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On blowups of vorticity for the homogeneous Euler equation

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Abstract

Blowups of vorticity for the three- and two-dimensional homogeneous Euler equations are studied. Two regimes of approaching a blow-up point, respectively, with variable or fixed time are analyzed. It is shown that in the n-dimensional ($n = 2, 3$) generic case the blowups of degrees $1, \cdots, n$ at the variable time regime and of degrees $1/2, \cdots, (n+1)/(n+2)$ at the fixed time regime may exist. Particular situations when the vorticity blows while the direction of the vorticity vector is concentrated in one or two directions are realizable.

KEYWORDS

gradient catastrophes, homogeneous Euler equations, vorticity evolution

1 | INTRODUCTION

Vorticity and associated phenomena are among the most studied subjects in hydrodynamics (see, e.g., Refs. [1–5] and the other papers[6–15]). A number of approaches and different techniques have been developed. Most of the studies of the blowups of vorticity have been performed for the ideal incompressible fluid. The compressible case is considered as the much more complicated one (see, e.g., Refs. [1–8]).

In the papers of Chefranov[9] and Kuznetsov,[14] it was observed that in the case of compressible fluid the behavior of vorticity for the Euler equation is intimately connected with that of the

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homogeneous Euler equation (HEE)

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = 0 . \tag{1}$$

without the constraint $\nabla \cdot \mathbf{u} = 0$. In papers,[9,10] an explicit integral-type formula for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ for Equation (1) has been presented. Another type of formula for the vorticity has been found in Kuznetsov's studies.[11,12] The blowup of vorticity as $t \to t_c > 0$ has been analyzed in Kuznetsov and Ruban[12,14] (see also Chefranov and Chefranov[10] and Kuznetsov and Mikhailov[15]).

HEE (1) is the most simplified version of the basic equations of the hydrodynamics when one can neglect all effects of pressure, viscosity, etc. Nevertheless, it has a number of applications in physics and represents itself as an excellent touchstone for an analysis of blowups of vorticity.

In this paper, we present some results concerning the blowups of vorticity for the three- and two-dimensional HEE (1). Our analysis is based in part on the previous study of the structure and hierarchies of blowups of derivatives for the n-dimensional HEE.[16,17]

We consider the behavior of vorticity in two different regimes of approaching the blow-up points at the blow-up hypersurface. The first regime is to approach such a point along the t axis, that is, $t \to t_b$ while the coordinates $\mathbf{u}$ in the hodograph space remain fixed. It is shown that, in the generic case, that is, for generic initial data for the three-dimensional (3D) HEE (1), the vorticity in this regime may have singularities of three different degrees

$$\omega_i \sim (t - t_b)^{-m} , \qquad t \to t_b, \qquad m = 1, 2, 3 . \tag{2}$$

Such blowups occur on the intersection of m branches of the blow-up hypersurface Γ. The existence of blowups of type (2) with $m = 1, 2$ has been observed earlier in Kuznetsov and Ruban.[12]

In the second regime, the time t_b is fixed while the coordinates $\mathbf{u}$ are varying. In this regime of approaching the blow-up point for 3D HEE (1) generically, there may exist four levels of blowups of the vorticity $\boldsymbol{\omega}$ with the behavior

$$\omega_i \sim \varepsilon^{-\frac{m}{m+1}} , \qquad m = 1, 2, 3, 4 , \tag{3}$$

where $\varepsilon \sim |\delta \mathbf{x}| \to 0$. Blowups (3) occur on the subspaces Γ_m of the blow-up hypersurface Γ and $\dim \Gamma_m = 4 - m$, $m = 1, 2, 3, 4$.

It may happen also that the components of the vorticity $\boldsymbol{\omega}$ behave differently on certain subspaces of Γ. In particular, at the first level $m = 1$ there may exist one-dimensional subspace Γ_1 at which the component ω_3 blows as $\varepsilon^{-1/2}$ when $\varepsilon \to 0$ while the components ω_1 and ω_2 remain bounded.

In such a case, the direction of the vorticity $\hat{\boldsymbol{\omega}}$ is a unit vector oriented along one axis, namely

$$\hat{\boldsymbol{\omega}} = (0, 0, 1) . \tag{4}$$

The calculations are performed both in the special coordinates introduced in Konopelchenko and Ortenzi[17] as well as in Cartesian coordinates $\mathbf{x}$ and $\mathbf{u}$.

For the two-dimensional (2D) HEE (1), the vorticity blows up as in (2) and (3) with m taking the values $m = 1, 2$ for (2) and $m = 1, 2, 3$ for (3), respectively. Three particular solutions of the 2D HEE with different blow-up behavior are considered.

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It is noted that we analyze the behavior of vorticity at certain points on the blow-up hypersurface Γ and at the time t_b which can be negative or positive. The realizability of blowups of different orders at positive time remains an open problem.

Similar results for the n-dimensional HEE are briefly discussed too.

The paper is organized as follows. Section 2 contains a brief exposition of the results of Konopelchenko and Ortenzi[17] for the 3D HEE. Blowups of vorticity in the first regime $t \to t_b$ are analyzed in Section 3. Blowups of vorticity for the 3D HEE in the regime with fixed t are studied in Sections 4 and 5. Similar results for the 2D HEE are presented in Section 6.

Three particular solutions of the 2D HEE with different blow-up behavior are described in detail in Section 7. The n-dimensional $n \geq 4$ case is discussed in Section 8. Conclusion 9 contains some indications on possible future developments.

2 | BLOWUPS OF DERIVATIVES

Here, for convenience, we report some results concerning the blowup of derivatives for the 3D HEE obtained in Konopelchenko and Ortenzi.[17] We also slightly change the notations in order to make the corresponding formulas more convenient for the further calculations.

The starting point of the analysis is the hodograph equations[9,16,18,19]

$$x_i = u_i t + f_i(\mathbf{u}), \qquad i = 1, 2, 3 \tag{5}$$

where $f_i(\mathbf{u})$ are arbitrary functions locally inverse to the initial data $u_i(t = 0, \mathbf{x})$. Any solution $\mathbf{u}(\mathbf{x}, t)$ of the system (5) is a solution of the 3D system HEE (1).

The matrix M with the elements

$$M_{ij} = t\delta_{ij} + \frac{\partial f_i}{\partial u_j}, \qquad i, j = 1, 2, 3, \tag{6}$$

plays a central role in the analysis of blowups of derivatives and possible gradient catastrophes. In particular,

$$\frac{\partial u_j}{\partial x_k} = (M^{-1})_{jk}, \qquad i, k = 1, 2, 3 \tag{7}$$

The blowups occur on the 3D hypersurface Γ defined by the equation

$$\det M(t; \mathbf{u}) = t^3 + a_2(\mathbf{u})t^2 + a_1(\mathbf{u})t + a_0(\mathbf{u}) = 0, \tag{8}$$

where $a_2(\mathbf{u})$, $a_1(\mathbf{u})$ are certain functions of $\mathbf{u}$ and $a_0(\mathbf{u}) = \det(M(t = 0, \mathbf{u})) \neq 0$ for generic initial data.

The blow-up hypersurface Γ is the union of the branches $t_\alpha = \phi_\alpha(\mathbf{u})$ corresponding to real roots of the cubic equation (8). In the 3D case, the number of branches can be one or three[17]

In the generic case, the rank r of the matrix M may assume two values $r = 2$ and $r = 1$. Equivalently, it means that there exists $3 - r$ vectors $\mathbf{R}^{(\alpha)}(\mathbf{u}_b)$ and $\mathbf{L}^{(\alpha)}(\mathbf{u}_b)$, $\alpha = 1, 3 - r$ such that

$(\mathbf{u}_b \in \Gamma)$

$$\sum_{j=1}^{3} M_{ij} R_j^{(\alpha)} = 0\,, \qquad i = 1,2,3\,, \quad \alpha = 1, 3-r\,,$$

$$\sum_{i=1}^{3} L_i^{(\alpha)} M_{ij} = 0\,, \qquad j = 1,2,3\,, \quad \alpha = 1, 3-r\,. \tag{9}$$

The existence of such vectors suggests the introduction of new dependent and independent variables v_1, v_2, v_3 and y_1, y_2, y_3 defined by the relations[17]

$$\delta \mathbf{u} \equiv \sum_{\alpha=1}^{3-r} \mathbf{R}^{(\alpha)} \delta v_\alpha + \sum_{\beta=1}^{r} \tilde{\mathbf{R}}^{(\beta)} \delta v_{\beta+3-r} \equiv \sum_{\alpha=1}^{3} \mathcal{R}^{(\alpha)} \delta v_\alpha\,,$$

$$\delta \mathbf{x} \equiv \sum_{\alpha=1}^{3-r} \mathbf{P}^{(\alpha)} \delta y_\alpha + \sum_{\beta=1}^{r} \tilde{\mathbf{P}}^{(\beta)} \delta y_{\beta+3-r} \equiv \sum_{\alpha=1}^{3} \mathcal{P}^{(\alpha)} \delta y_\alpha\,, \tag{10}$$

where the vectors $\tilde{\mathbf{R}}^{(\beta)}$ are r vectors complementary to the set of $3-r$ vectors $\mathbf{R}^{(\alpha)}$ and vectors $P^{(\alpha)}, \tilde{P}^{(\beta)}$ are defined by the relation

$$\sum_{\alpha=1}^{3-r} P_i^{(\alpha)} L_j^{(\alpha)} + \sum_{\beta=1}^{r} \tilde{P}_i^{(\beta)} \tilde{L}_j^{(\beta)} = \delta_{ij}\,, \qquad i,j = 1,2,3 \tag{11}$$

where $\tilde{\mathbf{L}}^{(\beta)}$ are r vectors complementary to the set of $3-r$ vectors $\mathbf{L}^{(\alpha)}$. One also has

$$\delta y_\beta = \mathcal{L}^{(\beta)} \cdot \delta \mathbf{x} = \sum_{i,j=1}^{3} \mathcal{L}_i^{(\beta)} M_{ij}(\mathbf{u}_b) \mathcal{R}_j^{(\alpha)} \delta v_\alpha + O(|\delta v|^2)\,, \tag{12}$$

where $\sum_{\alpha=1}^{3} \mathcal{P}_i^{(\alpha)} \mathcal{L}_j^{(\alpha)} = \delta_{ij}$.

The use of variational consequences of the hodograph Equations (5) shows that derivatives $\frac{\partial v_\alpha}{\partial y_\beta}(\mathbf{u}_b)$ behave differently in different subsectors of the independent and dependent variables.[16,17] For instance, for $r = 2$, on the first level of blowups, the derivatives

$$\frac{\partial v_1}{\partial y_1}, \frac{\partial v_1}{\partial y_2}, \frac{\partial v_1}{\partial y_3}, \frac{\partial v_2}{\partial y_1}, \frac{\partial v_3}{\partial y_1} \tag{13}$$

explode on the hypersurface Γ (8) while the derivatives

$$\frac{\partial v_2}{\partial y_2}, \frac{\partial v_2}{\partial y_3}, \frac{\partial v_3}{\partial y_2}, \frac{\partial v_3}{\partial y_3} \tag{14}$$

remain bounded. These blowups may happen both at a positive and negative time.

It is noted that all vectors given above and the behavior of derivatives $\frac{\partial v_\alpha}{\partial y_\beta}$ vary with the variation of the point $\mathbf{u}_b$ belonging to the hypersurface Γ (8).

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On the first level of blowups, the derivatives explode as $\varepsilon^{-1/2}$, $\varepsilon \sim |\delta y| \to 0$ and the behavior of derivatives at fixed time t_b presented in (13) and (14) can be resumed in the formula

$$\delta v_\alpha \sim \sum_{j=1}^{3} C_{\alpha\beta}\delta y_\beta\,, \qquad i = 1,2,3\,, \tag{15}$$

where

$$C = \begin{pmatrix} \varepsilon^{-1/2}\nu_{11} & \varepsilon^{-1/2}\nu_{12} & \varepsilon^{-1/2}\nu_{13} \\ \varepsilon^{-1/2}\nu_{12} & \nu_{22} & \nu_{23} \\ \varepsilon^{-1/2}\nu_{13} & \nu_{32} & \nu_{33} \end{pmatrix} \tag{16}$$

and $\nu_{ij}, i,j = 1,2,3$ are connected with the values of $\frac{\partial f_i}{\partial u_j}(\mathbf{u}_b)$ and $\frac{\partial^2 f_i}{\partial u_j \partial u_k}(\mathbf{u}_b)$ evaluated at the point $\mathbf{u}_b \in \Gamma_1$ (see Konopelchenko and Ortenzi[17]).

We emphasize that the formulae (16) represent the relations between the infinitesimal variations of the variables y_i and v_i around a point $\mathbf{u}_b \in \Gamma$ at fixed time t_b. Blow-up time t_b can be positive or negative. Blowup at $t_b > 0$ is referred to as the gradient catastrophe. In this paper, as in Konopelchenko and Ortenzi,[17] we will not discuss conditions that guarantee that $t_b > 0$.

It is also noted the domain $\mathcal{D}_{\mathbf{u}}$ of variations of $\mathbf{u}$ constructed via Equation (5) and, consequently, the domain of variations of variables $\mathbf{u}$ parameterizing the blow-up hypersurface Γ (8),

$$\mathcal{D}_{\mathbf{u}} \equiv \{\mathbf{u} \,:\, \det M(t,\mathbf{u}) = 0\}\,, \tag{17}$$

coincides with the domain $\mathcal{D}_{\mathbf{u}_0}$ of variations of the initial values $\mathbf{u}_0$, since $\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x} - \mathbf{u}t)$.

3 | BLOWUP OF VORTICITY

The formula (7) provides us with the explicit and useful expression for the vorticity vector in the original Cartesian coordinates in terms of the components u_i, $i = 1,2,3$ of the velocity. Namely,

$$\omega_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \sum_{j,k=1}^{3} \varepsilon_{ijk}(M^{-1})_{kj} = \frac{1}{\det(M(t,\mathbf{u}))}\sum_{j,k=1}^{3} \varepsilon_{ijk}\tilde{M}_{kj}(t,\mathbf{u})\,, \qquad i = 1,2,3 \tag{18}$$

where $\tilde{M}$ is the adjugate matrix.

We consider first the case $\mathrm{rank}(M(t_b,\mathbf{u}_b)) = 2$. Let us fix the point $\mathbf{u}_b$ on the blow-up hypersurface Γ (8) and take the corresponding real t_b, that is, the real root of the cubic Equation (8), which always exists for the 3D HEE.[16] The formula (18) implies that (Konopelchenko and Ortenzi[16])

$$\omega_i(t = t_b + \varepsilon)|_\Gamma = \frac{\sum_{j,k=1}^{3}\varepsilon_{ijk}\left(\tilde{M}_{kj}(t_b,\mathbf{u}_b) + \varepsilon\tilde{M}'_{kj}(t_b,\mathbf{u}_b) + O(\varepsilon^2)\right)}{\varepsilon D_1(t_b,\mathbf{u}_b) + \varepsilon^2 D_2(t_b,\mathbf{u}_b) + \varepsilon^3}\,, \qquad \varepsilon \equiv t - t_b \to 0 \tag{19}$$

where

$$D_1 \equiv \frac{\partial \det(M(t,\mathbf{u}))}{\partial t}|_{t_b,\mathbf{u}_b} = 3t_b^2 + 2a_2(\mathbf{u}_b)t_b + a_1(\mathbf{u}_b),$$

$$D_2 \equiv \frac{\partial^2 \det(M(t,\mathbf{u}))}{\partial t^2}|_{t_b,\mathbf{u}_b} = 3t_b + a_2(\mathbf{u}_b), \tag{20}$$

and $\tilde{M}'_{kj}(t_b, \mathbf{u}_b) \equiv \frac{\mathrm{d}\widetilde{M_{kj}}(t,\mathbf{u})}{\mathrm{d}t}|_{t_b,\mathbf{u}_b}$. Generically for $r = 2$ $\tilde{M}_{jk}(t_b, \mathbf{u}_b) \neq 0$ and $D_1(t_b, \mathbf{u}_b) \neq 0$. Hence, in the generic case, in the first regime the vorticity blows up on the full hypersurface Γ as

$$\omega_i(t, \mathbf{u}_b) \sim \sigma_i \varepsilon^{-1} \equiv \sigma_i (t - t_b)^{-1}, \qquad t \to t_b, \quad i = 1, 2, 3 \tag{21}$$

where $\sigma_i \equiv \sum_{j,k=1}^{3} \varepsilon_{ijk} \tilde{M}_{kj}(t_b, \mathbf{u}_b)/D_1(t_b, \mathbf{u}_b)$ for $i = 1, 2, 3$.

The existence of the higher order singularities is correlated with the structure of the blow-up hypersurface Γ. If it has a single branch (single real root of Equation (8)) then $M'(t_b, \mathbf{u}_b)$ cannot be zero. Hence, due to (19) and (20) in this case only the blowup of type (21) occurs.

Situation is different when Γ has three real branches, that is, all roots of Equation (8) are real. In this case, one has the formulae (19) and (20) and three different values of t_{b_α}, $\alpha = 1, 2, 3$ for the same value $\mathbf{u}_b$. Moreover, the condition

$$\frac{\partial \det(M(t_b, \mathbf{u}_b))}{\partial t} = 0, \tag{22}$$

that is, the condition that $\det(M(t_b, \mathbf{u}_b))$ has a double zero at t_b is now admissible.

Let the condition

$$D_1(t_b, \mathbf{u}_b) = 3t_b^2 + 2a_2(\mathbf{u}_b)t_b + a_1(\mathbf{u}_b) = 0 \tag{23}$$

be satisfied at one branch. It defines the 2D submanifold $\mathcal{D}_{\mathbf{u}}^{(2)}$ at $\mathcal{D}_{\mathbf{u}}$. At fixed $\mathbf{u}_b \in \mathcal{D}_{\mathbf{u}}^{(2)}$ and at the corresponding t_{b_α}, the vorticity blows up as

$$\omega_i(t, \mathbf{u}_b) \sim \varepsilon^{-2} \equiv (t - t_b)^{-2}, \qquad t \to t_b. \tag{24}$$

Moreover, condition (23) (cf. (20)) means that the root t_{b_α} is a double root, that is, coincides with another root t_{b_β}. So, the branches α and β of the blow-up hypersurface Γ intersect along the 2D surface Γ_2 corresponding to values of $\mathbf{u}_b \in \mathcal{D}_{\mathbf{u}}^{(2)}$ and on Γ_2 the vorticity blows up as in (24).

Hence, in the particular case (23), the vorticity $\boldsymbol{\omega}$ blows up as $(t - t_b)^{-2}$ on the intersection of two branches of Γ and blows up as $(t - t_b)^{-1}$ on the third branch.

Finally if, in addition to (23), the condition

$$D_2(t_b, \mathbf{u}_b) = 3t_b + a_2(\mathbf{u}_b) = 0, \tag{25}$$

is satisfied, but $\sum_{j,k=1}^{3} \varepsilon_{ijk} \tilde{M}_{kj}(t_b, \mathbf{u}_b) \neq 0$, with $i = 1, 2, 3$ then the vorticity $\boldsymbol{\omega}$ blows up as

$$\omega_i(t, \mathbf{u}_b) \sim \varepsilon^{-3} \equiv (t - t_b)^{-3}, \qquad t \to t_b. \tag{26}$$

The situation (26) happens on the curve Γ_3 in $\mathcal{D}_{\mathbf{u}}$ defined by the conditions (23) and (25). Since such conditions mean that the root t_{b_α} is a triple root, the behavior (26) occurs at the intersection of all three branches of the blow-up surface Γ.

Other possible situations, for instance, the condition $D_1(t_{b_\alpha}, \mathbf{u}_b) = 0$ for all $\alpha = 1, 2, 3$ are equivalent to (25) and (26).

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The existence of the blowups of the types (21), (24), and (26) becomes rather obvious if one rewrites the formula (8) as

$$\det(M(t,\mathbf{u})) = (t - t_{b_1})(t - t_{b_2})(t - t_{b_3})\,. \tag{27}$$

It is noted that one can treat the conditions (23) and (25) in a different manner, namely, to consider them as the equations for the functions $f_1(\mathbf{u}), f_2(\mathbf{u}), f_3(\mathbf{u})$. Within such a viewpoint, Equation (23) defines those functions $f_i(\mathbf{u}), i = 1,2,3$ for which two branches of the hypersurface Γ identically coincide. All three branches of Γ coincide in the particular case of initial data such that the functions $f_i(\mathbf{u}), i = 1,2,3$ are solutions of the pair of Equations (23) and (25).

The formulae (21) and (24) reproduce the results previously obtained in Kuznetsov and Ruban[12] with the use of the Lagrangian analog of the formula (18). The behavior of type (26) was not present in Kuznetsov and Ruban[12] due to the particular geometry of the vortex lines considered there.

An analysis of the behavior of vorticity and its integral characteristics has also been performed in Chefranov and Chefranov[10] with the use of an explicit integral representation of the Lagrangian type derived in Chefranov.[9]

The components ω_i behave according to (21), (24), and (26) in the general case when all $\sigma_i \neq 0$. In this case, the direction of the vorticity vector (see, e.g., Constantin and Fefferman[7])

$$\hat{\boldsymbol{\omega}} \equiv \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \tag{28}$$

is regular with components

$$\hat{\boldsymbol{\omega}} = \frac{1}{|\sigma|}(\sigma_1, \sigma_2, \sigma_3)\,, \qquad |\sigma|^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2\,. \tag{29}$$

Let us assume now that one of σ_i vanishes, for example, σ_3, that is,

$$\sum_{j,k=1}^{3} \varepsilon_{3jk}\tilde{M}_{kj}(t_b, \mathbf{u}_b) = 0\,. \tag{30}$$

This condition defines the 2D subspace $\mathcal{D}_2 \subset \mathcal{D}_{\mathbf{u}}$ in the hodograph space. At the points $\mathbf{u} \in \mathcal{D}_2$, one has $\sigma_3 = 0$ and, hence instead of (21), the vorticity vector direction blows up as as

$$\omega_1 \sim \sigma_1 (t - t_b)^{-m}\,, \qquad \omega_2 \sim \sigma_2 (t - t_b)^{-m}\,, \qquad \omega_3 \sim \sigma_3' (t - t_b)^{-m+1}\,, \qquad m = 1,2,3\,, \tag{31}$$

where $\sigma_i' \equiv \sum_{j,k=1}^{3} \varepsilon_{ijk}\tilde{M}_{kj}'(t_b, \mathbf{u}_b)$ for $i = 1,2,3$. Consequently, the vector $\hat{\boldsymbol{\omega}}$ is of the form

$$\hat{\boldsymbol{\omega}} = \frac{1}{|\boldsymbol{\sigma}|}(\sigma_1, \sigma_2, 0)\,. \tag{32}$$

Generically, for $m = 1$ such a situation may occur on the 2D subsurface of the blow-up hypersurface Γ. For $m = 2$, it may happen along the curve belonging to the 2D intersection of two branches of Γ. For $m = 3$, it may occur at the point belonging to the curve of the intersection of the three branches of Γ.

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In the very particular case of two vanishing components of σ_i, for example, $\sigma_1 = \sigma_2 = 0$, one has

$$\omega_1 \sim \sigma_1'(t-t_b)^{-m+1}, \qquad \omega_2 \sim \sigma_2'(t-t_b)^{-m+1}, \qquad \omega_2 \sim \sigma_3(t-t_b)^{-m}, \qquad m = 1,2,3, \tag{33}$$

and

$$\hat{\omega} = (0,0,1). \tag{34}$$

Generically, such behavior may exist only for $m = 1, 2$. For $m = 1$ it may happen along a curve on Γ, while for $m = 2$ it may occur at the point belonging to the intersection of two branches of Γ.

The behavior of vorticity described above corresponds to the case of rank $r = 2$ for the matrix M evaluated on the blow-up hypersurface Γ. It occurs on the whole blow-up hypersurface.[17] In contrast, the matrix $M(t_b, \mathbf{u}_b)$ may have rank 1 only on a set of points Γ_0 on Γ.[17] Moreover, for $r = 1$, the adjugate matrix $\tilde{M}$ vanishes identically:

$$\tilde{M}_{ij}|_{\Gamma_0} = 0, \qquad i,j = 1,2,3. \tag{35}$$

On the other hand, generically, $\tilde{M}'_{ij}|_{\Gamma_0}$ are different from zero. So, in such a situation the components of vorticity remain bounded when t is approaching t_b which corresponds to a point $\mathbf{u}_b$ belonging to Γ_0.

4 | BLOWUPS OF VORTICITY AT FIXED TIME

The formulae (21), (24), and (26) describe the behavior of the vorticity in the situation when time t approaches the blow-up time t_b along the t axis with fixed coordinate $\mathbf{u}_b$.

The approach presented in Konopelchenko and Ortenzi[17] and briefly reproduced in Section 2 looks more appropriate for the analysis of the blowups of vorticity in the regime when time t is fixed while the coordinates $\mathbf{u}$ are subject to variations.

The formulas presented in Section 2 (see also Konopelchenko and Ortenzi[17]) indicate that non-Cartesian coordinates y_i and v_i, $i = 1, 2, 3$, are rather convenient for the analysis of blowups of the derivatives. In order to use such coordinates for the analysis of blowups of vorticity, one has to consider its coordinate-independent definition as the differential two-form (see, e.g., Refs. [8, 20])

$$\omega = \mathrm{d}\theta = \mathrm{d}\mathbf{u} \wedge \mathrm{d}\mathbf{x}. \tag{36}$$

where $\theta = \mathbf{u} \cdot \mathrm{d}\mathbf{x}$.

We will use such definition in the form

$$\omega(\mathbf{u}_b) = \delta\mathbf{u} \wedge \delta\mathbf{x} \tag{37}$$

to study the behavior of vorticity at the point $\mathbf{u}_b$ of the blow-up hypersurface Γ.

Using the formulae (10), one gets

$$\omega(\mathbf{u}_b) \equiv \sum_{\alpha,\beta=1}^{3} q_{\alpha\beta}\delta v_\alpha \wedge \delta y_\beta, \tag{38}$$

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where

$$q_{\alpha\beta} \equiv \mathcal{R}^{(\alpha)} \cdot \mathcal{P}^{(\beta)}\,, \qquad \alpha, \beta = 1, 2, 3\,. \tag{39}$$

Then, due to the relation (15), at the blow-up point $\mathbf{u}_b$ one obtains

$$\omega(\mathbf{u}_b) = \sum_{\alpha,\beta=1}^{3} \omega_{\alpha\beta}(\mathbf{u}_b)\delta y_\alpha \wedge \delta y_\beta, \tag{40}$$

where

$$\omega_{\alpha\beta}(\mathbf{u}_b) \equiv \frac{1}{2}\sum_{\gamma=1}^{3}(C_{\gamma\alpha}q_{\gamma\beta} - C_{\gamma\beta}q_{\gamma\alpha})\,, \qquad \alpha, \beta = 1, 2, 3\,. \tag{41}$$

The components of the vorticity vector $\boldsymbol{\omega}$ in these coordinates are defined as usual as

$$\omega_\alpha = \sum_{\beta,\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma}\omega_{\beta\gamma}\,, \qquad \beta, \gamma = 1, 2, 3\,. \tag{42}$$

At the first level of blowup and rank $r = 2$, the matrix C is of the form (16). Consequently, the element of $\omega_{\alpha\beta}$, written in terms of the vorticity components ω_i, behaves as

$$\omega = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{43}$$

where

$$\begin{aligned} \omega_1 &= \varepsilon^{-1/2}S_1(\mathbf{u}_b) + T_1(\mathbf{u}_b)\,, \\ \omega_2 &= \varepsilon^{-1/2}S_2(\mathbf{u}_b) + T_2(\mathbf{u}_b)\,, \\ \omega_3 &= \varepsilon^{-1/2}S_3(\mathbf{u}_b) + T_3(\mathbf{u}_b)\,, \end{aligned} \tag{44}$$

as $\varepsilon \to 0$ and

$$\begin{aligned} S_1 &= (\nu_{12}q_{13} - \nu_{13}q_{12})\,, \qquad T_1 = (\nu_{22}q_{23} - \nu_{23}q_{22} + \nu_{32}q_{33} - \nu_{33}q_{32})\,, \\ S_2 &= (\nu_{11}\,(-q_{13}) + \nu_{13}q_{11} - \nu_{21}q_{23} - \nu_{31}q_{33})\,, \qquad T_2 = (\nu_{23}q_{21} + \nu_{33}q_{31})\,, \\ S_3 &= (\nu_{11}q_{12} - \nu_{12}q_{11} + \nu_{21}q_{22} + \nu_{31}q_{32})\,, \qquad T_3 = (-\nu_{22}q_{21} - \nu_{32}q_{31})\,. \end{aligned} \tag{45}$$

So, generically, that is, when all $S_\alpha \neq 0$, the vorticity $\boldsymbol{\omega}$ blows up as $\varepsilon^{-1/2}$, $\varepsilon \to 0$ at the point $\mathbf{u}_b$ of the 3D blow-up hypersurface Γ. In this case, the direction of the vorticity vector $\hat{\boldsymbol{\omega}}$ is regular with the components

$$\hat{\omega} = \frac{1}{|\mathbf{S}|}\,(S_1(\mathbf{u}_b), S_2(\mathbf{u}_b), S_3(\mathbf{u}_b))\,. \tag{46}$$

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However, particular situations are also admissible. Indeed, if there exists a point $\mathbf{u}_b \in \Gamma$ such that $S_3(\mathbf{u}_b) = 0$ then at this point the components ω_1 and ω_2 of the vorticity blow up while the component ω_3 remains finite. The condition $S_3(\mathbf{u}_b) = 0$ has a codimension one. So, such a situation is realizable, in principle, on the 2D subsurface of the blow-up hypersurface Γ and $\hat{\boldsymbol{\omega}}$ is of the form

$$\hat{\boldsymbol{\omega}} = \frac{1}{|\mathbf{S}|}\,(S_1(\mathbf{u}_b), S_2(\mathbf{u}_b), 0)\ . \tag{47}$$

Further, there may exist the points belonging to a certain curve on Γ at which

$$S_1(\mathbf{u}_b) = S_2(\mathbf{u}_b) = 0\,. \tag{48}$$

At these points, the components ω_1 and ω_2 remain bounded and only one component ω_3 of the vorticity blows up. Hence, the vorticity direction vector (28) assumes a particular form

$$\hat{\boldsymbol{\omega}} = (0, 0, 1)\ . \tag{49}$$

Such a situation when the vorticity vector $\boldsymbol{\omega}$ becomes very large in modulus, but concentrated in one direction looks rather special and of interest.

It may even happen at a certain point $\mathbf{u}_b \in \Gamma$ that

$$S_1(\mathbf{u}_b) = S_2(\mathbf{u}_b) = S_3(\mathbf{u}_b) = 0\,. \tag{50}$$

In such a case, the vorticity $\boldsymbol{\omega}$ remains bounded at the point of the first-level blowups of derivatives.

Finally, in order to analyze the blowup of vorticity in the Cartesian coordinates, it is sufficient to perform the change of coordinates $\mathbf{y} \to \mathbf{x}$ on the right-hand side of (40).

Performing the transformation (12) in (40), one obtains

$$\omega(\mathbf{u}_b) = \sum_{i,j=1}^{3} \omega_{ij}(\mathbf{u}_b)\delta x_i \wedge \delta x_j\ . \tag{51}$$

As a result, the components $\omega_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$ of the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ blow up on the whole hypersurface Γ, namely

$$\omega_i = \varepsilon^{-1/2}\tilde{S}_i(\mathbf{u}_b) + \tilde{T}_i(\mathbf{u}_b)\,, \qquad i = 1, 2, 3\,, \qquad \varepsilon \to 0\,, \tag{52}$$

where $\tilde{S}_i$ and $\tilde{T}_i$ are bounded functions obtained by a change of variables from (45).

The same result can be obtained directly, using the formulae (10), (15), and (12). Namely, one gets

$$\frac{\partial u_l}{\partial x_k} = \sum_{\beta=1}^{3} \frac{\partial u_l}{\partial y_\beta}\frac{\partial y_\beta}{\partial x_k} = \sum_{\alpha,\beta=1}^{3} \mathcal{R}_l^{(\alpha)}\frac{\partial v_\alpha}{\partial y_\beta}\mathcal{L}_k^{(\beta)} = \sum_{\alpha,\beta=1}^{3} \mathcal{R}_l^{(\alpha)} C_{\alpha\beta}\mathcal{L}_k^{(\beta)}\,, \qquad l, k = 1, 2, 3\,, \tag{53}$$

and, then, one obtains the formula (52).

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Again, it may happen that along certain curves Γ_1 belonging to Γ, one has

$$\tilde{S}_1(\mathbf{u}_b) = \tilde{S}_2(\mathbf{u}_b) = 0\,. \tag{54}$$

At the points on this curve, the components ω_1 and ω_2 remain bounded while the components $\omega_3 \to \infty$ and $\widehat{\omega} = (0,0,1)$. Such a situation, when the vorticity vector $\boldsymbol{\omega}$ becomes very large in modulus but concentrated in one direction, resembles somehow certain well-known physical phenomena.

5 | BLOWUPS AT RANK 1 AND HIGHER LEVELS

In the case of rank $r = 1$, which occurs at a set of points $\mathbf{u}_b \in \Gamma$ the matrix C is of the form (cf. Konopelchenko and Ortenzi[17])

$$C = \begin{pmatrix} \varepsilon^{-1/2}\mu_{11} & \varepsilon^{-1/2}\mu_{12} & \varepsilon^{-1/2}\mu_{13} \\ \varepsilon^{-1/2}\mu_{12} & \varepsilon^{-1/2}\mu_{22} & \varepsilon^{-1/2}\mu_{23} \\ \varepsilon^{-1/2}\mu_{13} & \varepsilon^{-1/2}\mu_{32} & \mu_{33} \end{pmatrix}. \tag{55}$$

The components of the vorticity vector $\boldsymbol{\omega}$ again are of the form (45) or (52).

However, in this case, one cannot impose any constraint of the type $S_3 = 0$ or (48), if one considers the situation with generic function $f_i(\mathbf{u})$ of initial data. Such constraints may be admissible for particular special initial data. Blowups of the second, third, and fourth levels for $r = 2$ occur on certain subspaces of the 3D blow-up hypersurface Γ.[17]

One of the subsections of the second level of blowups (in the rank 2 case) is characterized by the following behavior of derivatives[17]:

$$\frac{\partial v_1}{\partial y_1} \sim \varepsilon^{-2/3}\,, \qquad \frac{\partial v_1}{\partial y_2}, \frac{\partial v_2}{\partial y_1}, \frac{\partial v_1}{\partial y_3}, \frac{\partial v_3}{\partial y_1} \sim \varepsilon^{-1/2}\,, \qquad \frac{\partial v_2}{\partial y_2}, \frac{\partial v_3}{\partial y_3}, \frac{\partial v_2}{\partial y_3}, \frac{\partial v_3}{\partial y_3} \sim O(1)\,, \qquad \varepsilon \to 0\,, \tag{56}$$

which corresponds to a matrix C given by

$$C = \begin{pmatrix} \varepsilon^{-2/3}\eta_{11} & \varepsilon^{-1/2}\eta_{12} & \varepsilon^{-1/2}\eta_{13} \\ \varepsilon^{-1/2}\eta_{21} & \eta_{22} & \eta_{23} \\ \varepsilon^{-1/2}\eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix}, \tag{57}$$

where η_{ij} are certain coefficients depending on $\frac{\partial f_i}{\partial u_j}(\mathbf{u}_b)$ and $\frac{\partial^2 f_i}{\partial u_j \partial u_k}(\mathbf{u}_b)$ evaluated at the point $\mathbf{u}_b$. Consequently, the components ω_i of the vorticity have the following behavior at the blow-up point of the second level:

$$\begin{aligned} \omega_1 &= \varepsilon^{-1/2}\tilde{S}_1(\mathbf{u}_b) + \tilde{T}_1(\mathbf{u}_b)\,, \\ \omega_2 &= \varepsilon^{-2/3}Y_2(\mathbf{u}_b) + \varepsilon^{-1/2}\tilde{S}_2(\mathbf{u}_b) + \tilde{T}_2(\mathbf{u}_b)\,, \\ \omega_3 &= \varepsilon^{-2/3}Y_3(\mathbf{u}_b) + \varepsilon^{-1/2}\tilde{S}_3(\mathbf{u}_b) + \tilde{T}_3(\mathbf{u}_b)\,, \qquad \varepsilon \to 0, \end{aligned} \tag{58}$$

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where $Y_i, \tilde{S}_i$, and $\tilde{T}_i$ are certain bounded functions of $\mathbf{u}_b \in \Gamma$. In this case, the direction of vorticity vector (28) is

$$\hat{\omega} = \frac{1}{|\mathbf{Y}|}\left(0, Y_2(\mathbf{u}_b), Y_3(\mathbf{u}_b)\right) , \tag{59}$$

where $|\mathbf{Y}|^2 = Y_2(\mathbf{u}_b)^2 + Y_3(\mathbf{u}_b)^2$.

So, in contrast to the first level (45), the components of the vorticity vector generically blow up in a different manner. Such realization occurs in the 2D subspace of the blow-up hypersurface Γ.[17] So, one can impose at most two constraints.

Under the constraint

$$\tilde{S}_1(\mathbf{u}_b) = 0, \tag{60}$$

one has the following behavior:

$$\boldsymbol{\omega} \sim \left(O(1), \varepsilon^{-2/3}, \varepsilon^{-2/3}\right), \qquad \varepsilon \to 0. \tag{61}$$

If instead

$$Y_2(\mathbf{u}_b) = 0 \tag{62}$$

then

$$\boldsymbol{\omega} \sim \left(\varepsilon^{-1/2}, \varepsilon^{-1/2}, \varepsilon^{-2/3}\right), \qquad \varepsilon \to 0. \tag{63}$$

and

$$\hat{\omega} = (0, 0, 1) . \tag{64}$$

The situations (60) and (62) may happen on curves belonging to Γ_2.

Imposing two constraints, one may have essentially two different situations. Indeed if

$$Y_2(\mathbf{u}_b) = Y_3(\mathbf{u}_b) = 0 \tag{65}$$

all components of vorticity blow up in the same manner, namely,

$$\boldsymbol{\omega} \sim \left(\varepsilon^{-1/2}, \varepsilon^{-1/2}, \varepsilon^{-1/2}\right), \qquad \varepsilon \to 0. \tag{66}$$

and the vorticity direction vector is a generic one. On the other hand, if it happens that

$$\tilde{S}_1(\mathbf{u}_b) = Y_2(\mathbf{u}_b) = 0 , \tag{67}$$

then the components of vorticity behave quite differently since

$$\boldsymbol{\omega} \sim \left(O(1), \varepsilon^{-1/2}, \varepsilon^{-2/3}\right), \qquad \varepsilon \to 0. \tag{68}$$

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In this case, the vorticity direction vector $\hat{\omega}$ is oriented along the third axis, namely

$$\hat{\omega} = (0, 0, 1) \ . \tag{69}$$

Such a situation is realizable in principle at the points of intersection of the curves defined by (62) and (60).

One observes similar behaviors of vorticity in other subsectors of the second level of blowups.

The third level of blowups is realizable on a curve belonging to Γ. Derivatives $\frac{\partial v_\alpha}{\partial y_\beta}$ behave similarly to (56) except that

$$\frac{\partial v_1}{\partial y_1} \sim \varepsilon^{-3/4} , \tag{70}$$

and, as a consequence, one has the behavior of the type (58) with the substitution $\varepsilon^{-2/3} \to \varepsilon^{-3/4}$ in the Y_i terms. In this case, one can impose, generically, only one constraint. For instance, if $Y_3(\mathbf{u}_b) = 0$ one has the following behavior of component of vorticity:

$$\boldsymbol{\omega} = (\varepsilon^{-1/2}, \varepsilon^{-3/4}, \varepsilon^{-1/2}) , \qquad \varepsilon \to 0 . \tag{71}$$

and

$$\hat{\omega} = (0, 1, 0) \ . \tag{72}$$

Finally, the fourth level may occur at a point on Γ and this point (see also Konopelchenko and Ortenzi[17])

$$\frac{\partial v_1}{\partial y_1} \sim \varepsilon^{-4/5} , \qquad \varepsilon \to 0. \tag{73}$$

Again, one has formula (58) with the substitution $\varepsilon^{-2/3} \to \varepsilon^{-4/5}$ in the first term on the right-hand side, and, generically, no constraints are allowed.

6 | VORTICITY FOR 2D HEE

For the 2D HEE, an analog of the formula (18) for the vorticity $\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ is given by

$$\omega_3(t, \mathbf{u}) = \frac{\frac{\partial f_1}{\partial u_2} - \frac{\partial f_2}{\partial u_1}}{t^2 + \mathrm{tr}(M_0)t + \det(M_0)} , \tag{74}$$

where $M_0 \equiv M(t = 0, \mathbf{u})$ is the matrix with components $(M_0)_{ij} = \frac{\partial f_i}{\partial u_j}$, $i, j = 1, 2$. The quadratic equation $t^2 + \mathrm{tr}(M_0)t + \det(M_0) = 0$, defining the blow-up surface Γ,[16] may have, obviously, either two real roots or no one, depending on the sign of the discriminant

$$\Delta(\mathbf{u}) = \left(\frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2}\right)^2 - 4\left(\frac{\partial f_1}{\partial u_1}\frac{\partial f_2}{\partial u_2} - \frac{\partial f_1}{\partial u_2}\frac{\partial f_2}{\partial u_1}\right) = \left(\frac{\partial f_1}{\partial u_1} - \frac{\partial f_2}{\partial u_2}\right)^2 + 4\frac{\partial f_1}{\partial u_2}\frac{\partial f_2}{\partial u_1} . \tag{75}$$

So, in contrast to the 3D HEE, in two dimensions there are solutions with blowups free vorticity (cf. Konopelchenko and Ortenzi[16]).

It is natural to consider subdomains $\mathcal{D}^+_{\mathbf{u}} \subset \mathcal{D}_{\mathbf{u}}$, and $\mathcal{D}^-_{\mathbf{u}} \subset \mathcal{D}_{\mathbf{u}}$ defined as follows:

$$
\begin{aligned}
(u_1, u_2) &\in \mathcal{D}^+_{\mathbf{u}}\,, && \text{if} \quad \Delta(u_1, u_2) > 0\,,\\
(u_1, u_2) &\in \mathcal{D}^-_{\mathbf{u}}\,, && \text{if} \quad \Delta(u_1, u_2) < 0\,,\\
(u_1, u_2) &\in \mathcal{D}^0_{\mathbf{u}}\,, && \text{if} \quad \Delta(u_1, u_2) = 0\,,
\end{aligned}
\tag{76}
$$

then

$$
\mathcal{D}_{\mathbf{u}} = \mathcal{D}^+_{\mathbf{u}} \cup \mathcal{D}^-_{\mathbf{u}} \cup \mathcal{D}^0_{\mathbf{u}}, \tag{77}
$$

and the curve $\mathcal{D}^0$ is the boundary between $\mathcal{D}^+_{\mathbf{u}}$ and $\mathcal{D}^-_{\mathbf{u}}$. In the case $\mathcal{D}_{\mathbf{u}} = \mathcal{D}^-_{\mathbf{u}}$, one has the blow-up free situation.

In the rest of this section, we will assume that the subdomain $\mathcal{D}^+_{\mathbf{u}}$ is not empty and hence the blow-up surface has two branches Γ_+ and Γ_-.

Let $\mathbf{u}_b$ a point at $\mathcal{D}^+_{\mathbf{u}}$ and t_b be the corresponding value of time t on the first or the second branches of Γ. In the first regime, that is, when $t \to t_b$ with fixed $\mathbf{u}_b$, one has

$$
\omega_3(t_b + \varepsilon, \mathbf{u}_b) \sim \frac{\frac{\partial f_1}{\partial u_2}(\mathbf{u}_b) - \frac{\partial f_2}{\partial u_1}(\mathbf{u}_b) + O(\varepsilon)}{\left(2t_b + \frac{\partial f_1}{\partial u_1}(\mathbf{u}_b) + \frac{\partial f_2}{\partial u_2}(\mathbf{u}_b)\right)\varepsilon + \varepsilon^2}\,, \qquad \varepsilon \to 0\,. \tag{78}
$$

So, if

$$
2t_b + \frac{\partial f_1}{\partial u_1}(\mathbf{u}_b) + \frac{\partial f_2}{\partial u_2}(\mathbf{u}_b) = \sqrt{\Delta(\mathbf{u})}|_{\mathbf{u}=\mathbf{u}_b} \neq 0, \tag{79}
$$

the vorticity ω_3 blows up as

$$
\omega_3(t, \mathbf{u}_b) \sim \varepsilon^{-1} \equiv (t - t_b)^{-1}\,, \qquad t \to t_b\,. \tag{80}
$$

This happens at each point of the blow-up surface Γ.

If instead

$$
2t_b + \frac{\partial f_1}{\partial u_1}(\mathbf{u}_b) + \frac{\partial f_2}{\partial u_2}(\mathbf{u}_b) = \sqrt{\Delta(\mathbf{u})}|_{\mathbf{u}=\mathbf{u}_b} = 0, \tag{81}
$$

the vorticity ω_3 blows up as

$$
\omega_3(t, \mathbf{u}_b) \sim \varepsilon^{-2} \equiv (t - t_b)^{-2}\,, \qquad t \to t_b\,. \tag{82}
$$

Such a behavior occurs on the curve defined by the condition (81).

It is the condition of a coincidence for the values $t_{b\pm} = \frac{1}{2}$

$$
t_{b\pm} = \frac{-\mathrm{tr}M_0 \pm \sqrt{\Delta}}{2} \tag{83}
$$

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of the branches $\Gamma_{\pm}$, that is, $t_{b_+} = t_{b_-}$. Hence, the blowup of the type (82) occurs along the curve of the intersection of two branches of the blow-up surface Γ. The corresponding curve (81) in the hodograph space can be the border curve between two subdomains $\mathcal{D}^+_{\mathbf{u}}$ or $\mathcal{D}^-_{\mathbf{u}}$ when $\mathcal{D}^+_{\mathbf{u}} = \mathcal{D}_{\mathbf{u}}$ or $\mathcal{D}^-_{\mathbf{u}} = \mathcal{D}_{\mathbf{u}}$ respectively.

Similar to the 3D case, one can view the conditions (81) as the equation which defines those functions $f_1(\mathbf{u})$ and $f_2(\mathbf{u})$ for which two branches of Γ coincide.

In order to analyze the behavior of the vorticity ω_3 at fixed time t_b, similar to (10), one introduces the variables $\mathbf{y}$ and $\mathbf{v}$ (see also Konopelchenko and Ortenzi[17])

$$\delta\mathbf{u} = \sum_{\alpha=1}^{2} \mathcal{R}^{(\alpha)}\delta v_\alpha \,, \qquad \delta\mathbf{x} = \sum_{\alpha=1}^{2} \mathcal{P}^{(\alpha)}\delta y_\alpha \,. \tag{84}$$

At the first level of blowups, one has the following behavior of derivatives[17]:

$$\frac{\partial v_1}{\partial y_1}, \frac{\partial v_1}{\partial y_2}, \frac{\partial v_2}{\partial y_1} \sim \varepsilon^{-1/2}, \qquad \frac{\partial v_2}{\partial y_2} \sim O(1)\,. \tag{85}$$

So, one has the relation

$$\delta v_\alpha = \sum_{\beta=1}^{2} C_{\alpha\beta}\delta y_\beta \,, \qquad \alpha = 1, 2, \tag{86}$$

with the matrix

$$C = \begin{pmatrix} \varepsilon^{-1/2}\nu_{11} & \varepsilon^{-1/2}\nu_{12} \\ \varepsilon^{-1/2}\nu_{21} & \nu_{22} \end{pmatrix}. \tag{87}$$

In the 2D case, the vorticity is the differential two-form

$$\omega(\mathbf{u}_b) = \omega_{12}\delta y_1 \wedge \delta y_2 \,, \tag{88}$$

where

$$\omega_{12}(\mathbf{u}_b) = \varepsilon^{-1/2}S(\mathbf{u}_b) + T(\mathbf{u}_b)\,, \qquad \varepsilon \to 0 \tag{89}$$

and $S(\mathbf{u}_b)$ and $T(\mathbf{u}_b)$ are certain combinations of $\nu_{\alpha\beta}$ and $\mathcal{R}^\alpha \cdot \mathcal{P}^\beta$ (see in analogy the 3D case of the (39) and (40) relations).

In the Cartesian coordinates, the vorticity $\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ also is of the form (89). Along the curve defined by the condition

$$S(\mathbf{u}_b) = 0, \tag{90}$$

the vorticity is bounded.

Blowups of the second level occur on the curve contained in Γ and on this curve (see Konopelchenko and Ortenzi[17])

$$\frac{\partial v_1}{\partial y_1} \sim \varepsilon^{-2/3}, \qquad \frac{\partial v_1}{\partial y_2} \sim \varepsilon^{-1/2}, \qquad \frac{\partial v_2}{\partial y_1} \sim \varepsilon^{-1/2}, \qquad \frac{\partial v_2}{\partial y_2} \sim O(1), \qquad \varepsilon \to 0, \tag{91}$$

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and, consequently, the vorticity blows up as

$$\omega_{12} = \varepsilon^{-2/3} Y(\mathbf{u}_b) + \varepsilon^{-1/2} S(\mathbf{u}_b) + T(\mathbf{u}_b)\,. \tag{92}$$

Finally, at the third level which may occur at a point on Γ, one has $\frac{\partial v_1}{\partial y_1} \sim \varepsilon^{-3/4}$ and, hence, the vorticity blows up as $\omega_{12} = \varepsilon^{-3/4}$.

7 | EXAMPLES IN TWO DIMENSIONS

Here we will present three characteristic examples of the 2D HEE.

7.1 | Blow-up free solutions

Let the functions f_1 and f_2 be of the form

$$f_1 = \frac{\partial W}{\partial u_2}, \qquad f_2 = \frac{\partial W}{\partial u_1}, \tag{93}$$

where the real function $W(u_1, u_2)$ obeys the Laplace equation

$$\frac{\partial^2 W}{\partial u_1^2} + \frac{\partial^2 W}{\partial u_2^2} = 0\,. \tag{94}$$

It is easy to see that in this case

$$t_b = -\frac{\partial^2 W}{\partial u_1 \partial u_2} \pm \sqrt{\Delta} \quad \text{with} \quad \Delta = -\left(\frac{\partial^2 W}{\partial u_1^2}\right)^2 < 0 \tag{95}$$

for any function W except a linear one. So, the corresponding solutions u_1 and u_2 of the 2D HEE have no blowups.

The vorticity (74) is given by

$$\omega_3 = -2\frac{\frac{\partial^2 W}{\partial u_1^2}}{t^2 + 2\frac{\partial^2 W}{\partial u_1 \partial u_2} t + \left(\frac{\partial^2 W}{\partial u_1 \partial u_2}\right)^2 + \left(\frac{\partial^2 W}{\partial u_1^2}\right)^2}, \tag{96}$$

and it is blowup free too.

The particular choice

$$W = \frac{1}{2\alpha}\left(u_2^2 - u_1^2\right) \tag{97}$$

or $f_1 = \frac{u_2}{\alpha}$, $f_2 = -\frac{u_1}{\alpha}$ corresponds to initial velocities $u_1 = -\alpha x_2$ and $u_2 = \alpha x_1$ where α is an arbitrary real constant. Such an initial condition gives

$$u_1 = \frac{\alpha(\alpha x_1 t - x_2)}{\alpha^2 t^2 + 1}, \qquad u_2 = \frac{\alpha(\alpha x_2 t + x_1)}{\alpha^2 t^2 + 1}, \tag{98}$$

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and

$$\omega_3 = \frac{2\alpha}{\alpha^2 t^2 + 1} . \tag{99}$$

It is the rotational type vortex solution of the 2D HEE with the initial strength 2α and α^{-1} as the characteristic decaying time.

It is worth to note that the subclass of solutions of the 2D HEE corresponding to the choice (93) has a simple description in terms of complex coordinates[16]

$$Z = x_1 + ix_2 , \qquad V = u_1 + iu_2 , \qquad F = f_1 + if_2 . \tag{100}$$

Indeed, in these variables, the conditions (93) and (94) are given by

$$F = 2i \frac{\partial W}{\partial V} \tag{101}$$

and

$$\frac{\partial^2 W(V, \bar{V})}{\partial V \partial \bar{V}} = 0 . \tag{102}$$

Since

$$W(V, \bar{V}) = \mathcal{W}(V) + \bar{\mathcal{W}}(\bar{V}), \tag{103}$$

where $\mathcal{W}(V)$ is an arbitrary analytic function (note that (93) implies that W is real-valued), then

$$F = 2i \frac{\partial \mathcal{W}(V)}{\partial V} . \tag{104}$$

For such function F, the hodograph equation assumes the form

$$Z - Vt = F(V) . \tag{105}$$

Solutions of the hodograph Equation (105) obey the equation

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial Z} = 0 . \tag{106}$$

In the complex variables, the vorticity (96) is given by

$$\omega_3 = -2 \frac{\mathrm{Im}\left(\frac{\partial F}{\partial V}\right)}{|t + \frac{\partial F}{\partial V}|^2} . \tag{107}$$

For the solution (98) $F = -iV/\alpha$. For the generic analytic function $F(V)$. the corresponding solution $V(Z, t)$ of Equation (106) and its vorticity are blow-up free. In the trivial particular case $F = \beta V$, where β is an arbitrary real constant, the solution $V(Z, t) = \frac{Z}{t+\beta}$ of Equation (106) and its derivative exhibit the blowup at $t = -\beta$ while the vorticity $\omega_3 = 0$. In this case, the 2D HEE is decomposed into two one-dimensional Burgers–Hopf equations.

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The fact that for the generic analytic solutions of the 2D HEE, the derivatives are blowups free has been noted in Konopelchenko and Ortenzi[16] (Section 5). Indeed, in the complex variables (100) the full 2D HEE assumes the form

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial Z}+\bar{V}\frac{\partial V}{\partial \bar{Z}}=0\,, \tag{108}$$

and the blow-up surface is defined by

$$\left(\frac{\partial F}{\partial V}+t\right)\left(\frac{\partial \bar{F}}{\partial \bar{V}}+t\right)-|\frac{\partial F}{\partial \bar{V}}|^2=0\,. \tag{109}$$

For the analytic solutions ($\partial V/\partial\bar{Z}=0$, $\partial F/\partial\bar{V}=0$), 2D HEE (108) is reduced to (106) and the blow-up surface is defined by the equation (Konopelchenko and Ortenzi,[16] Section 5)

$$\det M=\left(\frac{\partial F}{\partial V}+t\right)\left(\frac{\partial \bar{F}}{\partial \bar{V}}+t\right)=0\,. \tag{110}$$

This equation has no real roots except the trivial case $F=\beta V$, mentioned above. Consequently, in nontrivial cases, the derivative $\partial V/\partial Z$ does not exhibit blowups for real-time t (negative or positive) in contrast to the classical one-dimensional case (real Z).

In different contexts, Equation (106) has been considered earlier[21–23]

7.2 | Nongeneric blowup

Let us choose

$$f_1=-\frac{u_1^3}{3}-\frac{2}{3}u_1u_2^2+2u_2\,,\qquad f_2=-\frac{u_2^3}{3}-\frac{1}{3}u_1^2u_2-u_1\,. \tag{111}$$

The corresponding initial data are

$$\begin{aligned} u_1(x_1,x_2,0)=&-x_2-\frac{x_1^3}{24}-\frac{1}{6}x_1x_2^2+\frac{x_2^5}{18}+\frac{1}{72}x_1^2x_2^3+\frac{1}{144}x_1^4x_2+\cdots\,,\\ u_2(x_1,x_2,0)=&\frac{x_1}{2}-\frac{x_2^3}{6}-\frac{1}{12}x_2x_1^2-\frac{x_1^5}{288}-\frac{1}{144}x_2^2x_1^3-\frac{1}{36}x_2^4x_1+\cdots\,. \end{aligned} \tag{112}$$

In this case, the matrix M is

$$M(t,\mathbf{u})=\begin{pmatrix} t-\left(u_1^2+\frac{2}{3}u_2^2\right) & 2-\frac{4}{3}u_1u_2\\ -\frac{2}{3}u_1u_2-1 & t-\left(\frac{1}{3}u_1^2+u_2^2\right)\end{pmatrix}, \tag{113}$$

and the blow-up surface Γ is defined by the equation

$$t^2-\left(\frac{4}{3}u_1^2+\frac{5}{3}u_2^2\right)t+\frac{1}{3}u_1^4+\frac{1}{3}u_1^2u_2^2+\frac{2}{3}u_2^4+2=0\,. \tag{114}$$

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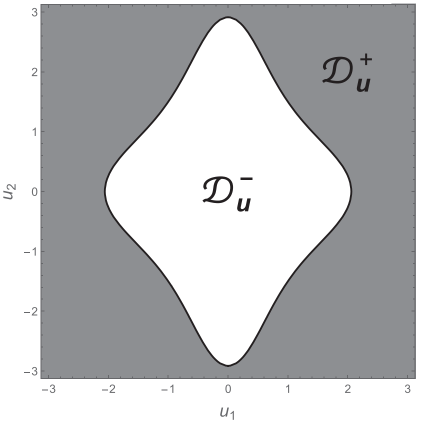

FIGURE 1 In the gray $\mathcal{D}_{\mathbf{u}}^{+}$ region, the discriminant $\Delta(u_1, u_2)$ (115) is positive and, therefore, blowups are possible. In the complementary region $\mathcal{D}_{\mathbf{u}}^{-}$, the discriminant $\Delta(u_1, u_2)$ is negative and, therefore, no blowups are possible.

The discriminant $\Delta(u_1, u_2)$ is

$$\Delta(u_1, u_2) \equiv 4u_1^4 + 28u_2^2u_1^2 + u_2^4 - 72\,. \tag{115}$$

So the subdomains $\mathcal{D}_{\mathbf{u}}^{+}$ and $\mathcal{D}_{\mathbf{u}}^{-}$ in $\mathcal{D}_{\mathbf{u}}$ are separated by the quartic curve

$$\Delta(u_1, u_2) = 4u_1^4 + 28u_2^2u_1^2 + u_2^4 - 72 = 0\,. \tag{116}$$

The subdomain $\mathcal{D}_{\mathbf{u}}^{-}$ is located around the origin $u_1 = u_2 = 0$ as shown in Figure 1.

The blow-up surface Γ has two branches

$$t_{\pm} = \frac{1}{6}\left(4u_1^2 + 5u_2^2 \pm \sqrt{4u_1^4 + 28u_2^2u_1^2 + u_2^4 - 72}\right)\,. \tag{117}$$

with $\mathbf{u} \in \mathcal{D}_{\mathbf{u}}^{+}$. It is easy to see that for both branches $t_+ \geq t_- > 0$ (see Figure 2).

The time of the gradient catastrophe is $t_{-\min} = 1.62019$ at the point $u_1 = \pm 1.59562$, $u_2 = \pm 1.17844$.

The vorticity is equal to

$$\omega_3 = \frac{3 - \frac{2}{3}u_1u_2}{\det M}\,. \tag{118}$$

In the first regime of approaching of generic blow-up point $(u_1, u_2) \in \mathcal{D}_{\mathbf{u}}^{+}$, the vorticity behaves as

$$\omega_3(t, \mathbf{u}_b) \sim \pm\frac{2u_1u_2 - 9}{\sqrt{4u_1^4 + 28u_2^2u_1^2 + u_2^4 - 72}\,(t - t_b)}\,, \qquad t \to t_b = \frac{1}{6}\left(4u_1^2 - 5u_2^2 \pm \sqrt{4u_1^4 + 28u_2^2u_1^2 + u_2^4 - 72}\right)\,. \tag{119}$$

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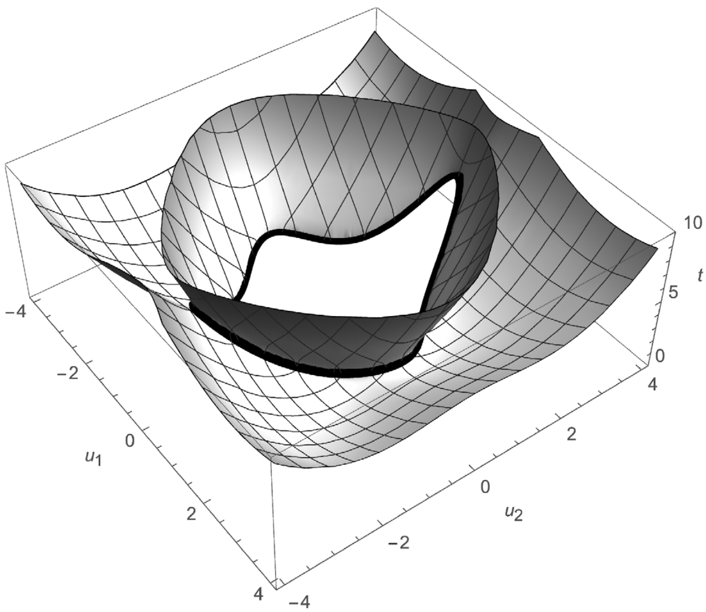

FIGURE 2 Blow-up Γ region (117) related to hodograph mappings (111). At the black curve (120), the vorticity behavior is nongeneric $\omega \sim (\Delta t)^{-2}$.

TABLE 1 The local inverses of the initial data (122).

	$x_1 \geq 0 x_2 \geq 0$	$x_1 \geq 0 x_2 < 0$	$x_1 < 0 x_2 \geq 0$	$x_1 < 0 x_2 < 0$
$x_1 = f_1(\mathbf{u})$	$\sqrt{\frac{1}{2}\log\frac{u_2}{u_1{}^3}}$	$\sqrt{\frac{1}{2}\log\frac{u_2}{u_1{}^3}}$	$-\sqrt{\frac{1}{2}\log\frac{u_2}{u_1{}^3}}$	$-\sqrt{\frac{1}{2}\log\frac{u_2}{u_1{}^3}}$
$x_2 = f_2(\mathbf{u})$	$\sqrt{\frac{1}{2}\log\frac{u_1}{u_2}}$	$-\sqrt{\frac{1}{2}\log\frac{u_1}{u_2}}$	$\sqrt{\frac{1}{2}\log\frac{u_1}{u_2}}$	$-\sqrt{\frac{1}{2}\log\frac{u_1}{u_2}}$

Approaching the points

$$t_{\pm b} = \frac{2}{3}u_1^2 + \frac{5}{6}u_2^2, \qquad \Delta(u_1, u_2) = 0, \tag{120}$$

which belongs to the curve of the intersection of two branches t_+ and t_-, the vorticity blows up as

$$\omega \sim \frac{\pm\frac{2}{3}\sqrt{2}u_1\sqrt{-7u_1^2 - \sqrt{6}\sqrt{8u_1^4 + 3} - 3}}{(t - t_b)^2}, \qquad t \to t_b = -11u_1^2 - 5\sqrt{\frac{16u_1^4}{3}} + 2. \tag{121}$$

In this case, the curve $\Delta(u_1, u_2) = 0$ is the boundary line between the subdomains $\mathcal{D}_{\mathbf{u}}^+$ and $\mathcal{D}_{\mathbf{u}}^-$.

7.3 | Gaussian initial data

Finally, we consider a solution of the HEE with the initial data

$$u_1(\mathbf{x}, 0) = e^{-x_1^2 - x_2^2}, \qquad u_2(\mathbf{x}, 0) = \exp^{-x_1^2 - 3x_2^2}. \tag{122}$$

Such initial values admit four different open sets of invertibility shown in Table 1. Where

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$f_i, i = 1, 2$ is the local inverse of (122). The hodograph Equations (5) assume the form of the system of four equations

$$G_{a,b} : \begin{cases} x_1 = u_1 t + a\sqrt{\frac{1}{2}\ln\left(\frac{u_2}{u_1^3}\right)}, \ a(x_1 - u_1 t) > 0 \\ x_2 = u_2 t + b\sqrt{\frac{1}{2}\ln\left(\frac{u_2}{u_1}\right)}, \ b(x_2 - u_2 t) > 0 \end{cases}, \qquad a = \pm, b = \pm. \tag{123}$$

Each pair of Equations (123) define a solution $\mathbf{u}_{ab}(\mathbf{x}, t)$ in the corresponding subdomain. So, the solution of the 2D HEE with the initial data (122) is a union

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_{++}(\mathbf{x}, t) \cup \mathbf{u}_{+-}(\mathbf{x}, t) \cup \mathbf{u}_{-+}(\mathbf{x}, t) \cup \mathbf{u}_{--}(\mathbf{x}, t). \tag{124}$$

In other words,

$$\mathbf{u}(\mathbf{x}, t) = \begin{cases} \mathbf{u}_{++}(\mathbf{x}, t), & \text{at} \quad x_1 - u_1(\mathbf{x}, t) > 0, \quad x_2 - u_2(\mathbf{x}, t) > 0, \\ \mathbf{u}_{+-}(\mathbf{x}, t), & \text{at} \quad x_1 - u_1(\mathbf{x}, t) > 0, \quad x_2 - u_2(\mathbf{x}, t) < 0, \\ \mathbf{u}_{-+}(\mathbf{x}, t), & \text{at} \quad x_1 - u_1(\mathbf{x}, t) < 0, \quad x_2 - u_2(\mathbf{x}, t) > 0, \\ \mathbf{u}_{--}(\mathbf{x}, t), & \text{at} \quad x_1 - u_1(\mathbf{x}, t) < 0, \quad x_2 - u_2(\mathbf{x}, t) < 0. \end{cases} \tag{125}$$

The function (125) is continuous on $\mathbb{R}^2 \times \mathbb{R}$ through the boundary $\mathbf{x} - \mathbf{u}t = 0$. Note that $\mathbf{u}_{--}(\mathbf{x}, t) = \mathbf{u}_{++}(-\mathbf{x}, -t)$, $\mathbf{u}_{-+}(\mathbf{x}, t) = \mathbf{u}_{+-}(-\mathbf{x}, -t)$. Moreover, the domain $\mathcal{D}_{\mathbf{u}}$ is the square $0 < u_1(\mathbf{x}, t), u_2(\mathbf{x}, t) \leq 1$. Using the standard formulae $\mathbf{u}(x, t) = \mathbf{u}_0(\xi_1, \xi_2)$ with $\xi_i = x_i - u_i t$, $i = 1, 2$, one can view the piecewise solution (125) as

$$\mathbf{u}_{ab}(x, t) = \mathbf{u}_0(a\xi_1, b\xi_2), \qquad a\xi_1 > 0, \quad b\xi_2 > 0. \tag{126}$$

Then four corresponding matrices M are of the form

$$M^{(ab)}(t, \mathbf{u}) = \begin{pmatrix} t - a\dfrac{3}{2\sqrt{2}u_1\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)}} & a\dfrac{1}{2\sqrt{2}u_2\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)}} \\ b\dfrac{1}{2\sqrt{2}u_1\sqrt{\log\left(\frac{u_1}{u_2}\right)}} & t - b\dfrac{1}{2\sqrt{2}u_2\sqrt{\log\left(\frac{u_1}{u_2}\right)}} \end{pmatrix}, \qquad a, b = \pm, \tag{127}$$

and the corresponding branches of the blow-up surface are defined by the equation

$$\det M = t^2 - \left(\frac{3a}{2\sqrt{2}u_1\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)}} + \frac{b}{2\sqrt{2}u_2\sqrt{\log\left(\frac{u_1}{u_2}\right)}}\right)$$
$$t + \frac{ab}{4u_1u_2\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)\log\left(\frac{u_1}{u_2}\right)}} = 0, \quad a, b = \pm. \tag{128}$$

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The values of the vorticity ω_3 for the branches (a, b) are given by

$$\omega_3 = \frac{1}{\det M_{pq}(t, \mathbf{u})}\left(\frac{a}{2\sqrt{2}u_2\sqrt{\log\left(\frac{u_1}{u_2{}^3}\right)}} - \frac{b}{2\sqrt{2}u_1\sqrt{\log\left(\frac{u_2}{u_1}\right)}}\right). \tag{129}$$

The discriminant Δ of Equation (128) is positive for all values of a and b since

$$\begin{aligned}\Delta_{ab}(\mathbf{u}) &= \left(\frac{3a}{2\sqrt{2}u_1\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)}} + \frac{b}{2\sqrt{2}u_2\sqrt{\log\left(\frac{u_1}{u_2}\right)}}\right)^2 - \frac{ab}{u_1 u_2\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)\log\left(\frac{u_1}{u_2}\right)}}\\ &= \left(\frac{3a}{2\sqrt{2}u_1\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)}} - \frac{b}{2\sqrt{2}u_2\sqrt{\log\left(\frac{u_1}{u_2}\right)}}\right)^2 + \frac{ab}{2u_1 u_2\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)\log\left(\frac{u_1}{u_2}\right)}}.\end{aligned} \tag{130}$$

So, for the solution (125) the blow-up surface Γ has two branches for all values of $\mathbf{u} \in \mathcal{D}_{\mathbf{u}}$

$$(t_\pm)_{ab} = \frac{3a}{2\sqrt{2}u_1\sqrt{\log\left(\frac{u_2}{u_1{}^3}\right)}} + \frac{b}{2\sqrt{2}u_2\sqrt{\log\left(\frac{u_1}{u_2}\right)}} \pm \sqrt{\Delta_{ab}}. \tag{131}$$

It is easy to see that for the $(a, b) = (+, +)$ piece

$$(t_\pm)_{++} > 0, \qquad (t_+)_{++} > (t_-)_{++}, \tag{132}$$

while

$$(t_\pm)_{--} < 0, \qquad (t_+)_{--} > (t_-)_{--}, \tag{133}$$

For the pieces $(a, b) = (+, -)$ and $(a, b) = (-, +)$, one has

$$(t_+)_{+-} > 0, \qquad (t_-)_{+-} < 0, \tag{134}$$

and

$$(t_+)_{-+} > 0, \qquad (t_-)_{-+} < 0. \tag{135}$$

Minimal values of $t_{\pm ab}$ for the positive pieces are

$$(t_-)_{++}|_{\min} = 0.642593, \qquad (t_+)_{+-}|_{\min} = 1.16582, \qquad (t_+)_{-+}|_{\min} = 0.673088. \tag{136}$$

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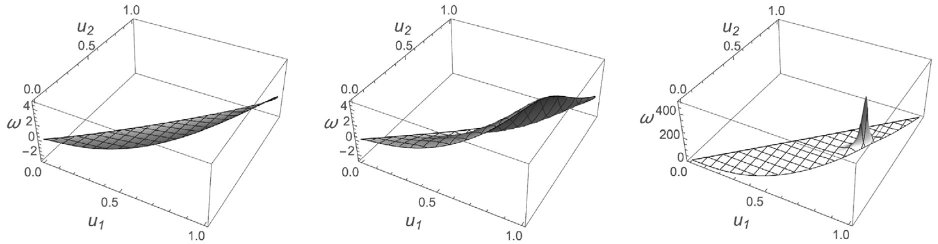

FIGURE 3 The time evolution of the vorticity depending on $\mathbf{u}$ with initial data given by (122). From left to right, the times are $t = 0$, $t = 0.85t_c$, $t = 0.999t_c$, where $t_c = 0.642593$ is the catastrophe time. Remark the change in the vertical scale in the last plot.

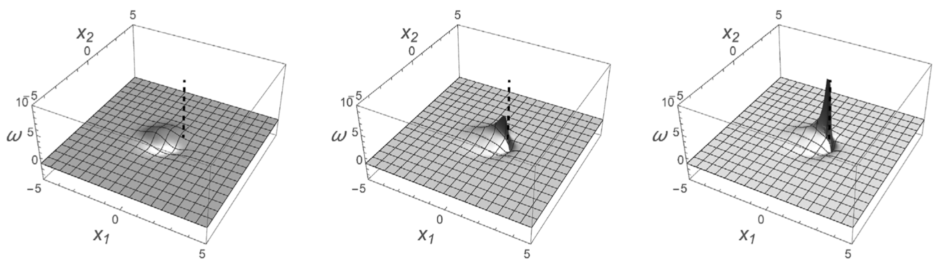

FIGURE 4 The time evolution of the vorticity depending on $\mathbf{x}$ with initial data given by (122). From left to right, the times are $t = 0$, $t = 0.85t_c$, $t = 0.999t_c$, where $t_c = 0.642593$ is the catastrophe time. The dashed vertical line indicates the catastrophe direction of the vorticity in the catastrophe place $\mathbf{x}_c$.

Thus, the gradient catastrophe occurs at

$$t_c \equiv (t_-)_{++}|_{\min} = 0.642593\,, \qquad \mathbf{u}_c = (0.803494, 0.584021)\,, \qquad \mathbf{x}_c = (0.759774, 0.77468)\,. \tag{137}$$

As expected, the behavior of the vorticity at $\mathbf{u} = \mathbf{u}_c$ in the first regime is

$$\omega(t, \mathbf{u}_c) = \frac{0.270466}{t_c - t} - 0.0747002 + 0.0206315(t_c - t) + \cdots\,. \tag{138}$$

The time evolution of the vorticity $\omega(t, \mathbf{u})$ is shown in Figure 3.

In Figure 4, it is shown the time evolution of the vorticity with respect to space variables, numerically computed using Mathematica. The behavior is in agreement with the analytical predictions (137).

Since $\Delta_{ab}(\mathbf{u}) \neq 0$ for all $\mathbf{u} \in \mathcal{D}_{\mathbf{u}}$, two branches (131) do not intersect. So, the blowup of the type $\omega \sim (t_c - t)^{-2}$ is absent in this case.

8 | BLOWUPS FOR n-DIMENSIONAL CASE

An extension of the results presented in this paper to the n-dimensional HEE is quite straightforward. Indeed, the components ω_{ij} of the vorticity two-form (36) in Cartesian coordinates are given

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by

$$\omega_{ij}(t,\mathbf{u}) = (M^{-1})_{ji}(t,\mathbf{u}) - (M^{-1})_{ij}(t,\mathbf{u}) = \frac{\tilde{M}_{ji}(t,\mathbf{u}) - \tilde{M}_{ij}(t,\mathbf{u})}{\det(M(t,\mathbf{u}))}, \qquad i,j = 1,\cdots,n. \tag{139}$$

In the n-dimensional case, $\det(M(t,\mathbf{u}))$ is a polynomial in t of degree n,[16] that is

$$\det(M(t,\mathbf{u})) = \prod_{k=1}^{n}(t - t_{b_k}). \tag{140}$$

So, in the first regime of approaching the blow-up point ω_{ij} may have the following behavior:

$$\omega_{ij} \sim (t - t_b)^{-m}, \qquad t \to t_b, \quad m = 1,\cdots,n. \tag{141}$$

As far as the second regime is concerned, it was shown in Konopelchenko and Ortenzi[17] that the derivatives $\partial u_i / \partial x_j$ may have singularities of the type $|\delta \mathbf{x}|^{-\frac{m}{m+1}}$, with $m = 1,\cdots,n+1$. Hence, in this regime, the vorticity two-form may blow up as

$$\omega_{ij} \sim |\delta \mathbf{x}|^{-\frac{m}{m+1}}, \qquad |\delta \mathbf{x}| \to 0, \qquad m = 1,\cdots,n+1. \tag{142}$$

Similar to the results described in Konopelchenko and Ortenzi,[17] blowups of the vorticity exhibit a rather rich fine structure.

The formulae (141) and (142) imply certain behavior of the characteristics of vorticity in different dimensions discussed in Arnold and Khesin.[20]

One obtains analogous results for the stress tensor

$$S_{ij} \equiv \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = (M^{-1})_{ij} + (M^{-1})_{ji} = \frac{\tilde{M}_{ij} + \tilde{M}_{ji}}{\det(M)}, \qquad i,j = 1,\cdots,n, \tag{143}$$

which is another important quantity in the theory of continuous media.[1–3]

9 | CONCLUSIONS

The results presented in this note are in part the consequences of those obtained in Konopelchenko and Ortenzi.[17] As in Konopelchenko and Ortenzi,[17] we are dealing with the most simplified version of the Navier–Stokes equation, namely with HEE (1) and do not discuss the possibility of blowups of vorticity of type (3) for positive values of time.

All that indicates at least two possible directions for further study. The first is the verification of the realizability of hierarchy of blowups (3) for positive times that is of most interest in physical applications.

An extension of such a type of analysis for more physical systems would be the second direction. In particular, it may be applicable to those hydrodynamical systems which are obtainable as the constraints of the multidimensional HEE.[24]

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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