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This paper aims at showing the potential of the synergistic use of algorithms as artifacts for the development of mathematical meanings with pupils of primary school. Specifically, we consider two division algorithms introduced in 4th grade and we show how a specific task design, that involves a comparison between the two different algorithms performing the same division, can generate a mathematical discussion. In such a discussion we can identify several situated signs potentially useful for the development of mathematical meanings related to the algorithms' functioning.

Keywords: Sinergy between algorithms; Semiotic Mediation; Artefacts; Canadian division algorithm; TI×- division algorithm.

Introduction

This article presents a study on teaching multiple algorithms for calculating divisions, carried out within a larger project (see www.percontare.it) aimed at providing Italian teachers with a repository of educational activities in Mathematics for primary school that pay particular attention to inclusiveness (Baccaglini-Frank, 2015). A fundamental design feature of the activities is their aim to help students reach a solid construction of mathematical meanings through the use of artefacts, following the Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008). This aim is prominent also within the set of activities concerning the introduction of different calculation algorithms for arithmetic operations. In this study we will focus on the teaching and learning of division between natural numbers through two algorithms in fourth grade (age 9-10).

The first algorithm we consider is the "Canadian algorithm" (Lisarelli, Baccaglini-Frank & Di Martino, 2021; Boero & Ferrero, 1988), consisting in a repeated subtraction of the divisor from the dividend. The other one is *"TI×- algorithm"* (Karagiannakis, 2018), that is similar, from a mathematical point of view, to the long division algorithm (see the *Methodology* section). The main difference between these two algorithms can be expressed through the *transparency* construct with respect to the meanings of the division between natural numbers. We want to extend to these algorithms the definition of transparent and opaque representations of numbers introduced by Zazkis & Gadowsky (2001): "A transparent representation has no more and no less meaning than the represented idea(s) or structure(s). An opaque representation emphasizes some aspects of the ideas and structures and de-emphasizes others" (p. 45). We can say that the Canadian algorithm is transparent with respect to the meaning of division, conceived as a progressive distribution, while TI×- algorithm is opaquer with respect to this meaning. Our research hypothesis is that children can make sense of the algorithms, understand *why* they work, and gain deep understanding of division of natural numbers, by becoming fluent with both of them and then comparing them and discovering what is behind the opacity of an algorithm like TI×-. Our main research interest is to test this hypothesis. In the study, we report on our attempt at promoting a mathematical discussion overcoming the opacity of the TI×- algorithm.

Theoretical framework

An *algorithm* can be considered as a cultural product, designed to solve a given class of problems. Schmittau (2003), discussing the role of algorithms within Davidov’s curriculum, expressly talks of them as a "symbolic trace of the meaningful mathematical actions required to solve a problem" (p. 240). In this perspective, an algorithm develops historically and is configured as a cultural tool that mediates an individual's knowledge and understanding of Mathematics (Ebby, 2005). The reliability of an algorithm rests on a body of knowledge that is not always visible to those who use it. Traditional algorithms, in fact, are the result of a historical-cultural evolution that has often favored the efficiency of algorithms in a mechanical sense rather than their transparency with respect to mathematical meanings underlying each step (Bass, 2003).

In this perspective, we can interpret didactical activities on algorithms for the arithmetic operations through the lens of a whole tradition of studies which have shown how it is possible for students, through the use of artefacts to accomplish a task, to develop meanings linked to the knowledge incorporated in the artefacts themselves (e.g., Bartolini Bussi & Mariotti, 2008). Starting from a Vygotskian perspective, Bartolini Bussi and Mariotti emphasize the crucial role of social interaction as an engine for student learning, with a focus on the semiotic processes that can occur in the classroom starting with an activity with an artefact, triggered and supported by the teacher. Nevertheless, unlike most of the studies informed by TSM - which concern the use of only one artefact - in this study we chose to use two different algorithms as artefacts. This choice is supported by recent studies that have begun to investigate the possibility of introducing more than one artefact having a common potential with respect to the development of the same mathematical knowledge (e.g., Faggiano et al. 2018; Maffia & Maracci, 2019). These studies confirm that in specifically designed didactical activities, the introduction of more than one artefact can result in a synergy, which can increase the didactic potential of the activity with respect to activities involving a single artefact (Faggiano et al., 2018; Maffia & Maracci, 2019).

Therefore, conceiving algorithms as artefacts can be useful to make visible to the students mathematical meanings related to the body of knowledge that make the algorithms reliable. From the perspective of TSM, starting from the use of algorithms to carry out specific tasks, and participating in an explicit discussion on this use - intentionally orchestrated by the teacher - students can develop knowledge on the nature of the algorithm and the properties of the operations to be performed. More specifically, we present a didactic intervention whose aim was precisely to develop students' knowledge of the mathematical meanings underlying two different division algorithms.

Research questions

1. What signs in the discussion can evolve in the direction of the discovery and understanding of the mathematical meanings underlying the functioning of TI×- algorithm?
2. Can (and if so, how can) the synergy of artefacts foster the emergence of signs related to mathematical meanings in common among the two algorithms involved?

Methodology

According to the TSM framework, the didactic activity was designed to include a part of interaction with artefacts and a subsequent part of mathematical discussion, to be developed in Distance Education (DE) due to the persistence of the SARS-COV-2 pandemic. As showed in other studies (e.g., Ramploud, Funghi & Mellone, 2021), in order to face the constraints of DE, teachers have sometimes chosen to adapt the framework of TSM to their new conditions. In this case, the teacher chose to separate the part of interaction with the artefacts from the discussion: in a first moment students had to calculate the division 874:7 with both algorithms, as homework; then, the class was divided into 4 heterogeneous groups of 7-8 students each, and the next day for each group a lesson in DE of 40 minutes was dedicated to the mathematical discussion starting from the different solutions of the students. The lessons were all video recorded and fully transcribed.

The two algorithms considered were specially chosen in analogy with other studies (e. g., Lisarelli et al., 2021) to allow "to discover various mathematical meanings behind the long division algorithm [in our case, the TI×- algorithm] and their role in unveiling the whys: the role of place value, the hidden powers of ten, [...] the meaning of each digit of the quotient, how each remainder is obtained." (ibidem, p. 3). We describe below the two algorithms for the division presented in the PerContare project.

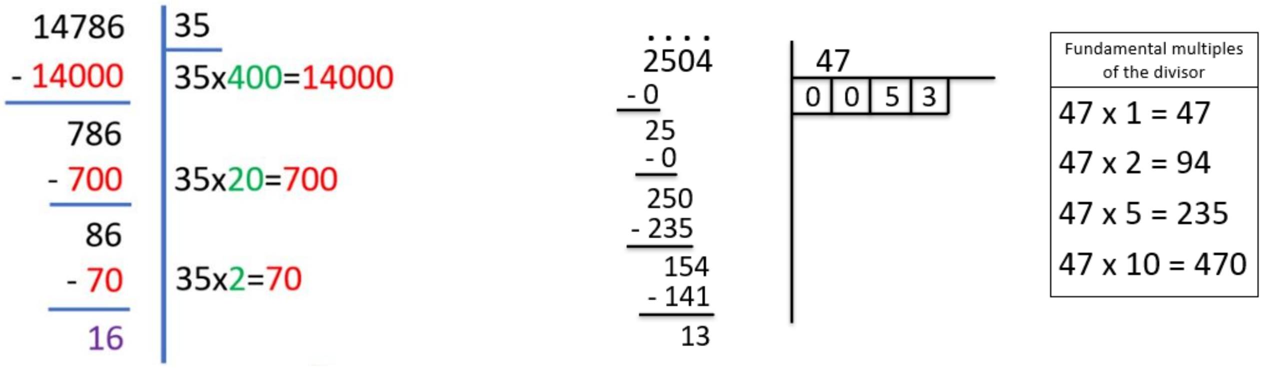

Figure 1: An Optimized Canadian algorithm applied to the division 14786:35 (on the left); TI×- algorithm applied to the division 2504:47 (on the right)

The first is the Canadian Algorithm in its non-optimized version (Boero & Ferrero, 1988). It works like this: the solver identifies a multiple of the divisor that is less than the dividend; then, s/he subtracts this multiple from the dividend. S/he repeats the same reasoning starting from the result of this subtraction (i.e., s/he identifies a multiple of the divisor, which is less than the result of the subtraction, and so on) until s/he obtains a number that is less than the divisor (it could be 0). The left diagram in Figure 1 shows an optimized version of the Canadian algorithm applied to the division 14786:35, where the multiple of the divisor to be subtracted at each step is chosen among those that are also multiples of the highest possible power of 10 (i.e., at the first step 35×400 is subtracted, then 35×20, then 35×2). The second division algorithm is TI×- algorithm. To illustrate it, we can start from this example: 2504:47 (see Figure 1 on the right). The first step to be carried out consists in writing some useful multiples of the divisor, which we call *fundamental multiples* (on the right in Figure 1). The multiples chosen are ×1, ×2, ×5, ×10, since they are the simplest multiples to calculate (the multiple ×5 can be obtained calculating half of the multiple ×10) and they are sufficient to obtain all the others applying the distributive property (e.g., 47×3=47+94). At this point, we can set up the diagram in Figure 1. We can observe that the number of spaces reserved to the quotient digits corresponds to the number of dividend digits, regardless of the development of the division (in our example, 2504 consists of 4 digits, so the diagram provides 4 spaces for the quotient digits). We therefore carry out the steps according to the acronym «TI×-», thus explicating its meaning:

- **T**ag the first dividend digit from the left, making a dot above it (in our example, the digit 2);
- **I**nsert the number of times the divisor is contained in the tagged digit in the first space dedicated to the quotient digits in the diagram. In our example, 47 is contained in 2 0 times, so we write 0 in the first space reserved for the quotient from the left;
- At this point (we are at the "×" in the acronym) we have to multiply the divisor with the number found, in our example 47×0 = 0;
- Finally, we subtract (we are at the "-" in the acronym) what we obtained from the tagged digit, in our example 2–0 = 2.

We then repeat the same procedure "TI×-". In our example, we now have to tag the digit 5, and transcribe next to the result of the last subtraction carried out, so that we now consider it as the number 25. Now, 47 in 25 is contained 0 times, so we transcribe 0 into the second space dedicated to the quotient, and then we calculate 47×0=0. At this point we subtract 0 from 25, obtaining still 25, and so on. We are therefore able to complete the division when we have tagged all the digits of the dividend, obtaining 53 as quotient, and the remainder 13 as the result of the last subtraction.

At the moment of the discussion we analyze here, participating students had worked since grade 2 on Canadian algorithm, while TI×- algorithm had been introduced for about a month: the students had learned to execute it correctly, but no time of the previous lessons was dedicated to the deepening of the mathematical meanings underlying its functioning.

This work focuses on the analysis of the signs that emerged in the mathematical discussion. We will distinguish between *situated signs, mathematical signs*, and *pivot signs* (Bartolini Bussi & Mariotti, 2008). Situated signs are signs that arise during the activity with the artefact, so they are contextual and understandable only to the participants to the activity at that time; mathematical signs, on the other hand, are the formal signs referring to the mathematical knowledge at the basis of the task designed by the teacher. Finally, pivot signs are particular artefact signs that the teacher can use to support a possible evolution from artefact signs to mathematical signs. We coded the transcripts classifying the signs with the labels “situated signs” (SS), “mathematical signs” (MS), “pivot signs” (PS). We focus especially on signs that could be related to mathematical meanings underlying the two algorithms, in particular those related to positional value of dividend digits and to the meaning of the sign to tag dividend digits in TI×- algorithm.

Data Analysis

To answer to our research questions, we chose to analyze two excerpts of the mathematical discussion of the first lesson, that we believe to be particularly significant to show the potential of synergy for the discovery of relationships between the two algorithms. The mathematical discussion was started by the teacher, who showed on her shared screen the operation 874:7 carried out with the two different algorithms (see Figure 2).

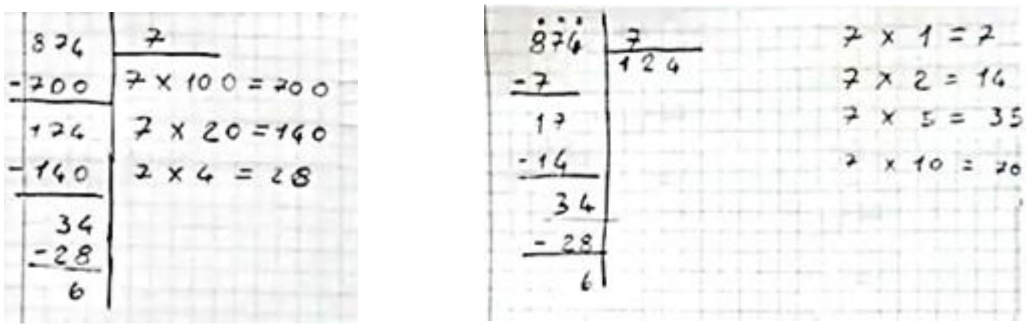

Figure 2: Optimized Canadian algorithm (on the left) and TI×- algorithm (on the right) to calculate the division 874:4

1	P1:	[...] I think that [...] it is not that we do faster in one algorithm, for example in the Canadian algorithm with respect to the TI×- algorithm, because in the Canadian algorithm we subtract bigger numbers, but in the TI×- algorithm we subtract the numbers... [...] 7, 14, 28, though... [the number 7] counts like the hundreds and then it is removed from the hundreds [of the dividend]...
2	Teacher:	Wait, I'll try to repeat what you said [...] in the TI×- algorithm, when I subtract 7, for example… what happens, to this 7 here? What do you mean, P1?
3	P1:	It's like... it's like a hundred [he means 7 hundreds], because ... it's below 8, but not because it is below 8, because it is used as... hundreds.
4	Teacher:	So, you are saying that this 7 is worth 7 hundreds... okay? And in the other algorithm what do we have in the first step, P1? […]
5	P1:	In the first step we have 700 [to subtract].[1]
6	Teacher:	Minus 700, ok.
7	P1:	It is because it involves all the numbers [the Canadian algorithm], but actually I think the TI×- algorithm is faster because you have to write fewer things.

In this first excerpt, the comparison of the two algorithms applied to the same operation allows P1 to make an interesting consideration, linked to positional value of dividend digits in TI×- algorithm (see the reference to MS "hundreds" in lines 1 and 3). In line 3, P1 tries to express something about the dependence of TI×- algorithm from some formal choices, such as tagging and respecting digits' vertical alignment. P1 seems to describe - still at an intuitive level - that the value of 7 subtracted from the dividend digit 8 should not be inferred from its vertical alignment, but from a certain "use" of 7 as hundreds, not better specified. It is only with teacher's intervention (line 4) that the identification of subtracted 7 in TI×- algorithm and subtracted 700 in Canadian algorithm becomes explicit (line 5), so that in line 7 P1 realizes a further step describing a more general difference between the two algorithms – namely, TI×- algorithm's articulation digit-by-digit, absent in Optimized Canadian algorithm.

[1] It is necessary to specify that, unlike in English where 7 hundreds and 700 are both pronounced as "Seven hundreds", in Italian 7 hundreds and 700 are pronounced as two different words, "sette centinaia" and "settecento" respectively. So, we note that in Italian the correspondence between 7 hundreds and 700 is not as transparent also for linguistic reasons.

8	P2:	I think they are both correct and I also noticed another thing: in the Canadian algorithm I see that you will always add a number... if you remove the two zeroes from 700 it becomes 7; if you remove the 4 from 174 it becomes 17. It is as if in Canadian algorithm you add a number [i.e. you write a digit at the end of each subtrahend].
[...]		
9	P2:	I was just thinking of the end [i.e., to the last subtraction 34 – 28 = 6 present as the last step in both algorithms], because when I thought that 700 becomes 7 when one removes the zeroes, it seemed strange to me that the end was identical [in both algorithms] and that you don't remove 4 from 34, so that it doesn't become 3.
10	Teacher:	Then you are saying: the last operation is the same.
11	P2:	Yes, because it is as if the division made different calculations but it has to arrive at the same point, where there is no other way to get to the result, so you have to do that ... that reasoning to get to the result.

In line 8 P2 describes a connection between the two algorithms, but differently from P1 she does not talk in terms of place value. Instead, she uses SSs "remove the zeros" and "add a number [digit]". In the expression "it is as if in Canadian algorithm you add a number", P2 formulates a simile that constitutes evidence of her attempt to describe something she has intuitively grasped but that she is not completely able to express. In this perspective, "add a number" and "remove the zeros" have the value of PS, with the potential to lead P2 to discover mathematical meanings at the basis of TI×- algorithm - in particular, positional reading of dividend digits and of numbers to be subtracted in the various steps. As we see in lines 9 and 11, this observation triggers P2 to deepen the relationships between the two algorithms. Indeed, P2 moves her attention to their last part. In the expression "it seemed strange to me that the end was identical and that you don't remove 4 from 34, so that it doesn't become 3", P2 identifies the last subtraction ("the end") as a common aspect of the two algorithms, but she also observes that this correspondence breaks her previous expectation on how TI×- algorithm works. The expression "you don't remove 4 from 34" is a PS, which recalls the previous sign "remove the zeroes" in line 8. Indeed, also this sign - "you don't remove 4 from 34" - has the potential to lead to the discover of positional reading within TI×- algorithm: the teacher can exploit this sign to support a reflection on the fact that the number 34 does not respect the correspondence identified by P2 because it must be interpreted in term of unities, differently from the other subtracted numbers in the previous steps of TI×- algorithm. SS "the end was identical", moreover, signals a further step towards a recognition of the common process at the basis of both algorithms, with respect to what P2 observed in line 8. In that line, she was still focusing on the perceived distance between the two algorithms; in line 9, instead, the identity of their last subtraction elicits a feeling of "strangeness", pusheing P2 to formulate in line 11 a more general conjecture on the similarities between the two algorithms, moving from a formal description towards meanings. Two significant SSs in this respect are "different calculations" and "same point": these are PSs with the potential to evolve towards a discovery of distributive process common to both algorithms, presented through different formal steps.

Discussion and conclusions

Data analysis shows how the designed task allowed the emergence, in the discussion among the involved children, of situated signs potentially significant for a progressive development of mathematical meanings that are crucial to explain the two algorithms' functioning, especially regarding TI×- algorithm. As we observed, in fact, this second algorithm is opaquer than Canadian one, with respect to the progressive distribution process underlying to both. Regarding the first research question, terms such as "hundreds", "remove the zeros" and "add a number" emerged as potentially crucial signs for this development, since they can lead to a reflection on the dependence of TI×- algorithm on place value of dividend digits. Situated signs as "different calculations" and "same point", instead, are significant because they were used to describe a similarity regarding the general mechanisms at the basis of both algorithms. These signs, in a TSM perspective, could be useful to the teacher to manage the discussion, in order to allow students discover the progressive distribution process underlying both algorithms. Regarding the second research question, the signs "different calculations" and "same point" emerged in relation with the issue of the identity of the last step of both algorithms. As shown by research with a similar approach to the discussion of division algorithms (see Lisarelli et al., 2021), this can be one of the key considerations to develop a "backward" reasoning to build a real argumentation of the two algorithms' functioning, using the more transparent algorithm to shed light on the steps of the opaquer one. This is particularly relevant for the argumentation of the opaquer algorithm's functioning (for Lisarelli et al. it was DMSB algorithm, for us is TI×- algorithm) and the discovery of mathematical meanings at its basis. Therefore, the designed task and the following discussion allowed the emergence of situated signs related to mathematical meanings common to both algorithms involved. It is necessary to underline that, both in our study and in that by Lisarelli et al., the synergy of artefacts could be seen as a substantial identity of distributive process underlying the involved algorithms: the opaquer algorithm can be seen as more refined and efficient on a formal level, through an appropriate recourse to the digits' vertical alignment and their reading according to their place value. Using a metaphor, we can say that this choice of algorithms as artefacts used in synergy transforms one algorithm into a sort of "can opener" of mathematical meanings for the other one. We believe that the analysis presented here contributes both to research concerning the introduction and the use of artefacts in mathematics classroom and to research concerning the teaching of algorithms in primary school. Our study contributes also to discussion about the potential of comparing algorithms and procedures as means of development of students' conceptual and procedural knowledge (e.g., Rittle-Johnson et al., 2017; Weber, 2019), since we highlighted the powerfulness of synergy of algorithms as artefacts, especially when among them intercours a relationship such as the one we described. Further studies are needed to confirm this, and to explore if there are other conditions determining which synergies are useful to develop mathematical meanings.

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