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# On the Monotonicity of the Stopping Boundary for Time-Inhomogeneous Optimal Stopping Problems

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**Abstract**
We consider a class of time-inhomogeneous optimal stopping problems and we provide sufficient conditions on the data of the problem that guarantee monotonicity of the optimal stopping boundary. In our setting, time-inhomogeneity stems not only from the reward function but, in particular, from the time dependence of the drift coefficient of the one-dimensional stochastic differential equation (SDE) which drives the stopping problem. In order to obtain our results, we mostly employ probabilistic arguments: we use a comparison principle between solutions of the SDE computed at different starting times, and martingale methods of optimal stopping theory. We also show a variant of the main theorem, which weakens one of the assumptions and additionally relies on the renowned connection between optimal stopping and free-boundary problems.

**Keywords** Optimal stopping · Monotone stopping boundary · Time-inhomogeneous diffusions · Partial information

**Mathematics Subject Classification** 60G07 · 60G40 · 60J60 · 49N30 · 35R35

## 1 Introduction

In this paper we consider a general class of time-inhomogeneous optimal stopping problems and we provide simple sufficient conditions on the data of the problem that guarantee monotonicity of the optimal stopping boundary. The novelty of our work is to prove this result when the underlying process is time-inhomogeneous. In our setting, the underlying process is the unique strong solution of a one-dimensional stochastic differential equation (SDE) whose drift coefficient may be time-dependent. We first show how to obtain monotonicity of the optimal stopping boundary when the

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reward function is time-homogeneous and then we extend the result to the case of a time-dependent reward function, when it is sufficiently regular to apply Ito's formula. We focus our attention on finite-horizon optimal stopping problems but our methods clearly apply also to infinite-horizon optimal stopping problems, as the latter do not carry an additional time-dependence in the domain of the admissible stopping times.

The behaviour of the optimal stopping boundary $t \mapsto b(t)$ is crucial in order to fully characterise an optimal stopping problem. In particular, continuity and monotonicity of the map $t \mapsto b(t)$ are two desirable properties. However, this regularity is usually studied on a case-by-case basis and the number of works that provide sufficient conditions to obtain these results in a general framework is limited. Classical tricks to show continuity of the stopping boundary are presented in [31] in various examples, whereas results in a general setting can be found in [4] (for one-dimensional diffusions) and [30] (for two-dimensional diffusions). Determining monotonicity of $t \mapsto b(t)$ can be even a more relevant turning point. First, it is a helpful result in order to obtain its continuity (as shown, e.g., in [4]). Furthermore, when the underlying process is strong Markov, it implies that the optimal stopping time $\tau^*_{t,x}$ is a continuous function of the starting point $(t, x)$ across the boundary[1] or, equivalently, that the boundary is regular for the interior of the stopping set in the sense of diffusions (a concept extensively illustrated in [7]). This yields global $C^1$-regularity of the value function, which is also a helpful result to characterise the stopping boundary (when continuous) as the unique continuous solution of a family of integral equations. An extensive probabilistic analysis of the geometry of a general class of optimal stopping problems, including continuity and monotonicity of the stopping boundary, is presented in [5] when the underlying diffusion and reward function are time-homogeneous. The shape of the continuation region is also studied under a general framework in [23]. However, their result on the monotonicity of $t \mapsto b(t)$ (see Proposition 4.4 therein) holds only for time-homogeneous diffusions. One contribution of this paper is to extend this result to a class of time-inhomogeneous diffusions. Regularity and characterisation of the value function are obtained for time-inhomogeneous Markov processes [27] and [37], and for time-inhomogeneous Poisson processes in [21]. To the best of our knowledge, no study of the properties of the stopping boundary has been developed in a general setting for time-inhomogeneous diffusions. It is also worth mentioning several theoretical works on the behaviour of the stopping boundary and of the value function in the context American options. We cite, among others, [2, 10, 22, 24, 26] and [36].

In order to obtain our results, we rely on probabilistic arguments. We first present a comparison principle between solutions of the underlying SDE computed at different starting times (see Lemma 3.2). Specifically, we show that if the drift coefficient $t \mapsto \mu(t, x)$ is monotone then the solutions of the SDE computed at different starting times are ordered. By means of this result and martingale methods of optimal stopping theory, we prove that if in addition a time-homogeneous reward function $x \mapsto g(x)$ is non-decreasing then $t \mapsto v(t, x)$ is also monotone for every $x \in \mathbb{R}$ (see Theorem 4.1). In a variant of the theorem we show that if monotonicity of $t \mapsto \mu(t, x)$ does not hold for every $x$ in the state space of the underlying process, we are able to weaken

[1] Here, we mean that if $(t, x) = (t, b(t))$ and $(t_n, x_n) \to (t, x)$ as $n \to \infty$, then $\tau^*_{t_n,x_n} \to \tau^*_{t,x}$ as $n \to \infty$, $\mathbb{P}$-a.s.

this condition and obtain the same result under a further assumption which involves the derivatives of the value function and it is implied by convexity of $x \mapsto v(t,x)$ (see Theorem 4.2). This proof additionally relies on the renowned connection between optimal stopping and free-boundary problems. An example of time-inhomogeneous diffusions which perfectly fits the weaker monotonicity assumption (of Theorem 4.2) on $t \mapsto \mu(t,x)$ is given by Brownian bridges. Several works have investigated optimal stopping problems involving Brownian bridges and we cite, among others, [6, 14, 16, 17, 19, 34] and [3]. Both Theorems 4.1 and 4.2 lead to the monotonicity of the optimal stopping boundary $t \mapsto b(t)$ (see Corollary 4.7). Then, we prove that monotonicity of $t \mapsto b(t)$ can be obtained even when the reward function $g$ depends on time (see Theorem 5.4). This extension holds when $g$ is sufficiently regular to apply Ito's formula and under the additional assumption of monotonicity of $t \mapsto \mathcal{L}g(t,x)$, where $\mathcal{L}$ denotes the infinitesimal generator of the underlying diffusion.

Our methods are particularly suited to study optimal stopping problems under incomplete information. The common feature of these problems is a random variable whose outcome is unknown to the optimiser and which affects the drift of the underlying process and/or the reward function. The literature is vast and diverse in this field and we cite, among others, [9, 11–15, 18–20, 35]. Our results apply, in particular, to models as in [13, 14] and [19] where a random variable affects the drift of the underlying process and, in a Bayesian formulation of the problem, only an arbitrary prior distribution of the random variable is known to the optimiser. As time evolves, the information obtained from observing the underlying process is used to update the initial beliefs about the unknown random variable. By filtering theory, the underlying process can be expressed as a time-inhomogeneous diffusion whose time-dependent drift is the conditional expectation of the unknown random variable given the observations of the process, which can be expressed the prior distribution. By means of these techniques, the original optimal stopping problem under incomplete information can be studied via an auxiliary stopping problem which fits into our framework, as we illustrate in Sect. 6.

The rest of the paper is organised as follows. In Sect. 2 we formulate the starting problem and we recall some standard results on optimal stopping theory. In Sect. 3 we provide a comparison principle between solutions of the underlying SDE starting at different times, which will be6m used in Sect. 4 to determine the monotonicity of the optimal stopping boundary. In Sect. 5 we extend the range of applicability for the results of Sect. 4 by considering stopping problems where also the reward functions may depend on time. Our methods are particularly suited to study a class of optimal stopping problems under partial information, which we describe in Sect. 6. We conclude by illustrating, in Sect. 7, some simple examples of optimal stopping problems where our results apply.

## 2 Starting Problem and Background Results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion which is $\mathbb{F}$-adapted. Let $T \in (0,\infty)$ be a finite time horizon. In this paper we treat finite-horizon optimal stopping problems, but it will be clear that our methods apply also to the infinite-horizon analogues, where the time-dependence of the value function stems only from the drift coefficient of the underlying SDE and not from the domain of the admissible stopping times.

Given an initial condition $X_t = x \in \mathbb{R}$ for $t \in [0, T)$, let $X = (X_s)_{s \geq t}$ be the time-inhomogeneous stochastic process described by

$$X_{t+s} = x + \int_0^s \mu(t+r, X_{t+r})\mathrm{d}r + \int_0^s \sigma(X_{t+r})\mathrm{d}W_r, \qquad s \in [0, T-t], \quad (2.1)$$

where the measurable functions $\mu : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are, respectively, the drift and diffusion coefficients. We assume that for every $t \in [0, T]$

$$|\mu(t, x) - \mu(t, y)| \leq K(t)|x - y|, \qquad x, y \in \mathbb{R}, \quad (2.2)$$

where the function $K : [0, T] \to [0, \infty)$ is integrable. Moreover, we assume that $x \mapsto \sigma(x)$ satisfies the standard Yamada–Watanabe condition which guarantees the strong existence and uniqueness of the solution for the SDE (2.1) (see, e.g., [33, Ch. V, Th. 40.1]). Namely, we assume that there exists an increasing function $\rho : [0, \infty) \to [0, \infty)$ such that

$$\int_0^\varepsilon \rho^{-1}(s)\mathrm{d}s = \infty, \qquad \forall\, \varepsilon > 0 \quad (2.3)$$

and

$$\big(\sigma(x) - \sigma(y)\big)^2 \leq \rho(|x - y|), \qquad \forall\, x, y \in \mathbb{R}. \quad (2.4)$$

In order to keep track of the initial condition $X_t = x$, we will sometimes denote the solution $X$ of the SDE (2.1) by $X^{t,x}$.

Given a (terminal) reward function $g : \mathbb{R} \to \mathbb{R}$, we define the optimal stopping problem

$$v(t, x) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\Big[g(X^{t,x}_{t+\tau})\Big], \qquad (t, x) \in [0, T] \times \mathbb{R}, \quad (2.5)$$

where $\mathcal{T}_t$ is the class of $\mathbb{F}$-stopping times $\tau$ such that $\tau \in [0, T-t]$, $\mathbb{P}$-a.s. To simplify the exposition, we start by considering stopping problems of the form (2.5). We then extend our results to stopping problems that include both a running reward function and a terminal reward function which may also depend on time (see Sect. 5).

Let $\mathcal{C}$ be the continuation region and its complement $\mathcal{D} := \mathcal{C}^c$ be the stopping region, respectively, defined by

$$\mathcal{C} := \{(t, x) \in [0, T] \times \mathbb{R} : v(t, x) > g(x)\}$$

and

$$\mathcal{D} := \{(t, x) \in [0, T] \times \mathbb{R} : v(t, x) = g(x)\}.$$

We now state some mild assumptions for the optimal stopping problem (2.5).

**Assumption 2.1** The reward function $g : \mathbb{R} \to \mathbb{R}$ is upper semi-continuous, the value function $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and we have that, for every $(t, x) \in [0, T] \times \mathbb{R}$,

$$\mathbb{E}\left[ \sup_{s\in[t,T]} \left|g(X_s^{t,x})\right| \right] < \infty. \tag{2.6}$$

***Remark 2.2*** In the optimal stopping literature continuity of the value function is usually proved on a case-by-case basis. Several case studies can be found in, e.g., [31, Ch. 6–7]. However, some general results exist. For instance, when $g(\cdot)$ is twice continuously differentiable and $\sigma(\cdot)$ is constant, [8, Th. 3.1] shows that the value function is locally Lipschitz continuous. If $g(\cdot)$ is bounded and continuous, then [28, Th. 4.3] guarantees continuity of the value function even when $X$ is just a Feller process (not necessarily a solution to an SDE).

Under Assumption 2.1, we obtain the next three propositions, which are standard results in optimal stopping.

**Proposition 2.3** *Let* $(t, x) \in [0, T] \times \mathbb{R}$*, then the stopping time*

$$\tau^* = \tau^*_{t,x} := \inf\{s \geq 0 : (t + s, X^{t,x}_{t+s}) \notin \mathcal{C}\} \wedge (T - t) \tag{2.7}$$

*is optimal for the stopping problem* (2.5)*.*

***Proof*** See, e.g., [31, Cor. 2.9].

**Proposition 2.4** *Let* $(t, x) \in [0, T] \times \mathbb{R}$*, then the process* $V = (V_s)_{s\in[0,T-t]}$*, defined by*

$$V_s = V^{t,x}_s := v(t + s, X^{t,x}_{t+s}),$$

*is a right-continuous supermartingale and the stopped process* $V^* = (V_{s\wedge\tau^*})_{s\in[0,T-t]}$ *is a right-continuous martingale.*

***Proof*** See, e.g., [31, Th. 2.4].

Let $\partial_t$, $\partial_x$ and $\partial_{xx}$ denote the time derivative, the spatial derivative and the second spatial derivative, respectively, and let $\partial\mathcal{C}$ denote the boundary of $\mathcal{C}$.

**Proposition 2.5** *We have that* $v \in C^{1,2}(\mathcal{C})$ *and it solves the free-boundary problem*

$$\begin{aligned} \Big(\partial_t + \mu(t, x)\partial_x + \tfrac{1}{2}(\sigma(x))^2\partial_{xx}\Big)v(t, x) &= 0, \qquad (t, x) \in \mathcal{C}, \\ v(t, x) &= g(x), \qquad (t, x) \in \partial\mathcal{C}. \end{aligned} \tag{2.8}$$

***Proof*** By Assumption 2.1, $\mathcal{C}$ is an open set. Then the free-boundary problem (2.8) follows, e.g., by the same arguments as in the proof of [22, Prop. 2.6].

***Remark 2.6*** Continuity of $v$ is not necessary to obtain Proposition 2.3 and Proposition 2.4 but lower semi-continuity would be sufficient. Moreover, these two propositions may hold with no continuity assumption on $v$: they still hold if, e.g., $g$ is continuous and non-negative and the integral condition (2.6) is satisfied (see, e.g., [25, App. D]). For the sake of simplicity, we assume continuity of $v$, which is necessary for Proposition 2.5.

To avoid further specific conditions on the data of the problem, we also introduce the following assumption.

**Assumption 2.7** There exists a (lower) optimal stopping boundary for the problem (2.5), i.e., a measurable function $b : [0, T] \to \mathbb{R}$ that separates $\mathcal{C}$ from $\mathcal{D}$. That is, we have

$$\mathcal{C} = \{(t, x) \in [0, T) \times \mathbb{R} : x > b(t)\}$$

and

$$\mathcal{D} = \{(t, x) \in [0, T) \times \mathbb{R} : x \leq b(t)\} \cup \{T\} \times \mathbb{R}.$$

Assumption 2.7 is usually proved by probabilistic arguments on a case-by-case basis (see, e.g., [22, Prop. 2.1]). It is easy to see that it holds if, e.g., $x \mapsto v(t, x) - g(x)$ is non-decreasing. More general sufficient conditions that guarantee the existence of an optimal stopping boundary are shown in, e.g., [23, Th. 4.3] and will be used later in Sect. 5. In this paper, we prove our results when a lower stopping boundary exists but it is clear that analogous arguments would follow when an upper stopping boundary exists instead. In particular, we would obtain a non-increasing upper stopping boundary instead of a non-decreasing lower stopping boundary.

## 3 A Comparison Principle

In this section we provide a comparison principle between solutions of the SDE (2.1) starting at different times, which will be applied in Sect. 4 to obtain monotonicity of the optimal stopping boundary.

Since we will make use of local times, we now recall their definition and some related properties. Given a real-valued, continuous semimartingale $Y$ and any $a \in \mathbb{R}$, the local time of $Y$ at $a$ is defined as the increasing process $L^a(Y)$ which satisfies

$$|Y_t - a| = |Y_0 - a| + \int_0^t \operatorname{sign}(Y_s - a)\mathrm{d}Y_s + L^a(Y)_t, \qquad \forall\, t \geq 0,$$

where $\operatorname{sign}(x) := 1$ if $x > 0$ and $\operatorname{sign}(x) := -1$ if $x \leq 0$. It is well known that there exists a modification of $L^a(Y)$ (which we still denote by $L^a(Y)$ for simplicity) such that the map $a \mapsto L^a(Y)$ is right-continuous with left limits (see, e.g., [33, Ch. IV, Th. 44.2]) and that $L^a(Y)$ satisfies the so-called *occupation time formula* (see, e.g., [32, Ch. VI, Cor. 1.6]), i.e.,

$$\int_0^t \Phi(Y_s)\mathrm{d}[Y]_s = \int_{-\infty}^{\infty} \Phi(a) L_t^a(Y)\mathrm{d}a, \tag{3.1}$$

for every $t \geq 0$ and for every non-negative Borel function $\Phi$. Then, we have the following lemma.

**Lemma 3.1** *Let $\rho : [0, \infty) \to [0, \infty)$ be increasing and satisfy condition (2.3). If*

$$\int_0^t \rho(Y_s)^{-1} \mathbb{1}_{\{Y_s > 0\}} \mathrm{d}[Y]_s < \infty, \qquad \forall\, t > 0, \tag{3.2}$$

*then $L_t^0(Y) = 0$ for every $t > 0$.*

***Proof*** Assume, by contradiction, that there exists $t > 0$ such that $L_t^0(Y) = 2\delta > 0$, for some $\delta > 0$. Then, by right-continuity, there exists $\varepsilon > 0$ such that $L_t^a(Y) > \delta$ for every $a \in (0, \varepsilon)$. Therefore, by the occupation time formula (3.1), we obtain

$$\int_0^t \rho(Y_s)^{-1} \mathbb{1}_{\{Y_s > 0\}} \mathrm{d}[Y]_s = \int_0^{\infty} \rho(a)^{-1} L_t^a(Y)\mathrm{d}a > \delta \int_0^{\varepsilon} \rho(a)^{-1}\mathrm{d}a = \infty,$$

where the last equality follows from (2.3). This contradicts the assumption (3.2).

We denote by $\mathcal{S} \subseteq \mathbb{R}$ the state space of the process $X$ defined in (2.1). For every $(t, x) \in [0, T] \times \mathbb{R}$, and for a non-empty Borel set $\mathcal{O} \subseteq [0, T] \times \mathcal{S}$, we define

$$\tau_{\mathcal{O}} = \tau_{\mathcal{O}}^{t,x} := \inf\{s \geq 0 : (t + s, X_{t+s}^{t,x}) \notin \mathcal{O}\} \wedge (T - t).$$

**Lemma 3.2** *Let $(t, x) \in [0, T] \times \mathbb{R}$ and let $\mathcal{O} \subseteq [0, T] \times \mathcal{S}$ be a non-empty Borel set. Assume that*

$$\mu(s, y) \leq \mu(u, y), \qquad \forall\, (s, y) \in \mathcal{O}, \quad \forall\, u \in [0, s]. \tag{3.3}$$

*Then, for every $u \in [0, t]$, we have that*

$$\mathbb{P}\Big(X_{t+s\wedge\tau_{\mathcal{O}}}^{t,x} \leq X_{u+s\wedge\tau_{\mathcal{O}}}^{u,x}, \quad \forall\, s \in [0, T - t]\Big) = 1.$$

***Proof*** Let $X_s^1 := X_{t+s}^{t,x}$, $X_s^2 := X_{u+s}^{u,x}$, $\mu_1(s, y) := \mu(t + s, y)$ and $\mu_2(s, y) := \mu(u + s, y)$ with $u \in [0, t]$. Thus, for $i = 1, 2$, we have

$$X_s^i = x + \int_0^s \mu_i(r, X_r^i)\mathrm{d}r + \int_0^s \sigma(X_r^i)\mathrm{d}W_r.$$

Then, for $Y := X^1 - X^2$ by assumption (2.4), we obtain

$$\int_0^s \rho(Y_r)^{-1} \mathbb{1}_{\{Y_r > 0\}} \mathrm{d}[Y]_r = \int_0^s \rho(|X_r^1 - X_r^2|)^{-1} \big(\sigma(X_r^1) - \sigma(X_r^2)\big)^2 \mathbb{1}_{\{Y_r > 0\}} \mathrm{d}r \leq s < \infty.$$

Therefore, by Lemma 3.1, we have that $L_s^0(Y) = 0$ for every $s \in [0, T]$. Thus, by Tanaka's formula, for every $s \in [0, T-t]$ we obtain

$$\begin{aligned}\left(X^1_{s\wedge\tau_{\mathcal{O}}} - X^2_{s\wedge\tau_{\mathcal{O}}}\right)^+ &= \int_0^{s\wedge\tau_{\mathcal{O}}} \left(\mu_1(r, X^1_r) - \mu_2(r, X^2_r)\right)\mathbb{1}_{\{X^1_r - X^2_r > 0\}}\mathrm{d}r \\ &\quad + \int_0^{s\wedge\tau_{\mathcal{O}}} \left(\sigma(X^1_r) - \sigma(X^2_r)\right)\mathbb{1}_{\{X^1_r - X^2_r > 0\}}\mathrm{d}W_r, \end{aligned} \tag{3.4}$$

where $\tau_{\mathcal{O}} = \tau_{\mathcal{O}}^{t,x}$ and $(x)^+ := \max\{x, 0\}$.

Now let $(\gamma_n)_{n\geq 0}$ be the localising sequence of stopping times defined by

$$\gamma_n := \inf\left\{s \geq 0 : \int_0^s \left(|\sigma(X^1_r)|^2 + |\sigma(X^2_r)|^2\right)\mathrm{d}r > n\right\} \wedge T, \qquad \forall\, n \in \mathbb{N},$$

so that the stochastic integrals in (3.4) are martingales when stopped at $\gamma_n$. Fix $s \in [0, T-t]$ and define $\tau_n := s \wedge \tau_{\mathcal{O}} \wedge \gamma_n$ for every $n \in \mathbb{N}$, so that $\lim_{n\to\infty} \tau_n = s \wedge \tau_{\mathcal{O}}$. Therefore, from (3.4), we obtain for every $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\Big[\left(X^1_{\tau_n} - X^2_{\tau_n}\right)^+\Big] &= \mathbb{E}\Big[\int_0^{\tau_n} \left(\mu(t+r, X^1_r) - \mu(u+r, X^2_r)\right)\mathbb{1}_{\{X^1_r - X^2_r > 0\}}\mathrm{d}r\Big] \\ &\leq \mathbb{E}\Big[\int_0^{\tau_n} \left(\mu(u+r, X^1_r) - \mu(u+r, X^2_r)\right)\mathbb{1}_{\{X^1_r - X^2_r > 0\}}\mathrm{d}r\Big] \\ &\leq \mathbb{E}\Big[\int_0^{\tau_n} K(u+r)\left(X^1_r - X^2_r\right)^+\mathrm{d}r\Big],\end{aligned}$$

where $K(t) > 0$ is the Lipschitz constant for $x \mapsto \mu(t, x)$ (recall (2.2)) and the second to last inequality follows from assumption (3.3). By Fatou's lemma, we obtain

$$\begin{aligned}\mathbb{E}\Big[\left(X^1_{s\wedge\tau_{\mathcal{O}}} - X^2_{s\wedge\tau_{\mathcal{O}}}\right)^+\Big] &\leq \liminf_{n\to\infty} \mathbb{E}\Big[\left(X^1_{\tau_n} - X^2_{\tau_n}\right)^+\Big] \\ &\leq \lim_{n\to\infty} \mathbb{E}\Big[\int_0^{\tau_n} K(u+r)\left(X^1_r - X^2_r\right)^+\mathrm{d}r\Big] \\ &= \mathbb{E}\Big[\int_0^{s\wedge\tau_{\mathcal{O}}} K(u+r)\left(X^1_r - X^2_r\right)^+\mathrm{d}r\Big],\end{aligned} \tag{3.5}$$

where the last equality follows from the monotone convergence theorem. Now notice that

$$\begin{aligned}\int_0^{s\wedge\tau_{\mathcal{O}}} K(u+r)\left(X^1_r - X^2_r\right)^+\mathrm{d}r &= \int_0^s K(u+r)\left(X^1_r - X^2_r\right)^+\mathbb{1}_{\{r\leq\tau_{\mathcal{O}}\}}\mathrm{d}r \\ &\leq \int_0^s K(u+r)\left(X^1_{r\wedge\tau_{\mathcal{O}}} - X^2_{r\wedge\tau_{\mathcal{O}}}\right)^+\mathrm{d}r.\end{aligned}$$

Then, continuing from (3.5) and using Fubini's theorem, we have that

$$0 \le \mathbb{E}\Big[\big(X^1_{s\wedge\tau_{\mathcal{O}}} - X^2_{s\wedge\tau_{\mathcal{O}}}\big)^+\Big] \le \int_0^s K(u+r)\mathbb{E}\Big[\big(X^1_{r\wedge\tau_{\mathcal{O}}} - X^2_{r\wedge\tau_{\mathcal{O}}}\big)^+\Big]\mathrm{d}r.$$

Therefore, since $t \mapsto K(t)$ is integrable, by Gronwall's lemma applied to the function $\varphi(s) := \mathbb{E}[(X^1_{s\wedge\tau_{\mathcal{O}}} - X^2_{s\wedge\tau_{\mathcal{O}}})^+]$ we obtain that

$$\mathbb{E}\Big[\big(X^1_{s\wedge\tau_{\mathcal{O}}} - X^2_{s\wedge\tau_{\mathcal{O}}}\big)^+\Big] = 0, \quad \forall\, s \in [0, T-t],$$

and by continuity of $Y = X^1 - X^2$ we reach the desired result.

***Remark 3.3*** Let $(t, x) \in [0, T] \times \mathbb{R}$. Notice that if $\mathcal{O} = [0, T] \times \mathcal{S}$, then $\tau^{t,x}_{\mathcal{O}} = T - t$ and so the result of Lemma 3.2 reads

$$\mathbb{P}\Big(X^{t,x}_{t+s} \le X^{u,x}_{u+s}, \quad \forall\, s \in [0, T-t]\Big) = 1.$$

## 4 Main Results

In this section we illustrate our main result for the optimal stopping problem (2.5), which provides monotonicity of $t \mapsto v(t, x)$ and in turn implies monotonicity of the stopping boundary. This is obtained by means of Lemma 3.2 and will be presented in two versions (Theorems 4.1 and 4.2) under different assumptions.

**Theorem 4.1** *Let Assumption 2.1 hold. Moreover, assume that*

(i) $x \mapsto g(x)$ *is non-decreasing.*
(ii) $t \mapsto \mu(t, x)$ *is non-increasing for every* $x \in \mathcal{S}$.

*Then,* $t \mapsto v(t, x)$ *is non-increasing for every* $x \in \mathbb{R}$.

***Proof*** Let $(t, x) \in [0, T] \times \mathbb{R}$ and $u \in [0, t]$. By assumption (ii), we can apply Lemma 3.2 with $\mathcal{O} = [0, T] \times \mathcal{S}$ and obtain that

$$\mathbb{P}\Big(X^{t,x}_{t+s} \le X^{u,x}_{u+s}, \quad \forall\, s \in [0, T-t]\Big) = 1. \tag{4.1}$$

By the (super)martingale property of the value function (recall Proposition 2.4) and since $\tau^* = \tau^*_{t,x}$ is optimal for $v(t, x)$ and sub-optimal for $v(u, x)$, we have that

$$\begin{aligned} v(t,x) - v(u,x) = V^{t,x}_0 - V^{u,x}_0 &\le \mathbb{E}\Big[V^{t,x}_{\tau^*} - V^{u,x}_{\tau^*}\Big] \\ &= \mathbb{E}\Big[v(t+\tau^*, X^{t,x}_{t+\tau^*}) - v(u+\tau^*, X^{u,x}_{u+\tau^*})\Big] \\ &\le \mathbb{E}\Big[g(X^{t,x}_{t+\tau^*}) - g(X^{u,x}_{u+\tau^*})\Big] \le 0, \end{aligned}$$

where to obtain the last inequality we have used assumption (i) and result (4.1). Hence, $t \mapsto v(t, x)$ is non-increasing for every $x \in \mathbb{R}$.

We now show that we can weaken the monotonicity assumption on $t \mapsto \mu(t,x)$ but still obtain, under an additional assumption, the same result as in Theorem 4.1. This alternative version partially relies on the free-boundary problem (2.8) and turns out to be useful in some optimal stopping problems, as we will illustrate in Sect. 7.

Let $\mathcal{M} := \{(t,x) \in [0,T] \times \mathcal{S} : \mu(t,x) < 0\}$ and let us denote by $\mathcal{M}^c$ its complement, i.e.,

$$\mathcal{M}^c := ([0,T] \times \mathcal{S}) \setminus \mathcal{M} = \{(t,x) \in [0,T] \times \mathcal{S} : \mu(t,x) \geq 0\}. \tag{4.2}$$

Throughout this paper we also assume that $\mathcal{M}$ is an open set, so that $(t+\tau_{\mathcal{M}}, X_{t+\tau_{\mathcal{M}}}) \in \mathcal{M}^c$ on $\{\tau_{\mathcal{M}} < T-t\}$, where recall that

$$\tau_{\mathcal{M}} = \tau_{\mathcal{M}}^{t,x} := \inf\{s \geq 0 : (t+s, X_{t+s}^{t,x}) \notin \mathcal{M}\} \wedge (T-t). \tag{4.3}$$

This holds if, e.g., $\mu$ is upper semi-continuous.

**Theorem 4.2** *Let Assumption 2.1 hold. Moreover, assume that*

(i) $x \mapsto g(x)$ *is non-decreasing.*
(ii) $\mu(t,x) \leq \mu(t-\varepsilon, x)$ *for every* $(t,x) \in \mathcal{M}$, $\varepsilon \in (0,t)$.
(iii) $\sigma^2(x)\partial_{xx}v(t,x) \geq -2\mu(t,x)\partial_x v(t,x)$ *for every* $(t,x) \in \mathcal{C} \cap \mathcal{M}^c$.

*Then,* $t \mapsto v(t,x)$ *is non-increasing for every* $x \in \mathbb{R}$.

***Remark 4.3*** Assumption (ii) of Theorem 4.2 is a weaker version of assumption (ii) of Theorem 4.1. Indeed, it does not require any monotonicity for $t \mapsto \mu(t,x)$ whenever $\mu$ is non-negative, but it only requires $t \mapsto \mu(t,x)$ to be non-increasing on the points where it is negative. Thus, it should be noticed that if the underlying drift is non-negative, then no monotonicity assumption on $\mu$ may be needed to obtain our results, as long as assumption (iii) of Theorem 4.2 holds.

***Remark 4.4*** Since $x \mapsto X^{t,x}$ is non-decreasing (see, e.g., [33, Ch. V, Th. 43.1]) and, under the assumptions of Theorem 4.2, $x \mapsto g(x)$ is non-decreasing, we also have that $x \mapsto v(t,x)$ is non-decreasing. Thus, notice that assumption (iii) holds, in particular, if $x \mapsto v(t,x)$ is convex. This is in turn implied by convexity of $x \mapsto X^{t,x}$ and of $x \mapsto g(x)$. Therefore, assumption (iii) of Theorem 4.2 can be substituted by convexity of $x \mapsto X^{t,x}$ and of $x \mapsto g(x)$. However, if $\sigma(x)$ is sufficiently small or if $\mu(t,x)\partial_x v(t,x)$ is sufficiently large on $\mathcal{M}^c$, then we may not need $x \mapsto v(t,x)$ to be convex in order to satisfy assumption (iii).

***Proof*** We prove the result of the theorem in two steps. We first show that $\partial_t v(t,x) \leq 0$ for every $(t,x) \notin \partial\mathcal{C}$ and we then prove that this implies that $t \mapsto v(t,x)$ is non-increasing for every $x \in \mathbb{R}$.

*Step 1.* If $(t,x) \in \mathcal{D}\backslash\partial\mathcal{C}$ then $v(t,x) = g(x)$ and so $\partial_t v(t,x) = 0$. If $(t,x) \in \mathcal{C}\cap\mathcal{M}^c$ (we can skip this step if $\mathcal{M}^c = \emptyset$), then by (2.8)

$$\partial_t v(t,x) + \mu(t,x)\partial_x v(t,x) + \tfrac{1}{2}(\sigma(x))^2\partial_{xx}v(t,x) = 0,$$

and, by assumption (iii), we obtain

$$\partial_t v(t,x) \le 0, \qquad \forall\,(t,x) \in \mathcal{C} \cap \mathcal{M}^c. \tag{4.4}$$

To conclude the proof we consider $(t,x) \in \mathcal{C} \cap \mathcal{M}$ (we can skip this step if $\mathcal{M} = \emptyset$). By assumption (ii) we can apply Lemma 3.2 with $\mathcal{O} = \mathcal{M}$ and, for every $\varepsilon \in (0,t)$, we obtain

$$\mathbb{P}\Big(X^{t,x}_{t+s\wedge\tau_{\mathcal{M}}} \le X^{t-\varepsilon,x}_{t-\varepsilon+s\wedge\tau_{\mathcal{M}}}, \quad \forall\, s \in [0, T-t]\Big) = 1, \tag{4.5}$$

where $\tau_{\mathcal{M}} = \tau^{t,x}_{\mathcal{M}}$ is defined in (4.3). Let $\varepsilon \in (0,t)$, $\tau^* = \tau^*_{t,x}$ (recall (2.7)) and $\rho := \tau^* \wedge \tau_{\mathcal{M}}$. By the (super)martingale property of the value function (recall Proposition 2.4) and since $\tau^*$ is optimal for $v(t,x)$ and $\rho$ is sub-optimal for $v(t-\varepsilon,x)$, we have that

$$\begin{aligned}
v(t,x) - v(t-\varepsilon,x) &\le \mathbb{E}\Big[v(t+\rho, X^{t,x}_{t+\rho}) - v(t-\varepsilon+\rho, X^{t-\varepsilon,x}_{t-\varepsilon+\rho})\Big] \\
&\le \mathbb{E}\Big[\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{M}}\}}\big(g(X^{t,x}_{t+\tau^*}) - g(X^{t-\varepsilon,x}_{t-\varepsilon+\tau^*})\big)\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{1}_{\{\tau_{\mathcal{M}} < \tau^*\}}\big(v(t+\tau_{\mathcal{M}}, X^{t,x}_{t+\tau_{\mathcal{M}}}) - v(t-\varepsilon+\tau_{\mathcal{M}}, X^{t-\varepsilon,x}_{t-\varepsilon+\tau_{\mathcal{M}}})\big)\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{\tau^* \le \tau_{\mathcal{M}}\}}\big(g(X^{t,x}_{t+\tau^*}) - g(X^{t-\varepsilon,x}_{t-\varepsilon+\tau^*})\big)\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{1}_{\{\tau_{\mathcal{M}} < \tau^*\}}\big(v(t+\tau_{\mathcal{M}}, X^{t,x}_{t+\tau_{\mathcal{M}}}) - v(t-\varepsilon+\tau_{\mathcal{M}}, X^{t,x}_{t+\tau_{\mathcal{M}}})\big)\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{1}_{\{\tau_{\mathcal{M}} < \tau^*\}}\big(v(t-\varepsilon+\tau_{\mathcal{M}}, X^{t,x}_{t+\tau_{\mathcal{M}}}) - v(t-\varepsilon+\tau_{\mathcal{M}}, X^{t-\varepsilon,x}_{t-\varepsilon+\tau_{\mathcal{M}}})\big)\Big] \\
&\le \mathbb{E}\Big[\mathbb{1}_{\{\tau_{\mathcal{M}} < \tau^*\}}\big(v(t+\tau_{\mathcal{M}}, X^{t,x}_{t+\tau_{\mathcal{M}}}) - v(t-\varepsilon+\tau_{\mathcal{M}}, X^{t,x}_{t+\tau_{\mathcal{M}}})\big)\Big],
\end{aligned}$$

where to obtain the last inequality we have used assumption (i) and result (4.5) for the first term; result (4.5) and the fact that $x \mapsto v(t,x)$ is non-decreasing (recall Remark 4.4) for the third term. Dividing by $\varepsilon$, letting $\varepsilon \to 0$ and applying dominated convergence theorem (by assumption (2.6)), we obtain

$$\partial_t v(t,x) \le \mathbb{E}\Big[\mathbb{1}_{\{\tau_{\mathcal{M}} < \tau^*\}} \partial_t v(t+\tau_{\mathcal{M}}, X^{t,x}_{t+\tau_{\mathcal{M}}})\Big] \le 0,$$

where the last inequality follows from (4.4). Hence, $\partial_t v(t,x) \le 0$ also for $(t,x) \in \mathcal{C} \cap \mathcal{M}$ and the proof of Step 1 is completed.

*Step* 2. If $(t,x) \in \mathcal{D}$, then $v(t,x) = g(x)$ and, since $v(s,x) \ge g(x)$ for every $(s,x) \in [0,T] \times \mathbb{R}$, then $v(s,x) \ge v(t,x)$ for every $s \in [0,t]$.

Now let $(t,x) \in \mathcal{C}$. We want to show that also $(s,x) \in \mathcal{C}$ for every $s \in [0,t]$, which by Step 1 would imply that $v(s,x) \ge v(t,x)$ for every $s \in [0,t]$ and would conclude the proof. Assume, by contradiction, that

$$D_{t,x} := \{s \in [0,t] : (s,x) \in \mathcal{D}\} \ne \emptyset$$

and let $t_0 := \sup D_{t,x}$. Recall that, since $(t,x) \in \mathcal{C}$, we have $v(t,x) > g(x)$. Since $v$ is continuous (by Assumption 2.1), then $t_0 < t$ and $t_0 \in D_{t,x}$, i.e., $(t_0,x) \in \mathcal{D}$ and so

$v(t_0, x) = g(x)$. Moreover, by definition of $t_0$, we have $(s, x) \in \mathcal{C}$ for every $s \in (t_0, t]$ and so

$$v(t, x) - v(t_0 + \varepsilon, x) = \int_{t_0+\varepsilon}^{t} \partial_t v(s, x) \mathrm{d}s \leq 0, \qquad \forall\, \varepsilon > 0,$$

where the last inequality follows from Step 1. Hence, by letting $\varepsilon \to 0$ and using continuity of $v$, we have that $v(t_0, x) \geq v(t, x)$. This leads to a contradiction, as we would obtain

$$g(x) = v(t_0, x) \geq v(t, x) > g(x).$$

***Remark 4.5*** If assumption (ii) in Theorem 4.1 (and similarly for assumptions (ii) and (iii) in Theorem 4.2) is substituted by a symmetric assumption (i.e., if $t \mapsto \mu(t, x)$ is increasing) then, in infinite-horizon problems, we would obtain a symmetric result, i.e., $t \mapsto v(t, x)$ would be increasing. However, this is, in general, not the case for finite-horizon problems. In that context we would have two opposite driving effects as time increases: the drift $\mu$ that increases and the stopping time domain $\mathcal{T}_t$ that shrinks. The former leads to an increase of the value function with respect to time, whereas the latter leads to a decrease of the value function with respect to time. In order to study the monotonicity of $t \mapsto v(t, x)$ in such problems, it would be necessary (and, perhaps, not sufficient) to have a quantitative information on the monotonicity of $t \mapsto \mu(t, x)$.

***Remark 4.6*** In some cases it is possible to apply a pure PDE approach, as in (4.4), and to derive monotonicity of $t \mapsto v(t, x)$ also when the diffusion coefficient may be time-dependent. Consider the same SDE as in (2.1) but when also $\sigma$ may be a function of time, i.e., $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$. If $\mu(t, x) \geq 0$ for every $(t, x) \in [0, T] \times \mathcal{S}$, then $\mathcal{M}^c = [0, T] \times \mathcal{S}$ (recall (4.2)). Thus, under assumptions (i) and (iii) of Theorem 4.2 and in the same way as in (4.4), we would obtain

$$\partial_t v(t, x) \leq 0, \qquad \forall\, (t, x) \notin \partial\mathcal{C}.$$

Monotonicity of $t \mapsto v(t, x)$ then follows as in Step 2 of the proof of Theorem 4.2.

Monotonicity of $t \mapsto v(t, x)$, which follows from either Theorems 4.1 or Theorem 4.2, then yields monotonicity of the optimal stopping boundary.

**Corollary 4.7** *If $t \mapsto v(t, x)$ is non-increasing for every $x \in \mathbb{R}$ and Assumption 2.7 holds, then the optimal stopping boundary $t \mapsto b(t)$ is non-decreasing.*

***Proof*** Let $(t, x) \in \mathcal{C}$. Then, $v(t, x) > g(x)$ and, since $t \mapsto v(t, x)$ is non-increasing, we obtain that $v(s, x) \geq v(t, x) > g(x)$ for every $s \in [0, t]$. Hence, also $(s, x) \in \mathcal{C}$ and thus $t \mapsto b(t)$ is non-decreasing.

***Remark 4.8*** Monotonicity of $t \mapsto v(t, x)$ is also a helpful result to obtain continuity of the stopping boundary (see, e.g., arguments as in [4, Sec. 3] and [6, Lem. 4]).

## 5 Extension to Time-Dependent Reward Functions

In this section we show how to obtain monotonicity of the stopping boundary for more general time-inhomogeneous optimal stopping problems, which include a running reward function and a terminal reward function that may also depend on time. We consider the same underlying framework of Sect. 2 but we study the optimal stopping problem

$$v(t,x) := \sup_{\tau\in\mathcal{T}_t} \mathbb{E}\Big[\int_0^\tau f(t+s, X_{t+s}^{t,x})\mathrm{d}s + g(t+\tau, X_{t+\tau}^{t,x})\Big], \qquad (t,x)\in[0,T]\times\mathbb{R}, \tag{5.1}$$

where $X = X^{t,x}$ is defined in (2.1), $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ is a running reward function and $g : [0,T]\times\mathbb{R}\to\mathbb{R}$ is a terminal reward function. For this problem the continuation region $\mathcal{C}$ and the stopping region $\mathcal{D}$ are defined, respectively, by

$$\mathcal{C} := \{(t,x)\in[0,T]\times\mathbb{R} : v(t,x) > g(t,x)\}$$

and

$$\mathcal{D} := \{(t,x)\in[0,T]\times\mathbb{R} : v(t,x) = g(t,x)\}.$$

Moreover, we define $\mathcal{L}$ by

$$\mathcal{L}\varphi(t,x) := \big(\partial_t + \mu(t,x)\partial_x + \tfrac{1}{2}(\sigma(x))^2\partial_{xx}\big)\varphi(t,x),$$

for any sufficiently regular function $\varphi : [0,T]\times\mathbb{R}\to\mathbb{R}$, and we introduce the following assumption.

**Assumption 5.1** We have that $g\in C^{1,2}([0,T]\times\mathbb{R})$, the value function $v : [0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous and, for every $(t,x)\in[0,T]\times\mathbb{R}$,

$$\mathbb{E}\Big[\sup_{s\in[t,T]}\Big|\int_t^s f(r, X_r^{t,x})\mathrm{d}r + g(s, X_s^{t,x})\Big|\Big] < \infty,$$

and

$$\mathbb{E}\Big[\sup_{s\in[t,T]}\Big|\int_t^s \mathcal{L}g(r, X_r^{t,x})\mathrm{d}r\Big|\Big] < \infty.$$

Notice that Assumption 5.1 is the analogous of Assumption 2.1 except for the stronger regularity of $g$. Under this regularity, we can apply Ito's formula and obtain that

$$\begin{aligned} g(t+s, X_{t+s}) &= g(t,x) + \int_0^s \mathcal{L}g(t+r, X_{t+r})\mathrm{d}r \\ &\quad + \int_0^s \sigma(X_{t+r})\partial_x g(t+r, X_{t+r})\mathrm{d}W_r, \quad \forall\, s\in[0,T-t]. \end{aligned}$$

Now define the localising sequence of stopping times

$$\tau_n := \inf\left\{s \geq 0 : \int_0^s |\sigma(X_{t+r})\partial_x g(t+r, X_{t+r})|^2 \mathrm{d}r > n\right\} \wedge (T-t), \qquad \forall\, n \in \mathbb{N}.$$

Then, for every $n \in \mathbb{N}$ and $\tau \in \mathcal{T}_t$, we have that

$$\mathbb{E}\big[g(t+\tau\wedge\tau_n, X_{t+\tau\wedge\tau_n})\big] = g(t,x) + \mathbb{E}\bigg[\int_0^{\tau\wedge\tau_n} \mathcal{L}g(t+r, X_{t+r})\mathrm{d}r\bigg].$$

Therefore, by Vitali's theorem given Assumption 5.1, we obtain for every $\tau \in \mathcal{T}_t$

$$\begin{aligned}\mathbb{E}\big[g(t+\tau, X_{t+\tau})\big] &= \lim_{n\to\infty} \mathbb{E}\big[g(t+\tau\wedge\tau_n, X_{t+\tau\wedge\tau_n})\big] \\ &= g(t,x) + \mathbb{E}\bigg[\int_0^{\tau} \mathcal{L}g(t+r, X_{t+r})\mathrm{d}r\bigg].\end{aligned}$$

The function $w(t,x) := v(t,x) - g(t,x)$ is, thus, the value function for the optimal stopping problem

$$w(t,x) = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}\bigg[\int_0^{\tau} h(t+s, X^{t,x}_{t+s})\mathrm{d}s\bigg], \qquad (t,x) \in [0,T]\times\mathbb{R}, \tag{5.2}$$

where we have defined $h(t,x) := f(t,x) + \mathcal{L}g(t,x)$. Then, notice that

$$\mathcal{C} = \{(t,x) \in [0,T]\times\mathbb{R} : w(t,x) > 0\}$$

and

$$\mathcal{D} = \{(t,x) \in [0,T]\times\mathbb{R} : w(t,x) = 0\}.$$

Analogously to Sect. 2, under Assumption 5.1, we have that the stopping time

$$\tau^* = \tau^*_{t,x} := \inf\{s \geq 0 : (t+s, X^{t,x}_{t+s}) \notin \mathcal{C}\} \wedge (T-t) \tag{5.3}$$

is optimal for the problem (5.1) (and thus also for the problem (5.2)). Moreover, we obtain that the process $V = (V_s)_{s\in[0,T-t]}$, defined by

$$V_s = V^{t,x}_s := \int_0^s h(t+r, X^{t,x}_{t+r})\mathrm{d}r + w(t+s, X^{t,x}_{t+s}), \tag{5.4}$$

is a right-continuous supermartingale and the process $V^* = (V_{s\wedge\tau^*})_{s\in[0,T-t]}$ is a right-continuous martingale.

***Remark 5.2*** For the sake of simplicity, we have assumed $g \in C^{1,2}([0,T]\times\mathbb{R})$ but one may consider different (perhaps weaker, see, e.g., [29]) conditions in order to apply Ito's formula and reformulate the stopping problem (5.1) into the stopping problem (5.2).

We then study the optimal stopping problem (5.1) by means of the equivalent problem (5.2). We have the following result on the existence of an optimal stopping boundary.

**Proposition 5.3** *Assume that* $x \mapsto h(t,x)$ *is non-decreasing for every* $t \in [0,T]$*, then there exists a lower optimal stopping boundary for the problem* (5.1)*, i.e., a function* $b : [0,T) \to \mathbb{R} \cup \{\pm\infty\}$ *such that*

$$\mathcal{C} = \{(t,x) \in [0,T) \times \mathbb{R} : x > b(t)\}$$

*and*

$$\mathcal{D} = \{(t,x) \in [0,T) \times \mathbb{R} : x \leq b(t)\} \cup \{T\} \times \mathbb{R}.$$

***Proof*** Since $x \mapsto h(t,x)$ is non-decreasing for every $t \in [0,T]$, then $x \mapsto w(t,x)$ is non-decreasing for every $t \in [0,T]$. Hence, if $(t,x_1) \in \mathcal{D}$, then $(t,x_2) \in \mathcal{D}$ for every $x_2 \in (-\infty, x_1]$. Therefore, for $t \in [0,T)$, the function

$$b(t) := \sup\{x \in \mathbb{R} : w(t,x) = 0\}$$

is a lower optimal stopping boundary for the stopping problem (5.2) and, thus, also for (5.1).

We can now obtain monotonicity of the optimal stopping boundary also for the more general class of time-inhomogeneous optimal stopping problems in (5.1). Recall that $\mathcal{S}$ denotes the state space of $X$, and that we define $h(t,x) := f(t,x) + \mathcal{L}g(t,x)$ and

$$\mathcal{L}g(t,x) := \big(\partial_t + \mu(t,x)\partial_x + \tfrac{1}{2}(\sigma(x))^2\partial_{xx}\big)g(t,x).$$

**Theorem 5.4** *Let Assumption 5.1 hold. Moreover, assume that*

(i) $x \mapsto h(t,x)$ *is non-decreasing for every* $t \in [0,T]$ *and* $t \mapsto h(t,x)$ *is non-increasing for every* $x \in \mathcal{S}$.
(ii) $t \mapsto \mu(t,x)$ *is non-increasing for every* $x \in \mathcal{S}$.

*Then,* $t \mapsto w(t,x)$ *is non-increasing for every* $x \in \mathbb{R}$ *and so the optimal stopping boundary* $t \mapsto b(t)$ *is non-decreasing.*

***Proof*** Let $(t,x) \in [0,T] \times \mathbb{R}$ and $u \in [0,t]$. By assumption (ii), we can apply Lemma 3.2 with $\mathcal{O} = [0,T] \times \mathcal{S}$ and obtain that

$$\mathbb{P}\Big(X^{t,x}_{t+s} \leq X^{u,x}_{u+s}, \quad \forall\, s \in [0, T-t]\Big) = 1. \tag{5.5}$$

By the (super)martingale property (5.4) of $V$ and since $\tau^* = \tau^*_{t,x}$ is optimal for $w(t,x)$ and sub-optimal for $w(u,x)$, we have that

$$
\begin{aligned}
w(t,x) - w(u,x) = V_0^{t,x} - V_0^{u,x} &\leq \mathbb{E}\Big[V_{\tau^*}^{t,x} - V_{\tau^*}^{u,x}\Big] \\
&= \mathbb{E}\bigg[\int_0^{\tau^*} \Big\{h(t+s, X_{t+s}^{t,x}) - h(u+s, X_{u+s}^{u,x})\Big\}\mathrm{d}s\bigg] \\
&\quad + \mathbb{E}\Big[w(t+\tau^*, X_{t+\tau^*}^{t,x}) - w(u+\tau^*, X_{u+\tau^*}^{u,x})\Big] \\
&\leq \mathbb{E}\bigg[\int_0^{\tau^*} \Big\{h(t+s, X_{t+s}^{t,x}) - h(u+s, X_{t+s}^{t,x})\Big\}\mathrm{d}s\bigg] \\
&\quad + \mathbb{E}\bigg[\int_0^{\tau^*} \Big\{h(u+s, X_{t+s}^{t,x}) - h(u+s, X_{u+s}^{u,x})\Big\}\mathrm{d}s\bigg] \leq 0,
\end{aligned}
$$

where the last inequality follows from assumption (i) and result (5.5). Hence, $t \mapsto w(t,x)$ is non-increasing for every $x \in \mathbb{R}$. The monotonicity of $t \mapsto b(t)$ (whose existence is guaranteed by Proposition 5.3) is, thus, obtained by the same arguments as in the proof of Corollary 4.7.

***Remark 5.5*** Notice that the proof of [23, Prop. 4.4], which provides monotonicity of the optimal stopping boundary, holds only if the underlying process is time-homogeneous. Our Theorem 5.4 extends that result to time-inhomogeneous optimal stopping problems, under the additional assumption that $t \mapsto \mu(t,x)$ is non-increasing for every $x \in \mathbb{R}$.

## 6 Optimal Stopping under Incomplete Information

Our methods are particularly suited to study optimal stopping problems under incomplete information. To this purpose, in this section, we provide some background material on this class of problems and in Sect. 7 we will look into a specific example.

The common feature of these stopping problems is a random variable whose outcome is unknown to the optimiser and which affects the drift of the underlying process and/or the payoff function. The literature is vast and diverse in this field and we cite, among others, [9, 11–15, 18–20, 35]. We focus, in particular, on models as in [13], [14] and [19] where a random variable affects the drift of the underlying process and, in a Bayesian formulation of the problem, only an arbitrary prior distribution of the random variable is known to the optimiser. As time evolves, the information obtained from observing the underlying process is used to update the initial beliefs about the unknown random variable.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which it is defined a standard Brownian motion $W = (W_t)_{t\geq 0}$, and a real-valued random variable $Y$ (in this context called *signal*) independent of W and with (prior) probability distribution $\nu$. Let $T \in (0,\infty)$ be a finite time horizon. The underlying process (in this context called *observation process*) $X$ evolves according to

$$
\mathrm{d}X_t = h(Y)\mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = 0, \tag{6.1}
$$

where $h$ is a measurable function. Given a reward function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, we can then define the stopping problem

$$V := \sup_{\tau \in [0,T]} \mathbb{E}\Big[g(\tau, X_\tau)\Big], \tag{6.2}$$

where $\tau$ is a stopping time with respect to $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\in[0,T]}$, the augmented filtration generated by $X$.

The optimal stopping problem (6.2) is not in the form of (5.1), as the observation process $X$ depends on the (random) signal $Y$. Nevertheless, by means of stochastic filtering techniques, we will be able to formulate an auxiliary stopping problem, in the form of (5.1), which is explicitly connected to (6.2). In order to show this, we now recall some results from stochastic filtering, and we refer to [1, Sec. 3] for further details. Let $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ be the augmented filtration generated by $(W, Y)$ and let $Z = (Z_t)_{t\geq 0}$ be the process defined by

$$Z_t := \exp\Big(-h(Y)W_t - \tfrac{1}{2}[h(Y)]^2 t\Big), \qquad t \geq 0. \tag{6.3}$$

Under the assumption[2]

$$\mathbb{E}\big[(h(Y))^2\big] < \infty \quad \text{and} \quad \mathbb{E}\Big[(h(Y))^2 \int_0^t Z_s \mathrm{d}s\Big] < \infty, \qquad t \geq 0, \tag{6.4}$$

we obtain that $Z$ is an $\mathbb{F}$-martingale (see, e.g., [1, Prop. 3.12]). Therefore, we can define a probability $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ by specifying its Radon-Nikodym derivative

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = Z_T,$$

and we denote by $\tilde{\mathbb{E}}$ the expectation taken with respect to the measure $\tilde{\mathbb{P}}$. Let $\tilde{Z} = (\tilde{Z}_t)_{t\geq 0}$ be the process defined as $\tilde{Z}_t := Z_t^{-1}$ for $t \geq 0$, i.e.,

$$\tilde{Z}_t = \exp\Big(h(Y)X_t - \tfrac{1}{2}[h(Y)]^2 t\Big), \qquad t \geq 0.$$

Finally, we define the so called *innovation process* $B = (B_t)_{t\geq 0}$ by

$$B_t := \int_0^t \big(h(Y) - \mathbb{E}[h(Y)|\mathcal{F}_s^X]\big)\mathrm{d}s + W_t, \qquad t \geq 0$$

and we have the following result.

**Proposition 6.1** *Under assumption (6.4), we have that* $\mathbb{E}[h(Y)|\mathcal{F}_t^X] = f(t, X_t)$ *where*

$$f(t,x) := \frac{\int_{\mathbb{R}} h(y)\mathrm{e}^{xh(y)-[h(y)]^2 t/2}\nu(\mathrm{d}y)}{\int_{\mathbb{R}} \mathrm{e}^{xh(y)-[h(y)]^2 t/2}\nu(\mathrm{d}y)}. \tag{6.5}$$

[2] This assumption is only needed to obtain martingality of $Z$ and then apply Girsanov's theorem. Thus, one may consider alternative assumptions, e.g., Novikov's condition (see [1, Sec. 3.3]).

*Moreover, $B$ is a $\mathbb{F}^X$-Brownian motion and*

$$dX_s = f(s, X_s)ds + dB_s. \tag{6.6}$$

***Proof*** By [1, Prop. 3.15−3.16] (see also [1, Rem. 3.20]), we have that

$$\mathbb{E}[h(Y)|\mathcal{F}_t^X] = \frac{\tilde{\mathbb{E}}[\tilde{Z}_t h(Y)|\mathcal{F}_t^X]}{\tilde{\mathbb{E}}[\tilde{Z}_t|\mathcal{F}_t^X]}.$$

Moreover, [1, Prop. 3.13] shows that $X$ and $Y$ are independent under $\tilde{\mathbb{P}}$ and that the signal $Y$ has the same distribution under $\mathbb{P}$ and under $\tilde{\mathbb{P}}$. Therefore, from the last equation, we obtain that $\mathbb{E}[h(Y)|\mathcal{F}_t^X] = f(t, X_t)$, where $f$ is defined in (6.5). Then, the representation (6.6) is straightforward and the fact that $B$ is a $\mathbb{F}^X$-Brownian motion follows from [1, Prop. 2.30].

By means of (6.6), we can solve the original stopping problem (6.2) by studying the Markovian stopping problem

$$v(t, x) := \sup_{\tau\in[0,T-t]} \mathbb{E}\Big[g(t+\tau, X_{t+\tau}^{t,x})\Big], \qquad (t, x) \in [0, T] \times \mathbb{R}, \tag{6.7}$$

where $\tau$ is a $\mathbb{F}^X$-stopping time and $X^{t,x}$ follows the dynamics in (6.6) with $X_t^{t,x} = x \in \mathbb{R}$. Indeed, we have $V = v(0, 0)$. The optimal stopping problem (6.7) is now in the form of (5.1) and we may, thus, apply Theorem 5.4 to obtain monotonicity of its stopping boundary. The form of the drift coefficient $f(t, x)$ defined in (6.5) highly depends on the prior distribution $\nu$ and in Sect. 7 we will look at a simple example where the monotonicity required in assumption (ii) of Theorem 5.4 holds.

## 7 Examples

In this section we show some simple examples of time-inhomogeneous optimal stopping problems where our results apply. We consider different underlying time-inhomogeneous diffusions of the form (2.1) that give rise to corresponding optimal stopping problems of the form (2.5). We then determine under which conditions on the data of the problems we can apply our theorems and obtain monotonicity of the stopping boundary. We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion which is $\mathbb{F}$-adapted and $T \in (0, \infty)$ be a finite time horizon. For simplicity, in the following examples we also let Assumptions 2.1 and 2.7 hold.

### 7.1 Applications of Theorem 4.1

We begin by considering two simple time-inhomogeneous diffusions and the corresponding optimal stopping problems of the form (2.5). For the reward function of these two examples we only assume that $x \mapsto g(x)$ is non-decreasing, as required in Theorem 4.1.

First, for $(t, x) \in [0, T] \times \mathbb{R}$, let $X = X^{t,x}$ be a Brownian motion with time-dependent drift, described by

$$X_{t+s} = x + \int_0^s \mu(t+r)\mathrm{d}r + \sigma W_s, \qquad s \in [0, T-t].$$

If $t \mapsto \mu(t)$ is non-increasing, then we can apply Theorem 4.1 and obtain that $t \mapsto v(t, x)$ is non-increasing for every $x \in \mathbb{R}$. Thus, Corollary 4.7 guarantees that the corresponding optimal stopping boundary $t \mapsto b(t)$ is non-decreasing.

Now, for $(t, x) \in [0, T] \times (0, \infty)$, let $X = X^{t,x}$ be a geometric Brownian motion with time-dependent drift, described by

$$X_{t+s} = x + \int_0^s \gamma(t+r)X_{t+r}\mathrm{d}r + \int_0^s \sigma X_{t+r}\mathrm{d}W_r, \qquad s \in [0, T-t].$$

Its state space is $\mathcal{S} = (0, \infty)$ and its drift is $\mu(t, x) := x\gamma(t)$. If $t \mapsto \gamma(t)$ is non-increasing, we can apply Theorem 4.1 and obtain that $t \mapsto v(t, x)$ is non-increasing for every $x \in \mathbb{R}$. Thus, Corollary 4.7 guarantees that the corresponding optimal stopping boundary $t \mapsto b(t)$ is non-decreasing.

Notice that the assumptions on the monotonicity of the drifts for the previous two examples could be weakened by considering assumption (ii) of Theorem 4.2 instead. However, this comes with the cost of adding assumption (iii) of Theorem 4.2 (which is implied by convexity of $x \mapsto X^{t,x}$ and of $x \mapsto g(x)$, recall Remark 4.4).

### 7.2 Applications of Theorem 4.2

For some underlying time-inhomogeneous diffusion it happens that assumption (ii) of Theorem 4.1 does not hold. In these cases it is useful to consider Theorem 4.2, which weakens this assumption. We now show an example of this situation. This application is even more suited to our techniques as no specific assumption on the drift of the underlying process is required in order to apply Theorem 4.2. This is the case when the underlying time-inhomogeneous diffusion is a Brownian bridge and an example of this optimal stopping problem, where $g(x) = \mathrm{e}^x$, is studied in [6]. For our example we assume that $x \mapsto g(x)$ is non-decreasing (as required by assumption (i) of Theorem 4.2) and, for simplicity, that it is convex (to satisfy assumption (iii) of Theorem 4.2, but convexity is not strictly necessary, as explained in Remark 4.4).

For $(t, x) \in [0, T] \times \mathbb{R}$, let $X = X^{t,x}$ be a Brownian bridge pinned at 0 at time $T \in (t, \infty)$, whose dynamics are described by

$$X_{t+s} = x - \int_0^s \frac{X_{t+r}}{T - t - r} \mathrm{d}r + \sigma W_s, \qquad s \in [0, T - t), \tag{7.1}$$

with $X_T = 0$. It is easy to check that the unique strong solution to this SDE is given by

$$X_{t+s} = (1 - t - s)\left(\frac{x}{1 - t} + \int_0^s \frac{1}{1 - t - r} \mathrm{d}W_r\right), \qquad s \in [0, T - t).$$

Hence, $x \mapsto X^{t,x}$ is linear (and thus convex) and together with convexity of $x \mapsto g(x)$ we have that assumption (iii) of Theorem 4.2 is satisfied (recall Remark 4.4). In order to apply Theorem 4.2, we are only left to check that assumption (ii) is satisfied. The drift coefficient of the SDE (7.1) is $\mu(t, x) := -x/(T - t)$. Thus, we have that

$$\mathcal{M} := \{(t, x) \in [0, T) \times \mathbb{R} : \mu(t, x) < 0\} = \{(t, x) \in [0, T) \times \mathbb{R} : x > 0\}$$

and

$$\mu(t, x) = -\frac{x}{T - t} \le -\frac{x}{T - t + \varepsilon} = \mu(t - \varepsilon, x), \qquad (t, x) \in \mathcal{M}, \quad \varepsilon \in (0, t).$$

Therefore, also assumption (ii) of Theorem 4.2 holds. Hence, we can apply Theorem 4.2 and obtain that $t \mapsto v(t, x)$ is non-increasing for every $x \in \mathbb{R}$. Thus, Corollary 4.7 guarantees that the corresponding optimal stopping boundary $t \mapsto b(t)$ is non-decreasing.

Further examples can be shown to satisfy the assumptions of Theorem 4.2. For instance, optimal stopping problems where the underlying diffusion follows the Vasicek model (i.e., an Ornstein–Ulhenbeck process) or the Cox-Ingersoll-Ross model can be studied in their time-inhomogeneous version, i.e., when the long-term mean is allowed to be time-dependent.

### 7.3 An Application to Optimal Stopping under Incomplete Information

To conclude, we consider an example of an optimal stopping problem under incomplete information as in Sect. 6, for which we want to apply Theorem 4.1. Assume that $X$ follows the dynamics as in (6.1) with $h(y) = y$, i.e.,

$$\mathrm{d}X_t = Y \mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = 0,$$

where $Y$ is a real-valued random variable independent of $W$ and with prior distribution $\nu$ such that (6.4) holds. Then, Proposition 6.1 guarantees that

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \mathrm{d}B_t,$$

where

$$f(t,x) := \frac{\int_{\mathbb{R}} y e^{xy - y^2 t/2} \nu(\mathrm{d}y)}{\int_{\mathbb{R}} e^{xy - y^2 t/2} \nu(\mathrm{d}y)},$$

and $B$ is a Brownian motion with respect to the filtration generated by $X$. We can then consider corresponding optimal stopping problems as in (6.7), where the reward function $x \mapsto g(x)$ is non-decreasing (so that it satisfies assumption (i) of Theorem 4.1). We now want to look at an example where also assumption (ii) of Theorem 4.1 is satisfied, i.e., where the drift $t \mapsto f(t,x)$ is non-increasing, so that we can obtain monotonicity of the optimal stopping boundary. Let us consider a simple example of a two-point prior distribution $\nu = p\delta_l + (1-p)\delta_r$, where $\delta_x$ is the Dirac delta centred in $x \in \mathbb{R}$, $-\infty < l < r < \infty$ and $p \in (0,1)$. Notice that (6.4) holds and so the results of Proposition 6.1 are guaranteed with

$$f(t,x) = \frac{pl e^{lx - l^2 t/2} + (1-p) r e^{rx - r^2 t/2}}{p e^{lx - l^2 t/2} + (1-p) e^{rx - r^2 t/2}}.$$

For assumption (ii) of Theorem 4.1 to hold, we would like to obtain $\partial_t f \le 0$. After some algebra, we have that

$$\partial_t f(t,x) = -\frac{1}{2} \frac{p(1-p) e^{(l+r)x - (l^2+r^2)t/2} (r-l)^2 (r+l)}{\left[p e^{lx - l^2 t/2} + (1-p) e^{rx - r^2 t/2}\right]^2}.$$

If $r \ge -l$, then $\partial_t f(t,x) \le 0$ for every $(t,x) \in [0,T] \times \mathbb{R}$. Therefore, we can apply Theorem 4.1, and obtain that $t \mapsto v(t,x)$ is non-increasing for every $x \in \mathbb{R}$. Thus, Corollary 4.7 guarantees monotonicity of the corresponding stopping boundary also for this example of optimal stopping problem under incomplete information. Notice that this is only one example of prior distribution and different priors may be investigated. For instance, Theorem 4.1 can also be applied to the optimal stopping of a Brownian bridge with unknown pinning point whose prior is normal $\mu \sim \mathcal{N}(m, \gamma^2)$ (see, [14, Sec. 5.1]).

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