds. Now we argue by contradiction and we assume $t_1 < b$. By continuity, it immediately follows that $x(t_1) = 0$ and therefore $x'(t_1) \le 0$. Now we notice that there exists an $\epsilon > 0$ such that $w(t) > 0$ in $(0, \epsilon)$. Indeed, if $w_0 > 0$, we obtain this by continuity, while if $w_0 = 0$, the equation yields

$$w'(0) = \delta x_0 y_0 + \delta x_0 z_0 > 0.$$

Now let us define $t_2 = \sup\{t \in (0, b) \mid w(s) > 0, \forall s \in [0, t)\}$. We have $t_2 \in (0, b]$ and $w(t) > 0$ in $(0, t_2)$. We prove now that $t_2 < t_1$. Indeed, if it were $t_2 \ge t_1$, then we would have $w(t_1) \ge 0$ and $x(t_1) = 0$. Let us define $y_1 = y(t_1)$, $z_1 = z(t_1)$, $w_1 = w(t_1)$, and $P_1 = (0, y_1, z_1, w_1)$. Let $\varphi(t)$ be the solution of the logistic equation

$$y'(t) = \gamma y(\lambda - y) \tag{5}$$

with $\varphi(t_1) = y_1$, a solution that locally exists and is unique. It is then easy to verify that the function $Y(t) = (0, \varphi(t), z_1, w_1)$ is a local solution of the Cauchy problem

$$\begin{cases} Y'(t) = F(Y) \\ Y(t_1) = (0, y_1, z_1, w_1). \end{cases} \tag{6}$$

We know that also $X(t_1) = (0, y_1, z_1, w_1)$, and of course X satisfies $X' = F(X)$, hence $X(t)$ and $Y(t)$ are two different solutions of the same Cauchy problem (6). This is a contradiction, which derives from the assumption $t_2 \ge t_1$. So we have proved that $t_2 < t_1$. From the definition of t_2 it is easy to obtain $w(t_2) = 0, w'(t_2) \le 0$. We also have $x(t_2) > 0, y(t_2) > 0, z(t_2) > 0$, so from the equation for w' we obtain

$$w'(t_2) = \delta x(t_2)y(t_2) + \delta x(t_2)z(t_2) > 0.$$

We have therefore arrived at a contradiction. It derives from the assumption $t_1 < b$, hence $t_1 = b$ and hence $x(t) > 0, \forall t \in [0, b)$. □

Lemma 6. *Let $X(t)$ be a solution of the system with the hypotheses of Proposition 2. Then $w(t) > 0$ for all $t \in [0, b)$.*

Proof. Let us define t_2 as in the previous lemma. We have to prove that $t_2 = b$. If $t_2 < b$, we can repeat the same argument of the previous lemma, in the case $t_2 < t_1$, because now we know $t_1 = b$. Hence we get a contradiction that proves $t_2 = b$, so that $w(t) > 0$ for all $t \in (0, b)$. □

Through the following lemmas we prove that $b = +\infty$, so that the solutions are defined in all the half-line $[0, +\infty)$.

Lemma 7. *$y(t)$ is bounded in $[0, b)$.*

Proof. Assume first $y_0 < \lambda$. and let us define

$$t_1 = \sup\{t \in (0, b) \mid y(s) < \lambda, \forall s \in [0, t)\}.$$

Then $t_1 > 0$ and $y(s) < \lambda$ for all $s \in [0, t_1)$. If $t_1 < b$ it holds $y(t_1) = \lambda$ and $y'(t_1) \ge 0$. On the other hand, the equation for y gives $y'(t_1) < 0$ because $x(t_1) > 0$, and the contradiction proves $t_1 = b$, hence $y(t) < \lambda$ for all $[0, b)$. Assume now $y_0 \ge \lambda$. If $y(t) \ge \lambda$ in all $[0, b)$ then we deduce from the equation that y in non increasing, so that $y(t) \le y_0$. If $y(t) < \lambda$ for some t, then there must be $t_1 \ge 0$ such that $y(t) \ge \lambda$ in $[0, t_1]$, $y(t) < \lambda$ in a right neighborhood of t_1. It is then easy to repeat the argument used in the first case and to obtain $y(t) < \lambda$ for all $t > t_1$. Hence again $y(t) \le y_0$ for all $t \in [0, b)$. Setting $m_0 = \max\{\lambda, y_0\}$, we get $0 < y(t) \le m_0$ for all $t \in [0, b)$, and the lemma is proved. □

Let us now prove that $b = +\infty$.

Lemma 8. *It holds $b = +\infty$.*

Proof. We argue by contradiction, so we assume $b < +\infty$. We prove that in this case the functions $x(t), z(t), w(t)$ are bounded in $(0, b)$. Indeed, from the equation we get that it holds, for all t,

$$x'(t) \le (\beta_0 - \alpha_m)x(t),$$

and it is easy to derive the thesis for $x(t)$ by an integration. In the same way the equation for z' gives $z'(t) \le kz$ and the thesis for $z(t)$ is easily deduced. The equation for w' gives $w' \le \delta xy + \delta xz$. We already know that x, y, z are bounded in $[0, b)$, so $w'(t) \le K$, $\forall t \in [0, b)$, and also w is bounded, if $b < +\infty$. So we have that the solution $X(t)$ is bounded on $[0, b)$ if $b < +\infty$. From standard results on the theory of maximal solutions for systems of ODE we know this is impossible, so we have a contradiction, and the thesis $b = +\infty$ is proved. □

The proof of Proposition 2 is now complete.

Now we state and prove a general results about the behavior of x and w as $t \to +\infty$.

Proposition 9. *Assume the hypotheses of Proposition 2. Then it cannot be* $\lim_{t\to+\infty} x(t) = +\infty$.

To prove Proposition 9 we argue by contradiction. The proof will be obtained by some lemmas. In all these lemmas we assume the hypotheses of Proposition 9, which are those of Proposition 2.

Lemma 10. *If* $\lim_{t\to+\infty} x(t) = +\infty$ *holds, then* $\lim_{t\to+\infty} y(t) = 0$.

Proof. We know that y is bounded. The equation for y is

$$y'(t) = y(\lambda\gamma - \gamma y - \delta x)$$

and since $x \to +\infty$, there exists a $T > 0$ such that for any $t > T$ it holds $\lambda\gamma - \gamma y - \delta x < 0$ and therefore $y'(t) < 0$. Thus, for $t > T$, y is decreasing and bounded from below, $y(t) > 0$ for any $t > 0$ and therefore there exists the limit $\ell_+ \geq 0$. If it were $\ell_+ > 0$, we would have $\lim_{t\to+\infty} y'(t) = -\infty$, a contradiction with standard results on the asymptotic behavior of functions on the real line. The contradiction proves that

$$\ell_+ = \lim_{t\to+\infty} y(t) = 0. \quad \square$$

Lemma 11. *If* $\lim_{t\to+\infty} x(t) = +\infty$ *holds, then* $\lim_{t\to+\infty} z(t) = 0$.

Proof. The reasoning is similar to the previous one. As

$$z'(t) = z\left(-\delta x + k\frac{xw}{xw+1}\right)$$

and $\frac{xw}{xw+1} \leq 1$, then there exists $T > 0$ such that for every $t > T$ we have $z'(t) < 0$ and thus for $t > T$, $z(t)$ is decreasing and bounded from below. Hence, it admits a limit $\ell_+ \geq 0$. If $\ell_+ > 0$, we would have $\lim_{t\to+\infty} z'(t) = -\infty$, and, again, this is a contradiction. The contradiction proves that

$$\ell_+ = \lim_{t\to+\infty} z(t) = 0. \quad \square$$

Lemma 12. *If* $\lim_{t\to+\infty} x(t) = +\infty$ *holds, then* $\lim_{t\to+\infty} w(t) = 0$.

Proof. We know that $\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} z(t) = 0$. Fix $\epsilon > 0$ and let $T_\epsilon > 0$ be such that

$$\delta y(t) + \delta z(t) < \epsilon/2, \qquad t > T_\epsilon.$$

From the equation for w' we have that, for all $t \geq 0$, it holds

$$w'(t) \leq x(\delta y + \delta z - w). \tag{7}$$

We can then infer the following thesis: it cannot be $w(t) \geq \epsilon$ for every $t > T_\epsilon$. Indeed, if this were true, we would have

$$w > \delta y + \delta z \qquad \forall t > T_\epsilon$$

and therefore $w'(t) < 0$ for any $t > T_\epsilon$. Thus, we would have $w(t) \geq \epsilon$ and $w'(t) < 0$ for any $t > T_\epsilon$, hence there would exist $\lim_{t\to+\infty} w(t) = w_+ \geq \epsilon$. From (7) we then obtain $\lim_{t\to+\infty} w'(t) = -\infty$, contradicting standard results. Hence, we have proved the thesis above.

Since it cannot be $w(t) \geq \epsilon$ $\forall t > T_\epsilon$, there must exist $t_\epsilon > T_\epsilon$ such that $w(t_\epsilon) < \epsilon$. Let us show now that $w(t) \leq \epsilon$ for every $t \geq t_\epsilon$. If it were not true, there would exist $s_\epsilon > t_\epsilon$ such that $w(s_\epsilon) > \epsilon$. We then define $\bar{s_\epsilon} = \inf\left\{s \in (t_\epsilon, s_\epsilon) \mid w(s) > \epsilon, \forall s \in (s, s_\epsilon]\right\}$. Then we have $\bar{s_\epsilon} \in (t_\epsilon, s_\epsilon)$ and $w(\bar{s_\epsilon}) = \epsilon$, $w(\sigma) > \epsilon$ for $s \in (\bar{s_\epsilon}, s_\epsilon)$. Then $w'(\bar{s_\epsilon}) \geq 0$, but on the other hand

$$w'(\bar{s_\epsilon}) \leq x(\bar{s_\epsilon})(\delta y(\bar{s_\epsilon}) + \delta z(\bar{s_\epsilon}) - w(\bar{s_\epsilon})) < 0$$

The contradiction proves the thesis. We have then shown that

$$0 < w(t) \leq \epsilon, \qquad t \geq t_\epsilon.$$

But this argument can be repeated for any $\epsilon > 0$. That is to say, for any $\epsilon > 0$ we can determine $t_\epsilon > 0$ such that, for any $t > t_\epsilon$ it holds $0 < w(t) \leq \epsilon$. Of course this means exactly that

$$\lim_{t\to+\infty} w(t) = 0. \quad \square$$

We can now complete the proof of Proposition 9

Proof (*End of Proof of Proposition 9*)**.** As $x \to +\infty$ and $w \to 0$, there exists $T > 0$ such that for every $t > T$, $x > w$ holds, and therefore the equation for x' becomes

$$x'(t) = (\beta - \alpha_M)x + (\alpha_M - \alpha_m)w.$$

Since $x \to +\infty$, $w \to 0$ and $\beta < \alpha_M$, there exists $T_1 > T$ such that $x'(t) < 0$ for every $t \geq T_1$. But then

$$x(t) \leq x(T_1), \qquad t > T_1,$$

and therefore it cannot be $x \to +\infty$. In the end, assuming $x(t) \to +\infty$ we arrive at a contradiction, so Proposition 9 is proved. □

We now prove a similar result for w.

Proposition 13. *Assume the hypotheses of Proposition 2. Then it cannot be* $\lim_{t\to+\infty} w(t) = +\infty$.

Proof. We will prove that $\lim_{t\to+\infty} w(t) = +\infty$ implies $\lim_{t\to+\infty} x(t) = +\infty$. As we have already prove that this cannot be, of course we obtain the desired result. So, let us assume $\lim_{t\to+\infty} w(t) = +\infty$. We want to prove that $\forall c > 0$ there exists $T_c > 0$ such that $t > T_c$ implies $x(t) > c$. This means exactly that $\lim_{t\to+\infty} x(t) = +\infty$. Let us fix $c > 0$. Since $\lim_{t\to+\infty} w(t) = +\infty$, there exists $T_c > 0$ such that $w(t) > c$ for $t > T_c$. If $x(t) \leq w(t)$ for $t > T_c$ then $x'(t) = (\beta - \alpha_m)x$ in $[T_c, +\infty)$. Therefore $x(t) = x(T_c)e^{(\beta-\alpha_m)(t-T_c)}$ for $t \geq T_c$ and $\lim_{t\to+\infty} x(t) = +\infty$, so we have concluded. Let us then assume that there is $t_0 > T_c$ such that $x(t_0) > w(t_0) > c$. We then prove that

$$x(t) > c, \quad \forall t \geq t_0.$$

Indeed, let $t > t_0$. If $x(t) \geq w(t)$ then obviously $x(t) > c$ since $w(t) > c$ for $t > t_0 > T_c$. If instead $x(t) < w(t)$ we define $t_1 = \inf\{\tau \in (t_0, t) \mid x(s) - w(s) < 0, \forall s \in (\tau, t)\}$. From the continuity of x and w we easily obtain that $t_1 \in (t_0, t)$, $x(t_1) - w(t_1) = 0$, $x(s) - w(s) < 0$, for every $s \in (t_1, t)$. This means that, for $s \in (t_1, t)$ we have $x'(s) = (\beta - \alpha_m)x(s) > 0$, therefore $x(t) > x(t_1) = w(t_1) > c$. We have then proven that, for every $t > t_0$ it holds that $x(t) > c$, and we have completed the proof. □

We now look for the equilibrium points of the system. We will determine all such points and we will study their stability. An important remark about our study of stability is the following: we are interested to solution that stay in C and we look for equilibrium points in $\overline{C}$, so when we perturb the initial conditions to study stability, we use initial conditions that stay in $\overline{C}$ or in C, and all our results about stability of equilibria refer to such perturbations.

About the stability we will find several results, showing different interesting behaviors of the system, even if we are not able to give a complete picture of the stability properties of all the equilibria. So, there are still some open problems, for which we will see some numerical simulations that gives interesting suggestions.

To find the equilibrium points, we have to distinguish the cases $x \leq w$ and $x > w$.

3. Fixed points and stability: case $x \leq w$

In this subset of the state space, the system becomes:

$$\begin{cases} x'(t) = (\beta - \alpha_m)x \\ y'(t) = \gamma y(\lambda - y) - \delta xy \\ z'(t) = -\delta xz + k\dfrac{xw}{xw+1}z \\ w'(t) = \delta xy + \delta xz - xw - sx, \end{cases} \tag{8}$$

As $\beta > \alpha_m$, equilibrium points exist if and only if:

$$\begin{cases} x = 0 \\ \gamma y(\lambda - y) = 0 \end{cases} \tag{9}$$

There are no conditions on z and w. Then in $\overline{C} \cap \{x \leq w\}$, the set of equilibrium points is given by points of the form $Q = (0, 0, Z, W)$, or $R = (0, \lambda, Z, W)$ which represent two family of points varying with two parameters Z, W with $Z \geq 0, W \geq 0$.

Now that we have all the possible equilibria in the subset $\{x \leq w\}$, we can study their stability. If $W > 0$, the function F is differentiable in a neighborhood of the equilibrium point, so we can apply the standard linearization technique. On the other hand, if $W = 0$ we are on a point of non differentiability for F, hence we have to find out some other ideas for the study of stability. We distinguish several cases.

3.1. Case $W > 0$

Proposition 14. *If $W > 0$ all the equilibrium points Q, R are unstable.*

Proof. When $w > x$ and $xw \neq -1$, the Jacobian matrix of the system is given by:

$$J(x,y,z,w) = \begin{pmatrix} \beta - \alpha_m & 0 & 0 & 0 \\ -\delta y & \gamma\lambda - 2\gamma y - \delta x & 0 & 0 \\ -\delta z + k\dfrac{wz}{(xw+1)^2} & 0 & -\delta x + k\dfrac{xw}{xw+1} & k\dfrac{xz}{(xw+1)^2} \\ \delta y + \delta z - w - s & \delta x & \delta x & -x \end{pmatrix}$$

Hence, if we calculate $J(Q)$ with $W > 0$, we obtain:

$$J(Q) = J(0,0,Z,W) = \begin{pmatrix} \beta - \alpha_m & 0 & 0 & 0 \\ 0 & \gamma\lambda & 0 & 0 \\ -\delta Z + kZW & 0 & 0 & 0 \\ \delta Z - W - s & 0 & 0 & 0 \end{pmatrix}$$

If we calculate $J(R)$ we obtain:

$$J(R) = J(0,\lambda,Z,W) = \begin{pmatrix} \beta - \alpha_m & 0 & 0 & 0 \\ -\delta\lambda & -\gamma\lambda & 0 & 0 \\ -\delta Z + kZ\,W & 0 & 0 & 0 \\ \delta\lambda + \delta Z - W - s & 0 & 0 & 0 \end{pmatrix}$$

In both cases $\beta - \alpha_m > 0$ is a strictly positive eigenvalue, so all these equilibria are unstable. □

3.2. Case W=0 and equilibrium point Q= (0,0,Z,0)

In this case, as we said above, we cannot apply the standard linearization technique.

Proposition 15. *If $W = 0$ the equilibrium points $Q = (0,0,Z,0)$ (with $Z \geq 0$) are unstable.*

Proof. We recall that the equation for y' is given by $y'(t) = y(t)(\gamma\lambda - \gamma y - \delta x)$. Let $\epsilon > 0$ such that $\epsilon < \frac{\gamma\lambda}{2(\gamma+\delta)}$ and let $B_\epsilon(Q)$ be the ball centered at Q with radius ϵ. Let $P_\epsilon = (x_\epsilon, y_\epsilon, z_\epsilon, w_\epsilon)$ be any point in $B_\epsilon(Q)$ with $y_\epsilon > 0$, and let $u(t) = u(t, P_\epsilon)$ be the trajectory starting from P_ϵ, that is, $u(t) = (x(t), y(t), z(t), w(t))$ solves the system and $u(0) = P_\epsilon$. If $u(t) \in B_\epsilon(Q)$ it must be that $|x(t)| < \epsilon$, $|y(t)| < \epsilon$ and thus

$$\gamma\lambda - \gamma y(t) - \delta x(t) \geq \gamma\lambda - \epsilon(\gamma+\delta) \geq \gamma\lambda - \frac{\gamma\lambda}{2(\gamma+\delta)}(\gamma+\delta) = \frac{1}{2}\gamma\lambda.$$

Obviously, $u(t) \in B_\epsilon(Q)$ at least in a neighborhood of $t = 0$. If $u(t) \in B_\epsilon(Q)$ for any $t \geq 0$, we would have $\gamma\lambda - \gamma y(t) - \delta x(t) \geq \frac{1}{2}\gamma\lambda$ for any $t \geq 0$, from which we get $y'(t) \geq \frac{1}{2}\gamma\lambda y(t)$ and therefore $y(t) \geq y_\epsilon e^{\frac{1}{2}\gamma\lambda t}$ for every $t \geq 0$. As $y_\epsilon > 0$ this implies $y(t) \to +\infty$ which is absurd as we are assuming that $u(t) \in B_\epsilon(Q)$, $\forall t \geq 0$. This means that if $P_\epsilon = (x_\epsilon, y_\epsilon, z_\epsilon, w_\epsilon) \in B_\epsilon(Q)$ and $y_\epsilon > 0$, $u(t, P_\epsilon) \in B_\epsilon(Q)$, $\forall t \geq 0$ cannot be true, and therefore there exists a $\bar{t} > 0$ such that $u(\bar{t}, P_\epsilon) \notin B_\epsilon(Q)$. This implies that for any $\delta \in (0,\epsilon)$, the neighborhood $B_\delta(Q)$ contains points P_ϵ such that the trajectory $u(t, P_\epsilon)$ exits from $B_\epsilon(Q)$ and this is indeed the definition of instability. □

3.3. Case W=0 and equilibrium point R= (0, λ,Z,0)

In this case we have different results, depending on the parameters. To simplify the formulas we put

$$\alpha = \frac{\alpha_M - \beta}{\alpha_M - \alpha_m} \in (0,1)$$
$$\zeta = \alpha_M - \alpha_m > 0,$$

so that $\alpha_M - \beta = \alpha(\alpha_M - \alpha_m) = \alpha\zeta > 0$ and $\beta - \alpha_m = \zeta(1-\alpha) > 0$.

For the stability of these equilibria, a crucial point seems to be the sign of $\delta(\lambda + Z) - \alpha s$.

Proposition 16. *If $\delta(\lambda + Z) > \alpha s$, the critical points $R = (0,\lambda,Z,0)$ are unstable.*

Proof. From the first and fourth equations in (3) we have

$$sx'(t) = s\zeta(1-\alpha)x - s\zeta(x-w)^+$$
$$\zeta w'(t) = \delta\zeta xy + \delta\zeta xz - \zeta xw - s\zeta x + s\zeta(x-w)^+,$$

hence

$$sx'(t) + \zeta w'(t) = x[s\zeta(1-\alpha) + \delta\zeta y + \delta\zeta z - \zeta w - s\zeta] = \zeta x[-\alpha s + \delta(y+z) - w]$$

From the hypothesis we can determine ϵ such that $\delta(\lambda + Z - 2\epsilon) > \alpha s + 2\epsilon$. Let $B_\epsilon(R)$ be the ball of center R and radius ϵ. If R is stable, there must exist $\eta \in (0,\epsilon)$ such that for any $P \in B_\eta(R)$, the trajectory $u(t) = u(t,P)$ stays in $B_\epsilon(R)$. Let us now see that this leads to a contradiction. Let $u(t) = (x(t), y(t), z(t), w(t))$ be the trajectory that starts from P and solves the system. Assume also that the x and the w coordinate of P are strictly positive. If $u(t) \in B_\epsilon(R)$ then $|x(t)| < \epsilon$, $|y(t) - \lambda| < \epsilon$, $|z(t) - Z| < \epsilon$, and $|w(t)| < \epsilon$. Therefore, for any $t \geq 0$ we have

$$-\alpha s + \delta(y+z) - w > -\alpha s + \delta(\lambda - \epsilon + Z - \epsilon) - \epsilon > \epsilon.$$

Then $sx'(t) + \zeta w'(t) > \epsilon\zeta x > 0$ for every $t \geq 0$. The function $g(t) = sx(t) + \zeta w(t)$ is therefore strictly increasing and, being bounded, admits limit $\ell > 0$ i.e.,

$$\ell = \lim_{t\to+\infty} (sx(t) + \zeta w(t)) > 0$$

Let now $T > 0$ be such that $g(t) > \ell/2 > 0$ for every $t \geq T$. We prove that there exists $t_0 \geq T$ such that

$$x(t) > \frac{\ell}{2(s+\zeta)}, \qquad \text{for any } t \geq t_0. \tag{10}$$

To prove (10) we observe that it cannot be $x(t) \leq w(t), \forall t \geq T$. In fact in this case we would have $x'(t) = \zeta(1-\alpha)x, \forall t \geq T$ and therefore $x(t) \to +\infty$ which is impossible. Then there exists $t_0 \geq T$ such that $x(t_0) > w(t_0)$. This implies $x(t_0)(s+\zeta) \geq sx(t_0) + \zeta w(t_0) > \ell/2$ and therefore $x(t_0) > \frac{\ell}{2(s+\zeta)}$. If it were $x(t) \geq w(t), \forall t \geq t_0$ we would have, in the same way $x(t) > \frac{\ell}{2(s+\zeta)}, \forall t \geq t_0$ and (10) would be proven. Let then $t_1 > t_0$ be a value such that $x(t_1) < w(t_1)$. Let $t_2 = \inf\{t \in (t_0, t_1) \mid x(s) < w(s), \forall s \in (t, t_1)\}$. By continuity we have $t_2 \in (t_0, t_1)$, $x(t_2) = w(t_2)$, $x(s) < w(s)$, $\forall s \in (t_2, t_1]$. Then $x'(s) = \zeta(1-\alpha)x(s) > 0$ in $(t_2, t_1]$ therefore $x(t_1) > x(t_2) = w(t_2)$ and it follows $x(t_2) > \frac{\ell}{2(s+\zeta)}$. Then (10) is verified. However we have seen that $sx'(t) + \zeta w'(t) > \epsilon\zeta x$ for every $t \geq 0$ therefore $sx'(t) + \zeta w'(t) > \frac{\ell\epsilon\zeta}{2(s+\zeta)}, \forall t \geq t_0$ which obviously implies that $sx(t) + \zeta w(t) \to +\infty$ for $t \to +\infty$ and this contradicts that $u(t) \in B_\epsilon(R)$. We have then proven that for any $P \in B_\epsilon(R)$, with strictly positive x and w coordinate of P, it cannot be $u(t,P) \in B_\epsilon(R)$ for all $t \geq 0$. This proves that R is unstable. □

We now consider the case $\delta(\lambda + Z) < \alpha s$. In this case we do not have a complete answer about the stability of the equilibria $R = (0, \lambda, Z, 0)$, but we are able to study the asymptotic behavior of the components x, y, w of the trajectories, see Proposition 17 below for a precise statement. We underline here that this Proposition says, roughly speaking, that there is indeed some kind of stability for what concerns the components x, y, w of the trajectories, and it is asymptotic stability for x, w: that is, if a trajectory starts in C near enough the critical point R, then the components x, y, w stay near the correspondent values of R, and x, w tend to those values as $t \to +\infty$. However, we cannot state a standard stability result because we do not control the behavior of the z component: we only know it is decreasing.

In the proof of Proposition 17, to make some arguments a little bit simpler, it will be helpful to use "rectangular" neighborhoods of R of the following form

$$I_\epsilon(R) = (-\epsilon, \epsilon) \times (\lambda - \epsilon, \lambda + \epsilon) \times (Z - \epsilon, Z + \epsilon) \times (-\epsilon, \epsilon),$$

As we assume $\delta(\lambda + Z) < \alpha s$, we can fix $\epsilon_1 > 0$ such that $\forall \epsilon \in (0, \epsilon_1)$ it holds that $\delta(\lambda + Z + 2\epsilon) < \alpha s - \epsilon$. Also, let $\epsilon_2 = \frac{\delta}{2k}$ and $\epsilon_3 = \delta\lambda$.

We are now ready to state and prove Proposition 17.

Proposition 17. *Assume $\delta(\lambda + Z) < \alpha s$, and let $R = (0, \lambda, Z, 0)$ be an equilibrium point, with $\lambda > 0$ and $Z \geq 0$. For any positive $\epsilon < \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ there is $\eta \in (0, \epsilon)$ such that for any $P \in I_\eta(R)$, where $P = (x_p, y_p, z_p, w_p)$ with $x_p > 0$, $y_p > 0$, $z_p > 0$, $w_p \geq 0$, if $u(t,P) = (x(t), y(t), z(t), w(t))$ is the trajectory starting from P, then it holds:*

(i) $0 < x(t) < \epsilon$, $|y(t) - \lambda| < \epsilon$, $0 < w(t) < \epsilon$, $0 < z(t) < Z + \epsilon$ *for all* $t \geq 0$.
(ii) $\lim_{t\to+\infty} x(t) = 0$, $\lim_{t\to+\infty} w(t) = 0$.
(iii) $z(t)$ *is decreasing.*

The proof of Proposition 17 will be obtained by the following Lemmas 18, 20 and Corollary 19. Firstly, we notice that we will have to work also in $\mathbb{R}^3$, because we want to study the behavior of $(x(t), y(t), w(t))$. Hence we introduce suitable neighborhoods in $\mathbb{R}^3$, that is, for any $\eta > 0$ we define

$$J_\eta = \{(x, y, w) \in \mathbb{R}^3 \mid |x| < \eta, |y - \lambda| < \eta, |w| < \eta\}.$$

Also we define $u_0(t,P)$ be the function in $\mathbb{R}^3$ obtained by taking the components x, y, w of $u(t,P)$ i.e., $u_0(t,P) = (x(t), y(t), w(t))$.

Lemma 18. *Take ϵ as in Proposition 17. There exists $0 < \bar{\eta} < \epsilon$ such that for any $\eta \in (0, \bar{\eta})$ and any $P \in I_\eta(R)$, with the assumptions on P of Proposition 17, we obtain $u_0(t,P) \in J_\epsilon$ for any $t \geq 0$.*

Proof. The value $\bar{\eta}$ will be fixed at the end of the proof. Let us start by fixing $0 < \bar{\eta} < \epsilon$. So, if $\eta < \bar{\eta}$ and $P \in I_\eta(R)$, we have $(x_p, y_p, w_p) \in J_\epsilon$ and therefore, by continuity, $u_0(t,P) \in J_\epsilon$ in a right neighborhood of $t = 0$. We then define $t_1 = \sup\{t > 0 \mid u_0(s,P) \in J_\epsilon, \forall s \in [0,t)\}$. We have $u(s,P) \in J_\epsilon$ for any $s \in [0, t_1)$. If $t_1 = +\infty$ we have concluded. We then argue by contradiction and we suppose $0 < t_1 < +\infty$. Thanks to the hypotheses, from $0 < t_1 < +\infty$ we deduce that $u_0(t_1, P) \in \partial J_\epsilon$ while, as stated above, $u_0(s,P) \in J_\epsilon$ for any $s \in [0, t_1)$.

Let us then study $u_0(t,P)$ in $[0, t_1)$. From the first equation in (3) we obtain, as already seen,

$$sx'(t) + \zeta w'(t) = \zeta x[-\alpha s + \delta(y + z) - w]$$

On the other hand, if $u_0(t,P) \in J_\epsilon$ we have $w < \epsilon$, therefore $kw < k\epsilon < \frac{1}{2}\delta$ hence

$$-\delta + k\frac{w}{xw+1} < -\delta + kw < -\delta + \frac{\delta}{2} < -\frac{\delta}{2}$$

and therefore

$$z'(t) = xz\left(-\delta + \frac{kw}{xw+1}\right) < -\frac{\delta}{2}xz < 0$$

Therefore, z is decreasing in $[0, t_1)$. Since $z(0) = z_p < Z + \epsilon$ we obtain $z(t) < Z + \epsilon$ in $[0, t_1)$. From this, using $y(t) < \lambda + \epsilon$ and $\epsilon < \epsilon_1$, we obtain

$$sx'(t) + \zeta w'(t) < \zeta x[-\alpha s + \delta(\lambda + \epsilon + Z + \epsilon)] < -\epsilon\zeta x < 0.$$

We then have that the function $t \to sx(t)+\zeta w(t)$ is decreasing in $[0, t_1)$. Therefore, for any $t \in [0, t_1)$ we have $sx(t)+\zeta w(t) < sx_p+\zeta w_p < (s+\zeta)\eta$ as $P \in I_\eta(R)$. We then get

$$x(t) = \frac{1}{s}sx(t) < \frac{1}{s}(sx(t) + \zeta w(t)) < \frac{s+\zeta}{s}\eta,$$

so that

$$x(t) < \frac{s+\zeta}{s}\eta \qquad \forall t \in [0, t_1),$$

and similarly

$$w(t) < \frac{s+\zeta}{\zeta}\eta \qquad \forall t \in [0, t_1).$$

We can then assume that $\frac{s+\zeta}{s}\bar\eta < \frac{\epsilon}{2}$ and $\frac{s+\zeta}{\zeta}\bar\eta < \frac{\epsilon}{2}$. In this way we have $0 < x(t) < \frac{\epsilon}{2}$ and $0 < w(t) < \frac{\epsilon}{2}$ for any $t \in [0, t_1)$ and therefore $0 \le x(t_1) \le \frac{\epsilon}{2}$ and $0 \le w(t_1) \le \frac{\epsilon}{2}$.

Let us now study $y(t)$. We know that $\lambda - \eta < y_p < \lambda + \eta$, where $\eta < \bar\eta$. We notice that the hypotheses above on $\bar\eta$ imply $\bar\eta < \frac{\epsilon}{2}$. From the equation for y

$$y'(t) = \gamma y(\lambda - y) - \delta xy$$

we have already obtained that for any $t \ge 0$ it holds $y(t) \le \max\{y_p, \lambda\}$ and therefore, in our case,

$$y(t) < \lambda + \eta < \lambda + \frac{\epsilon}{2}.$$

We now want to obtain an inequality in the opposite sense. For this, remembering that in $[0, t_1)$ it is $x(t) < \frac{s+\zeta}{s}\eta$, we can assume that

$$\frac{\delta}{\gamma}\frac{s+\zeta}{s}\bar\eta < \frac{\epsilon}{2}.$$

Then $-\delta xy > -\delta\frac{s+\zeta}{s}\eta\, y$ and therefore

$$y'(t) > \gamma y(\lambda - y) - \delta\frac{s+\zeta}{s}\eta\, y = \gamma y\left(\lambda - y - \frac{\delta}{\gamma}\frac{s+\zeta}{s}\eta\right) > \gamma y\left(\lambda - \frac{\epsilon}{2} - y\right).$$

We now fix $\sigma \in (0, y_p)$ and let v_σ be the solution of

$$\begin{cases} v_\sigma'(t) = \gamma v_\sigma(t)\left(\lambda - \frac{\epsilon}{2} - v_\sigma(t)\right) \\ v_\sigma(0) = y_p - \sigma. \end{cases}$$

By the comparison principle we have

$$y(t) \ge v_\sigma(t), \qquad \forall t \in [0, t_1).$$

But $v_\sigma(t)$ is the solution of a usual logistic equation and therefore

$$v_\sigma(t) \ge \min\left\{v_\sigma(0), \lambda - \frac{\epsilon}{2}\right\} = \min\left\{y_q - \sigma, \lambda - \frac{\epsilon}{2}\right\}.$$

Since $y_p > \lambda - \eta > \lambda - \frac{\epsilon}{2}$ we can choose σ such that $y_p - \sigma > \lambda - \frac{\epsilon}{2}$ so that $v_\sigma(t) \ge \lambda - \frac{\epsilon}{2}$, hence $y(t) \ge \lambda - \frac{\epsilon}{2}$ in $[0, t_1)$. We have then obtained

$$\lambda - \frac{\epsilon}{2} \le y(t) \le \lambda + \frac{\epsilon}{2}$$

in $[0, t_1)$ and therefore also

$$\lambda - \frac{\epsilon}{2} \le y(t_1) \le \lambda + \frac{\epsilon}{2}.$$

Hence, if we assume

$$\bar\eta < \min\left\{\epsilon,\ \frac{s}{s+\zeta}\frac{\epsilon}{2},\ \frac{\zeta}{s+\zeta}\frac{\epsilon}{2},\ \frac{\gamma}{\delta}\frac{s}{s+\zeta}\frac{\epsilon}{2}\right\}$$

and $\eta < \bar{\eta}$, we obtain $0 \le x(t_1) \le \frac{\epsilon}{2}, 0 \le w(t_1) \le \frac{\epsilon}{2}; \lambda - \frac{\epsilon}{2} \le y(t_1) \le \lambda + \frac{\epsilon}{2}$. This means $(x(t_1), y(t_1), w(t_1)) \in J_\epsilon$ that is $u_0(t_1, P) \in J_\epsilon$, which contradicts $u_0(t_1, P) \in \partial J_\epsilon$. The contradiction follows from the assumption $t_1 < +\infty$. Hence $t_1 = +\infty$ and the proposition is proven. □

Within the previous proof it was also proved the following corollary:

Corollary 19. *$z(t)$ is decreasing in $[0, +\infty)$.*

Now let us obtain the asymptotic properties of x and w.

Lemma 20. *In the hypotheses of Proposition 17, we have $x(t) \to 0, w(t) \to 0$ for $t \to +\infty$.*

Proof. Arguing as in Lemma 18 we obtain that $sx + \zeta w$ is decreasing in $[0, +\infty)$. As the functions x, w are positive, there exists $\ell \ge 0$ such that $\ell = \lim_{t\to+\infty}(sx + \zeta w)$. We prove now that $\ell = 0$, arguing by contradiction. If we suppose $\ell > 0$ we can repeat the same arguments in the proof of Proposition 16, and we obtain that there exists $t_0 > 0$ such that, for any $t > t_0$ it holds

$$x(t) > \frac{\ell}{2(s+\zeta)}.$$

But then

$$sx' + \zeta w' < -\frac{\epsilon\ell}{2(s+\zeta)}$$

for any $t \ge t_0$. Therefore, it would be $sx + \zeta w \to -\infty$ which is absurd. Then $sx + \zeta w \to 0$ for $t \to +\infty$ and therefore $x(t) \to 0, w(t) \to 0$ for $t \to +\infty$. □

Remark 1 (*A Comparison with the HANDY Model*)**.** It is possible to compare these results with those obtained in [5], Section 3, for the egalitarian society in the original HANDY model. In that paper there are three equilibrium points: the "desert state" with $x = 0$ and $y = 0$, the "nature state" with $x = 0$ and $y = \lambda = 100$, and the "sustainable state" with $x \ne 0$ and $y \ne 0$. In [5] the desert state is unstable because the logistic term in the equation for y implies that nature recovers. This is the same that happens in the present papers for the family of equilibrium points of type Q, which correspond to desert state. On the other hand, in [5] the nature state is stable for $\delta < 1/30$. In the present paper the nature states are the equilibrium points of the type R. We have obtained a sort of stability when $\delta(\lambda + Z) < \alpha s$, and instability when $\delta(\lambda + Z) > \alpha s$. In [5] it is $\lambda = 100$, $s = 5$, and a straightforward computation gives $\alpha = 2/3$. Hence, if $Z = 0$, the condition $\delta(\lambda + Z) < \alpha s$ becomes $100\delta < 10/3$ that is $\delta < 1/30$, the same condition of [5]. So our results recover in part those of [5], and in particular they show that the presence of non renewable resources ($Z \ne 0$) makes smaller the range of δ's for which there is stability, hence non renewable resources tend to create instability of the nature state.

As to the "sustainable state" of [5], Section 3, we find equilibrium points of this type when $x > w$, hence in the next section. However, it seems that here the situation is more complicated: there can be many sustainable states or none, depending on the parameters, so a direct comparison with the result of [5] seems difficult and we will not try any.

4. Fixed points and stability: case $x > w$

In the open set $\{x > w\}$, the system has the following form

$$\begin{cases} x'(t) = (\beta - \alpha_M)x + (\alpha_M - \alpha_m)w \\ y'(t) = y(\gamma\lambda - \gamma y - \delta x) \\ z'(t) = xz\left(-\delta + k\dfrac{w}{xw+1}\right) \\ w'(t) = \delta xy + \delta xz - xw - sw. \end{cases} \tag{11}$$

It is then easy (recalling that now $x > w \ge 0$) to see that the equilibrium points are given by

$$\begin{cases} w = \dfrac{\alpha_M - \beta}{\alpha_M - \alpha_m}x = \alpha x \\ y = 0 \qquad \text{or} \qquad y = -\dfrac{\delta}{\gamma}x + \lambda \\ z = 0 \qquad \text{or} \qquad x_{1,2} = \dfrac{\alpha k \pm \sqrt{\alpha^2k^2 - 4\alpha\delta^2}}{2\alpha\delta} \\ \delta xy + \delta xz - xw - sw = 0. \end{cases} \tag{12}$$

Of course, to get real values for $x_{1,2}$ in the third equation we have to assume $\alpha k^2 \ge 4\delta^2$. From the last equation, it can be noticed that it is not possible to have both $y = z = 0$ Indeed, in this case we would obtain either $x = -s$, which is impossible since $x > 0$ and $s > 0$, or we would get $w = 0$, which impossible because from the first equation we would get also $x = 0$, while we are now working in the set $\{x > w\}$. Therefore, if $z = 0$, we must necessarily have $y = -\frac{\delta}{\gamma}x + \lambda$, and thus we obtain from the fourth equation

$$x = \gamma\frac{\lambda\delta - s\alpha}{\delta^2 + \alpha\gamma}. \tag{13}$$

Since $x > 0$, we have the additional constraint $\lambda\delta > s\alpha$. Substituting (13) into the expressions for y and w, we obtain the critical point P_0 with coordinates

$$P_0 = \begin{cases} x_0 = \gamma\dfrac{\lambda\delta - s\alpha}{\delta^2 + \alpha\gamma} \\ y_0 = \alpha\dfrac{s\delta + \gamma\lambda}{\delta^2 + \alpha\gamma} \\ z_0 = 0 \\ w_0 = \alpha\gamma\dfrac{\lambda\delta - s\alpha}{\delta^2 + \alpha\gamma} = \alpha x_0, \end{cases} \tag{14}$$

with $\lambda\delta > s\alpha$.

If $z \neq 0$ and $y = 0$, solving for z in the fourth equation, we obtain the two critical points P_j, $j = 1, 2$

$$P_j = \begin{cases} x_j = \dfrac{\alpha k \pm \sqrt{\alpha^2 k^2 - 4\alpha\delta^2}}{2\alpha\delta} \\ y_j = 0 \\ z_j = \dfrac{\alpha}{\delta}x_j + \dfrac{\alpha s}{\delta} = \dfrac{\alpha}{\delta}(x_j + s) \\ w_j = \alpha x_j, \end{cases} \quad j = 1, 2 \tag{15}$$

with $\alpha k^2 \geq 4\delta^2$.

Finally, if $z \neq 0$ and $y \neq 0$, solving for z in the fourth equation, we obtain the two critical points P_j, $j = 3, 4$

$$P_j = \begin{cases} x_j = \dfrac{\alpha k \pm \sqrt{\alpha^2 k^2 - 4\alpha\delta^2}}{2\alpha\delta} \\ y_j = -\dfrac{\delta}{\gamma}x_j + \lambda \\ z_j = \dfrac{\alpha\gamma + \delta^2}{\delta\gamma}x_j + \dfrac{\alpha s}{\delta} - \lambda = \dfrac{\alpha}{\delta}(x_j + s) - y_j \\ w_j = \alpha x_j, \end{cases} \quad j = 3, 4 \tag{16}$$

with $\alpha k^2 \geq 4\delta^2$ and $x_j \leq \frac{\lambda\gamma}{\delta}$.

These are all the critical points of the system in $\{x > w\}$. We notice that, when $\lambda\delta \leq s\alpha$ and $\alpha k^2 < 4\delta^2$, there are no fixed points in the set $C \cap \{x > w\}$. Numerical simulations (see Section 5, Fig. 9) support the guess that at least some of the trajectories starting in $C \cap \{x > w\}$ converge to stable equilibria in $\{x \leq w\}$, but we have no theoretical results on the global behavior of the solutions in $\{x > w\}$. To get such a result is an open problem for future research.

We want now to study their stability. We will use the standard linearization technique, as in the case $\{x < w\}$. Unfortunately, the study of the Jacobian matrix is now much more difficult, and in some cases, that we will see below, the problem of the stability remains open. We will try to obtain sufficient conditions for stability or instability, at least for some ranges of the parameters, but, as far as we know, the conditions we find are not always necessary.

As first thing, we compute the Jacobian matrix at the generic point $P = (x, y, z, w)$, when $x > w \geq 0$, and we find that it is given by

$$J(x, y, z, w) = \begin{pmatrix} -\alpha\zeta & 0 & 0 & \zeta \\ -\delta y & \gamma\lambda - 2\gamma y - \delta x & 0 & 0 \\ -\delta z + k\dfrac{zw}{(xw+1)^2} & 0 & -\delta x + k\dfrac{xw}{xw+1} & k\dfrac{xz}{(xw+1)^2} \\ \delta y + \delta z - w & \delta x & \delta x & -x - s \end{pmatrix}.$$

4.1. Stability for P_0

Let us initially study the stability at the point P_0 with coordinates given by (14). In this case, the Jacobian matrix is

$$J(P_0) = \begin{pmatrix} -\alpha\zeta & 0 & 0 & \zeta \\ -\delta y_0 & -\gamma y_0 & 0 & 0 \\ 0 & 0 & n_0 & 0 \\ \delta y_0 - \alpha x_0 & \delta x_0 & \delta x_0 & -x_0 - s \end{pmatrix}$$

where

$$n_0 = -\frac{x_0}{\alpha x_0^2+1}(\alpha\delta x_0^2 - k\alpha x_0 + \delta). \tag{17}$$

It is easy to compute that the characteristic polynomial of $J(P_0)$ is given by $(\rho - n_0)(a_0\rho^3 + a_1\rho^2 + a_2\rho + a_3)$ with

$$\begin{aligned} a_0 &= 1 \\ a_1 &= x_0 + s + \alpha\zeta + \gamma y_0 \\ a_2 &= (x_0+s)(\alpha\zeta+\gamma y_0) + \alpha\gamma\zeta y_o - \delta\zeta y_0 + \alpha\zeta x_0 \\ a_3 &= (x_0+s)\alpha\gamma\zeta y_0 - \zeta y_0(-\delta^2 x_0 + \alpha\gamma s). \end{aligned}$$

Hence, n_0 is an eigenvalue of $J(P_0)$, so if $n_0 > 0$, P_0 is unstable. If $n_0 < 0$, we apply the Routh–Hurwitz criterion (see [11]) to the polynomial $\rho^3 + a_1\rho^2 + a_2\rho + a_3$. The Routh–Hurwitz conditions for stability in this case are given by

$$\begin{aligned} &a_1 > 0 \\ &a_1 a_2 > a_3 > 0. \end{aligned} \tag{18}$$

Clearly, $a_1 > 0$ as sum of positive quantities, while

$$a_3 = \zeta y_0(\alpha\gamma x_0 + \alpha\gamma s + \delta^2 x_0 - \alpha\gamma s) = \zeta x_0 y_0(\alpha\gamma + \delta^2) > 0.$$

Finally, $a_1 a_2 > a_3$ leads to

$$\begin{aligned} &(x_0+s)^2(\alpha\zeta+\gamma y_0) + (x_0+s)\alpha\zeta(\gamma y_0 + x_0) + \\ &(x_0+s)(\alpha\zeta+\gamma y_0)^2 + (\alpha\zeta+\gamma y_0)\alpha\gamma\zeta y_0 + \alpha^2\zeta^2 x_0 > \\ &\delta\zeta y_0(x_0+s) + \delta\zeta y_0(\alpha\zeta+\gamma y_0) + \delta^2\zeta x_0 y_0. \end{aligned} \tag{19}$$

Hence we got that, if $n_0 < 0$, the inequality (19) gives a necessary and sufficient condition for the asymptotic stability of P_0. However, (19) is clearly not easy to be dealt with, so we now look for sufficient conditions for this inequality to be true. With some computations, it is not difficult to verify that (19) holds if the following inequalities are satisfied

$$\begin{cases} (x_0+s)(\alpha\zeta+\gamma y_0) > \delta\zeta y_0. \\ (x_0+s)\alpha\zeta(\gamma y_0 + x_0) + (\alpha\zeta+\gamma y_0)\alpha\gamma\zeta y_0 + \alpha^2\zeta^2 x_0 > \delta^2\zeta x_0 y_0. \end{cases}$$

Explicitly stating the values of x_0 and y_0 and carrying out the calculations, the first inequality is equivalent to

$$2\alpha\delta^2\gamma^2\lambda s + \alpha\delta\gamma^3\lambda^2 + \alpha\delta^3\gamma s^2 > 0$$

which is always verified. The last inequality leads to

$$\alpha\gamma\zeta x_0 y_0 + \alpha\zeta x_0^2 + \alpha\gamma\zeta s y_0 + \alpha\zeta s x_0 + \alpha^2\gamma\zeta^2 y_0 + \alpha\gamma^2\zeta y_0^2 + \alpha^2\zeta x_0 > \delta^2\zeta x_0 y_0.$$

Considering the quadratic form

$$\alpha x_0^2 + (\alpha\gamma - \delta^2)x_0 y_0 + \alpha\gamma^2 y_0^2 > 0$$

we have that it is positive definite if and only if

$$\alpha^2\gamma^2 - \frac{1}{4}(\alpha\gamma-\delta^2)^2 > 0 \iff \delta^4 - 2\alpha\gamma\delta^2 - 3\alpha^2\gamma^2 < 0 \iff \delta^2 < 3\alpha\gamma.$$

Hence $\delta^2 < 3\alpha\gamma$ is a sufficient condition for $a_1 a_2 > a_3$.

Another condition can be obtained considering the inequality

$$\alpha\zeta s x_0 + \alpha\zeta x_0^2 > \delta^2\zeta x_0 y_0 \iff \alpha(s + x_0) > \delta^2 y_0$$

which, inserting the values of x_0 and y_0, leads to the condition $\delta < 1$.

We have then proved Proposition 21, that collects all the above results about the equilibrium point P_0. In the statement we will use the reverse of inequality (19), that is the following

$$\begin{aligned} &(x_0+s)^2(\alpha\zeta+\gamma y_0) + (x_0+s)\alpha\zeta(\gamma y_0 + x_0) + \\ &(x_0+s)(\alpha\zeta+\gamma y_0)^2 + (\alpha\zeta+\gamma y_0)\alpha\gamma\zeta y_0 + \alpha^2\zeta^2 x_0 < \\ &\delta\zeta y_0(x_0+s) + \delta\zeta y_0(\alpha\zeta+\gamma y_0) + \delta^2\zeta x_0 y_0. \end{aligned} \tag{20}$$

Proposition 21. *Assume $\lambda\delta > s\alpha$. Let n_0 defined as (17). Then*

(i) If either $n_0 > 0$ or the inequality (20) holds, then P_0 is unstable.
(ii) If $n_0 < 0$ and the inequality (19) holds, then the equilibrium point P_0 is asymptotically stable. A sufficient condition for (19) is $\delta < \max\{1, \sqrt{3\alpha\gamma}\}$.

4.2. Some remarks about P_1, P_2, P_3 and P_4

The expression for $x_j,\ j=1,2,3,4$ in (15) and (16), was derived from the third equation of the system, which gives $\delta = \frac{kw}{xw+1}$. We will often use, for the following discussions, the identity

$$\delta = \frac{k\,\alpha\,x_j}{\alpha x_j^2+1}, \qquad j=1,2,3,4.$$

We note that the points $P_{1,2}$ differ from the points $P_{3,4}$ only in the expression of their y_j and hence of their z_j. Moreover, when $\alpha k^2-4\delta^2>0$, it holds for $j=1,3$,

$$x_j^2 < \frac{1}{\alpha}, \qquad j=1,3. \tag{21}$$

Indeed, recalling that of course $\delta>0$, we have

$$\left(\frac{\alpha k-\sqrt{\alpha^2k^2-4\alpha\delta^2}}{2\alpha\delta}\right)^2 < \frac{1}{\alpha} \iff 2\alpha k\sqrt{\alpha^2k^2-4\alpha\delta^2} > 2\alpha^2k^2-8\alpha\delta^2 \iff$$
$$\alpha k^2(\alpha k^2-4\delta^2) > (\alpha k^2-4\delta^2)^2 \iff \alpha k^2 > \alpha k^2-4\delta^2 \iff \delta^2>0.$$

On the other hand, for $j=2,4$ it holds

$$x_j^2 > \frac{1}{\alpha}, \qquad j=2,4. \tag{22}$$

Indeed this means

$$\left(\frac{\alpha k+\sqrt{\alpha^2k^2-4\alpha\delta^2}}{2\alpha\delta}\right)^2 > \frac{1}{\alpha} \iff k\sqrt{\alpha^2k^2-4\alpha\delta^2} > 4\delta^2-\alpha k^2$$

which is verified when $4\delta^2-\alpha k^2<0$. Recall that $4\delta^2-\alpha k^2\le 0$ is the condition for the existence of the critical points we are dealing with, and that $4\delta^2-\alpha k^2=0$ implies $P_1=P_2$ and $P_3=P_4$.

For the following analysis, let us now compute some of the entries of the Jacobian $(J_{i,j})_{i,j=1}^4$ in the equilibrium points. We first study the value of the elements $J_{3,1}(P_j), J_{3,3}(P_j)$ and $J_{3,4}(P_j)$ for $j=1,2,3,4$. For these entries the variable y_j does not appear and we do not explicit the z_j.

$$\begin{aligned}
J_{3,1}(P_j) &= -\delta z_j + k\frac{w_jz_j}{(x_jw_j+1)^2} = -\delta z_j + \frac{k\alpha x_j}{\alpha x_j^2+1}\frac{z_j}{\alpha x_j^2+1}\\
&= -\delta z_j + \frac{\delta z_j}{\alpha x_j^2+1} = -\frac{\alpha\delta x_j^2z_j}{\alpha x_j^2+1} = -\frac{\delta^2x_jz_j}{k}\\
J_{3,3}(P_j) &= -\delta x_j + k\frac{x_jw_j}{(x_jw_j+1)} = -\delta x_j + \frac{k\alpha x_j^2}{\alpha x_j^2+1} = -\delta x_j+\delta x_j = 0\\
J_{3,4}(P_j) &= \frac{kx_jz_j}{(\alpha x_j^2+1)^2} = \frac{z_j}{\alpha^2kx_j}\frac{k^2x_j^2\alpha^2}{(\alpha x_j^2+1)^2} = \frac{\delta^2z_j}{\alpha^2kx_j}.
\end{aligned}$$

4.3. Stability for P_1 and P_2

To study the stability of P_1 and P_2, we go on with the computations of the entries of the Jacobian. We compute $J_{4,1}(P_j)$ (with $j=1,2$). We obtain:

$$J_{4,1}(P_j) = \delta y_j + \delta z_j - w_j = \delta\left(\frac{\alpha}{\delta}x_j + \frac{\alpha s}{\delta}\right) - \alpha x_j = \alpha s.$$

Hence, the Jacobian at the points P_j, $j=1,2$ is given by

$$J(P_j) = A = \begin{pmatrix} -\alpha\zeta & 0 & 0 & \zeta\\ 0 & \gamma\lambda-\delta x_j & 0 & 0\\ -\dfrac{\delta^2x_jz_j}{k} & 0 & 0 & \dfrac{\delta^2z_j}{\alpha^2kx_j}\\ \alpha s & \delta x_j & \delta x_j & -x_j-s\end{pmatrix}.$$

The characteristic polynomial is

$$\det(\rho I-A) = (\rho-\gamma\lambda+\delta x_j)\begin{vmatrix} \rho+\alpha\zeta & 0 & -\zeta\\ \dfrac{\delta^2x_jz_j}{k} & \rho & -\dfrac{\delta^2z_j}{\alpha^2kx_j}\\ -\alpha s & -\delta x_j & \rho+x_j+s\end{vmatrix}.$$

So we have an eigenvalue

$$\rho_1 = \gamma\lambda - \delta x_{1,2}.$$

From the expression of $x_{1,2}$ in (15) we have

$$x_{1,2} < \frac{\alpha k + \sqrt{\alpha^2 k^2}}{2\alpha\delta} = \frac{k}{\delta}$$

so

$$\rho_1 = \gamma\lambda - \delta x_{1,2} > \gamma\lambda - k.$$

We then get a first result

Proposition 22. *If $\gamma\lambda - k \geq 0$, the critical points P_1 and P_2 are unstable.*

In the following part of this subsection, we will assume $\gamma\lambda - k < 0$. We then study the third order equation

$$\begin{vmatrix} \rho + \alpha\zeta & 0 & -\zeta \\ \dfrac{\delta^2 x_j z_j}{k} & \rho & -\dfrac{\delta^2 z_j}{\alpha^2 k x_j} \\ -\alpha s & -\delta x_j & \rho + x_j + s \end{vmatrix} = 0,$$

that is

$$\rho^3 + (x_j + s + \alpha\zeta)\rho^2 + \left(-\frac{\delta^3 z_j}{\alpha^2 k} + \alpha\zeta x_j\right)\rho + \frac{\delta^3 z_j}{k}\zeta\left(-\frac{1}{\alpha} + x_j^2\right) = 0. \tag{23}$$

1. First we deal with the point $P_1 = (x_1, y_1, z_1, w_1)$ namely

$$P_1 = \left(\frac{\alpha k - \sqrt{\alpha^2 k^2 - 4\alpha\delta^2}}{2\alpha\delta},\ 0,\ \frac{\alpha}{\delta}x_1 + \frac{\alpha s}{\delta},\ \alpha x_1\right) \tag{24}$$

in the case $\alpha k^2 - 4\delta^2 > 0$.

The value of the characteristic polynomial for $\rho = 0$ is given by

$$\frac{\delta^3 z_1}{k}\zeta\left(-\frac{1}{\alpha} + x_1^2\right) < 0$$

so there is a positive root, that is a positive eigenvalue. The equilibrium point P_1 is then unstable, and we have obtained the following proposition.

Proposition 23. *If $\alpha k^2 > 4\delta^2$ then the equilibrium point P_1 is unstable.*

2. Now we deal with the point $P_2 = (x_2, y_2, z_2, w_2)$ given by

$$P_2 = \left(\frac{\alpha k + \sqrt{\alpha^2 k^2 - 4\alpha\delta^2}}{2\alpha\delta},\ 0,\ \frac{\alpha}{\delta}x_2 + \frac{\alpha s}{\delta},\ \alpha x_2\right) \tag{25}$$

when $\alpha k^2 - 4\delta^2 > 0$. In this case we have not a general result, but we will show that P_2 can be stable or unstable at least for some ranges of parameter values. Our results are stated in the following Propositions 24, 25, 26.

We will apply the Hurwitz technique to the polynomial in (23), which we write as $a_0\rho^3 + a_1\rho^2 + a_2\rho + a_3$, so that

$$\begin{aligned} a_0 &= 1 \\ a_1 &= x_2 + s + \alpha\zeta \\ a_2 &= \alpha\zeta x_2 - \frac{\delta^3 z_2}{\alpha^2 k} \\ a_3 &= \frac{\delta^3 \zeta z_2}{k}\left(x_2^2 - \frac{1}{\alpha}\right) \end{aligned}$$

and to have roots with strictly negative real part, the condition (18) must hold. The condition $a_1 > 0$ is obviously verified. The condition $a_3 > 0$ leads us to $x_2^2 > \frac{1}{\alpha}$, which is always verified as we have seen above (see (22)). Now, we analyze the condition $a_1 a_2 > a_3$. It leads to

$$\alpha\zeta x_2^2 + \alpha\zeta s x_2 + \alpha^2\zeta^2 x_2 > \frac{\delta^3 x_2 z_2}{\alpha^2 k} + \frac{\delta^3 s z_2}{\alpha^2 k} + \frac{\delta^3 \zeta x_2^2 z_2}{k}.$$

Explicitly stating z_2 and x_2^2 remembering that $x_2^2 = \frac{k}{\delta}x_2 - \frac{1}{\alpha}$ and $z_2 = \frac{\alpha}{\delta}x_2 + \frac{\alpha s}{\delta}$ we obtain

$$\begin{aligned}&\left(\frac{\alpha\zeta k}{\delta} + \alpha\zeta s + \alpha^2\zeta^2 + \frac{\delta^2\zeta}{k}\right)x_2 + \frac{\delta^2}{\alpha^2 k} + \delta\zeta + \frac{\delta^2\zeta s}{k} > \\ &\left(\frac{\delta}{\alpha} + 2\frac{\delta^2 s}{\alpha k} + \alpha\zeta k + \alpha\delta\zeta s\right)x_2 + \zeta + \frac{\delta^2 s^2}{\alpha k}\end{aligned} \tag{26}$$

where, multiplying both sides by δ^2 we get

$$\begin{aligned}&\left(\alpha\zeta k + \alpha\delta\zeta s + \alpha^2\delta\zeta^2 + \frac{\delta^3\zeta}{k}\right)\delta x_2 + \delta^4\left(\frac{1}{\alpha^2 k} + \frac{\zeta s}{k}\right) + \delta^3\zeta > \\ &\left(\frac{\delta^2}{\alpha} + 2\frac{\delta^3 s}{\alpha k} + \alpha\delta\zeta k + \alpha\delta^2\zeta s\right)\delta x_2 + \delta^2\zeta + \frac{\delta^4 s^2}{\alpha k}.\end{aligned} \tag{27}$$

Note that

$$\delta x_2 = \frac{k}{2}\left(1 + \sqrt{1 - \frac{4\delta^2}{\alpha k^2}}\right)$$

so $\delta x_2 \to k$ as $\delta \to 0^+$. If in (27) we take the limit for both sides as $\delta \to 0^+$ we get $\alpha\zeta k > 0$ which is always verified. Also we get $\rho_1(\delta) \to \gamma\lambda - k < 0$ as $\delta \to 0^+$. We can therefore state the following proposition.

Proposition 24. *If $\alpha k^2 > 4\delta^2$ then P_2 is an equilibrium point. If $\gamma\lambda < k$ then there exists $\delta^* > 0$ such that for any $\delta \in (0, \delta^*)$ the equilibrium point P_2 is asymptotically stable.*

Note that, to verify the conditions required by the proposition, it must be $\delta^* < k\sqrt{\alpha}/2$. When δ takes on the maximum value, that is $\delta = \delta_c = k\sqrt{\alpha}/2$, we obtain $P_1 \equiv P_2 = \left(\frac{1}{\sqrt{\alpha}}, 0, \frac{2}{k}(1 + s\sqrt{\alpha}), \sqrt{\alpha}\right)$ and the coefficients of the characteristic polynomial are

$$\begin{aligned}a_0 &= 1\\ a_1 &= x_0 + s + \alpha\zeta = \frac{1}{\sqrt{\alpha}} + s + \alpha\zeta\\ a_2 &= \sqrt{\alpha}\zeta - \frac{k}{4\sqrt{\alpha}}(1 + s\sqrt{\alpha})\\ a_3 &= 0\end{aligned}$$

so that the characteristic polynomial is

$$\rho(\rho^2 + a_1\rho + a_2).$$

Since $a_1 > 0$, if $a_2 < 0$ then the polynomial will have a positive real root and therefore the system is unstable. We have $a_2 < 0$ if $4\alpha\zeta < k(1 + s\sqrt{\alpha})$. Since all a_i depend continuously on δ, we easily get the following proposition:

Proposition 25. *Let $4\alpha\zeta < k(1 + s\sqrt{\alpha})$ and let $\delta_c = \frac{k}{2}\sqrt{\alpha}$. Then there exists $\delta_* < \delta_c$ such that for any $\delta \in (\delta_*, \delta_c]$ the critical point P_2 is unstable.*

In a similar way we can prove that if $4\alpha\zeta > k(1 + s\sqrt{\alpha})$ the equilibrium P_2 is stable.

Proposition 26. *Assume $4\alpha\zeta > k(1 + s\sqrt{\alpha})$ and $\gamma\lambda < k/2$. Let $\delta_c = \frac{k}{2}\sqrt{\alpha}$. Then there exists $\delta_* < \delta_c$ such that for any $\delta \in (\delta_*, \delta_c)$ the equilibrium point P_2 is asymptotically stable.*

Proof. As first thing we notice that $\rho_1(\delta_c) = \gamma\lambda - k/2 < 0$, by assumption, so obviously $\rho_1(\delta) < 0$ when δ is near δ_c. Then we observe that

$$x_2 = \frac{\alpha k + \sqrt{\alpha^2 k^2 - 4\alpha\delta^2}}{2\alpha\delta} = \frac{k}{2\delta} + \frac{1}{2\alpha}\sqrt{\frac{\alpha^2 k^2}{\delta^2} - 4\alpha}$$

is decreasing in δ so if $\delta < \delta_c$ then $x_2(\delta) > x_2(\delta_c) = \frac{1}{\sqrt{\alpha}}$. Hence for $\delta < \delta_c$ we have $x_2^2 - \frac{1}{\alpha} > 0$ i.e. $a_3(\delta) > 0$ while $a_3(\delta_c) = 0$. Since $4\alpha\zeta > k(1 + s\sqrt{\alpha})$, we have $a_2(\delta_c) > 0$ and $a_1(\delta_c)a_2(\delta_c) > a_3(\delta_c) = 0$. Therefore, by continuity, there exists $\delta_* \in (0, \delta_c)$ such that for any $\delta \in (\delta_*, \delta_c)$ it holds $a_1(\delta)a_2(\delta) > a_3(\delta) > 0$ and by Routh–Hurwitz theorem the proposition is proven. □

Notice that in the last case we do not know what happens when $\delta = \delta_c$.

4.4. Stability for P_3 and P_4

Now let us study the equilibrium points P_3 and P_4 in (16). As first thing we notice that we are now dealing with $y_j > 0$, hence we must have $\gamma\lambda > \delta x_j$. It is easy to verify that it is always $\delta x_j < k$, while $\lim_{\delta\to 0^+} \delta x_4 = k$, so a reasonable sufficient condition to have is the inequality $\gamma\lambda > k$, and it will be assumed throughout this subsection.

We start our study by computing, for $j = 3, 4$:

$$J_{2,2}(P_j) = \gamma\lambda - 2\gamma y_j - \delta x_j = \gamma\lambda - 2\gamma y_j + \delta\left(y_j - \lambda\right)\frac{\gamma}{\delta} = -\gamma y_j$$

and

$$\begin{aligned} J_{4,1}(P_j) &= \delta y_j + \delta z_j - w_j = \delta y_j + \frac{1}{x_j}(x_j w_j + s w_j - \delta x_j y_j) - w_j \\ &= \delta y_j + w_j + \frac{s w_j}{x_j} - \delta y_j - w_j = s\alpha. \end{aligned}$$

The Jacobian matrix at points P_j, $j = 3, 4$, is then given by

$$J(P_j) = A = \begin{pmatrix} -\alpha\zeta & 0 & 0 & \zeta \\ -\delta y_j & -\gamma y_j & 0 & 0 \\ -\frac{\delta^2 x_j z_j}{k} & 0 & 0 & \frac{\delta^2 z_j}{\alpha^2 k x_j} \\ \alpha s & \delta x_j & \delta x_j & -x_j - s \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} \rho + \alpha\zeta & 0 & 0 & -\zeta \\ +\delta y_j & \rho + \gamma y_j & 0 & 0 \\ \frac{\delta^2 x_j z_j}{k} & 0 & \rho & -\frac{\delta^2 z_j}{\alpha^2 k x_j} \\ -\alpha s & -\delta x_j & -\delta x_j & \rho + x_j + s \end{vmatrix} = 0$$

and the characteristic polynomial is

$$\begin{aligned} &\rho^4 + (x_j + s + \gamma y_j + \alpha\zeta)\rho^3 + \\ &\left[-\frac{\delta^3 z_j}{\alpha^2 k} + (\gamma y_j + \alpha\zeta)(x_j + s) + \alpha\gamma\zeta y_j - \alpha s\zeta\right]\rho^2 + \\ &\left[-(\alpha\zeta + \gamma y_j)\frac{\delta^3 z_j}{\alpha^2 k} + \alpha\gamma\zeta(x_j + s)y_j + \delta^2\zeta x_j y_j + \frac{\delta^3\zeta x_j^2 z_j}{k} - \alpha s\gamma\zeta y_j\right]\rho + \\ &-\frac{\delta^3\gamma\zeta y_j z_j}{\alpha k} + \frac{\delta^3\gamma\zeta}{k} x_j^2 y_j z_j. \end{aligned} \tag{28}$$

We write it in the standard way as $a_0\rho^4 + a_1\rho^3 + a_2\rho^2 + a_3\rho a_0$, where the a_i's are given by (28). As we have seen in the study of P_1, it holds $a_3 < 0$ when $x_j^2 < \frac{1}{\alpha}$, and this happens in the equilibrium P_3.

Hence we have

Proposition 27. *If $\alpha k^2 > 4\delta^2$ then P_3 is an unstable equilibrium point.*

Let us now look at the point P_4. As in the case of the equilibrium P_2, we will not get a general results, but we have several results for different ranges of the parameters. Applying Routh–Hurwitz criterion, the necessary and sufficient conditions for all roots of (28) to have a strictly negative real part are the following

$$\begin{cases} a_i > 0 \qquad i = 0, \dots, 4 \\ a_1 a_2 - a_0 a_3 > 0 \\ a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 > 0. \end{cases} \tag{29}$$

We want to study what happens for $\delta \to 0^+$, so we look at the asymptotic behavior of the a_i's. We start noticing that

$$\delta x_4(\delta) = \frac{k}{2} + \frac{1}{2\alpha}\sqrt{\alpha^2 k^2 - 4\alpha\delta^2}$$

from which $\lim_{\delta\to 0^+}\delta x_4(\delta)=k$ and therefore $x_4(\delta)=\frac{k}{\delta}+o(\delta^{-1})$. Also we have $y_4(\delta)=-\frac{1}{\gamma}\delta x_4(\delta)+\lambda$, hence $\lim_{\delta\to 0^+} y_4(\delta)=-\frac{k}{\gamma}+\lambda$. As to z_4 we have $z_4(\delta)=\frac{\alpha}{\delta}x_4(\delta)+\frac{\alpha s}{\delta}-y_4(\delta)=\frac{\alpha k}{\delta^2}+\frac{\alpha s}{\delta}+o(\delta^{-1})$. Up to know we have obtained

$$
\begin{aligned}
x_4(\delta) &= \frac{k}{\delta}+o(\delta^{-1})\\
y_4(\delta) &= -\frac{k}{\gamma}+\lambda+o(1)\\
z_4(\delta) &= \frac{\alpha k}{\delta^2}+\frac{\alpha s}{\delta}+o(\delta^{-1})\\
w_4 &= \frac{\alpha k}{\delta}+o(\delta^{-1}).
\end{aligned}
\tag{30}
$$

Now let us examine the a_i.

$$
\begin{aligned}
a_0 &= 1>0\\
a_1 &= x_j+s+\gamma y_j+\alpha\zeta=\frac{k}{\delta}+o(\delta^{-1})>0\\
a_2 &= -\frac{\delta^3 z_j}{\alpha^2 k}+\gamma y_j(x_j+s)+\alpha\gamma\zeta y_j+\alpha x_j\zeta=\frac{(\alpha\zeta+\gamma\lambda-k)k}{\delta}+o(\delta^{-1})\\
a_3 &= -(\alpha\zeta+\gamma y_j)\frac{\delta^3 z_j}{\alpha^2 k}+\alpha\gamma\zeta(x_j+s)y_j+\delta^2\zeta x_j y_j\\
&\quad+\frac{\delta^3\zeta x_j^2 z_j}{k}-\alpha s\gamma\zeta y_j=\frac{\alpha\gamma\zeta\lambda k}{\delta}+o(\delta^{-1})\\
a_4 &= \frac{\delta^3\gamma\zeta}{k}y_4 z_4\left(x_4^2-\frac{1}{\alpha}\right)=\frac{\alpha\zeta k^2}{\delta}(-k+\gamma\lambda)+o(\delta^{-1})>0.
\end{aligned}
\tag{31}
$$

Obviously $a_0>0$ and $a_1>0$. As for a_4 we know that $a_4=\frac{\delta^3\gamma\zeta}{k}y_4 z_4(x_4^2-\frac{1}{\alpha})>0$. We note also that $a_3=\frac{\alpha\gamma\zeta\lambda k}{\delta}+o(\delta^{-1})$ and thus it tends to $+\infty$ for $\delta\to 0^+$. As for a_2, it is $a_2=\frac{(\alpha\zeta+\gamma\lambda-k)k}{\delta}+o(\delta^{-1})$. We are assuming $\gamma\lambda>k$, so of course it is for $\alpha\zeta+\gamma\lambda>k$, hence $a_2\to+\infty$ as $\delta\to 0^+$. We conclude that there is $\delta^*>0$ such that $a_i>0$, $\forall\delta\in(0,\delta^*)$, $i=0,1,2,3,4$.

We study the condition $a_1a_2-a_0a_3>0$. From (31) we see that

$$
a_1a_2-a_0a_3=\frac{(\alpha\zeta+\gamma\lambda-k)k^2}{\delta^2}+o(\delta^{-2})>0
$$

so, as above, the assumption $\gamma\lambda>k$ gives $a_1a_2-a_0a_3>0$ for small δ's.

Let us now study the inequality $a_1a_2a_3-a_1^2a_4-a_0a_3^2>0$ We have

$$
\begin{aligned}
a_1a_2a_3-a_1^2a_4-a_0a_3^2 &= \frac{(\alpha\zeta+\gamma\lambda-k)\alpha\gamma\lambda\zeta k^3}{\delta^3}-\frac{k^4\alpha\zeta(\gamma\lambda-k)}{\delta^3}+o(\delta^{-3})\\
&= \frac{\alpha\zeta k^3}{\delta^3}(\alpha\gamma\lambda\zeta+\gamma^2\lambda^2+k^2-2\gamma\lambda k)+o(\delta^{-3})\\
&= \frac{\alpha\zeta k^3}{\delta^3}\left[\alpha\gamma\lambda\zeta+(\gamma\lambda-k)^2\right]+o(\delta^{-3})>0
\end{aligned}
$$

for small δ's. Using the Routh–Hurwitz criterion, we can conclude that we have proven the following proposition.

Proposition 28. *If $\alpha k^2>4\delta^2$ and $\gamma\lambda>k$ then P_4 is an equilibrium point and there exists $\delta^*>0$ such that, for any $\delta\in(0,\delta^*)$, P_4 is asymptotically stable.*

The range of parameter δ for the existence of the equilibrium points we are dealing with is $(0,k\sqrt{\alpha}/2)$. As we have studied what happens as $\delta\to 0^+$, it would be interesting to see what happens as $\delta\to\delta_c^-$, where $\delta_c=k\sqrt{\alpha}/2$. We have not a general result in this case, but it is possible to analyze some particular examples, as we see in what follows. Here we fix arbitrary values for all the parameters except δ and α, and we see what happens as $\alpha\to 0^+$ or $\alpha\to 1^-$. The proof are simple computations using the Routh–Hurwitz criterion, so we skip them.

Example 1. Fix $\zeta=3$, $\gamma=1/2$, $\lambda=10$, $s=5$, $k=1$ (so that $\gamma\lambda>k$ and $\delta_c=\sqrt{\alpha}/2$). Then there are $\alpha^*\in(0,1)$ and $\delta^*\in(0,\delta_c)$ such that for all $\alpha\in(\alpha^*,1)$ and $\delta\in(\delta^*,\delta_c)$ the equilibrium point P_4 is asymptotically stable.

Example 2. Fix $\zeta=3$, $\gamma=1/2$, $\lambda=3$, $s=5$, $k=2$ (so that $\gamma\lambda<k$ and $\delta_c=\sqrt{\alpha}$). As before, we get that there are $\alpha^*\in(0,1)$ and $\delta^*\in(0,\delta_c)$ such that for all $\alpha\in(\alpha^*,1)$ and $\delta\in(\delta^*,\delta_c)$ the equilibrium point P_4 is asymptotically stable. Let us see a last example for the case $\alpha\to 0^+$.

Example 3. Let us fix the parameters as in Example 2. If $\delta=\delta_c$ a short computation gives $x_4=1/\sqrt{\alpha}$, $y_4=\lambda-k/2\gamma$, $z_4=\sqrt{\alpha}$. Using these values we easily get $a_2\to-\infty$ as $\alpha\to 0^+$, hence we can conclude that there is $\alpha^*\in(0,1)$ such that for all $\alpha\in(0,\alpha^*)$ there is $\delta^*=\delta^*(\alpha)\in(0,\delta_c)$ such that for all $\delta\in(\delta^*,\delta_c)$ the equilibrium point $P_4=P_4(\alpha,\delta)$ is unstable.

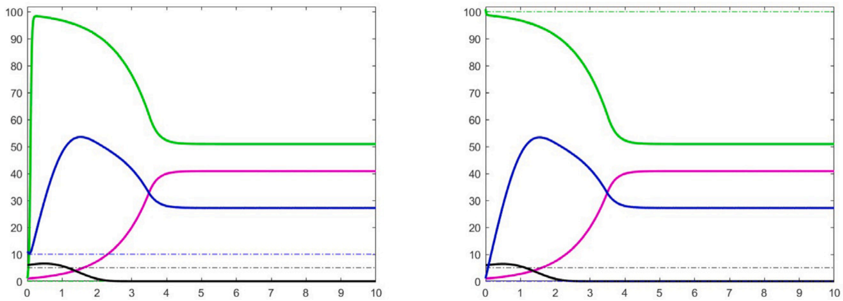

Fig. 1. On the left: situation near $Q = (0, 0, Z, W)$, with $Z = 5$, $W = 10$. On the right: situation near $R = (0, \lambda, Z, 0)$, with $\lambda = 100$ and $Z = 5$.

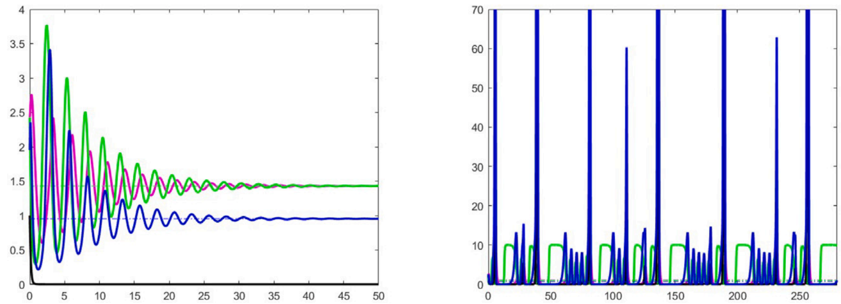

Fig. 2. Scenario around the equilibrium point P_0. On the left: a situation of stability. On the right: a situation of instability with a non-periodic solution.

5. Simulation results

We now introduce the results obtained through simulations carried out using the Matlab solver `ode45` that implements an explicit adaptive Runge–Kutta. In all the figures, the evolution of the variables are plotted in different colors, specifically the population in magenta, the renewable resources in green, the non-renewable resources in black, and the accumulated wealth in blue. The dashed lines represent the coordinate of the equilibrium point under study, and have the same colors for the same components (that is, magenta for the population, and so on).

We have tested the theoretical studies by uniformly perturbing the data from the previously calculated critical points, varying the parameters, and analyzing the evolution of the scenario. In all the tests carried out, we kept the values of some parameters fixed, and in particular, we considered, as seen in [5], $\gamma = 0.5$, $s = 5$, $\zeta = 3$ and $\beta - \alpha_m = 1$, the latter almost everywhere except in cases where we wanted to test the behavior of the parameter α.

We first consider the case where $x \leq w$, that is when the wealth is greater than the current population. The critical points are given by two families of points indicated with Q and R (see Section 3). These critical points can be stable or unstable, depending on the parameters. Here, we choose values for which our theoretical results give instability, and this is confirmed by simulations. In Fig. 1 we reported on the left the test made on the Q type points with $Z = 5$ and $W = 10$, on the right that on the R points in the particular case where $x = w = 0$ with $Z = 5$. In both cases we set $\lambda = 100$, $k = 1$, $\alpha = 2/3$, $\delta = 0.6$.

We note that after some time, both critical points converge to the point $P_0 = (40.8654, 50.9615, 0, 27.2436)$ which, with these parameters, is a stable critical point. In particular, it is the only additional critical point present in this scenario since $\alpha k^2 - 4\delta^2 < 0$.

We have then some simulations for the case $x > w$, starting from the P_0 type points. In Fig. 2 we set $\lambda = 10$. On the left, we chose $\delta = 3$ and $k = 1$ obtaining $P_0 = (1.4286, 1.4286, 0, 0.9524)$. We are in the situation where P_0 is a stable critical point as $n_0 < 0$ and the condition (19) is satisfied.

On the right, we set $\delta = 4$ and $k = 10$ obtaining $P_0 = (1.1224, 1.0204, 0, 0.7483)$. Neither of the two conditions for stability is satisfied; in fact we have $n_0 = 0.0752 > 0$ and the stability condition (19) is not satisfied. In this figure, a precise periodicity does not appear, but rather a sequence of oscillations without a precise pattern. This leads us to think of a chaotic trend, which will be an interesting subject for future research.

The simulations in Fig. 3 seem to suggest the existence of periodic solutions, for which we have no theoretical result. In this case we have $k = 1$, $\lambda = 10$ and $\delta = 4$. In both the left and right simulations we start with a small perturbation with respect to P_0. In this case it is an unstable equilibrium point because the stability condition (19) is not satisfied (while $n_0 < 0$). In particular, on the left we set $\beta - \alpha_m = 2.7$ and consequently $\alpha = 0.1$. On the right, we set $\beta - \alpha_m = 1$ with $\alpha = 2/3$. We have numerically analyzed the behavior of the eigenvalues of the Jacobian matrix calculated at the fixed point P_0 as the δ parameter varies. We tested multiple values of the parameter and found that, in all cases, the Jacobian matrix presents two negative real eigenvalues and two conjugate complex eigenvalues with a real part that changes sign as δ varies. Since we have continuity with respect to the parameters, this

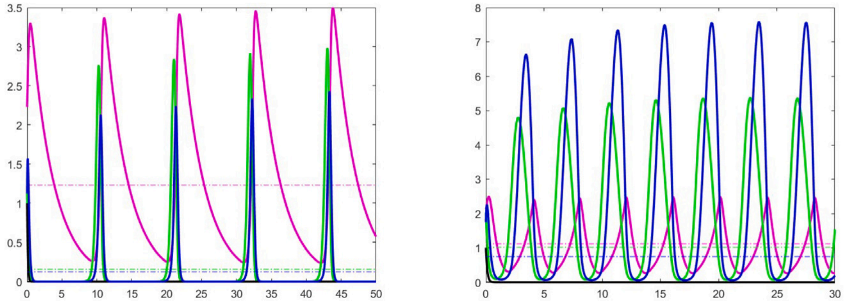

Fig. 3. Periodic solutions near P_0.

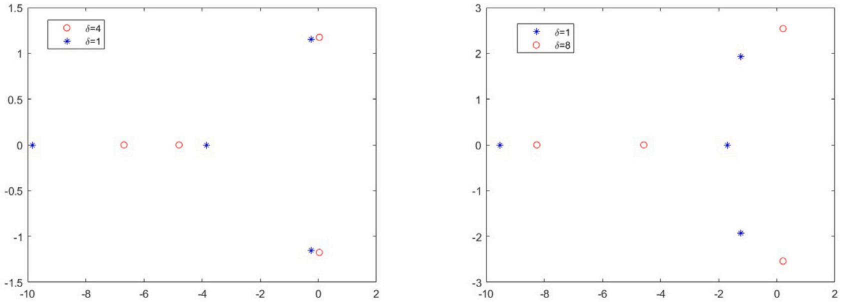

Fig. 4. Numerical analysis of eigenvalues. On the left: eigenvalues of the Jacobian matrix computed in the fixed point P_0 as the δ parameter varies in the situation of periodic solutions of Fig. 3 on the left. On the right: eigenvalues of the Jacobian matrix computed in the fixed point P_0 as the δ parameter varies in the situation of periodic solutions of Fig. 3 on the right.

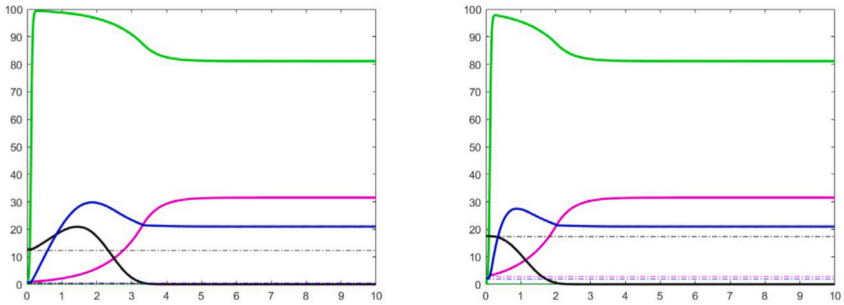

Fig. 5. Case $\gamma\lambda - k \geq 0$ for P_1 and P_2. On the left: situation near P_1. On the right: situation near P_2.

behavior suggests that we might be in the presence of a Hopf bifurcation of periodic solutions. In Fig. 4, we have reported the value of these eigenvalues for two different values of δ. We can observe that the real eigenvalues are always negative as δ varies, while the conjugate complex eigenvalues exhibit a change of sign in the real part.

In Figs. 5 and 6, we have studied the behavior near the points P_1 and P_2. As for Fig. 5, in both cases we set $\lambda = 100$, $k = 1$, $\alpha = 2/3$, and $\delta = 0.3$. In this situation we have $\delta_c = 0.4082$ and $\gamma\delta - k > 0$. With these data, both $P_1 = (0.5363, 0, 12.3028, 0.3575)$ and $P_2 = (2.7971, 0, 17.3268, 1.8647)$ are unstable critical points. From the two graphs, we can notice a convergence to a point, and we have verified that this point is P_0, which with these parameters is an asymptotically stable equilibrium point.

In Fig. 6, we present the results obtained for P_2 under the stability conditions of Proposition 24. We set $\lambda = 10$, $k = 10$, and, as usual, $\gamma = 0.5$ and $\alpha = 2/3$, so that $\gamma\lambda - k < 0$ and $\delta_c = 4.0825$. From numerical experiments, we believe it holds $\delta^* \sim 1.1$. We verified this assertion by varying δ and we reported on the left the graph obtained for δ values lower than 1.1, specifically $\delta = 0.9$. The numerical simulation suggests stability of P_2, in this case. On the right, we set $\delta = 1.1$. The simulation seems to indicate a situation of instability. Considering that with these parameters the condition $4\alpha\zeta - k(1 + s\sqrt{\alpha}) = -42.8248$ is satisfied, this result is compatible with Proposition 25, with $\delta_* \leq 1.1$.

In Fig. 7, we used the parameters $\lambda = 10$, $\delta = 0.2$, $k = 1$ with $\delta_c = 0.4082$ to test the behavior near the points P_3 and P_4. We recall that P_3 is always an unstable equilibrium point while P_4, with the chosen data, is stable for small values of δ (Proposition 28). We can notice that, after a short time interval, the solution obtained starting from the perturbed data of P_3 converges to the point P_4.

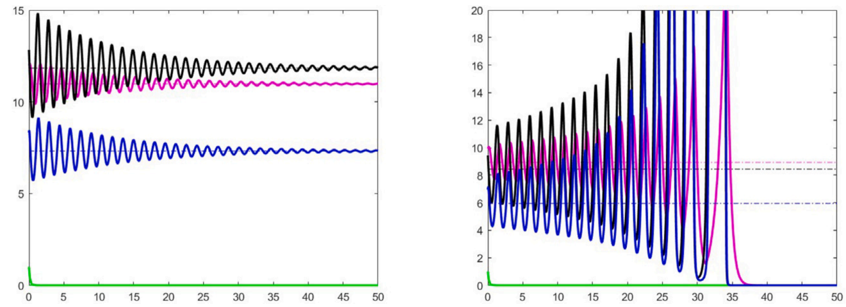

Fig. 6. Situation near P_2 in the case $\gamma\lambda < k$. On the left: small δ. On the right: large δ and $4\alpha\zeta - k(1 + s\sqrt{\alpha}) = -42.8248$.

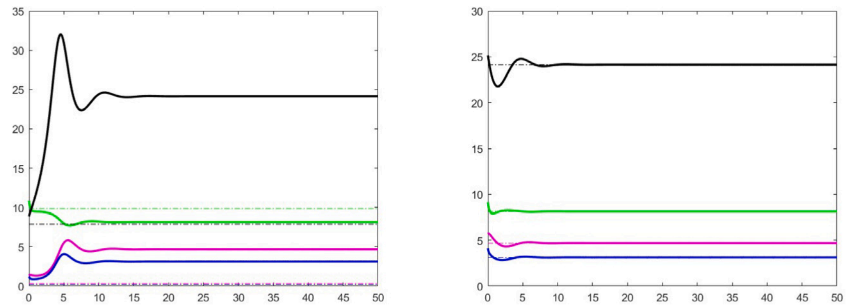

Fig. 7. On the left: situation near P_3. The solution tends to P_4 that satisfied stability conditions. On the right: situation near P_4.

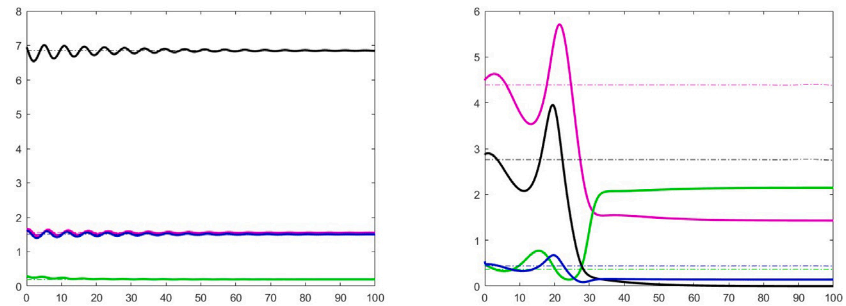

Fig. 8. Situation near P_4. Case $\gamma\lambda < k$. On the left: large α. On the right: small α.

Then, in Fig. 8, we wanted to verify the behavior of the solution starting from perturbed data related to P_4 in cases where $\gamma\lambda - k < 0$ with δ close to δ_c, cases in which we do not have a precise theoretical result. The parameters used are those indicated in Examples 2 and 3 at the end of Section 4. On the left, we analyzed the behavior in the case where α is large and specifically we set $\beta - \alpha_m = 0.1$ in order to obtain $\alpha = 0.9667$ and $\delta = 0.9$ close to $\delta_c = 0.9832$.

On the right, we instead set $\beta - \alpha_m = 2.7$ in order to obtain a small α, $\alpha = 0.1$, and $\delta = 0.3$ close to $\delta_c = 0.3162$. The behaviors seem to follow the analyses made in Examples 2 and 3, and in particular for $\gamma\lambda - k < 0$ and $\delta \to \delta_c$ we have instability for small α and stability for large α. Specifically, we verified that the solution in the left figure converges to the point P_0, which is asymptotically stable, with the data taken into consideration.

In Fig. 9, we wanted to analyze the case when the parameters do not meet the required conditions $\alpha k^2 \geq 4\delta^2$ and $\lambda\delta \geq s\alpha$, so there no fixed points in $x > w$. We test with $\lambda = 1$, $s = 5$, $k = 0.5$, $\delta = 0.3$, $\alpha = 2/3$. We have reported the results of two numerical tests with different initial points and in both cases the solution converges to the fixed point $R = (0, \lambda, Z, 0)$ under stability conditions $\delta(\lambda + Z) < \alpha s$.

Finally, in Figs. 10 and 11, we wanted to revisit the scenarios proposed in Fig. 2 on the left, related to a stability situation of point P_0, and Fig. 5 on the right, related to an instability situation for point P_2, in the case where $\delta_1 \neq \delta_2$. From this and other similar simulations, it seems to us that different choices for these parameters do not influence too much the results we have obtained.

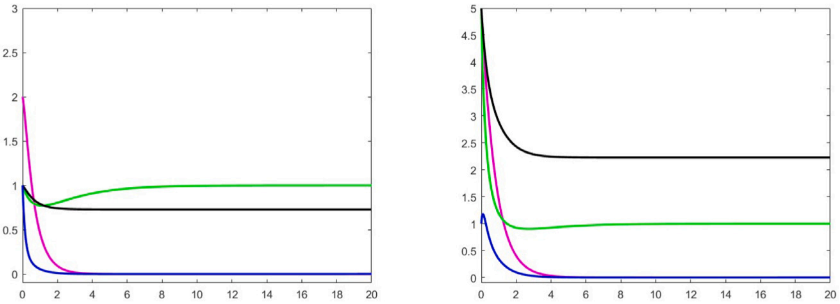

Fig. 9. $\alpha k^2 < 4\delta^2$ and $\lambda\delta < s\alpha$. No fixed points in $\{x > w\}$. On the left: initial point $P = [2, 1, 1, 1]$. On the right: initial point $P = [5, 5, 5, 1]$..

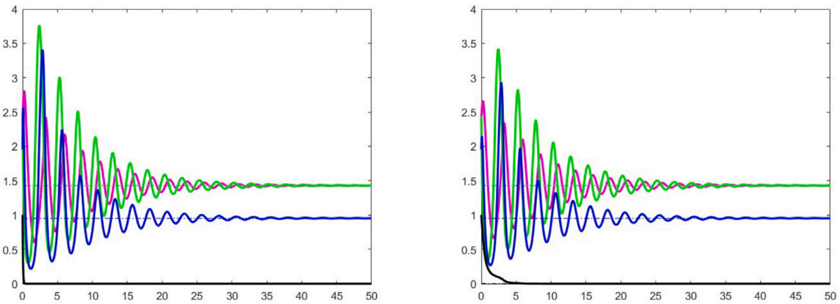

Fig. 10. Study of the influences of the parameters. Case $\delta_1 \neq \delta_2$. Scenario around the equilibrium point P_0 in the situation of stability of Fig. 2 (case on the left). On the left: $\delta_1 = 3$, $\delta_2 = 7$. On the right: $\delta_1 = 3$, $\delta_2 = 1$.

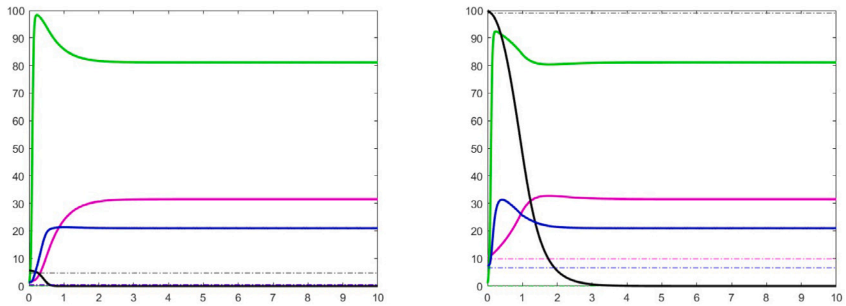

Fig. 11. Study of the influences of the parameters. Case $\delta_1 \neq \delta_2$. Scenario around the equilibrium point P_2 in the situation of non stability of Fig. 5 (case on the right). On the left: $\delta_1 = 0.3$, $\delta_2 = 0.8$. On the right: $\delta_1 = 0.3$, $\delta_2 = 0.1$.

6. Open problems and future research

The work we have done in this paper can be pursued in several directions, as in the following suggestions:

- A first step would be to divide the population in Commoners and Elites, as in the original HANDY model.
- Another interesting problem would be the search for general stability results for all of the equilibria, This analysis could start with some hypotheses suggested by the results obtained above. For example, Propositions 25 and 26 suggest the guess that the equilibrium P_2 is stable when $4\alpha\zeta > k(1 + s\sqrt{\alpha})$, while, if $4\alpha\zeta < k(1 + s\sqrt{\alpha})$, there exists $\delta_1 \in (0, \delta_c)$ such that P_2 is stable if $\delta \in (0, \delta_1)$, unstable if $\delta \in (\delta_1, \delta_c)$. Some similar guess could be done for P_4, starting from the examples in the last part of Section 4: one could try to prove that P_4 is always stable for large α's and δ's, unstable for small α's and large δ's.
- When we have a result of unstability of an equilibrium point, it would be interesting to know what is the asymptotic behavior of the single components of trajectories, in particular population and wealth. In the present paper, we were able to do this just in some particular cases.
- Another interesting research line would be to change the equation for z', inserting some more pessimistic hypotheses, for example a term like $\frac{xw}{(xw)^2+1}$ instead of $\frac{xw}{xw+1}$.

- The numerical simulations (figure 3) suggest the existence of periodical solutions, and the numerical computations of the complex eigenvalues show a change of sign of real parts. This facts point to the possibility of a Hopf bifurcation. It would be interesting to prove it rigorously.
- Fig. 2 seems to suggest the onset of chaotic dynamics, the study of which should be developed.

Acknowledgments

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