Centralized Allocation in Multiple Markets*†

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Abstract

We provide a characterization result for the problem of centralized allocation of indivisible objects in multiple markets. Each market may be interpreted either as a different type of object or as a different period. We show that every allocation rule that is strategy-proof, Pareto-efficient and nonbossy is a sequential dictatorship. The result holds for an arbitrary number of agents and for any preference domain that contains the class of lexicographical preferences.

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1 Introduction

A central planner often faces the task of distributing indivisible objects to the agents. For example, municipalities assign public houses to families, education departments allocate students to public schools, and firms allocate projects among workers. The problem of assigning indivisible objects to agents when monetary transfers are not allowed has been widely studied from many different perspectives. Pápai (2000), in particular, shows that the only way to implement a Pareto-efficient allocation with a rule that is strategy-proof,

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nonbossy and reallocation-proof is through the use of a *hierarchical exchange rule*.[1] Pycia and Ünver (2011) further show that the only rules that are strategy-proof, nonbossy and Pareto-efficient are the trading cycles rules. Not only are these results of theoretical importance, but they also provide important guidance for practitioners and policy makers.[2]

We study the centralized allocation problem that takes place in multiple markets, where each market may be interpreted either as a different type of object or as a different period. Indeed, in reality agents are often involved in more than one assignment problem at one time; people who participate in the allocation of public housing, for example, might also have their children enrolled in public schools. In the US there are more than one thousand federally-funded benefit and assistance programs, many of which involve the assignment of indivisible objects. Moreover, a single family may be eligible for many of these programs at the same time.[3] In Brazil there is a comprehensive social program called *Bolsa Familia*, which is a conditional cash transfer program that benefits over 11 million families. The *Bolsa Familia* unified several separate clearinghouses that were already in place. Precisely, it unified the following existing social programs National School Allowance Program, the Food Card Program, the Food Allowance Program, and the Child Labor Eradication Program.[4]

Additionally, our results apply to dynamic matching problems. One example of dynamic matching previously studied in the literature is the allocation of new physicians in the United Kingdom, where each young doctor applies for two successive positions: a medical post and a surgical post (Roth, 1991; Irving, 1998). Another illustrative example is the allocation of courses among the faculty of a department in which each professor teaches one undergraduate and one graduate course. The important market design problem known as the school choice problem might be studied as a dynamic matching problem if (i) student mobility is taken

[1] A hierarchical exchange rule is a generalization of the top trading cycles allocation rule and can be described as follows. In the first stage, the planner distributes the objects to the agents; in particular, some agents might receive multiple objects while others might receive none. Then, the top trading cycles algorithm is applied, with each agent pointing to her preferred object and each object pointing to its owner. The agents who form a cycle receive the objects they pointed to. The non allocated objects whose owners left in the first stage are inherited by the remaining agents and the top trading cycles algorithm is applied again. The procedure is repeated until all agents are assigned an object.

[2] For example, on April 16, 2012, it was announced that the New Orleans Recovery School District would utilize a version of the top trading cycles allocation rule as the allocation rule for the centralized enrollment of children in public schools (Vanacore, 2012).

[3] The website *benefits.gov* (formerly GovBenefits.gov) is a partnership of seventeen federal agencies as well as other governmental agencies that provides a centralized source of information for many of these assistance programs.

[4] More details can be obtained directly from the official website: http://www.planalto.gov.br/ccivil_03/_ato2004-2006/2004/lei/l10.836.htm

into account, or (ii) sibling priorities are considered. Finally, Kennes et al. (2014) introduced the dynamic matching problem of allocating young children to public day care centers.

In our model, there are n agents and two (or more) markets, and each agent must be assigned at most one object from each market. Agents have preferences over the different bundles, where a bundle is a vector consisting of one object per market. We restrict our attention to the cases in which markets are *independent*, by which we mean that the set of objects available in a particular market is exogenous and not affected by the other markets.

In environments with multiple markets, there might be scope for a mutually beneficial trade between agents even if the allocation is Pareto-efficient within each market. This raises the question of our paper: how to implement a Pareto-efficient outcome in a multiple-market problem?

In our main result (Theorem 2) we show that the set of rules that are strategy-proof and nonbossy and that implement a Pareto-efficient allocation are the sequential dictatorships. These rules generalize the serial dictatorship rule in that the order of the agents who choose the objects might be a function of the choices made previously by the other agents.[5] Despite its wide use in the literature, we feel that it is important to justify the use of the nonbossyness axiom in our formulation. First, a rule that fails this axiom is susceptible to coalitional deviations, implying that it might be problematic for implementing it. In addition, from a normative point of view, a bossy mechanism might be considered as unfair, since an agent might be able to dictate others' allocations without changing her own allocation.

We first introduce a novel class of preferences, which we call *(generalized) lexicographic preferences*. We then prove that our result holds for any preference domain that includes the domain of lexicographic preferences. In particular, it holds for the class of separable preferences, a widely used domain when agents demand more than one object. By using the class of lexicographic preferences we are able to contrast our results more sharply with Pycia and Ünver (2011)'s result on single market allocations. In our preference domain, all objects from all markets are ranked for each individual under a single ranking. This means that the cardinality of the set of preferences in our multi-market environment is slightly smaller than the one in a single market case which pools the objects from all the markets. Thus, the fact that our set of rules is much narrower than in the single market case is not due to the increased cardinality of the domain of preferences, but it follows from the specific feature of

[5] In the single-market case, the sequential dictatorship rules are special cases of Pápai's (2000) hierarchical exchange rules.

multiple markets that each individual demands more than one object. From the technical perspective, we believe that the domain of lexicographical preferences will be useful in other studies in which agents demand more than one objects, due to the tractability of this class of preferences.

This paper is related to the literature on centralized allocation of multiple objects. Pápai (2001), Ehlers and Klaus (2003), and Hatfield (2009) also obtain the same characterization result as ours, but in different settings. Hatfield (2009) studies a model in which each agent must be allocated an exact number of objects, which he refers to as fixed *quotas*, from one pool of objects. Our model is related to Hatfield's work in the sense that two markets in our model might be interpreted as a quota of two goods for every individual. However, in our setting any two objects that an individual can be allocated must be drawn from different pool of objects. In addition, the smallest domain the above-mentioned studies consider is the one of separable preferences. Our domain of lexicographical preferences is smaller than that of separable preferences.

To the best of our knowledge, this is the first paper that provides a complete characterization of centralized allocation in multiple markets without an endowment structure. Konishi et al. (2001) considered the multi-type allocation problem,[6] but in their work each agent is initially endowed with one object– as in the economy proposed by Shapley and Scarf (1974). Konishi et al. (2001) show that the core may be empty in these multi-type Shapley-Scarf economies and also that there are no Pareto-efficient, individually rational, and strategy-proof rules. Here, since we do not assume an initial endowment structure, we do not impose the individual rationality constraint, which plays a crucial role in their results.

Nguyen et al. (2014) work on a similar problem of allocating multiple objects–from different markets– to different agents. They provide a mechanism, which generalizes the Probabilistic Serial mechanism (see Bogomolnaia and Moulin (2001)) by having the mechanism return probability shares over the different bundles. They prove that their mechanism is efficient, envy-free and asymptotically strategy-proof. While their matching model is more general than ours, their axioms are not the same as ours, in particular they focus on asymptotic strategy-proofness, for example.

This paper is organized as follows. In the following section we describe the model and state its main assumptions. In Section 3, we describe and define an allocation rule and its

[6]See Klaus (2008) for further reference.

main properties. In Section 4, we describe the sequential dictatorship. We prove our main result (Theorem 2) in Section 5. Finally, we conclude the paper in Section 6. In the Appendix we include the proof of the special case of two agents and two goods in each market as well as the general proof of Theorem 2 for the case in which the number of agents is greater than two.

2 Model

Let $N = \{1, \cdots, n\}$, where $n < \infty$, be the set of agents. There are two types of indivisible objects, A and B, which also stand for the respective sets of objects types. We refer to a pair $(a, b) \in A \times B$ as a bundle. For convenience, we assume that an artificial null object, 0, is in both sets A and B. Throughout the paper, we assume that $|A \setminus \{0\}| \geq n$ and $|B \setminus \{0\}| \geq n$, i.e., there are enough A- and B-objects to distribute to the agents. An allocation $x = (x_1, \cdots, x_n)$ is a list of the assignments for the n agents, where $x_i \in A \times B$. If $x_i = (a, b)$, then agent i is assigned the bundle (a, b). We write x_i^A (x_i^B) to denote the A-object (B-object) that agent i obtains under allocation x. We refer to $x^A = (x_i^A)_{i \in N}$ and $x^B = (x_i^B)_{i \in N}$ as the A- and B-allocation, respectively. An allocation x is feasible if no object (except the null object) is assigned to more than one agent. Similarly, we define feasible A- and B-allocations. Let X stand for the set of all feasible allocations. The notations X^A and X^B stand for the sets of feasible A- and B -allocations, respectively.

Each agent i has a preference relation R_i over $A \times B$, and $R = (R_i)_{i \in N}$ is the preference profile of the agents. We use the conventional notation R_{-i} to denote $(R_j)_{j \neq i}$. Throughout the paper we will maintain the following two assumptions on preferences:

Assumption 1 (Strictness)**.** *Each agent's preference relation R_i is strict, i.e., the conditions $(a, b) R_i (\hat{a}, \hat{b})$ and $(\hat{a}, \hat{b}) R_i (a, b)$ together imply that $(a, b) = (\hat{a}, \hat{b})$.*

Assumption 2 (Desirability)**.** *For any $(a, b) \neq (0, 0)$ and $i \in N$, $(a, b) R_i (0, 0)$.*

We use the notation $\mathcal{R} = \Pi_{i \in N} \mathcal{R}_i$, where $\mathcal{R}_i$ stands for the set of all preference relations for agent i that satisfy Assumptions 1 and 2.[7] Clearly, $\mathcal{R}$ is a very big preference domain, and we will later place some restrictions and concentrate on preference domains that are

[7] Although $\mathcal{R}_i$ is common for all the agents, we are using the notations $\mathcal{R}_i$ and $\mathcal{R}$ to keep them consistent with the notations R_i and R.

subsets of $\mathcal{R}$. Any domain $\bar{\mathcal{R}}$ we will consider will be assumed to be a Cartesian product of agents' preference domains, i.e., $\bar{\mathcal{R}}$ satisfies that $\bar{\mathcal{R}} = \Pi_{i\in N}\bar{\mathcal{R}}_i$ where $\bar{\mathcal{R}}_i \subseteq \mathcal{R}_i$.

The following preference domain–that of separable preferences– is widely used in the literature when agents demand more than one object.

Definition 1 (Separability)**.** *A preference relation, R_i, is separable if there exists a function $u_i : A \cup B \to \mathbb{R}$ such that*

$$(a, b)R_i(\hat{a}, \hat{b}) \text{ for some } a, \hat{a} \in A \text{ and } b, \hat{b} \in B \text{ if and only if } u_i(a) + u_i(b) \geq u_i(\hat{a}) + u_i(\hat{b}).$$

The domain of separable preference profiles, denoted by $\mathcal{R}^{SP}$, consist of all the separable preference profiles.

The domain of separable preferences rules out complementarity between A- and B-objects. For any separable preference relation $R_i \in \mathcal{R}_i^{SP}$, one can define two (strict) preference relations, R_i^A and R_i^B. The A-preference relation R_i^A is defined over A, and aR_i^Aa' holds only if $(a, b)\, R_i\, (a', b)$, for $\forall b \in B$. The B-preference relation R_i^B is defined in a similar manner. For any $\bar{\mathcal{R}} \subseteq \mathcal{R}^{SP}$, we denote the corresponding sets of A- and B-preference relations by $\bar{\mathcal{R}}^A$ and $\bar{\mathcal{R}}^B$, respectively.

Although the domain of separable preferences has a very specific structure, it is significantly larger than the domain of preferences over single objects in $A \cup B$. We seek a domain of preferences in which its each member is unambiguously defined by a list of objects in $A \cup B$. Below we define the class of preferences that we will concentrate.

Definition 2 (Generalized lexicographical preference)**.** *A preference relation of agent i, R_i, is a (generalized) lexicographical preference if there exists a bijection $\eta_i : A \cup B \to \{1, 2, \cdots, |A \cup B|\}$, which we call a lexicographical ordering, such that whenever $(a, b)R_i(\bar{a}, \bar{b})$ for some $(a, b), (\bar{a}, \bar{b}) \in A \times B$, one of the following conditions is satisfied:*

(i) $\min\{\eta_i(a), \eta_i(b)\} < \min\{\eta_i(\bar{a}), \eta_i(\bar{b})\}$; *or*

(ii) $\min\{\eta_i(a), \eta_i(b)\} = \min\{\eta_i(\bar{a}), \eta_i(\bar{b})\}$ & $\max\{\eta_i(a), \eta_i(b)\} \leq \max\{\eta_i(\bar{a}), \eta_i(\bar{b})\}$.

If the preference relation of agent i is lexicographical, then we use the notation L_i to denote i's preference relation. The notation $(a, b)L_i(\bar{a}, \bar{b})$ means that either agent i (strictly) prefers (a, b) to $(\bar{a}, \bar{b})$ or $(a, b) = (\bar{a}, \bar{b})$. When L_i is a lexicographical preference relation associated with ordering η_i, we usually write $L_i : \eta_i^{-1}(1), \eta_i^{-1}(2), \cdots, \eta_i^{-1}(|A \cup B|)$. Each lexicographical preference relation can therefore be represented by a single list of objects in $A \cup B$, which is itself a huge simplification in that even separable preferences have to be represented by a list of pairs in $A \times B$. We use the notation $\mathcal{L}$ to denote the set of all lexicographical preferences.

Before we move on, let us consider an example of lexicographical preferences. Let $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_m\}$. Consider $L_i : a_1, b_1, a_2, b_2, \cdots, a_m, b_m$. Then

$$\begin{gathered}(a_1, b_1)L_i(a_1, b_2)L_i \cdots L_i(a_1, b_m)L_i \\ (a_2, b_1)L_i(a_3, b_1)L_i \cdots L_i(a_m, b_1)L_i \\ (a_2, b_2)L_i(a_2, b_3)L_i \cdots L_i(a_2, b_m)L_i \\ (a_3, b_2)L_i(a_4, b_2)L_i \cdots L_i(a_m, b_2)L_i \\ \vdots\end{gathered}$$

As we mentioned above, each list of objects on $A \cup B$ defines a unique lexicographical preference relation. Consequently, the cardinality of lexicographical preference domain is smaller than the domain of the preferences over objects in $A \cup B$. Below we show that the lexicographical preferences are also separable.

Lemma 1 (Separability)**.** *Any lexicographical preferences L_i is separable.*

Proof. Let η_i be an ordering associated with L_i. For all $c \in A \cup B$, set $u_i(c) = 2^{-\eta_i(c)}$. Consider any $a, \hat{a} \in A$ and $b, \hat{b} \in B$. One can easily verify that $(a, b)L_i(\hat{a}, \hat{b})$ if and only if $u_i(a) + u_i(b) \geq u_i(\hat{a}) + u_i(\hat{b})$. □

The domain of lexicographical preferences can be too narrow in applications, but we will later show that our characterization result holds for all preference domains that contain all lexicographical preferences. In this sense, we are also identifying a narrow domain in which our main characterization result holds.

3 Allocation Rule and Its Properties

An allocation rule (or a direct mechanism) on domain $\bar{\mathcal{R}} = \Pi_{i \in N} \bar{\mathcal{R}}_i$, denoted by $\varphi = (\varphi^A, \varphi^B)$, is a mapping from $\bar{\mathcal{R}}$ to the set of feasible allocations X. For a given allocation rule on domain $\bar{\mathcal{R}}$, the agents play a revelation game in which each agent's strategy set is $\bar{\mathcal{R}}_i$.

We now turn our attention to the properties of the allocation rules that we will consider in this paper. First, we say that an allocation rule is efficient if it returns an efficient allocation for each preference profile.

Definition 3 (Pareto Efficiency)**.** *An allocation rule $\varphi: \bar{\mathcal{R}} \to X$ is* Pareto-efficient *if for all $R \in \bar{\mathcal{R}}$, there does not exist an allocation $x \neq \varphi(R)$ such that $x_i R_i \varphi_i(R)$, for all $i \in N$.*

An allocation rule is strategy-proof if, in its associated revelation game, reporting one's true preferences is a weakly dominant strategy for every agent. Below we present the formal definition of strategy-proofness.

Definition 4 (Strategy-Proofness)**.** *An allocation rule $\varphi: \bar{\mathcal{R}} \to X$ is* strategy-proof *if, for all $i \in N$, all $R \in \bar{\mathcal{R}}$, and all $\hat{R}_i \in \bar{\mathcal{R}}_i$,*

$$\varphi_i(R) \, R_i \varphi_i\left(\hat{R}_i, R_{-i}\right).$$

An allocation rule is nonbossy if no agent can change the others' allocations without changing her own allocation.

Definition 5 (Nonbossiness)**.** *An allocation rule $\varphi: \bar{\mathcal{R}} \to X$ is nonbossy if, for all $R \in \mathcal{R}$, all $i \in N$, and all $\hat{R}_i \in \bar{\mathcal{R}}_i$,*

$$\varphi_i(R_i, R_{-i}) = \varphi_i\left(\hat{R}_i, R_{-i}\right) \Longrightarrow \varphi(R_i, R_{-i}) = \varphi\left(\hat{R}_i, R_{-i}\right).$$

Finally, an allocation rule $\varphi: \bar{\mathcal{R}} \to X$ is an φ-monotonic allocation rule if it satisfies following property: if each agent's lower contour set of $\varphi(R)$ expands (weakly) when going from preference profile R to R^1, then φ prescribes the same allocation for both $R \in \bar{\mathcal{R}}$ and $R^1 \in \bar{\mathcal{R}}$.

Definition 6 (Monotonicity)**.** *For a given allocation rule $\varphi: \bar{\mathcal{R}} \to X$, we say that a preference profile $R^1 \in \bar{\mathcal{R}}$ is an φ-monotonic change of $R \in \bar{\mathcal{R}}$ if, for each agent i, the relative ranking of the allocation $\varphi_i(R)$ weakly improves under R^1, specifically,*

$$\{(a,b) \in A \times B : \varphi_i(R) R_i (a,b)\} \subseteq \left\{(a,b) \in A \times B : \varphi_i(R) R_i^1 (a,b)\right\}.$$

An allocation rule $\varphi: \bar{\mathcal{R}} \to X$ is monotonic if, for each $R \in \bar{\mathcal{R}}$ and for any of its φ-monotonic changes $R^1 \in \bar{\mathcal{R}}$, φ yields the same allocation for both R and R^1, i.e., $\varphi(R^1) =$

$\varphi(R)$.

The next lemma, which is from Svensson (1999), establishes that each nonbossy and strategy-proof allocation rule φ is monotonic.

Lemma 2 (Lemma 1 of Svensson (1999))**.** *If an allocation rule φ is nonbossy and strategy-proof, then φ is monotonic.*

Recall that for any subdomain of separable preferences, $\bar{\mathcal{R}} \subseteq \mathcal{R}^{SP}$, we defined the corresponding domains of A- and B-preference relations, $\bar{\mathcal{R}}^A$ and $\bar{\mathcal{R}}^B$. We now define two market specific allocation rules that depend on the market specific preferences, $f^A : \bar{\mathcal{R}}^A \to X^A$ and $f^B : \bar{\mathcal{R}}^B \to X^B$. Furthermore, the market specific counterparts of Pareto efficiency, nonbossiness, and strategy-proofness can be defined for market specific allocation rules in a similar manner to how these notions were defined for allocation rules.

Before we move on, we consider a class of allocation rules defined on a subdomain of separable preferences such that the allocation in each market only depends on the market specific preference relations.

Definition 7 (Market-Independent Rule)**.** *An allocation rule $\varphi : \bar{\mathcal{R}} \to X$, where $\bar{\mathcal{R}} \subseteq \mathcal{R}^{SP}$, is market-independent if there exists two market-specific allocation rules $f^A : \bar{\mathcal{R}}^A \to X^A$ and $f^B : \bar{\mathcal{R}}^B \to X^B$ such that*

$$\varphi^A(R) = f^A(R^A), \text{ and}$$
$$\varphi^B(R) = f^B(R^B) \text{ for all } R \in \bar{\mathcal{R}}.$$

Market-independent allocation rules turn out to satisfy some interesting properties that we note in the following remark.

Remark 1. *Observe here that market-independent allocation rules that consist of two market-specific strategy-proof rules is also strategy-proof in our multi-market setting. This differs from the problems with multi-unit goods (see, for example, Pápai (2001), Ehlers and Klaus (2003), and Hatfield (2009)). In that literature, it is shown that natural adaptations of allocation rules that are strategy-proof in the single-object case might not be strategy-proof in the multi-unit case. For concreteness, consider the HBS draft allocation rule described by Budish and Cantillon (2012), in which the choosing order of the agents is reversed at every* *round. That allocation rule is not strategy-proof. Now consider a version of that allocation rule in our setting: Agents choose in market A according to some exogenous ordering, but in market B they choose according to the exact opposite ordering of market A. This allocation rule is strategy-proof, but fails efficiency. The same intuition applies to the setting of Manea (2007), in which he shows that the serial dictatorship is manipulable.*

Furthermore, observe here that the same thing can be said for nonbossiness. In this sense, as long as the preferences are separable, achieving nonbossiness and strategy-proofness in multiple-market settings is no more difficult than achieving them in single-market settings.

Below we present a simple example that demonstrates that an market-independent allocation rule consisting of two market specific Pareto-efficient rules might fail efficiency when we consider the joint-allocation problem. We will return to this example in the following section.

Example 1 (Failure of Pareto Efficiency)**.** *Let* $n = 2$, $A = \{a_1, a_2\}$, *and* $B = \{b_1, b_2\}$. *Consider the preference profile* $L \in \mathcal{L}$ *such that*

$$L_1 : b_2, a_1, b_1, a_2 \text{ and}$$
$$L_2 : a_1, b_2, b_1, a_2.$$

The allocation $((a_1, b_1), (a_2, b_2))$ *is Pareto-efficient within each market, but it is clearly Pareto-dominated by* $((a_2, b_2), (a_1, b_1))$.

In Remark 1, we concluded that one can design a strategy-proof or nonbossy allocation rule on any subdomain of separable preferences by combining two market specific strategy-proof and nonbossy rules. However, as the example above shows Pareto efficiency is much harder to achieve in a multiple-market setting than in a single-market setting. Therefore, we conclude that efficiency is the driving force for why the set of allocation rules that are strategy-proof, nonbossy and Pareto-efficient narrows in multiple-market settings.

4 Sequential Dictatorship

In this section, we define a sequential-dictatorship allocation rule, which was studied in multi-unit allocation settings (see, for example, Pápai (2001), Ehlers and Klaus (2003), and Hatfield (2009)). In this allocation rule, there is a first agent exogenously chosen who will be allocated her most preferred bundle from the set of all bundles. The first agent's allocation determines who will be the second agent, and this agent will be allocated her most preferred bundle from the set of available bundles, which excludes the bundle allocated to the first agent. Then the first two agents' allocations determine who will be the third agent. This agent will be allocated her most preferred bundle from the set of bundles available, which excludes the bundles allocated to the first and second agents. The process continues until all agents are allocated to a bundle. Below we define the sequential-dictatorship algorithm formally.

For any nonempty subsets $\bar{A} \subseteq A$ and $\bar{B} \subseteq B$ and any preferences of agent i, R_i, we define $\tau(R_i, \bar{A}, \bar{B})$ as the most preferred bundle of agent i (under preferences R_i) in the set $\bar{A} \times \bar{B}$.

Let $\pi : N \to \{1, \cdots, n\}$, be a bijective function that defines an order over the agents, where $\pi(i) = j$ means that agent i is the j^{th} agent in the order. To simplify notation, we will denote by i_j the agent i for whom $\pi(i) = j$.

For any given $R \in \bar{\mathcal{R}}$ and a given π we construct sets of A- and B-objects recursively as follows. Let $A_j(R, \pi) = A$ if $j = 1$ and

$$A_j(R,\pi) = A_{j-1}(R,\pi) \setminus \left\{\tau^A\left(R_{i_{j-1}}, A_{j-1}(R,\pi), B_{j-1}(R,\pi)\right)\right\}, \text{for all } j > 1,$$

and $B_j(R, \pi) = B$ for $j = 1$, while

$$B_j(R,\pi) = B_{j-1}(R,\pi) \setminus \left\{\tau^B\left(R_{i_{j-1}}, A_{j-1}(R,\pi), B_{j-1}(R,\pi)\right)\right\}, \text{ for all } j > 1.$$

Using this notation, and given an ordering π, the most preferred bundle of agent i_j given that the sets of available bundles are $A_j(R, \pi)$ and $B_j(R, \pi)$ is $\tau\left(R_{i_j}, A_j(R,\pi), B_j(R,\pi)\right)$.

We are now ready to define the sequential-dictatorship allocation rule.

Definition 8 (Sequential Dictatorship)**.** *An allocation rule $\varphi : \bar{\mathcal{R}} \to X$ is a* ***sequential-dictatorship*** *allocation rule on domain $\bar{\mathcal{R}}$ if for each $R \in \bar{\mathcal{R}}$, there is a bijective function $\pi_R : N \to \{1, \cdots, n\}$, such that:*

1. If $i = \pi_R^{-1}(1)$ for some $R \in \bar{\mathcal{R}}$, then $i = \pi_{R'}^{-1}(1)\ \forall R' \in \bar{\mathcal{R}}$.

2. For all $j \in \{1, 2, ..., n\}$, $\varphi_{i_j}(R) = \tau\left(R_{i_j}, A_j(R,\pi), B_j(R,\pi)\right)$.

3. Consider any $R, R' \in \bar{\mathcal{R}}$, if we have that $\varphi_{i_k}(R) = \varphi_{i_k}(R')$ for all $k \leq j-1$, then it must be true that $\pi_R^{-1}(j) = \pi_{R'}^{-1}(j)$.

The first item in the definition above requires that there is only one agent who is always allocated her most preferred bundle. The second item means that each agent must be assigned to her most preferred bundle among the available bundles. The third item requires that if there are two different preference profiles under which each of the first $j-1$ agents are allocated to the same bundles in either one of the profiles, then the j^{th} agent who makes a choice under each one of the two different profiles must be the same agent. To implement a sequential dictatorship, we need a mechanism in which the same agent always makes the first choice among all available bundles, this choice determines the second agent who then makes her choice among the remaining bundles. The choices of these two agents determine the third agent to make the choice and so on.

The standard serial-dictatorship allocation rule is a special case of a sequential dictatorship, in which π_R is constant for all $R \in \bar{\mathcal{R}}$. That is, the order in which the agents make their choices is the same, regardless of the preference profile.

Example 2 (Example 1 revisited)**.** *For preference profile L defined in example 1, the sequential dictatorship allocation rule yields the allocation $((a_1, b_2), (a_2, b_1))$ if agent 1 is the first to choose, and $((a_2, b_1), (a_1, b_2))$ if agent 1 is the second to choose. Clearly, in both cases the final allocation is efficient.*

Remark 2. *In the example above, observe that for preference profile L, the sequential-dictatorship allocation rule never yields the allocation $((a_2, b_2), (a_1, b_1))$ which is also Pareto-efficient. This result, which has been obtained for the multi-unit setting by (Manea, 2007), contrasts with the result in the allocation problem in single markets, in which all Pareto-efficient allocations are reached through some serial dictatorship (Abdulkadiroğlu and Sönmez, 1999).*

The sequential-dictatorship rule is strategy-proof, nonbossy and Pareto-efficient in the multiple market allocation setting, as will be discussed in the next section. Moreover, from example 1 above we conclude that the sequential dictatorship does not span the entire set of Pareto-efficient allocations in the joint problem. Therefore, a natural question is whether there is any other rule that is strategy-proof, nonbossy and Pareto-efficient. The main contribution of our paper is to show that sequential dictatorships are the *only* rules that satisfy strategy-proofness, nonbossiness and Pareto efficiency on the preference domains that contain the lexicographical preferences.

5 Efficiency, Nonbossiness and Strategy-Proofness

In this section, we characterize the allocation rules that are strategy-proof, nonbossy and Pareto-efficient. First, let us note that any sequential-dictatorship allocation rule is strategy-proof, nonbossy and Pareto-efficient, which we state as a theorem below.

Theorem 1. *The sequential dictatorship allocation rules on any domain $\bar{\mathcal{R}}$ are strategy-proof, nonbossy and Pareto-efficient.*

Now we turn our attention to the main result of the paper: *only* the sequential-dictatorship allocation rules satisfy nonbossiness, strategy-proofness and Pareto-efficiency on the domain of lexicographical preferences.

We first state a pair of properties of the lexicographical preferences that are useful for our main result. If one has lexicographical preferences, then it is easy to identify one's most preferred bundle in any given nonempty subset $\bar{A} \times \bar{B} \subseteq A \times B$.

Lemma 3. *Let L_i be a lexicographical preference relation associated with ordering η. Then, for any nonempty subsets $\bar{A} \subseteq A$ and $\bar{B} \subseteq B$,*

$$\tau(L_i, \bar{A}, \bar{B}) = \left(\arg\min_{a \in \bar{A}} \eta_i(a), \arg\min_{b \in \bar{B}} \eta_i(b) \right).$$

Proof. The proof follows directly from the definition of lexicographical preferences. □

With lexicographical preferences it is also easy to determine whether a preference profile is an φ-monotonic change of another.

Lemma 4. *Consider an allocation rule $\varphi: \mathcal{L} \to X$ and a lexicographical preference profile $L = (L_i)_{i \in N}$ associated with $\eta = (\eta_i)_{i \in N}$. Let $\bar{L}$ be a lexicographical preference profile associated with $\bar{\eta} = (\bar{\eta}_i)_{i \in N}$ that satisfies the following three conditions:*

(i) If $\bar{\eta}_i(a) < \bar{\eta}_i(\varphi_i^A(L))$ for any $a \in A$ and $i \in N$, then $\eta_i(a) < \eta_i(\varphi_i^A(L))$.

(ii) If $\bar{\eta}_i(b) < \bar{\eta}_i(\varphi_i^B(L))$ for any $b \in B$ and $i \in N$, then $\eta_i(b) < \eta_i(\varphi_i^B(L))$.

(iii) If $\eta(\varphi_i^A(L)) < \eta(\varphi_i^B(L))$ for any i, then $\bar{\eta}(\varphi_i^A(L)) \leq \bar{\eta}(\varphi_i^B(L)) + 1$. Similarly, if $\eta(\varphi_i^B(L)) < \eta(\varphi_i^A(L))$ for any i, then $\eta(\varphi_i^B(L)) \leq \eta(\varphi_i^A(L)) + 1$.

Then $\bar{L}$ is an φ-monotonic change of L.

Proof. The proof follows directly from the definitions of lexicographical preferences and φ-monotonic change. □

We are now ready to present the main result of our paper. We note here that the first part of the proof is similar to the proof in Theorem 1 of Svensson (1999).

Theorem 2. *If an allocation rule on the lexicographical preference domain is strategy-proof, nonbossy and Pareto-efficient then it must be a sequential dictatorship.*

Proof. For now assume that $n = 2$.[8]
Claim 1. For any $(a, b) \in A \times B$, there exists $i \in N$ such that $\varphi_i(L) = (a, b)$ for all L in which $(a, b) = \tau(L_i, A, B)$.
Proof of Claim 1. Without loss of generality let $a = a_1$ and $b = b_1$. Fix two lexicographical preferences, L_1^1 and L_2^1, such that

$$L_1^1 : b_1, a_1, b_2, a_2, \cdots \text{ and}$$
$$L_2^1 : b_2, a_1, b_1, a_2, \cdots .$$

Because φ is efficient, it must be true that either (1) $\varphi(L_1^1, L_1^2) = ((a_1, b_1), (a_2, b_2))$, or (2) $\varphi(L_1^1, L_2^1) = ((a_2, b_1), (a_1, b_2))$.

Suppose that Case (1) occurs. We claim that if $(a_1, b_1) = \tau(L_1, A, B)$ for some L_1, then $\varphi_1(L_1, L_2) = (a_1, b_1)$ for any L_2. Consider two more lexicographical preferences, L_1^2 and L_2^2, such that

$$L_1^2 : a_1, b_1, b_2, a_2, \cdots \text{ and}$$
$$L_2^2 : a_1, b_2, b_1, a_2, \cdots .$$

We now show that

$$\varphi(L_1^1, L_2^1) = \varphi(L_1^2, L_2^1) = \varphi(L_1^1, L_2^2) = \varphi(L_1^2, L_2^2). \tag{1}$$

Because (L_1^2, L_2^1) is a φ-monotonic change of (L_1^1, L_2^1) (Lemma 4), we obtain the first equality above due to Lemma 2. Consider now $\varphi(L_1^1, L_2^2)$. Observe that $\varphi_2(L_1^1, L_2^2) \neq (a_1, b_2)$; other-

[8]For the special case of two markets with two goods in each market and two agents, there is an alternative proof that makes use of the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) or (**?**) if the preferences are separable. This alternative proof is shown in Appendix A.

wise, $\varphi_2(L_1^1, L_2^2) L_2^1 \varphi_2(L_1^1, L_2^1)$, which contradicts the strategy-proofness of φ. This combined with the efficiency of φ imply that $\varphi_2(L_1^1, L_2^2) = (a_2, b_2)$. Now the efficiency of φ implies that $\varphi(L_1^1, L_2^2) = \varphi(L_1^1, L_2^1)$, the second equality in (1). Consider now (L_1^2, L_2^2), which is a φ-monotonic change of (L_1^1, L_2^2). Thus, $\varphi(L_1^2, L_2^2) = \varphi(L_1^1, L_2^1)$, due to Lemma 2, the third equality in (1).

Now consider lexicographical preference relation L_2^3 such that:

$$L_2^3 : a_1, b_1, b_2, a_2, \cdots .$$

We now show that $\varphi(L_1^1, L_2^3) = \varphi(L_1^1, L_2^2)$. Observe here that $\varphi_2^A(L_1^1, L_2^3) \neq a_1$; otherwise, $\varphi_2(L_1^1, L_2^3) L_2^2 \varphi_2(L_1^1, L_2^2)$, which is a contradiction with the strategy-proofness of φ. Then, due to the efficiency of φ, it must be the case that $\varphi_2(L_1^1, L_2^3)$ is either (a_2, b_1) or (a_2, b_2). If $\varphi_2(L_1^1, L_2^3) = (a_2, b_1)$, then $((a_2, b_1), (a_1, b_2))$ Pareto-dominates $\varphi(L_1^1, L_2^3)$ under (L_1^1, L_2^3), which is a contradiction. Hence, $\varphi_2(L_1^1, L_2^3) = (a_2, b_2)$. Then the efficiency of φ implies that $\varphi(L_1^1, L_2^3) = \varphi(L_1^1, L_2^1)$.

Consider any L_2. We claim that $\varphi_2^A(L_1^1, L_2) \neq a_1$. Suppose otherwise, i.e., suppose that $\varphi_2(L_1^1, L_2) = (a_1, b)$ for some $b \in B$. Then, $\varphi_2(L_1^1, L_2) L_2^3 \varphi_2(L_1^1, L_2^3)$, which contradicts the strategy proofness of φ. Thus, $\varphi_2^A(L_1^1, L_2) \neq a_1$. Similarly, we can show that $\varphi_2^B(L_1^1, L_2) \neq b_1$. Thus, by efficiency, $\varphi_1(L_1^1, L_2) = (a_1, b_1)$.

Finally, consider any L_1 in which $(a_1, b_1) = \tau(L_1, A, B)$. Clearly, (L_1, L_2) is a φ-monotonic change of (L_1^1, L_2). Thus, it must be true that $\varphi_1(L_1, L_2) = (a_1, b_1)$, thanks to Lemma 2. This proves that if Case (1) occurs, then $\varphi_1(L) = (a_1, b_1)$ for all L in which $(a_1, b_1) = \tau(L_1, A, B)$.

Suppose now that Case (2) occurs. We claim that if $(a_1, b_1) = \tau(\bar{L}_2, A, B)$ for some $\bar{L}_2$, then $\varphi_2(L_1, \bar{L}_2) = (a_1, b_1)$ for any L_1.

For Case (2), a proof similar to that used in Case (1) implies that for all L_2 with $\tau(L_2, A, B) = (a_1, b_2)$, it must be that $\varphi_2(L_1, L_2) = (a_1, b_2)$ for all L_1. Fix any $\bar{L}_2$ in which $(a_1, b_2) = \tau(\bar{L}_2, A, B)$. Then $\varphi_2(L_1, \bar{L}_2) = (a_1, b_2)$ for all L_1.

Now consider two lexicographical preferences, $\bar{L}_1^1$ and $\bar{L}_2^1$, such that

$$\bar{L}_1^1 : b_2, a_1, b_1, a_2, \cdots \; and$$
$$\bar{L}_2^1 : b_1, a_1, b_2, a_2, \cdots .$$

By efficiency, it must be the case that either (i) $\varphi(\bar{L}_1^1, \bar{L}_2^1) = ((a_1, b_2), (a_2, b_1))$ or (ii)

$\varphi(\bar{L}_1^1, \bar{L}_2^1) = ((a_2, b_2), (a_1, b_1))$. In Case (i), using the same arguments made in Case (1), we obtain the result that for all L_1 in which $(a_1, b_2) = \tau(L_1, A, B)$, it must be true that $\varphi_1(L_1, L_2) = (a_1, b_2)$. Fix $\bar{L}_1$ such that $(a_1, b_2) = \tau(\bar{L}_1, A, B)$. Consider now $(\bar{L}_1, \bar{L}_2)$. Then the agents cannot both obtain (a_1, b_2) which means that Case (i) cannot occur. In Case (ii), the arguments used in Case (1) yield that $\varphi_2(L) = (a_1, b_1)$ for all L in which $(a_1, b_1) = \tau(L_2, A, B)$. This completes the proof that if Case 2 occurs, then $\varphi_2(L) = (a_1, b_1)$ for all L in which $(a_1, b_1) = \tau(\bar{L}_2, A, B)$.

This completes the proof of Claim 1.

Claim 2. There must exists an agent i such that $\varphi_i(L) = \tau(L_i, A, B)$ for all L.

Proof of Claim 2. Fix any (a, b). By Claim 1 there exists an agent i such that $\varphi_i(L) = (a, b)$ for all L with $\tau(L_i, A, B) = (a, b)$. Fix any $(a, \bar{b})$. By Claim 1 there must exist j such that $\varphi_j(L) = (a, \bar{b})$ for all R with $\tau(L_j, A, B) = (a, \bar{b})$. If $(a, \bar{b}) = (a, b)$, then clearly $i = j$. If $(a, \bar{b}) \neq (a, b)$, we need to show that $i = j$. If $i \neq j$, consider $\bar{L}$ such that $(a, b) = \tau(\bar{L}_i, A, B)$ and $(a, \bar{b}) = \tau(\bar{L}_j, A, B)$. Then it must be the case that $\varphi_i^A(\bar{L}) = \varphi_j^A(\bar{L}) = a$, which is a contradiction. Thus, $i = j$.

A similar proof shows that for any $(\bar{a}, \bar{b})$, it must be true that $\varphi_i(L) = (\bar{a}, \bar{b})$ for all L with $\tau(L_i, A, B) = (\bar{a}, \bar{b})$. Given that we picked arbitrary (a, b) and $(\bar{a}, \bar{b})$, it must be the case that $\varphi_i(L) = \tau(L_i, A, B)$ for all L.

By combining Claim 2 with the fact that φ is Pareto efficient, we complete the proof that φ is a sequential dictatorship for the $n = 2$ cases. The proof for the $n \geq 3$ cases are in the Appendix. □

In Theorem 2 we restricted our attention to the domain of lexicographic preferences. This means that even if the preference domain is so small that each of its member is completely determined by a single list of objects in $A \cup B$, the requirements of strategy-proofness, efficiency and nonbossiness together lead to a very negative result. We next prove that the characterization general result holds for *any* domain that contains lexicographic preferences. In this proof, we use the following conventional notations: $R_S \equiv (R_i)_{i \in S}$ and $R_{-S} \equiv (R_i)_{i \notin S}$, for all $S \subset N$.

Proposition 1. *Let $\bar{\mathcal{R}} \supseteq \mathcal{L}$. If an allocation rule $\varphi : \bar{\mathcal{R}} \to X$ is strategy-proof, nonbossy and efficient then it must be a sequential dictatorship.*

Proof. From Theorem 2, we know that if we restrict our attention to the domain of lexicographical preferences, then $\varphi : \mathcal{L} \to X$ must be a sequential dictatorship.

Claim 1. Fix any $L \in \mathcal{L}$. Without loss of generality, let i be the ith agent to make a choice if the reported preference profile is L. Fix any $S \subseteq N$ and let $\bar{i}$ be the agent with the lowest index in S. Then for any R_S, it must be that

$$\varphi_i(R_S, L_{-S}) = \tau\left(L_i, A \setminus \cup_{j<i}\varphi_j^A(R_S, L_{-S}), B \setminus \cup_{j<i}\varphi_j^B(R_S, L_{-S})\right) \;\; foralli < \bar{i}$$

and

$$\varphi_{\bar{i}}(R_S, L_{-S}) = \tau\left(R_{\bar{i}}, A \setminus \cup_{j<\bar{i}}\varphi_j^A(R_S, L_{-S}), B \setminus \cup_{j<\bar{i}}\varphi_j^B(R_S, L_{-S})\right).$$

Proof of Claim 1. Let us prove the claim when $|S| = 1$. Let $\{i\} = S$. Here, obviously $\bar{i} = i$ because $S = \{i\}$. Claim 1 for the $i = 1$ case is a consequence of the strategy-proofness of φ. Now we prove Claim 1 for any random i assuming that Claim 1 is true for all $j < \bar{i}$. The strategy-proofness of φ yields that i cannot obtain any of $\{\varphi_1^A(L), \cdots, \varphi_{i-1}^A(L), \varphi_1^B(L), \cdots, \varphi_{i-1}^B(L)\}$ as long as the others report L_{-i}. Because φ is a sequential dictatorship mechanism on $\mathcal{L}$, i must be the ith agent to choose as long as she reports lexicographical preferences when the others report L_{-i}. Thus, i should be able to obtain $\tau\left(R_i, A \setminus \cup_{j<i}\varphi_j^A(L), B \setminus \cup_{j<i}\varphi_j^B(L)\right)$ by reporting some L_i'. Now due to the strategy-proofness of φ, it must be that

$$\varphi_i(R_i, L_{-i}) = \varphi_i(L_i', L_{-i}) = \tau\left(R_i, A \setminus \cup_{j<i}\varphi_j^A(L), B \setminus \cup_{j<i}\varphi_j^B(L)\right). \tag{2}$$

Now the nonbossiness of φ yields that $\varphi(R_i, L_{-i}) = \varphi(L_i', L_{-i})$. Using φ is a sequential dictatorship on $\mathcal{L}$, we know that $\varphi_j(L) = \varphi_j(L_i', L_{-i})$ for all $j < i$. This in turn gives that $\varphi_j(R_i, L_{-i}) = \varphi_j(L)$ for all $j < i$, the first item of the claim. Combining this with (2), we obtain the second item of the claim.

Now we prove the claim for any S with $|S| > 1$. We argue this by induction. Specifically, we assume that the claim is true for all $\bar{S}$ with size $1 < |\bar{S}| < n$. We now prove the claim for any S with size $|S| = |\bar{S}| + 1$.

Fix any S with $2 \leq |S| \leq n$ and R_S. If $\bar{i} = 1$, then the claim is a consequence of the strategy proofness of φ, the induction assumption and the fact that φ is a sequential dictatorship on $\mathcal{L}$. Suppose now $\bar{i} \neq 1$. Let $\bar{S} = S \setminus \{\bar{i}\}$. The induction assumption and the fact that φ is a sequential dictatorship mechanism on $\mathcal{L}$ give that $\bar{i}$ is the $\bar{i}$th agent to choose her bundle if she reports some lexicographical preference by deviating from the preference profile (R_S, L_{-S}). In such cases, the agents indexed below $\bar{i}$ must pick their allocation according to the order of their indices. Combining these properties with the strategy proofness of φ, we obtain that

$$\varphi_{\bar{i}}(R_S, L_{-S}) = \tau(R_i, A \setminus \cup_{i<\bar{i}} \varphi_i^A(R_{\bar{S}}, L_{-\bar{S}}), B \setminus \cup_{i<\bar{i}} \varphi_i^B(R_{\bar{S}}, L_{-\bar{S}})). \tag{3}$$

Now consider the following preference of $\bar{i}$:

$$L'_{\bar{i}} : \varphi_{\bar{i}}^A(R_S, L_{-S}), \varphi_{\bar{i}}^B(R_S, L_{-S}), \cdots .$$

Suppose that $\bar{i}$ deviates from (R_S, L_{-S}) and reports $L'_{\bar{i}}$. Now the induction assumption and φ being a sequential dictatorship mechanism on $\mathcal{L}$ imply that

$$\varphi_i(R_{\bar{S}}, L'_{\bar{i}}, L_{-S}) = \tau\left(L_i, A \setminus \cup_{j<\bar{i}} \varphi_j^A(R_{\bar{S}}, L'_{\bar{i}}, L_{-S}), B \setminus \cup_{j<\bar{i}} \varphi_j^B(R_{\bar{S}}, L'_{\bar{i}}, L_{-S})\right) \forall\, i < \bar{i}. \tag{4}$$

and

$$\begin{aligned} \varphi_{\bar{i}}(R_{\bar{S}}, L'_{\bar{i}}, L_{-S}) &= \tau\left(L'_{\bar{i}}, A \setminus \cup_{j<\bar{i}} \varphi_j^A(R_{\bar{S}}, L'_{\bar{i}}, L_{-S}), B \setminus \cup_{j<\bar{i}} \varphi_j^B(R_{\bar{S}}, L'_{\bar{i}}, L_{-S})\right) \\ &= \varphi_{\bar{i}}(R_S, L_{-S}). \end{aligned} \tag{5}$$

Combining this with the nonbossiness of φ it must be that

$$\varphi(R_{\bar{S}}, L'_{\bar{i}}, L_{-S}) = \varphi(R_S, L_{-S}). \tag{6}$$

Finally, because each agent with a strictly lower index than $\bar{i}$ has the same preferences under both $(R_{\bar{S}}, L'_{\bar{i}}, L_{-S})$ and (R_S, L_{-S}), the proof is complete thanks to (4), (5) and (6).

Claim 1 and the fact that φ is a sequential dictatorship on $\mathcal{L}$ yield that there exists an agent who obtains her most preferred bundle under each preference profile in $\bar{\mathcal{R}}$. Call this agent i_1. Once we fix a preference for i_1, then there must exist some other agent, say i_2, who selects second under φ. In fact, due to the nonbossiness, i_2 is the same agent as long as i_1 selects the same bundle under different preferences. By continuing with the same argument, we obtain that φ is a sequential dictatorship on $\bar{\mathcal{R}}$. □

Given that the result holds for any domain that contains the lexicographic preferences, it must also hold for separable preferences (Lemma 1).

Corollary 1. *If any allocation rule* $\varphi : \mathcal{R}^{SP} \to X$ *is efficient, strategy-proof and nonbossy then* φ *is a sequential dictatorship.*

Remark 3. *For the two-agent case, efficiency and bossiness are not compatible with each other. To see this, observe that for each preference profile, if the allocation that one of the agents receives is fixed then there is at most one Pareto efficient allocation because the agents have strict preferences. Consequently, no one agent should be able to change the other's allocation without changing her own allocation if the allocation rule is efficient. However, when there are more than two agents this result is not valid: in fact, later we present an example in which the allocation rule is Pareto efficient, strategy-proof and yet bossy.*

Remark 4. *Theorem 2 and Proposition 1 remains valid if there are more than 2 markets. We provide the proof of these statements at an online appendix.*

Remark 5. *We always assumed that in each market there are enough objects. If this assumption is violated, our main characterization result is not valid any more. To see this, suppose that* $|A \setminus 0| + |B \setminus 0| \leq n$. *Then one can focus on rules in which each agent is assigned at most one object. Once this restriction is in place, we can treat our multi-market allocation problem as a standard house allocation problem in which each agent demands only one object as long as our desirability assumption (Assumption 2) is satisfied. Given that the trading cycles mechanism of Pycia and Ünver (2011) is efficient, strategy-proof and nonbossy in the standard house allocation problem, it will be also efficient, strategy-proof and nonbossy in our setting.*

We now turn our attention to the question of whether any non-sequential-dictatorship rule defined on some domain of preferences satisfies all of strategy-proofness, nonbossiness and Pareto efficiency. Due to Proposition 1, we know that any such rule cannot be defined on a domain of preferences that contains the domain of lexicographical preferences. In Remark 1, we already noted that on the domain of separable preferences any market independent rule consisting of two market specific strategy-proof and nonbossy rules is strategy-proof and nonbossy. This suggests that non-sequential-dictatorship rule satisfying strategy-proofness, nonbossiness and Pareto efficiency can be found on a subdomain of separable preferences in which (market specific) Pareto efficiency in both markets is equivalent to Pareto efficiency in the entire market. We define a such domain below.

Definition 9 (*A*-favored lexicographical preference)**.** *We say that a preference relation of agent i, $L_i, \in \mathcal{L}_i$ is market A favored if its lexicographical ordering ranks all the A-objects ahead of B-objects. The notation $\mathbb{L}$ stands for the set of A-favored lexicographical preference profiles.*

We now show that on the domain of A-favored lexicographical preferences any market independent rule consisting of two market specific Pareto efficient rules is Pareto efficient.

Lemma 5. *Any market independent allocation rule $\varphi : \mathbb{L} \to X$ consisting of two market specific Pareto efficient rules is Pareto efficient.*

Proof. In contradiction to the lemma, suppose that there exist $L \in \mathbb{L}$ and $x \neq \varphi(L)$ such that $x_i L_i \varphi_i(L)$ for all $i \in N$. Suppose first $x^A = \varphi^A(L)$. Then it must be that $x^B \neq \varphi^B(L)$. In addition, because any lexicographical preference is separable, we must have $x_i^B L_i^B \varphi^B(L)$ for all $i \in N$. This is clearly a contradiction as φ consists of two market specific Pareto efficient rules. If $x^A \neq \varphi^A(L)$ then there must exist an agent i who strictly prefers $\varphi_i^A(L)$ to x_i^A in terms of her A-preference relation L_i^A because φ consists of two market specific Pareto efficient rules. Thus, $\varphi_i^A(L)$ is ranked higher than x_i^A in i's lexicographical ordering. Moreover, recall that $\varphi_i^A(L)$ is ranked higher than and B-object in i's lexicographical ordering. Consequently, i strictly prefers $\varphi(i)$ to x. This contradicts that x Pareto dominates $\varphi(L)$. □

Now by combining the lemma above and Remark 1, we obtain the following result.

Proposition 2. *Any market independent allocation rule $\varphi : \mathbb{L} \to X$ consisting of two market specific strategy-proof, nonbossy and Pareto efficient rules is strategy-proof, nonbossy and Pareto efficient.*

Remark 6. *This result has some implications for the school choice problem (Abdulkadiroğlu and Sönmez, 2003). This problem has been modeled as a static matching problem, but it has dynamic features if the schools have so called sibling priorities: each school gives a priority to students with older siblings who attend that particular school.*[9] *We interpret market A as the first period and market B as the subsequent period, and for the expositional simplicity we assume that there are n families with two children, one in market A and one in market B.*

[9]See Dur (2011) for a recent working paper on this topic.

Both children in each family has a preference relation over the schools, but here assume that each family's preferences over pairs of schools is A-favored lexicographical. This assumption seems reasonable if the older sibling cares about her well-beings first and her sibling's well-beings next. Indeed in this case Proposition 2 means that any allocation rule which is strategy-proof, nonbossy and efficient rule in each period is strategy-proof, nonbossy and efficient.[10]

Remark 7. *The results in the dynamic school choice problem are mostly negative if one concentrate on general preference domain. For example, Kennes et al. (2014) show that there are no strategy-proof and stable allocation rules and also that the top trading cycles is neither Pareto-efficient nor strategy-proof. Dur (2011) shows that there are no fair and stable allocation rules in the dynamic school choice problem.*

However, Proposition 2 implies that the top trading cycles rule achieves both strategy-proofness and Pareto efficiency in the school choice problem with sibling priorities if the families have A- or initial-period favored lexicographical preferences. In the exact same setting Dur (2011) shows that the deferred acceptance allocation rule is not strategy-proof. The main reason why the TTC is strategy-proof while the DA is manipulable is that the TTC is nonbossy while the DA is not. Thus, TTC has an edge over DA in terms of non-manipulability.

Independence of Axioms

We conclude this section by showing that each of nonbossiness, strategy-proofness, and Pareto efficiency plays an indispensable role for Theorem 2.[11] Below we present three examples in which a non-sequential dictatorship rule satisfies two of the three properties.

Example 3 (Allocation Rule that is Strategy-proof and Nonbossy but not Efficient)**.** *Consider a constant allocation rule, (i.e., a rule that does not depend on the preference profiles of the agents). Clearly, this rule is both strategy-proof and nonbossy, but not necessarily Pareto-efficient.*

Example 4 (Allocation Rule that is Efficient and Nonbossy but not Strategy-proof)**.** *Recall Example 1 and consider an allocation rule $\varphi : \mathcal{L} \rightarrow X$ which differs from the serial dictator-*

[10]In the school choice problem, the allocation rule for the older siblings is a function of their reported preferences (and some exogenous priorities which we do not need to specify). On the other hand, the allocation rule for the younger siblings is a function of the older siblings' allocations (which determine the priorities of their younger siblings) and the younger siblings' reported preferences. Accommodating these assumptions into our model does not affect Proposition 2.

[11]See Remark 3 for the special case of two agents.

ship rule in which agent 1 is the first agent to select only in that $\varphi(L) = ((a_2, b_2), (a_1, b_1))$. *One can easily check that* φ *is both Pareto-efficient and nonbossy, but it is not a serial dictatorship rule. Thus, Theorem 2 implies that* φ *is not strategy-proof.*

Example 5 (Allocation Rule that is Efficient and Strategy-proof but not Nonbossy)**.** *Let* $n \geq 3$ *and consider the following allocation rule,* φ*, which is a slight modification of a sequential dictatorship rule: agent 1 is the first agent to select, and agent 2 (agent 3) is the second agent to make a selection only if agent 1's second most preferred bundle contains* a_1 *(*a_2 *or* a_3*). One can easily check that* φ *is both Pareto-efficient and strategy-proof but not nonbossy.*

6 Conclusion

We have studied the problem of centralized assignment in multiple markets, which includes the class of dynamic matching problems.[12] In our main result, we showed that the set of rules that are strategy-proof, nonbossy and implement a Pareto-efficient allocation is the set of sequential dictatorship rules.

One interesting question that remains to be answered is the characterization of Pareto-efficient and strategy-proof rules. This problem seems to be much more challenging than ours, since Pareto-efficient and strategy-proof rules are not necessarily monotonic – the key property that allowed us to divide the preference profiles to classes such that each class is "big" and the same allocation is prescribed to each preference within a class. To illustrate, consider the simple case of three players, two markets and three goods in each market. In this example, there are 1440 possible preference rankings for each agent, totaling 2,985,984,000 possible preference profiles. Thus, even in this rather simple example, computing the strategy-proof and Pareto-efficient rules is a daunting task. Nevertheless, we conjecture that the class of Pareto-efficient and strategy-proof rules is only a slight generalization of the sequential dictatorships in which the order of selection is determined by the reported preferences of the previous agent in the order. However, we were not able to prove this result.

Another interesting open question is to characterize the smallest preference domain for which our characterization result holds. We have proved that it holds for any preference do-

[12] Our main theorem, which is proved for the cases with two markets, can be generalized for the cases with more than 2 markets. The proof can be provided upon request.

main that contains lexicographic preferences and we have also constructed a domain smaller than the lexicographic domain in which Pycia and Ünver (2011)'s result on single market allocations holds for the multiple market framework as well. It remains to be proven a sufficient condition on the domain for our result to hold.

One implication of our result is that when considering the allocation of several markets together, there is a trade-off between Pareto efficiency and fairness. Given that the sequential dictatorship is usually considered to be an "unfair" allocation rule, we conclude by suggesting that a possible direction for future research might be to work with a solution concept other than Pareto efficiency. As we have argued in the text, many well-known allocation rules that are strategy-proof and nonbossy in single markets remain strategy-proof and nonbossy in the multiple-markets case if applied separately and independently to each different market. In this sense, Pareto efficiency seems to be a very demanding concept for the class of multiple-market problems.

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7 Appendix

7.1 Appendix A: The 2x2 Cases and the Gibbard-Satterthwaite Theorem

In this section, we present an alternative proof for our main result, when we restrict attention to the specific case of two markets, two objects in each market and two agents only. That is, $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $N = 2$. The proof of this result in this specific environment follows from the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) or **?**. The key aspect here is that the allocation of one agent fully determines the allocation of the other agent. For example, when the allocation of agent 1 is $x_1 = (a_1, b_1)$, the allocation of agent 2 must be $x_2 = (a_2, b_2)$ and so on.[13] The strict preference ordering of each agent over the set of her own final allocations induces a strict ordering over the set of agent 1's allocations. In this newly interpreted setting, an allocation rule maps the agents' preferences to agent 1's allocations. Perhaps the most important observation here is that in the reinterpreted setting, an allocation rule is a social choice function as used in the implementation literature. Now, using the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975), one obtains that if the allocation rule is strategy-proof and efficient, then it must be a dictatorship, or in our setting a sequential dictatorship allocation rule (which is also a serial dictatorship as there are only 2 objects of each type and 2 agents).

Theorem 3. *Any strategy-proof, nonbossy and Pareto efficient allocation rule for the* $|N| = |A| = |B| = 2$ *case is a sequential dictatorship.*

Proof. Fix an efficient and strategy-proof allocation rule φ, where, recall, $\varphi : \mathcal{R}_1 \times \mathcal{R}_2 \to X$.

Let us use the following notations: $t_1 = (a_1, b_1)$, $t_2 = (a_1, b_2)$, $t_3 = (a_2, b_1)$, $t_4 = (a_2, b_2)$ and let $\mathcal{T} = \{t_1, t_2, t_3, t_4\}$. First let us show that φ_1 is an onto function. Fix any $t = (a, b) \in A \times B$. Consider $R_1 \in \mathcal{R}_1$ and $R_2 \in \mathcal{R}_2$ such that (a, b) is agent 1's most preferred bundle in $A \times B$ while the remaining pair in $A \times B$ is agent 2's top choice. Because φ is efficient, $\varphi_1(R) = (a, b)$. This means that φ_1 is an onto function. Now we will show that $\varphi_1 : \mathcal{R} \to \mathcal{T}$ must be dictatorial.

We will view $\varphi_1 : \mathcal{R} \to \mathcal{T}$ as a social choice function that assigns agent 1 some object t. Specifically, $t \in \mathcal{T}$ stands for the objects that agent 1 obtains. On the other hand, if

[13] The same is not true if there are more than 2 goods even when there are only 2 agents or if there are (strictly) more than 2 agents.

agent 1 is assigned $t_1/t_2/t_3/t_4$ then agent 2 is assigned $t_4/t_3/t_2/t_1$ by feasibility. Agent 1's preferences rank alternatives assuming that these are the alternatives she would obtain, while agent 2's preferences rank alternatives based on what is left after agent 1 is allocated some alternative. With this relabeling, one can view $\varphi_1 : \mathcal{R} \to \mathcal{T}$ as a social choice function. Then the Gibbard-Satterthwaite theorem yields the desired result (Gibbard, 1973; Satterthwaite, 1975).[14] □

7.2 Appendix B: Proof of Theorem 2 for $N > 2$.

Proof of Theorem 2. Let $n \geq 2$, $|A| \geq n$ and $|B| \geq n$. Without loss of generality we assume that $|B| \geq |A|$. First we will prove that for each efficient, nonbossy and strategy-proof allocation rule φ there exists an agent i such that $\varphi_i(L) = \tau(L_i, A, B)$ for all $L \in \mathcal{L}$. Because we have proved this for the $n = 2$ case (in the main text of the paper), our proof will be by induction:

Induction Assumption: For each multi-market allocation problem in which $2 \leq n \leq m - 1$, $|A| \geq n$, and $|B| \geq n$ and for each efficient, strategy-proof, and nonbossy allocation rule of this market, there exists an agent who is assigned her most preferred bundle for each preference profile.

Fix any multi-market allocation problem in which $n = m$ and $|B| \geq |A| \geq n$. Fix any efficient, strategy-proof, and nonbossy allocation rule φ for this market. We will now show that there exists an agent i such that $\varphi_i(L) = \tau(L_i, A, B)$ for all $L \in \mathcal{L}$.

The proof is completed in several steps.

Claim 1. Consider any nonempty and (strict) subset $S \subset N$ and any preference profile $L \in \mathcal{L}$. Let $A_S = \{a \in A : \varphi_i^A(L) = a \; for some i \in S\}$. Similarly, define B_S. Then there must exist an agent $i \notin S$ such that $\varphi_i(L) = \tau(L_i, A \setminus A_S, B \setminus B_S)$.

Proof of Claim 1. To the contrary of Claim 2, suppose that $\varphi_i(L) \neq \tau(L_i, A \setminus A_S, B \setminus B_S)$ for all $i \notin S$. Let L^1 be a φ-monotonic change of L satisfying the following two conditions:

1. if $j \in S$, then $\tau(L_j^1, A, B) = \varphi_j(L)$.

2. if $i \notin S$, then i's preferences satisfy that

[14] The precise statement of the Gibbard-Satterthwaite theorem is the following: In any environments with at least three social alternatives, any strategy-proof and onto social choice function is a dictatorship (The proof is well-known and can be found, for example, in Mas-Colell et al. (1995), Proposition 23.C.3).

(a) the lexicographical ordering of i ranks all objects in $A \setminus A_S \cup B \setminus B_S$ ahead of the objects in $A_S \cup B_S$.

(b) the relative lexicographical ordering of the objects in $A \setminus A_S \cup B \setminus B_S$ (and in $A_S \cup B_S$) under L_i^1 is the same as under L.

By construction, $\varphi_i(L) \in A \setminus A_S \times B \setminus B_S$ for all $i \notin S$. By Lemma 4, L^1 is a φ-monotonic change of L. Consequently, Lemma 2 gives that $\varphi(L^1) = \varphi(L)$. Thus, the claim is proved once we show that there is an agent $i \notin S$ such that $\varphi_i(L^1) = \tau(L_i^1, A \setminus A_S, B \setminus B_S)$. With this in mind, consider the class of preferences $\mathcal{L}^S(L)$ such that each $L' \in \mathcal{L}^S(L)$ satisfies the following conditions:

1. If $j \in S$, then $\tau(L'_j, A, B) = \varphi_j(L)$.

2. If $i \notin S$, then i's preferences satisfy that

   (a) the lexicographical ordering of i ranks all objects in $A \setminus A_S \cup B \setminus B_S$ ahead of the objects in $A_S \cup B_S$.

Observe that $L^1 \in \mathcal{L}^S(L)$. For each preference profile in $\mathcal{L}^S(L)$, each agent $j \in S$ must obtain $\varphi_i(L)$ due to the efficiency of φ. Consequently, for $\mathcal{L}^S(L)$, we can treat φ as the allocation rule that allocates $A \setminus A_S \times B \setminus B_S$ among the agents in $N \setminus S$. Then by the induction assumption, there must exist an agent $i \in N \setminus S$ such that $\varphi_i(\bar{L}) = \tau(\bar{L}_i, A \setminus A_S, B \setminus B_S)$ for all $\bar{L} \in \mathcal{R}^S(R)$. Consequently, because $L^1 \in \mathcal{L}^S(L)$, it must be that $\varphi_i(L^1) = \tau\left(L_i^1, A \setminus A_S, B \setminus B_S\right)$, reaching a contradiction.

In fact, we can strengthen Claim 1 as follows:

Claim 2. Consider any nonempty and (strict) subset $S \subset N$ and consider the set of preference profiles $\mathcal{L}^S$ such that $\varphi_i(L) = \varphi_i(\bar{L})$ for all $i \in S$ and $L, \bar{L} \in \mathcal{L}^S$. Then there must exist an agent $j \notin S$ such that $\varphi_j(L) = \tau(L_j, A \setminus A_S, B \setminus B_S)$ for all $L \in \mathcal{L}^S$.

Proof of Claim 2. Recall that how $\mathcal{L}^S(L)$ is defined in the proof of Claim 1. Observe here that $\mathcal{L}^S(L) = \mathcal{L}^S(\bar{L})$ for all $L, \bar{L} \in \mathcal{L}^S$. This and the proof of Claim 1 complete the proof of Claim 2.

In the next 3 claims (3-5), we prove that for any $(a, b) \in A \times B$, there exists an agent i such that $\varphi_i(L) = (a, b)$ whenever $\tau(L_i, A, B) = (a, b)$. Without loss of generality, let us set $a = a_1$ and $b = b_1$.

Claim 3. Let L be a lexicographical preference profile in which each agent's lexicographical ordering of the objects is the same and as follows:

$$L_i : a_1, b_1, a_2, b_2, \cdots, a_{|A|}, b_{|A|}, b_{|A|+1}, \cdots, b_{|B|}.$$

Then φ allocates each (a_k, b_k) where $k \leq n$ to some agent under L.

Proof of Claim 3. Since $n \geq 3$ there must exist an agent i for whom $\varphi_i^A(L) \neq a_1$ and $\varphi_i^B(L) \neq b_1$. Set $S = \{i\}$, and observe that $(a_1, b_1) \in A \setminus A_S \times B \setminus B_S$. Then by Claim 2, there must exist an agent $i_1 \notin S$ for whom $\varphi_{i_1}(L) = (a_1, b_1)$ because $(a_1, b_1) = \tau(L_j, A \setminus A_S, B \setminus B_S)$ for all $j \in N \setminus S$, due to Lemma 3. Now set $S_1 = i_1$ and observe that $(a_2, b_2) = \tau(L_j, A \setminus A_{S_1}, B \setminus B_{S_1})$ for all $j \in N \setminus S_1$, due to Lemma 3. Then by Claim 2, there exists an agent $i_2 \notin S_1$ for whom $\varphi_{i_2}(L) = (a_2, b_2)$. Next set $S_2 = \{i_1, i_2\}$. Using a similar argument as before we obtain that there exists an agent $i_3 \notin S_2$ for whom $\varphi_{i_3}(L) = (a_3, b_3)$. We complete the proof of this claim by applying the same argument repeatedly.

Without loss of generality, let us assume $\varphi_i(L) = (a_i, b_i)$.

Claim 4. Consider any lexicographical preference profile in which all agents' lexicographical ordering of the objects is the same and starts by listing a_1 and b_1 and then alternates the remaining elements of A and B. Then for this lexicographical preference profile φ must assign (a_1, b_1) to agent 1.

Proof of Claim 4. To prove this claim it suffices to prove the following claim.

Consider a lexicographical preference profile $\hat{L}$ in which each agent's lexicographical ordering of the objects is the same and that this is obtained from L (considered in Claim 4) by reversing the lexicographical orderings of only two neighboring B-objects (except b_1), i.e., each agent's lexicographical ordering of the objects is:

$$\hat{L}_i : a_1, b_1, a_2, b_2, \cdots, a_{j-1}, b_{j-1}, a_j, \mathbf{b_{j+1}}, a_{j+1}, \mathbf{b_j}, a_{j+2}, b_{j+2}, a_{j+3}, b_{j+3}, \cdots, b_{|B|},$$

where $j \geq 2$. Then for each $i < j$, $\varphi_i(\hat{L}) = \varphi_i(L)$ and $\varphi_j(\hat{L}) = (a_j, b_{j+1})$.

If $j > n$, then $\hat{L}$ is a φ-monotonic change of L (Lemma 4). Hence, Lemma 2 yields the statement above. Thus, let us concentrate on the $j \leq n$ cases.

Fix any j such that $j \leq n$, and consider a lexicographical preference profile L^1 such that

$$
\begin{array}{llllll}
L_i^1: & a_1, b_1, \cdots, a_i, b_i, & \mathbf{a_j}, \mathbf{b_j}, & a_{i+1}, b_{i+1}, & \cdots, & a_{|A|}, b_{|A|}, \cdots, b_{|B|} \text{ if } i < j \\
L_j^1: & a_1, b_1, \cdots, a_j, b_j, & \mathbf{b_{j+1}}, \mathbf{a_{j+1}}, & a_{j+2}, b_{j+2}, & \cdots, & a_{|A|}, b_{|A|}, \cdots, b_{|B|} \\
L_i^1: & \mathbf{a_j}, a_1, b_1, \cdots, a_{j-1}, & b_{j-1}, b_j, & a_{j+1}, b_{j+1}, & \cdots, & a_{|A|}, b_{|A|}, \cdots, b_{|B|} \text{ if } i > j.
\end{array}
$$

Clearly L^1 is a φ-monotonic change of L. Hence, $\varphi(L) = \varphi(L^1)$. Now let L^2 be the lexicographical preference obtained from L^1 by changing only agent j's lexicographical ordering of the objects as follows:

$$L_j^2 : a_1, b_1, \cdots, a_j, \mathbf{b_{j+1}}, \mathbf{b_j}, a_{j+1}, a_{j+2}, b_{j+2}, \cdots, a_{|A|}, b_{|A|}, \cdots, b_{|B|}.$$

Observe here that there is only one bundle, (a_j, b_{j+1}), such that j prefers it to $\varphi_j(L^1) = (a_j, b_j)$ under L_j^2 but not under L_j^1. As φ is strategy-proof, $\varphi_j(L^2)$ is either (a_j, b_j) or (a_j, b_{j+1}). In the former case, thanks to nonbossiness, $\varphi(L^2) = \varphi(L^1)$. But by Claim 2, it must be that $\varphi_j(L^2) = \tau(L_j^2, A \setminus \{a_1, \cdots, a_{j-1}\}, B \setminus \{b_1, \cdots, b_{j-1}\}) = (a_j, b_{j+1})$, a contradiction. Hence, $\varphi_j(L^2) = (a_j, b_{j+1})$. Because $(a_1, b_1) = \tau(L_i^2, A \setminus \{a_j\}, B \setminus \{b_{j+1}\})$ for all $i \neq j$, some agent other than j must obtain (a_1, b_1) under $\varphi(L^2)$ by Claim 1. In addition, when $j > 2$, because $(a_2, b_2) = \tau(L_i^2, A \setminus \{a_j, a_1\}, B \setminus \{b_{j+1}, b_1\})$, some agent other than j must obtain (a_2, b_2) under $\varphi(L^2)$ by Claim 1. A similar logic yields that each of the $\{(a_1, b_1), \cdots, (a_{j-1}, b_{j-1})\}$ is allocated to some agent under $\varphi(L^2)$. However, observe that (a_1, b_1) cannot be allocated to any agent $i > j$ (if such i exists) under $\varphi(L^2)$. Otherwise, by swapping their allocations agents j and i Pareto improve. Similarly, we obtain that none of the $\{(a_1, b_1), \cdots, (a_{j-1}, b_{j-1})\}$ are allocated to agents $\{j+1, \cdots, n\}$ under $\varphi(L^2)$. Now let us show that agent 1 obtains (a_1, b_1) under $\varphi(L^2)$. Otherwise, she obtains one of the $\{(a_2, b_2), \cdots, (a_{j-1}, b_{j-1})\}$. But then agents 1 and j can swap their allocations and Pareto improve. Then agent 2 must obtain (a_2, b_2) under $\varphi(L^2)$; otherwise agents 2 and j can swap their allocations and Pareto improve. A similar logic yields that all agents $i \leq j - 1$, $\varphi_i(L^2) = (a_i, b_i)$ and $\varphi_j(L^2) = (a_i, b_{j+1})$.

If $j = n$, then observe that $\hat{L}$ is a φ-monotonic change of L^1. Thus, we obtain the desired result, due to Lemma 2. Let $j < n$. We now need to show that $\varphi_i(L^1) = \varphi_i(\hat{L})$ for all $i = 1, \cdots, j$. To prove this, we need some extra steps. First, observe that by Claim 1 there exists an agent $k > j$ for whom $\varphi_k(L^2) = \tau(L_k^2, A \setminus \{a_1, \cdots, a_j\}, B \setminus \{b_1, \cdots, b_{j-1}, b_{j+1}\}) = (a_{j+1}, b_j)$. Also, due to Claim 2, each of the $\{(a_{j+2}, b_{j+2}), (a_{j+3}, b_{j+3}), \cdots, (a_{|A|}, b_{|B|})\}$ is allocated to some agent $i \neq k$ $(i > j)$. Now consider a lexicographical preference L^3 such that $L^3_k = L^2_k$ and $L^3_i = \hat{L}_i$, for all $i \neq k$. Observe that L^3 is a φ-monotonic change of L^2, hence $\varphi(L^3) = \varphi(L^2)$.

Consider a lexicographical preference profile L^4 in which

$$\begin{array}{lllllll}
L^4_i: & a_1, b_1, \cdots, a_i, b_i, & \mathbf{a_{j+1}}, \mathbf{b_{j+1}}, & a_{i+1}, b_{i+1}, & \cdots, & a_{|A|}, b_{|A|}, \cdots, b_{|B|} & \text{if } i < j \\
L^4_j: & a_1, b_1, \cdots, a_j, \mathbf{b_{j+1}}, & \mathbf{a_{j+1}}, \mathbf{b_j}, & a_{j+2}, b_{j+2}, & \cdots, & a_{|A|}, b_{|A|}, \cdots, b_{|B|} & \\
L^4_k: & a_1, b_1, \cdots, a_j, b_j, & \mathbf{b_{j+1}}, \mathbf{a_{j+1}}, & a_{j+2}, b_{j+2}, & \cdots, & a_{|A|}, b_{|A|}, \cdots, b_{|B|} & \\
L^4_i: & \mathbf{a_{j+1}}, a_1, b_1, \cdots, & a_j, b_j, b_{j+1}, & a_{j+2}, b_{j+2}, & \cdots, & a_{|A|}, b_{|A|}, \cdots, b_{|B|} & \text{if } i \neq k \,\&\, i > j.
\end{array}$$

Clearly, L^4 is a φ-monotonic change of L^3. Hence, $\varphi(L^4) = \varphi(L^3)$. Now let L^5 be a lexicographical preference obtained from L^4 by changing agent k's order of the objects as follows:

$$L^5_k : a_1, b_1, \cdots, a_j, \mathbf{b_{j+1}}, \mathbf{b_j}, a_{j+1}, a_{j+2}, b_{j+2}, \cdots, a_{|A|}, b_{|A|}, \cdots, b_{|B|}.$$

Going from L^4 to L^5 only the relative ranking of (a_{j+1}, b_{j+1}) improves with respect to $\varphi_k(L^4) = (a_{j+1}, b_j)$ for agent k. As φ is strategy-proof, $\varphi_j(L^5)$ is either (a_{j+1}, b_j) or (a_{j+1}, b_{j+1}). Now we rule out the latter case. Suppose the latter case occurs. Using the same steps as we used to prove that $\{(a_1, b_1), \cdots, (a_{j-1}, b_{j-1})\}$ is allocated among the agents $\{1, \cdots, j-1\}$ under L^2, we obtain that $\{(a_1, b_1), \cdots, (a_j, b_j)\}$ is allocated among the agents $\{1, \cdots, j\}$. If agent 1 does not obtain (a_1, b_1), by swapping the allocations of 1 and k, we can Pareto improve. Similarly, each agent $i \leq j-1$ must obtain (a_i, b_i). Therefore, agent j obtains (a_j, b_j). But this contradicts Claim 2 because j is not obtaining her most preferred bundle in $A \setminus \{a_1, \cdots, a_{j-1}\} \times B \setminus \{b_1, \cdots, b_{j-1}\})$ under L^5. Hence, $\varphi_k(L^5) = \varphi_k(L^4) = (a_{j+1}, b_j)$. Now the nonbossiness of φ yields that $\varphi(L^5) = \varphi(L^4)$. Finally, observe that $\hat{L}$ is an φ-monotonic change of L^5. Thus, $\varphi(\hat{L}) = \varphi(L^5)$. This completes the proof of Claim 4.

We now show that $\varphi_1(L) = (a_1, b_1)$ for all L in which $(a_1, b_1) = \tau(L_1, A, B)$.

Claim 5. For any preference profile L^* in which $(a_1, b_1) = \tau(L^*_1, A, B)$ it must be that $\varphi_1(L^*) = (a_1, b_1)$.

Proof of Claim 5. Pick any preference profile L^* such that $\tau(L^*_1, A, B) = (a_1, b_1)$. Now let us construct a lexicographical preference L^n in n iterative rounds.

Round 1. Set $i_1 = 1$. Pick any lexicographical preference L^1 in which everyone's order of the objects is the same and starts with (a_1, b_1) and alternates the remaining A and B-objects. Set $I_1 = \{i_1\}$ and $A_1 = A \setminus \{a_1\}$ and $B_1 = B \setminus \{b_1\}$. Observe that $\varphi_{i_1}(L^1) = (a_1, b_1)$ by Claim 4.

Round 2. Let $i_2 \in N \setminus I_1$ be the agent for whom $\varphi_{i_2}(L^1) = \tau(L^1_{i_2}, A_1, B_1)$. This is always feasible thanks to Claim 1.[15] Set $I_2 = I_1 \cup \{i_2\}$. Let $\tau(L^*_{i_2}, A_1, B_1) := (\hat{a}_2, \hat{b}_2)$. Set $A_2 = A_1 \setminus \{\hat{a}_2\}$ and $B_2 = B_1 \setminus \{\hat{b}_2\}$. Pick a lexicographical preference L^2 in which the order of the objects is the same for everyone, starts with $(a_1, b_1, \hat{a}_2, \hat{b}_2)$, and alternates the remaining A and B-objects. Observe that $\varphi_{i_1}(L^2) = (a_1, b_1)$ and $\varphi_{i_2}(L^2) = (\hat{a}_2, \hat{b}_2)$.

Round $k \leq n$. Let $i_k \in N \setminus I_{k-1}$ be the agent for whom $\varphi_{i_k}(L^{k-1}) = \tau(L^{k-1}_{i_k}, A_{k-1}, B_{k-1})$. Set $I_k = I_{k-1} \cup \{i_k\}$. Let $\tau(L^*_{i_k}, A_{k-1}, B_{k-1}) := (\hat{a}_k, \hat{b}_k)$. Set $A_k = A_{k-1} \setminus \{\hat{a}_k\}$ and $B_k = B_{k-1} \setminus \{\hat{b}_k\}$. Pick a lexicographical preference L^k in which the order of the objects is the same, starts with $(a_1, b_1, \hat{a}_2, \hat{b}_2, \cdots, \hat{a}^k, \hat{b}^k)$, and then alternates the remaining A and B-objects. Observe that $\varphi_{i_1}(L^k) = (a_1, b_1)$ and $\varphi_{i_j}(L^k) = (\hat{a}_j, \hat{b}_j)$ where $j \leq k$.

Consider L^n and we now show that L^* is a φ-monotonic change of L^n. In other words, we need to show that $\varphi_i(L^n) L^*_i (a, b)$ for each i and $(a, b) \in A \times B$ satisfying $\varphi_i(L^n) L^n_i (a, b)$. Clearly, this is true for $i = i_1$ because $\varphi_{i_1}(L^n) = (a_1, b_1) = \tau(L^*_{i_1}, A, B) = \tau(L^n_{i_1}, A, B)$. Consider now the $i = i_2$ case. Fix any $(a, b) \in A \times B$ such that $\varphi_{i_2}(L^n) = (\hat{a}_2, \hat{b}_2) L_{i_2} (a, b)$. Then the definition of lexicographical preferences and the construction of L^n yield that $a \neq a_1$ and $b \neq b_1$. Recall that $(\hat{a}_2, \hat{b}_2) = \tau(L^*_{i_2}, A \setminus \{a_1\}, B \setminus \{b_1\})$. Thus, in the lexicographical ordering under $L^*_{i_2}$, $\hat{a}_2$ must be ranked ahead of a (if $\hat{a}_2 \neq a$) and $\hat{b}_2$ ahead of b (if $\hat{b}_2 \neq b$). Thus, $(\hat{a}_2, \hat{b}_2) L^*_{i_2} (a, b)$, our desired result. A similar proof applies for all $i = i_k$ where $k \leq n$. Thus, L^* is a φ-monotonic change of L^n. Consequently, by Lemma 2, $\varphi(L^*) = \varphi(L^n)$ and $\varphi_{i_1}(L^*) = (a_1, b_1)$.

Claim 6. There exists an agent such that $\varphi_i(L) = \tau(L_i, A, B)$ for all L.
Proof of Claim 6. This proof is the replica of the proof of Claim 2 for the $n = 2$ case.

Claim 7. Any strategy-proof, Pareto-efficient allocation rule φ is a sequential serial dictatorship.
Proof of Claim 7. This claim is a consequence of Claims 2 and 6. □

[15] In fact, this agent is the second agent.

Appendix: Proof upon Request

Now we extend our model to m market cases. Here O_l denotes the set of objects in lth market. We use all the notations and concepts used in the two-market case in the m market case. We use φ^l denotes the allocation in lth market.

Proposition 3. *Let there be m separate markets. Then any allocation rule $\varphi : \mathcal{L} \to X$ which is strategy-proof, nonbossy and efficient must be a sequential dictatorship.*

Proof. Let $\mathcal{L}^2$ be a domain of lexicographical preferences such that for any $L \in \mathcal{L}^2$, no two agents have the same top-ranked object in all markets except two.

Claim 1. Fix any $L \in \mathcal{L}^2$. Let α and β be the markets in which agents do not necessarily have different top-ranked object. Then the allocation rule φ restricted to the domain of lexicographical preference profiles in which the relative rankings of objects in each market $\gamma \neq \alpha, \beta$ stay the same as under L is a sequential dictatorship.

Proof of Claim 1. Given that lexicographical preferences are separable and φ is efficient, each agent must get her top ranked object in each market $\gamma \neq \alpha, \beta$ under $\varphi(L)$. We now can vary lexicographical preferences by only changing the relative rankings of the objects in markets α and β. Now following the proof of the main theorem in two markets, we obtain that there must be an agent who obtains her top-ranked object in both markets α and β as long as the relative-rankings of objects in each of the remaining market stay the same as under L.

Claim 2. $\varphi : \mathcal{L}^2 \to X$ is a sequential dictatorship.

Proof of Claim 2. Consider any two preference profiles $L \in \mathcal{L}^2$ and $L' \in \mathcal{L}^2$. Then we can find a sequence of profiles $\{L^1, L^2, \cdots, L^h\}$ such that

- $L^1 = L$ and $L^k = L'$;
- any two neighboring profiles L^k and L^{k+1} satisfies the following condition: For L^k and L^{k+1} there exist three markets α, β and γ such that:
  1. each agent has a different top-ranked object in each market $\delta \neq \alpha, \beta$ under L^k;
  2. each agent has a different top-ranked object in each market $\delta \neq \alpha, \gamma$ under L^{k+1};
  3. each agent's top-ranked object in each market $\delta \neq \alpha\beta, \gamma$ is the same under L^k and L^{k+1}.

Now we show that for any L^k and L^{k+1} there exists an agent who always obtains her most preferred bundle under φ. Suppose otherwise, and let agents i and j be the agents who obtain their most preferred bundles under L^k and L^{k+1}, respectively. Then consider a profile L^* such that

1. each agent has the same top-ranked object in market α under L^*;
2. each agent's top-ranked object in each market β is the same under L^k and L^*;
3. each agent's top-ranked object in each market β is the same under L^{k+1} and L^*;
4. each agent's top-ranked object in each market $\delta \neq \alpha\beta, \gamma$ is the same under L^k, L_{k+1} and L^*.

Under both L^k and L^*, i must be the agent who obtains her most preferred bundle. Under both L^{k+1} and L^*, j must be the agent who obtains her most preferred bundle. However, observe that the most preferred bundle of i and j under L^* contains the top ranked object in market α. Thus, we reach a contradiction.

This and the fact that $L, L' \in \mathcal{L}^2$ are selected randomly prove that there exists one agent who obtains her most preferred bundle under $\varphi : \mathcal{L}^2 \to X$. Once this agent's most preferred bundle is fixed, using similar arguments we can find another agent who obtains her most preferred bundle under $\varphi : \mathcal{L}^2 \to X$. Continuing in this fashion, we prove Claim 2.

Claim 3. Fix any $L \in \mathcal{L}^2$. Without loss of generality let i be the ith agent to make a choice if the reported preference profile is L. Let L'_i such that $(L'_i, L_{-i}) \notin \mathcal{L}^2$. Then

$$\varphi_i(R_i, L_{-i}) = \tau\left(L'_i, O_1 \setminus \cup_{j<i}\varphi^1_j(L), O_2 \setminus \cup_{j<i}\varphi^2_j(L)\right) \cdots, O_m \setminus \cup_{j<i}\varphi^m_j(L),$$

and

$$\varphi_j(L'_i, L_{-i}) = \varphi_j(L), \text{ for all } j < i.$$

Proof of Claim 3. Claim 1 when $i = 1$ is a consequence of the strategy-proofness of φ. Now we prove Claim 3 for any random i assuming that Claim 3 is true for all $j < i$. The strategy-proofness of φ yields that i cannot obtain any of

$$\{\varphi^1_1(L), \cdots, \varphi^1_{i-1}(L), \varphi^2_1(L), \cdots, \varphi^2_{i-1}(L), \cdots \varphi^m_1(L), \cdots, \varphi^m_{i-1}(L)\},$$

as long as the others report L_{-i}. Because agent i is the ith one to choose as long as she reports lexicographical preferences when the others report L_{-i}. Thus, i should be able to obtain $\tau\left(L'_i, O_1 \setminus \cup_{j<i}\varphi^1_j(L), O_2 \setminus \cup_{j<i}\varphi^2_j(L)\cdots, O_m \setminus \cup_{j<i}\varphi^2_j(L)\right)$ by reporting some L'_i. Now due to the strategy-proofness of φ, it must be that

$$\varphi_i(L'_i, L_{-i}) = \varphi_i(L'_i, L_{-i}) = \tau\left(L'_i, A \setminus \cup_{j<i}O_1 \setminus \cup_{j<i}\varphi^1_j(L), O_2 \setminus \cup_{j<i}\varphi^2_j(L)\cdots, O_m \setminus \cup_{j<i}\varphi^2_j(L)\right).$$

This is the first item in Claim 3 for agent i. Now the nonbossiness of φ yields that $\varphi(L'_i, L_{-i}) = \varphi(L'_i, L_{-i})$. Using $\varphi : \mathcal{L} \to X$ is a sequential dictatorship, we know that $\varphi_j(L) = \varphi_j(L'_i, L_{-i})$ for all $j < i$. This in turn gives that $\varphi_j(L'_i, L_{-i}) = \varphi_j(L)$ for all $j < i$.

Claim 4. Fix any $L' \in \mathcal{L}$. Consider the lexicographical preference profile, $L \in \mathcal{L}^2$, such that

$$L_i : \varphi^1_i(L'), \varphi^2_i(L'), \cdots \varphi^m_i(L') \cdots \text{ for all } i.$$

If i is the ith agent to choose her bundle when the reported preference profile is L then

$$\varphi_i(L') = \tau\left(L'_i, O_1 \setminus \{\cup_{j<i}\varphi^1_j(L')\}, O_2 \setminus \{\cup_{j<i}\varphi^2_1(L'), \cdots, O_m \setminus \{\cup_{j<i}\varphi^m_1(L')\}\right).$$

Proof of Claim 2. Clearly, L is a φ-monotonic transformation of L'. Thus, $\varphi(L) = \varphi(L')$. In addition, (L'_i, L_{-i}) where $i \in N$ is a φ-monotonic transformation of L'. Consequently,

$$\varphi(L'_i, L^2_{-i}) = \varphi(L'). \tag{7}$$

We now prove the claim for agent 1. Because agent 1 is the first one to select when any lexicographical preference profile in $\mathcal{L}^2$ is reported, Claim 3 gives that

$$\varphi_1(L'_1, L_{-1}) = \tau(L'_1, O_1, \cdots O_m). \tag{8}$$

Combining (7) and (8) we obtain that

$$\varphi_1(L') = \tau(L'_1, O_1, \cdots, O_m).$$

Fix any agent i, and suppose that Claim 4 is true for all $j < i$. Now we prove the claim for agent i. Given that agent i is the ith one to select when the agents report a lexicographical preference profile in $\mathcal{L}^2$ in which each $j < i$ reports L_j, Claim 3 gives that

$$\varphi_i(L'_i, L_{-i}) = \tau\left(L'_i, O_1 \setminus \{\cup_{j<i}\varphi^1_j(L)\}, O_2 \setminus \{\cup_{j<i}\varphi^2_1(L) \cdots, O_m \setminus \{\cup_{j<i}\varphi^m_1(L)\}\right). \quad (9)$$

Combining this with (7) and the assumption that Claim 2 is true for $j < i$, we obtain that

$$\varphi_i(L') = \tau\left(L'_i, O_1 \setminus \{\cup_{j<i}\varphi^1_j(L')\}, O_2 \setminus \{\cup_{j<i}\varphi^2_1(L'), \cdots, O_m \setminus \{\cup_{j<i}\varphi^m_1(L')\}\right).$$

Claim 4 and the fact that φ is a sequential dictatorship on $\mathcal{L}^2$ complete the proof. $\square$