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PH. D. THESIS

ON THE NODAL SET OF SOLUTIONS OF DEGENERATE - SINGULAR AND NONLOCAL EQUATIONS

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Men fear thought
as they fear nothing else on earth,
more than ruin,
more even than death.
Thought is subversive and revolutionary.

— Bertrand Russel

Guarda che luna,
guarda che ..

— Fred Buscaglione

Esatto!

— Dogui Nicheli

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CONTENTS

INTRODUCTION

In this thesis we are concerned with the study of the nodal set of nontrivial solutions of different types of elliptic equations strictly related to a problem of singularly perturbed systems of nonlocal elliptic equations.
The manuscript is divided in three different parts, corresponding to the main problems treated during the last years. The first one is devoted to the study of the nodal set of a segregated critical configuration arising as singular limit of a system of elliptic nonlocal equations with strongly competing interaction terms, while in the second one we consider the problem of s-harmonic functions on cones when the parameter s approaches 1, wondering whether solutions of the problem do converge to harmonic functions in the same cone or not.
Finally, in the last part we focus the attention on the nodal set of solutions of a class of degenerate-singular elliptic equation trying to understand how the presence of degeneracy and singularity in the coefficient affects the structure and the regularity of the solutions. Moreover, in this last part, we find a remarkable link with the problem of the nodal set of s-harmonic functions.

Before moving on, we would like to stress that all the Chapters are not only centred on the research topic of the nodal set of solutions of partial differential equations, but they represent three key points in the analysis of patterns formation through spatial segregation in some models of enhanced anomalous diffusion.
Several physical phenomena can be described by a certain number of densities, populations or probabilities distributed in a domain and subject to laws of diffusion, reaction, and competitive interaction. In the pioneering work [53] of Georgii Gause of the 1932, has been introduced the so called "*competitive exclusion principle*" which states that whenever the competitive interaction is the prevailing phenomenon, the densities can not coexist simultaneously and tend to segregate, hence determining a partition of the domain itself.
As a model problem, let us start with the system of stationary equations

$$\begin{cases} -\Delta u_{i,\beta} = f_{i,\beta}(u_{i,\beta}) - \beta u_{i,\beta} \displaystyle\sum_{j \neq i} g_{ij}(u_{j,\beta}) \\ u_{i,\beta} > 0. \end{cases}$$

In particular, the cases $g_{ij}(t) = \beta_{ij} t$ (Lotka-Volterra competitive interactions) and $g_{ij}(t) = \beta_{ij} t^2$ (focusing-defocusing Gross-Pitaevskii system) are of particular interest in the applications to population dynamics [75] and theoretical physics [54, 61] respectively. For the case of standard diffusion, the regularity of solutions and the asymptotic analysis is fairly well understood, starting from [11, 20, 31] for the Lotka-Volterra case and from [17, 26, 30, 33] for the Gross-Pitaevskii system, in a series of recent papers [19, 32, 35, 72, 90], also in the parabolic case [36, 34, 35, 89]. Since, from a modeling point of view, the limiting configurations as $\beta \to +\infty$ describe an approximation of highly competing systems, one crucial step of this analysis is the study of the qualitative properties of the singular limit. Indeed, in the classic case has been shown that the limit vector $\mathbf{u} = (u_1, \dots, u_h)$ has densities with mutually disjoint supports, i.e. the segregated states u_i satisfy

$$\begin{cases} -\Delta u_i = f_i(u_i) & \text{in } \{u_i > 0\} \\ u_i \cdot u_j \equiv 0 & \text{for } j \neq i. \end{cases} \tag{1}$$

The natural subjects of this analysis concern the optimal regularity of the limiting profiles, equilibrium principle at the arising interfaces and regularity of the free boundary itself. In the mentioned papers, the authors in [30, 33] studied singularly perturbed systems relating them to some optimal partition problem for nonlinear eigenvalues. For this latter problem, we remark that in [25] the authors have proved the regularity of free interfaces of optimal partition problems for the eigenvalues of the Laplacian operator with Dirichlet boundary conditions. Moreover, in [32, 29] they proved Lipschitz regularity of the limiting solutions as well as the regularity of the free boundaries in the case of two dimension.
On the other hand, in [71, 72] has been deeply studied that the limits of a system of Gross-Pitaevskii equations relying the proof on elliptic estimates, blow-up technique, the monotonicity formula by Almgren [1] and Alt-Caffarelli-Friedman type formula [2, 3]. We mention the book of Caffarelli-Salsa [21] for a complete picture of the application of the monotonicity formulas for a larger class of free boundaries problem and to [16] for the case of more general nonlinearities.
The common characteristic of all these problems is that in the singular limit, the components of the solutions of these systems group in different blocks and the supports of the different blocks become disjoint. In particular, these are specific cases of free boundary problems and they are strictly connected to the problem of nodal and critical point sets of solution for PDEs, which is itself a research topic that has attracted a great deal of attention in the last decades (see e.g. [15, 41, 56, 57, 58, 66]). The philosophy is that both in the case of energy minimizing solutions and critical ones, the limiting segregated configurations satisfy a reflection law which represents the only interaction between the different densities through the common free boundary. Thanks to this reflection property, the free boundary is locally described as the nodal set of a scalar valued solution of some PDE.

From this perspective, before to compare this results with their nonlocal counterpart, we would like to give more attention to two recent papers [18, 81] which include all the previous cases and summarize the most recent results achieved in this research topic. In the first one, the authors studied the local structure and the smoothness of singularities of the nodal set of a constrained harmonic maps into a singular space, i.e. given $\Omega \subset \mathbb{R}^n$ a bounded smooth domain and

$$\Sigma = \Big\{x \in \mathbb{R}^h \colon F(x) = 0\Big\} \quad \text{with } F(x) = \sum_{i \neq j} x_i^2 x_j^2,$$

the problem is to find a minimizer $\mathbf{v} = (v_1, \dots, v_h) \in H_0^1(\Omega, \Sigma)$ such that

$$\int_\Omega |\nabla \mathbf{u}|^2 \,\mathrm{d}X = \min \left\{ \int_\Omega |\nabla \mathbf{v}|^2 \,\mathrm{d}X \colon \mathbf{u} \in H_0^1(\Omega, \Sigma) \text{ such that } \int_\Omega \mathbf{u}^2 \mathrm{d}X = 1 \right\}.$$

We remark that this problem is strictly related to the one of optimal partition for the Dirichlet eigenvalue and contains the class of limiting profile arising as $\beta \to \infty$. In particular they proved a stratification result for the singular set, as was done in the classic case [79], by a convexity argument deeply based on the validity of Weiss type monotonicity formula [91]. Moreover, with a convexity argument, they proved uniqueness of tangent maps as well as the local structures of the singular sets.
Instead, in [81] the authors dealt with the nodal set of segregated critical configurations under a weak reflection law, i.e. they considered the class of functions $\mathbf{u} \in (H^1(\Omega))^h$ whose components are all nonnegative and Lipschitz continuous in the interior of Ω and such that

$$\begin{cases} -\Delta u_i = f_i(u_i) - \mu_i & \text{in } \mathcal{D}'(\Omega) \\ u_i \cdot u_j \equiv 0 & \text{for } j \neq i, \end{cases}$$

where f_i are a suitable collection of differentiable functions and $\mu_i \in \mathcal{M}(\Omega)$ some nonnegative Radon measures, each supported on the nodal set $\Gamma(\mathbf{u}) = \{x \in \Omega \colon \mathbf{u}(x) = \mathbf{0}\}$. Moreover, they impose the validity of a weak reflection principle based on some Pohožaev type identities, which implies that the absolute value of the gradient is the same when we approach the regular set from opposite sides. The importance of this class is due to the fact that it collects the singular limit to competition-diffusion systems, both those possessing a variational structure and those with Lotka-Volterra type interaction.

In the recent years has been given much attention on the case of anomalous diffusion, when the Gaussian statistics of the classical Brownian motion is replaced by a different one, giving rise to the so called "Lévy jumps". Since such operators are of real interest both in population dynamics (see [59]) and in relativistic quantum electrodynamics (see [64, 65]), we plan to extend the previous analysis of the nodal set of segregated configuration to the nonlocal context.
Since the asymptotic analysis and the study of the nodal set in case of fractional Laplacians are

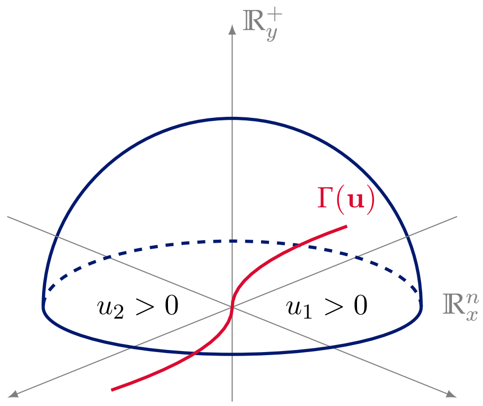


Figure 1: Prototype of segregated configurations in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n_x \times \mathbb{R}^+_y$

very challenging issue, the only known results are contained in [83, 84, 85, 88, 86]. In [83, 84, 86], the authors considered the class of stationary systems of semilinear equations

$$\begin{cases} (-\Delta)^s u_{i,\beta} = f_{i,\beta}(u_{i,\beta}) - \beta u_{i,\beta} \sum_{j\neq i} g_{ij}(u_j) \\ u_{i,\beta} \in H^s(\mathbb{R}^n), \end{cases}$$

focusing on the case $g_{ij}(t) = \beta_{ij} t$ (Lotka-Volterra competitive interactions [86]) and $g_{ij}(t) = \beta_{ij} t^2$ (relativistic Gross-Pitaevskii system [83, 84]). In both cases, they provide some uniform estimates in Hölder spaces with respect to the parameter of competition β. This results can be obtained considering the local realisation of the fractional Laplacian due to the so called Caffarelli-Silvestre extension popularized in [23], which characterize the fractional Laplacian in $\mathbb{R}^n$ as the Dirichlet-to-Neumann map for a variable v depending on one more space dimension. With this formulation, the competition-diffusion problem in $\mathbb{R}^n$ translates into a degenerate-singular elliptic equation in $\mathbb{R}^{n+1}_+$ with a Neumann type condition on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$, which allows to introduce the fractional versions of the Alt-Caffarelli-Friedman and Almgren type monotonicity formulas.

As a byproduct, up to subsequences, there is convergence of the above solutions to a limiting profile, which components are segregated. Because of the genuinely nonlocal nature of the problem, many difficulties and technicalities arise in the asymptotic analysis and in the study of the nodal set. First of all, since these problems have been studied using the extension technique, both in the Lotka-Volterra and in the variational case, the segregation occurs only in the n-dimensional space and it is natural to expect free boundaries of codimension 2 (see Figure 1). Secondly, the Gross-Pitaevskii competition and the Lotka-Volterra one exhibit deep differences not only from the point of view of the optimal regularity exponent, but also with the one of the segregated limiting profiles, which is in deep contrast with the local case $s = 1$, as we previously pointed out. More precisely, in the first case the limiting profiles satisfy a natural extension to the fractional setting of (1), that is

$$\begin{cases} (-\Delta)^s u_i = f_i(u_i) & \text{in } \{u_i > 0\} \\ u_i \cdot u_j \equiv 0 & \text{for } j \neq i, \end{cases}$$

while in the second one

$$\begin{cases} (-\Delta)^s \left(u_i - \sum_{j\neq i} u_j\right) = f_i(u_i) - \sum_{j\neq i} f_j(u_j) & \text{in } \{u_i > 0\} \\ u_i \cdot u_j \equiv 0 & \text{for } j \neq i. \end{cases}$$

Moreover, in the Gross-Pitaevskii case, where the structure of the nodal set is wilder than the Lotka-Volterra one, the nonlocal nature of the problem affects the ideas and techniques developed in [18, 81]. In the latter, the definition of the nonlocal operator $(-\Delta)^s$ does not allow to relate the structure of the free boundary to the nodal set of s-harmonic function since, roughly speaking, the linear combination of s-harmonic functions with disjoint supports is no more s-harmonic in their union. Secondly, the nonlocal counterpart of the formulation via constrained harmonic maps into singular space introduced in [18], is intimately related to the problem of harmonic maps with "partially free boundary". Unfortunately, this strategy turn out to be inefficient since the segregated condition on $\mathbb{R}^n$ translates into the problem of fractional harmonic maps into singular space which implies the occurrence of a "singular partially free boundary"(see [69] for an application in the context of fractional Ginzburg-Landau equations).
Last but not least, we remark that in [83, 84] the most challenging issue lies in the lack of the validity of an exact Alt-Caffarelli-Friedman monotonicity formula, which reflects, at the spectral level, the lack of convexity of the eigenvalues with respect to domain variations.

In Chapter 1 we tried to give a better picture of the limiting profiles in the context of variational competition. In this analysis, the main difficulties are the problem of codimension between the segregation and the degenerate-singular elliptic equation introduced with the extension technique and the lack of validity of a reflection principle that allows to compare our problem to the one of the nodal set of some nonlocal elliptic equation. In order to overcame this problem, we consider the case of planar segregated configurations, in order to exploit the topology of S^1. Nevertheless, as pointed out in [84], we need to take care of the presence of self-segregation for $s \geq 1/2$. This phenomenon was also discussed in [81] for the local case with critical segregated configurations, but the nonlocal attitude of our problem prevents to apply the same reduction used for the classical Laplacian. We mention that in [35] the authors proved, when we consider segregated profiles arising as limit of competition-diffusion systems, that the self-segregation can be ruled out using an improvement of flatness.

In our final result, we split the nodal set into its segregated and self-segregated part, proving a local regularity result near the two strata. Moreover, in the first case, we show the existence of a regular set $\mathcal{R}(\mathbf{u})$ relatively open in $\Gamma(\mathbf{u})$ which satisfy a vanishing Reifenberg flatness condition and a singular set $\mathcal{S}(\mathbf{u})$ which consists in a locally finite collection of singular points.
All the result contained in Chapter 1 are obtained in collaboration with Susanna Terracini and Alessandro Zilio.

As already mention, in Chapter 2, we consider the problem of s-harmonic function on cone, i.e. given C an open cone in $\mathbb{R}^n$ with vertex in 0 and $s \in (0,1)$, we consider the problem of the classification of nontrivial functions which are s-harmonic inside the cone and vanish identically outside, that is

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in} \quad C, \\ u_s \geq 0 & \text{in} \quad \mathbb{R}^n \\ u_s \equiv 0 & \text{in} \quad \mathbb{R}^n \setminus C. \end{cases} \tag{2}$$

By [5], it is known that there exists a homogeneous, nonnegative and nontrivial solution of the form

$$u_s(x) = |x|^{\gamma_s} u_s\left(\frac{x}{|x|}\right),$$

where $\gamma_s := \gamma_s(C)$ is a definite homogeneity degree (characteristic exponent of the cone C).
This problems is actually deeply connected to the one of Chapter 1, since it consists on the study of such conic s-harmonic functions that appear as limiting blow-up profiles and play a major role in many free boundary problems with fractional diffusions and in the study of the geometry of nodal sets, also in the case of partition problems (see, e.g. [7, 14, 39, 51] and the blow-up analysis of Chapter 1). Moreover, as we shall see later, they are strongly involved with the possible extensions of the Alt-Caffarelli-Friedman monotonicity formula to the case of fractional diffusion.
The problem of homogeneous s-harmonic functions on cones has been deeply studied in [5, 8, 9, 67]. and since not many qualitative properties are known for the s-harmonic functions on cones, we decided to focus our attention on the limiting behaviour as $s \nearrow 1$ wondering whether solutions of the problem do converge to a harmonic function in the same cone and, in case, which are the suitable spaces for convergence in order to deduce. In such a way, we wanted to deduce some qualitative properties of the s-harmonic function for s sufficiently near 1.
We therefore addressed the problem of the asymptotic behavior of the solutions of problem (2) for $s \nearrow 1$, obtaining a rather unexpected result: our analysis shows high sensitivity to the opening solid angle ω of the cone C_ω, as evaluated by the value of the homogeneity degree $\gamma(C_\omega) = \gamma_1(C_\omega)$ of the harmonic function on C:

1. in the case of "*wide cones*", when $\gamma(C) < 2$ (that is, $\theta \in (\pi/4, \pi)$ for spherical caps), our solutions do converge to the harmonic homogeneous function of the cone;

2. instead, in the case of "*narrow cones*", when $\gamma(C) > 2$ (that is, $\theta \in (0, \pi/4]$ for spherical caps), then limit of the homogeneity degree will be always two and the limiting profile will be something different, though related through a correction term.

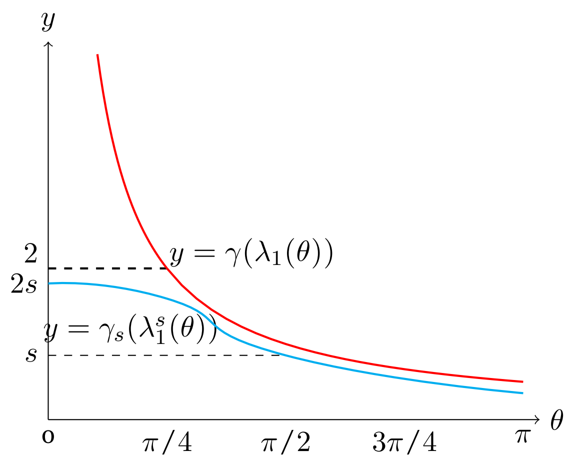


Figure 2: Characteristic exponents of spherical caps of aperture 2θ for $s < 1$ and $s = 1$.

This surprisedly result yields different nontrivial improvement in the context of segregated critical configurations. First of all, as shown in [84, 83], estimates in Hölder spaces can be obtained by the use of fractional versions of the Alt-Caffarelli-Friedman and Almgren monotonicity formulas. In particular, one could prove a deep connection with the optimal partition problem among the class $\mathcal{P}^s$ of 2-partitions on S^{n-1}

$$\nu_s^{ACF} := \frac{1}{2} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \gamma_s(\lambda_1^s(\omega_i)),$$

where $\gamma_s(\lambda_1^s(\omega))$ is equal to the characteristic exponent of the cone spanned by $\omega \subset S^{n-1}$.
A classical result by Friedland and Hayman, [47], yields $\nu^{ACF} = 1$ (case $s = 1$), and the minimal value is achieved for two half spheres; this equality is the core of the proof of the classical Alt-Caffarelli-Friedman monotonicity formula.
It [84] was also conjectured that $\nu_s^{ACF} = s$ for every $s \in (0, 1)$. Unfortunately, the exact value of ν_s^{ACF} is still unknown, but as a byproduct of our asymptotic analysis we have

$$\lim_{s\to 1} \nu_s^{ACF} = 1 \; .$$

In the end, we remark that this asymptotic analysis suggests that even the segregated configurations are affected by this unexpect phenomenon since the trace of the blow-up limits introduced in Chapter 1 belong to the class of s-harmonic functions on cones. We believe that this asymptotic result will push our research in a new challenging direction. All these results are obtained in collaboration with Susanna Terracini and Stefano Vita (see [82]).

Finally, in the last Chapter we conclude the thesis dealing with the nodal set of solutions of degenerate-singular elliptic equations, which put an end to our glimpse into the study of segregated configurations and free boundary problems ruled by anomalous diffusion.
In literature, the subject of nodal sets, or level sets in general, is an important research topic for solutions of PDEs. While in some cases, this topics are themselves the primary concern, in many others they provide an important tool in the study of qualitative properties of solutions of PDEs. Initially, in [15, 58] the authors respectively proved an optimal bound on the Hausdorff dimension of the singular set of solutions of linear and superlinear elliptic equations and, in the second work, the first estimate on the $(n-1)$-Hausdorff measure of the nodal set in a neighbourhood of a point with vanishing order. We remark that in the second paper, the estimate is explicit and based only on the existence of a finite order of vanishing, which suggests that the validity of a strong unique continuation property is the starting point of this kind of analysis.
Recently in [41, 56, 57, 66] they proved several results on the structure of the singular set and even some estimate on the $(n-2)$-Hausdorff measure of the singular set.
In all these cases, as pointed out in [48, 49, 66], the class of solution of PDEs, of which we want to study the nodal set, must satisfy a strong unique continuation principle, in order to ensure the existence of a finite vanishing order. Many improvement have been done in this topic, using on one side the Carleman estimates approach and on the other one the monotonicity approach (see e.g. [48, 49]), deeply based on the existence of an Almgren type monotonicity formulas and a geometrical reduction, first introduced in [4]. These last results prove the validity of the strong unique continuation principle for solutions of divergence form elliptic equations of the second order with Lipschitz leading coefficients and suitable lower order terms. In particular, in [68] the author proved the optimality of the Lipschitz condition with an Hölder continuous counterexample.
At the same time, in their pioneering papers [44, 43] the authors introduced a general class of degenerate operators $L=\mathrm{div}(A(X)\nabla\cdot)$ whose coefficient $A(X)=(a_{ij}(X))$ are defined starting from a symmetric matrix valued function such that

$$\lambda\omega(X)\left|\xi\right|^2 \leq (A(X)\xi,\xi) \leq \Lambda\omega(X)\left|\xi\right|^2, \quad \text{for some } \lambda,\Lambda>0,$$

where ω may either vanish, or be infinite, or both. In particular they focus the attention on the case $\omega\in A_2$-Muckenhoupt class, i.e.

$$\sup_{B\subset\mathbb{R}^{n+1}}\left(\frac{1}{|B|}\int_B \omega(X)\mathrm{d}X\right)\left(\frac{1}{|B|}\int_B \omega^{-1}(X)\mathrm{d}X\right)<\infty.$$

While in the recent years these operators are quite commonly used since they are strictly related to the local realisation of fractional powers of operators (see [23, 27, 80] for different application in the extension of fractional operator), the authors initiated this research motivated by some result on the boundary behaviour of harmonic functions in non-tangentially accessible domains in [60].
Inspired by [43, Section 3], given $X = (x, y) \in \mathbb{R}^n_x \times \mathbb{R}_y$ we consider the cases of $\omega(X) = |y|^a$, with $a \in (-1, 1)$, and the associated degenerate-singular operator L_a defined as $L_a = \mathrm{div}(|y|^a \nabla)$, with div and ∇ respectively the divergence and the gradient operator in $\mathbb{R}^{n+1}$.
Now, given Σ the "*characteristic manifold*" associated to our weight, as the set of points where the coefficient either vanishes or blows up, we studied the properties of the nodal set $\Gamma(u)$ of solutions to

$$-L_a u = 0 \quad \text{in } B_1 \subset \mathbb{R}^{n+1},$$

focusing the attention on the restriction of the nodal set $\Gamma(u)$ on the characteristic manifold Σ. Following the philosophy explained in the third paragraph of the Introduction, one motivation of our analysis is the application of this results on a competition-diffusion system with variational competition and degenerate-singular diffusion: one can imagine that the characteristic manifold Σ is playing an active role in the diffusion phenomenon, indeed we expect that the diffusion across the manifold Σ is penalized or encouraged accordingly to the value of $a \in (-1, 1)$.
On the other hand, the choice to study this class of 1-dimensional homogeneous weights, allows to extend our analysis to the cases when Σ is an n-dimensional manifold properly embedded in $\mathbb{R}^{n+1}$ and the weights take the form $\omega(X) = \mathrm{dist}(X, \Sigma)^a$ and even then to a wilder class of monomial weights(see for example [12, 62]).

In Chapter 3 we discuss the local properties of L_a-harmonic functions and their nodal set near the characteristic manifold Σ. In particular, using some Almgren and Weiss type monotonicity formulas, we classify the possible blow-up limit and we prove the uniqueness of a nondegenerate tangent map at every point of the nodal set.
The main feature of this class of degenerate-singular equations is that any L_a-harmonic function can be decomposed with respect to the direction orthogonal to the characteristic manifold Σ, in the sense that given u an L_a-harmonic function in $H^{1,a}(B_1)$ there exist $u^a_e \in H^{1,a}(B_1), u^{2-a}_e \in H^{1,2-a}(B_1)$ two unique functions symmetric with respect to Σ respectively L_a and L_{2-a} harmonic in B_1 and locally smooths, such that

$$u(X) = u^a_e(X) + u^{2-a}_e(X) y |y|^{-a} \quad \text{in } B_1.$$

Therefore, local properties of the solutions, as their exponent of optimal regularity, their Taylor expansion near nodal set and the structure of the nodal set itself, are fulled comprehended by knowing the local behaviour of their even (symmetric with respect to orthogonal direction of Σ)

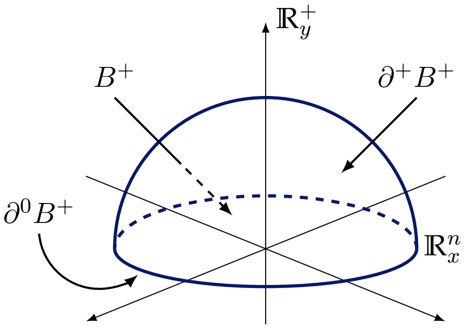
and odd (antisymmetric with respect to orthogonal direction of Σ) parts.
This specific "conduct" of the solutions is due to the presence of the characteristic manifold Σ, where either the vanishing or the blowing up of the weights imposes a quantization of the possible ways in which the nodal set can diffuse across Σ.
With the previous decomposition in mind, we restrict our blow-up analysis to the symmetric case and finally we introduce the new notion of "*tangent field*" Φ^{X_0} of u at a nodal point $X_0 \in \Sigma$, which takes care of the different behaviour of both the symmetric and antisymmetric part of u. Moreover, we introduce the regular $\mathcal{R}(u)$ and the singular part $\mathcal{S}(u)$ as

$$\mathcal{R}(u) = \left\{X \in \Gamma(u)\colon\ |\nabla_x u(X)|^2 + \left|\partial^a_y u(X)\right|^2 \neq 0\right\}, \quad \mathcal{S}(u) = \Gamma(u) \setminus \mathcal{R}(u)$$

and we developed a blow-up analysis in order to fully understand the structure in $\mathbb{R}^{n+1}$ and its restriction on Σ.

Since our approach seems to be quite flexible, in the last part of Chapter 3 we present an applications of our theory in the context of nonlocal elliptic equations. In particular, inspired by [23, 27, 80], we exploit the local realisation of the fractional Laplacian, and more generally of fractional power of divergence form operator L with Lipschitz leading coefficient, in order to study the structure and the regularity of the nodal set of $(-L)^s$-harmonic functions, for $s \in (0,1)$. Moreover, this last Section allows to extend our analysis to fractional powers $(-\Delta_M)^s$ of the Laplace-Beltrami operator on a Riemannian manifold M, also for the case of Lipschitz metric, and moreover to conformal fractional Laplacian on conformally compact Einstein manifolds and asymptotically hyperbolic manifold, thanks to the extension technique developed in [27] and the asymptotic expansion of their geodesic boundary defining function.
As suggested in [86], we would like to stress that our analysis on the nodal set of s-harmonic function allows to fully understand the limiting profile arising from the case of Lotka-Volterra competition, showing a different behaviour with respect to the one presented in Chapter 1.
Finally, our results show some purely nonlocal feature on the possible local expansion of s-harmonic map near their zero set and on the structure of the nodal set itself. On one side we prove that first term of the Taylor expansion of an $(-L)^s$-harmonic function is either an homogeneous harmonic polynomial or any possible homogeneous polynomial. In particular, we exhibit the stratification of the singular set $\mathcal{S}(u)$, showing the existence of an unexpected stratum $\mathcal{S}^s(u)$ contained in a $(n-1)$-dimensional C^1 manifolds, in deep contrast with the local case. In the end, we prove what could be seen as the nonlocal counterpart of a conjecture that Lin proposed in [66]. Following his strategy, we give an explicit estimate on the $(n-1)$-Hausdorff measure of the nodal set $\Gamma(u)$ in terms of the Almgren monotonicity formula previously introduced.
This Chapter is part of a bigger project in collaboration with Yannick Sire, Susanna Terracini and Stefano Vita.

Notations and general results. Throughout the manuscript, we will consider $X = (x, y) \in \mathbb{R}^{n+1}$, with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. With this notation in mind, we define the subspace $\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1} \cap \{y > 0\}$ with $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^N \times \{0\}$. Now, given $D \subset \mathbb{R}^{n+1}_+$ we write



$$D^+ = D \cap \{y > 0\}$$
$$\partial^+ D = \partial D \cap \{y > 0\}$$
$$\partial^0 D = \partial D \cap \{y = 0\}$$

In the picture, we use this notation with $D = B_r(x_0, 0)$ for the $(n+1)$-dimensional ball centered in $X_0 = (x_0, 0) \in \mathbb{R}^n \times \{0\}$.
For any vector valued function $\mathbf{u} = (u_1, \dots, u_h) \in \mathbb{R}^h$, we define $\mathbf{u}^2 = \sum_{i=1}^h u_i^2$, $\nabla\mathbf{u} := (\nabla u_1, \dots, \nabla u_h)$ and $\partial_\nu \mathbf{u} := (\partial_\nu u_1, \dots, \partial_\nu u_h)$, for every $\nu \in \mathbb{R}^{n+1}$. In particular, through Chapter 1 we will deeply use the notation $\langle \cdot, \cdot \rangle$ for the scalar product in $\mathbb{R}^h$.
Through the paper, for $a \in (-1, 1)$ we will always consider the weighted Sobolev spaces $H^{1,a}(B_1)$ deeply studied in [44, 43, 70] as the closure of $C^\infty(\overline{B_1})$ with respect the norm

$$\|u\|^2_{H^{1,a}(B_1)} = \int_{B_1} |y|^a u^2 \mathrm{d}X + \int_{B_1} |y|^a |\nabla u|^2 \mathrm{d}X.$$

In this setting, we will always denote with $L_a = \mathrm{div}(|y|^a \nabla)$ the divergence form operator associated to the weight $\omega(y) = |y|^a$. More precisely, following the idea in the mentioned [23], for every $u \in H^s(\mathbb{R}^n)$, we consider $v \in H^{1,a}(\mathbb{R}^{n+1}_+)$ satisfying

$$\begin{cases} \mathrm{div}(|y|^a \nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v(x, 0) = u(x) & \text{in } \mathbb{R}^n\ , \end{cases}$$

with $a = 1 - 2s \in (-1, 1)$. In this setting, the nonlocal operator $(-\Delta)^s$ translates into

$$(-\Delta)^s \colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n), \quad u \longmapsto -\frac{C(n,s)}{\gamma(n,s)} \lim_{y \to 0^+} y^{1-2s} \partial_y v(x, y).$$

Such an extension exists unique for a suitable class of functions u, and it is given by the formula

$$v(x, y) = \gamma(n, s) \int_{\mathbb{R}^n} \frac{y^{2s} u(\eta)}{(|x - \eta|^2 + y^2)^{n/2+s}} \mathrm{d}\eta \quad \text{where } \gamma(n, s)^{-1} := \int_{\mathbb{R}^n} \frac{1}{(|\eta|^2 + 1)^{n/2+s}} \mathrm{d}\eta\ .$$

In the introduction of Chapter 2 and in Chapter 3, we will give more details on this content.

1

NODAL SET OF SEGREGATED CRITICAL CONFIGURATIONS

1.1 INTRODUCTION

Several physical phenomena can be described by a certain number of densities or populations distributed in a domain and subject to laws of diffusion, reaction, and competitive interaction, starting from biological models for competing species in population dynamics [75] to the phase-segregation phenomenon in Bose-Einstein condensation and theoretical physics [54, 61].

In the recent years has been given much attention on the fractional Laplacians, since such operators are of real interest both in population dynamics (see [59]) and in relativistic quantum electrodynamics (see [64, 65]). Inspired by this physical motivations, we plan to extend the known results on the nodal set of segregated configuration to their nonlocal counterpart.
Hence, as pointed out in [83, 84], exploiting the local realization of the fractional Laplacian $(-\Delta)^s$ as a Dirichlet-to-Neumann map (see for instance [23]), several asymptotic results can be proved in the context of competition-diffusion problems with internal reactions, fractional diffusion and strong variational competition.

Theorem 1.1.1 ([84]). *Let $\beta > 0$, $(f_{i,\beta})_\beta$ be a collection of continuous functions uniformly bounded with respect to β on bounded sets and let $(\mathbf{u}_\beta)_\beta \in H^{1,a}(B_1^+;\mathbb{R}^h)$ be a family of solutions $\mathbf{u}_\beta = (u_{1,\beta},\dots,u_{h,\beta})$ of the problems*

$$\begin{cases} -L_a u_{i,\beta} = 0 & \text{in } B_1^+ \\ -\partial_y^a u_{i,\beta} = f_{i,\beta}(u_{i,\beta}) - \beta u_{i,\beta}\sum_{j\neq i} a_{ij}u_{j,\beta}^2 & \text{on } \partial^0 B_1^+. \end{cases} \tag{3}$$

Let us assume that

$$\|\mathbf{u}_\beta\|_{L^\infty(B_1^+)} \leq M$$

for some constant $M > 0$ independent on β. Then, there exists $\alpha = \alpha(n,s) > 0$, non depending on β, such that for $\alpha \in (0,\alpha^*)$

$$\|\mathbf{u}_\beta\|_{C^{0,\alpha}(\overline{B_{1/2}^+})} \leq C,$$

with $C = C(M,\alpha)$*. Moreover,* $(\mathbf{u}_\beta)_\beta$ *is relatively compact in* $H^{1,a}(B^+_{1/2}) \cap C^{0,\alpha}(\overline{B^+_{1/2}})$*, for* $\alpha \in (0,\alpha^*)$*.*

The above result allows to prove its natural global counterpart, either on the whole of $\mathbb{R}^n$ or on domains with suitable boundary conditions.

Theorem 1.1.2 ([84]). *Let* $\beta > 0$*,* $(f_{i,\beta})_\beta$ *be a collection of continuous functions uniformly bounded with respect to* β *on bounded sets and let* $(\mathbf{u}_\beta)_\beta \in H^s(\mathbb{R}^n;\mathbb{R}^h)$ *be a family of solutions* $\mathbf{u}_\beta = (u_{1,\beta},\dots,u_{h,\beta})$ *of the problems*

$$\begin{cases} (-\Delta)^s u_{i,\beta} = f_{i,\beta}(u_{i,\beta}) - \beta u_{i,\beta}\sum_{j\neq i} a_{ij}u^2_{j,\beta} & \text{in } \Omega \\ u_{i,\beta} \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4}$$

where Ω *is a bounded domain of* $\mathbb{R}^n$ *with smooth boundary. Let us assume that*

$$\|\mathbf{u}_\beta\|_{L^\infty(\Omega)} \le M$$

for some constant $M > 0$ *independent on* β*. Then, there exists* $\alpha^* = \alpha^*(n,s) > 0$*, non depending on* β*, such that for* $\alpha \in (0,\alpha^*)$

$$\|\mathbf{u}_\beta\|_{C^{0,\alpha}(\mathbb{R}^n)} \le C,$$

with $C = C(M,\alpha)$*.*

As a byproduct of these results, can be proved that, up to subsequences, we have convergence of the above solutions to a limiting profile, which components are segregated on the boundary $\partial^0 B^+_1$. Actually, if furthermore $f_{i,\beta} \to f_i$, uniformly on compact sets, we can prove that this $\mathbf{u} = (u_1,\dots,u_h)$ limiting configuration satisfies

$$\begin{cases} -L_a u_i = 0 & \text{in } B^+_1 \\ u_i\left(\partial^a_y u_i + f_i(u_i)\right) = 0 & \text{on } \partial^0 B^+_1 \\ u_i \cdot u_j = 0 & \text{on } \partial^0 B^+_1, \text{ for every } i \neq j \end{cases}.$$

Since in the singular limit one finds a vector $\mathbf{u} \in H^{1,a}(B^+_1)$ of limiting profiles with mutually disjoint supports, it is a natural question to understand the regularity and the structure of the nodal set

$$\Gamma(\mathbf{u}) = \{X \in \partial^0 B^+_1 \colon \mathbf{u}(X) = \mathbf{0}\},$$

where all the components of $\mathbf{u}$ takes zero value. Hence, we focus our attention on the following class of vector valued configurations with segregated supports. At this point we postpone the discussion on the order $\alpha^* \in (0,1)$ of Hölder regularity of the segregated configurations for several reasons that will be show later.

Definition 1.1.3. Let $s \in (0,1), a = 1-2s \in (-1,1)$ and $\alpha^* \in (0,1)$, we define the class $\mathcal{G}^s(B_1^+)$ as the set of vector valued function $\mathbf{u} = (u_1, \dots, u_h) \in H^{1,a}_{\text{loc}}(\overline{B_1^+}; \mathbb{R}^h)$ whose components are all non negative, continuous functions such that

(1) $\mathbf{u} \in H^{1,a}(K \cap B_1^+) \cap C^{0,\alpha}(\overline{K \cap B_1^+})$, for every compact set $K \subset B$ and every $\alpha \in (0, \alpha^*)$;

(2) $u_i \cdot u_j|_{y=0} \equiv 0$ for every $i \neq j$ and $\mathbf{u} \neq \mathbf{0}$ on $B_1 \cap \Sigma$. Moreover, for $i = 1, \dots, h$ it satisfies

$$\begin{cases} -L_a u_i = 0 & \text{in } B_1^+ \\ u_i \left(\partial_y^a u_i + f_i(u_i)\right) = 0 & \text{on } \partial^0 B_1^+ \end{cases} \tag{5}$$

where $f_i \colon \mathbb{R}^+ \to \mathbb{R}$ are nonnegative $\mathcal{C}^1$ functions such that $f_i(s) = O(s)$ for $s \to 0$;

(3) for every $X_0 = (x_0, 0) \in \partial^0 B_1^+$ and $r \in (0, \text{dist}(X_0, \partial B))$, the following Pohožaev type identity holds

$$(1-a-n)\int_{B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}X + r\int_{\partial B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}\sigma +$$
$$+2n\int_{\partial^0 B_r^+(X_0)} \sum_{i=1}^h F_i(u_i)\mathrm{d}x - 2r\int_{S_r^{n-1}(X_0)} \sum_{i=1}^h F_i(u_i)\mathrm{d}x = 2r\int_{\partial^+ B_r^+(X_0)} |y|^a (\partial_r \mathbf{u})^2 \mathrm{d}\sigma \tag{6}$$

where $\mathbf{F}(s) = (F_1(s), \dots, F_h(s))$ with $F_i(s) = \int_0^s f_i(t)\,\mathrm{d}t$ for every $i = 1, \dots, h$.

First, in [83] the authors proved for the case $s = 1/2$, i.e. $a = 0$, that for solutions $\mathbf{u} \in \mathcal{G}^{1/2}(B^+)$ the highest possible regularity correspond to the Hölder exponent $\alpha^* = 1/2$. This result is based on a blow-up analysis based on an Almgren type monotonicity formula and an optimal Liouville type theorem for segregated configuration.
Instead, for the general case $s \in (0,1)$ in [84] the authors proved, with a combination of a blow-up analysis and a Liouville type theorem based on the validity of an Alt-Caffarelli-Friedman type monotonicity formula, that the highest possible regularity of the limiting profile correspond to the Hölder exponent $\alpha^* = \alpha^*(n,s)$ such that

$$\alpha^* = \begin{cases} \nu_s^{ACF}, & 0 < s \leq \frac{1}{2}, \\ \min\{\nu_s^{ACF}, 2s-1\}, & \frac{1}{2} < s < 1, \end{cases}$$

where ν_s^{ACF} corresponds to the exponent associated to the Alt-Caffarelli-Friedman formula (see [83, 84]). The threshold $s = 1/2$ is due to the presence of the phenomenon of self-segregation of nonlocal problem where $s \in (1/2, 1)$, which consists in the existence of a ball $\widetilde{B}^+ \subset \mathbb{R}^{n+1}_+$ centered on the nodal set $\Gamma(\mathbf{u})$ and an index $i = 1, \dots, h$ such that all the components u_j of $\mathbf{u}$

with $j \neq i$ are identically zero on the ball make exception of u_i which is not identically zero and such that

$$\partial^0 \widetilde{B}^+ \setminus \Gamma(\mathbf{u}) = \{X \in \partial^0 \widetilde{B}^+ \colon u_i(X) > 0\}.$$

More precisely, following the idea of the optimal Liouville exponent in [83], in Section 1.6 we easily improve the previous results finding the following bound

$$\alpha^* = \begin{cases} s, & 0 < s \leq \frac{1}{2}, \\ 2s - 1, & \frac{1}{2} < s < 1. \end{cases}$$

In particular, this improvement emphasizes the deep relation between the different α-Hölder regularity of the solutions near the nodal set and the structure of the nodal set itself. Inspired by this connection, we decompose the nodal set into its "segregated" and "self-segregated" part. Finally, under the previous notations, we have

Segregated nodal set	Self-Segregated nodal set
$\Gamma(\mathbf{u}) = \partial^0 B_1^+ \setminus \bigcup_{i=1}^h \mathrm{int}\left(\overline{\{u_i > 0\}}\right)$	$\Gamma(\mathbf{u}) = \bigcup_{i=1}^h \partial\{u_i > 0\} \setminus \mathrm{int}\left(\overline{\{u_i > 0\}}\right)$.
α - Hölder continuous, for $\alpha \in (0, s)$	α - Hölder continuous, for $\alpha \in (0, 2s-1)$

Since our main result does not concern the self-segregated portion of the nodal set, we remark that through this Chapter we will always consider the class of segregated profiles $\mathcal{G}^s(B^+)$ locally α-Hölder continuous for every $\alpha \in (0, s)$. Just in Section , we will give more details in the context of self-segregation.

Our approach is deeply based on the validity of an Almgren's type monotonicity formula and on a blow-up analysis of the critical configurations. More precisely, for every $X_0 \in \partial^0 B_1^+$ and $r > 0$ such that $B_r^+(X_0) \subset B_1^+$, we define the functionals

$$E(X_0, \mathbf{u}, r) = \frac{1}{r^{n-1+a}} \left(\int_{B_r^+(X_0)} |y|^a \, |\nabla \mathbf{u}|^2 \, \mathrm{d}X - \int_{\partial^0 B_r^+} \langle \mathbf{u}, F(x, \mathbf{u}) \rangle \mathrm{d}x \right),$$
$$H(X_0, \mathbf{u}, r) = \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+(X_0)} |y|^a \, \mathbf{u}^2 \mathrm{d}\sigma,$$

and, the Almgren's monotonicity formula as

$$N(x_0, \mathbf{u}, r) = \frac{E(x_0, \mathbf{u}, r)}{H(x_0, \mathbf{u}, r)}.$$

Unfortunately, as we anticipate in the Introduction of the manuscript, in order to overcame some technical problem due to problem of codimension between the free boundary $\Gamma(\mathbf{u})$ and the space where the degenerate-singular equation is satisfied, we restrict our attention on the planar case $n = 2$. In this case, we are able to prove

Proposition 1.1.4. *Given $s \in (0,1), n = 2$ and $u \in \mathcal{G}^s(B_1^+)$, then for $X_0 \in \Gamma(\mathbf{u})$ either*

$$N(X_0, \mathbf{u}, 0^+) = s \quad \text{or} \quad N(X_0, \mathbf{u}, 0^+) \geq s + \delta,$$

for some universal constant $\delta > 0$. Moreover, the possible values of the Almgren frequency formula $N(X_0, \mathbf{u}, 0^+)$ are a discrete set in $[s, 2s)$ with $2s$ as point of accumulation.

This result, combined with the convergence of the blow-up sequence both with respect to the strong topologies in $H^{1,a}_{\text{loc}}, C^{0,\alpha}_{\text{loc}}$ and to the Hausdorff distance $d_{\mathcal{H}}$, allows to prove the main result of this Chapter.

Theorem 1.1.5. *Let $s \in (0,1), n = 2$ and $\mathbf{u} \in \mathcal{G}^s(B^+)$. Then the nodal set $\Gamma(\mathbf{u})$ splits into its regular and singular part defined by*

$$\mathcal{R}(\mathbf{u}) = \{X_0 \in \Gamma(\mathbf{u}) \colon N(X_0, \mathbf{u}, 0^+) = s\},$$
$$\mathcal{S}(\mathbf{u}) = \{X_0 \in \Gamma(\mathbf{u}) \colon N(X_0, \mathbf{u}, 0^+) > s\},$$

where $\mathcal{S}(\mathbf{u})$ is a locally finite collection of points and $\mathcal{R}(\mathbf{u})$ a set relatively open in $\Gamma(\mathbf{u})$ which satisfies a vanishing Reifenberg flatness condition.

This Chapter is organized as follows. In Section 1.2 we prove that elements in $\mathcal{G}^s(B^+)$ satisfy an Almgren's type monotonicity formula; by exploiting this fact, in Section 1.3 we prove convergence of blow-up sequences as well as some closure properties of the class $\mathcal{G}^s(B_+)$. In Section 1.4 we use the Federer's Reduction Principle in order to prove some Hausdorff estimates for the nodal sets and we introduce the notion of regular and singular set. Moreover, in Section 1.5 we prove that the regular part of the nodal set satisfies a vanishing Reifenberg condition. Finally in Section 1.6 we present some useful remark and the relation between the class $\mathcal{G}^s(B_1^+)$ and the singular limit of competition-diffusion problem with fractional diffusion and variational competition.
Through this Chapter we will substitute the assumption on the dimension $n = 2$ only in the "clean-up" type result, in order to stress which results hold for every dimensions.

1.2 ALMGREN'S TYPE MONOTONICITY FORMULA

The functions belonging to $\mathcal{G}^s(B_1^+)$ have a very rich structure, mainly thanks to the validity of the Pohožaev identities we are able to prove the validity of the Almgren's monotonicity formula. The most challenging feature of this Section is that the segregation occurs only in the n-dimensional space, which it implies that, when dealing with Pohožaev type identities, integrals on the "boundary of the boundary" appear.

Let us recall the definition of the class $\mathcal{G}^s(B_1^+)$ that we will use through the paper.

Definition 1.2.1. Let $a \in (-1,1)$, we define the class $\mathcal{G}^s(B_1^+)$ as the set of vector valued function $\mathbf{u} = (u_1, \dots, u_h) \in H^{1,a}_{\text{loc}}(\overline{B_1^+}; \mathbb{R}^h)$ whose components are all non negative, continuous functions such that

(1) $\mathbf{u} \in H^{1,a}(K \cap B_1^+) \cap C^{0,\alpha}(\overline{K \cap B_1^+})$, for every compact set $K \subset B$ and every $\alpha \in (0,s)$;

(2) $u_i \cdot u_j|_{y=0} \equiv 0$ for every $i \neq j$, $\mathbf{u} \neq \mathbf{0}$ on Σ and for every $i = 1, \dots, h$ it satisfies

$$\begin{cases} -L_a u_i = 0 & \text{in } B_1^+ \\ u_i \left(\partial_y^a u_i + f_i(u_i)\right) = 0 & \text{on } \partial^0 B_1^+ \end{cases} \tag{7}$$

where $f_i \colon \mathbb{R}^+ \to \mathbb{R}$ are nonnegative $\mathcal{C}^1$ functions such that $f_i(s) = O(s)$ for $s \to 0$;

(3) for every $X_0 = (x_0, 0) \in \partial^0 B_1^+$ and $r \in (0, \text{dist}(X_0, \partial B))$, the following Pohožaev type identity holds

$$\begin{aligned}(1-a-n)\int_{B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \, \mathrm{d}X + r\int_{\partial B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \, \mathrm{d}\sigma + \\ +2n\int_{\partial^0 B_r^+(X_0)} \sum_{i=1}^h F_i(u_i)\mathrm{d}x - 2r\int_{S_r^{n-1}(X_0)} \sum_{i=1}^h F_i(u_i)\mathrm{d}x = 2r\int_{\partial^+ B_r^+(X_0)} |y|^a (\partial_r \mathbf{u})^2 \mathrm{d}\sigma\end{aligned} \tag{8}$$

where $\mathbf{F}(s) = (F_1(s), \dots, F_h(s))$ with $F_i(s) = \int_0^s f_i(t)\,\mathrm{d}t$ for every $i = 1, \dots, h$.

Now, for every $X_0 \in \partial^0 B_1^+$ and $r > 0$ such that $B_r^+(X_0) \subset B_1^+$, we define the functionals

$$\begin{aligned} E(X_0, \mathbf{u}, r) &= \frac{1}{r^{n-1+a}}\left(\int_{B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \, \mathrm{d}X - \int_{\partial^0 B_r^+} \langle \mathbf{u}, F(x, \mathbf{u})\rangle \mathrm{d}x\right) \\ H(X_0, \mathbf{u}, r) &= \frac{1}{r^{n+a}}\int_{\partial^+ B_r^+(X_0)} |y|^a \mathbf{u}^2 \mathrm{d}\sigma \end{aligned}$$

and, whenever the average $H(x_0, \mathbf{u}, r) \neq 0$, the Almgren's frequency formula by

$$N(x_0, \mathbf{u}, r) = \frac{E(x_0, \mathbf{u}, r)}{H(x_0, \mathbf{u}, r)}.$$

Since $\mathbf{u} \in H^{1,a}(B_1; \mathbb{R}^h)$ both $r \mapsto E(X_0, \mathbf{u}, r)$ and $r \mapsto H(X_0, \mathbf{u}, r)$ are locally absolutely continuous functions for $r \in (0, \text{dist}(X_0, \partial B_1^+))$. As usual, integrating by parts on $B_r^+(X_0)$ every component u_i and summing over $i = 1, \dots, h$ we get

$$E(X_0, \mathbf{u}, r) = \frac{1}{r^{n-1+a}}\int_{\partial^+ B_r^+} |y|^a \langle \mathbf{u}, \partial_r \mathbf{u}\rangle \mathrm{d}\sigma = \frac{r}{2}\frac{d}{dr} H(X_0, \mathbf{u}, r). \tag{9}$$

The presence of internal reaction terms in the definition of the energy $E(X_0, \mathbf{u}, r)$ has to be dealt with. For this reason, we introduce the following lemmata to provide a crucial estimate in order to bound the Almgren quotient.
Let $a \in (-1, 1)$ and $u \in H^{1,a}(B_r^+(X_0))$ for some $X_0 \in \partial^0 B_1^+$ and $r \in (0, \mathrm{dist}(X_0, \partial B_1^+))$. Then, for every $p \in [2, p^\#]$, where $p^\# = 2n/(n-2s)$ there exists a constant $C(n, p, s)$ such that

$$\left[\frac{1}{r^n}\int_{\partial^0 B_r^+(X_0)} |u|^p \,\mathrm{d}x\right]^{\frac{2}{p}} \le C(n,p,s)\left[\frac{1}{r^{n-1+a}}\int_{B_r^+(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}X + \frac{1}{r^{n+a}}\int_{\partial^+ B_r^+(X_0)} |y|^a u^2 \mathrm{d}X\right] \tag{10}$$

This result is a direct consequence of the characterization of the class of trace of $H^{1,a}(B_1^+)$ in [70] and the critical Sobolev exponent for the trace embedding in the context of fractional Sobolev-Slobodeckij spaces $W^{s,2}(K)$, with $s \in (0, 1)$ and $K \subset \mathbb{R}^n$.

Lemma 1.2.2. *Let $s \in (0, 1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$. Then, for every $p \in [2, p^\#]$ and $X_0 \in \partial^0 B_1^+$ there exist constants $C > 0, \overline{r} > 0$ such that*

$$\left[\frac{1}{r^n}\int_{\partial^0 B_r^+(X_0)} |\mathbf{u}|^p \,\mathrm{d}x\right]^{\frac{2}{p}} \le C\left(E(X_0, \mathbf{u}, r) + H(X_0, \mathbf{u}, r)\right),$$

for every $r \in (0, \overline{r})$.

Proof. Since $\mathbf{u} \in L^\infty(B_1^+)$, and each components of $\mathbf{F} = (f_1, \dots, f_h)$ is locally Lipschitz continuous with $f_i(0) = 0$, we obtain

$$\begin{aligned}\left|\frac{1}{r^{n-1+a}}\int_{\partial^0 B_r^+} \langle \mathbf{u}, \mathbf{F}(\mathbf{u})\rangle \mathrm{d}x\right| &\le \frac{C}{r^{n-1+a}}\int_{\partial^0 B_r^+} \mathbf{u}^2 \mathrm{d}x \\ &\le C_2 r^{1-a}\left[\frac{1}{r^{n-1+a}}\int_{B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}X + \frac{1}{r^{n+a}}\int_{\partial^+ B_r^+(X_0)} |y|^a \mathbf{u}^2 \mathrm{d}X\right],\end{aligned}$$

where we used the trace inequality in the case $p = 2$. Finally, since $a \in (-1, 1)$ we get

$$E(X_0, \mathbf{u}, r) + H(X_0, \mathbf{u}, r) \ge (1 - C_2 r^{1-a})\left[\frac{1}{r^{n-1+a}}\int_{B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}X + \frac{1}{r^{n+a}}\int_{\partial^+ B_r^+(X_0)} |y|^a \mathbf{u}^2 \mathrm{d}X\right], \tag{11}$$

the result follows by taking into account the trace inequality and choosing $\overline{r} > 0$ sufficiently small. □

Following the same idea in [83] for the case $s = 1/2$, let introduce for $p \in (2, p^\#]$ the auxiliary function

$$\psi(X_0, \mathbf{u}, r) = \left(\frac{1}{r^n}\int_{\partial^0 B_r^+(X_0)} |\mathbf{u}|^2 \,\mathrm{d}X\right)^{1-\frac{2}{p}}$$

which is bounded for $r \in (0, \mathrm{dist}(X_0, \partial B_1^+))$. Under this notations, for $a \in (-1,1)$ consider

$$\Psi(X_0, \mathbf{u}, r) = C(n,s) \int_0^r t^{-a} \left(1 + \frac{d}{dt}\left(t\psi(X_0, \mathbf{u}, t)\right)\right) \mathrm{d}t,$$

which is well defined on $r \in (0, \mathrm{dist}(X_0, \partial B_1^+))$ such that $\lim_{r\to 0^+} \Psi(X_0, \mathbf{u}, r) = 0$, since $\psi(X_0, \mathbf{u}, r)$ is bounded for r sufficiently small. In order to simplify the notations, through the Section we will just use the notation $\psi(r)$ and $\Psi(r)$ for the auxiliary functions previously defined.

Lemma 1.2.3. *Let $s \in (0,1)$ and $u \in \mathcal{G}^s(B_1^+)$. Then, for every $p \in (2, p^\#]$ and $X_0 \in \partial^0 B_1^+$ there exist constants $C > 0, \overline{r} > 0$ such that*

$$\frac{1}{r^{n-1}} \int_{S_r^{n-1}} |\mathbf{u}|^p \,\mathrm{d}\sigma \leq C\left(E(X_0, \mathbf{u}, r) + H(X_0, \mathbf{u}, r)\right) \frac{d}{dr}\left(r\psi(r)\right),$$

for every $r \in (0, \overline{r})$.

Proof. The proof follows it is the same of [83, Lemma 9.5] make exception in our case is based on the generalized Poincarè inequality (10). Hence, a direct computation yields the identity

$$\frac{d}{dr}\left(r\psi(r)\right) = \psi(r)\left[r\left(1 - \frac{2}{p}\right)\frac{\displaystyle\int_{S_r^{n-1}} |\mathbf{u}|^p \,\mathrm{d}\sigma}{\displaystyle\int_{\partial^0 B_r^+} |\mathbf{u}|^p \,\mathrm{d}\sigma} + \left(1 - n\left(1 - \frac{2}{p}\right)\right)\right],$$

and, since $p \leq p^\#$ implies $n(1 - 2/p) \leq 1$, we infer

$$\frac{d}{dr}\left(r\psi(r)\right) \geq r\psi(r)\left(1 - \frac{2}{p}\right)\frac{\displaystyle\int_{S_r^{n-1}} |\mathbf{u}|^p \,\mathrm{d}\sigma}{\displaystyle\int_{\partial^0 B_r^+} |\mathbf{u}|^p \,\mathrm{d}\sigma}.$$

Finally, recalling the definition of ψ and using Lemma 1.2.2, we deduce

$$\left(E(X_0, \mathbf{u}, r) + H(X_0, \mathbf{u}, r)\right)\frac{d}{dr}\left(r\psi(r)\right) \geq C\frac{1}{r^{n-1}} \int_{S_r^{n-1}} |\mathbf{u}|^p \,\mathrm{d}\sigma.$$

□

We are now ready to prove the boundedness of the Almgren quotient, rather than its monotonicity, considering a modified version of the quotient.

Proposition 1.2.4. *Given $s \in (0,1)$, $u \in \mathcal{G}^s(B_1^+)$ and $\Omega^+ \subset\subset B_1^+$, there exist constants $C, \overline{r} > 0$ such that, for every $X_0 \in \partial^0\Omega^+ \subset \partial^0 B_1^+$ and $r \in (0,\overline{r})$ such that $B_r^+(X_0) \subset B_1^+$, we have that $H(X_0,u,r) > 0$ and $N(X_0,u,r) > 0$ for every $r \in (0,\overline{r})$. Moreover, the map*

$$r \mapsto e^{C\Psi(X_0,\mathbf{u},r)}\left(N(X_0,\mathbf{u},r)+1\right)$$

is monotone non decreasing on $(0,\overline{r})$, which ensures the existence of limit

$$N(X_0,\mathbf{u},0^+) = \lim_{r\to 0^+} N(X_0,\mathbf{u},r),$$

which is finite and called the Almgren frequency of $\mathbf{u}$ at X_0.

Proof. Let $X_0 \in \Gamma(\mathbf{u})$ and $\overline{r} > 0$ be such that $\overline{r} < \mathrm{dist}(\Omega^+, B_1^+)$ and Lemma 1.2.2 and Lemma 1.2.3 hold true. First, let us consider the following modified Almgren frequency formula

$$\widetilde{N}(X_0,\mathbf{u},r) = \frac{E(X_0,\mathbf{u},r)}{H(X_0,\mathbf{u},r)} + 1 = N(X_0,\mathbf{u},r) + 1. \tag{12}$$

Under this notations, we get by Lemma 1.2.2

$$E(X_0,\mathbf{u},r) + H(X_0,\mathbf{u},r) \geq 0 \longrightarrow \widetilde{N}(X_0,\mathbf{u},r) = \frac{E(X_0,\mathbf{u},r)}{H(X_0,\mathbf{u},r)} + 1 \geq 0,$$

whenever $H(X_0,\mathbf{u},r) \neq 0$. By continuity of $r \mapsto H(X_0,\mathbf{u},r)$ we can consider a reasonable neighborhood of r where it does not vanish. Since $\mathbf{u} \in L^\infty(B_1^+)$, and each components of $\mathbf{F} = (f_1,\dots,f_h)$ is locally Lipschitz continuous with $f_i(0) = 0$, there exists a positive constant $C > 0$ such that

$$|\langle \mathbf{u}, \mathbf{F}(x,\mathbf{u})\rangle| \leq C\mathbf{u}^2 \quad \text{and} \quad |\mathbf{F}(x,\mathbf{u})| \leq C\mathbf{u}^2,$$

for every $i = 1,\dots,h$. Now, taking into account the Pohožaev identity (8), if we differentiate the map $r \mapsto E(X_0,\mathbf{u},r)$ we obtain

$$\begin{aligned}\frac{d}{dr}E(X_0,\mathbf{u},r) = &-\frac{n-1+a}{r^{n+a}}\left(\int_{B_r^+} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}X - \int_{\partial^0 B_r^+} \langle \mathbf{u}, \mathbf{F}(\mathbf{u})\rangle \mathrm{d}x\right) + \\ &+ \frac{1}{r^{n-1+a}}\int_{\partial^+ B_r^+} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}\sigma - \frac{1}{r^{n-1+a}}\int_{S_r^{N-1}} \langle \mathbf{u}, \mathbf{F}(\mathbf{u})\rangle \mathrm{d}\sigma \\ = &\frac{2}{r^{n-1+a}}\int_{\partial^+ B_r^+} |y|^a |\partial_r \mathbf{u}|^2 \,\mathrm{d}\sigma + R(x_0,\mathbf{u},r),\end{aligned}$$

where the remainder can be estimated as

$$
\begin{aligned}
|R(X_0,\mathbf{u},r)| \leq & \frac{n-1+a}{r^{n+a}}\int_{\partial^0 B_r^+(X_0)} |\langle \mathbf{u},\mathbf{F}(\mathbf{u})\rangle|\,\mathrm{d}x + \frac{2n}{r^{n+a}}\int_{\partial^0 B_r^+(X_0)} \sum_{i=1}^h |F_i(u_i)|\,\mathrm{d}x+\\
&+\frac{2}{r^{n+a-1}}\int_{S_r^{n-1}(X_0)}\sum_{i=1}^h |F_i(u_i)|\,\mathrm{d}x + \frac{1}{r^{n+a-1}}\int_{S_r^{n-1}(X_0)} |\langle \mathbf{u},\mathbf{F}(\mathbf{u})\rangle|\,\mathrm{d}\sigma\\
&\leq C(n,s)\left[\frac{1}{r^{n+a}}\int_{\partial^0 B_r^+(X_0)} \mathbf{u}^2\mathrm{d}x + \frac{1}{r^{n+a-1}}\int_{S_r^{n-1}(X_0)}\mathbf{u}^2\mathrm{d}\sigma\right]\\
&\leq C(n,s)r^{-a}\left(E(X_0,\mathbf{u},r)+H(X_0,\mathbf{u},r)\right)\left(1+\frac{d}{dr}(r\psi(r))\right)
\end{aligned}
$$

where in the third inequality we used Lemma 1.2.2 and Lemma 1.2.3. Therefore, differentiating the Almgren quotient and using the Cauchy-Schwarz inequality on $\partial^+ B_r^+$, we obtain

$$
\begin{aligned}
\frac{d}{dr}\widetilde{N}(X_0,\mathbf{u},r) =& \frac{\frac{d}{dr}E(X_0,\mathbf{u},r)+\frac{d}{dr}H(X_0,\mathbf{u},r)}{E(X_0,\mathbf{u},r)+H(X_0,\mathbf{u},r)} - \frac{\frac{d}{dr}H(X_0,\mathbf{u},r)}{H(X_0,\mathbf{u},r)}\\
\geq & \frac{2H(X_0,\mathbf{u},r)}{r^{2n+2a-1}}\left[\int_{\partial^+ B_r^+}|y|^a\,|\partial_r \mathbf{u}|^2\,\mathrm{d}\sigma\int_{\partial^+ B_r^+}|y|^a\,\mathbf{u}^2\mathrm{d}\sigma - \left(\int_{\partial^+ B_r^+}|y|^a\,\langle \mathbf{u},\partial_r\mathbf{u}\rangle\mathrm{d}\sigma\right)^2\right]+\\
&- C(n,s)\widetilde{N}(X_0,\mathbf{u},r)r^{-a}\left(1+\frac{d}{dr}(r\psi(r))\right)\\
\geq & - C(n,s)\widetilde{N}(X_0,\mathbf{u},r)r^{-a}\left(1+\frac{d}{dr}(r\psi(r))\right).
\end{aligned}
$$

which implies that the function

$$
r\mapsto e^{C\Psi(X_0,\mathbf{u},r)}\widetilde{N}(X_0,\mathbf{u},r)
$$

is nondecreasing as far as $H(X_0,\mathbf{u},r)\neq 0$. Passing to the logarithmic derivative of $r\mapsto H(X_0,\mathbf{u},r)$ we infer from (9) that for $r\in(r_1,r_2)$ we get

$$
\frac{d}{dr}\log H(X_0,\mathbf{u},r) = \frac{2}{r}N(X_0,\mathbf{u},r). \tag{13}
$$

More precisely, we can choose $r_1=0, r_2=+\infty$. On one hand, the above equation provides that, if $\log H(X_0,\mathbf{u},R) > -\infty$ then $\log H(X_0,\mathbf{u},r) > -\infty$ for every $r > R$, so that $r_2 = \mathrm{dist}(X_0,\partial B_1^+)$. Now, on the other hand assume by contradiction that

$$
r_1 = \inf\{r\colon H(X_0,\mathbf{u},r) > 0 \text{ on } (r,r_2)\} > 0.
$$

By the monotonicity result on the modified Almgren quotient (12), we have that

$$N(X_0, \mathbf{u}, r) < e^{C\Psi(2r_1)}\left(N(X_0, \mathbf{u}, 2r_1) + 1\right) - 1,$$

for every $r_1 < r \leq 2r_1$. Hence, integrating (13) between r and $2r_1$, we get

$$\frac{H(X_0, \mathbf{u}, 2r_1)}{H(X_0, \mathbf{u}, r)} \leq \left(\frac{2r_1}{r}\right)^{2\left(e^{C\Psi(2r_1)}(N(X_0,\mathbf{u},2r_1)+1)-1\right)}$$

and, since $r \mapsto H(X_0, \mathbf{u}, r)$ is continuous we deduce the absurd $H(X_0, \mathbf{u}, r_1) > 0$. □

The first Corollary of the Almgren monotonicty result is the following lower bound for the Almgren frequency formula of $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ at $X_0 \in \Gamma(\mathbf{u})$.

Corollary 1.2.5. *Given $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$, for any $X_0 \in \Gamma(\mathbf{u})$ we have*

$$N(X_0, \mathbf{u}, 0^+) \geq s.$$

Proof. Let $\overline{r} > 0$ be such that Proposition 1.2.4 holds true and suppose by contradiction the existence of $0 < \widetilde{r} < \overline{r}$ and $\varepsilon > 0$ such that

$$e^{C\Psi(\widetilde{r})}\left(N(X_0, \mathbf{u}, \widetilde{r}) + 1\right) \leq 1 + s - \varepsilon.$$

By the above bound, we obtain for every $r \in (0, \widetilde{r})$ that

$$\frac{d}{dr} \log H(X_0, \mathbf{u}, r) \leq 2\,\frac{e^{C\Psi(\widetilde{r})}\left(N(X_0, \mathbf{u}, \widetilde{r}) + 1\right) - 1}{r} \leq \frac{2(s-\varepsilon)}{r}.$$

Integrating this inequality between r and $\widetilde{r}$ yields

$$\frac{H(X_0, \mathbf{u}, \widetilde{r})}{H(X_0, \mathbf{u}, r)} \leq \left(\frac{\widetilde{r}}{r}\right)^{2(s-\varepsilon)}$$

which, together with the fact that $\mathbf{u} \in C^{0,\alpha}_{\mathrm{loc}}(B_1^+)$ for every $\alpha \in (0, s)$ and that $\mathbf{u}(X_0) = \mathbf{0}$, implies

$$Cr^{2(s-\varepsilon)} \leq H(X_0, \mathbf{u}, r) \leq Cr^{2\alpha},$$

for every $\alpha \in (0, s)$. Hence, the contradiction follows for r sufficiently small. □

Corollary 1.2.6. *For every $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ the map from $X_0 \mapsto N(X_0, \mathbf{u}, 0^+)$ is upper semi-continuous on $\partial^0 B_1^+$.*

Proof. Fixed a vector valued function $\mathbf{u} \in \mathcal{G}^s(B_1^+)$, let us take a sequence $X_k \to X$ in $\partial^0 B_1^+$. By Proposition 1.2.4 there exists a constant $C > 0$ and $\overline{r} > 0$ such that, for $r \in (0, \overline{r})$

$$N(X_k, \mathbf{u}, r) = e^{-C\Psi(r)} e^{C\Psi(r)} \left(N(X_k, \mathbf{u}, r) + 1\right) - 1 \geq e^{-C\Psi(r)} \left(N(X_k, \mathbf{u}, 0^+) + 1\right) - 1.$$

By taking the limit superior in k and afterwards the limit as $r \to 0^+$ we obtain

$$N(X, \mathbf{u}, 0^+) \geq \limsup_{k\to\infty} N(X_k, \mathbf{u}, 0^+).$$

□

Another simple consequence of the monotonicity result is the following comparison property which, with $r_2 = 2r_1$ is the so called doubling property.

Proposition 1.2.7. *Given $s \in (0,1), \mathbf{u} \in \mathcal{G}^s(B_1^+)$ and $\Omega^+ \subset\subset B_1^+$, there exists $\overline{C} > 0$ and $\overline{r} > 0$ such that*

$$H(X_0, \mathbf{u}, r_2) \leq H(X_0, \mathbf{u}, r_1) \left(\frac{r_2}{r_1}\right)^{2\overline{C}}$$

for every $X_0 \in \partial^0\Omega^+$ and $0 < r_1 < r_2 \leq \overline{r}$.

Proof. Fixed $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ and $\Omega^+ \subset\subset B_1^+$, let $C > 0$ and $\overline{r} > 0$ be such that Proposition 1.2.4 holds true for every $X_0 \in \partial^0\Omega^+$. Hence, given

$$\overline{C} = \sup_{X_0 \in \partial^0\Omega^+} N(X_0, \mathbf{u}, \overline{r}) < +\infty, \tag{14}$$

we get

$$\begin{aligned}\frac{d}{dr}\log H(X_0, \mathbf{u}, r) =& \frac{2}{r} N(X_0, \mathbf{u}, r)\\ \leq& \frac{2}{r}\left(e^{-C\Psi(\overline{r})} e^{C\Psi(\overline{r})}\left(N(X_0, \mathbf{u}, \overline{r}) + 1\right) - 1\right)\\ \leq& \frac{2}{r}\left((\overline{C}+1)e^{-C\Psi(\overline{r})} - 1\right),\end{aligned}$$

for every $0 < r < \overline{r}$. Now, by integrating the previous inequality between r_1 and r_2, for $0 < r_1 < r_2 \leq \overline{r}$, we obtain

$$\frac{H(X_0, \mathbf{u}, r_2)}{H(X_0, \mathbf{u}, r_1)} \leq \left(\frac{r_2}{r_1}\right)^{2\overline{C}},$$

as we previously claimed. □

1.3 COMPACTNESS OF THE BLOW-UP SEQUENCES

All techniques presented in this Chapter involve a local analysis of the nodal set of the solution, which will be performed via a blow-up and blow-down procedure. In this Section we study the behaviour of the class $\mathcal{G}^s(B_1^+)$ under rescaling and translations with respect to point on Σ, in order to apply the blow-up analysis of $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ near the nodal set $\Gamma(\mathbf{u})$.

Given $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ let $\mathbf{F} = (F_1, \dots, F_h)$ be the vector valued associated to Definition 1.2.1. For every $\rho, t > 0$ and $X_0 = (x_0, 0) \in \partial^0 B_1^+$ we define the rescaled function

$$\mathbf{v}(X) = \frac{\mathbf{u}(X_0 + tX)}{\rho}, \quad \text{for } X \in B^+_{X_0,t} := \frac{B_1^+ - X_0}{t}, \tag{15}$$

where obviously the previous equality holds for every component of the vector valued function. It is easy to check that each components of $\mathbf{v}$ solves the system

$$\begin{cases} -L_a v_i = 0 & \text{in } B^+_{X_0,t} \\ v_i\Big(\partial^a_y v_i + g_i(v_i)\Big) = 0 & \text{on } \partial^0 B^+_{X_0,t} \end{cases} \tag{16}$$

where

$$g_i(s) = \frac{t^{1-a}}{\rho} f_i(\rho s). \tag{17}$$

In this setting, if we define for any $Z_0 \in B^+_{X_0,t}$ and $r \in (0, \mathrm{dist}(Z_0, \partial B^+_{X_0,t}))$

$$\begin{aligned} E(Z_0, \mathbf{v}, r) &= \frac{1}{r^{n-1+a}}\left(\int_{B_r^+(X_0)} |y|^a |\nabla \mathbf{v}|^2 \,\mathrm{d}X - \int_{\partial^0 B_r^+} \langle \mathbf{v}, \mathbf{G}(\mathbf{v})\rangle \mathrm{d}x\right) \\ H(Z_0, \mathbf{v}, r) &= \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+(X_0)} |y|^a \,\mathbf{v}^2 \mathrm{d}\sigma \end{aligned}$$

and the following identities hold

$$E(Z_0, \mathbf{v}, r) = \frac{1}{\rho^2} E(X_0 + tZ_0, \mathbf{u}, tr) \;\text{ and }\; H(Z_0, \mathbf{v}, r) = \frac{1}{\rho^2} H(X_0 + tZ_0, \mathbf{u}, tr) \tag{18}$$

and hence $N(Z_0, \mathbf{v}, r) = N(X_0 + tZ_0, \mathbf{u}, tr)$.

Proposition 1.3.1. *Let $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ be fixed. Then, for every $\rho, t > 0$ and $X_0 \in \partial^0 B_1^+$, given $\mathbf{v}$ as in (1.3.1) we have $\mathbf{v} \in \mathcal{G}^s(B^+_{x_0,t})$.*

Proof. Fixed $\mathbf{u} \in \mathcal{G}^s(B_1^+)$, by the previous remarks, the last thing left to prove is the validity of the Pohožaev identity (8) for $\mathbf{v}$, in every ball $B_r(Z_0) \subset B^+_{X_0,t}$, with $Z_0 \in \partial^0 B^+_{X_0,t}$ and

$r \in (0, \mathrm{dist}(Z_0, \partial B^+_{X_0,t}))$. We check itby using (18) and by performing the change of variables $(x, y) = (x_0 + tz, tw)$ in the expression of the derivative of the energy $r \mapsto E(Z_0, \mathbf{v}, r)$. More precisely, from (17) let us define $\mathbf{G} = (G_1, \dots, G_h)$ where

$$G_i(s) = \int_0^s g_i(\tau) \mathrm{d}\tau = \frac{t^{1-a}}{\rho^2} \int_0^{\rho s} f_i(\tau) \mathrm{d}\tau = \frac{t^{1-a}}{\rho^2} F_i(\rho s),$$

then

$$\begin{aligned}
\frac{d}{dr} E(Z_0, \mathbf{v}, r) &= \frac{d}{dr} \frac{1}{\rho^2} E(X_0 + tZ_0, \mathbf{v}, tr) = \frac{t}{\rho^2} \frac{dE}{dr}(X_0 + tZ_0, \mathbf{u}, tr) \\
&= \frac{2t}{\rho^2 (tr)^{n-1+a}} \int_{\partial^+ B^+_{tr}(X_0+tZ_0)} |y|^a \, |\partial_r \mathbf{u}|^2 \, \mathrm{d}\sigma + \frac{t}{\rho^2} R(X_0 + tZ_0, \mathbf{u}, tr) \\
&= \frac{2}{r^{n-1+a}} \int_{\partial^+ B^+_r(Z_0)} |y|^a \, |\partial_r \mathbf{v}|^2 \, \mathrm{d}\sigma + R(Z_0, \mathbf{v}, r),
\end{aligned}$$

where

$$\begin{aligned}
\frac{t}{\rho^2} R(X_0 + tZ_0, \mathbf{u}, tr) &= \frac{t}{\rho^2 (tr)^{n+a}} \int_{\partial^0 B^+_{tr}(X_0+tZ_0)} (n + a - 1) \langle \mathbf{u}, \mathbf{F}(\mathbf{u}) \rangle - 2n \sum_{i=1}^h F_i(u_i) \mathrm{d}x + \\
&\quad + \frac{t}{\rho^2 (tr)^{n+a-1}} \int_{S^{n-1}_{tr}(X_0+tZ_0)} 2 \sum_{i=1}^h F_i(u_i) - \langle \mathbf{u}, \mathbf{F}(\mathbf{u}) \rangle \mathrm{d}x \\
&= \frac{1}{r^{n+a}} \int_{\partial^0 B^+_r(Z_0)} (n + a - 1) \langle \mathbf{u}, \mathbf{G}(\mathbf{v}) \rangle - 2n \sum_{i=1}^h G_i(v_i) \mathrm{d}x + \\
&\quad + \frac{1}{r^{n+a-1}} \int_{S^{n-1}_r(Z_0)} 2 \sum_{i=1}^h G_i(v_i) - \langle \mathbf{v}, \mathbf{G}(\mathbf{v}) \rangle \mathrm{d}x \\
&= R(Z_0, \mathbf{v}, r).
\end{aligned}$$

□

Now, we turn our attention to the convergence of the blow-up sequences, using the previous results about the rescaled functions. Given $\Omega^+ \subset\subset B^+_1$ compactly supported in B^+_1 and $(X_k)_k \in \Omega^+$and $r_k \searrow 0$, let us consider the following normalized blow-up sequence

$$\mathbf{u}_k(X) = \frac{\mathbf{u}(X_k + r_k X)}{\rho_k} \qquad \text{for } X \in B^+_{X_k, r_k} \tag{19}$$

with

$$\rho_k^2 = \|\mathbf{u}(X_k + r_k \cdot)\|^2_{L^{2,a}(\partial^+ B^+_1)} = \frac{1}{r_k^{n+a}} \int_{\partial^+ B^+_{r_k}(X_k)} |y|^a \, \mathbf{u}^2 \, d\sigma = H(X_k, \mathbf{u}, r_k).$$

Hence we have that $\|\mathbf{u}_k\|_{L^{2,a}(\partial^+ B_1^+)} = 1$ and, by Proposition 1.3.1, $\mathbf{u}_k \in \mathcal{G}^s(B^+_{X_k,r_k})$ since every component solves

$$\begin{cases} -L_a u_{i,k} = 0 & \text{in } B^+_{X_k,r_k} \\ u_{i,k}\left(\partial^a_y u_{i,k} + f_{i,k}(u_{i,k})\right) = 0 & \text{on } \partial^0 B^+_{X_k,r_k} \end{cases} \tag{20}$$

where

$$f_{i,k}(s) = \frac{t_k^{1-a}}{\rho_k} f_i(\rho_k s). \tag{21}$$

In order to simplify some notations, we introduce the following class of functions which corresponds to the one introduced in [83, 84] in the context of entire segregated profiles.

Definition 1.3.2. For every $s \in (0,1)$ we define with $\mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$ the collection of $\mathbf{u} \in H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1};\mathbb{R}^h)$ with $\mathbf{u} = (u_1,\dots,u_h)$ continuous, and such that

- $u_i \cdot u_j|_{y=0} \equiv 0$ for every $i \neq j$ and $\mathbf{u} \not\equiv \mathbf{0}$ on Σ. Moreover, for every $i = 1,\dots,h$ it satisfies

$$\begin{cases} -L_a u_i = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u_i \partial^a_y u_i = 0 & \text{on } \Sigma; \end{cases} \tag{22}$$

- for every $X_0 = (x_0, 0) \in \Sigma$ and $r > 0$, the following Pohožaev type identity holds

$$\frac{1-a-n}{r}\int_{B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}X + \int_{\partial B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}\sigma = 2\int_{\partial^+ B_r^+(X_0)} |y|^a (\partial_r \mathbf{u})^2 \mathrm{d}\sigma. \tag{23}$$

In the remaining part of the Section we will prove the following convergence result for blow-up sequences, and present some of its main consequences. First, roughly speaking, we observe that $(B_1^+ - X_k)/r_k$ approaches the whole $\mathbb{R}^{n+1}$ as $k \to +\infty$ since the distance $\text{dist}(X_k, \partial B_1^+) \geq \text{dist}(\Omega^+, \partial B_1^+) > 0$ for every k.

Theorem 1.3.3. *Given $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$, let us consider a sequence $(X_k)_k \subset \partial^0 B_1^+$ and $(\mathbf{u}_k)_k$ the associated blow-up sequence defined in (19). Thus, there exists a vector valued function $\overline{\mathbf{u}} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$ such that, up to a subsequence, $\mathbf{u}_k \to \overline{\mathbf{u}}$ in $C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for every $\alpha \in (0,s)$ and strongly in $H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1})$. In particular, the blow-up limit $\overline{\mathbf{u}} = (\overline{u}_1,\dots,\overline{u}_h)$ satisfies*

$$\begin{cases} -L_a \overline{u}_i = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\overline{u}_i \partial^a_y \overline{u}_i = 0 & \text{on } \Sigma \end{cases} \quad \text{in } \mathcal{D}'(\mathbb{R}^{n+1}), \tag{24}$$

for every $i = 1,\dots,h$.

The proof will be presented in a series of lemmata, since some results follow directly from the ideas and techniques presented in [83, 84].

Lemma 1.3.4. *Let $s \in (0,1)$, $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ and $\Omega^+ \subset\subset B_1^+$ compactly supported in B_1^+. Then, there exist $\tilde{C} > 0$ and $\tilde{r} > 0$ such that for every $X \in \partial^0\Omega^+$ and $0 < r \leq \tilde{r}$ we have*

$$\frac{1}{r^{n-1+a}} \int_{B_r^+(X_0)} |y|^a |\nabla \mathbf{u}|^2 \,\mathrm{d}X + \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+(X_0)} |y|^a \,\mathbf{u}^2 \mathrm{d}X \leq C\left(E(X_0, \mathbf{u}, r) + H(X_0, \mathbf{u}, r)\right).$$

The proof of the previous result is a direct consequence of (11) in the proof of Lemma 1.2.2.

Lemma 1.3.5. *Under the previous notations, for any given $R > 0$ we have*

$$\|\mathbf{u}_k\|_{H^{1,a}(B_R^+)} \leq C \quad \text{and} \quad \|\mathbf{u}_k\|_{L^\infty(\overline{B_R^+})} \leq C,$$

where $C > 0$ is a constant independent on $k > 0$.

Proof. Let $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ be such that $\mathbf{u} = (u_1, \ldots, u_h)$, with $u_i \in H^{1,a}_{\mathrm{loc}}(B_+)$. First, in order to prove the uniform bound with respect to the $H^{1,a}$-norm, by the Poincarè inequality (10) we consider the weighted Sobolev space $H^{1,a}(B_R^+)$ endowed with the norm

$$\|v\|^2_{H^{1,a}(B_R^+)} := \frac{1}{r^{n-1+a}} \int_{B_R^+} |y|^a |\nabla v|^2 \,\mathrm{d}X + \frac{1}{r^{n+a}} \int_{\partial^+ B_R^+} |y|^a \, v^2 \mathrm{d}\sigma.$$

By definition of the blow-up sequence $(\mathbf{u}_k)_k$, given $C > 0$ and $\overline{r} > 0$ be constants such that Proposition 1.2.4, Proposition 1.2.7 and Lemma 1.3.4 hold true. Then, up to taking k so large that $r_k, r_k R \leq \overline{r}$, we get

$$\begin{aligned}
\int_{\partial^+ B_R^+} |y|^a \,\mathbf{u}_k^2 \mathrm{d}\sigma &= \frac{1}{\rho_k^2} \int_{\partial^+ B_R^+} |y|^a \,\mathbf{u}^2(X_k + r_k X) \mathrm{d}\sigma \\
&= \frac{1}{\rho_k^2 r_k^{n+a}} \int_{\partial^+ B_{Rr_k}^+(X_k)} |y|^a \,\mathbf{u}^2 \mathrm{d}\sigma \\
&= R^{n+a} \frac{H(X_k, \mathbf{u}, Rr_k)}{H(X_k, \mathbf{u}, r_k)} \\
&\leq R^{n+a} \left(\frac{Rr_k}{r_k}\right)^{2\overline{C}}
\end{aligned}$$

where $\overline{C} > 0$ is defined in (14). Since we proved $\|\mathbf{u}_k\|^2_{L^{2,a}(\partial^+ B^+_R)} \leq C(R)R^{n+a}$, passing to the second term we conclude

$$\begin{aligned}\int_{B_R} |y|^a |\nabla \mathbf{u}_k|^2 \mathrm{d}\sigma &= N(0, \mathbf{u}_k, R)\frac{1}{R}\int_{\partial B_R} |y|^a \mathbf{u}_k^2 \mathrm{d}\sigma \\ &\leq C(R)R^{n-1+a} N(X_k, \mathbf{u}, Rr_k) \\ &\leq C(R)R^{n-1+a} e^{C\Psi(\overline{r})} \left(N(X_k, \mathbf{u}, \overline{r}) + 1\right) \\ &\leq \overline{C}(R)R^{n-1+a}\end{aligned}$$

where in the second inequality we used the monotonicity result of Proposition 1.2.4. Since by (14) we obtain $\|\nabla \mathbf{u}_k\|^2_{L^{2,a}(B^+_R)} \leq C(R)R^{n+a-1}$, it remains to prove the uniform bound with respect to the $L^\infty(B^+_R)$-norm.
Fixed $R > 0$, let $\mathbf{v}_k \in H^{1,a}(B^+_R)$ be the symmetric extension of $\mathbf{u}_k$ with respect to Σ to the whole B_1. Since $-\partial^a_y \mathbf{u}_k \leq 0$ on $\partial^0 B^+_1$, the map $\mathbf{v}_k$ is L_a-subharmonic, i.e. $-L_a \mathbf{v}_k \leq 0$, by [88, Lemma A.2] we get

$$\begin{aligned}\sup_{B^+_{R/2}} \mathbf{u}_k = \sup_{B_{R/2}} \mathbf{v}_k &\leq C(n,s)\left(\frac{1}{R^{n+1+a}}\int_{B_R} |y|^a \mathbf{v}_k^2 \mathrm{d}X\right)^{1/2} \\ &= 2C(n,s)\left(\frac{1}{R^{n+1+a}}\int_{B^+_R} |y|^a \mathbf{u}_k^2 \mathrm{d}X\right)^{1/2} \\ &\leq 2C(n,s)\left(\frac{H(0,\mathbf{u}_k,R)}{n+a+1}\right)^{1/2},\end{aligned}$$

where in the third inequality we used the monotonicity of $r \mapsto H(0, \mathbf{u}_k, r)$ in $(0, R)$. Finally, the estimate follows directly from the one the $L^{2,a}(\partial^+ B^+_R)$-norm. □

So far we have proved the existence of a nontrivial function $\overline{\mathbf{u}} \in H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+}; \mathbb{R}^h) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ such that, up to a subsequence, we have $u_{i,k} \rightharpoonup \overline{u}_i$ weakly in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, for every $i = 1, \dots, h$.
Moreover, since by Definition 1.2.1 and (21) there exists $M > 0$ such that, for every $i = 1, \dots, h$ and $k > 0$

$$\|f_{i,k}(u_{i,k})\|_{L^\infty(\partial^0 B^+_R)} \leq M r_k^{1-a} \|u_{i,k}\|_{L^\infty(\partial^0 B^+_R)} \to 0, \tag{25}$$

since $a \in (-1, 1)$ and $r_k \to 0^+$. Hence we deduce

$$\begin{cases} -L_a \overline{u}_i = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \overline{u}_i \partial^a_y \overline{u}_i = 0 & \text{on } \Sigma \end{cases} \quad \text{in } \mathcal{D}'(\mathbb{R}^{n+1}), \tag{26}$$

for every $i = 1, \ldots, h$. The next step is to prove that the convergence of $\mathbf{u}_k \to \overline{\mathbf{u}}$ is indeed strong in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and $C^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ for $\alpha \in (0, s)$.

Lemma 1.3.6. *Let $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ be fixed. Given $(\mathbf{u}_k)_k$ a blow-up sequences of the form (19), then for every $R > 0$, up to a subsequence, $\mathbf{u}_k \to \overline{\mathbf{u}}$ strongly in $H^{1,a}(B_R^+)$.*

Proof. We already know, by compactness, the existence of a blow-up limit $\overline{\mathbf{u}} \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^n)$, which solves (26) in $\mathcal{D}'(\mathbb{R}^{n+1}_+)$. To prove the strong convergence in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ let us consider $\varphi \in C^\infty_c(B_{2R})$ a cut-off function such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in B_R. First, since it holds

$$-L_a\left(u_{i,k} - \overline{u}_i\right) = 0 \quad \text{in } \mathcal{D}'(B_{2R}^+),$$

testing it with $(u_{i,k} - \overline{u}_i)\varphi$ and integrating by parts, we get

$$\begin{aligned}\int_{B_{2R}^+} |y|^a \varphi \left|\nabla(u_{i,k} - \overline{u}_i)\right|^2 \mathrm{d}X + \int_{B_{2R}^+} |y|^a \left(u_{i,k} - \overline{u}_i\right)\langle\nabla(u_{i,k} - \overline{u}_i), \nabla\varphi\rangle \mathrm{d}X = \\ = -\int_{\partial^0 B_{2R}^+} \varphi\left(u_{i,k} - \overline{u}_k\right)\partial^a_y\left(u_{i,k} - \overline{u}_k\right)\mathrm{d}x.\end{aligned} \tag{27}$$

Now, we can conclude just by observing that

$$\begin{aligned}\left|\int_{B_{2R}^+} |y|^a \left(u_{i,k} - \overline{u}_i\right)\langle\nabla(u_{i,k} - \overline{u}_i), \nabla\varphi\rangle \mathrm{d}X\right| &\leq C\left\|u_{i,k} - \overline{u}_i\right\|_{L^\infty(\overline{B_{2R}^+})}\left\|\nabla u_{i,k}\right\|_{L^{2,a}(B_{2R}^+)} \to 0,\\ \left|\int_{\partial^0 B_{2R}^+} \varphi\left(u_{i,k} - \overline{u}_k\right)\partial^a_y\left(u_{i,k} - \overline{u}_k\right)\mathrm{d}x\right| &\leq \left\|u_{i,k} - \overline{u}_i\right\|_{L^\infty(\overline{B_{2R}^+})}\int_{\partial^0 B_{2R}^+}\varphi\partial^a_y u_{i,k}\mathrm{d}x + \\ &+ \left\|u_{i,k} - \overline{u}_i\right\|_{L^\infty(\overline{B_{2R}^+})}\int_{\partial^0 B_{2R}^+}\varphi\partial^a_y\overline{u}_i\mathrm{d}x\\ &+ C(R)\left\|u_{i,k} - \overline{u}_i\right\|_{L^\infty(\overline{B_{2R}^+})} \to 0,\end{aligned}$$

where in the right hand side of (27) we used [88, Lemma A.2], since $-\partial^a_y u_{i,k} \leq 0$ and $-\partial^a_y u_{i,k} \leq 0$ on Σ. □

Similarly, given $\mathbf{v}_k \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $\overline{\mathbf{v}} \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ respectively the symmetric extensions of $\mathbf{u}_k$ and $\overline{\mathbf{u}}$ through Σ, one could relate the system (26) to a system of degenerate elliptic equation with a boundary measure data on Σ. More precisely, for every $k > 0$ there exists a collection of non negative Radon measures $\mu_{i,k} \in \mathcal{M}(B_1^+)$ for $i = 1, \ldots, h$, each one supported on $\partial^0 B_1^+$, such that

$$-L_a v_{i,k} = -\mu_{i,k} \quad \text{in } \mathcal{D}'\left(B^+_{X_k, r_k}\right),$$

for every $i = 1, \ldots, h$. Then, following the strategy in [81, Lemma 3.7] and [81, Lemma 3.11], one could obtain the same strong convergence in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ by using the uniform L^∞ and $H^{1,a}$ estimates in B_R^+, for every $R > 0$.

Lemma 1.3.7. *Under the previous notations, for every $R > 0$ there exists $C > 0$, independent of k, such that*

$$[\mathbf{u}_k]_{C^{0,\alpha}(B_R)} = \sup_{X_1, X_2 \in \overline{B}_R} \frac{|\mathbf{u}(X_1) - \mathbf{u}(X_2)|}{|X_1 - X_2|^\alpha} \leq C$$

for every $\alpha \in (0, \alpha^*)$.

Proof. The proof follows essentially the ideas of the similar results in [83, 84]. Without loss of generality, let $R = 1$ and suppose by contradiction that up to a subsequence

$$L_k = \max_{i=1,\dots,h} \sup_{X_1, X_2 \in \overline{B_1^+}} \frac{|\eta(X_1) u_{i,k}(X_1) - \eta(X_2) u_{i,k}(X_2)|}{|X_1 - X_2|^\alpha} \to \infty$$

where $\eta \in C_c^\infty(B_1)$ is a smooth function such that

$$\begin{cases} \eta(X) = 1, & 0 \leq |X| \leq 1/2 \\ 0 < \eta(X) \leq 1, & 1/2 \leq |X| \leq 1 \\ \eta(X) = 0, & |X| = 1. \end{cases}$$

Since we may assume that L_k is achieved by the first component of $\mathbf{u}_k$ and a sequence of points $(X_{1,k}, X_{2,k}) \in \overline{B_1^+} \times \overline{B_1^+}$, given $r_k = |X_{1,k} - X_{2,k}|$ we can prove, as $k \to \infty$, that

- $r_k \to 0$
- $\dfrac{\mathrm{dist}(X_{1,k}, \partial^+ B_1^+)}{r_k} \to \infty, \ \dfrac{\mathrm{dist}(X_{2,k}, \partial^+ B_1^+)}{r_k} \to \infty.$

Before to continue, let us fix the notations $X_{1,k} = (x_{1,k}, y_{1,k})$ and $X_{2,k} = (x_{2,k}, y_{2,k})$. Now, since by Lemma 1.3.5 the norm $\|\mathbf{u}_k\|_{L^\infty(B_1^+)}$ is uniformly bounded, we have

$$L_k \leq \frac{\|\mathbf{u}_k\|_{L^\infty(B_1^+)}}{r_k^\alpha} \left(\eta(X_{1,k}) - \eta(X_{2,k})\right), \tag{28}$$

which immediately implies that $r_k \to 0$. Now, since η is compactly supported in B_1 and it vanishes on $\partial^+ B_1^+$, for every $X \in \overline{B_1^+}$ we have

$$\eta(X) \leq \mathrm{dist}(X, \partial^+ B_1^+) \mathrm{Lip}(\eta),$$

where obviously $\mathrm{Lip}(\eta)$ denotes the Lipschitz constant of η. Finally, the inequality (28) becomes

$$\frac{\mathrm{dist}(X_{1,k}, \partial^+ B_1^+)}{r_k} + \frac{\mathrm{dist}(X_{2,k}, \partial^+ B_1^+)}{r_k} \geq \frac{L_k r_k^{\alpha-1}}{\mathrm{Lip}(\eta) \, \|\mathbf{u}_k\|_{L^\infty(B_1^+)}} \to \infty$$

and the result follows by recalling that $\alpha < 1$. As in [83, 84], our proof is based on two different blow-up sequences, indeed for every $i = 1, \dots, h$ we introduce the auxiliary sequences

$$w_{i,k}(X) = \eta(P_k)\frac{u_{i,k}(P_k + r_k X)}{L_k r_k^\alpha} \quad \text{and} \quad \overline{w}_{i,k}(X) = \frac{(\eta u_{i,k})(P_k + r_k X)}{L_k r_k^\alpha}$$

for $X \in B^+_{P_k,r_k}$ and $P_k = (p_{x,k}, p_{y,k})$ a suitable sequence of points that will be choose later. On one hand the sequence $(\overline{\mathbf{w}}_k)_k$ has an uniform bound on the α - Hölder seminorm, i.e.

$$\sup_{X_1 \neq X_2 \in \overline{B^+_{P_k,r_k}}} \frac{|\overline{w}_{i,k}(X_1) - \overline{w}_{i,k}(X_2)|}{|X_1 - X_2|^\alpha} \leq \left| \overline{w}_{1,k}\left(\frac{X_1 - P_k}{r_k}\right) - \overline{w}_{1,k}\left(\frac{X_2 - P_k}{r_k}\right)\right| = 1,$$

while on the other hand $(\mathbf{w}_k)_k \in \mathcal{G}^s(B^+_{P_k,r_k})$, where each components satisfy

$$\begin{cases} -L_a^k w_{i,k} = 0 & \text{in } B^+_{P_k,r_k} \\ w_{i,k}\left(\partial_y^{a,k} w_{i,k} + g_{i,k}(w_{i,k})\right) = 0 & \text{on } \partial^0 B^+_{P_k,r_k} \end{cases} \tag{29}$$

with the new operators

$$L_a^k = \mathrm{div}\left(\left(y + \frac{p_{y,k}}{r_k}\right)^a \nabla\right), \quad \partial_y^{a,k} = \lim_{y \to -\frac{p_{y,k}}{r_k}} \left(y + \frac{p_{y,k}}{r_k}\right)^a \partial_y,$$

and

$$g_{i,k}(t) = \eta(P_k)\frac{r_k^{2s-\alpha}}{L_k} f_{i,k}\left(\frac{L_k r_k^\alpha}{\eta(P_k)} t\right).$$

By Lemma 1.3.5 and (25) , we infer

$$\sup_{\partial^0 B^+_{P_k,r_k}} |g_{i,k}(w_{i,k})| = \eta(P_k)\frac{r_k^{2s-\alpha}}{L_k} \sup_{\partial^0 B_1^+} |f_{i,k}(u_{i,k})| \to 0^+$$

as $k \to +\infty$.
The importance of these two sequences lies in the fact that they have asymptotically equivalent behaviour. Namely, since

$$\begin{aligned} |w_{i,k}(X) - \overline{w}_{i,k}(X)| &\leq \frac{\|\mathbf{u}_k\|_{L^\infty(B_1)}}{r_k^\alpha L_k} |\eta(P_k + r_k X) - \eta(P_k)| \\ &\leq \frac{\mathrm{Lip}(\eta) r_k^{1-\alpha}}{L_k} \|\mathbf{u}_k\|_{L^\infty(B_1)} |X| \end{aligned} \tag{30}$$

we get, for any compact $K \subset \mathbb{R}^{n+1}$, that

$$\max_{X \in K \cap B^+_{P_k, r_k}} |\mathbf{w}_k(X) - \overline{\mathbf{w}}_k(X)| \longrightarrow 0. \tag{31}$$

Moreover, since $\mathbf{w}_k(0) = \overline{\mathbf{w}}_k(0)$ we note by (30) that

$$\begin{aligned} |w_{i,k}(X) - w_{i,k}(0)| &\leq |w_{i,k}(X) - \overline{w}_{i,k}(X)| + |\overline{w}_{i,k}(X) - \overline{w}_{i,k}(0)| \\ &\leq C\left(\frac{r_k^{1-\alpha}}{L_k}|X| + |X|^\alpha\right) \end{aligned}$$

and consequently, there exists $C = C(K)$ such that $|\mathbf{w}_k(X) - \mathbf{w}_k(0)| \leq C$, for every $X \in K$.
Let us prove that it is not restrictive to choose $P_k \in \Sigma$ in the definitions of the sequences $(\mathbf{w}_k)_k$ $(\overline{\mathbf{w}}_k)_k$, showing that $X_{1,k}, X_{2,k}$ must converge to $\partial^0 B_1^+$, i.e. there exists $C > 0$ such that, for k sufficiently large,

$$\frac{\mathrm{dist}(X_{1,k}, \partial^0 B_1^+) + \mathrm{dist}(X_{2,k}, \partial^0 B_1^+)}{r_k} \leq C.$$

The following proof follows directly the one of [84, Lemma 4.5]) but for the sake of complexness we report some details. Arguing by contradiction, suppose that

$$\frac{\mathrm{dist}(X_{1,k}, \partial^0 B_1^+) + \mathrm{dist}(X_{2,k}, \partial^0 B_1^+)}{r_k} \longrightarrow \infty$$

and let us choose $P_k = X_{1,k}$ in the definition of $\mathbf{w}_k, \overline{\mathbf{w}}_k$ so that $B^+_{P_k, r_k} \to \mathbb{R}^{n+1}$ and $p_{y,k}^{-1} r_k \to 0^+$.
Given $\mathbf{W}_k = \mathbf{w}_k - \mathbf{w}_k(0)$ and $\overline{\mathbf{W}}_k = \overline{\mathbf{w}}_k - \overline{\mathbf{w}}_k(0)$, by construction $\overline{\mathbf{W}}_k$ is a sequence of functions which share the same bound on the α - Hölder seminorm and they are uniformly bounded in every compact $K \subset \mathbb{R}^{n+1}$ since $\overline{\mathbf{W}}_k(0) = \mathbf{0}$. Thus, by the Ascoli-Arzelá theorem, there exists $\mathbf{W} \in C(K)$ which, up to a subsequence, is the uniform limit of $\overline{\mathbf{W}}_k$. By (**??**), we also find that $\mathbf{W}_k \to \mathbf{W}$ uniformly con compact sets.
In order to reach a contradiction we can prove that $\mathbf{W}$ is a nonconstant globally Hölder harmonic function with $\alpha \in (0, \alpha^*)$.
Since we already know that $\mathbf{W} \in C^{0,\alpha}(\mathbb{R}^{n+1})$ it reamins to prove the harmonicity of the limit function. To this purpose, let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ be a compactly supported smooth function and $\overline{k}$ be sufficiently large so that $\operatorname{supp} \varphi \subset B^+_{P_k, r_k}$, for all $k \geq \overline{k}$. Fixed $i = 1, \ldots, h$, by testing the first equation in (29) with φ we get

$$\int_{\mathbb{R}^{n+1}} \mathrm{div}\left(\left(1 + y\frac{r_k}{p_{y,k}}\right)^a \nabla\varphi\right) w_{i,k} \mathrm{d}X = 0.$$

Passing to the uniform limit and observing that

$$\left(1 + y\frac{r_k}{p_{y,k}}\right)^a \to 1 \quad \text{in } C^\infty(\operatorname{supp}\varphi),$$

we deduce that $\mathbf{W}$ is indeed harmonic. The contradiction follows by the classical Liouville Theorem once we show that $\mathbf{W}$ is globally α - Hölder continuous and not constant. Hence, since $P_k = X_{1,k}$ then, up to a subsequence,

$$\frac{X_{2,k} - P_k}{r_k} = \frac{X_{2,k} - X_{1,k}}{|X_{2,k} - X_{1,k}|} \to X_2 \in \partial B_1.$$

Finally, by the equicontinuity and the uniform convergence, we conclude

$$\left|\overline{W}_{1,k}\left(\frac{X_1 - P_k}{r_k}\right) - \overline{W}_{1,k}\left(\frac{X_2 - P_k}{r_k}\right)\right| = 1 \longrightarrow \left|\overline{W}_1(0) - \overline{W}_1(X_2)\right| = 1.$$

At this point, the choice $P_k = (x_{1,k}, 0)$ for every $k \in \mathbb{N}$ guarantees the convergence of the rescaled domains $B^+_{P_k,r_k} \to \mathbb{R}^{n+1}_+$, while for any compact set $K \subset \mathbb{R}^{n+1}$

$$\max_{X \in K \cap B^+_{P_k,r_k}} |\mathbf{w}_k(X) - \overline{\mathbf{w}}_k(X)| \longrightarrow 0.$$

Hence, we are left with two possibilities:

- for any compact set $K \subset \Sigma$ we have $w_{1,k}(X) \neq 0$ for every $k \geq k_0$ and $X \in K$;
- there exists a sequence $(X_k)_k \subset \Sigma$ such that $\mathbf{w}_k(X_k) = \mathbf{0}$, for every $k \in \mathbb{N}$.

In the first case, if we define again $\mathbf{W}_k = \mathbf{w}_k - \mathbf{w}_k(0)$ and $\overline{\mathbf{W}}_k = \overline{\mathbf{w}}_k - \overline{\mathbf{w}}_k(0)$ we obtain that the last sequence is uniformly bounded in $C^{0,\alpha}$ and hence $(\mathbf{W}_k)_k$ converges uniformly on compact set to a nonconstant globally α - Hölder continuous L_a-harmonic function $\mathbf{W}$, with $\partial^a_y W_1 \equiv 0$ and $W_i \equiv 0$ for $i > 1$, on Σ. Now, extending properly the vector $\mathbf{W}$ to the whole $\mathbb{R}^{n+1}$, we find a contradiction with the Liouville theorem for entire L_a-harmonic function, since $\alpha < \min\{1, 1-a\}$.
Similarly, in the second case $(\mathbf{w}_k)_k$ itself does converge uniformly on compact sets to a nonconstant globally α - Hölder continuous function $\mathbf{w}$. In particular, by Lemma 1.3.6, we already know that $(\mathbf{w}_k)_k$ itself converge strongly in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and consequently $\mathbf{w} \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. The contradiction follows by the Liouville theorem for entire segregated configurations in $\mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. □

We remark that the class of entire segregated profiles $\mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ has been introduced in [84, 84], where the authors proved several properties and monotonicity formulas in order to better understand the asymptotic behaviour of solutions of a competition-diffusion problem with anomalous diffusion and variational competition.

The following is an improved version of a compactness result concerning entire segregated profiles in [84, Proposition 4.7]. Moreover, this result provides a compactness criterion for suitable blow-up sequences, and will be useful in the proof of the gap condition on the possible values of the Almgren frequency formula.

Proposition 1.3.8. *Let $(\mathbf{u}_k)_k \subset \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1}_+) \cap C^{0,\alpha}(\overline{B^+_1})$, for some $\alpha \in (0,\nu)$, such that*

$$\|\mathbf{u}_k\|_{L^\infty(B^+_1)} \leq M,$$

with M independent on k. Then, for every $\alpha' \in (0,\alpha)$, there exists a constant $C = C(M,\alpha')$ such that

$$\|\mathbf{u}_k\|_{C^{0,\alpha'}(\overline{B^+_{1/2}})} \leq C.$$

Furthermore, the subset $(\mathbf{u}_k)_k$ is relatively compact in $H^{1,a}(B^+_{1/2}) \cap C^{0,\alpha'}(\overline{B^+_{1/2}})$, for every $\alpha' \in (0,\alpha)$.

Furthermore, in the context of entire segregated profiles, we can improve Proposition 1.2.4 and Corollary 1.2.6 with the following result.

Proposition 1.3.9. *[84, Proposition 2.11] Let $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Then, for every $X_0 \in \Sigma$, the Almgren frequency function*

$$N(X_0,\mathbf{u},r) = \frac{E(X_0,\mathbf{u},r)}{H(X_0,\mathbf{u},r)} = \frac{\displaystyle\frac{1}{r^{n+a-1}}\int_{B^+_r(X_0)} |y|^a \left|\nabla \mathbf{u}\right|^2 \mathrm{d}X}{\displaystyle\frac{1}{r^{n+a}}\int_{\partial^+ B^+_r(X_0)} |y|^a \,\mathbf{u}^2 \mathrm{d}\sigma}$$

well define on $(0,+\infty)$ and monotone non decreasing and it satisfies

$$\frac{d}{dr}\log H(X_0,\mathbf{u},r) = 2\frac{N(X_0,\mathbf{u},r)}{r}. \tag{32}$$

Moreover, if $N(X_0,\mathbf{u},r) \equiv k$ on an open interval, then $N(X_0,\mathbf{u},r) \equiv k$ for every r, and $\mathbf{u} = (u_1,\dots,u_h)$ is k-homogeneous function in $\mathbb{R}^{n+1}$.

The following is a generalization for $s \in (0,1)$ of [83, Lemma 3.4], that will be crucial in the study of the structure of the nodal set $\Gamma(\mathbf{u})$.

Corollary 1.3.10. *Let $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Then*

- *$X \mapsto N(X,\mathbf{u},0^+)$ is a non negative upper semi-continuous function on Σ,*
- *$X \mapsto N(X,\mathbf{u},0^+)$ is constant, even infinite.*

Proof. The first part of the Corollary follows because in the case of entire configurations $N(X,\mathbf{u},0^+)$ is defined, by monotonicity, as the infimum of continuous functions $X \mapsto N(X,\mathbf{u},r)$. On the other hand, following the reasoning in Lemma 3.7 in [83], given

$$k = \lim_{r\to+\infty} N(0,\mathbf{u},r) > 0$$

let us prove the second assertion in the case $k < +\infty$, otherwise it follows with minor changes. By contradiction, suppose there exists $X_0 \in \Sigma$ such that $\sup_{r>0} N(X_0, \mathbf{u}, r) = k - 2\varepsilon$, for some $\varepsilon > 0$. Let moreover $r_0 > 0$ be such that

$$N(0, \mathbf{u}, r_0) \geq k - \varepsilon.$$

Up to taking R_1 and R_2 sufficiently large, integrating by parts (32) we get from the previous assumption

$$H(X_0, \mathbf{u}, R_1) \leq H(X_0, \mathbf{u}, 1) R_1^{2(k-2\varepsilon)} \quad \text{and} \quad H(0, \mathbf{u}, R_2) \geq H(0, \mathbf{u}, 1) R_2^{2(k-\varepsilon)}.$$

By definition

$$\int_{B^+_{R_1}(X_0)\setminus B^+_{r_0}(X_0)} |y|^a \, \mathbf{u}^2 \mathrm{d}X = \int_{r_0}^{R_1} \rho^{n+a} H(X_0, \mathbf{u}, \rho) \mathrm{d}\rho \leq C R_1^{n+a+2(k-2\varepsilon)}$$

and similarly

$$\int_{B^+_{R_2}\setminus B^+_{r_0}} |y|^a \, \mathbf{u}^2 \mathrm{d}X = \int_{r_0}^{R_2} \rho^{n+a} H(0, \mathbf{u}, \rho) \mathrm{d}\rho \geq C R_2^{n+a+2(k-\varepsilon)}.$$

Now, if we let $|X_0| = R_1 - R_2$, we get

$$\begin{aligned} C R_2^{n+a+2(k-\varepsilon)} &\leq \int_{B^+_{R_2}\setminus B^+_{r_0}} |y|^a \, \mathbf{u}^2 \mathrm{d}X \\ &\leq \int_{B^+_{r_0}(X_0)} |y|^a \, \mathbf{u}^2 \mathrm{d}X - \int_{B^+_{r_0}} |y|^a \, \mathbf{u}^2 \mathrm{d}X + \int_{B^+_{R_1}(X_0)\setminus B^+_{r_0}(X_0)} |y|^a \, \mathbf{u}^2 \mathrm{d}X \\ &\leq C + C(R_2 + |X_0|)^{n+a+2(k-2\varepsilon)}, \end{aligned}$$

and we find a contradiction for R_2 sufficiently large. □

Up to now we have dealt with blow-up sequences with arbitrary moving centers $(X_k)_k \subset \Sigma$: the following result emphasizes how some particular choices of X_k provide additional informational on the blow-up limit and how it is correlated to the Almgren frequency formula. More precisely, we have

Proposition 1.3.11. *Let $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$. Fixed a blow-up sequence $(\mathbf{u}_k)_k$ associated to $(X_k)_k \subset \Gamma(\mathbf{u})$, suppose that one of these situations occurs:*

- *$X_k = X_0$ for every $k \in \mathbb{N}$,*

- *$X_k \in \Gamma(\mathbf{u})$ and $X_k \to X_0 \in \Gamma(\mathbf{u})$ with $N(X_0, \mathbf{u}, 0^+) = s$.*

Then $N(0, \overline{\mathbf{u}}, r) = N(X_0, \mathbf{u}, 0^+) =: \alpha$ for every $r > 0$ and the blow-up limit $\overline{\mathbf{u}}(r, \theta) = r^\alpha \mathbf{g}(\theta)$, where (r, θ) are the generalized polar coordinates centered at the origin in $\mathbb{R}^{n+1}$.

Proof. First of all we prove that $N(0, \overline{\mathbf{u}}, r)$ is constant for every $r \in (0, +\infty)$. Let us recall that $N(0, \mathbf{u}_k, r) = N(X_k, \mathbf{u}, r_k r)$ and that Theorem 1.3.3 yields that

$$N(0, \overline{\mathbf{u}}, r) = \lim_{k\to\infty} N(0, \mathbf{u}_k, r) = \lim_{k\to\infty} N(X_k, \mathbf{u}, r_k r).$$

If $X_k = X_0$, for some $X_0 \in \Gamma(\mathbf{u})$, then $\lim_k N(X_0, \mathbf{u}, r_k r) = N(X_0, \mathbf{u}, 0^+)$ by Proposition 1.2.4.

In the second case, i.e. $X_k \in \Gamma(\mathbf{u})$ and $X_k \to X_0 \in \Gamma(\mathbf{u})$ with $N(X_0, \mathbf{u}, 0^+) = s$, our purpose is to prove $\lim_k N(X_k, \mathbf{u}, r_k r) = s$.

Denoting with $\overline{r} > 0, C > 0$ the constants associated to Proposition 1.2.4, for any given $\varepsilon > 0$ let us take $0 < \widetilde{r} = \widetilde{r}(\varepsilon) \le \overline{r}$ such that

$$N(X_0, \mathbf{u}, r) \le s + \frac{\varepsilon}{2} \quad \text{for every } 0 < r \le \widetilde{r} \quad \text{such that } e^{C\Psi(\widetilde{r})} \le \frac{s+1+2\varepsilon}{s+1+\varepsilon}.$$

Furthermore there exists $\delta > 0$ such that

$$N(X, \mathbf{u}, \widetilde{r}) \le s + \varepsilon \quad \text{for } X \in \partial^0 B^+_\delta(X_0).$$

Hence, using Proposition 1.2.4 we obtain

$$N(X, \mathbf{u}, r) \le (s+1+\varepsilon)e^{C\Psi(\widetilde{r})} - 1 \le s + 2\varepsilon, \quad \text{for } X \in \partial^0 B^+_\delta(X_0)$$

and the claim follows by taking into account Corollary 1.2.5.

Finally, let us compute the derivative of $r \mapsto N(0, \overline{\mathbf{u}}, r)$, in order to prove that $\overline{\mathbf{u}}$ is α-homogeneous in $\overline{\mathbb{R}^{n+1}_+}$, i.e. for every $X \in \overline{\mathbb{R}^{n+1}_+} \neq 0$

$$\overline{\mathbf{u}}(X) = |X|^\alpha \, \overline{\mathbf{u}} \left(\frac{X}{|X|} \right).$$

An previously remarked in (9)

$$\frac{d}{dr} H(0, \overline{\mathbf{u}}, r) = \frac{2}{r^{n+a}} \int_{\partial^+ B^+_r} |y|^a \langle \overline{\mathbf{u}}, \partial_r \overline{\mathbf{u}} \rangle \, dx = \frac{2}{r} E(0, \overline{\mathbf{u}}, r)$$

which, together with Theorem 1.3.3, readily implies

$$0 = \frac{1}{2} \frac{d}{dr} N(0, \overline{\mathbf{u}}, r) = \frac{\displaystyle\int_{\partial^+ B^+_r} |y|^a \, \overline{\mathbf{u}}^2 \, d\sigma \int_{\partial^+ B^+_r} |\partial_\nu \overline{\mathbf{u}}|^2 \, d\sigma - \left(\int_{\partial^+ B^+_r} |y|^a \langle \overline{\mathbf{u}}, \partial_r \overline{\mathbf{u}} \rangle \, d\sigma \right)^2}{r^{2n+2a-2} H^2(0, \mathbf{u}, r)}$$

for $r>0$. This equality yields the existence of $C=C(r)>0$ such that $\partial_r \overline{\mathbf{u}} = C(r)\overline{\mathbf{u}}$ for $r>0$. Using this fact we get

$$2C(r) = \frac{2\displaystyle\int_{\partial^+ B_r^+} |y|^a \langle \overline{\mathbf{u}}, \partial_r \overline{\mathbf{u}}\rangle \, d\sigma}{\displaystyle\int_{\partial^+ B_r^+} |y|^a \, \overline{\mathbf{u}}^2 \, d\sigma} = \frac{d}{dr}\log H(0,\overline{\mathbf{u}},r) = \frac{2}{r}N(0,\overline{\mathbf{u}},r) = \frac{2}{r}\alpha$$

and thus $C(r)=\alpha/r$ and $\overline{\mathbf{u}}(r,\theta) = r^\alpha \mathbf{g}(\theta)$ as we claimed. □

Moreover, in the case of a blow-up sequences centered we can further improve the convergence result in the following way

Corollary 1.3.12. *Given $s\in(0,1)$ and $\mathbf{u}\in\mathcal{G}^s(B_1^+)$, let $(\mathbf{u}_k)_k$ be a blow-up sequence centered in $X_0\in\Gamma(\mathbf{u})$ and $\Gamma(\mathbf{u}_k)$ the associated nodal sets. Then $\Gamma(\mathbf{u}_k)\to\Gamma(\overline{\mathbf{u}})$ locally with respect to the Hausdorff distance $d_{\mathcal{H}}$ in Σ, i.e. for every $R>0$*

$$\mathrm{d}_{\mathcal{H}}\left(\Gamma(\mathbf{u}_k)\cap\partial^0 B_R^+, \Gamma(\overline{\mathbf{u}})\cap\partial^0 B_R^+\right)\to 0.$$

In the previous statement we denoted with

$$\mathrm{d}_{\mathcal{H}}(A,B) := \max\left\{\sup_{a\in A}\mathrm{dist}(a,B), \sup_{b\in B}\mathrm{dist}(A,b)\right\}, \quad A,B\subseteq\mathbb{R}^N \tag{33}$$

the Hausdorff distance in $\mathbb{R}^n$. Notice that $\mathrm{d}_{\mathcal{H}}(A,B)\le\varepsilon$ if and only if $A\subseteq N_\varepsilon(B)$ and $B\subseteq N_\varepsilon(A)$, where $N_\varepsilon(\cdot)$ is the closed ε-neighborhood of a set, more precisely

$$N_\varepsilon(A) = \{x\in\mathbb{R}^N : \mathrm{dist}(x,A)\le\varepsilon\}, \quad A\subseteq\mathbb{R}^N.$$

Proof of Corollary 1.3.12. It is not restrictive to consider the case $R=1$. By the definition of Hausdorff distance, the claimed result is equivalent to prove that for every $\varepsilon>0$ there exists $\overline{k}>0$ such that for every $k\ge\overline{k}$

$$\Gamma(\mathbf{u}_k)\cap\partial^0 B_1^+ \subseteq N_\varepsilon\left(\Gamma(\overline{\mathbf{u}})\cap\partial^0 B_1^+\right)$$
$$\Gamma(\overline{\mathbf{u}})\cap\partial^0 B \subseteq N_\varepsilon\left(\Gamma(\mathbf{u}_k)\cap\partial^0 B_1^+\right).$$

Supposing by contradiction that the first inclusion is not true, then there exist $\overline{\varepsilon}>0$ and a sequence $X_k\in\Gamma(\mathbf{u}_k)\cap\partial^0 B_1^+$ such that $\mathrm{dist}\left(X_k,\Gamma(\overline{\mathbf{u}})\cap\partial^0 B_1^+\right)>\overline{\varepsilon}$. Moreover, up to a subsequence, $X_k\to\overline{X}\in\Gamma(\overline{\mathbf{u}})\cap\overline{\partial^0 B_1^+}$ by the L^∞_{loc} convergence of $\mathbf{u}_k\to\overline{\mathbf{u}}$. Since by Proposition 1.3.11 the nodal set $\Gamma(\overline{\mathbf{u}})$ is a conical set, i.e. for every $\lambda>0$ and $\overline{X}\in\Gamma(\overline{\mathbf{u}})$ we have $\lambda\overline{X}\in\Gamma(\overline{\mathbf{u}})$ and $0\in\Gamma(\overline{\mathbf{u}})$, we deduce that $\mathrm{dist}(\overline{X},\Gamma(\overline{\mathbf{u}})\cap\partial^0 B_1^+)=0$, which provides the contradiction.

Finally, we have to prove that for every $\varepsilon > 0$ there exists $\bar{k} > 0$ such that

$$\Gamma(\overline{\mathbf{u}}) \cap \partial^0 B \subseteq N_\varepsilon\left(\Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+\right) \quad \text{for every } k \geq \bar{k}.$$

We start by proving that given $\overline{X} \in \Gamma(\overline{\mathbf{u}}) \in \partial^0 B_1^+$ and $\delta > 0$, the vector valued function $\mathbf{u}_k$ must have a zero in $\partial^0 B_\delta^+(\overline{X})$, for k sufficiently large. If not, by recalling that $u_{i,k} \cdot u_{j,k}|_{y=0} \equiv 0$ for every $i \neq j$, we would have that there exists an index $0 < i < h$ such that

$$\begin{cases} -L_a u_{i,k} = 0 & \text{in } B_\delta^+(\overline{X}) \\ -\partial_y^a u_{i,k} = f_i(u_{i,k}) & \text{on } \partial^0 B_\delta^+(\overline{X}) \end{cases}, \quad u_{i,k} > 0 \text{ on } \partial^0 B_\delta^+(\overline{X})$$

and $u_{j,k} \equiv 0$ in $\partial^0 B_\delta^+(\overline{X})$, for every $j \neq i$. Passing to the limit, this would imply that

$$\begin{cases} -L_a \overline{u}_i = 0 & \text{in } B_\delta^+(\overline{X}) \\ -\partial_y^a \overline{u}_i = 0 & \text{on } \partial^0 B_\delta^+(\overline{X}) \end{cases}, \quad \overline{u}_i \geq 0 \text{ on } \partial^0 B_\delta^+(\overline{X})$$

and $\overline{u}_j \equiv 0$ on $\partial^0 B_\delta^+(\overline{X})$ for every $j \neq i$. Since $\overline{X} \in \Gamma(\overline{\mathbf{u}})$ it follows from the Hopf principle (see [43, 13]) that $\overline{\mathbf{u}} \equiv 0$ in $\partial^0 B_\delta^+(\overline{X})$, a contradiction with the fact the $\Gamma(\overline{\mathbf{u}})$ has empty interior.

Now, arguing by contradiction, suppose the existence of $\bar{\varepsilon} > 0$ and $(X_k)_k \in \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+$ such that $X_k \to \overline{X} \in \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+$ and $\text{dist}(X_k, \Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+) > \bar{\varepsilon}$. Since $\Gamma(\overline{\mathbf{u}})$ is a conical set passing through the origin, let us take $\widetilde{X} \in \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+$ such that $\left|\widetilde{X} - \overline{X}\right| \leq \bar{\varepsilon}/4$. Furthermore, we can take, by using the result proved in the previous paragraph, a sequence $(\overline{X}_k)_k \in \Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+$ such that $\left|\overline{X}_k - \widetilde{X}\right| \leq \bar{\varepsilon}/4$ for sufficiently large k. The final contradiction follows noticing that

$$\text{dist}(X_k, \Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+) \leq \left|X_k - \overline{X}_k\right| \leq \left|X_k - \overline{X}\right| + \left|\overline{X} - \widetilde{X}\right| + \left|\widetilde{X} - \overline{X}_k\right| \leq \frac{3\bar{\varepsilon}}{4} \leq \bar{\varepsilon},$$

for sufficiently large k. □

Finally, we define the following class of blow-up which contains all the possible blow-up limit of u centered in a fixed point $X_0 \in \mathbf{u}$.

Definition 1.3.13. Given $s \in (0,1)$ we define the set $\mathfrak{B}^s(\mathbb{R}^{n+1})$ of all possible blow-up limit of $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ centered in $X_0 \in \Gamma(\mathbf{u})$ as the collection of homogenous entire segregated profile in $\mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$.

In particular, given $\mathbf{u} \in \mathfrak{B}^s(\mathbb{R}^{n+1})$ a k-homogenous entire segregated profile, then there exists $\overline{\mathbf{g}} \in H^{1,a}(S^n_+)$ such that

$$\mathbf{u}(X) = |X|^k \, \mathbf{g}\left(\frac{X}{|X|}\right). \tag{34}$$

We remark that for $a \in (-1,1)$, given (r,θ) the generalized spherical coordinates in $\mathbb{R}^{n+1}$ with $r > 0$ and $\theta \in S^n_+$, the weighted Sobolev space $H^{1,a}(S^n_+)$ is defined as the closure of $C^\infty_c(S^n_+)$ with respect to the norm

$$\|g\|^2_{H^{1,a}(S^n_+)} = \int_{S^n_+} |\sin\theta_n|^a \, g^2 \mathrm{d}\sigma + \int_{S^n_+} |\sin\theta_n|^a \, |\nabla_{S^n} g|^2 \, \mathrm{d}\sigma,$$

where θ_n is the spherical coordinate associated to the y-direction and ∇_{S^n} the tangential gradient on S^n. Moreover, under the previous notations, we can find a spherical decomposition of the L_a-operator. More precisely,

$$L_a u = \sin^a(\theta_n) \frac{1}{r^n} \partial_r \left(r^{n+a} \partial_r u\right) + \frac{1}{r^a} L^{S^n}_a u \tag{35}$$

where $y = r\sin(\theta_n)$ and the Laplace-Beltrami type operator is defined as

$$L^{S^n}_a u = \mathrm{div}_{S^n}(\sin^a(\theta_n) \nabla_{S^n} u), \tag{36}$$

with div_{S^n} the tangential divergence on S^n. Inspired by the previous spherical decomposition, we can find a simple characterization of the blow-up limit $\mathbf{u} \in \mathfrak{B}^s(\mathbb{R}^{n+1})$ in term of its trace on the upper sphere S^n_+.

Proposition 1.3.14. *Let $s \in (0,1)$ and $\mathbf{u} \in \mathfrak{B}^s(\mathbb{R}^{n+1})$ be a γ-homogeneous blow-up limit, i.e. such that $N(0,\mathbf{u},1) = \gamma$. Then, there exists $\mathbf{g} = (g_1,\dots,g_h) \in H^{1,a}(S^n_+)$ such that, for every $i = 1,\dots,h$ we get*

$$\begin{cases} -L^{S^n}_a g_i = \lambda(\gamma) g_i \sin^a(\theta_n) & \text{in } S^n_+ \\ \partial^a_{\theta_n} g_i = 0 & \text{on } \omega_i \subset S^{n-1} \\ g_i = 0 & \text{on } S^{n-1} \setminus \omega_i, \end{cases} \tag{37}$$

where $\lambda(\gamma) = \gamma(\gamma + n + a - 1)$, $(\omega_i)_i \subset S^{n-1} \times \{0\}$ and

$$\partial^a_{\theta_n} g(\theta', 0) = \lim_{\theta_n \to 0^+} \sin^a(\theta_n) \partial_{\theta_n} g(\theta', \theta_n) \quad \textit{for } \theta' \in S^{n-1}.$$

Proof. Let us consider initially a general case: let $u \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ be a γ-homogeneous L_a-harmonic function in $\mathbb{R}^{n+1}_+$ such that $u\partial^a_y u = 0$ on Σ. By the homogeneity of u, there exists $\omega \subset S^{n-1}$ such that

$$C_\omega = \left\{X \in \Sigma \colon \frac{X}{|X|} \in \omega\right\} = \left\{X \in \Sigma \colon \partial^a_y u(X) = 0\right\}, \tag{38}$$

where C_ω is the cone in Σ spanned by ω with vertex at zero. Since u is γ-homogeneous, i.e. $u(r,\theta) = r^\gamma g(\theta)$ for $r \in (0,+\infty)$ and $\theta \in S^{n-1}$, we get by (35) and (36) that

$$-L_a^{S^n} g = \lambda_1^s(\gamma) g \sin^a(\theta_n) \quad \text{on } S^n_+,$$

with $\lambda_1^s(\gamma) = \gamma(\gamma+n+a-1)$, and similarly by (38) we get

$$\partial^a_{\theta_n} g(\theta',0) = 0 \quad \text{on } \omega.$$

Hence, with a slight abuse of notations, for every open region $\omega \subseteq S^{n-1}$, we can define the eigenvalue

$$\lambda_1^s(\omega) = \inf \left\{ \frac{\displaystyle\int_{S^n_+} y^{1-2s} \left|\nabla_{S^n} u\right|^2 \mathrm{d}\sigma}{\displaystyle\int_{S^n_+} y^{1-2s} u^2 \mathrm{d}\sigma} \; : \; \begin{array}{l} u \in H^{1,a}(S^n_+)\setminus\{0\} \\ u \equiv 0 \text{ in } S^{n-1}\setminus \omega \end{array} \right\} \tag{39}$$

and similarly the characteristic exponent of the cone C_ω spanned by ω as the quantity

$$\gamma_s(C_\omega) = \gamma_s(\lambda_1^s(\omega)) \;,$$

where the function $\gamma_s(t)$ is defined by

$$\gamma_s(t) = \sqrt{\left(\frac{n-2s}{2}\right)^2 + t} - \frac{n-2s}{2}.$$

Now, given $\mathbf{u} \in \mathfrak{B}^s(\mathbb{R}^{n+1})$ let $\mathbf{g} \in H^{1,a}(S^n_+;\mathbb{R}^h)$ be its spherical part defined by (34). Since all the components $u_i \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ share the same homogeneity degree γ, we directly get that the eigenvalue $\lambda(g_i)$ is the same for every component of $\mathbf{g}$. □

Using the variational formulation of the eigenvalue problem associated to $\lambda_1^s(\omega)$ defined in (39), for every $\omega \subset S^{n-1}$ we easily get that

$$0 = \lambda_s^1(S^{n-1}) \leq \lambda_s^1(\omega) \leq \lambda_s^1(\emptyset) = 2s$$

and more generally, for $\omega_1 \subset \omega_2$ it holds $\lambda_s^1(\omega_2) < \lambda_s^1(\omega_1)$ (see [85, 82] for further properties of this eigenvalue problem). We will exhibit several connection between this two formulations of blow-up limits in Chapter 2 finding a different connection with their interpretation on the traces space as s-harmonic function on cones. However, by the previous characterization, at this point we can improve the bound on the Almgren frequency for the segregated profile.

Corollary 1.3.15. *Let $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$. Then, for every $X_0 \in \Gamma(\mathbf{u})$ we get*

$$N(X_0, \mathbf{u}, 0^+) < 2s.$$

Proof. By the definition of the class of segregated profiles, given $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ we have $\mathbf{u} \not\equiv \mathbf{0}$ on Σ. Given $X_0 \in \mathbf{u}$ and $\overline{\mathbf{u}} \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$ a blow-up limit of $\mathbf{u}$ at X_0, we get by the uniform convergence that $\overline{\mathbf{u}} \not\equiv 0$ on Σ. By the characterization of Proposition 1.3.14 we get that necessary $\omega \neq \emptyset$ or, in other words, $\lambda_s^1(\omega) < 2s$. The previous inequality implies

$$N(X_0, \mathbf{u}, 0^+) = N(0, \overline{\mathbf{u}}, 1) = \lambda^1_s(\omega) < 2s,$$

as we claimed. □

Actually, we remark that combining the previous bound and the monotonicity result Proposition 1.2.4 we can prove that a segregated function in $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ has identically zero trace on $B_1 \cap \Sigma$ if and only if $N(X_0, \mathbf{u}, 0^+) = 2s$ on some point $X_0 \in \Gamma(\mathbf{u})$.

1.4 HAUSDORFF DIMENSION ESTIMATES FOR REGULAR AND SINGULAR SETS

In the same spirit of [19, 18, 36, 81] we prove that there exists a gap in the possible values of the Almgren frequency formula $N(X, \mathbf{u}, 0^+)$ for $X_0 \in \Gamma(\mathbf{u})$. We remark that as in [81], our analysis is not restricted to solutions of minimal energy as in [19].

Proposition 1.4.1. *Given $s \in (0,1), n = 2$ and $u \in \mathcal{G}^s(B_1^+)$, then for $X_0 \in \Gamma(\mathbf{u})$ either*

$$N(X_0, \mathbf{u}, 0^+) = s \quad \textit{or} \quad N(X_0, \mathbf{u}, 0^+) \geq s + \delta,$$

for some universal constant $\delta > 0$.

Proof. By contradiction, given $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ suppose there exist two sequences $\varepsilon_k \searrow 0^+$ and $(X_k)_k \subset \Gamma(\mathbf{u}) \cap \partial^0\Omega^+$, for some $\Omega^+ \subset\subset B_1^+$, such that

$$N(X_k, \mathbf{u}, 0^+) \leq s + \varepsilon_k.$$

Moreover, it is not restrictive to suppose that $\varepsilon_k \leq s/2$, in order to always have $s + \varepsilon_k < 2s$. Since $\Gamma(\mathbf{u})$ has empty interior in $\mathbb{R}^{n+1}$, up to a subsequence there exists $X_0 \in \Gamma(\mathbf{u})$ such that $X_k \to X_0$ and, by Corollary 1.2.5 and Corollary 1.2.6 we get $N(X_0, \mathbf{u}, 0^+) = s$.
Therefore, let us construct a sequence of blow-up limit $(\overline{\mathbf{u}}_k)_k \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ in order to translate the absurd hypothesis in the context of entire segregated configurations in $\mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. Hence, for every $k \in \mathbb{N}$ let

$$\mathbf{u}_{X_k,i}(X) = \frac{\mathbf{u}(X_k + r_i X)}{\rho_i}$$

be the blow-up sequence centered in X_k associated to $r_i \searrow 0^+$ and $\rho_i^2 = H(X_k, \mathbf{u}, r_i)$. By Theorem 1.3.3, there exists a family of blow-up limits $(\overline{\mathbf{u}}_k)_k \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$ of homogenous function in $\mathbb{R}^{n+1}$ such that $\mathbf{u}_{X_k,i} \to \overline{\mathbf{u}}_k$ and

$$\int_{\partial^+ B_1^+} |y|^a \, \overline{\mathbf{u}}_k^2 \mathrm{d}\sigma = 1 \quad \text{and} \quad N(0, \overline{\mathbf{u}}_k, r) = s + \varepsilon_k \quad \text{for every } r > 0, \tag{40}$$

namely $\overline{\mathbf{u}}_k$ is $(s+\varepsilon_k)$-homogeneous in $\mathbb{R}^{n+1}$, i.e.

$$\overline{\mathbf{u}}_k(X) = |X|^{s+\varepsilon_k} \, \mathbf{g}_k\left(\frac{X}{|X|}\right),$$

with $\mathbf{g}_k \in H^{1,a}(S^n_+)$. Following the same idea in the proof of Lemma 1.3.5, since for every $R > 0$ we have $-\partial^a_y \overline{\mathbf{u}}_k \leq 0$ on $\partial^0 B^+_R$, if we consider $\overline{\mathbf{v}}_k \in H^{1,a}(B^+_R)$ the symmetric extension of $\mathbf{u}_k$ with respect to Σ to the whole B_1 we get

$$-L_a \overline{\mathbf{v}}_k \leq 0 \quad \text{in } \mathbb{R}^{n+1}.$$

Hence, by [88, Lemma A.2] it follows for every $R > 0$

$$\begin{aligned}
\sup_{B^+_{R/2}} \overline{\mathbf{u}}_k = \sup_{B_{R/2}} \overline{\mathbf{v}}_k &\leq C(n,s)\left(\frac{1}{R^{n+1+a}} \int_{B_R} |y|^a \, \overline{\mathbf{u}}_k^2 \mathrm{d}X\right)^{1/2} \\
&= 2C(n,s)\left(\frac{1}{R^{n+1+a}} \int_{B^+_R} |y|^a \, \overline{\mathbf{u}}_k^2 \mathrm{d}X\right)^{1/2} \\
&= 2C(n,s) R^{s+\varepsilon_k}\left(\frac{1}{n+2+2\varepsilon_k} \int_{\partial^+ B^+_1} |y|^a \, \overline{\mathbf{u}}_k^2 \mathrm{d}\sigma\right)^{1/2},
\end{aligned}$$

where in the last equality we used the $(s+\varepsilon_k)$-homogeneity of $\overline{\mathbf{u}}_k$. By (40) it follows that for every $R > 0$ the sequence $(\overline{\mathbf{u}}_k)_k \subset \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1}) \cap C^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ with $\alpha \in (0, \alpha^*)$ is uniformly bounded in $L^\infty(B^+_R)$, for every $R > 0$, which implies by Theorem 1.3.8 the existence of $\overline{\mathbf{u}} \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Moreover, by the strong convergence in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ it follows that $N(0, \overline{\mathbf{u}}, r) = s$ for every $r > 0$, i.e. up to a rotation $\overline{\mathbf{u}} = (\overline{u}_1, \overline{u}_2, 0, \dots, 0)$ where

$$\overline{u}_1(x,y) = C_1\left(\frac{\sqrt{x_1^2 + y^2} - x_1}{2}\right)^s \quad \text{and} \quad \overline{u}_2(x,y) = C_1\left(\frac{\sqrt{x_1^2 + y^2} + x_1}{2}\right)^s, \tag{41}$$

for some positive constant $C_1 > 0$ such that $\|\overline{\mathbf{u}}\|_{L^{2,a}(\partial^+ B^+_1)} = 1$. Up to relabeling the components of $\overline{\mathbf{u}}_k$, let us suppose that the components $\overline{u}_{i,k} \to 0$ strongly in $H^{1,a}_{\mathrm{loc}} \cap C^{0,\alpha}_{\mathrm{loc}}$ for every $i = 2, \dots, h$, while $(\overline{u}_{1,k}, \overline{u}_{2,k}) \to (\overline{u}_1, \overline{u}_2)$ strongly in $H^{1,a}_{\mathrm{loc}} \cap C^{0,\alpha}_{\mathrm{loc}}$. By (41), since

$$\begin{aligned}
\{\overline{u}_1 = 0\} \cap \Sigma &= \{X = (x_1, x_2, y) \in \Sigma \colon x_1 \geq 0\} \\
\{\overline{u}_2 = 0\} \cap \Sigma &= \{X = (x_1, x_2, y) \in \Sigma \colon x_1 \leq 0\}
\end{aligned} \tag{42}$$

for every $k > 0$ there exist nonempty $\omega_k \subset \{X \in \Sigma\colon x_1 > 0\}$ such that ω_k is a connected component in S^{n-1} of $\{g_{1,k} > 0\}$. By contradiction, let us suppose there exists $\overline{k} > 0$ such that $g_{1,\overline{k}} \equiv 0$ on $S^{n-1} \cap \{X \in \Sigma\colon x_1 \geq 0\}$. Then, necessary we must have $g_{2,\overline{k}} \equiv 0$ on $S^{n-1} \cap \{X \in \Sigma\colon x_1 \leq 0\}$, otherwise by the monotonicity of the eigenvalue (37) we would obtain $\lambda(g_{1,\overline{k}}) > \lambda(g_{2,\overline{k}})$ in contradiction with the definition of $\overline{\mathbf{u}} \in \mathfrak{B}^s(\mathbb{R}^{n+1})$. Hence, since $\overline{\mathbf{u}}_{\overline{k}}$ and $\overline{\mathbf{u}}$ satisfy the same boundary condition, by uniqueness their homogeneities must be equal, in contradiction with the fact that $\overline{\mathbf{u}}_{\overline{k}}$ is $(s+\varepsilon_{\overline{k}})$-homogeneous in $\mathbb{R}^{n+1}$. The same contradiction follows from the sequence $(B_k)_k$.

Now, since we are working in dimension $n = 2$, there exist two sequences $(P_k)_k, (Q_k)_k \subset S^1 \cap \{X \in \Sigma\colon x_1 > 0\}$ such that $\omega_k = (P_k, Q_k)$ can be seen as an arc of S^1 between the endpoints P_k and Q_k. Since by compactness of S^1 there exist, up to a subsequence, $P, Q \in S^1 \cap \{X \in \Sigma\colon x_1 \geq 0\}$ respectively limit of $(P_k)_k$ and $(Q_k)_k$, let us consider separately the cases $|P_k - Q_k| \to 0$ and $P \neq Q$.
If $P \neq Q$, let $\omega = (P, Q)$ be the limit of the sequence $(\omega_k)_k$ and C_ω be the cone in Σ spanned by ω, i.e.

$$C_\omega = \left\{X \in \Sigma\colon \frac{X}{|X|} \in \omega \subset S^1\right\}.$$

One one hand, by definition of ω_k, since $u_{1,k}$ and $u_{2,k}$ are segregated on Σ, we get passing $\overline{u}_{2,k} \equiv 0$ on ω_k and then, passing to the limit for $k \to \infty$, $\overline{u}_2 \equiv 0$ on ω. On the other hand, by the L^∞_{loc}-convergence of the sequence $(\overline{\mathbf{u}}_k)_k$ we get from (42) that $\left\|\overline{g}_{1,k}\right\|_{L^\infty(\omega_k)} \to 0$ which implies, passing to its homogeneous extension, that $\overline{u}_1 \equiv 0$ on every compact set $K \subset C_\omega$, and similarly $\overline{\mathbf{u}} \equiv 0$ on every compact set in C_ω, in contradiction with (41).

Hence in the other case, given $r_k = |P_k - Q_k| \searrow 0$ and $P_k \in \Sigma$ let introduce the blow-up sequence $(\overline{\mathbf{w}}_k)_k \subset \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$ centered in $(P_k)_k \in \Gamma(\mathbf{u}_k) \cap S^{n-1}$ and associated to $r_k > 0$, i.e.

$$\overline{\mathbf{w}}_k(X) = \frac{\overline{\mathbf{u}}_k(P_k + r_k X)}{\rho_k} \quad \text{with } \rho_k = \sqrt{H(P_k, \overline{\mathbf{u}}_k, r_k)},$$

such that $\|\overline{\mathbf{w}}_k\|_{L^{2,a}(\partial^+ B_1^+)} = 1$. By construction, for every $k \in \mathbb{N}$ we have

$$\overline{\mathbf{w}}_k(0) = \mathbf{0} = \overline{\mathbf{w}}_k\left(\frac{Q_k - P_k}{r_k}\right), \tag{43}$$

where $(Q_k - P_k)/r_k \to \nu \in S^{n-1} \times \{0\}$ by compactness in S^{n-1}. Now, following the same ideas in the proof of Lemma 1.3.5, given $C > 0$ and $\overline{r} > 0$ be such that Proposition 1.2.4, Proposition

1.2.7 and Lemma 1.3.4 hold true then, up to taking k so large that $r_k, r_k R \leq \overline{r}$, we get for every $R > 0$ that

$$\begin{aligned}\int_{\partial^+ B_R^+} |y|^a \, \overline{\mathbf{w}}_k^2 \mathrm{d}\sigma &= \frac{1}{\rho_k^2 r_k^{n+a}} \int_{\partial^+ B^+_{Rr_k}(P_k)} |y|^a \, \overline{\mathbf{u}}_k^2 \mathrm{d}\sigma \\ &= R^{n+a} \frac{H(P_k, \overline{\mathbf{u}}_k, Rr_k)}{H(P_k, \overline{\mathbf{u}}_k, r_k)} \\ &\leq R^{n+a} \left(\frac{Rr_k}{r_k}\right)^{2\overline{C}},\end{aligned}$$

which implies that

$$\sup_{B^+_{R/2}} \overline{\mathbf{w}}_k \leq 2C(n,s) \left(\frac{1}{R^{n+1+a}} \int_{B_R^+} |y|^a \, \overline{\mathbf{w}}_k^2 \mathrm{d}X\right)^{1/2} \leq 2C(n,s) \left(\frac{H(0, \overline{\mathbf{w}}_k, R)}{n+a+1}\right)^{1/2}$$

is uniformly bounded for $k > 0$. By Proposition 1.3.8, there exists a blow-up limit $\overline{\mathbf{w}} \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ for every $\alpha \in (0,s)$ such that $\overline{\mathbf{w}} \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$ with $\|\overline{\mathbf{w}}\|_{L^{2,a}(\partial^+ B_1^+)} = 1$.

Since the blow-up sequence $(\overline{\mathbf{w}}_k)_k$ is constructed starting from a family of homogeneous entire segregated profiles in $\mathfrak{B}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$, we can prove that $\overline{\mathbf{w}}$ is constant along the direction parallel to $P \in S^{n-1}$ and that its restriction on the orthogonal half plane belongs to $\mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^n)$.
Hence, let $X \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}$ be fixed. By the homogeneity of $\overline{\mathbf{u}}_k$ we obtain

$$\begin{aligned}\overline{\mathbf{w}}_k(X + \lambda P_k) &= \frac{\overline{\mathbf{u}}_k(P_k + r_k(X + \lambda P_k))}{\rho_k} = \frac{\overline{\mathbf{u}}_k((1 + r_k\lambda)P_k + r_k X)}{\rho_k} \\ &= \frac{(1 + r_k\lambda)^{s+\varepsilon_k}}{\rho_k} \overline{\mathbf{u}}_k \left(P_k + \frac{r_k}{1 + r_k\lambda} X\right) \\ &= (1 + r_k\lambda)^{s+\varepsilon_k} \, \overline{\mathbf{w}}_k \left(\frac{r_k}{1 + r_k\lambda} X\right)\end{aligned}$$

and then

$$\begin{aligned}|\overline{\mathbf{w}}_k(X + \lambda P_k) - \overline{\mathbf{w}}_k(X)| &\leq \left|(1 + r_k\lambda)^{s+\varepsilon_k} \, \overline{\mathbf{w}}_k \left(\frac{r_k}{1 + r_k\lambda} X\right) - \overline{\mathbf{w}}_k \left(\frac{r_k}{1 + r_k\lambda} X\right)\right| + \\ &\quad + \left|\overline{\mathbf{w}}_k \left(\frac{r_k}{1 + r_k\lambda} X\right) - \overline{\mathbf{w}}_k(X)\right| \\ &\leq \left(|(1 + r_k\lambda)^{s+\varepsilon_k} - 1| + \left|\frac{r_k}{1 + r_k\lambda} - 1\right|^{\alpha} |X|^{\alpha}\right) \|\overline{\mathbf{w}}_k\|_{C^{0,\alpha}(\overline{B_1^+})}\end{aligned}$$

with $\alpha \in (0, s)$. Thus, as $r_k \searrow 0$, by the uniform convergence $(\overline{\mathbf{w}}_k)_k$ on every compact set, we get

$$|\overline{\mathbf{w}}(X + \lambda P) - \overline{\mathbf{w}}(X)| = 0 \quad \text{for every } \lambda \in \mathbb{R},$$

where $P = \lim_k P_k \in S^{n-1} \times \{0\}$.

Now, given the section of $\overline{\mathbf{w}}$ with respect to the direction λP, with $\lambda \in \mathbb{R}$, we observe that the equations and the segregation conditions are trivially satisfied and the Pohožaev identities on every ball $B_r(X_0)$ on $\mathbb{R}^n_+$ follow immediately by the ones for $\overline{\mathbf{w}}$ on the corresponding ball in $\mathbb{R}^{n+1}_+$ having $B_r(X_0)$ as n-dimensional section. Hence, with slight abuse of notation we still denote the n-dimensional section by $\overline{\mathbf{w}} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^n)$. By Corollary 1.3.10, since for every $k > 0$ and $t \in (0, +\infty)$

$$\begin{aligned} N(P_k, \overline{\mathbf{u}}_k, t) &\leq N(P_k, \overline{\mathbf{u}}_k, +\infty) \\ &= N(0, \overline{\mathbf{u}}_k, +\infty) = s + \varepsilon_k \end{aligned}$$

we get from Proposition 1.2.4 and Proposition 1.3.9 that for every $R > 0$

$$N(0, \overline{\mathbf{w}}, R) = \lim_{r_k \to 0} N(0, \overline{\mathbf{w}}_k, R) = \lim_{r_k \to 0} N(P_k, \overline{\mathbf{u}}_k, r_k R) \leq s + \varepsilon_k. \tag{44}$$

Finally, we reach the contradiction applying a blow-down analysis on the limit function $\overline{\mathbf{w}} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^n)$. With a slight abuse of notations, for $r_k \to +\infty$, consider the blow-down sequence $(\overline{\mathbf{v}}_k)_k$ centered in the origin defined as

$$\overline{\mathbf{v}}_k(X) = \frac{\overline{\mathbf{w}}(r_k X)}{\rho_k} \qquad \text{for } X \in B^+_{X_k, r_k}$$

with

$$\rho_k^2 = \|\overline{\mathbf{w}}(r_k \cdot)\|^2_{L^{2,a}(\partial^+ B_1^+)} = \frac{1}{r_k^{n+a}} \int_{\partial^+ B^+_{r_k}} |y|^a\, \overline{\mathbf{w}}^2 \, d\sigma = H(0, \overline{\mathbf{w}}, r_k).$$

Fixed $R > 1$, since $\|\overline{\mathbf{w}}\|_{L^{2,a}(\partial^+ B_1^+)} = 1$, we get from integrating (32) between 1 and R that

$$H(0, \overline{\mathbf{w}}, R) \leq R^{2(s+\varepsilon_k)},$$

and consequently, up to relabeling the constant, we get

$$\sup_{\overline{B^+_R}} \overline{\mathbf{v}}_k \leq 2C(n,s) \left(\frac{H(0, \overline{\mathbf{v}}_k, 2R)}{n + a + 1} \right)^{1/2} \leq C(n,s) R^{s+\varepsilon_k}.$$

Since $(\overline{\mathbf{v}}_k)_k \subset \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$, by the previous uniform bound in $L^\infty_{\text{loc}}(\mathbb{R}^{n+1})$ there exists a blow-down limit $\overline{\mathbf{v}} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$ such that $\overline{\mathbf{v}}_k \to \overline{\mathbf{v}}$ strongly in $H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1})$ and uniformly in

$C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n+1})$, for $\alpha \in (0,s)$.
Moreover, the uniform convergence and Proposition 1.3.9 yield that

$$N(0,\overline{\mathbf{v}},R) = \lim_{k\to\infty} N(0,\overline{\mathbf{v}}_k,R) = \lim_{k\to\infty} N(0,\overline{\mathbf{w}},r_kR) = N(0,\overline{\mathbf{w}},+\infty),$$

for every $R>0$, which implies by Proposition 1.3.9 that $\overline{\mathbf{v}} \in \mathfrak{B}^s(\mathbb{R}^{n+1})$ with degree homogeneity $\gamma(\overline{\mathbf{v}}) \leq s+\varepsilon_k$. By the gap condition in the lower dimensional case $\mathbb{R}^n$, we get that necessary $\gamma(\overline{\mathbf{v}}) = s$, which it implies, going back to the function $\overline{\mathbf{w}}$, that

$$s \leq N(0,\overline{\mathbf{w}},r) \leq N(0,\overline{\mathbf{w}},+\infty) = s.$$

In other words, by the monotonicity result Proposition 1.3.9, the Almgren monotonicity formula satisfies $N(0,\overline{\mathbf{w}},r) = s$ for every $r>0$, i.e. up to a rotation $\overline{\mathbf{w}} = (\overline{w}_1,\overline{w}_2,0,\dots,0)$ where

$$\overline{w}_1(x,y) = C_1\left(\frac{\sqrt{x_1^2+y^2}-x_1}{2}\right)^s \quad \text{and} \quad \overline{w}_2(x,y) = C_1\left(\frac{\sqrt{x_1^2+y^2}+x_1}{2}\right)^s, \tag{45}$$

for some positive constant $C_1>0$ such that $\|\overline{\mathbf{w}}\|_{L^{2,a}(\partial^+B_1^+)} = 1$. The contradiction follows immediately by (43). □

We remark that in general, the latter statement is equivalent to the following one:

$$\text{for every } \alpha \text{ - homogeneous } \overline{\mathbf{u}} \in \mathfrak{B}^s(\mathbb{R}^{n+1}), \text{ either } \alpha = s \text{ or } \alpha \geq s+\delta_n,$$

for some universal constant $\delta_n>0$. Moreover, we can actually generalize the previous result by proving that the possible values of the Almgren frequency formula are a discrete subset of the interval $[s,2s)$.

Proposition 1.4.2. *Given $s\in(0,1), n=2$ and $u \in \mathcal{G}^s(B_1^+)$, then for $X_0 \in \Gamma(\mathbf{u})$ the possible values of the Almgren frequency formula $N(X_0,\mathbf{u},0^+)$ are a discrete set in $[s,2s)$ with $2s$ as point of accumulation.*

Proof. The proof of the first part follows the one of Proposition 1.4.1 since it was based on a contradiction argument due to the a gap in dimension $n=1$ for the possible values of the Almgren frequency formula. Hence, let us prove the result by induction on the Almgren frequency. Since the first step is prove in Proposition 1.4.1, let us consider the inductive step. Let $A\in(0,1)$ be a possible value of the Almgren frequency and $\mathbf{u}\in\mathcal{G}^s(B_1^+)$. By induction we already know that $A = B+\delta$, for some $\delta>0$ and $B\in[s,2s)$ lower Almgren frequency. By contradiction, suppose there exist two sequences $\varepsilon_k \searrow 0^+$ and $(X_k)_k \subset \Gamma(\mathbf{u})\cap\partial^0\Omega^+$, for some $\Omega^+ \subset\subset B_1^+$, such that

$$N(X_k,\mathbf{u},0^+) \leq A+\varepsilon_k.$$

with $\varepsilon_k \leq s - A/2$, in order to always have $A + \varepsilon_k < 2s$. Since $\Gamma(\mathbf{u})$ has empty interior in $\mathbb{R}^{n+1}$, up to a subsequence there exists $X_0 \in \Gamma(\mathbf{u})$ such that $X_k \to X_0$ and, by the induction hypothesis and Corollary 1.2.6 we get $N(X_0, \mathbf{u}, 0^+) = A$.
Therefore, let us construct a sequence of blow-up limit $(\overline{\mathbf{u}}_k)_k \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1}_+)$ in order to translate the absurd hypothesis in the context of entire segregated configurations in $\mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1}_+)$. Hence, following the details in Proposition 1.4.1, we can construct a family of blow-up limits $(\overline{\mathbf{u}}_k)_k \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$ of homogenous function in $\mathbb{R}^{n+1}$ such that $\mathbf{u}_{X_k,i} \to \overline{\mathbf{u}}_k$ and

$$\int_{\partial^+ B_1^+} |y|^a \, \overline{\mathbf{u}}_k^2 \mathrm{d}\sigma = 1 \quad \text{and} \quad N(0, \overline{\mathbf{u}}_k, r) = A + \varepsilon_k \quad \text{for every } r > 0, \tag{46}$$

namely $\overline{\mathbf{u}}_k$ is $(A + \varepsilon_k)$-homogeneous in $\mathbb{R}^{n+1}$., i.e.

$$\overline{\mathbf{u}}_k(X) = |X|^{A+\varepsilon_k} \mathbf{g}_k \left(\frac{X}{|X|} \right),$$

with $\mathbf{g}_k \in H^{1,a}(S^n_+)$. By (46) it follows that for every $R > 0$ the sequence $(\overline{\mathbf{u}}_k)_k \subset \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1}) \cap C^{0,\alpha}_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$ with $\alpha \in (0, s)$ is uniformly bounded in $L^\infty(B_R^+)$, for every $R > 0$, which implies by Theorem 1.3.8 the existence of $\overline{\mathbf{u}} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$. Moreover, by the strong convergence in $H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1})$ it follows that $N(0, \overline{\mathbf{u}}, r) = A$ for every $r > 0$.
Since $\overline{\mathbf{u}} \in \mathfrak{B}^s(\mathbb{R}^{n+1})$, we get that $\Gamma(\overline{\mathbf{u}})$ is a conic set and its domain of positivity are totally defined by its spherical section $\mathbf{g} \in H^{1,a}(S^n_+)$ (this is a generalisation of the previous case in Proposition 1.4.1, since in this case we do not know the explicit formulation of the blow-up limit $\overline{\mathbf{u}}$).

Now, let us prove the existence of an index $i = 1, \dots, h$ such that, given $\overline{\omega} = \{\theta \in S^{n-1} \colon g_i \equiv 0\}$, then for every $k > 0$ there exist a nonempty $\omega_k \subset \overline{\omega}$ such that ω_k is a connected component in S^{n-1} of $\{g_{i,k} > 0\}$.
By contradiction, suppose that for every index $i = 1, \dots, h$ there exists $\overline{k} > 0$ such that $g_{i,\overline{k}} \equiv 0$ on the zero set of g_i on S^{n-1}. Then, by uniqueness of the eigenvalue problem (37) we must obtain that $\lambda(\mathbf{g}_k) = \lambda(\mathbf{g})$, in contradiction with the definition of $\overline{\mathbf{u}}_k$ and its homogeneity.

Now, since we are working in dimension $n = 2$, there exist two sequences $(P_k)_k, (Q_k)_k \subset S^1$ such that $\omega_k = (P_k, Q_k)$ can be seen as an arc of S^1 between the endpoints P_k and Q_k. Since by compactness of S^1 there exist, up to a subsequence, $P, Q \in S^1$ respectively limit of $(P_k)_k$ and $(Q_k)_k$, let us consider separately the cases $|P_k - Q_k| \to 0$ and $P \neq Q$.

If $P \neq Q$, let $\omega = (P, Q)$ be the limit of the sequence $(\omega_k)_k$ and C_ω be the cone in Σ spanned by ω, i.e.

$$C_\omega = \left\{ X \in \Sigma \colon \frac{X}{|X|} \in \omega \subset S^1 \right\}.$$

One one hand, by definition of ω_k, since $u_{i,k} \cdot u_{j,k} = 0$ on Σ, we get $\overline{u}_{j,k} \equiv 0$ on $\omega_k \subseteq \overline{\omega}$ for every æ $\neq i$ and then, passing to the limit for $k \to \infty$, $\overline{u}_j \equiv 0$ on ω, for every $j \neq i$. The contradiction follows from the fact that since, by definition $g_i \equiv 0$ on $\overline{\omega}$ and necessary must exist a component of $\mathbf{g}$ non identically zero on $\omega \subset \overline{\omega}$.

Hence in the other case, given $r_k = |P_k - Q_k| \searrow 0$ and $P_k \in \Sigma$ let introduce the blow-up sequence $(\overline{\mathbf{w}}_k)_k \subset \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$ centered in $(P_k)_k \in \Gamma(\mathbf{u}_k) \cap S^{n-1}$ and associated to $r_k > 0$, i.e.

$$\overline{\mathbf{w}}_k(X) = \frac{\overline{\mathbf{u}}_k(P_k + r_k X)}{\rho_k} \quad \text{with } \rho_k = \sqrt{H(P_k, \overline{\mathbf{u}}_k, r_k)},$$

such that $\|\overline{\mathbf{w}}_k\|_{L^{2,a}(\partial^+ B_1^+)} = 1$. By construction, for every $k \in \mathbb{N}$ we have

$$\overline{\mathbf{w}}_k(0) = \mathbf{0} = \overline{\mathbf{w}}_k\left(\frac{Q_k - P_k}{r_k}\right), \tag{47}$$

where $(Q_k - P_k)/r_k \to \nu \in S^{n-1} \times \{0\}$ by compactness in S^{n-1}. Now, following the same ideas of the proof of Proposition 1.4.1, by Proposition 1.3.8 there exists a blow-up limit $\overline{\mathbf{w}} \in H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1}) \cap C^{0,\alpha}_{\text{loc}}(\mathbb{R}^{n+1})$ for every $\alpha \in (0,s)$ such that $\overline{\mathbf{w}} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1})$ with $\|\overline{\mathbf{w}}\|_{L^{2,a}(\partial^+ B_1^+)} = 1$.

Since the blow-up sequence $(\overline{\mathbf{w}}_k)_k$ is constructed starting from a family of homogeneous entire segregated profiles in $\mathfrak{B}^s_{\text{loc}}(\mathbb{R}^{n+1})$, we can prove that $\overline{\mathbf{w}}$ is constant along the direction parallel to $P \in S^{n-1}$ and that its restriction on the orthogonal half plane belongs to $\mathcal{G}^s_{\text{loc}}(\mathbb{R}^n)$.
Hence, with slight abuse of notation we still denote the n-dimensional section by $\overline{\mathbf{w}} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^n)$.
By Corollary 1.3.10, since for every $k > 0$ and $t \in (0, +\infty)$

$$\begin{aligned} N(P_k, \overline{\mathbf{u}}_k, t) &\leq N(P_k, \overline{\mathbf{u}}_k, +\infty) \\ &= N(0, \overline{\mathbf{u}}_k, +\infty) = A + \varepsilon_k \end{aligned}$$

we get from Proposition 1.2.4 and Proposition 1.3.9 that for every $R > 0$

$$N(0, \overline{\mathbf{w}}, R) = \lim_{r_k \to 0} N(0, \overline{\mathbf{w}}_k, R) = \lim_{r_k \to 0} N(P_k, \overline{\mathbf{u}}_k, r_k R) \leq A + \varepsilon_k. \tag{48}$$

Finally, as in Proposition 1.4.1, we reach the contradiction applying a blow-down analysis on the limit function $\overline{\mathbf{w}} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^n)$. Indeed, applying the same procedure, we get by the gap condition in the lower dimensional case $\mathbb{R}^n$ that

$$s \leq N(0, \overline{\mathbf{w}}, r) \leq N(0, \overline{\mathbf{w}}, +\infty) = s.$$

In other words, by the monotonicity result Proposition 1.3.9, the Almgren monotonicity formula satisfies $N(0, \overline{\mathbf{w}}, r) = s$ for every $r > 0$, i.e. up to a rotation $\overline{\mathbf{w}} = (\overline{w}_1, \overline{w}_2, 0, \dots, 0)$ where

$$\overline{w}_1(x,y) = C_1 \left(\frac{\sqrt{x_1^2 + y^2} - x_1}{2}\right)^s \quad \text{and} \quad \overline{w}_2(x,y) = C_1 \left(\frac{\sqrt{x_1^2 + y^2} + x_1}{2}\right)^s, \tag{49}$$

for some positive constant $C_1 > 0$ such that $\|\overline{\mathbf{w}}\|_{L^{2,a}(\partial^+ B_1^+)} = 1$. The contradiction follows immediately by (47). The second part of the proof is a direct consequence of the results in [85]. In this paper, the authors proved in the case $n = 2$ the existence of some segregated profiles possessing some natural symmetry. Such solutions are constructed as limit of a competition-diffusion problem and starting from the eigenvalue problem (39) and in particular they proved that they have growth rate at infinity which is arbitrarily close to the critical one, that is, $2s$. □

1.5 REGULARITY AND FLATNESS OF THE REGULAR SET

In this Section, we will provide an estimate of the Hausdorff dimension of the whole nodal set

$$\Gamma(\mathbf{u}) = \left\{X \in \partial^0 B_1^+ \colon \mathbf{u}(X) = \mathbf{0}\right\},$$

and, regarding its regularity, we will split $\Gamma(\mathbf{u})$ in two parts:

- the singular part $\mathcal{S}(\mathbf{u})$, which we will show to be a local finite collection point in $\partial^0 B_1^+$;
- the regular part $\mathcal{R}(\mathbf{u})$, which is relatively open in $\Gamma(\mathbf{u})$ and satisfies a flatness type condition.

Hence, let us start introducing the notion of regular and singular set in the planar case.

Definition 1.5.1. Given $s \in (0,1), n = 2$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$, we define its regular and singular sets respectively as

$$\mathcal{R}(\mathbf{u}) = \{X_0 \in \Gamma(\mathbf{u}) \colon N(X_0, \mathbf{u}, 0^+) = s\} \quad \text{and} \quad \mathcal{S}(\mathbf{u}) = \{X_0 \in \Gamma(\mathbf{u}) \colon N(X_0, \mathbf{u}, 0^+) > s\}.$$

Corollary 1.5.2. *For $s \in (0,1)$ and $n = 2$ the set $\mathcal{R}(\mathbf{u})$ is relatively open in $\Gamma(\mathbf{u})$ and $\mathcal{S}(\mathbf{u})$ is closed in $\partial^0 B_1^+$, whenever $\mathbf{u} \in \mathcal{G}^s(B_1^+)$.*

Proof. This result is a direct consequence of Proposition 1.4.1 together with the upper semi-continuity of the Almgren frequency function $X \mapsto N(X, \mathbf{u}, 0^+)$ stated in Corollary 1.2.6. □

Next we state and prove some estimates regarding the Hausdorff dimensions of two strata previously defined by using the version of the Federer's Reduction principle in [78]. Hence, let us take a class of functions $\mathcal{F}$ invariant under rescaling and translation and $\mathcal{S}$ a map which associate to each function $\Phi \in \mathcal{S}$ a subset of $\mathbb{R}^{n+1}$. Thus, this principle establishes conditions on $\mathcal{F}$ and $\mathcal{S}$ which imply that to control the Hausdorff dimension of $\mathcal{S}(\Phi)$ for every $\Phi \in \mathcal{S}$, we just need to control the Hausdorff dimension of $\mathcal{S}(\Phi)$ for elements which are homogeneous of some degree. In Chapter 3, we will state the Federer's Reduction principle in its most general version.

Theorem 1.5.3. *Given $s \in (0,1)$ and $n = 2$, let $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ and $\Gamma(\mathbf{u})$ be its nodal set. Then if $\Gamma(\mathbf{u}) \not\equiv \Sigma$, then $\dim_{\mathcal{H}}(\Gamma(\mathbf{u})) \le 1$ and*

$$\dim_{\mathcal{H}}(\mathcal{R}(\mathbf{u})) = 1 \quad \dim_{\mathcal{H}}(\mathcal{S}(\mathbf{u})) = 0.$$

Moreover for any given compact $K \subset\subset \partial^0 B_1^+$ we that $\mathcal{S}(\mathbf{u}) \cap K$ is a finite set.

Proof. A preliminary remark is that we only need to prove the Hausdorff dimensional estimates for the localization of the sets in $K \subset\subset B_1^+$, since the general statement follows because a countable union of sets with Hausdorff dimension less than or equal to some $n \in \mathbb{R}_0^+$ also has Hausdorff dimension less than or equal to n. Moreover, since the Hausdorff dimension of the nodal set of a function is invariant under rescaling, in order to simplify the notations we claim that

$$\dim_{\mathcal{H}}(\Gamma(\mathbf{u}) \cap \partial^0 B_1^+) \le 1 \quad \dim_{\mathcal{H}}(\mathcal{S}(\mathbf{u}) \cap \partial^0 B_1^+) = 0.$$

Let us consider the class of functions $\mathcal{F}$ defined as

$$\mathcal{F} = \left\{ \mathbf{u} \in \left(L^\infty_{\text{loc}}(\mathbb{R}^{2+1}_+)\right)^h : \mathbf{u} \in \mathcal{G}^s(B_r^+(X_0)), \text{ for } r \in \mathbb{R},\ X_0 \in \Sigma \text{ such that } B_r^+(X_0) \subset B_1^+ \right\}.$$

By the linearity of the L_a operator, we already know that the closure under rescaling, translation and normalization and assumption $(\mathrm{F1})$ are all satisfied.
On the other hand, let $X_0 \in \partial^0 B_1^+, r_k \downarrow 0^+$ and $u \in \mathcal{F}$, and choose $\rho_k = \|\mathbf{u}(X_0 + r_k \cdot x)\|_{L^{2,a}(\partial^+ B_1^+)}$.
Theorem 1.3.3 and Proposition 1.3.11 yield the existence of a blow-up limit $\overline{\mathbf{u}} \in \mathcal{F}$, i.e. up to a subsequence $\mathbf{u}_k \to \overline{\mathbf{u}}$ in $\mathcal{F}$ and $\overline{\mathbf{u}}$ is a homogeneous entire segregated profile of degree $k = N(X_0, \mathbf{u}, 0+) \ge s$. Hence also $(\mathrm{F2})$ holds.
Next we choose the map $\overline{\mathcal{S}}$ in $(\mathrm{F3})$ according to our needs.

1. Dimensional estimate of the nodal set $\Gamma(\mathbf{u})$
First, let us consider $\overline{\mathcal{S}} \colon \mathbf{u} \mapsto \Gamma(\mathbf{u})$. By the continuity of $\mathbf{u}$, we already know that the set $\Gamma(\mathbf{u}) \cap B_1^+$ is obviously closed in B_1^+ and it is quite straightforward to check the two hypotheses in $(\mathrm{F3})$.
Hence, in order to conclude the analysis, the only thing left to prove is that the integer d in (131) is equal to 1. Suppose by contradiction that $d = 2$, then this would imply the existence of $\mathbf{v} \in \mathcal{F}$ with $\mathcal{S}(\mathbf{v}) = \mathbb{R}^2$ i.e., $\mathbf{v} \equiv 0$ on Σ, which contradicts the definition of $\mathcal{G}^s$.
Actually, taking $V = \mathbb{R}^1 \times \{(0,0)\}$ and $\overline{\mathbf{v}} = (\overline{v}_1, \overline{v}_2, 0, \dots, 0)$ where

$$\overline{v}_1(x_1,x_2,y) = \left(\frac{\sqrt{x_1^2 + y^2} - x_1}{2} \right)^s \quad \text{and} \quad \overline{v}_2(x_1,x_2,y) = \left(\frac{\sqrt{x_1^2 + y^2} + x_1}{2} \right)^s,$$

we obtain the claimed estimate on d.

2. Dimensional estimate of the regular set $\mathcal{R}(\mathbf{u})$
Let us consider $\overline{\mathcal{S}}\colon \mathbf{u} \mapsto \mathcal{R}(\mathbf{u})$. By the inclusion in $\Gamma(\mathbf{u})$, we already know that

$$\dim_{\mathcal{H}}(\mathcal{R}(\mathbf{u}) \cap \partial^0 B_1^+) \leq 1.$$

Finally, we can apply the Reduction principle since (F3) is completely satisfied. More precisely, for $X_0 \in \partial^0 B_1^+, \rho > 0$ and $t > 0$ if $X \in \mathcal{R}(\rho \mathbf{u}_{X_0,t})$ then obviously $X_0 + tX \in \mathcal{R}(\mathbf{u})$, i.e. $N(X_0 + tX, \mathbf{u}, 0^+) = 1$. Secondly, given $\mathbf{u}_k, \overline{\mathbf{u}} \in \mathcal{F}$ as in (F3), suppose by contradiction that there exists a sequence $X_{\mathbf{i}} \in \partial^0 B_1^+$ and $\overline{\varepsilon} > 0$ such that

$$N(X_k, \mathbf{u}_k, 0^+) = s$$

and $\operatorname{dist}(X_k, \overline{\mathcal{S}}(\overline{\mathbf{u}})) \geq \overline{\varepsilon}$. Since, up to a subsequence, $X_k \to \overline{X}$, by the upper semi-continuity of the Almgren frequency formula, we already know that $N(\overline{X}, \overline{\mathbf{u}}, 0^+) \geq 1$. Moreover, up to a subsequence, $X_k \to \overline{X} \in \Gamma(\overline{\mathbf{u}}) \cap \Sigma \cap \overline{B_1}$ by the L^∞_{loc} convergence of $u_i \to \overline{u}$.

Now, since $\Gamma(\overline{\mathbf{u}})$ is a conical set, i.e. for every $\lambda > 0$ and $\overline{X} \in \Gamma(\overline{\mathbf{u}})$ we have $\lambda \overline{X} \in \Gamma(\overline{\mathbf{u}})$, we deduce that if we can prove $\overline{X} \in \mathcal{R}(\overline{\mathbf{u}}) \cap \overline{\partial^0 B_1^+}$ we provide a contradiction, more precisely we get $\operatorname{dist}(\overline{X}, \overline{\mathcal{S}}(\overline{\mathbf{u}}) \cap \partial^0 B_1^+) = 0$. Since there exists $\Omega \subset\subset \partial^0 B_1^+$ such that $(X_0 + r_i X_i)_i \subset \Omega$, if we consider

$$\begin{aligned} R_1 &= \min_{p \in \overline{\partial^0 \Omega^+}} \operatorname{dist}(p, S^1 \times \{0\}), \\ \overline{C} &= \sup_{p \in \overline{\partial^0 \Omega^+}} N(p, \mathbf{u}, R_1), \end{aligned}$$

we easily get from Corollary 1.2.7 that there exists $\overline{C} > 0$ and $\overline{r} > 0$ such that for $p \in \Omega \cap \mathcal{R}(u)$ and $r < \min\{\overline{r}, R_1\}$ we have

$$N(p, \mathbf{u}, r) \leq N(p, \mathbf{u}, R_1) \left(\frac{R_1}{r}\right)^{2+a-1+2\overline{C}} \leq \overline{C} \frac{1}{r^{2+a-1+2\overline{C}}}.$$

In particular, from the previous inequality we get that there exists $\overline{R} = \overline{R}(a) > 0$ sufficiently small, such that for $r < \overline{R}$ we have

$$s \leq N(X_k, \mathbf{u}_k, r) \leq s + \frac{\delta}{2}.$$

Since $\lim_k N(X_k, \mathbf{u}_k, r) = N(\overline{X}, \overline{\mathbf{u}}, r)$ for sufficiently small r, we directly obtain from Proposition 1.4.1 that $N(\overline{X}, \overline{\mathbf{u}}, 0^+) = s$, as we claimed.

As before, let us suppose now that there exist $\overline{\mathbf{v}} \in \mathcal{F}$ and a d-dimensional subspace $V \subset \Sigma$, with $d \leq 1$, and $k \geq 0$ such that

$$\overline{\mathbf{v}}_{Y,r} = r^k \overline{\mathbf{v}} \text{ for all } Y \in V, r > 0 \text{ and } \mathcal{R}(\varphi) \cap \partial^0 B_1^+ = V \cap \partial^0 B_1^+$$

Since $\overline{\mathbf{v}} \in \mathfrak{B}^s(\mathbb{R}^{2+1})$ is homogenous of degree k with respect to any $Y \in V = \mathcal{R}(\varphi)$, namely $N(Y, \overline{\mathbf{v}}, 0^+) = k$, we get that necessary $k = s$ and that $\mathcal{R}(\varphi)$ is d-dimensional. Since the only entire segregated profiles with degree s is, up to rotation, of the form $\overline{\mathbf{v}} = (\overline{v}_1, \overline{v}_2, 0, \dots, 0)$ where

$$\overline{v}_1(x_1, x_2, y) = \left(\frac{\sqrt{x_1^2 + y^2} - x_1}{2}\right)^s \quad \text{and} \quad \overline{v}_2(x_1, x_2, y) = \left(\frac{\sqrt{x_1^2 + y^2} + x_1}{2}\right)^s,$$

we get that $\mathcal{R}(\varphi)$ must be 1-dimensional, and consequently that

$$\dim_{\mathcal{H}}(\mathcal{R}(u) \cap \partial^0 B_1^+) = 1.$$

3. Dimensional estimate of the singular set $\mathcal{S}(\mathbf{u})$
Let us focus on the singular strata, namely given $\overline{\mathcal{S}}\colon \mathbf{u} \mapsto \mathcal{S}(\mathbf{u})$, the map satisfies the first part of (F3) thanks to (18), since for $X_0 \in \partial^0 B_1^+, \rho > 0$ and $t > 0$, if $X \in \overline{\mathcal{S}}(\rho \mathbf{u}_{X_0,t})$ we get

$$N(X, \rho \mathbf{u}_{X_0,t}, 0^+) > s \longleftrightarrow N(X_0 + tX, \mathbf{u}, 0^+) > s,$$

which is equivalent to $X_0 + tX \in \mathcal{S}(\mathbf{u})$. Now, given $\mathbf{u}_k = \rho_k \mathbf{u}_{X_0,r_k}, \overline{\mathbf{u}} \in \mathcal{F}$ as in (F3), suppose by contradiction that there exists a sequence $X_k \in \partial^0 B_1^+$ and $\overline{\varepsilon} > 0$ such that, up to a subsequence, $X_k \to \overline{X}$ and

$$N(X_k, \mathbf{u}_k, 0^+) \geq s + \delta \tag{50}$$

and $\operatorname{dist}(X_k, \overline{\mathcal{S}}(\overline{\mathbf{u}})) \geq \overline{\varepsilon}$. Then, following the same reasoning in Corollary 1.2.6, by Proposition 1.2.4 there exists a constant $C > 0$ and $\overline{r} > 0$ such that, for $r \in (0, \overline{r})$

$$N(X_k, \mathbf{u}, r) = e^{-C\Psi(r)} e^{C\Psi(r)} (N(X_k, \mathbf{u}, r)) - 1 > e^{-C\Psi(r)}(s + 1 + \delta) - 1$$

and hence, since for $r \in (0, \overline{r})$ it holds $N(X_k, \mathbf{u}_k, r) \to N(\overline{X}, \overline{\mathbf{u}}, r) \to$, we get $N(\overline{X}, \overline{\mathbf{u}}, 0^+) \geq s + \delta$, which implies a contradiction.

Since $\mathcal{S}(\mathbf{u}) \subseteq \Gamma(\mathbf{u})$, we already know that

$$\dim_{\mathcal{H}}(\mathcal{S}(\mathbf{u}) \cap \partial^0 B_1^+) \leq 1, \tag{51}$$

which is not the optimal bound for the singular set. Indeed, suppose $d = 1$, then must exist $\mathbf{v} \in \mathcal{F}$ homogeneous with respect to every point in $\mathbb{R} \times \{(0,0)\}$, i.e. there exists $k > 0$ such that

$$\mathbf{v}(Y + \lambda X) = \lambda^k \mathbf{v}(X) \text{ for all } Y \in \mathbb{R} \times \{(0,0)\},\ X \in \mathbb{R}^{2+1}_+,$$

such that $\mathcal{S}(\mathbf{v}) = \mathbb{R} \times \{(0,0)\}$. Hence, given $Y_0 \in \mathcal{S}(\mathbf{v})$, we get for every $\mu > 0$ that $\mathbf{v}(\mu Y_0 + X) = \mathbf{v}(X)$, which implies that $\mathbf{v} \in \mathfrak{B}^s(\mathbb{R}^{1+1}_+)$ with $N(0,\mathbf{v},0^+) = k > s$ and $\mathcal{S}(\mathbf{v}) = 0$. The absurd follows from the fact that necessary $N(0,\mathbf{v},0^+) = 2s$ and hence $\mathbf{v} \equiv 0$ on Σ, in contradiction with the definition of $\mathcal{F}$ and the upper bound (51). □

At this point, combining Corollary 1.3.12 and Corollary 1.5.2 we can state the following results about the flatness of the regular part $\mathcal{R}(\mathbf{u})$ of the nodal set $\Gamma(\mathbf{u})$. More precisely, the following result prove that $\mathcal{R}(\mathbf{u})$ verifies the so called vanishing Reifenberg flat condition, i.e. the (δ, R)-Reifenberg flat condition for every $\delta \in (0,1)$ and some $R = R(\delta) > 0$.

Proposition 1.5.4. *Given $s \in (0,1)$ and $n = 2$ consider $\mathbf{u} \in \mathcal{G}^s(B_1^+)$. Then, fixed $\Omega^+ \subset\subset B_1^+$, for any given $\delta \in (0,1)$ there exists $R > 0$ such that $X \in \mathcal{R}(\mathbf{u}) \cap \partial^0\Omega^+$ and $0 < r < R$ there exists an hyper-plane $H = H_{X,r}$ passing through X such that*

$$d_{\mathcal{H}}(\Gamma(\mathbf{u}) \cap \partial^0 B_r^+(X), H_{X,r} \cap \partial^0 B_r^+(X)) \leq \delta r,$$

where $\mathrm{d}_{\mathcal{H}}$ is the Hausdorff distance defined in (33).

The idea of the proof is similar to the one in Corollary 1.3.12 since, roughly speaking, they are both a consequence of Proposition 1.3.11. Indeed, by Theorem 1.3.3 we already know the topology in which the blow-up sequence converges and Proposition 1.3.11 ensure the existence of an homogeneous blow-up limit for a specific choices of X_k, which are the ones considered in Proposition 1.3.12 and in this result.

Proof of Proposition 1.5.4. Arguing by contradiction, let us suppose there exists $\overline{\delta} > 0$ and a sequence $(X_k)_k \subset \mathcal{R}(\mathbf{u}) \cap \partial^0\Omega^+, r_k \to 0$ such that

$$\mathrm{d}_{\mathcal{H}}(\Gamma(\mathbf{u}) \cap \partial^0 B_{r_k}^+(X_k), H \cap \partial^0 B_{r_k}^+(X_k)) > \overline{\delta} r_k.$$

for every hyper-plane H passing through X_k. If we consider the blow-up sequence centered in $(X_k)_k$ associated to r_k, i.e. let

$$\mathbf{u}_k(X) = \frac{\mathbf{u}(X_k + r_k X)}{\sqrt{H(X_k, \mathbf{u}, r_k)}},$$

then "rescaling" the contradiction statement we get

$$\mathrm{d}_{\mathcal{H}}(\Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+, H \cap \partial^0 B_1^+) > \overline{\delta},$$

whenever H is a hyper-plane that passes through the origin.
Hence, since up to a subsequence $X_k \to \overline{X} \in \Gamma(\mathbf{u}) \in \partial^0\Omega^+$, Theorem 1.3.3 together with

property Proposition 1.3.11 implies the existence of a blow-up limit $\overline{\mathbf{u}}$ whose nodal set $\Gamma(\overline{\mathbf{u}})$ is a hyper-plane containing the origin. Hence we obtain a contradiction once we are able to prove that

$$d_{\mathcal{H}}(\Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+, \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+) \to 0.$$

Equivalently, the claimed result is to prove that for every $\varepsilon > 0$ there exists $\overline{k} > 0$ such that for every $k \geq \overline{k}$

$$\Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+ \subseteq N_\varepsilon \left(\Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+\right)$$
$$\Gamma(\overline{\mathbf{u}}) \cap \partial^0 B \subseteq N_\varepsilon \left(\Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+\right).$$

Supposing by contradiction that the first inclusion is not true, then there exist $\overline{\varepsilon} > 0$ and a sequence $X_k \in \Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+$ such that dist $(X_k, \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+) > \overline{\varepsilon}$. Moreover, up to a subsequence, $X_k \to \overline{X} \in \Gamma(\overline{\mathbf{u}}) \cap \overline{\partial^0 B_1^+}$ by the L^∞_{loc} convergence of $\mathbf{u}_k \to \overline{\mathbf{u}}$. Since by Proposition 1.3.11 the nodal set $\Gamma(\overline{\mathbf{u}})$ is a conical set, i.e. for every $\lambda > 0$ and $\overline{X} \in \Gamma(\overline{\mathbf{u}})$ we have $\lambda\overline{X} \in \Gamma(\overline{\mathbf{u}})$ and $0 \in \Gamma(\overline{\mathbf{u}})$, we deduce that $\mathrm{dist}(\overline{X}, \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+) = 0$, which provides the contradiction.

Finally, we have to prove that for every $\varepsilon > 0$ there exists $\bar{k} > 0$ such that

$$\Gamma(\overline{\mathbf{u}}) \cap \partial^0 B \subseteq N_\varepsilon \left(\Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+\right) \quad \text{for every } k \geq \bar{k}.$$

We start by proving that given $\overline{X} \in \Gamma(\overline{\mathbf{u}}) \in \partial^0 B_1^+$ and $\delta > 0$, the vector valued function $\mathbf{u}_k$ must have a zero in $\partial^0 B_\delta^+(\overline{X})$, for k sufficiently large. If not, by recalling that $u_{i,k} \cdot u_{j,k}|_{y=0} \equiv 0$ for every $i \neq j$, we would have that there exists an index $0 < i < h$ such that

$$\begin{cases} -L_a u_{i,k} = 0 & \text{in } B_\delta^+(\overline{X}) \\ -\partial_y^a u_{i,k} = f_i(u_{i,k}) & \text{on } \partial^0 B_\delta^+(\overline{X}) \end{cases}, \quad u_{i,k} > 0 \text{ on } \partial^0 B_\delta^+(\overline{X})$$

and $u_{j,k} \equiv 0$ in $\partial^0 B_\delta^+(\overline{X})$, for every $j \neq i$. Passing to the limit, this would imply that

$$\begin{cases} -L_a \overline{u}_i = 0 & \text{in } B_\delta^+(\overline{X}) \\ -\partial_y^a \overline{u}_i = 0 & \text{on } \partial^0 B_\delta^+(\overline{X}) \end{cases}, \quad \overline{u}_i \geq 0 \text{ on } \partial^0 B_\delta^+(\overline{X})$$

and $\overline{u}_j \equiv 0$ on $\partial^0 B_\delta^+(\overline{X})$ for every $j \neq i$. Since $\overline{X} \in \Gamma(\overline{\mathbf{u}})$ it follows from the Hopf principle (see [13, 43]) that $\overline{\mathbf{u}} \equiv 0$ in $\partial^0 B_\delta^+(\overline{X})$, a contradiction with the fact the $\Gamma(\overline{\mathbf{u}})$ has empty interior.

Now, arguing by contradiction, suppose the existence of $\overline{\varepsilon} > 0$ and $(X_k)_k \in \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+$ such that $X_k \to \overline{X} \in \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+$ and $\mathrm{dist}(X_k, \Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+) > \overline{\varepsilon}$. Since $\Gamma(\overline{\mathbf{u}})$ is a conical set passing through the origin, let us take $\widetilde{X} \in \Gamma(\overline{\mathbf{u}}) \cap \partial^0 B_1^+$ such that $\left|\widetilde{X} - \overline{X}\right| \leq \overline{\varepsilon}/4$.

Furthermore, we can take, by using the result proved in the previous paragraph, a sequence $(\overline{X}_k)_k \in \Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+$ such that $\left|\overline{X}_k - \widetilde{X}\right| \leq \bar{\varepsilon}/4$ for sufficiently large k. The final contradiction follows noticing that

$$\mathrm{dist}(X_k, \Gamma(\mathbf{u}_k) \cap \partial^0 B_1^+) \leq \left|X_k - \overline{X}_k\right| \leq \left|X_k - \overline{X}\right| + \left|\overline{X} - \widetilde{X}\right| + \left|\widetilde{X} - \overline{X}_k\right| \leq \frac{3\bar{\varepsilon}}{4} \leq \bar{\varepsilon},$$

for sufficiently large k. □

With the vanishing Reifenberg property we are able to prove a local separation result. We remark that the following result follows the idea of Proposition 5.4 in [81].

Proposition 1.5.5. *Given $s \in (0,1)$ and $n = 2$ consider $\mathbf{u} \in \mathcal{G}^s(B_1^+)$. Then, given $X_0 \in \Gamma^*$ there exists a radius $R_0 > 0$ such that $\partial^0 B_{R_0}^+(X_0) \cap \mathcal{R}(\mathbf{u}) = \partial^0 B_{R_0}^+(X_0) \cap \Gamma(\mathbf{u})$ and $\partial^0 B_{R_0}^+(X_0) \setminus \Gamma(\mathbf{u}) = \partial^0 B_{R_0}^+(X_0) \cap \{\mathbf{u} > 0\}$ has exactly two connected components, i.e.*

$$\partial^0 B_{R_0}^+(X_0) \cap \mathcal{R}(\mathbf{u}) = \Omega_+ \cup \Omega_-.$$

More precisely, there exists $\delta > 0$ such that, given $Y \in \Gamma(\mathbf{u}) \in \partial^0 B_R^+(X_0)$ and $r \in (0, R_0 - |Y - X_0|)$ there exists a hyperplane $H_{Y,r}$ and a vector $\nu_{Y,r} \in S^{n-1}$ orthogonal to $H_{Y,r}$ such that

$$\{X \pm t\nu_{Y,r} \in \partial^0 B_r^+(Y) : X \in H_{Y,r}, t \geq \delta r\} \subset \Omega_\pm.$$

Proof. Fixed $X_0 \in \mathcal{R}(\mathbf{u})$, since $\mathcal{R}(\mathbf{u})$ is a relatively open set in $\Gamma(\mathbf{u})$, there exists τ such that $\partial^0 B_{2\tau}^+(X_0) \cap \mathcal{R}(\mathbf{u}) = \partial^0 B_{2\tau}^+(X_0) \cap \Gamma(\mathbf{u})$ and fix $\delta < 1/6$. Using the notations of Proposition 1.5.4, for $\Omega^+ = B_\tau^+(X_0)$ there exists $R > 0$ such that $\Gamma(\mathbf{u}) \cap \partial^0 B_\tau^+(X_0)$ satisfies the (δ, R)-Reifenberg flatness condition. Let us prove our result with $R_0 := \min\{R, \tau\}$.
By Proposition 1.5.4, there exists a hyperplane H_{X_0,R_0} containing X_0 and such that

$$d_{\mathcal{H}}(\Gamma(\mathbf{u}) \cap \partial^0 B_{R_0}^+(X_0), H_{X_0,R_0} \cap \partial^0 B_{R_0}^+(X_0)) \leq \delta R_0. \tag{52}$$

Hence the subset $\partial^0 B_{R_0}^+(X_0) \setminus N_{2\delta R_0}(H_{X_0,R_0})$ has exactly two connected components, namely D_1 and D_2, which do not intersect the nodal set $\Gamma(\mathbf{u})$. Hence, let us define the function

$$\sigma(X) = \begin{cases} 1 & \text{if } X \in D_1, \\ -1 & \text{if } X \in D_2. \end{cases}$$

Take now a point $X_1 \in \Gamma(\mathbf{u}) \cap \partial^0 B_{R_0}^+(X_0) \subseteq N_{\delta R_0}(H_{X_0,R_0}) \cap \partial^0 B_{R_0}^+(X_0)$. As we did before, by using Proposition 1.5.4, considering a ball of radius $R_0/2$ centered at X_1 there exists a hyperplane $H_{X_1,R_0/2}$ such that

$$d_{\mathcal{H}}(\Gamma(\mathbf{u}) \cap \partial^0 B_{R_0/2}^+(X_1), H_{X_1,R_0/2} \cap \partial^0 B_{R_0/2}^+(X_1)) \leq \delta \frac{R_0}{2}.$$

This inequality combined with (52) yields that

$$N_{\delta R_0/2}(H_{X_1,R_0/2}) \cap \partial^0 B^+_{R_0/2}(X_1) \cap \partial^0 B^+_{R_0}(X_0) \subseteq N_{2\delta R_0}(H_{X_0,R_0}) \cap \partial^0 B^+_{R_0}(X_0).$$

Hence $\partial^0 B^+_{R_0}(X_0) \cap \partial^0 B^+_{R_0/2}(X_1) \setminus N_{\delta R_0}(H_{X_1,R_0/2})$ has exactly two connected components, each one intersecting D_1 or D_2 and not both. Thus the set

$$\left(\bigcup_{X_1 \in \Gamma(\mathbf{u}) \cap \partial^0 B^+_{R_0}(X_0)} \partial^0 B^+_{R_0}(X_0) \cap \partial^0 B^+_{R_0/2}(X_1) \setminus N_{\delta R_0}(H_{X_1,R_0/2}) \right) \cup D_1 \cup D_2 \tag{53}$$

has exactly two connected components which do not interest the nodal set $\Gamma(\mathbf{u})$ and hence we can continuously extend σ to this set.
Iterating this argument to a ball of radius $R_0/2^k$ centered at a point of $\Gamma(\mathbf{u})$ we find two connected and disjoint set Ω_+, Ω_- such that $\partial^0 B^+_{R_0}(X_0) \setminus \Gamma(\mathbf{u}) = \Omega_+ \cup \Omega_-$, with $D_1 \subseteq \Omega_+$ and $D_2 \subseteq \Omega_-$. Furthermore, the map $\sigma\colon \partial^0 B^+_1 \setminus \Gamma(\mathbf{u}) \to \{+1, -1\}$ such that

$$\sigma\colon X \mapsto \chi_{\Omega_+}(X) - \chi_{\Omega_-}(X)$$

is continuous and thus $\partial^0 B^+_{R_0}(X_0) \setminus \Gamma(\mathbf{u})$ has exactly two connected components.
In order to check the continuity, take $X \in \partial^0 B^+_{R_0}(X_0)$ such that $\mathrm{dist}(X, \Gamma(\mathbf{u}) \cap \partial^0 B^+_{R_0}(X_0)) = \gamma > 0$, with $\overline{X} \in \Gamma(\mathbf{u}) \cap \partial^0 B^+_{R_0}(X_0)$ the point of minimum distance, and k so large that $R_0/2^{k+1} \leq \gamma < R_0/2^k$; then $X \in B_{R_0/2^k}(\overline{X}) \setminus N_{\delta R_0/2^k}(H_{\overline{X},R_0/2^{k-1}})$, due to the choice of δ, and hence σ is constant in a small neighborhood of X. □

1.6 SINGULAR LIMIT OF A COMPETITION-DIFFUSION PROBLEM

In this last Section we consider the class of segregated profiles arising from a competition-diffusion problem with nonlocal diffusion and variational competition. In particular, we motivate our definition of the class $\mathcal{G}^s(B^+_1)$ and we give some result in the context of self-segregation, comparing them with the ones known for the local case.

As we mentioned in the introduction, in the papers [83, 84] the authors proved that given $\beta > 0$ and $(f_{i,\beta})_\beta$ a collection of continuous functions uniformly bounded with respect to β on bounded sets, the sequence $(\mathbf{u}_\beta)_\beta \in H^{1,a}(B^+_1; \mathbb{R}^h)$ of solutions $\mathbf{u}_\beta = (u_{1,\beta}, \dots, u_{h,\beta})$ of the problems

$$\begin{cases} -L_a u_{i,\beta} = 0 & \text{in } \ B^+_1 \\ -\partial^a_y u_{i,\beta} = f_{i,\beta}(u_{i,\beta}) - \beta u_{i,\beta} \sum_{j \neq i} a_{ij} u^2_{j,\beta} & \text{on } \ \partial^0 B^+_1. \end{cases}$$

uniformly bounded in $L^\infty(B_1^+)$ with respect to β, does converge uniformly on compact sets and strongly in $H^{1,a}(K\cap B_1^+)$, for every $K\subset B_1$, to a vector valued function $\mathbf{u}=(u_1,\dots,u_h)\in H^{1,a}_{\mathrm{loc}}(\overline{B_1^+};\mathbb{R}^h)$ whose components are all non negative, continuous functions such that

(1) $\mathbf{u}\in H^{1,a}(K\cap B_1^+)\cap C^{0,\alpha}(\overline{K\cap B_1^+})$, for every compact set $K\subset B$ and every $\alpha\in(0,\alpha^*)$;

(2) $u_i\cdot u_j|_{y=0}\equiv 0$ for every $i\neq j$ and $\mathbf{u}\neq \mathbf{0}$ on $B_1\cap\Sigma$. Moreover, for $i=1,\dots,h$ it satisfies

$$\begin{cases}-L_a u_i=0 & \text{in } B_1^+\\ u_i\left(\partial^a_y u_i+f_i(u_i)\right)=0 & \text{on } \partial^0 B_1^+\end{cases}$$

where $f_i\colon \mathbb{R}^+\to\mathbb{R}$ are the nonnegative $\mathcal{C}^1$ limits of $f_{i,\beta}$, such that $f_i(s)=O(s)$ for $s\to 0$;

(3) for every $X_0=(x_0,0)\in\partial^0 B_1^+$ and $r\in(0,\mathrm{dist}(X_0,\partial B))$, the following Pohožaev type identity holds

$$\begin{aligned}(1-a-n)&\int_{B_r^+(X_0)}|y|^a|\nabla\mathbf{u}|^2\,\mathrm{d}X+r\int_{\partial B_r^+(X_0)}|y|^a|\nabla\mathbf{u}|^2\,\mathrm{d}\sigma+\\ &+2n\int_{\partial^0B_r^+(X_0)}\sum_{i=1}^h F_i(u_i)\mathrm{d}x-2r\int_{S_r^{n-1}(X_0)}\sum_{i=1}^h F_i(u_i)\mathrm{d}x=2r\int_{\partial^+B_r^+(X_0)}|y|^a(\partial_r\mathbf{u})^2\mathrm{d}\sigma\end{aligned}$$

where $\mathbf{F}(s)=(F_1(s),\dots,F_h(s))$ with $F_i(s)=\int_0^s f_i(t)\,\mathrm{d}t$ for every $i=1,\dots,h$.

In particular, they proved that for $s=1/2$ the limit profile are $C^{0,1/2}_{\mathrm{loc}}(B_1^+)$ while in general they estimate the solutions in the Hölder spaces by the use of a fractional versions of the Alt-Caffarelli-Friedman and Almgren monotonicity formula. More precisely, let us recall the fractional version of the spectral problem beyond the Alt-Caffarelli-Friedman formula used in [84, 83]. Consider the set of 2-partitions of S^{n-1} as

$$\mathcal{P}^2:=\left\{(\omega_1,\omega_2):\omega_i\subseteq S^{n-1}\text{ open}, \omega_1\cap\omega_2=\emptyset,\ \overline{\omega_1}\cup\overline{\omega_2}=S^{n-1}\right\},$$

the optimal partition value ν_s^{ACF} is defined as

$$\nu_s^{ACF}:=\frac{1}{2}\inf_{(\omega_1,\omega_2)\in\mathcal{P}^2}\sum_{i=1}^2\gamma_s(\lambda_1^s(\omega_i)). \tag{54}$$

It is easy to see, by a Schwarz symmetrization argument, that ν_s^{ACF} is achieved by a pair of complementary spherical caps $(\omega_\theta,\omega_{\pi-\theta})\in\mathcal{P}^2$ with aperture 2θ and $\theta\in(0,\pi)$ (for a detailed proof of this kind of symmetrization we refer to [85]), that is:

$$\nu_s^{ACF}=\min_{\theta\in[0,\pi]}\Gamma^s(\theta)=\min_{\theta\in[0,\pi]}\frac{\gamma_s(\theta)+\gamma_s(\pi-\theta)}{2}.$$

Then, for $s \in (0,1)$ the previous statement for the limiting profile holds true for

$$\alpha^* = \begin{cases} \nu_s^{ACF}, & 0 < s \leq \frac{1}{2}, \\ \min\{\nu_s^{ACF}, 2s-1\}, & \frac{1}{2} < s < 1. \end{cases}$$

As we mentioned, the threshold $s = 1/2$ is due to the presence of the phenomenon of self-segregation of nonlocal problem with $s \in (1/2,1)$, namely centered on the nodal set $\Gamma(\mathbf{u})$ there exists a ball $\widetilde{B}^+ \subset \mathbb{R}^{n+1}_+$ sufficiently small and an index $i = 1,\dots,h$ such that all the components u_j of $\mathbf{u}$ with $j \neq i$ are identically zero on the ball make exception of u_i which is not identically zero and such that

$$\partial^0 \widetilde{B}^+ \setminus \Gamma(\mathbf{u}) = \{X \in \partial^0 \widetilde{B}^+ \colon u_i(X) > 0\}.$$

The presence of self-segregation in the context of competition-diffusion problem is a phenomenon well known in the literature, even in the local case. Indeed, in [81] the authors dealt with the case of self-segregated profile relabeling the restrictions of the profile on each connected component of the positive set. Since in that case the operator is local, the restriction itself satisfies the assumption of the segregated profile, but unfortunately the nonlocal attitude of our operator does not allow this strategy. Moreover, we mention the work [35], where the authors proved that the self-segregation is a phenomenon that does not appear as singular limit of the local counterpart of our competition-diffusion problem (3).
In the context of the fractional Laplacian $(-\Delta)^s$, as pointed out in [84], the main point is that the fundamental solution turns out to be bounded near 0 and in $H^{1,a}(B_1^+)$ whenever $s > 1/2, n = 1$. This implies that, when $s \in [1/2,1)$ and $n \geq 2$, the function

$$u(x,y) = \left(x_1^2 + y^2\right)^{\frac{2s-1}{2}}$$

is a positive L_a-harmonic function in $\mathbb{R}^{n+1}_+$ with non trivial trace on Σ and $\partial^a_y u = 0$ on Σ. In particular, its trace on Σ is self-segregated since it has two disconnected positivity regions.

The following result is a refinement of the Liouville theorem [83, Theorem 7.1] for every $s \in (0,1)$ and it is based on the division of the nodal set in its segregated and self-segregated part, in such a way we can apply our result obtained in the previous Sections to the segregated part of $\Gamma(\mathbf{u})$.

Definition 1.6.1. Let $s \in (0,1)$ and $u \in \mathcal{G}^s(B_1)$. Then the nodal set $\Gamma(\mathbf{u})$ is said to be either "segregated" in $K \cap \Sigma$ if

$$\Gamma(\mathbf{u}) = \partial^0 K^+ \setminus \bigcup_{i=1}^h \mathrm{int}\left(\overline{\{u_i > 0\}}\right)$$

or "self-segregated" in $K \cap \Sigma$ if

$$\Gamma(\mathbf{u}) = \bigcup_{i=1}^h \partial\{u_i > 0\} \setminus \mathrm{int}\left(\overline{\{u_i > 0\}}\right).$$

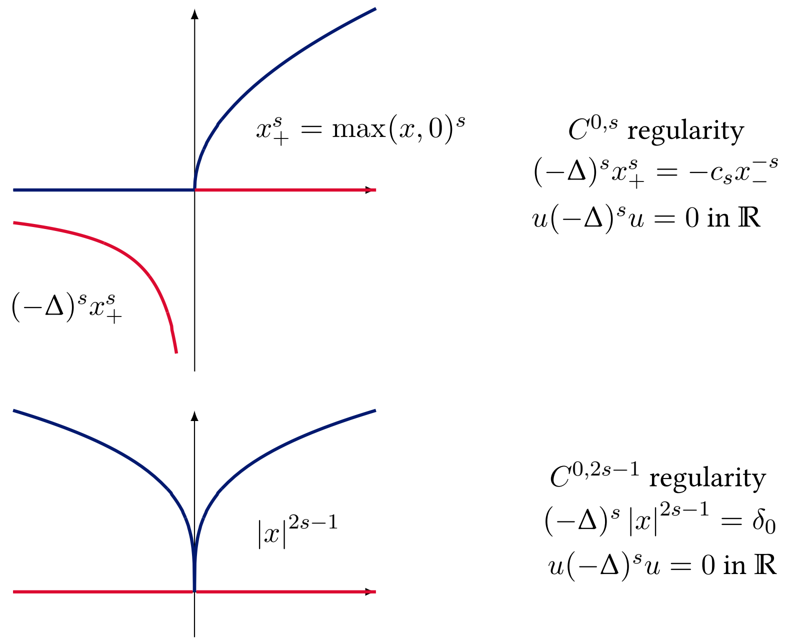


Figure 3: One-dimensional configurations in $\mathcal{G}^s_{\text{loc}}(\mathbb{R})$ respectively segregated and self-segregated

In general, the nodal set is said to be segregated (self-segregated) in Σ if it is segregated (self-segregated) on every compact set $K \cap \Sigma$.

Theorem 1.6.2. *Let $s \in (0,1)$. If*

- *either $\Gamma(\mathbf{u})$ is segregated in Σ and $\mathbf{u} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1}_+) \cap C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ for $\alpha \in (0,s)$,*
- *or $\Gamma(\mathbf{u})$ is self-segregated in Σ and $\mathbf{u} \in \mathcal{G}^s_{\text{loc}}(\mathbb{R}^{n+1}_+) \cap C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ for $\alpha \in (0,2s-1)$,*

then $\mathbf{u}$ is constant.

In the remaining part of the Chapter we prove this result following the procedure in [83]. This result implies a refinement in theory of the regularity of segregated profiles near the nodal set.

Corollary 1.6.3. *Let $s \in (0,1)$ and $\mathbf{u} \in \mathcal{G}^s(B_1^+)$ be limit of a sequence $(\mathbf{u}_\beta)_\beta$ of solutions of (3). Then $\mathbf{u} \in C^{0,\alpha}_{\text{loc}}(\overline{B_1^+})$ for every $\alpha \in (0,s)$ if and only if $\Gamma(\mathbf{u})$ is segregated in $B_1 \cap \Sigma$.*

The proof of this Corollary is based on our Liouville type theorem and on the techniques developed in the mentioned works [83, 84].

Proof of Theorem 1.6.2. Since by assumption $\mathbf{u} \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1}_+) \cap C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$, we easily get by [84, Corollary 2.12] that $\mathbf{u}$ is homogeneous of degree γ with respect to any of its possible zeros. Thus, for $\alpha > 0$ and for every dimension n, as in [83], let us introduce the critical value

$$\nu_s^{\mathrm{Liouv}}(n) = \inf\left\{\alpha > 0\colon \mathfrak{B}^s(\mathbb{R}^{n+1}) \cap C^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}) \text{ is non empty}\right\}.$$

Since for every $s \in (0,1)$ and $n \geq 1$ the function $(y^{2s}, 0, \cdot, 0) \in \mathfrak{B}^s(\mathbb{R}^{n+1}) \cap C^{0,2s}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$, we get $\nu_s^{\mathrm{Liouv}} \leq 2s$. However, since we need to take care of the structure of the nodal set we introduce the following critical value

$$\nu_s^1(n) = \inf\left\{\alpha > 0 \,\middle|\, \begin{array}{l} \mathfrak{B}^s(\mathbb{R}^{n+1}) \cap C^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}) \text{ is non empty} \\ \Gamma(\mathbf{u}) \text{ is segregated in } \Sigma \end{array}\right\}$$

such that $\nu_s^1(n) \leq \nu_s^{\mathrm{Liouv}}(n)$. The main idea is to reduce such problem to the ones of estimating $\nu_s^{\mathrm{Liouv}}(1)$ and $\nu_1^1(1)$, which can be computed explicitly: let us prove that for any dimension $n \geq 2$ it holds

$$\nu_s^{\mathrm{Liouv}}(n) \geq \nu_s^{\mathrm{Liouv}}(n-1).$$

Since $\mathbf{u} \in \mathfrak{B}^s(\mathbb{R}^{n+1})$, as we previously remarked, we have that $\Gamma(\mathbf{u})$ is a cone with vertex at the origin and $N(0, \mathbf{u}, r) = \alpha$, for every $r \in (0, +\infty)$.

We can easily exclude the case $\Gamma(\mathbf{u}) = \Sigma$, since in that case all the components of $\mathbf{u}$ have trivial trace on Σ. As a consequence, the odd extension of $\mathbf{u}$ through Σ is a nontrivial vector of harmonic functions on $\mathbb{R}^{n+1}$, forcing $\alpha \geq 2s \geq \nu_s^{\mathrm{Liouv}}(n-1)$.

Similarly, since $n \geq 2$ and for $\dim \Gamma(\mathbf{u}) \leq n - 2s$ we obtain that $\Gamma(\mathbf{u})$ has null L_a-capacity, we can exclude the case $\Gamma(\mathbf{u}) = \{0\}$.

Now, given $X_0 \in \Gamma(\mathbf{u}) \cap S^{n-1}$, let us introduce the blow-up sequence of $\mathbf{u}$ associated to $r_k > 0$ as

$$\mathbf{u}_k(X) = \frac{\mathbf{u}(X_0 + r_k X)}{\rho_k} \quad \text{with } \rho_k^2 = H(X_0, \mathbf{u}, r_k).$$

Now, following the same ideas in the proof of Lemma 1.3.5, given $C > 0$ and $\overline{r} > 0$ be such that Proposition 1.2.4, Proposition 1.2.7 and Lemma 1.3.4 hold true then, up to taking k so large that $r_k, r_k R \leq \overline{r}$, we get for every $R > 0$ that

$$\begin{aligned}
\int_{\partial^+ B^+_R} |y|^a\, \mathbf{u}_k^2 \mathrm{d}\sigma &= \frac{1}{\rho_k^2 r_k^{n+a}} \int_{\partial^+ B^+_{Rr_k}(X_0)} |y|^a\, \mathbf{u}_k^2 \mathrm{d}\sigma \\
&= R^{n+a} \frac{H(X_0, \mathbf{u}, Rr_k)}{H(X_0, \mathbf{u}, r_k)} \\
&\leq R^{n+a} \left(\frac{Rr_k}{r_k}\right)^{2\overline{C}},
\end{aligned}$$

which implies, by the L_a-subharmonicity of the odd extension of $\mathbf{u}_k$ through Σ, that

$$\sup_{B^+_{R/2}} \mathbf{u}_k \leq 2C(n,s)\left(\frac{1}{R^{n+1+a}}\int_{B^+_R}|y|^a\,\mathbf{u}_k^2 \mathrm{d}X\right)^{1/2} \leq 2C(n,s)\left(\frac{H(0,\mathbf{u}_k,R)}{n+a+1}\right)^{1/2}$$

is uniformly bounded for $k>0$. By Proposition 1.3.8, there exists a blow-up limit $\overline{\mathbf{u}} \in \mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$ with $\|\overline{\mathbf{u}}\|_{L^{2,a}(\partial^+ B^+_1)} = 1$.

Since the blow-up sequence $(\mathbf{u}_k)_k$ is constructed starting from a family of homogeneous entire segregated profiles in $\mathfrak{B}^s_{\mathrm{loc}}(\mathbb{R}^{n+1})$, we can prove that $\overline{\mathbf{u}}$ is constant along the direction parallel to $X_0 \in S^{n-1}$ and that its restriction on the orthogonal half plane belongs to $\mathcal{G}^s_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, since the blow-up sequence $(\mathbf{u}_k)_k$ is centered in a fixed point X_0, by Proposition 1.2.4, Corollary 1.2.6 and Proposition 1.3.11 we get that $\overline{\mathbf{u}} \in \mathfrak{B}^s(\mathbb{R}^n) \cap C^{0,\gamma}_{\mathrm{loc}}(\mathbb{R}^{n+1})$, with $N(0,\overline{\mathbf{u}},1) = \gamma \leq \alpha$. Finally, since α is arbitrary choose in $\nu_s^{\mathrm{Liouv}}(n)$, the thesis follows from the bound $\nu_s^{\mathrm{Liouv}}(n-1) \leq \alpha$.
If instead, we consider the critical values $\nu^1_s(n)$, the result of dimensional descent still holds since the uniform convergence on compact sets of the blow-up sequence $(\mathbf{u}_k)_k$ ensures that the nodal set $\Gamma(\overline{\mathbf{u}})$ is still segregated in Σ, and hence

$$\nu^1_s(n) \geq \nu^1_s(n-1),$$

for every $s \in (0,1)$ and $n \geq 2$. Thus, by a complete classification of the elements in $\mathfrak{B}^s(\mathbb{R}^2_+)$, see [88, Section 5], we can finally reach the claimed result since

$$\nu_s^{\mathrm{Liouv}}(1) = \begin{cases} s, & \text{if } \left(0,\frac{1}{2}\right) \\ 2s-1, & \text{if } s \in \left[\frac{1}{2},1\right) \end{cases} \quad \text{and} \quad \nu^1_s(1) = s \text{ for any } s \in (0,1).$$

□

2

ON S-HARMONIC FUNCTIONS ON CONES

2.1 INTRODUCTION

Let $n \geq 2$ and C be an open cone in $\mathbb{R}^n$ with vertex in 0, for a given $s \in (0,1)$, we consider the problem of the classification of nontrivial functions which are s-harmonic inside the cone and vanish identically outside, that is:

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in} \quad C, \\ u_s \geq 0 & \text{in} \quad \mathbb{R}^n \\ u_s \equiv 0 & \text{in} \quad \mathbb{R}^n \setminus C. \end{cases} \tag{55}$$

By [5, Theorem 3.2], it is known that there exists a homogeneous, nonnegative and nontrivial solution to (55) of the form

$$u_s(x) = |x|^{\gamma_s} u_s\left(\frac{x}{|x|}\right),$$

where $\gamma_s := \gamma_s(C)$ is a definite homogeneity degree (characteristic exponent), which depends on the cone. Moreover, such a solution is continuous in $\mathbb{R}^n$ and unique, up to multiplicative constants. We can normalize it in such a way that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. We consider the case when s approaches 1, wondering whether solutions of the problem do converge to a harmonic function in the same cone and, in case, which are the suitable spaces for convergence.

Our problem (55) can be linked to a specific spectral problem of local nature in the upper half sphere by using the extension technique popularized in [23] by Caffarelli and Silvestre, which characterize the fractional Laplacian in $\mathbb{R}^n$ as the Dirichlet-to-Neumann map for a variable v depending on one more space dimension.

Hence, let us consider an open region $\omega \subseteq S^{n-1} = \partial S^n_+$, with $S^n_+ = S^n \cap \{y > 0\}$, and define the eigenvalue

$$\lambda_1^s(\omega) = \inf\left\{ \frac{\displaystyle\int_{S^n_+} y^{1-2s}|\nabla_{S^n} u|^2 \mathrm{d}\sigma}{\displaystyle\int_{S^n_+} y^{1-2s} u^2 \mathrm{d}\sigma} \ : \ u \in H^1(S^n_+; y^{1-2s}\mathrm{d}\sigma) \setminus \{0\} \text{ and } u \equiv 0 \text{ in } S^{n-1} \setminus \omega \right\}.$$

Next, define the *characteristic exponent* of the cone C_ω spanned by ω (see Definition 2.2.1) as

$$\gamma_s(C_\omega) = \gamma_s(\lambda_1^s(\omega)) \, , \tag{56}$$

where the function $\gamma_s(t)$ is defined by

$$\gamma_s(t) := \sqrt{\left(\frac{n-2s}{2}\right)^2 + t} - \frac{n-2s}{2} \, .$$

We recall the existence of a remarkable link between the nonnegative $\lambda_1^s(\omega)$-eigenfunctions and the $\gamma_s(\lambda_1^s(\omega))$-homogeneous L_a-harmonic functions. As pointed out in Proposition 1.3.14, given φ_s the first nonnegative eigenfunction to $\lambda_1^s(\omega)$ and v_s its $\gamma_s(\lambda_1^s(\omega))$-homogeneous extension to $\mathbb{R}^{n+1}_+$, i.e.

$$v_s(r,\theta) = r^{\gamma_s(\lambda_1^s(\omega))}\varphi_s(\theta),$$

we easily get that v_s is L_a-harmonic in the upper half-space, with $a = 1-2s \in (-1,1)$. Moreover its trace $u_s(x) = v_s(x,0)$ is s-harmonic in the cone C_ω spanned by ω, vanishing identically outside: in other words u_s is a solution of our problem (55). In a symmetric way, for the standard Laplacian, we consider the problem of γ-homogeneous functions which are harmonic inside the cone spanned by ω and vanish outside:

$$\begin{cases} -\Delta u_1 = 0 & \text{in} \quad C_\omega, \\ u_1 \geq 0 & \text{in} \quad \mathbb{R}^n \\ u_1 = 0 & \text{in} \quad \mathbb{R}^n \setminus C_\omega. \end{cases} \tag{57}$$

Is is well known that the associated eigenvalue problem on the sphere is that of the Laplace-Beltrami operator with Dirichlet boundary conditions:

$$\lambda_1(\omega) = \inf\left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \mathrm{d}\sigma}{\displaystyle\int_{S^{n-1}} u^2 \mathrm{d}\sigma} \ : \ u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus \omega \right\},$$

and the *characteristic exponent* of the cone C_ω is

$$\gamma(C_\omega) = \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(\omega)} - \frac{n-2}{2} = \gamma_{s|s=1}(\lambda_1(\omega)) \ . \tag{58}$$

In the classical case, the characteristic exponent enjoys a number of nice properties: it is minimal on spherical caps among sets having a given measure. Moreover for the spherical caps, the eigenvalues enjoy a fundamental convexity property with respect to the colatitude θ (see the results in [3, 47]). We remark that the convexity plays a major role in the proof of the Alt-Caffarelli-Friedman monotonicity formula, a key tool in the Free boundary theory (see [21] for a general excursus on the subject).

Since the standard Laplacian can be viewed as the limiting operator of the family $(-\Delta)^s$ as $s \nearrow 1$, some questions naturally arise:

Problem 2.1.1. Is it true that

(1) $\lim_{s\to 1} \gamma_s(C) = \gamma(C)$?

(2) $\lim_{s\to 1} u_s = u_1$ uniformly on compact sets, or better, in Hölder local norms?

(3) for spherical caps of opening θ is there any convexity of the map $\theta \mapsto \lambda_1^s(\theta)$ at least, for s near 1?

We therefore addressed the problem of the asymptotic behavior of the solutions of problem (55) for $s \nearrow 1$, obtaining a rather unexpected result: our analysis shows high sensitivity to the opening solid angle ω of the cone C_ω, as evaluated by the value of $\gamma(C)$. In the case of wide cones, when $\gamma(C) < 2$ (that is, $\theta \in (\pi/4, \pi)$ for spherical caps of colatitude θ), our solutions do converge to the harmonic homogeneous function of the cone; instead, in the case of narrow cones, when $\gamma(C) \geq 2$ (that is, $\theta \in (0, \pi/4]$ for spherical caps), then limit of the homogeneity degree will be always two and the limiting profile will be something different, though related through a correction term. Similar transition phenomena have been detected in other contexts for some types of free boundary problems on cones (see [74]). Moreover, we will see that an important quantity which appears in this estimates and plays a fundamental role is

$$\frac{C(n,s)}{2s - \gamma_s(C)},$$

where $C(n,s) > 0$ is the normalization constant given in (62). It will be therefore very important to bound this quantity uniformly in s. Our main result is the following Theorem.

Theorem 2.1.2. *Let C be an open cone with vertex at the origin. There exist finite the following limits:*

$$\overline{\gamma}(C) := \lim_{s\to 1^-} \gamma_s(C) = \min\{\gamma(C), 2\}$$

and

$$\mu(C) := \lim_{s\to 1^-} \frac{C(n,s)}{2s-\gamma_s(C)} = \begin{cases} 0 & \text{if } \gamma(C) \le 2, \\ \mu_0(C) & \text{if } \gamma(C) \ge 2, \end{cases}$$

where $C(n,s)$ is defined in (62) and

$$\mu_0(C) := \inf \left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 - 2nu^2 \mathrm{d}\sigma}{\left(\displaystyle\int_{S^{n-1}} |u| \mathrm{d}\sigma\right)^2} \ :\ u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus C \right\}.$$

Let us consider the family (u_s) of nonnegative solutions to (55) such that $||u_s||_{L^\infty(S^{n-1})} = 1$. Then, as $s \nearrow 1$, up to a subsequence, we have

1. *$u_s \to \overline{u}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ to some $\overline{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(S^{n-1})$.*
2. *The convergence is uniform on compact subsets of C, $\overline{u}$ is nontrivial with $||\overline{u}||_{L^\infty(S^{n-1})} = 1$ and is $\overline{\gamma}(C)$-homogeneous.*
3. *The limit $\overline{u}$ solves*

$$\begin{cases} -\Delta \overline{u} = \mu(C) \displaystyle\int_{S^{n-1}} \overline{u} \mathrm{d}\sigma & \text{in } C, \\ \overline{u} = 0 & \text{in } \mathbb{R}^n \setminus C\,. \end{cases} \tag{59}$$

Uniqueness of the limit $\overline{u}$ and therefore existence of the limit of u_s as $s \nearrow 1$ holds in the case of connected cones and, in any case, whenever $\gamma(C) > 2$. We will see in Remark 2.4.2 that under symmetry assumptions on the cone C, the limit function $\overline{u}$ is unique and hence it does not depend on the choice of the subsequence.

A further motivation to our study of (55), as shown in [83, 84] and in the blow-up analysis in Chapter 1, is its deep relation with the exponent of the optimal Hölder regularity of segregated profiles and the geometric analysis of the segregation phenomenon. In [84, 83], estimates in Hölder spaces have been obtained by the use of fractional versions of the Alt-Caffarelli-Friedman and Almgren monotonicity formulas. Let us state here the fractional version of the spectral problem beyond the first monotonicity formula: consider the set of 2-partitions of S^{n-1} as

$$\mathcal{P}^2 := \left\{(\omega_1, \omega_2) : \omega_i \subseteq S^{n-1} \text{ open}, \omega_1 \cap \omega_2 = \emptyset,\ \overline{\omega_1} \cup \overline{\omega_2} = S^{n-1}\right\}$$

and define the optimal partition value as:

$$\nu_s^{ACF} := \frac{1}{2} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \gamma_s(\lambda_1^s(\omega_i)). \tag{60}$$

It is easy to see, by a Schwarz symmetrization argument, that ν_s^{ACF} is achieved by a pair of complementary spherical caps $(\omega_\theta, \omega_{\pi-\theta}) \in \mathcal{P}^2$ with aperture 2θ and $\theta \in (0,\pi)$ (for a detailed proof of this kind of symmetrization we refer to [85]), that is:

$$\nu_s^{ACF} = \min_{\theta\in[0,\pi]} \Gamma^s(\theta) = \min_{\theta\in[0,\pi]} \frac{\gamma_s(\theta)+\gamma_s(\pi-\theta)}{2}.$$

This gives a further motivation to our study of (55) for spherical caps. A classical result by Friedland and Hayman, [47], yields $\nu^{ACF} = 1$ for the case $s = 1$, and the minimal value is achieved for two half spheres; this equality is the core of the proof of the classical Alt-Caffarelli-Friedman monotonicity formula.

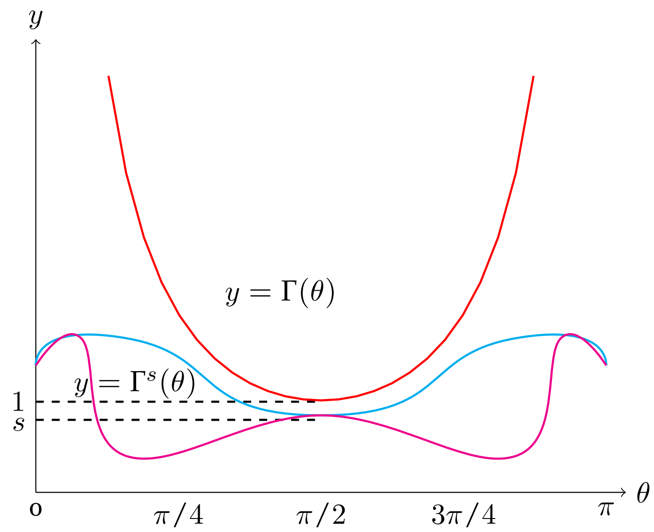


Figure 4: Possible values of $\Gamma^s(\theta) = \Gamma^s(\omega_\theta, \omega_{\pi-\theta})$ for $s < 1$ and $s = 1$ and $n = 2$.

It was proved in [84] that ν_s^{ACF} is linked to the threshold for uniform bounds in Hölder norms for competition-diffusion systems, as the interspecific competition rate diverges to infinity, as well as the exponent of the optimal Hölder regularity for their limiting profiles. It was also conjectured that $\nu_s^{ACF} = s$ for every $s \in (0,1)$. Unfortunately, the exact value of ν_s^{ACF} is still unknown, and we only know that $0 < \nu_s^{ACF} \leq s$ (see [84, 83]). Our contribution to this open problem is a byproduct of Theorem 2.1.2.

Corollary 2.1.3. *In any space dimension we have*

$$\lim_{s\to 1} \nu_s^{ACF} = 1 \ .$$

Moreover, exploiting the connection between s-harmonic functions on cones and the traces on $\mathbb{R}^n$ of the blow-up limits in $\mathfrak{B}^s(\mathbb{R}^{n+1})$, discussed in Proposition 1.3.14 in Chapter 1, we can reasonably state the following remark on the asymptotic limit of the segregated configurations in $\mathcal{G}^s_{\text{loc}}(B_1^+)$.
More precisely, given for $s \in (0,1)$ the class $\mathfrak{B}^s(\mathbb{R}^n)$ of the traces of the blow-up limit in $\mathfrak{B}^s(\mathbb{R}^{n+1})$, we directly get, from Theorem 2.1.2 that

$$\sup\left\{N(0,\overline{u},1)\colon \overline{u}\in \mathfrak{B}^s(\mathbb{R}^{n+1})\right\} \le \sup_{\theta\in(0,\pi)} \gamma_s(\theta) \le 2s \le 2$$

for every $s \in (0,1)$. This simple bound suggests that the possible blow-up limits of the segregated profiles arising from a competition-diffusion problem with nonlocal diffusion and variational competition can not converge to the ones of the segregated solutions studied in [18, 81]. This remark suggests that even for the case of segregated configurations we have to expect, for $s \nearrow 1$, a rather unexpected result.

The Chapter is organized as follows. In Section 2.2 we introduce our setting and we state the relevant known properties of homogeneous s-harmonic functions on cones. After this, we will obtain local $C^{0,\alpha}$-estimates in compact subsets of C and local H^s-estimates in compact subsets of $\mathbb{R}^n$ for solutions of (55). In Section 2.3 we analyze the asymptotic behaviour of $\gamma_s(C)$ as s converges to 1, in order to understand the quantities $\overline{\gamma}(C)$ and $\mu(C)$. To do this, we will establish a distributional semigroup property for the fractional Laplacian for functions which grow at infinity. In Section 2.4 we prove Theorem 2.1.2 and Corollary 2.1.3. Eventually, in Section 2.5, we prove a nontrivial improvement of the main Theorem concerns uniform bounds in Hölder spaces holding uniformly for $s \to 1$.

2.2 HOMOGENOUS s-HARMONIC FUNCTIONS ON CONES

In this Section, we focus our attention on the local properties of homogeneous s-harmonic functions on *regular cones*. Since in Section 2.3 we will study the behaviour of the characteristic exponent as s approaches 1, in this section we recall some known results related to the boundary behaviour of the solution of (55) restricted to the unitary sphere S^{n-1} and some estimates of the Hölder and H^s seminorm.

Definition 2.2.1. Let $\omega \subset S^{n-1}$ be an open set, that may be disconnected. We call *unbounded cone* with vertex in 0, spanned by ω the open set

$$C_\omega = \{rx \ : \ r>0, \ x\in\omega\}.$$

Moreover we say that $C = C_\omega$ is narrow if $\gamma(C) \geq 2$ and wide if $\gamma(C) < 2$. We call C_ω *regular cone* if ω is connected and of class $C^{1,1}$. Let $\theta \in (0, \pi)$ and $\omega_\theta \subset S^{n-1}$ be an open spherical cap of colatitude θ. Then we denote by $C_\theta = C_{\omega_\theta}$ the *right circular cone* of aperture 2θ.

Hence, let C be a fixed unbounded open cone in $\mathbb{R}^n$ with vertex in 0 and consider

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in} \quad C, \\ u_s \equiv 0 & \text{in} \quad \mathbb{R}^n \setminus C. \end{cases}$$

with the condition $\|u_s\|_{L^\infty(S^{n-1})} = 1$. By Theorem 3.2 in [5] there exists, up to a multiplicative constant, a unique nonnegative function u_s smooth in C and $\gamma_s(C)$-homogenous, i.e.

$$u_s(x) = |x|^{\gamma_s(C)} \, u_s\left(\frac{x}{|x|}\right)$$

where $\gamma_s(C) \in (0, 2s)$. As it is well know (see for example [8, 76]), the fractional Laplacian $(-\Delta)^s$ is a nonlocal operator well defined in the class of integrability $\mathcal{L}^1_s := \mathcal{L}^1\left(\mathrm{d}x/(1+|x|)^{n+2s}\right)$, namely the normed space of all Borel functions u satisfying

$$\|u\|_{\mathcal{L}^1_s} := \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}} \mathrm{d}x < +\infty. \tag{61}$$

Hence, for every $u \in \mathcal{L}^1_s, \varepsilon > 0$ and $x \in \mathbb{R}^n$ we define

$$(-\Delta)^s_\varepsilon u(x) = C(n,s) \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \mathrm{d}y,$$

where

$$C(n,s) = \frac{2^{2s} s \Gamma(\frac{n}{2} + s)}{\pi^{n/2} \Gamma(1-s)} \in \left(0, 4\Gamma\left(\frac{n}{2} + 1\right)\right]. \tag{62}$$

and we can consider the fractional Laplacian as the limit

$$(-\Delta)^s u(x) = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_\varepsilon u(x) = C(n,s) \, \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \mathrm{d}y.$$

We remark that $u \in \mathcal{L}^1_s$ is such that $u \in \mathcal{L}^1_{s+\delta}$ for any $\delta > 0$, which will be an important tool in this part of the manuscript, in order to compute high order fractional Laplacians. Another definition of the fractional Laplacian, which can be constructed by a double change of variables as in [38], is

$$(-\Delta)^s u(x) = \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \mathrm{d}y$$

which emphasize that given $u \in C^2(D) \cap \mathcal{L}^1_s$, we obtain that $x \mapsto (-\Delta)^s u(x)$ is a continuous and bounded function on D, for some bounded $D \subset \mathbb{R}^n$.

By [67, Lemma 3.3], if we consider a regular unbounded cone C symmetric with respect to a fixed axis, there exists two positive constant $c_1 = c_1(n, s, C)$ and $c_2 = c_2(n, s, C)$ such that

$$c_1 |x|^{\gamma_s - s} \text{dist}(x, \partial C)^s \leq u_s(x) \leq c_2 |x|^{\gamma_s - s} \text{dist}(x, \partial C)^s \tag{63}$$

for every $x \in C$. We remark that this result can be easily generalized to regular unbounded cones C_ω with $\omega \subset S^{n-1}$ which is a finite union of connected $C^{1,1}$ domain ω_i, such that $\overline{\omega}_i \cup \overline{\omega}_j = \emptyset$ for $i \neq j$, since the reasonings in [67] rely on a Boundary Harnack principle and on sharp estimates for the Green function for bounded $C^{1,1}$ domain non necessary connected (for more details [28]).

Through the paper we will call the coefficient of homogeneity γ_s as "*characteristic exponent*", since it is strictly related to an eigenvalue partition problem.
As we already mentioned, our solutions are smooth in the interior of the cone and locally $C^{0,s}$ near the boundary $\partial C \setminus \{0\}$ (see for example [67]), but we need some quantitative estimates in order to better understand the dependence of the Hölder seminorm on the parameter $s \in (0, 1)$.

Before showing the main result of Hölder regularity, we need the following estimates about the fractional Laplacian of smooth compactly supported functions: this result can be found in [8, Lemma 3.5] and [37, Lemma 5.1], but here we compute the formula with a deep attention on the dependence of the constant with respect to $s \in (0, 1)$.

Proposition 2.2.2. *Let $s \in (0, 1)$ and $\varphi \in C^2_c(\mathbb{R}^n)$. Then*

$$|(-\Delta)^s \varphi(x)| \leq \frac{c}{(1 + |x|)^{n+2s}}, \qquad \forall x \in \mathbb{R}^n, \tag{64}$$

where the constant $c > 0$ depends only on n and the choice of φ.

Proof. Let $K \subset \mathbb{R}^n$ be the compact support of φ and $k = \max_{x \in K} |\varphi(x)|$. There exists $R > 1$ such that $K \subset B_{R/2}(0)$.
Let $|x| > R$.

$$\begin{aligned}
|(-\Delta)^s \varphi(x)| &= \left| C(n, s) \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+2s}} \mathrm{d}y \right| = \left| C(n, s) \int_K \frac{\varphi(y)}{|x - y|^{n+2s}} \mathrm{d}y \right| \\
&\leq \frac{C(n, s) k}{|x|^{n+2s}} \int_K \frac{1}{(1 - |\frac{y}{x}|)^{n+2s}} \mathrm{d}y \leq \frac{C(n, s) k 2^{n+2s} |K|}{|x|^{n+2s}} \\
&\leq \frac{C(n, s) k 2^{2(n+2s)} |K|}{(1 + |x|)^{n+2s}} \leq \frac{c}{(1 + |x|)^{n+2s}},
\end{aligned}$$

where $c > 0$ depends only on n and the choice of φ.

Let now $|x| \leq R$. We use the fact that any derivative of φ of first and second order is uniformly continuous in the compact set K and the fact that in $B_R(0)$ the function $(1+|x|)^{n+2s}$ has maximum given by $(1+R)^{n+2s}$. Hence there exist $0 < \delta < 1$ and a constant $M > 0$, both depending only on n and the choice of φ such that

$$|\varphi(x+z) + \varphi(x-z) - 2\varphi(x)| \leq M|z|^2 \qquad \forall z \in B_\delta(0).$$

Hence

$$\begin{aligned}
|(-\Delta)^s\varphi(x)| &= \left| C(n,s) \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}} \mathrm{d}y + C(n,s) \int_{B_\delta(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}} \mathrm{d}y \right| \\
&\leq 2kC(n,s) \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{1}{|x-y|^{n+2s}} \mathrm{d}y + \frac{C(n,s)}{2} \int_{B_\delta(0)} \frac{|\varphi(x+z)+\varphi(x-z)-2\varphi(x)|}{|z|^{n+2s}} \mathrm{d}z \\
&\leq 2kC(n,s)\omega_{n-1} \int_\delta^{+\infty} r^{-1-2s} \mathrm{d}r + \frac{C(n,s)\omega_{n-1}M}{2} \int_0^\delta r^{1-2s} \mathrm{d}r \\
&= \frac{kC(n,s)\omega_{n-1}}{s\delta^{2s}} + \frac{C(n,s)\omega_{n-1}M\delta^{2-2s}}{4(1-s)} \\
&\leq \frac{c}{\delta^2} + c = c\frac{(1+|x|)^{n+2s}}{(1+|x|)^{n+2s}} \leq \frac{c(1+R)^{n+2}}{(1+|x|)^{n+2s}} = \frac{c}{(1+|x|)^{n+2s}},
\end{aligned}$$

where $c > 0$ depends only on n and the choice of φ. This concludes the proof. □

By the previous calculations we have also the following result.

Remark 2.2.3. Let $s \in (0,1)$ and $\varphi \in C_c^2(\mathbb{R}^n)$. Then there exists a constant $c = c(n,\varphi) > 0$ and a radius $R = R(\varphi) > 0$ such that

$$|(-\Delta)^s\varphi(x)| \leq c\frac{C(n,s)}{(1+|x|)^{n+2s}}, \qquad \forall x \in \mathbb{R}^n \setminus B_R(0). \tag{65}$$

The following result provides interior estimates for the Hölder norm of our solutions.

Proposition 2.2.4. *Let C be a cone and $K \subset C$ be a compact set and $s_0 \in (0,1)$. Then there exist a constant $c > 0$ and $\overline{\alpha} \in (0,1)$, both dependent only on s_0, K, n, C, such that*

$$||u_s||_{C^{0,\alpha}(K)} \leq c\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right),$$

for any $\alpha \in (0, \overline{\alpha}]$ and any $s \in [s_0, 1)$.

By a standard covering argument, there exists a finite number of balls such that $K \subset \bigcup_{j=1}^{k} B_r(x_j)$, for a given radius $r > 0$ such that $\bigcup_{j=1}^{k} \overline{B_{2r}(x_j)} \subset C$. Thus, it is enough to prove

Proposition 2.2.5. *Let $\overline{B_{2r}(\overline{x})} \subset C$ be a closed ball and $s_0 \in (0,1)$. Then there exist a constant $c > 0$ and $\overline{\alpha} \in (0,1)$, both dependent only on $s_0, r, \overline{x}, n, C$, such that*

$$||u_s||_{C^{0,\alpha}(\overline{B_r(\overline{x})})} \leq c\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right),$$

for any $\alpha \in (0, \overline{\alpha}]$ and any $s \in [s_0, 1)$.

In order to achieve the desired result, we need to estimate locally the value of the fractional Laplacian of u_s in a ball compactly contained in the cone C.

Lemma 2.2.6. *Let $\eta \in C_c^\infty(B_{2r}(\overline{x}))$ be a cut-off function such that $0 \leq \eta \leq 1$ with $\eta \equiv 1$ in $B_r(\overline{x})$. Under the same assumptions of Proposition 2.2.5,*

$$||(-\Delta)^s(u_s\eta)||_{L^\infty(B_{2r}(\overline{x}))} \leq C_0\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right)$$

for any $s \in [s_0, 1)$, where $C_0 > 0$ depends on $s_0, n, \overline{x}, r, C$, and the choice of the function η.

Proof. Let $R > 1$ such that $\overline{B_{2r}(\overline{x})} \subset B_{R/2}(0)$. Hence, let fix a point $x \in B_{2r}(\overline{x})$. We can express the fractional Laplacian of $u_s\eta$ in the following way

$$\begin{aligned}
(-\Delta)^s(u_s\eta)(x) &= \eta(x)(-\Delta)^s u_s(x) + C(n,s)\int_{\mathbb{R}^n} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y \\
&= C(n,s)\int_{B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y + \\
&\quad + C(n,s)\int_{\mathbb{R}^n \setminus B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y.
\end{aligned}$$

We recall that $u_s(x) = |x|^{\gamma_s(C)} u_s(x/|x|)$ and that for any $s \in (0,1)$ the functions u_s are normalized such that $||u_s||_{L^\infty(S^{n-1})} = 1$. Moreover we remark that $\eta(x) - \eta(y) = \eta(x) \geq 0$ in $B_{2r}(\overline{x}) \times (\mathbb{R}^n \setminus B_R(0))$. Hence, using Proposition 2.2.2 and the fact that $\gamma_s(C) < 2s$, we obtain

$$\begin{aligned}
|(-\Delta)^s(u_s\eta)(x)| &\leq C(n,s)\left|\int_{B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y\right| + \\
&\quad + C(n,s)\left|\int_{\mathbb{R}^n\setminus B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y\right| \\
&\leq R^{\gamma_s(C)}|(-\Delta)^s\eta(x)| + C(n,s)2^{n+2s}\int_{\mathbb{R}^n\setminus B_R(0)}\frac{1}{|y|^{n+2s-\gamma_s(C)}}\mathrm{d}y \\
&\leq \frac{cR^2}{(1+|x|)^{n+2s}} + C(n,s)2^{n+2}\omega_{n-1}\int_R^{+\infty} r^{-1-2s+\gamma_s(C)}\mathrm{d}r \\
&\leq \frac{cR^2}{(1+|x|)^{n+2s}} + \frac{cC(n,s)}{R^{2s-\gamma_s(C)}(2s-\gamma_s(C))} \\
&\leq C_0\left(1+\frac{C(n,s)}{2s-\gamma_s(C)}\right).
\end{aligned}$$

□

Proof of Proposition 2.2.5. Let as before $\eta \in C_c^\infty(B_{2r}(\overline{x}))$ be a cut-off function such that $0 \leq \eta \leq 1$ with $\eta \equiv 1$ in $B_r(\overline{x})$. First, we remark that there exists a constant $c_0 > 0$ such that for any $s \in (0,1)$, it holds

$$||u_s\eta||_{L^\infty(\mathbb{R}^n)} \leq c_0, \tag{66}$$

where c_0 depends only on $n, \overline{x}, r$. In fact, let $R > 0$ be such that $\overline{B_{2r}(\overline{x})} \subset B_R(0)$. Then, for any $x \in \mathbb{R}^n$, we have $0 \leq u_s\eta(x) \leq R^{\gamma_s(C)} \leq R^2$. Using the bound (66) and the previous Lemma, we can apply [24, Theorem 12.1] obtaining the existence of $\overline{\alpha} \in (0,1)$ and $C > 0$, both depending only on n, s_0 and the choice of $B_r(\overline{x})$ such that

$$\begin{aligned}
||u_s\eta||_{C^{0,\alpha}(\overline{B_r(\overline{x})})} &\leq C(||u_s\eta||_{L^\infty(\mathbb{R}^n)} + ||(-\Delta)^s(u_s\eta)||_{L^\infty(B_{2r}(\overline{x}))}) \\
&\leq C\left(c_0 + C_0\left(1+\frac{C(n,s)}{2s-\gamma_s(C)}\right)\right),
\end{aligned}$$

for any $s \in [s_0, 1)$ and any $\alpha \in (0, \overline{\alpha}]$. Since $\eta \equiv 1$ in $B_r(\overline{x})$ we obtain the result. □

Similarly, now we need to construct some estimate related to the H^s seminorm of the solution u_s, Since the functions do not belong to $H^s(\mathbb{R}^n)$, we need to truncate the solution with some cut off function in order to avoid the problems related to the growth at infinity. In such a way, we can use

$$[v]^2_{H^s(\mathbb{R}^n)} = \left\|(-\Delta)^{s/2}v\right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} v(-\Delta)^s v\mathrm{d}x. \tag{67}$$

which holds for every $v \in H^s(\mathbb{R}^n)$. So, let $\eta \in C_c^\infty(B_2)$ be a radial cut off function such that $\eta \equiv 1$ in B_1 and $0 \le \eta \le 1$ in B_2, and consider $\eta_R(x) = \eta(\frac{x-x_0}{R})$ the rescaled cut off function defined in $B_{2R}(x_0)$, for some $R > 0$ and $x_0 \in \mathbb{R}^n$.

Proposition 2.2.7. *Let $s_0 \in (0,1)$ and $\eta_R \in C_c^\infty(B_{2R}(x_0))$ previously defined. Then*

$$[u_s\eta_R]^2_{H^s(\mathbb{R}^n)} \le c\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right)$$

for any $s \in [s_0, 1)$, where $c > 0$ is a constant that depends on x_0, R, C, s_0 and η.

Proof. Let $\eta \in C_c^\infty(B_2)$ be a radial cut off function such that $\eta \equiv 1$ in B_1 and $0 \le \eta \le 1$ in B_2, and consider the collection of $(\eta_R)_R$ with $R > 0$ defined by $\eta_R(x) = \eta(\frac{x-x_0}{R})$ with some $x_0 \in \mathbb{R}^n$. By (67), for every $R > 0$ we obtain

$$[u_s\eta_R]^2_{H^s(\mathbb{R}^n)} = \left\|(-\Delta)^{s/2}(u_s\eta_R)\right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u_s\eta_R(-\Delta)^s(u_s\eta_R)\mathrm{d}x.$$

By definition of the fractional Laplacian we have

$$\begin{aligned}\int_{\mathbb{R}^n} u_s\eta_R(-\Delta)^s(u_s\eta_R)\mathrm{d}x &= C(n,s)\int_{\mathbb{R}^n\times\mathbb{R}^n} u_s(x)\eta_R(x)\frac{u_s(x)\eta_R(x) - u_s(y)\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \eta_R^2 u_s(-\Delta)^s u_s\mathrm{d}x + C(n,s)\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}u_s(x)u_s(y)\eta_R(x)\mathrm{d}y\mathrm{d}x \\ &= \frac{C(n,s)}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x\end{aligned}$$

where the last equation is obtained by the symmetrization of the previous integral with respect to the variable $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Before splitting the domain of integration into different subset, it is easy to see that

$$\begin{aligned}\eta_R(x) - \eta_R(y) \equiv 0 \quad &\text{in } B_R(x_0)\times B_R(x_0) \cup (\mathbb{R}^n \setminus B_{2R}(x_0)) \times (\mathbb{R}^n \setminus B_{2R}(x_0)) \\ |\eta_R(x) - \eta_R(y)| \equiv 1 \quad &\text{in } B_R(x_0) \times (\mathbb{R}^n \setminus B_{2R}(x_0)) \cup (\mathbb{R}^n \setminus B_{2R}(x_0)) \times B_R(x_0).\end{aligned}$$

where all the previous balls are centered at the point x_0. Hence, given the sets $\Omega_1 = B_{3R}(x_0) \times B_{3R}(x_0)$ and $\Omega_2 = B_{2R}(x_0) \times (\mathbb{R}^n \setminus B_{3R}(x_0)) \cup (\mathbb{R}^n \setminus B_{3R}(x_0)) \times B_{2R}(x_0)$ we have

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x \le &\int_{\Omega_1}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x + \\ &+\int_{\Omega_2}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x.\end{aligned}$$

In particular

$$\begin{aligned}\int_{\Omega_1} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x &\le \sup_{B_{3R}(x_0)} u_s^2 \int_{B_{3R}(x_0)\times B_{3R}(x_0)} \frac{\|\nabla\eta_R\|^2_{L^\infty(\mathbb{R}^n)}}{|x-y|^{n+2s-2}}\mathrm{d}y\mathrm{d}x \\ &\le \|\nabla\eta_R\|^2_{L^\infty} \sup_{B_{3R}(x_0)} u_s^2 \int_{B_{3R}(0)}\mathrm{d}x \int_{B_{6R}(x)} \frac{1}{|x-y|^{n+2s-2}}\mathrm{d}y \\ &\le \frac{\|\nabla\eta\|^2_{L^\infty}}{R^2} \sup_{B_{3R}(x_0)} u_s^2 \,|B_{3R}|\left|S^{n-1}\right| \frac{(6R)^{2-2s}}{2(1-s)} \\ &\le C\,\|\nabla\eta\|^2_{L^\infty} \frac{R^{n-2s}}{2(1-s)} \max\{|x_0|^{2\gamma_s}, (3R)^{2\gamma_s}\}\,\|u_s\|_{L^\infty(S^{n-1})}\end{aligned}$$

where in the second inequality we use the changes of variables $x-x_0$ and $y-x_0$ and the fact that $B_{3R}(0)\times B_{3R}(0)\subset B_{3R}(0)\times B_{6R}(x)$ for every $x\in B_{3R}(0)$. Similarly we have

$$\begin{aligned}\int_{\Omega_2} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x &\le 2\int_{B_{2R}(x_0)} u_s(x)\left(\int_{\mathbb{R}^n\setminus B_{3R}(x_0)} \frac{u_s(y)}{|x-y|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\ &\le 2\int_{B_{2R}(0)} u_s(x+x_0)\left(\int_{\mathbb{R}^n\setminus B_{3R}(0)} \frac{u_s(y+x_0)}{|y|^{n+2s}\left(1-\frac{|x|}{|y|}\right)^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\ &\le 2\cdot 3^{n+2s}\int_{B_{2R}(0)} u_s(x+x_0)\left(\int_{\mathbb{R}^n\setminus B_{3R}(0)} \frac{C(|y|+|x_0|)^{\gamma_s}}{|y|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\ &\le C\sup_{B_{2R}(x_0)} u_s\,|B_{2R}|\left|S^{n-1}\right| 2^{\gamma_s} G(x_0,R)\end{aligned}$$

with

$$G(x_0,R)=\begin{cases}\dfrac{|x_0|^{\gamma_s}}{2s-\gamma_s}(3R)^{-2s} & \text{if } |x_0|\ge 3R\\ \dfrac{(3R)^{\gamma_s-2s}}{2s-\gamma_s} & \text{if } |x_0|\le 3R\end{cases} \le \frac{(3R)^{-2s}}{2s-\gamma_s}\max\{|x_0|,3R\}^{\gamma_s}.$$

Finally, we obtain the desired bound for the seminorm $[u_s\eta_R]^2_{H^s(\mathbb{R}^n)}$ summing the two terms and recalling that $\|u_s\|_{L^\infty(S^{n-1})}=1$. □

2.3 CHARACTERISTIC EXPONENT $\gamma_s(c)$: PROPERTIES AND ASYMPTOTIC BEHAVIOUR

In this Section we start the analysis of the asymptotic behaviour of the homogeneity degree $\gamma_s(C)$ as s converges to 1. The main results are two: first we get a monotonicity result for the map $s\mapsto\gamma_s(C)$, for a fixed regular cone C, which ensures the existence of the limit and, using some comparison result, a bound on the possible value of the limit exponent. Secondly we study the asymptotic behaviour of the quotient $\frac{C(n,s)}{2s-\gamma_s(C)}$.

In order to prove the first result and compare different order of s-harmonic functions for different power of $(-\Delta)^s$, we need to introduce some results which give a natural extension of the classic semigroup property of the fractional Laplacian, for function defined on cones which grow at infinity.

2.3.1 *Distributional semigroup property*

It is well known that if we deal with smooth functions with compact support, or more generally with functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, a semigroup property holds for the fractional Laplacian, i.e. $(-\Delta)^{s_1} \circ (-\Delta)^{s_2} = (-\Delta)^{s_1+s_2}$, where $s_1, s_2 \in (0,1)$ with $s_1 + s_2 < 1$. Since we have to deal with functions in $\mathcal{L}^1_s$ that grow at infinity, we have to construct a distributional counterpart of the semigroup property, in order to compute high order fractional Laplacians for solutions of the problem given in (55).

First of all, we remark that a solution u_s to (55) for a fixed cone C belongs to $\mathcal{L}^1_s$ since $0 \leq u_s(x) \leq |x|^{\gamma_s(C)}$ in $\mathbb{R}^n$ with $\gamma_s(C) \in (0, 2s)$. Moreover, by the homogeneity one can rewrite the norm (61) in the following way

$$\begin{aligned}\|u_s\|_{\mathcal{L}^1_s} = \int_{\mathbb{R}^n} \frac{u_s(x)}{(1+|x|)^{n+2s}} \mathrm{d}x &= \int_{S^{n-1}} u_s \mathrm{d}\sigma \int_0^\infty \frac{\rho^{n-1+\gamma_s(C)}}{(1+\rho)^{n+2s}} \mathrm{d}\rho \\ &= \frac{\Gamma(n+\gamma_s(C))\Gamma(2s-\gamma_s(C))}{\Gamma(n+2s)} \int_{S^{n-1}} u_s \mathrm{d}\sigma.\end{aligned}$$

In the recent paper [40] the authors introduced a new notion of fractional Laplacian applying to a wider class of functions which grow more than linearly at infinity. This is achieved by defining an equivalence class of functions modulo polynomials of a fixed order. However, it can be hardly exploited to the solutions of (55) as they annihilate on a set of nonempty interior.

As shown in [8, Definition 3.6], if we consider a smooth function with compact support $\varphi \in C^\infty_c(\mathbb{R}^n)$(or $\varphi \in C^2_c(\mathbb{R}^n)$), we can define the distribution k^{2s} by the formula

$$(-\Delta)^s \varphi(0) = (k^{2s}, \varphi).$$

By this definition, it follows that $(-\Delta)^s \varphi(x) = k^{2s} * \varphi(x)$.

Definition 2.3.1. [8, Definition 3.7] For $u \in \mathcal{L}^1_s$ we define the *distributional fractional Laplacian* $(-\widetilde{\Delta})^s u$ by the formula

$$((-\widetilde{\Delta})^s u, \varphi) = (u, (-\Delta)^s \varphi), \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

In particular, since given an open subset $D \subset \mathbb{R}^n$ and $u \in C^2(D) \cap \mathcal{L}^1_s$, the fractional Laplacian exists as a continuous function of $x \in D$ and $(-\widetilde{\Delta})^s u = (-\Delta)^s u$ as a distribution in D [8, Lemma 3.8], through the Chapter we will always use $(-\Delta)^s$ both for the classic and the distributional fractional Laplacian. The following is a useful tool to compute the distributional fractional Laplacian.

Lemma 2.3.2. *[8, Lemma 3.3] Assume that*

$$\iint_{|y-x|>\varepsilon} \frac{|f(x)g(y)|}{|y-x|^{n+2s}} \mathrm{d}x\mathrm{d}y < +\infty \quad \textit{and} \quad \int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x < +\infty, \tag{68}$$

then $((-\Delta)^s_\varepsilon f, g) = (f, (-\Delta)^s_\varepsilon g)$*. Moreover if* $f \in \mathcal{L}^1_s$ *and* $g \in C_c(\mathbb{R}^n)$ *the assumptions* (68) *are satisfied for every* $\varepsilon > 0$.

Before proving the semigroup property, we prove the following lemma which ensures the existence of the δ-Laplacian of the s-Laplacian, for $0 < \delta < 1$.

Lemma 2.3.3. *Let* u_s *be solution of* (55) *with* C *a regular cone. Then we have* $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ *for any* $\delta > 0$*, i.e.*

$$\int_{\mathbb{R}^n} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}} \mathrm{d}x < +\infty.$$

Proof. Since the function u_s is s-harmonic in C, namely $(-\Delta)^s u_s(x) = 0$ for all $x \in C$, we can restrict the domain of integration to $\mathbb{R}^n \setminus C$.
By homogeneity and the results in [8], we have that the function $(-\Delta)^s u_s$ is $(\gamma_s - 2s)$-homogeneous and in particular $x \mapsto (-\Delta)^s u_s(x)$ is a continuous negative function, for every $x \in D \subset\subset \mathbb{R}^n \setminus C$.
In order to compute the previous integral, we focus our attention on the restriction of the fractional Laplacian to the sphere S^{n-1}, in particular, we prove that there exists $\bar{\varepsilon} > 0$ and $C > 0$ such that

$$|(-\Delta)^s u_s(x)| \leq \frac{C}{\mathrm{dist}(x, \partial C)^s} \quad \forall x \in N_{\bar{\varepsilon}}(\partial C) \cap S^{n-1}, \tag{69}$$

where $N_\varepsilon(\partial C) = \{x \in \mathbb{R}^n \setminus C : \mathrm{dist}(x, \partial C) \leq \varepsilon\}$ is the tubular neighborhood of ∂C .
Hence, fixed $R > 0$ small enough, consider initially $\varepsilon < R$ and $x \in S^{n-1} \cap N_\varepsilon(\partial C)$: since $u_s(y) \leq |y|^{\gamma_s}$ in $\mathbb{R}^n$ and by (63) there exists a constant $C > 0$ such that for every $y \in C$ we have

$$u_s(y) \leq C\,|y|^{\gamma_s - s}\,\mathrm{dist}(y, \partial C)^s,$$

it follows, defining $\delta(x) := \text{dist}(x, \partial C) > 0$, that

$$\begin{aligned}|(-\Delta)^s u_s(x)| &= C(n,s)\int_{C\cap B_R(x)} \frac{u_s(y)}{|x-y|^{n+2s}}\mathrm{d}y + C(n,s)\int_{C\setminus B_R(x)} \frac{u_s(y)}{|x-y|^{n+2s}}\mathrm{d}y\\ &\leq C(n,s)\int_{C\cap B_R(x)} \frac{C\,|y|^{\gamma_s - s}\,\text{dist}(y,\partial C)^s}{|x-y|^{n+2s}}\mathrm{d}y + C(n,s)\int_{C\setminus B_R(x)} \frac{|y|^{\gamma_s}}{|x-y|^{n+2s}}\mathrm{d}y.\end{aligned}$$

Since $C \cap B_R(x) \subset B_R(x) \setminus B_{\delta(x)}(x)$, we have

$$\begin{aligned}|(-\Delta)^s u_s(x)| &\leq C\int_{R\geq|x-y|\geq\delta(x)} \frac{|y|^{\gamma_s - s}}{|x-y|^{n+s}}\mathrm{d}y + \int_{|x-y|\geq R} \frac{(|x-y|+1)^{\gamma_s}}{|x-y|^{n+2s}}\mathrm{d}y\\ &\leq C\int_{R\geq|x-y|\geq\delta(x)} \frac{1}{|x-y|^{n+s}}\mathrm{d}y + \omega_{n-1}\int_R^\infty \frac{(t+1)^{\gamma_s}}{t^{1+2s}}\mathrm{d}t\\ &\leq C\int_{\delta(x)}^R \frac{1}{r^{1+s}}\mathrm{d}r + M\\ &\leq C\frac{1}{\text{dist}(x,\partial C)^s} + M.\end{aligned}$$

Moreover, again since $s \in (0,1)$, up to consider a smaller neighborhood $N_\varepsilon(\partial C)$, we obtain that there exists a constant $\bar{\varepsilon} > 0$ small enough and $C > 0$ such that

$$|(-\Delta)^s u_s(x)| \leq \frac{C}{\text{dist}(x,\partial C)^s} \quad \text{for every } x \in N_{\bar{\varepsilon}}(\partial C) \cap S^{n-1}.$$

Now, fixed $\delta > 0$ and considered $\bar{\varepsilon} > 0$ of (69), we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus C} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\mathrm{d}x &= \int_{\mathbb{R}^n\setminus C} \frac{|x|^{\gamma_s - 2s}\left|(-\Delta)^s u_s\left(\frac{x}{|x|}\right)\right|}{(1+|x|)^{n+2\delta}}\mathrm{d}x\\ &= \int_0^\infty \int_{S^{n-1}\cap(\mathbb{R}^n\setminus C)} \frac{r^{\gamma_s-2s}\,|(-\Delta)^s u_s(z)|}{(1+r)^{n+2\delta}} r^{n-1}\mathrm{d}\sigma(z)\mathrm{d}r\\ &= \int_0^\infty \frac{r^{n-1+\gamma_s-2s}}{(1+r)^{n+2\delta}}\mathrm{d}r \int_{S^{n-1}\cap(\mathbb{R}^n\setminus C)} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma.\end{aligned}$$

Since $\gamma_s \in (0, 2s)$ and $s \in (0,1)$, it follows

$$\begin{aligned}\int_{\mathbb{R}^n\setminus C} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\mathrm{d}x &\leq C\int_{S^{n-1}\cap N_{\bar{\varepsilon}}(\partial C)} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma + C\int_{((\mathbb{R}^n\setminus C)\setminus N_{\bar{\varepsilon}}(\partial C))\cap S^{n-1}} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma\\ &\leq C\int_{S^{n-1}\cap N_{\bar{\varepsilon}}(\partial C)} \frac{1}{\text{dist}(z,\partial C)^s}\mathrm{d}\sigma + M\\ &< +\infty\end{aligned}$$

where in the second inequality we used that $z \mapsto (-\Delta)^s u_s(z)$ is continuous in every $A \subset\subset S^{n-1} \cap (\mathbb{R}^n \setminus C)$ and in the last one that $\text{dist}(x, \partial C)^{-s} \in L^1(S^{n-1} \cap N_{\bar{\varepsilon}}(\partial C), \text{d}\sigma)$. □

Proposition 2.3.4 (Distributional semigroup property). *Let u_s be a solution of (55) with C a regular cone and consider $\delta \in (0, 1-s)$. Then*

$$(-\Delta)^{s+\delta} u_s = (-\Delta)^{\delta}[(-\Delta)^s u_s] \quad \text{in } \mathcal{D}'(C)$$

or equivalently

$$((-\Delta)^{s+\delta} u_s, \varphi) = ((-\Delta)^{\delta}[(-\Delta)^s u_s], \varphi), \quad \forall \varphi \in C_c^{\infty}(C).$$

Proof. Since $|u_s(x)| \leq |x|^{\gamma_s}$, with $\gamma_s \in (0, 2s)$, it is easy to see that $u_s \in \mathcal{L}^1_s \cap C^2(C)$. Moreover, as we have already remarked, if $u_s \in \mathcal{L}^1_s$ then $u_s \in \mathcal{L}^1_{s+\delta}$ for every $\delta > 0$. In particular, $(-\Delta)^{s+\delta} u_s$ does exist and it is a continuous function of $x \in C$, for every $\delta \in (0, 1-s)$. By definition of the distributional fractional Laplacian, we obtain

$$((-\Delta)^{s+\delta} u_s, \varphi) = (u_s, (-\Delta)^{s+\delta} \varphi),$$

and since for $\varphi \in C_c^{\infty}(C) \subset \mathcal{S}(\mathbb{R}^n)$ in the Schwarz space, the classic semigroup property holds, we obtain that

$$((-\Delta)^{s+\delta} u_s, \varphi) = (u_s, (-\Delta)^s [(-\Delta)^{\delta} \varphi]).$$

On the other hand, since by Lemma 2.3.3 we have $(-\Delta)^s u_s \in \mathcal{L}^1_{\delta}$, it follows

$$((-\Delta)^{\delta}_{\varepsilon}[(-\Delta)^s u_s], \varphi) = ((-\Delta)^s u_s, (-\Delta)^{\delta}_{\varepsilon} \varphi) \tag{70}$$

for every $\varepsilon > 0$. Since $(-\Delta)^s u_s \in \mathcal{L}^1_{\delta}$ and $\varphi \in C_c^{\infty}(\mathbb{R}^n)$, the δ-Laplacian of $(-\Delta)^s u_s$ does exists in a distributional sense and hence the left hand side in (70) does converge to $((-\Delta)^{\delta}[(-\Delta)^s u_s], \varphi)$ as $\varepsilon \to 0$. Moreover the right hand side in (70) does converge to $((-\Delta)^s u_s, (-\Delta)^{\delta} \varphi)$ by the dominated convergence theorem, using Proposition 2.2.2 and Lemma 2.3.3 which give

$$\int_{\mathbb{R}^n} (-\Delta)^s u_s(x) (-\Delta)^{\delta}_{\varepsilon} \varphi(x) \text{d}x \leq \int_{\mathbb{R}^n} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}} \text{d}x < +\infty.$$

By the previous remarks,

$$((-\Delta)^{\delta}[(-\Delta)^s u_s], \varphi) = ((-\Delta)^s u_s, (-\Delta)^{\delta} \varphi).$$

In order to conclude the proof of the distributional semigroup property, we need to show that

$$(u_s, (-\Delta)^s [(-\Delta)^{\delta} \varphi]) = ((-\Delta)^s u_s, (-\Delta)^{\delta} \varphi), \tag{71}$$

which is not a trivial equality, since $(-\Delta)^\delta\varphi \in C^\infty(\mathbb{R}^n)$ is no more compactly supported.

Let $\eta \in C_c^\infty(B_2(0))$ be a radial cutoff function such that $\eta \equiv 1$ in $B_1(0)$ and $0 \le \eta \le 1$ in $B_2(0)$, and define $\eta_R(x) = \eta(x/R)$, for $R > 0$. Obviously, since $u_s\eta_R \in C_c(\mathbb{R}^n)$ and $(-\Delta)^\delta\varphi \in \mathcal{L}_s^1$, by Lemma 2.3.2 we have

$$(u_s\eta_R, (-\Delta)_\varepsilon^s[(-\Delta)^\delta\varphi]) = ((-\Delta)_\varepsilon^s(u_s\eta_R), (-\Delta)^\delta\varphi) \tag{72}$$

for every $\varepsilon, R > 0$. First, for $R > 0$ fixed, we want to pass to the limit for $\varepsilon \to 0$. For the left hand side in (72), we get the convergence to $(u_s\eta_R, (-\Delta)^s[(-\Delta)^\delta\varphi])$ since we can apply the dominated convergence theorem. In fact

$$\int_{\mathbb{R}^n} u_s\eta_R(-\Delta)_\varepsilon^s[(-\Delta)^\delta\varphi] \le c\int_K (-\Delta)^{s+\delta}\varphi < +\infty,$$

where K denotes the support of $u_s\eta_R$. For the right hand side in (72) we observe that, for any $x \in \mathbb{R}^n$

$$(-\Delta)_\varepsilon^s(u_s\eta_R)(x) = \eta_R(x)(-\Delta)_\varepsilon^s u_s(x) + u_s(x)(-\Delta)_\varepsilon^s\eta_R(x) - I_\varepsilon(u_s,\eta_R)(x),$$

where

$$I_\varepsilon(u_s,\eta_R)(x) = C(n,s)\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{(u_s(x)-u_s(y))(\eta_R(x)-\eta_R(y))}{|x-y|^{n+2s}}\mathrm{d}y.$$

Obviously the first term $((-\Delta)_\varepsilon^s u_s, \eta_R(-\Delta)^\delta\varphi) \to ((-\Delta)^s u_s, \eta_R(-\Delta)^\delta\varphi)$ by definition of the distributional s-Laplacian, since $u_s \in \mathcal{L}_s^1$ and $\eta_R(-\Delta)^\delta\varphi \in C_c^\infty(\mathbb{R}^n)$. The second term $(u_s(-\Delta)_\varepsilon^s\eta_R, (-\Delta)^\delta\varphi) \to (u_s(-\Delta)^s\eta_R, (-\Delta)^\delta\varphi)$ by dominated convergence, since

$$\int_{\mathbb{R}^n} u_s(-\Delta)_\varepsilon^s\eta_R(-\Delta)^\delta\varphi\mathrm{d}x \le c\int_{\mathbb{R}^n}\frac{u_s(x)}{(1+|x|)^{n+2s}}\mathrm{d}x.$$

Finally, the last term $(I_\varepsilon(u_s,\eta_R), (-\Delta)^\delta\varphi) \to (I(u_s,\eta_R), (-\Delta)^\delta\varphi)$ by dominated convergence, since

$$\int_{\mathbb{R}^n} I_\varepsilon(u_s,\eta_R)(-\Delta)^\delta\varphi\mathrm{d}x \le C\int_{\mathbb{R}^n}\left|(-\Delta)^\delta\varphi\right|\mathrm{d}x,$$

which is integrable by Proposition 2.2.2. Finally, passing to the limit for $\varepsilon \to 0$, from (72) we get

$$(u_s\eta_R, (-\Delta)^s[(-\Delta)^\delta\varphi]) = ((-\Delta)^s(u_s\eta_R), (-\Delta)^\delta\varphi), \tag{73}$$

for every $R > 0$.

Now we want to prove (71), concluding this proof, by passing to the limit in (73) for $R \to +\infty$. Since we know, by dominated convergence, that the left hand side converges to $(u_s, (-\Delta)^s(-\Delta)^\delta \varphi)$ for $R \to \infty$, we focus our attention on the other one. At this point, we need to prove that for any $\varphi \in C_c^\infty(C)$,

$$\int_{\mathbb{R}^n} (-\Delta)^s(u_s\eta_R)(-\Delta)^\delta\varphi \longrightarrow \int_{\mathbb{R}^n} (-\Delta)^s u_s(-\Delta)^\delta\varphi, \tag{74}$$

as $R \to +\infty$. First of all, we remark that $(-\Delta)^s(u_s\eta_R) \to (-\Delta)^s u_s$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. In fact, let $K \subset \mathbb{R}^n$ be a compact set. There exists $\overline{r} > 0$ such that $K \subset B_{\overline{r}}$. Then, considering any radius $R > \overline{r}$, $\eta_R(x) = 1$ for any $x \in K$. Hence, for any $R > \overline{r}$, using the fact that $u_s(x) = |x|^{\gamma_s}u_s(x/|x|)$, we obtain

$$\begin{aligned}
\int_K |(-\Delta)^s(u_s\eta_R)(x) - (-\Delta)^s u_s(x)|\mathrm{d}x &= \int_K \mathrm{d}x \left| C(n,s)\text{P.V}\int_{\mathbb{R}^n} \frac{u_s(x)\eta_R(x) - u_s(y)\eta_R(y) + u_s(y) - u_s(x)}{|x-y|^{n+2s}} \mathrm{d}y \right. \\
&= C(n,s)\int_K \mathrm{d}x \left(\text{P.V}\int_{C\setminus B_R} \frac{u_s(y)[1-\eta_R(y)]}{|x-y|^{n+2s}}\mathrm{d}y \right) \\
&\leq C(n,s)\int_K \mathrm{d}x \left(\text{P.V}\int_{C\setminus B_R} \frac{|y|^{\gamma_s}}{(|y|-\overline{r})^{n+2s}}\mathrm{d}y \right) \\
&\leq C(n,s)\int_K \mathrm{d}x \left(\text{P.V}\int_{C\setminus B_R} \frac{|y|^{\gamma_s}}{|y|^{n+2s}(1-\frac{\overline{r}}{R})^{n+2s}}\mathrm{d}y \right) \\
&= C\left(\frac{R}{R-\overline{r}}\right)^{n+2s} \lim_{\rho\to+\infty}\int_R^\rho \frac{1}{r^{2s-\gamma_s+1}}\mathrm{d}r \\
&= C\left(\frac{R}{R-\overline{r}}\right)^{n+2s} \frac{1}{R^{2s-\gamma_s}} \longrightarrow 0,
\end{aligned}$$

as $R \to +\infty$. Hence we obtain also pointwise convergence almost everywhere. Moreover, we can give the following expression

$$(-\Delta)^s(u_s\eta_R)(x) = \eta_R(x)(-\Delta)^s u_s(x) + C(n,s)\text{P.V.}\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y. \tag{75}$$

We remark that $\eta_R(x)(-\Delta)^s u_s(x) \to (-\Delta)^s u_s(x)$ and $\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y \to 0$ pointwisely. Moreover we can dominate the first term in the following way

$$\eta_R(x)(-\Delta)^s u_s(x) \leq (-\Delta)^s u_s(x),$$

and

$$\int_{\mathbb{R}^n} (-\Delta)^s u_s(x)(-\Delta)^\delta\varphi(x)\mathrm{d}x < +\infty$$

since $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ and using Proposition 2.2.2 over $\varphi \in C_c^\infty(C)$. In order to prove (74), we want to apply the dominated convergence theorem, and hence we need the following condition for any $R > 0$

$$I := \left| \int_{\mathbb{R}^n} (-\Delta)^\delta \varphi(x) \left(\text{P.V.} \int_{\mathbb{R}^n} u_s(y) \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{n+2s}} \mathrm{d}y \right) \mathrm{d}x \right| \leq c.$$

Therefore, we will obtain a stronger condition; that is, the existence of a value $k > 0$ such that for any $R > 1$

$$I \leq \frac{c}{R^k}.$$

We split the region of integration $\mathbb{R}^n \times \mathbb{R}^n$ into five different parts; that is,

$$\Omega_1 := (\mathbb{R}^n \setminus B_{2R}) \times \mathbb{R}^n,\ \Omega_2 := B_{2R} \times B_{2R},\ \Omega_3 := (B_{2R} \setminus B_R) \times (B_{3R} \setminus B_{2R}),$$

$$\Omega_4 := (B_{2R} \setminus B_R) \times (\mathbb{R}^n \setminus B_{3R}),\ \Omega_5 := B_R \times (\mathbb{R}^n \setminus B_{2R}).$$

First of all, we remark that $(-\Delta)^s \eta_R(x) = R^{-2s}(-\Delta)^s \eta(x/R)$ and also that $||(-\Delta)^s \eta||_{L^\infty(\mathbb{R}^n)} < +\infty$. For the first term, using the fact that $\eta_R(x) - \eta_R(y) = 0$ if $(x, y) \in (\mathbb{R}^n \setminus B_{2R}) \times (\mathbb{R}^n \setminus B_{2R})$

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^n \setminus B_{2R}} |(-\Delta)^\delta \varphi(x)| \left| \int_{\mathbb{R}^n} u_s(y) \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{n+2s}} \mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n \setminus B_{2R}} \left|(-\Delta)^\delta \varphi(x)\right| \left| \int_{B_{2R}} u_s(y) \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{n+2s}} \right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n \setminus B_{2R}} |(-\Delta)^\delta \varphi(x)| \left(\sup_{B_{2R}} u_s \right) |(-\Delta)^s \eta_R(x)| \mathrm{d}x \\
&\leq \frac{c}{R^{2s-\gamma_s}} \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}} \mathrm{d}x \leq \frac{c}{R^{2s-\gamma_s}}.
\end{aligned}$$

For the second term, using the fact that $\eta_R(x) - \eta_R(y) \geq 0$ if $(x, y) \in B_{2R} \times (\mathbb{R}^n \setminus B_{2R})$, we obtain as before

$$\begin{aligned}
I_2 &:= \int_{B_{2R}} |(-\Delta)^\delta \varphi(x)| \left| \int_{B_{2R}} u_s(y) \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{n+2s}} \mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_{B_{2R}} |(-\Delta)^\delta \varphi(x)| \left(\sup_{B_{2R}} u_s \right) |(-\Delta)^s \eta_R(x)| \mathrm{d}x \\
&\leq \frac{c}{R^{2s-\gamma_s}} \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}} \mathrm{d}x \leq \frac{c}{R^{2s-\gamma_s}}.
\end{aligned}$$

For the third part

$$I_3 := \int_{B_{2R}\setminus B_R} |(-\Delta)^\delta \varphi(x)| \left| \int_{B_{3R}\setminus B_{2R}} u_s(y) \frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}} \mathrm{d}y \right| \mathrm{d}x,$$

we consider the following change of variables $\xi = x/R \in B_2 \setminus B_1$ and $\zeta = y/R \in B_3 \setminus B_2$. Hence, using the γ_s-homogeneity of u_s and the definition of our cut-off functions, we obtain

$$I_3 \leq \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| u_s(\zeta) \frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}} \mathrm{d}\xi \mathrm{d}\zeta.$$

We use the fact that $u_s \in C^{0,s}(B_3 \setminus B_1)$ (see (63) proved in [67]) and the cut off function $\eta \in \mathrm{Lip}(B_3 \setminus B_1)$; that is, there exists a constant $c > 0$ such that

$$|u_s(\xi) - u_s(\zeta)| \leq c|\xi-\zeta|^s \qquad \text{and} \qquad |\eta(\xi)-\eta(\zeta)| \leq c|\xi-\zeta|, \tag{76}$$

for every $\xi, \zeta \in B_3 \setminus B_1$. Hence,

$$\begin{aligned} I_3 \quad &\leq \quad \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| \frac{|u_s(\zeta)-u_s(\xi)|\,|\eta(\xi)-\eta(\zeta)|}{|\xi-\zeta|^{n+2s}} \mathrm{d}\xi \mathrm{d}\zeta \\ &\quad + \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| u_s(\xi) \frac{|\eta(\xi)-\eta(\zeta)|}{|\xi-\zeta|^{n+2s}} \mathrm{d}\xi \mathrm{d}\zeta \\ &= \quad J_1 + J_2. \end{aligned}$$

By (76), we obtain

$$\begin{aligned} J_1 \quad &\leq \quad c\frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| \frac{|\xi-\zeta|^{s+1}}{|\xi-\zeta|^{n+2s}} \mathrm{d}\xi \mathrm{d}\zeta \\ &\leq \quad c\frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} \frac{1}{(1+R|\xi|)^{n+2\delta}} \frac{1}{|\xi-\zeta|^{n+s-1}} \mathrm{d}\xi \mathrm{d}\zeta \\ &\leq \quad \frac{c}{R^{2s+2\delta-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} \frac{1}{|\xi-\zeta|^{n+s-1}} \mathrm{d}\xi \mathrm{d}\zeta \leq \frac{c}{R^{2s+2\delta-\gamma_s}}. \end{aligned}$$

Moreover, using other two changes of variable $(\xi,\zeta)\mapsto(\xi,\xi+h)$ and $(\xi,\zeta)\mapsto(\xi,\xi-h)$, we obtain

$$\begin{aligned}
J_2 &\leq \frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)}|(-\Delta)^\delta\varphi(R\xi)|u_s(\xi)\frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)}\frac{1}{(1+R\,|\xi|)^{n+2\delta}}u_s(\xi)\frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)}\frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\left(c+\iint_{(B_2\setminus B_1)\times B_\varepsilon}\frac{<\nabla^2\eta(\xi)h,h>}{|h|^{n+2s}}\mathrm{d}\xi\mathrm{d}h\right)\leq\frac{c}{R^{2s+2\delta-\gamma_s}}.
\end{aligned}$$

For the fourth part

$$I_4 := \int_{B_{2R}\setminus B_R}|(-\Delta)^\delta\varphi(x)|\left|\int_{\mathbb{R}^n\setminus B_{3R}}u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y\right|\mathrm{d}x,$$

we consider, as before, the following change of variables $\xi=x/R\in B_2\setminus B_1$ and $\zeta=y/R\in\mathbb{R}^n\setminus B_3$. Hence,

$$\begin{aligned}
I_4 &\leq c\frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)}|(-\Delta)^\delta\varphi(R\xi)|\frac{|\zeta|^{\gamma_s}}{|\zeta-\xi|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq c\frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)}\frac{1}{(1+R|\xi|)^{n+2\delta}}\frac{|\zeta|^{\gamma_s}}{|\zeta-\frac{2\zeta}{|\zeta|}|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)}\frac{|\zeta|^{\gamma_s}}{|\zeta|^{n+2s}(1-\frac{2}{|\zeta|})^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)}\frac{1}{|\zeta|^{n+2s-\gamma_s}}\mathrm{d}\xi\mathrm{d}\zeta\leq\frac{c}{R^{2s+2\delta-\gamma_s}}.
\end{aligned}$$

Eventually, we consider the last term

$$I_5 := \int_{B_R}|(-\Delta)^\delta\varphi(x)|\left|\int_{\mathbb{R}^n\setminus B_{2R}}u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y\right|\mathrm{d}x.$$

Hence we obtain

$$\begin{aligned} I_5 &\leq c\int_{B_R}|(-\Delta)^\delta\varphi(x)|\left(\int_{\mathbb{R}^n\setminus B_{2R}}\frac{|y|^{\gamma_s}}{|y-x|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\ &\leq c\int_{B_R}|(-\Delta)^\delta\varphi(x)|\left(\int_{\mathbb{R}^n\setminus B_{2R}}\frac{|y|^{\gamma_s}}{|y-\frac{Ry}{|y|}|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\ &\leq c\int_{B_R}|(-\Delta)^\delta\varphi(x)|\left(\int_{\mathbb{R}^n\setminus B_{2R}}\frac{1}{|y|^{n+2s-\gamma_s}}\mathrm{d}y\right)\mathrm{d}x \\ &\leq c\left(\int_{\mathbb{R}^n}\frac{1}{(1+|x|)^{n+2\delta}}\mathrm{d}x\right)\left(\int_{2R}^{+\infty}\frac{1}{r^{1+2s-\gamma_s}}\mathrm{d}r\right)\frac{c}{R^{2s-\gamma_s}}, \end{aligned}$$

which it implies the desired result. □

At this point, fixed $s \in (0,1)$, by the distributional semigroup property we can compute easily high order fractional Laplacians $(-\Delta)^{s+\delta}$ viewing it as the δ-Laplacian of the s-Laplacian.

Corollary 2.3.5. *Let C be a regular cone. For every $\delta \in (0, 1-s)$, the solution u_s of (55) is $(s+\delta)$-superharmonic in C in the sense of distribution, i.e.*

$$((-\Delta)^{s+\delta}u_s, \varphi) \geq 0$$

for every test function $\varphi \in C_c^\infty(C)$ nonnegative in C.
Moreover, u_s is also superharmonic in C in the sense of distribution, i.e.

$$(-\Delta u_s, \varphi) \geq 0$$

for every test function $\varphi \in C_c^\infty(C)$ nonnegative in C.

Proof. As said before, the facts that $u_s \in \mathcal{L}^1_{s+\delta}$ and $u_s \in C^2(A)$ for every $A \subset\subset C$ ensure the existence of the $(-\Delta)^{s+\delta}u_s$ and the continuity of the map $x \mapsto (-\Delta)^{s+\delta}u_s(x)$ for every $x \in A \subset\subset C$. Hence at this point, the only part we need to prove is the positivity of the $(s+\delta)$-Laplacian in the sense of the distribution, which is a direct consequence of the previous result. Indeed, since u_s is a solution of the problem (55), by Proposition 2.3.4 we know that for every $\varphi \in C_c^\infty(C)$ we have

$$\begin{aligned} ((-\Delta)^{s+\delta}u_s, \varphi) &= ((-\Delta)^\delta[(-\Delta)^s u_s], \varphi) \\ &= \int_C \varphi(x)\ \text{P.V.}\int_{\mathbb{R}^n}\frac{(-\Delta)^s u_s(x) - (-\Delta)^s u_s(y)}{|x-y|^{n+2\delta}}\mathrm{d}y\mathrm{d}x. \end{aligned}$$

where $(-\Delta)^\delta[(-\Delta)^s u_s]$ is well defined since that $(-\Delta)^s u_s \equiv 0 \in C^2(A)$ for every $A \subset\subset C$ and, by Lemma 2.3.3, $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ for every $\delta \in (0, 1-s)$.
Consider now nonnegative test function $\varphi \geq 0$ in C, since $(-\Delta)^s u_s(x) = 0$ for every $x \in C$, we have for every $x \in \mathbb{R}^n \setminus \overline{C}$

$$(-\Delta)^s u_s(x) = -\int_C \frac{u_s(y)}{|x-y|^{n+2s}} \mathrm{d}y \leq 0.$$

Similarly,

$$((-\Delta)^\delta[(-\Delta)^s u_s], \varphi) = \int_C \varphi(x) \int_{\mathbb{R}^n} \frac{-(-\Delta)^s u_s(y)}{|x-y|^{n+2\delta}} \mathrm{d}y\mathrm{d}x \geq 0,$$

since the support of φ is compact in the cone C, and so there exists $\varepsilon > 0$ such that $|x-y| > \varepsilon$ in the above integral. We have obtained that for any $\delta \in (0, 1-s)$ and any nonnegative $\varphi \in C_c^\infty(C)$

$$((-\Delta)^{s+\delta} u_s, \varphi) \geq 0,$$

then, passing to the limit for $\delta \to 1-s$, the function u_s is superharmonic in the distributional sense

$$0 \leq \lim_{\delta \to 1-s} ((-\Delta)^{s+\delta} u_s, \varphi) = \lim_{\delta \to 1-s} (u_s, (-\Delta)^{s+\delta}\varphi) = (u_s, -\Delta\varphi) = (-\Delta u_s, \varphi).$$

□

2.3.2 *Monotonicity of $s \mapsto \gamma_s(C)$*

The following proposition is a consequence of Corollary 2.3.5 and it follows essentially the proof of Lemma 2 in [9].

Proposition 2.3.6. *For any fixed regular cone C with vertex in 0, the map $s \mapsto \gamma_s(C)$ is monotone non decreasing in $(0, 1)$.*

Proof. Fixed the cone C, let us denote with γ_s and $\gamma_{s+\delta}$ respectively the homogeneities of u_s and $u_{s+\delta}$. Let us suppose by contradiction that $\gamma_s > \gamma_{s+\delta}$ for a $\delta \in (0, 1-s)$, and let us consider the function

$$h(x) = u_{s+\delta}(x) - u_s(x) \quad \text{in } \mathbb{R}^n,$$

where u_s is the homogeneous solution of (55) and $u_{s+\delta}$ is the unique, up to multiplicative constants, nonnegative nontrivial homogeneous and continuous in $\mathbb{R}^n$ solution for

$$\begin{cases} (-\Delta)^{s+\delta} u = 0, & \text{in } \quad C, \\ u = 0, & \text{in } \quad \mathbb{R}^n \setminus C, \end{cases}$$

of the form

$$u_{s+\delta}(x) = |x|^{\gamma_{s+\delta}} u_{s+\delta}\left(\frac{x}{|x|}\right).$$

The function h is continuous in $\mathbb{R}^n$ and $h(x) = 0$ in $\mathbb{R}^n \setminus C$. We want to prove that $h(x) \leq 0$ in $\mathbb{R}^n \setminus (C \cap B_1)$. Since $h = 0$ outside the cone, we can consider only what happens in $C \setminus B_1$. As we already quoted, we have

$$c_1(s)|x|^{\gamma_s - s}\text{dist}(x, \partial C)^s \leq u_s(x) \leq c_2(s)|x|^{\gamma_s - s}\text{dist}(x, \partial C)^s, \tag{77}$$

for any $x \in \overline{C} \setminus \{0\}$, and there exist two constants $c_1(s+\delta), c_2(s+\delta) > 0$ such that

$$c_1(s+\delta)|x|^{\gamma_{s+\delta}-(s+\delta)}\text{dist}(x, \partial C)^{s+\delta} \leq u_{s+\delta}(x) \leq c_2(s+\delta)|x|^{\gamma_{s+\delta}-(s+\delta)}\text{dist}(x, \partial C)^{s+\delta}.$$

We can choose u_s and $u_{s+\delta}$ so that $c := c_1(s) = c_2(s+\delta)$ since they are defined up to a multiplicative constant. Then, for any $x \in C \setminus B_1$, since $|x|^{\gamma_{s+\delta}} \leq |x|^{\gamma_s}$, we have

$$h(x) \leq c|x|^{\gamma_s}\text{dist}(x, \partial C)^s \left[\frac{\text{dist}(x, \partial C)^\delta}{|x|^\delta} - 1\right] \leq 0. \tag{78}$$

In fact, if we take x such that $\text{dist}(x, \partial C) \leq 1$, then (78) follows by

$$\frac{\text{dist}(x, \partial C)^\delta}{|x|^\delta} - 1 \leq \text{dist}(x, \partial C)^\delta - 1 \leq 0.$$

Instead, if we consider x so that $\text{dist}(x, \partial C) > 1$, then $\text{dist}(x, \partial C)^\delta < |x|^\delta$ and hence (78) follows.

Now we want to show that there exists a point $x_0 \in C \cap B_1$ such that $h(x_0) > 0$. Let us take a point $\overline{x} \in S^{n-1} \cap C$ and let $\alpha := u_{s+\delta}(\overline{x}) > 0$ and $\beta := u_s(\overline{x}) > 0$. Hence, there exists a small $r > 0$ so that $\alpha r^{\gamma_{s+\delta}} > \beta r^{\gamma_s}$, and so, taking x_0 with $|x_0| = r$ and so that $\frac{x_0}{|x_0|} = \overline{x}$, we obtain $h(x_0) > 0$.

If we consider the restriction of h to $\overline{C \cap B_1}$, which is continuous on a compact set, for the considerations done before and for the Weierstrass Theorem, there exists a maximum point $x_1 \in C \cap B_1$ for the function h which is global in $\mathbb{R}^n$ and is strict at least in a set of positive measure. Hence,

$$(-\Delta)^{s+\delta}h(x_1) = C(n,s)\ \text{P.V.} \int_{\mathbb{R}^n} \frac{h(x_1) - h(y)}{|x_1 - y|^{n+2(s+\delta)}}\, dy > 0,$$

and since $(-\Delta)^{s+\delta}h$ is a continuous function in the open cone, there exists an open set $U(x_1)$ with $\overline{U(x_1)} \subset C$ such that

$$(-\Delta)^{s+\delta}h(x) > 0 \quad \forall x \in U(x_1).$$

But thanks to Corollary 2.3.5 we obtain a contradiction since for any nonnegative $\varphi \in C_c^\infty(U(x_1))$

$$((-\Delta)^{s+\delta} h, \varphi) = ((-\Delta)^{s+\delta} u_{s+\delta}, \varphi) - ((-\Delta)^{s+\delta} u_s, \varphi) = -((-\Delta)^{s+\delta} u_s, \varphi) \leq 0.$$

□

With the same argument of the previous proof we can show also the following useful upper bound.

Proposition 2.3.7. *For any fixed regular cone C with vertex in 0 and any $s \in (0,1)$, $\gamma_s(C) \leq \gamma(C)$.*

Proof. Seeking a contradiction, we suppose that there exists $s \in (0,1)$ such that $\gamma_s > \gamma$. Hence we define the function

$$h(x) = u(x) - u_s(x) \quad \text{in } \mathbb{R}^n,$$

where u_s and u are respectively solutions to (55) and

$$\begin{cases} -\Delta u = 0, & \text{in} \quad C, \\ u = 0, & \text{in} \quad \mathbb{R}^n \setminus C. \end{cases} \tag{79}$$

We recall that these solutions are unique, up to multiplicative constants, nonnegative nontrivial homogeneous and continuous in $\mathbb{R}^n$ of the form

$$u(x) = |x|^\gamma u\left(\frac{x}{|x|}\right), \qquad u_s(x) = |x|^{\gamma_s} u_s\left(\frac{x}{|x|}\right).$$

for some $\gamma_s \in (0, 2s)$ and $\gamma \in (0, +\infty)$. The function h is continuous in $\mathbb{R}^n$ and $h(x) = 0$ in $\mathbb{R}^n \setminus C$. We want to prove that $h(x) \leq 0$ in $\mathbb{R}^n \setminus (C \cap B_1)$. Since $h = 0$ outside the cone, we can consider only what happens in $C \setminus B_1$. So, there exist two constants $c_1(s), c_2(s) > 0$ such that, for any $x \in \overline{C} \setminus \{0\}$, it holds (77). Moreover there exist two constants $c_1, c_2 > 0$ such that,

$$c_1|x|^{\gamma-1}\text{dist}(x, \partial C) \leq u(x) \leq c_2|x|^{\gamma-1}\text{dist}(x, \partial C).$$

We can choose u_s and u so that $c := c_1(s) = c_2$ since they are defined up to a multiplicative constant. Then, for any $x \in C \setminus B_1$, since $|x|^\gamma \leq |x|^{\gamma_s}$, we have

$$h(x) \leq c|x|^{\gamma_s}\text{dist}(x, \partial C)^s \left[\frac{\text{dist}(x, \partial C)^{1-s}}{|x|^{1-s}} - 1\right] \leq 0,$$

with the same arguments of the previous proof.

Now we want to show that there exists a point $x_0 \in C \cap B_1$ such that $h(x_0) > 0$. Let us take a point $\overline{x} \in S^{n-1} \cap C$ and let $\alpha := u(\overline{x}) > 0$ and $\beta := u_s(\overline{x}) > 0$. Hence, there exists a small $r > 0$ so that $\alpha r^{\gamma} > \beta r^{\gamma_s}$, and so, taking x_0 with $|x_0| = r$ and so that $\frac{x_0}{|x_0|} = \overline{x}$, we obtain $h(x_0) > 0$.

If we consider the restriction of h to $\overline{C \cap B_1}$, which is continuous on a compact set, for the considerations done before and for the Weierstrass Theorem, there exists at least a maximum point in $C \cap B_1$ for the function h which is global in $\mathbb{R}^n$. Moreover, since h cannot be constant on $C \cap B_1$ and it is of class C^2 inside the cone, there exists a global maximum $y \in C \cap B_1$ such that, up to a rotation, $\partial^2_{x_i x_i} h(y) \leq 0$ for any $i = 1, ..., n$ and $\partial^2_{x_j x_j} h(y) < 0$ for at least a coordinate direction. Hence

$$\Delta h(y) = \sum_{i=1}^{n} \partial^2_{x_i x_i} h(y) < 0.$$

By the continuity of Δh in the open cone, there exists an open set $U(y)$ with $\overline{U(y)} \subset C$ such that

$$\Delta h(x) < 0 \quad \forall x \in U(y).$$

Since, by Corollary 2.3.5 for any nonnegative $\varphi \in C_c^{\infty}(U(y))$

$$(-\Delta u_s, \varphi) \geq 0,$$

hence

$$(\Delta h, \varphi) = (\Delta u, \varphi) - (\Delta u_s, \varphi) = (-\Delta u_s, \varphi) \geq 0,$$

and this is a contradiction. □

2.3.3 *Asymptotic behavior of* $\frac{C(n,s)}{2s-\gamma_s(C)}$

Let us define for any regular cone C the limit

$$\mu(C) = \lim_{s \to 1^-} \frac{C(n,s)}{2s - \gamma_s(C)} \in [0, +\infty].$$

Obviously, thanks to the monotonicity of $s \mapsto \gamma_s(C)$ in $(0, 1)$, this limit does exist, but we want to show that $\mu(C)$ can not be infinite. At this point, this situation can happen since $2s - \gamma_s(C)$ can converge to zero and we do not have enough information about this convergence. The study of this limit depends on the cone C itself and so we will consider separately the case of wide cones and narrow cones, which are respectively when $\gamma(C) < 2$ and when $\gamma(C) \geq 2$. In this Section, we prove this result just for regular cones, while in Section 2.4 we will extend the existence of a finite limit $\mu(C)$ to any unbounded cone, without the monotonicity result of Proposition 2.3.6.

Wide cones: $\gamma(C) < 2$

We remark that, fixed a wide cone $C \subset \mathbb{R}^n$, then there exists $\varepsilon > 0$ and $s_0 \in (0,1)$, both depending on C, such that for any $s \in [s_0, 1)$

$$2s - \gamma_s(C) \geq \varepsilon > 0.$$

In fact we know that $s \mapsto \gamma_s(C)$ is monotone non decreasing in $(0,1)$ and $0 < \gamma_s(C) \leq \gamma(C) < 2$. Hence, defining $\overline{\gamma}(C) = \lim_{s\to 1} \gamma_s(C) \in (0,2)$ we can choose

$$s_0 := \frac{\overline{\gamma}(C) - 2}{4} + 1 \in (1/2, 1) \quad \text{and} \quad \varepsilon := \frac{2 - \overline{\gamma}(C)}{2} > 0,$$

obtaining

$$2s - \gamma_s(C) \geq 2s_0 - \overline{\gamma}(C) = \varepsilon > 0.$$

As a consequence we obtain $\mu(C) = 0$ for any wide cone.

Narrow cones: $\gamma(C) \geq 2$

Before addressing the asymptotic analysis for any regular cone, we focus our attention on the spherical caps ones with "small" aperture. Hence, let us fix $\theta_0 \in (0, \pi/4)$ and for any $\theta \in (0, \theta_0]$, let

$$\lambda_1(\theta) := \lambda_1(\omega_\theta) = \min_{\substack{u \in H_0^1(S^{n-1} \cap C_\theta) \\ u \neq 0}} \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \, \mathrm{d}\sigma}{\int_{S^{n-1}} u^2 \mathrm{d}\sigma}.$$

We have that $\lambda_1(\theta) > 2n$, and hence the following problem is well defined

$$\mu_0(\theta) := \min_{\substack{u \in H_0^1(S^{n-1} \cap C_\theta) \\ u \neq 0}} \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 - 2nu^2 \mathrm{d}\sigma}{\left(\int_{S^{n-1}} |u| \mathrm{d}\sigma\right)^2}. \tag{80}$$

This number $\mu_0(\theta)$ is strictly positive and achieved by a nonnegative $\varphi \in H_0^1(S^{n-1} \cap C_\theta) \setminus \{0\}$ which is strictly positive on $S^{n-1} \cap C_\theta$ and is obviously solution to

$$\begin{cases} -\Delta_{S^{n-1}} \varphi = 2n\varphi + \mu_0(\theta) \displaystyle\int_{S^{n-1}} \varphi \mathrm{d}\sigma & \text{in} \quad S^{n-1} \cap C_\theta, \\ \varphi = 0 & \text{in} \quad S^{n-1} \setminus C_\theta, \end{cases} \tag{81}$$

where $-\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on the unitary sphere S^{n-1}.
Let now v be the 0-homogeneous extension of φ to the whole of $\mathbb{R}^n$ and $r(x) := |x|$. Such a function will be solution to

$$\begin{cases} -\Delta v = \frac{2nv}{r^2} + \frac{\mu_0(\theta)}{r^2} \displaystyle\int_{S^{n-1}} v \mathrm{d}\sigma & \text{in} \quad C_\theta, \\ v = 0 & \text{in} \quad \mathbb{R}^n \setminus C_\theta. \end{cases} \tag{82}$$

Since the spherical cap $C_\theta \cap S^{n-1}$ is an analytic submanifold of S^{n-1} and the data $(\partial C_\theta \cap S^{n-1}, 0, \partial_\nu \varphi)$ are not characteristic, by the classic theorem of Cauchy-Kovalevskaya we can extend the solution φ of (81) to a function $\tilde{\varphi}$, which is defined in a enlarged cone and it satisfies

$$\begin{cases} -\Delta_{S^{n-1}}\tilde{\varphi} = 2n\tilde{\varphi} + \mu_0(\theta)\displaystyle\int_{S^{n-1}} \varphi \mathrm{d}\sigma & \text{in } S^{n-1} \cap C_{\theta+\varepsilon}, \\ \tilde{\varphi} = \varphi & \text{in } S^{n-1} \cap C_\theta, \end{cases}$$

for some $\varepsilon > 0$. As in (82), we can define $\tilde{v}$ as the 0-homogenous extension of $\tilde{\varphi}$. Finally, we introduce the following function

$$v_s(x) := r(x)^{\gamma_s^*(\theta)} v(x),$$

where the choice of the homogeneity exponent $\gamma_s^*(\theta) \in (0, 2s)$ will be suggested by the following important result.

Theorem 2.3.8. *Let $\theta \in (0, \theta_0]$, then there exists $s_0 = s_0(\theta) \in (0,1)$ such that*

$$(-\Delta)^s v_s(x) \le 0 \qquad \text{in } C_\theta,$$

for any $s \in [s_0, 1)$.

Proof. By the $\gamma_s^*(\theta)$-homogeneity of v_s, it is sufficient to prove that $(-\Delta)^s v_s \le 0$ on $C_\theta \cap S^{n-1}$, since $x \mapsto (-\Delta)^s v_s$ is $(\gamma_s^*(\theta) - 2s)$-homogenous. In order to ease the notations, through the following computations we will simply use γ instead of $\gamma_s^*(\theta)$ and $o(1)$ for the terms which converge to zero as s goes to 1. Hence, for $x \in S^{n-1} \cap C_\theta$, we have

$$(-\Delta)^s v_s(x) = |x|^\gamma (-\Delta)^s v(x) + v(x)(-\Delta)^s r^\gamma(x) - C(n,s)\int_{\mathbb{R}^n} \frac{(r^\gamma(x) - r^\gamma(y))(v(x) - v(y))}{|x-y|^{n+2s}}\mathrm{d}y.$$

First for $R > 0$,

$$\begin{aligned} (-\Delta)^s r^\gamma(x) &= C(n,s)\int_{B_R(x)} \frac{|x|^\gamma - |y|^\gamma}{|x-y|^{n+2s}}\mathrm{d}y + C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{|x|^\gamma - |y|^\gamma}{|x-y|^{n+2s}}\mathrm{d}y \\ &= \frac{C(n,s)}{2}\int_{B_R(0)} \frac{2|x|^\gamma - |x+z|^\gamma - |x-z|^\gamma}{|z|^{n+2s}}\mathrm{d}z + C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{1 - |y|^\gamma}{|x-y|^{n+2s}}\mathrm{d}y \\ &= -\frac{C(n,s)}{2}\int_0^R \frac{\rho^2\rho^{n-1}}{\rho^{n+2s}}\mathrm{d}\rho \int_{S^{n-1}} \langle \nabla^2 |x|^\gamma z, z\rangle \mathrm{d}\sigma + o(1) + \\ &\quad + C(n,s)\left|S^{n-1}\right| \int_R^\infty \frac{1}{\rho^{1+2s}}\mathrm{d}\rho - C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{|y|^\gamma}{|x-y|^{n+2s}}\mathrm{d}y \\ &= -\frac{C(n,s)}{2}\frac{R^{2-2s}}{2-2s}\int_{S^{n-1}} \langle \nabla^2 |x|^\gamma z, z\rangle \mathrm{d}\sigma + \\ &\quad - C(n,s)\int_R^\infty \frac{\rho^{n-1+\gamma}}{\rho^{n+2s}}\int_{S^{n-1}} \left|\frac{x}{\rho} - \vartheta\right|^\gamma \mathrm{d}\sigma(\vartheta)\mathrm{d}\rho + o(1). \end{aligned}$$

Since for every symmetric matrix A we have

$$\int_{S^{n-1}} \langle Az, z\rangle \mathrm{d}\sigma = \frac{\mathrm{tr}A}{n}\omega_{n-1}$$

where ω_{n-1} is the Lebesgue measure of the $(n-1)$-sphere S^{n-1}, we can simplify the first term since $\mathrm{tr}\nabla^2 |x|^\gamma = \Delta(|x|^\gamma)$ and checking that $\left|\frac{x}{\rho} - \vartheta\right|^\gamma = 1 + \gamma\rho^{-1}\langle\vartheta, x\rangle + o(\rho^{-1})$ as $\rho \to \infty$ it follows

$$\begin{aligned}
(-\Delta)^s r^\gamma(x) =& -\frac{C(n,s)}{2}\frac{R^{2-2s}}{2-2s}\frac{\Delta(|x|^\gamma)\omega_{n-1}}{n} - C(n,s)\omega_{n-1}\int_R^\infty \frac{\rho^{n-1+\gamma}}{\rho^{n+2s}}\mathrm{d}\rho + o(1)\\
=& -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma)\,|x|^{\gamma-2} R^{2-2s} - \frac{C(n,s)}{2s-\gamma}\omega_{n-1}R^{\gamma-2s} + o(1)\\
=& -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma)R^{2-2s} - \frac{C(n,s)}{2s-\gamma}\omega_{n-1}R^{\gamma-2s} + o(1)\\
=& -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma) - \frac{C(n,s)}{2s-\gamma}\omega_{n-1} + o(1),
\end{aligned}$$

where in the last equality we choose $\gamma = \gamma_s^*(\theta)$ such that $\gamma_s^*(\theta) - 2s \to 0$ as s goes to 1.
Similarly, if $\tilde{v}$ is the 0-homogenous extension of v in an enlarged cone, which is such that $v \geq \tilde{v}$ and $v = \tilde{v}$ on $C_\theta \cap S^{n-1}$, it follows

$$\begin{aligned}
(-\Delta)^s v(x) =& \frac{C(n,s)}{2}\int_{|z|<1} \frac{2v(x) - v(x+z) - v(x-z)}{|z|^{n+2s}}\mathrm{d}z + C(n,s)\int_{|x-y|>1}\frac{v(x)-v(y)}{|x-y|^{n+2s}}\mathrm{d}y\\
\leq& \frac{C(n,s)}{2}\int_{|z|<1} \frac{2\tilde{v}(x) - \tilde{v}(x+z) - \tilde{v}(x-z)}{|z|^{n+2s}}\mathrm{d}z + C(n,s)\int_1^\infty \frac{\rho^{n-1}}{\rho^{n+2s}}\int_{S^{n-1}} v(x) - v(y)\mathrm{d}\sigma\mathrm{d}\rho\\
=& -\frac{C(n,s)}{2}\int_0^1 \frac{\rho^{n-1}\rho^2}{\rho^{n+2s}}\int_{S^{n-1}}\langle\nabla^2\tilde{v}(x)z, z\rangle\mathrm{d}\sigma\mathrm{d}\rho + o(1)\\
=& \frac{C(n,s)\omega_{n-1}}{4n(1-s)}(-\Delta)\tilde{v}(x) + o(1),
\end{aligned}$$

where we can use that $\tilde{v}$ solves

$$-\Delta\tilde{v} = 2n\tilde{v} + \mu_0\int_{S^{n-1}} v\mathrm{d}\sigma$$

in the enlarged cap $S^{n-1} \cap C_{\theta+\varepsilon}$. Finally,

$$\begin{aligned}
C(n,s)\int_{\mathbb{R}^n}\frac{(|x|^\gamma - |y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y =& C(n,s)\left[\int_{|y|<1}\frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y + \right.\\
&\left. + \int_{|y|>1}\frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y\right]
\end{aligned}$$

where the first term is $o(1)$ since

$$\int_0^1 (1-\rho^\gamma)\rho^{n-1}\int_{S^{n-1}}\frac{v(x)-v(y)}{|x-\rho y|^{n+2s}}\mathrm{d}\sigma\mathrm{d}\rho = \int_0^1 (1-\rho^\gamma)\rho^{n-1}\int_{S^{n-1}}(v(x)-v(y))(1+o(\rho))\mathrm{d}\sigma\mathrm{d}\rho + \int_0^R (1-\rho^\gamma)\rho^{n-1}\int_{S^{n-1}}(v(x)-v(y))(n+2s)\rho\langle x,y\rangle\mathrm{d}\sigma\mathrm{d}\rho.$$

Hence, we obtain

$$\begin{aligned}C(n,s)\int_{\mathbb{R}^n}\frac{(|x|^\gamma-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y =& C(n,s)\int_{|y|>1}\frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y+o(1)\\ =& o(1)-C(n,s)\int_{|y|>1}\frac{|y|^\gamma(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y+o(1)\\ =& o(1)-C(n,s)\int_1^\infty \rho^\gamma\rho^{n-1}\int_{S^{n-1}}\frac{v(x)-v(y)}{|x-\rho y|^{n+2s}}\mathrm{d}\sigma\mathrm{d}\rho\\ =& o(1)-C(n,s)\int_1^\infty \rho^{-1+\gamma-2s}\int_{S^{n-1}}(v(x)-v(y))(1+o(\rho^{-1}))\mathrm{d}\sigma\mathrm{d}\\ &-C(n,s)\int_1^\infty \rho^{-1+\gamma-2s}\int_{S^{n-1}}(v(x)-v(y))(n+2s)\langle y,x\rangle\rho^{-1}\mathrm{d}\sigma\\ =& o(1)-\frac{C(n,s)\omega_{n-1}}{2s-\gamma}v(x)+\frac{C(n,s)}{2s-\gamma}\int_{S^{n-1}}v(y)\mathrm{d}\sigma.\end{aligned}$$

Hence, recalling that $\gamma=\gamma_s^*(\theta)$, for $x\in S^{n-1}\cap C_\theta$ we have

$$\begin{aligned}(-\Delta)^s v_s(x) &\le \left(\mu_0(\theta)\frac{C(n,s)\omega_{n-1}}{4n(1-s)}-\frac{C(n,s)}{2s-\gamma_s^*(\theta)}\right)\int_{S^{n-1}}v_s\mathrm{d}\sigma+\frac{C(n,s)\omega_{n-1}}{4n(1-s)}(n+\gamma_s^*(\theta))(2-\gamma_s^*(\theta))v_s\\ &\le \left(\mu_0(\theta)-\frac{C(n,s)}{2s-\gamma_s^*(\theta)}\right)\int_{S^{n-1}}v_s\mathrm{d}\sigma+o(1)\end{aligned}$$

where $o(1)$ is uniform with respect to $\gamma_s^*(\theta)$ as $s\to 1$. In order to obtain a negative right hand side, it is sufficient to choose $\gamma_s^*(\theta)<2s$ in such a way to make the denominator $2s-\gamma_s^*(\theta)$ small enough and the quotient $\frac{C(n,s)}{2s-\gamma_s^*(\theta)}$ still bounded. □

The previous result suggestes the following choice of the homogeneity exponent

$$\gamma_s^*(\theta):=2s-s\frac{C(n,s)}{\mu_0(\theta)}.$$

We can finally prove the main result of this Section.

Corollary 2.3.9. *For any regular cone C we get $\mu(C)<+\infty$.*

Proof. We will show that $\mu(\theta) < +\infty$ for any $\theta \in (0, \theta_0]$. Then, fixed an unbounded regular cone C, there exists a spherical cone C_θ such that $\theta \in (0, \theta_0]$ and $C_\theta \subset C$. Since by inclusion $\gamma_s(C) < \gamma_s(\theta)$, we obtain

$$\mu(C) \leq \mu(\theta) < +\infty.$$

We want to show that fixed $\theta \in (0, \theta_0]$, $\gamma_s(\theta) \leq \gamma_s^*(\theta)$ for any $s \in [s_0(\theta), 1)$, where the choice of $s_0(\theta) \in (0, 1)$ is given in Theorem 2.3.8. The proof of this fact is based on considerations done in Proposition 2.3.6. By contradiction, $\gamma_s(\theta) > \gamma_s^*(\theta)$. Let

$$h(x) = v_s(x) - u_s(x).$$

The function h is continuous in $\mathbb{R}^n$ and $h(x) = 0$ in $\mathbb{R}^n \setminus C_\theta$. We want to prove that $h(x) \leq 0$ in $\mathbb{R}^n \setminus (C_\theta \cap B_1)$. Since $h = 0$ outside the cone, we can consider only what happens in $C_\theta \setminus B_1$. By (77), there exist two constants $c_1(s), c_2(s) > 0$ such that, for any $x \in \overline{C_\theta} \setminus \{0\}$,

$$c_1(s)|x|^{\gamma_s - s}\text{dist}(x, \partial C_\theta)^s \leq u_s(x) \leq c_2(s)|x|^{\gamma_s - s}\text{dist}(x, \partial C_\theta)^s,$$

and there exist two constants $c_1, c_2 > 0$ such that

$$c_1|x|^{\gamma_s^* - 1}\text{dist}(x, \partial C_\theta) \leq v_s(x) \leq c_2|x|^{\gamma_s^* - 1}\text{dist}(x, \partial C_\theta).$$

We can choose v_s so that $c := c_1(s) = c_2$ since it is defined up to a multiplicative constant. Then, for any $x \in C_\theta \setminus B_1$, since $|x|^{\gamma_s^*} \leq |x|^{\gamma_s}$, we have

$$h(x) \leq c|x|^{\gamma_s}\text{dist}(x, \partial C_\theta)^s \left[\frac{\text{dist}(x, \partial C_\theta)^{1-s}}{|x|^{1-s}} - 1 \right] \leq 0.$$

Now we want to show that there exists a point $x_0 \in C_\theta \cap B_1$ such that $h(x_0) > 0$. Let us consider for example the point $\overline{x} \in S^{n-1} \cap C_\theta$ determined by the angle $\vartheta = \theta/2$, and let $\alpha := v_s(\overline{x}) > 0$ and $\beta := u_s(\overline{x}) > 0$. Hence, there exists a small $r > 0$ so that $\alpha r^{\gamma_s^*} > \beta r^{\gamma_s}$, and so, taking x_0 with angle $\vartheta = \theta/2$ and $|x_0| = r$, we obtain $h(x_0) > 0$.

If we consider the restriction of h to $\overline{C_\theta \cap B_1}$, which is continuous on a compact set, for the considerations done before and for the Weierstrass Theorem, there exists a maximum point $x_1 \in C_\theta \cap B_1$ for the function h which is global in $\mathbb{R}^n$ and is strict at least in a set of positive measure. Hence,

$$(-\Delta)^s h(x_1) = C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{h(x_1) - h(y)}{|x_1 - y|^{n+2s}} \, dy > 0,$$

and since $(-\Delta)^s h$ is a continuous function in the open cone, there exists an open set $U(x_1)$ with $\overline{U(x_1)} \subset C_\theta$ such that

$$(-\Delta)^s h(x) > 0 \quad \forall x \in U(x_1).$$

But thanks to Theorem 2.3.8 we obtain a contradiction since for any nonnegative $\varphi \in C_c^\infty(U(x_1))$

$$((-\Delta)^s h, \varphi) = ((-\Delta)^s v_s, \varphi) - ((-\Delta)^s u_s, \varphi) = ((-\Delta)^s v_s, \varphi) \leq 0,$$

where the last inequality holds for any $s \in [s_0(\theta), 1)$. Hence, for any $\theta \in (0, \theta_0]$

$$\mu(\theta) = \lim_{s \to 1^-} \frac{C(n,s)}{2s - \gamma_s(\theta)} \leq \lim_{s \to 1^-} \frac{C(n,s)}{2s - \gamma_s^*(\theta)} = \mu_0(\theta) < +\infty. \tag{83}$$

□

2.4 THE LIMIT FOR $s \nearrow 1$

In this Section we prove the main result, Theorem 2.1.2, emphasizing the difference between wide and narrow cones. Then we improve the asymptotic analysis proving uniqueness of the limit under assumptions on the geometry and the regularity of C.

Let $C \subset \mathbb{R}^n$ be an open cone and consider the minimization problem

$$\lambda_1(C) = \inf \left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 d\sigma}{\displaystyle\int_{S^{n-1}} u^2 d\sigma} \; : \; u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus C \right\}, \tag{84}$$

which is strictly related to the homogeneity of the solution of (79) by $\lambda_1(C) = \gamma(C)(\gamma(C) + n - 2)$.
Moreover, if $\gamma(C) > 2$, equivalently if $\lambda_1(C) > 2n$, the problem

$$\mu_0(C) := \inf \left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 - 2nu^2 d\sigma}{\left(\displaystyle\int_{S^{n-1}} |u| d\sigma\right)^2} \; : \; u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus C \right\} \tag{85}$$

is well defined and the number $\mu_0(C)$ is strictly positive.

By a standard argument due to the variational characterization of the previous quantities, we already know the existence of a nonnegative eigenfunction $\varphi \in H_0^1(S^{n-1} \cap C) \setminus \{0\}$ associated to the minimization problem (84) and a nonnegative function $\psi \in H_0^1(S^{n-1} \cap C) \setminus \{0\}$ that achieves the minimum (85), since the numerator in (85) is a coercive quadratic form equivalent to the one in (84).

Since the cone C may be disconnected, it is well known that φ is not necessarily unique. Instead, the function ψ is unique up to a multiplicative constant, since it solves

$$\begin{cases} -\Delta_{S^{n-1}}\psi = 2n\psi + \mu_0(C)\displaystyle\int_{S^{n-1}}\psi \mathrm{d}\sigma & \text{in} \quad S^{n-1}\cap C, \\ \psi = 0 & \text{in} \quad S^{n-1}\setminus C. \end{cases} \tag{86}$$

In fact, due to the integral term in the equation, the solution ψ must be strictly positive in every connected component of C and localizing the equation in a generic component we can easily get uniqueness by maximum principle.

A fundamental toll in order to reach as $s \to 1$ the space H^1_{loc}, is the following result

Proposition 2.4.1. *[10, Corollary 7] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For $1 < p < \infty$, let $f_s \in W^{s,p}(\Omega)$, and assume that*

$$[f_s]_{W^{s,p}(\Omega)} \le C_0.$$

Then, up to a subsequence, (f_s) converges in $L^p(\Omega)$ as $s \to 1$(and, in fact, in $W^{t,p}(\Omega)$, for all $t < 1$) to some $f \in W^{1,p}(\Omega)$.

In [10] the authors used a different notation since in our manuscript the normalization constant $C(n,s)$ is incorporate in the seminorm $[\cdot]_{H^s}$, in order to obtain a continuity of the norm $\|\cdot\|_{H^s}$ for $s \in (0,1]$.

Proof of Theorem 2.1.2. Let C be an open cone and C_R be a regular cone with Section on S^{n-1} of class $C^{1,1}$ such that $C_R \subset C$ and $\partial C_R \cap \partial C = \{0\}$.

By monotonicity of the homogeneity degree $\gamma_s(\cdot)$ with respect to the inclusion, we directly obtain $\gamma_s(C) < \gamma_s(C_R)$ and consequently, up to consider a subsequence, we obtain the existence of the following finite limits

$$\overline{\gamma}(C) = \lim_{s\to 1}\gamma_s(C), \quad \mu(C) = \lim_{s\to 1}\frac{C(n,s)}{2s - \gamma_s(C)}. \tag{87}$$

Since $\gamma_s(C) < 2s$, then $\overline{\gamma}(C) \le 2$ and similarly $\mu(C) \in [0, +\infty)$.

Let $K \subset \mathbb{R}^n$ be a compact set and consider $x_0 \in K$ and $R > 0$ such that $K \subset B_R(x_0)$. Given $\eta \in C_c^\infty(B_2)$, a radial cut off function such that $\eta \equiv 1$ in B_1 and $0 \le \eta \le 1$ in B_2, consider the rescaled function $\eta_K(x) = \eta(\frac{x-x_0}{R})$ which satisfies $\eta_K \equiv 1$ on K.
By Proposition 2.2.7, we have

$$[u_s\eta_K]^2_{H^s(B_{2R}(x_0))} \le [u_s\eta_K]^2_{H^s(\mathbb{R}^n)} \le M(n,K)\left[\frac{C(n,s)}{2(1-s)} + \frac{C(n,s)}{2s-\gamma_s}\right],$$

and similarly

$$\begin{aligned}\|u_s\eta_K\|^2_{H^s(B_{2R}(x_0))} &\leq \|u_s\eta_K\|^2_{L^2(\mathbb{R}^n)} + [u_s\eta_K]^2_{H^s(\mathbb{R}^n)} \\ &\leq M(n,K)\left[\frac{C(n,s)}{2(1-s)} + \frac{C(n,s)}{2s-\gamma_s} + 1\right] \\ &\leq M(n,K)\left[\frac{2n}{\omega_{n-1}} + c\mu(C) + 1\right].\end{aligned}$$

By applying Proposition 2.4.1 with $\Omega = B_{2R}(x_0)$, we obtain that, up to a subsequence, $u_s\eta_K \to \overline{u}\eta_K$ in $L^2(B_{2R}(x_0))$ and

$$\|\overline{u}\eta_K\|^2_{H^1(B_{2R}(x_0))} \leq M(n,K)$$

up to relabeling the constant $M(n,K)$.
By construction, since $\eta_K \equiv 1$ on K and $\eta_K \in [0,1]$, we obtain that $u_s \to \overline{u}$ in $L^2(K)$ and similarly

$$\|\overline{u}\|_{H^1(K)} \leq \|\overline{u}\eta_K\|_{H^1(K)} \leq \|\overline{u}\eta_K\|_{H^1(B_{2R}(x_0))} < \infty,$$

which gives us the local integrability in $H^1(\mathbb{R}^n)$.

By Proposition 2.2.4 and Corollary 2.3.9 we obtain, up to pass to a subsequence, uniform in s bound in $C^{0,\alpha}_{\mathrm{loc}}(C)$ for (u_s). Then, since we obtain uniform convergence on compact subsets of C, the limit must be necessary nontrivial with $||\overline{u}||_{L^\infty(S^{n-1})} = 1$, nonnegative and $\overline{\gamma}(C)$-homogeneous.

Let $\varphi \in C^\infty_c(C)$ be a positive smooth function compactly supported such that supp $\varphi \subset B_\rho$, for some $\rho > 0$. By definition of the distributional fractional Laplacian

$$0 = \int_{\mathbb{R}^n} \varphi(-\Delta)^s u_s \mathrm{d}x = \int_{\mathbb{R}^n} u_s(-\Delta)^s\varphi \mathrm{d}x = \int_{\mathbb{R}^n\setminus B_\rho} u_s(-\Delta)^s\varphi\mathrm{d}x + \int_{B_\rho} u_s(-\Delta)^s\varphi\mathrm{d}x.$$

Since

$$\frac{1}{|x-y|^{n+2s}} = \frac{1}{|x|^{n+2s}}\left(1-(n+2s)\frac{y}{|x|}\int_0^1 \frac{\frac{x}{|x|} - t\frac{y}{|x|}}{\left|\frac{x}{|x|} - \frac{ty}{|x|}\right|^{n+2s+2}}\mathrm{d}t\right),$$

by definition of the fractional Laplacian for regular functions, it follows

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_\rho} u_s(-\Delta)^s\varphi\mathrm{d}x =& C(n,s)\int_{\mathbb{R}^n\setminus B_\rho} u_s(x)\int_{\text{supp }\varphi}\frac{-\varphi(y)}{|y-x|^{n+2s}}\mathrm{d}y\mathrm{d}x \\ =& C(n,s)\int_{\mathbb{R}^n\setminus B_\rho}\frac{u_s(x)}{|x|^{n+2s}}\int_{\text{supp }\varphi} -\varphi(y)\mathrm{d}y\mathrm{d}x + \\ &+ C(n,s)(n+2s)\int_{\mathbb{R}^n\setminus B_\rho}\frac{u_s(x)}{|x|^{n+2s+1}}\psi(x)\mathrm{d}x,\end{aligned}$$

for some $\psi \in L^\infty$. Moreover, since u_s is $\gamma_s(C)$-homogeneous with $\gamma_s(C) < 2s$, we have

$$C(n,s)\int_{\mathbb{R}^n\setminus B_\rho}\frac{u_s(x)}{|x|^{n+2s}}\mathrm{d}x = \frac{C(n,s)}{2s-\gamma_s(C)}\rho^{\gamma_s(C)-2s}\int_{S^{n-1}}u_s(\theta)\mathrm{d}\sigma$$

and similarly

$$C(n,s)\left|\int_{\mathbb{R}^n\setminus B_\rho}\frac{u_s(x)}{|x|^{n+2s+1}}\psi(x)\mathrm{d}x\right| \le \frac{C(n,s)\,\|\psi\|_{L^\infty}}{2s-\gamma_s(C)+1}\rho^{\gamma_s(C)-2s-1}\int_{S^{n-1}}u_s(\theta)\mathrm{d}\sigma = o(1).$$

Hence, for each $s \in (0,1)$

$$\begin{aligned}\int_{B_\rho}u_s(-\Delta)^s\varphi\mathrm{d}x &= \int_{\mathbb{R}^n\setminus B_\rho}u_s(-\Delta)^s\varphi\mathrm{d}x\\ &= C(n,s)\int_{\mathbb{R}^n\setminus B_\rho}u_s(x)\int_{\text{supp }\varphi}\frac{\varphi(y)}{|x-y|^{n+2s}}\mathrm{d}y\mathrm{d}x\\ &= \frac{C(n,s)}{2s-\gamma_s(C)}\int_{\text{supp }\varphi}\varphi(x)\mathrm{d}x\int_{S_{n-1}}u_s\mathrm{d}\sigma + o(1)\end{aligned}$$

and passing through the limit, up to a subsequence, we obtain

$$\begin{aligned}\int_{B_\rho}\overline{u}(-\Delta)\varphi\mathrm{d}x &= \mu(C)\int_{S_{n-1}}\overline{u}\mathrm{d}\sigma\int_{\text{supp }\varphi}\varphi(x)\mathrm{d}x\\ &= \int_{B_\rho}\left(\mu(C)\int_{S_{n-1}}\overline{u}\mathrm{d}\sigma\right)\varphi(x)\mathrm{d}x,\end{aligned}$$

which implies, integrating by parts, that

$$-\Delta\overline{u} = \mu(C)\int_{S^{n-1}}\overline{u}\mathrm{d}\sigma \quad \text{in } \mathcal{D}'(C).$$

Since the function $\overline{u}$ is $\overline{\gamma}(C)$-homogenous, we get

$$-\Delta_{S^{n-1}}\overline{u} = \overline{\lambda}\overline{u} + \mu(C)\int_{S^{n-1}}\overline{u}\mathrm{d}\sigma \quad \text{on } S^{n-1}\cap C, \tag{88}$$

where $\overline{\lambda} = \overline{\gamma}(C)(\overline{\gamma}(C)+n-2)$ is the eigenvalue associated to the critical exponent $\overline{\gamma}(C) \le 2$.

Consider now a nonnegative $\varphi \in H_0^1(S^{n-1}\cap C)\setminus\{0\}$, strictly positive on $S^{n-1}\cap C$ which achieves (84). Then

$$-\Delta_{S^{n-1}}\varphi = \lambda_1(C)\varphi, \quad \text{in } H^{-1}(S^{n-1}\cap C). \tag{89}$$

By testing this equation with $\overline{u}$ and integrating by parts, we obtain

$$\left(\lambda_1(C)-\overline{\lambda}\right)\int_{S^{n-1}}\overline{u}\varphi\mathrm{d}\sigma = \mu(C)\int_{S^{n-1}}\overline{u}\mathrm{d}\sigma\int_{S^{n-1}}\varphi\mathrm{d}\sigma \ge 0 \tag{90}$$

which implies that in general $\gamma(C) \ge \overline{\gamma}(C)$ and $\gamma(C) = \overline{\gamma}(C)$ if and only if $\mu(C) = 0$.

Wide cones: $\gamma(C) < 2$

By the previous remark we have $\overline{\gamma}(C) < 2$ and by definition of $\mu(C)$, it follows $\mu(C) = 0$. Since φ is the trace on S^{n-1} of an homogenous harmonic function on C, we obtain that $\overline{\gamma}(C) = \gamma(C)$ and $\overline{u}$ is an homogeneous nonnegative harmonic function on C such that $\|\overline{u}\|_{L^\infty(S^{n-1})} = 1$.

Narrow cones: $\gamma(C) \geq 2$

If $\overline{\gamma}(C) < 2$ we have $\mu(C) = 0$ and consequently $\lambda_1(C) = \overline{\lambda}$, which is a contradiction since $\gamma(C) \geq 2 > \overline{\gamma}(C)$. Hence, if C is a narrow cone we get $\overline{\gamma}(C) = 2$. Since $\gamma(C) = 2$ is trivial and it follows directly from the previous computations, consider now $\mu_0(C)$ as the minimum defined in (85), which is well defined and strictly positive since we are focusing on the remaining case $\gamma(C) > 2$. We already remarked that it is achieved by a nonnegative $\psi \in H_0^1(S^{n-1} \cap C) \setminus \{0\}$ which is strictly positive on $S^{n-1} \cap C$ and solution of

$$-\Delta_{S^{n-1}}\psi = 2n\psi + \mu_0(C)\int_{S^{n-1}} \psi \mathrm{d}\sigma \quad \text{in } H^{-1}(S^{n-1} \cap C).$$

As we already did in the previous cases, by testing this equation with $\overline{u}$ we obtain $\mu(C) = \mu_0(C)$. By uniqueness of the limits $\overline{\gamma}(C)$ and $\mu(C)$, the result in (87) holds for $s \to 1$ and not just up to a subsequence. □

Remark 2.4.2. The possible obstruction to the existence of the limit of u_s as s converge s to one lies in the possible lack of uniqueness of nonnegative solutions to (59) such that $\|\overline{u}\|_{L^\infty(S^{n-1})} = 1$. This is the reason why we need to extract subsequences in the asymptotic analysis of Theorem 2.1.2. More precisely, uniqueness of (84) implies uniqueness of the limit $\overline{u}$ in the case $\gamma(C) \leq 2$ and uniqueness of (85) in the case $\gamma(C) > 2$. When C is connected (84) is attained by a unique normalized nonnegative solution via a standard argument based upon the maximum priciple. On the other hand, as we already remarked, when $\gamma(C) > 2$, problem (85) always admits a unique solution. Ultimately, the main obstacle in this analysis is the disconnection of the cone C when $\gamma(C) \leq 2$: in this case we cannot always ensure the uniqueness of the solution of the limit problem and even the positivity of the limit function $\overline{u}$ on every connected components of C.

The following example shows uniqueness of the limit function $\overline{u}$ due to the nonlocal nature of the fractional Laplacian under a symmetry assumption on the cone C.

Proposition 2.4.3. *Let $C = C_1 \cup \cdots \cup C_m$ be a union of disconnected cones such that C_1 is connected and there are orthogonal maps $\Phi_2, \ldots, \Phi_m \in O(n)$ (e.g. reflections about hyperplanes) such that $C_i = \Phi_i(C_1)$ and and $\Phi_i(C) = (C)$ for $i = 2, \ldots, m$. Let (u_s) be the family of nonnegative solutions to (55) such that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. Then there exists the limit of u_s as $s \nearrow 1$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and uniformly on compact subsets of C.*

Proof. We remark that, for any element of the orthogonal group $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$(-\Delta)^s (u \circ \Phi)(x) = C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{u(\Phi(x)) - u(y)}{|\Phi(x) - y|^{n+2s}} \mathrm{d}y = (-\Delta)^s u(\Phi(x)) \ .$$

By the uniqueness result [5, Theorem 3.2] of s-harmonic functions on cones, we infer that $u_s \equiv u_s \circ \Phi_i$, for every $i = 2, \dots, m$. Therefore, there holds convergence to $\overline{u}$, where satisfies $\|\overline{u}\|_{L^\infty(S^{n-1})} = 1$, and it is a solution of

$$\begin{cases} -\Delta \overline{u} = \mu(C) \displaystyle\int_{S^{n-1}} \overline{u} \mathrm{d}\sigma & \text{in } C, \\ \overline{u} \geq 0 & \text{in } C, \\ \overline{u} = 0 & \text{in } \mathbb{R}^n \setminus C \ , \end{cases} \tag{91}$$

such that $\overline{u} \equiv \overline{u} \circ \Phi_i$ for every $i = 2, \dots, m$. Finally, connectedness of C_1 yields uniqueness of such solution also for narrow cones. □

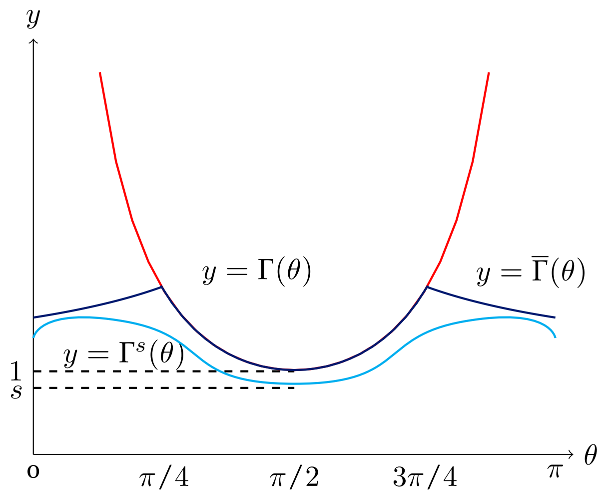


Figure 5: Values of the limit $\overline{\Gamma}(\theta) = \lim_{s \to 1} \Gamma^s(\theta)$ and $\Gamma(\theta)$, for $n = 2$.

Proof of Corollary 2.1.3. Corollary 2.1.3 is an easy application of our main Theorem 2.1.2, since it is a consequence of the Dini's Theorem for a monotone sequence of continuous functions which converges pointwisely to a continuous function on a compact set. In fact, fixed $s \in (0,1)$, the function $\theta \mapsto \gamma_s(\theta)$ is continuous in $[0,\pi)$ with $\gamma_s(0) = 2s$ and $\gamma_s(\pi) = 0$. Moreover this function is also monotone decreasing in $[0,\pi]$ and since there exists the limit

$$\lim_{\theta\to\pi^-} \gamma_s(\theta) = \begin{cases} \frac{2s-1}{2} & \text{if } n=2 \text{ and } s>\frac{1}{2}, \\ \gamma_s(\pi)=0 & \text{otherwise,} \end{cases}$$

we can extend $\theta \mapsto \gamma_s(\theta)$ to a continuous function in $[0,\pi]$ (see [67]). Nevertheless, the limit $\overline{\gamma}(\theta) = \lim_{s\to 1}\gamma_s(\theta) = \min\{\gamma(\theta), 2\}$ is continuous on $[0,\pi]$ with

$$\overline{\gamma}(\pi) = \begin{cases} \frac{1}{2} & \text{if } n=2, \\ 0 & \text{otherwise.} \end{cases}$$

Eventually, for any fixed $\theta \in [0,\pi]$, the function $s \mapsto \gamma_s(\theta)$ is monotone nondecreasing in $(0,1)$. By the Dini's Theorem the convergence is uniform on $[0,\pi]$. This fact obviously implies the uniform convergence

$$\Gamma^s(\theta) = \frac{\gamma_s(\theta)+\gamma_s(\pi-\theta)}{2} \longrightarrow \overline{\Gamma}(\theta) = \frac{\overline{\gamma}(\theta)+\overline{\gamma}(\pi-\theta)}{2}$$

in $[0,\pi]$, and hence

$$\nu_s^{ACF} = \min_{\theta\in[0,\pi]} \Gamma^s(\theta) \longrightarrow \min_{\theta\in[0,\pi]} \overline{\Gamma}(\theta) = \nu^{ACF}.$$

□

2.5 UNIFORM ESTIMATES IN $C^{0,\alpha}$ ON ANNULI

We have already remarked in Section 2.2 that, if you take a cone $C = C_\omega$ with $\omega \subset S^{n-1}$ a finite union of connected $C^{1,1}$ domain ω_i, such that $\overline{\omega}_i \cup \overline{\omega}_j = \emptyset$ for $i \neq j$, by [67, Lemma 3.3] we have (63).
Hence solutions u_s to (55) are $C^{0,s}(S^{n-1})$ and for any fixed $\alpha \in (0,1)$, any solution u_s with $s \in (\alpha, 1)$ is $C^{0,\alpha}(S^{n-1})$; that is, there exists $L_s > 0$ such that

$$\sup_{x,y\in S^{n-1}} \frac{|u_s(x)-u_s(y)|}{|x-y|^\alpha} = L_s.$$

Let us consider an annulus $A = A_{r_1,r_2} = B_{r_2} \setminus \overline{B_{r_1}}$ with $0 < r_1 < r_2 < +\infty$. We have the following result.

Lemma 2.5.1. *Let $\alpha \in (0,1)$, $s_0 \in (\max\{1/2, \alpha\}, 1)$ and A an annulus centered at zero. Then there exists a constant $c > 0$ such that any solution u_s to (55) with $s \in [s_0, 1)$ satisfies*

$$\sup_{x,y \in A} \frac{|u_s(x) - u_s(y)|}{|x-y|^\alpha} \le cL_s.$$

Proof. First of all we remark that

$$\sup_{x,y \in S_r^{n-1}} \frac{|u_s(x) - u_s(y)|}{|x-y|^\alpha} \le cL_s, \tag{92}$$

for any $r \in (r_1, r_2)$. In fact, by the γ_s-homogeneity of our solutions, we have

$$\sup_{x,y \in S_r^{n-1}} \frac{|u_s(x) - u_s(y)|}{|x-y|^\alpha} = L_s r^{\gamma_s - \alpha},$$

and since $(2s_0 - 1)/2 \le \gamma_s(C) < 2$ for any $s \in [s_0, 1)$ by the inclusion $C \subset \mathbb{R}^n \setminus \{\text{half} - \text{line from } 0\}$, we obtain (92).

Now we can show what happens considering $x, y \in A$ which are not on the same sphere. We can suppose without loss of generality that $x \in S_R^{n-1}$, $y \in S_r^{n-1}$ with $r_1 < r < R < r_2$. Hence let us take the point z obtained by the intersection between S_r^{n-1} and the half-line connecting 0 and x (z may be y itself). Hence

$$\begin{aligned} |u_s(x) - u_s(y)| &\le |u_s(x) - u_s(z)| + |u_s(z) - u_s(y)| \\ &\le u_s(x/|x|)||x|^{\gamma_s} - |z|^{\gamma_s}| + cL_s|z-y|^\alpha \\ &\le cL_s|x-y|^\alpha. \end{aligned}$$

In fact we remark that $||u_s||_{L^\infty(S^{n-1})} = 1$. Moreover, since the angle $\beta = \widehat{xzy} \in (\pi/2, \pi]$, obviously $|z-y|^\alpha \le |x-y|^\alpha$. Moreover by the α-Hölder continuity of $t \mapsto t^{\gamma_s}$ in (r_1, r_2) and the bounds $(2s_0 - 1)/2 \le \gamma_s(C) < 2$, one can find a universal constant $c > 0$ such that

$$||x|^{\gamma_s} - |z|^{\gamma_s}| \le c||x| - |z||^\alpha \le c|x-z|^\alpha \le c|x-y|^\alpha,$$

where the last inequality holds since z is the point on S_r^{n-1} which minimizes the distance $\mathrm{dist}(x, S_r^{n-1})$. □

A nontrivial improvement of the main Theorem concerns uniform bounds in Hölder spaces holding uniformly for $s \to 1$.

Theorem 2.5.2. *Assume the cone is $C^{1,1}$. Let $\alpha \in (0,1)$, $s_0 \in (\max\{1/2, \alpha\}, 1)$ and A an annulus centered at zero. Then the family of solutions u_s to (55) is uniformly bounded in $C^{0,\alpha}(A)$ for any $s \in [s_0, 1)$.*

Proof. Seeking a contradiction,

$$\max_{x,y\in S^{n-1}} \frac{|u_{s_k}(x)-u_{s_k}(y)|}{|x-y|^\alpha} = L_{s_k} = L_k \to +\infty, \qquad \text{as } s_k \to 1. \tag{93}$$

We can consider the sequence of points $x_k, y_k \in S^{n-1}$ which realizes L_k at any step. It is easy to see that this couple belongs to $\overline{C} \cap S^{n-1}$. Moreover we can always think x_k as the one closer to the boundary $\partial C \cap S^{n-1}$. Therefore, to have (93), we have $r_k = |x_k - y_k| \to 0$. Hence, without loss of generality, we can assume that x_k, y_k belong defenetively to the same connected component of C and

$$\frac{|u_{s_k}(y_k)-u_{s_k}(x_k)|}{r_k^\alpha} = L_k, \qquad \frac{y_k - x_k}{r_k} \to e_1.$$

Let us define

$$u^k(x) = \frac{u_{s_k}(x_k + r_k x) - u_{s_k}(x_k)}{r_k^\alpha L_k}, \qquad x \in \Omega_k = \frac{C - x_k}{r_k}.$$

We remark that $u^k(0) = 0$ and $u^k((y_k - x_k)/r_k) = 1$.

Moreover we can have two different situations.

Case 1 : If

$$\frac{r_k}{\mathrm{dist}(x_k, \partial C)} \to 0,$$

then the limit of Ω_k is $\mathbb{R}^n$.

Case 2 : If

$$\frac{r_k}{\mathrm{dist}(x_k, \partial C)} \to l \in (0, +\infty],$$

then the limit of Ω_k is an half-space $\mathbb{R}^n \cap \{x_1 > 0\}$.

In any case let us define Ω_∞ this limit set. Let us consider the annulus $A^* := B_{3/2} \setminus \overline{B_{1/2}}$. By Lemma 2.5.1 and the definition of u^k, we obtain, for any k,

$$\sup_{x,y\in A_k^*} \frac{|u^k(x)-u^k(y)|}{|x-y|^\alpha} \le c, \tag{94}$$

where $A_k^* := \frac{A^* - x_k}{r_k} \to \mathbb{R}^n$ and the constant $c > 0$ depends only on α and A^*. Let us consider a compact subset K of Ω_∞. Since for k large enough $K \subset A_k^*$, functions u^k are $C^{0,\alpha}(K)$ uniformly in k. This is due also to the fact that they are uniformly in $L^\infty(K)$, since $|u^k(x) - u^k(0)| \le c|x|^\alpha$

on K. Hence $u^k \to \overline{u}$ uniformly on compact subsets of Ω_∞. Moreover $\overline{u}$ is globally α-Hölder continuous and it is not constant, since $\overline{u}(e_1) - \overline{u}(0) = 1$. To conclude, we will show that $\overline{u}$ is harmonic in the limit domain Ω_∞; that is, for any $\phi \in C_c^\infty(\Omega_\infty)$

$$\int_{\Omega_\infty} \phi(-\Delta)\overline{u}\mathrm{d}x = 0,$$

and this fact will be a contradiction with the global Hölder continuity. In fact we can apply Corollary 2.3 in [72], if $\Omega_\infty = \mathbb{R}^n$ directly on the function $\overline{u}$ and if $\Omega_\infty = \mathbb{R}^n \cap \{x_1 > 0\}$, since $\overline{u} = 0$ in $\partial\Omega_\infty$, we can use the same result over its odd reflection. Hence we want to prove

$$\int_{\Omega_\infty} \phi(-\Delta)\overline{u}\mathrm{d}x = \int_{\Omega_\infty} \overline{u}(-\Delta)\phi\mathrm{d}x = \lim_{k\to+\infty} \int_{B_R} u^k(-\Delta)^{s_k}\phi\mathrm{d}x = 0,$$

where B_R contains the support of ϕ and the second equality holds by the uniform convergences $u^k \to \overline{u}$ and $(-\Delta)^{s_k}\phi \to (-\Delta)\phi$ on compact subsets of Ω_∞, since ϕ is a smooth function compactly supported. Moreover, since u^k is s_k-harmonic on Ω_k, and for k large enough the support of ϕ is contained in this domain, we have

$$\int_{\mathbb{R}^n} u^k(-\Delta)^{s_k}\phi\mathrm{d}x = \int_{\mathbb{R}^n} \phi(-\Delta)^{s_k}u^k\mathrm{d}x = 0.$$

In order to conclude we want

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n\setminus B_R} u^k(-\Delta)^{s_k}\phi\mathrm{d}x = 0.$$

Hence, defining $\eta = x_k + r_k x$ and using Remark 2.2.3, we obtain

$$\left|\int_{\mathbb{R}^n\setminus B_R} u^k(-\Delta)^{s_k}\phi\mathrm{d}x\right| \le \frac{C(n,s_k)}{L_k} r_k^{2s_k-\alpha} \int_{|\eta-x_k|>Rr_k} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}}\mathrm{d}\eta.$$

For k large enough, we notice that we can choose $\varepsilon > 0$ such that the set $\{\eta \in \mathbb{R}^n : Rr_k < |\eta - x_k| < \varepsilon\}$ is contained in A^*. So, we can split the integral obtaining

$$\int_{|\eta-x_k|>Rr_k} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}}\mathrm{d}\eta \le \int_{Rr_k<|\eta-x_k|<\varepsilon} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}}\mathrm{d}\eta + \int_{|\eta-x_k|>\varepsilon} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}}\mathrm{d}\eta$$

where we have

$$\begin{aligned}\frac{C(n,s_k)r_k^{2s_k-\alpha}}{L_k} \int_{Rr_k<|\eta-x_k|<\varepsilon} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}}\mathrm{d}\eta &\le C(n,s_k) r_k^{2s_k-\alpha} c\omega_{n-1} \int_{Rr_k}^{\varepsilon} t^{-1+\alpha-2s_k}\mathrm{d}t \\ &= \frac{C(n,s_k)c\omega_{n-1}}{2s_k-\alpha}\left(R^{\alpha-2s_k} - \frac{r_k^{2s_k-\alpha}}{\varepsilon^{2s_k-\alpha}}\right)\end{aligned}$$

and similarly

$$\frac{C(n,s_k)r_k^{2s_k-\alpha}}{L_k}\int_{|\eta-x_k|>\varepsilon}\frac{|u_{s_k}(\eta)-u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}}\mathrm{d}\eta \leq \frac{C(n,s_k)r_k^{2s_k-\alpha}c\omega_{n-1}}{L_k}\int_{\varepsilon}^{\infty}\frac{(1+t)^{\gamma_{s_k}}}{t^{1+2s_k}}\mathrm{d}t$$
$$=\frac{C(n,s_k)r_k^{2s_k-\alpha}c\omega_{n-1}}{L_k}\left(1+\frac{\varepsilon^{\gamma_{s_k}-2s_k}}{2s_k-\gamma_{s_k}}\right).$$

Finally, recalling that $r_k \to 0$, $C(n,s_k)\to 0$, $L_k\to\infty$ and $2s_k-\alpha>0$ taking $s_0>1/2$, we obtain

$$\left|\int_{\mathbb{R}^n\setminus B_R} u^k(-\Delta)^{s_k}\phi\mathrm{d}x\right| \leq \left(C(n,s_k)+\frac{C(n,s_k)}{2s_k-\gamma_{s_k}}\frac{r_n^{2s_k-\alpha}}{L_k}\right)M$$

which converges to zero as we claimed, since

$$\frac{C(n,s_k)}{2s_k-\gamma_{s_k}(C)} \to \mu(C)\in[0,+\infty)$$

in any regular cone $C\subset\mathbb{R}^n$. □

3

NODAL SET OF SOLUTIONS OF DEGENERATE - SINGULAR EQUATIONS

3.1 INTRODUCTION

In literature, the subject of nodal sets, or level sets in general, is an important research topic for solutions of PDEs. Recently in [41, 56, 57, 66] much attention has been paid on the structure of the singular set and on its $(n-2)$-Hausdorff measure, and, as pointed out in [48, 49, 66], a starting point of this analysis is the validity of a strong unique continuation principle, in order to ensure the existence of a finite vanishing order.

In this Chapter we consider the nodal set in $\mathbb{R}^{n+1}$ of solution of a peculiar class of degenerate-singular operator, firstly studied in the pioneering works [44, 43]. In the 80s Fabes, Jerison, Kening and Serapioni introduced a general class of degenerate operators $L = \mathrm{div}(A(X)\nabla\cdot)$ whose coefficient $A(X) = (a_{ij}(X))$ are defined starting from a symmetric matrix valued function such that

$$\lambda\omega(X)\left|\xi\right|^2 \leq (A(X)\xi,\xi) \leq \Lambda\omega(X)\left|\xi\right|^2, \quad \text{for some } \lambda,\Lambda>0,$$

where ω may either vanish, or be infinite, or both. In particular, the prototypes of weights considered in their analysis where in the Muckenhoupt A_2-class, i.e. such that

$$\sup_{B\subset\mathbb{R}^{n+1}}\left(\frac{1}{|B|}\int_B \omega(X)\mathrm{d}X\right)\left(\frac{1}{|B|}\int_B \omega^{-1}(X)\mathrm{d}X\right) < \infty.$$

Given $a\in(-1,1)$ and $X=(x,y)\in\mathbb{R}^n_x\times\mathbb{R}_y$ we consider the cases of $\omega(X) = \left|y\right|^a$, with

$$L_a = \mathrm{div}(\left|y\right|^a\nabla),$$

where obviously we denote with div and ∇ respectively the divergence and the gradient operator in $\mathbb{R}^{n+1}$. Our main purpose is to fully understand the local behaviour of L_a-harmonic function near their nodal set and to develop a geometric analysis of its structure and regularity in order to comprehend how the presence of a nontrivial set where the coefficients of an elliptic equation

may either vanishes or be infinite can affect the local picture of its solution.
Inspired by this last claim, we introduce the notion of "*characteristic manifold*" Σ associated to the operator L_a, as the set of points where the coefficient either vanishes or blows up, and we studied the properties of the nodal set $\Gamma(u)$ of solutions of

$$-L_a u = 0 \quad \text{in } B_1 \subset \mathbb{R}^{n+1}.$$

In particular, since the operator L_a is locally uniformly elliptic on $\mathbb{R}^{n+1} \setminus \Sigma$, we restrict our attention on the structure of the nodal set in a neighbourhood of the manifold Σ, trying to understand the difference between the whole nodal set $\Gamma(u)$ and its restriction on the characteristic manifold Σ.

At first sight, our approach seems to be based upon the validity of an Almgren and Weiss type monotonicity formulas, which guarantee the uniqueness of a non trivial tangent map at every point of the nodal set, and on a complete classification of the possible homogenous configurations appearing at the blow-up limit. Instead, the crucial result of our analysis relies in the decomposition of L_a-harmonic function with respect to the orthogonal direction to the characteristic manifold Σ. More precisely, we prove

Proposition 3.1.1. *Given $a \in (-1,1)$ and u an L_a-harmonic function in B_1, there exist two unique functions $u_e^a \in H^{1,a}(B_1), u_e^{2-a} \in H^{1,2-a}(B_1)$ symmetric with respect to Σ respectively L_a and L_{2-a} harmonic in B_1 and locally smooth, such that*

$$u(X) = u_e^a(X) + u_e^{2-a}(X) y \, |y|^{-a} \quad \textit{in } B_1.$$

Heuristically, the presence of a characteristic manifold Σ imposes a quantization of the possible ways in which the nodal set can diffuse across Σ.
With the previous decomposition in mind, we can reduce the classification of the possible blow-up limits to the symmetric ones ad finally recover all the possible cases. In particular, in our analysis we introduce the new notion of "*tangent field*" Φ^{X_0} of u at a nodal point, which takes care of the different behaviour of both the symmetric and antisymmetric part of an L_a-harmonic function. Namely, by the decomposition and the Definition 3.5.7 of the notion of tangent map, i.e. the unique nonzero map $\varphi^{X_0} \in \mathfrak{B}_k^a(u)$ such that

$$u_{X_0,r}(X) = \frac{u(X_0 + rX)}{r^k} \longrightarrow \varphi^{X_0}(X),$$

with k the vanishing order of u at X_0, we introduce the following concept.

Definition 3.1.2. Let $a \in (-1,1)$, u be an L_a-harmonic function in B_1 and $X_0 \in \Gamma_k(u) \cap \Sigma$, for some $k \geq \min\{1, 1-a\}$. We define as *tangent field* of u at X_0 the unique nontrivial vector field $\Phi^{X_0} \in (H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1}))^2$ such that

$$\Phi^{X_0} = (\varphi_e^{X_0}, \varphi_o^{X_0}),$$

where $\varphi_e^{X_0}$ and $\varphi_o^{X_0}$ are respectively the tangent map of the symmetric part u_e of u and of the antisymmetric one u_o.

This new object allows to overcame the obstacle of the degeneracy-singularity of the coefficient and it allows to understand the topology of the nodal set by proving in Proposition 3.5.19 a "*vectorial*" counterpart of the classic result of upper semi-continuity of the vanishing order. Hence, given now the regular $\mathcal{R}(u)$ and singular part $\mathcal{S}(u)$ as

$$\mathcal{R}(u) = \left\{X \in \Gamma(u)\colon\ |\nabla_x u(X)|^2 + \left|\partial_y^a u(X)\right|^2 \neq 0\right\},$$
$$\mathcal{S}(u) = \left\{X \in \Gamma(u)\colon\ |\nabla_x u(X)|^2 + \left|\partial_y^a u(X)\right|^2 = 0\right\},$$

we developed a blow-up analysis in order to fully understand the structure of $\Gamma(u)$ in $\mathbb{R}^{n+1}$ and its restriction on Σ. The following is a summary of the main result on the regular and singular set.

Theorem 3.1.3. *Let $a \in (-1,1), a \neq 0$ and u be an L_a-harmonic function in B_1. Then the regular set $\mathcal{R}(u)$ is locally a $C^{k,r}$ hypersurface on $\mathbb{R}^{n+1}$ in the variable $(x, y\,|y|^{-a})$ with*

$$k = \left\lfloor \frac{2}{1-a} \right\rfloor \quad \textit{and} \quad r = \frac{2}{1-a} - \left\lfloor \frac{2}{1-a} \right\rfloor.$$

On the other hand, it holds

$$\mathcal{S}(u) \cap \Sigma = \mathcal{S}^*(u) \cup \mathcal{S}^a(u)$$

where $\mathcal{S}^*(u)$ is contained in a countable union of $(n-2)$-dimensional C^1 manifolds and $\mathcal{S}^a(u)$ is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. Moreover

$$\mathcal{S}^*(u) = \bigcup_{j=0}^{n-2} \mathcal{S}_j^*(u) \quad \textit{and} \quad \mathcal{S}^a(u) = \bigcup_{j=0}^{n-1} \mathcal{S}_j^a(u),$$

where both $\mathcal{S}_j^*(u)$ and $\mathcal{S}_j^a(u)$ are contained in a countable union of j-dimensional C^1 manifolds.

In the last part of Chapter 3 we present an applications of our theory in the context of nonlocal elliptic equations. In particular, inspired by [23, 27, 80], we exploit the local realisation of the fractional Laplacian, and more generally of fractional power of divergence form operator L with Lipschitz leading coefficient, in order to study the structure and the regularity of the nodal set of $(-L)^s$-harmonic functions, for $s \in (0,1)$. More precisely, we combine the extension developed in [80] with a geometric reduction introduced in [4] and deeply popularized in [48, 49].
This last Section allows to extend our analysis to fractional powers $(-\Delta_M)^s$ of the Laplace-Beltrami operator on a Riemannian manifold M, also for the case of Lipschitz metric, and moreover to conformal fractional Laplacian on conformally compact Einstein manifolds and

asymptotically hyperbolic manifold, thanks to the extension technique developed in [27] and the asymptotic expansion of their geodesic boundary defining function. These examples suggest that our choice of weight collects a wilder class of degenerate-singular elliptic problems.

Moreover, our results show some purely nonlocal feature on the possible local expansion of $(-L)^s$-harmonic map near their zero set and on the structure of the nodal set itself. One one side we prove that first term of the Taylor expansion of an $(-L)^s$-harmonic function is either an homogeneous harmonic polynomial or any possible homogeneous polynomial. In particular, it implies

Theorem 3.1.4. *Given $s \in (0,1)$ and L a divergence form operator with Lipschitz leading coefficients, let u be $(-L)^s$-harmonic in B_1. Then it holds*

$$\mathcal{S}(u) = \mathcal{S}^*(u) \cup \mathcal{S}^s(u)$$

where $\mathcal{S}^*(u)$ is contained in a countable union of $(n-2)$-dimensional C^1 manifolds and $\mathcal{S}^s(u)$ is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. Moreover

$$\mathcal{S}^*(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^*_j(u) \quad \text{and} \quad \mathcal{S}^s(u) = \bigcup_{j=0}^{n-1} \mathcal{S}^s_j(u),$$

where both $\mathcal{S}^*_j(u)$ and $\mathcal{S}^s_j(u)$ are contained in a countable union of j-dimensional C^1 manifolds.

In the end, we prove what could be seen as the nonlocal counterpart of a conjecture that Lin proposed in [66]. Following his strategy, we give an explicit estimate on the $(n-1)$-Hausdorff measure of the nodal set $\Gamma(u)$ of s-harmonic functions in terms of the Almgren monotonicity formula previously introduced. Finally, we propose an interesting direction of research in order to improve that stated result.

This Chapter is organized as follows. In Section 3.2 we prove some general result about L_a-harmonic function, first of all the decomposition with respect to the direction orthogonal to Σ. After that in Section 3.3 we prove the validity of an Almgren's type monotonicity formula which allows in Section 3.4 to prove the existence of blow-up limit in every point of the nodal set $\Gamma(u)$. Finally, in Section 3.5 we prove a Weiss type monotonicity formula, which allows to introduce the notion of tangent map and tangent field at every point of the nodal set. In Section 3.6 we present some useful result on the stratification of the nodal set and finally in Section 3.7 we prove a general result on the regularity of the whole nodal set $\Gamma(u)$ and on its restriction on the characteristic manifold Σ. In the last two Sections we consider an application of the previous results for solutions of fractional powers of divergence form operator, with Lipschitz leading coefficient. In particular, in Section 3.8 we apply our technique in order to study the nodal set of

s-harmonic function and, more generally, of solutions of $(-L)^s$ operators, and in Section 3.9 we give a new estimate of the Hausdorff measure of the nodal set of s-harmonic functions.

3.2 DECOMPOSITION OF L_a-HARMONIC FUNCTIONS

In this Section we states some general results on L_a-harmonic function and we introduce some basic additional concept that will be often use through this Chapter in order to better understand the structure of the nodal set $\Gamma(u)$.
In particular, we give a definition of characteristic manifold Σ for a degenerate-singular operator and we prove a crucial decomposition of L_a-harmonic function with respect to the orthogonal direction to Σ. Thanks to this property, we can state a general regularity result on L_a-harmonic functions.

As already remarked, in the pioneering works [44, 43] the authors introduced a class of degenerate-singular operator strictly correlated to some weighted Sobolev spaces with Muckenhoupt A_p-weights. In [43, Section 2] they gave six general properties that the weight must satisfy in order to have existence of weak solutions, Sobolev embeddings, Poincaré inequality, Harnack inequality, local solvability in Hölder spaces and estimates on the Green's function and in particular they found a sufficient condition in the definition of the Muckenhoupt A_2-class. Hence, they introduced for $a \in (-1,1)$ the weighted Sobolev spaces $H^{1,a}(B_1)$ as the closure of $C^\infty(\overline{B_1})$ functions under the norm

$$\|u\|^2_{H^{1,a}(B_1)} = \int_{B_1} |y|^a\, u^2 \mathrm{d}X + \int_{B_1} |y|^a \left|\nabla u\right|^2 \mathrm{d}X.$$

Anyway, as the authors in [43] pointed out in the study of a special classes of elliptic problem associated to quasi-conformal maps, properties as the Sobolev embeddings, Poincaré inequality, Harnack inequality and local solvability in Hölder spaces still hold for every $a \in (-1,+\infty)$. Thus, the following definition is well defined for every $a \in (-1,+\infty)$.

Definition 3.2.1 ([43]). Given $F = (f_1, \cdots, f_n)$ on B_1 such that $|F| \in L^{2,-a}(B_1)$, we say that $u \in H^{1,a}(B_1)$ is a solution of $L_a u = \mathrm{div} F$ if for every $\varphi \in C^\infty_c(B_1)$ we have

$$\int_{B_1} |y|^a \left\langle \nabla u, \nabla\varphi\right\rangle \mathrm{d}X = \int_{B_1} \left\langle F, \nabla\varphi\right\rangle \mathrm{d}X.$$

Similarly, a function $u \in H^{1,a}(B_1)$ is said to be L_a-harmonic in B_1 if for every $\varphi \in C^\infty_c(B_1)$ we have

$$\int_{B_1} |y|^a \left\langle \nabla u, \nabla\varphi\right\rangle \mathrm{d}X = 0.$$

Now, we can finally state the concept of characteristic manifold associated to the operator L_a. We obviously remark that the following definition can be easily generalized to the whole class of

Muckenhoupt A_2-weights, where in general Σ can be any possible non-smooth subset of $\mathbb{R}^{n+1}$ with dimension $0 \leq d \leq n$.

Definition 3.2.2. Let $a \in (-1,1)$ and L_a the weighted divergence form operator in $\mathbb{R}^{n+1}$ associated to the weight $\omega(X) = |y|^a$. Then we call as "*characteristic manifold*" associated to L_a the collection of points $\Sigma \subset \mathbb{R}^{n+1}$ where the weight takes value zero (degeneracy, $a > 0$) or infinite (singularity, $a < 0$).

We remark that in a more general case, on the characteristic manifold the weight could attains both zero and infinite values.
Since the operator L_a is uniformly elliptic on every compact subset of $\mathbb{R}^{n+1} \setminus \Sigma$, the challenging part of our work is the one related to the study of the nodal set near the characteristic manifold Σ associated to L_a. Inspired by this remark, through the Chapter we will focus on the case $X_0 \in \Sigma$ and we will simply compare the result on Σ with the case $\mathbb{R}^{n+1} \setminus \Sigma$, avoiding all the technical details.

In order to better understand the structure of the nodal set and the local behaviour of the L_a-harmonic function, we decided to decompose these functions with respect to the characteristic manifold Σ. More precisely, we construct, starting from an L_a-harmonic function, its parts respectively symmetric and antisymmetric with respect to the orthogonal direction to Σ, since we can imagine that the latter affects the way this functions cross the space of degeneracy-singularity.

Definition 3.2.3. Let $a \in (-1,1)$ and $u \in H^{1,a}(B_1)$ be an L_a-harmonic function in B_1. Then, u is said to be *symmetric* with respect to Σ if

$$u(x,-y) = u(x,y) \quad \text{in } \mathbb{R}^{n+1}.$$

Conversely, the function u is said to be *antisymmetric* with respect to Σ if

$$u(x,-y) = u(x,y) \quad \text{in } \mathbb{R}^{n+1}.$$

It is easy to see that given an L_a-harmonic function u in B_1, the functions

$$u_e(x,y) = \frac{u(x,y) + u(x,-y)}{2} \quad \text{and} \quad u_o(x,y) = \frac{u(x,y) - u(x,-y)}{2}$$

are respectively symmetric and antisymmetric with respect to Σ and such that

$$u(X) = u_e(X) + u_o(X).$$

At first sight, the previous decomposition seems to be innocuous and independent on the occurrence of degeneracy of the operator, but with the following Propositions would be clear the

complete picture of how the presence of a set where the coefficients take value zero of infinite affect the local behaviour of the solutions.
First, the following result allows us to focus the characterization of the blow-up limits to just the symmetric L_a-harmonic function

Proposition 3.2.4. *Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1 antisymmetric with respect to Σ. Thus, there exists a unique L_{2-a}-harmonic function v symmetric with respect to Σ such that*

$$u(x,y) = v(x,y)y\,|y|^{-a} \quad \text{in } \mathbb{R}^{n+1}.$$

Proof. Given $v(x,y) = u(x,y)\,|y|^a\,y^{-1}$, let us first prove that $v \in H^{1,2-a}(B_1)$, where $2-a \in (1,3)$. By direct computations we get

$$\int_{B_1} |y|^{2-a}\,v^2 \mathrm{d}X = \int_{B_1} |y|^a\,u^2 \mathrm{d}X, \tag{95}$$

and similarly

$$\begin{aligned}\int_{B_1} |y|^{2-a}\,|\nabla v|^2\,\mathrm{d}X &= \int_{B_1} |y|^a\,|\nabla u|^2\,\mathrm{d}X + (a-1)^2\int_{B_1} |y|^a\,\frac{u^2}{y^2}\mathrm{d}X\\ &\le C\left(\int_{B_1} |y|^a\,u^2\mathrm{d}X + \int_{B_1} |y|^a\,|\nabla u|^2\,\mathrm{d}X\right),\end{aligned}$$

where in the last inequality we used the validity of an Hardy type inequality (see [42]). Since that for a.e. $X \in B_1$ we have

$$L_{2-a}v = \mathrm{div}(|y|^{2-a}\,\nabla v) = (a-1)\partial_y u + \mathrm{div}(y\nabla u) = y\,|y|^{-a}\,L_a u, \tag{96}$$

let us prove v is L_{2-a}-harmonic in B_1 in the sense of Definition 3.2.1.
For every $\varphi \in C_c^\infty(B_1)$ and $0 < \delta < 1$ let $\eta_\delta \in C^\infty(B_1)$ be a family of compactly supported cut-off functions such that $0 \le \eta_\delta \le 1$ and

$$\eta_\delta(x,y) = \begin{cases} 0 & \text{on } \{(x,y) \in B_1\colon\ |y| \le \delta\},\\ 1 & \text{on } \{(x,y) \in B_1\colon\ |y| \ge 2\delta\},\end{cases}$$

with $|\nabla \eta_\delta| \le 1/\delta$. Thus, by testing (96) with $\varphi\eta_\delta$ we get for every $\delta \in (0,1)$

$$\begin{aligned}\int_{B_1} |y|^{2-a}\,\langle \nabla v, \nabla(\eta_\delta\varphi)\rangle \mathrm{d}X &= -\int_{B_1} \eta_\delta\varphi L_{2-a}v\mathrm{d}X\\ &= -\int_{B_1}\left(y\,|y|^{-a}\,\eta_\delta\varphi\right) L_a u\mathrm{d}X = 0,\end{aligned}$$

where in the last equality we used that $y\,|y|^{-a}\,\eta_\delta\varphi \in C^\infty_c(B_1)$. Now, by integration by parts

$$\int_{B_1} |y|^{2-a}\,\langle \nabla v, \nabla(\eta_\delta\varphi)\rangle \mathrm{d}X = \int_{B_1} |y|^{2-a}\,\eta_\delta\langle \nabla v, \nabla\varphi\rangle \mathrm{d}X + \int_{B_1} |y|^{2-a}\,\varphi\langle \nabla v, \nabla\eta_\delta\rangle \mathrm{d}X, \quad (97)$$

where by Dominated convergence we get that

$$\lim_{\delta\to 0^+}\int_{B_1} |y|^{2-a}\,\eta_\delta\langle \nabla v, \nabla\varphi\rangle \mathrm{d}X = \int_{B_1} |y|^{2-a}\,\langle \nabla v, \nabla\varphi\rangle \mathrm{d}X$$

and by Hölder inequality

$$\begin{aligned}
\int_{B_1} |y|^{2-a}\,\varphi\langle \nabla v, \nabla\eta_\delta\rangle \mathrm{d}X &\leq \|\varphi\|_{L^\infty(B_1)}\left(\int_{B_1} |y|^{2-a}\,|\nabla v|^2\,\mathrm{d}X\right)^{1/2}\left(\int_{B_1} |y|^{2-a}\,|\nabla \eta_\delta|^2\,\mathrm{d}X\right)^{1/2}\\
&\leq C\,\|\varphi\|_{L^\infty(B_1)}\,\|v\|_{H^{1,a}(B_1)}\,\frac{1}{\delta}\left(\int_\delta^{2\delta} |y|^{2-a}\,\mathrm{d}y\right)^{1/2}\\
&\leq C\,\|\varphi\|_{L^\infty(B_1)}\,\|v\|_{H^{1,a}(B_1)}\left(\frac{2^{3-a}-1}{3-a}\right)^{1/2}\delta^{\frac{1-a}{2}},
\end{aligned}$$

which imply, passing through $\delta \to 0$ in (97), that

$$\int_{B_1} |y|^{2-a}\,\langle \nabla v, \nabla\varphi\rangle \mathrm{d}X = 0 \quad \text{for } \varphi \in C^\infty_c(B_1),$$

since we are dealing with $a < 1$. □

Hence, for $a \in (-1,1)$ and every L_a-harmonic function $u \in H^{1,a}(B_1)$ there exist $u^a_e \in H^{1,a}(B_1)$ and $u^{2-a}_e \in H^{1,2-a}(B_1)$ two symmetric function with respect to Σ respectively L_a and L_{2-a} harmonic in B_1 such that

$$u(X) = u^a_e(X) + u^{2-a}_e(X)y\,|y|^{-a} \quad \text{in } B_1. \quad (98)$$

Thus, through the following Sections we will restrict the classification of the blow-up limit, i.e. the entire homogenous L_a-harmonic functions, to the symmetric with respect to Σ and in the final part of the work we will collect all the result for a generic L_a-harmonic function.
Secondly, the previous decomposition combined with the following result gives a complete picture of the regularity of an L_a-harmonic function.

Proposition 3.2.5 ([87]). *Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. Then it holds:*

- *if u is symmetric with respect to Σ, we get $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1)$, for any $\alpha \in (0,1)$;*

- *if u is antisymmetric with respect to Σ, we get $u \in C^{0,\alpha}_{\rm loc}(B_1)$, for any $\alpha \in (0, \alpha^*)$ with $\alpha^* = \min\{1, 1-a\}$.*

Moreover, if $a \in (-1, +\infty)$ and u is symmetric with respect to Σ, we even get that $u \in C^\infty_{\rm loc}(B_1)$.

Proposition 3.2.6 ([87]). *Let $a \in (-1, 1)$ and u be L_a-harmonic in B_1. Then for every $i = 1, \dots, n$ we get that $\partial_{x_i} u$ is L_a-harmonic in B_1 and $\partial^a_y u$ is L_{-a}-harmonic in B_1, where*

$$\partial^a_y u = \begin{cases} |y|^a \, \partial_y u & \text{if } X \notin \Sigma \\ \lim_{y \to 0} |y|^a \, \partial_y u(x, y) & \text{if } X \in \Sigma \end{cases}.$$

These results have been recently obtained in [87] using some new approximation technique and Liouville type theorem for a wilder class of degenerate-singular elliptic problems. The main idea is to consider degenerate-singular operator as asymptotic limit of a specific class of uniformly elliptic operator, where the exponent of Hölder regularity can be reached by a blow-up argument combined with some Almgren's type monotonicity formula.

We recall here some general result about L_a-harmonic functions. First we introduce the following Caccioppoli inequality, which enables us to give a priori estimates of the $L^{2,a}$ norm of the derivatives of the solution u in terms of the $L^{2,a}$-norm of u.

Proposition 3.2.7. *Let $a \in (-1, 1)$ and u an L_a-harmonic function in B_1. Then, for each $X_0 \in B_1 \cap \Sigma$ and $0 < r < R \leq 1 - |X_0|$ we have*

$$\int_{B_r(X_0)} |y|^a \, |\nabla u|^2 \, \mathrm{d}X \leq \frac{C}{(R-r)^2} \int_{B_R(X_0) \setminus B_r(X_0)} |y|^a \, |u - \lambda|^2 \, \mathrm{d}X, \tag{99}$$

for every $\lambda \in \mathbb{R}$.

Proof. Fix $0 < r < R \leq 1 - |X_0|$ and consider a smooth cut-off function $\eta \in C^\infty_c(B_1)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $B_r(X_0)$ and $\eta \equiv 0$ on $B_R(X_0) \setminus B_r(X_0)$. Moreover, it is not restrictive to suppose that

$$|\nabla \eta| \leq \frac{2}{R - r} \quad \text{in } B_R(X_0).$$

Now, by testing the equation $-L_a u = 0$ with the test function $\varphi = (u - \lambda)\eta^2$ and integrating by parts, we get

$$\int_{B_1} |y|^a \, \eta^2 \, |\nabla u|^2 \, \mathrm{d}X + 2 \int_{B_1} |y|^a \, \eta \langle \nabla \eta, \nabla u \rangle (u - \lambda) \mathrm{d}X = 0,$$

and using the Hölder inequality

$$\int_{B_R(X_0)} |y|^a \eta^2 |\nabla u|^2 \, \mathrm{d}X \leq \left(\int_{B_R(X_0)} |y|^a \eta^2 |\nabla u|^2 \, \mathrm{d}X \right)^{1/2} \left(\int_{B_R(X_0)} 4 |y|^a |u-\lambda|^2 |\nabla \eta|^2 \, \mathrm{d}X \right)^{1/2}.$$

Hence, dividing by the first term in the right hand side and taking into account the properties of η we obtain

$$\begin{aligned} \int_{B_r(X_0)} |y|^a |\nabla u|^2 \, \mathrm{d}X &\leq \int_{B_R(X_0)} |y|^a \eta^2 |\nabla u|^2 \, \mathrm{d}X \\ &\leq \frac{16}{(R-r)^2} \int_{B_R(X_0) \setminus B_r(X_0)} |y|^a |u-\lambda|^2 \, \mathrm{d}X. \end{aligned}$$

□

Now, for $a \in (-1,1)$ let us fix

$$|S^n|_a = \int_{\partial B_1} |y|^a \, \mathrm{d}\sigma,$$

which implies

$$|S^n_r|_a = r^{n+a} |S^n|_a \quad \text{and} \quad \left| B^{n+1}_r \right|_a = \frac{r^{n+a+1}}{n+a+1} |S^n|_a .$$

Lemma 3.2.8 ([88, Lemma A.1]). *Let $a \in (-1,1)$ ad u be an L_a-harmonic function in B_1. Then, for each ball $B_r(X_0)$, with $X_0 \in \Sigma$ and $r \in (0, 1-|X_0|)$, we have*

$$u(X_0) = \frac{1}{|S^n|_a r^{n+a}} \int_{\partial B_r(X_0)} |y|^a u \mathrm{d}\sigma = \frac{1}{|B^{n+1}|_a r^{n+a+1}} \int_{B_r(X_0)} |y|^a u \mathrm{d}X.$$

Proof. Let us consider the case $X_0 = 0$ since the problem is invariant under translation on $B_1 \cap \Sigma$. Set

$$\Phi(r) = \frac{1}{r^{n+a}} \int_{\partial B_r} |y|^a u \mathrm{d}\sigma = \int_{\partial B_1} |y|^a u(rx) \mathrm{d}\sigma,$$

then

$$\frac{d}{dr}\Phi(r) = \frac{1}{r^{n+a}} \int_{\partial B_r} |y|^a \partial_r u \mathrm{d}\sigma.$$

Since $L_a u = 0$ on B_1, by a Gauss-Green formula we get

$$\int_{\partial B_r} |y|^a \partial_r u \mathrm{d}\sigma = \int_{B_r} L_a u \mathrm{d}X = 0$$

which directly implies that $r \mapsto \Phi(r)$ is constant and consequently

$$\frac{1}{r^{n+a}} \int_{\partial B_r} |y|^a u \mathrm{d}\sigma = \lim_{r \to 0} \int_{\partial B_1} |y|^a \lim_{r \to 0} u(rx) \mathrm{d}\sigma$$
$$= u(0) |S^n|_a .$$

Similarly, by integrating from 0 to r the function $\Phi(r)$, we get

$$\int_0^r \int_{\partial B_t} |y|^a u \mathrm{d}\sigma \mathrm{d}t = \int_{B_r} |y|^a u \mathrm{d}x$$

and secondly

$$\int_0^r \left(\int_{\partial B_t} |y|^a u \mathrm{d}\sigma \right) \mathrm{d}t = u(0) |S^n|_a \int_0^r t^{n+a} \mathrm{d}t,$$

from which we get the claimed result. □

We remark that in the case of L_a-subharmonic function, i.e. $-L_a u \leq 0$, the previous result holds true in the form of inequality. Finally, by standard Moser's iteration, we also have the following bound

Lemma 3.2.9 ([88, Lemma A.2.]). *Let $a \in (-1, 1)$ and u be a L_a-subharmonic function in B_1. Then, for $X_0 \in B_1 \cap \Sigma$ and $r \in (0, 1 - |X_0|)$ we get*

$$\|u\|_{L^\infty(B_{r/2}(X_0))} \leq C(n, a) \left(\frac{1}{r^{n+1+a}} \int_{B_r(X_0)} |y|^a u^2 \mathrm{d}X \right)^{1/2},$$

where $C(n, a)$ is a constant depending only on n and a.

3.3 ALMGREN TYPE MONOTONICITY FORMULA

In this Section we introduce the degenerate-singular counterpart of the classical Almgren monotonicity formula for harmonic functions. This computations are more manageable with respect to the ones in Chapter 1 since peculiar phenomena like the problem of the codimension of the nodal set does not manifest in the case of L_a-harmonic functions.
Since we want to understand the structure and regularity of the nodal set of L_a-harmonic function near the characteristic manifold Σ, let us consider $X_0 = (x_0, 0) \in \Sigma$. Hence, for every $r \in (0, R)$, where $R > 0$ will be defined later, consider

$$E(X_0, u, r) = \frac{1}{r^{n+a-1}} \int_{B_r(X_0)} |y|^a |\nabla u|^2 \mathrm{d}X$$
$$H(X_0, u, r) = \frac{1}{r^{n+a}} \int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma$$

and the Almgren type monotonicity formula

$$N(X_0,u,r) = \frac{E(X_0,u,r)}{H(X_0,u,r)} = \frac{r\displaystyle\int_{B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}X}{\displaystyle\int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma}. \tag{100}$$

Since $u \in H^{1,a}_{\mathrm{loc}}(B_1)$, both the functional $r \mapsto E(X_0,u,r)$ and $r \mapsto H(X_0,u,r)$ are locally absolutely continuous on $(0,+\infty)$, that is that both them derivatives are in $L^1_{\mathrm{loc}}((0,+\infty))$.

Proposition 3.3.1. *Let $a \in (-1,1)$ and u be an L_a-harmonic function on B_1. Then, for every $X_0 \in B_1 \cap \Sigma$ we have that the map $r \mapsto N(X_0,u,r)$ is absolutely continuous and monotone nondecreasing on $(0,1-|X_0|)$.*
Hence, there always exists finite the limit

$$N(X_0,u,0^+) = \lim_{r\to 0^+} N(X_0,u,r) = \inf_{r>0} N(X_0,u,r),$$

which we will call as the Almgren frequency formula.

Proof. Obviously the denominator is nonnegative and at least strictly positive on a nonempty interval (r_1,r_2), otherwise we get $u \equiv 0$. First, passing to the logarithmic derivatives, the monotonicity of $r \mapsto N(X_0,u,r)$ is a direct consequence of the claim

$$\frac{d}{dr}\log N(X_0,u,r) = \frac{1}{r} + \frac{\displaystyle\frac{d}{dr}\int_{B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}X}{\displaystyle\int_{B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}X} - \frac{\displaystyle\frac{d}{dr}\int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma}{\displaystyle\int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma} \geq 0$$

for $r \in (r_1,r_2)$. Deriving the numerator and using the Pohožaev identity, i.e. for any $X_0 \in \mathbb{R}^{n+1}$ and $r > 0$

$$\begin{aligned}\frac{1-n-a}{2}\int_{B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}X + \frac{r}{2}\int_{\partial B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}\sigma = r\int_{\partial B_r(X_0)} |y|^a (\partial_r u)^2 \mathrm{d}\sigma +\\ - \frac{a y_0}{2}\int_{B_r(X_0)} |y|^a \frac{|\nabla u|^2}{y} \mathrm{d}X\end{aligned}$$

we easily get

$$\begin{aligned}\frac{d}{dr}\int_{B_r(X_0)}|y|^a|\nabla u|^2\,\mathrm{d}X &= \int_{\partial B_r(X_0)}|y|^a|\nabla u|^2\,\mathrm{d}\sigma\\ &= 2\int_{\partial B_r(X_0)}|y|^a(\partial_r u)^2\mathrm{d}\sigma+\frac{n+a-1}{r}\int_{B_r(X_0)}|y|^a|\nabla u|^2\,\mathrm{d}X\\ &\quad-\frac{ay_0}{r}\int_{B_r(X_0)}|y|^a\frac{|\nabla u|^2}{y}\mathrm{d}X\end{aligned} \tag{101}$$

and similarly

$$\begin{aligned}\frac{d}{dr}\int_{\partial B_r(X_0)}|y|^a u^2\mathrm{d}\sigma &= \frac{d}{dr}\left(r^n\int_{\partial B_1}|y_0+ry|^a u^2(X_0+rX)\mathrm{d}\sigma\right)\\ &= 2\int_{\partial B_r(X_0)}|y|^a u\partial_r u\mathrm{d}\sigma-\frac{ay_0}{r}\int_{\partial B_r(X_0)}\frac{|y|^a}{y}u^2\mathrm{d}\sigma+\\ &\quad+\frac{n+a}{r}\int_{\partial B_r(X_0)}|y|^a u^2\mathrm{d}\sigma.\end{aligned} \tag{102}$$

As a consequence, by the Cauchy-Schwarz inequality, if $X_0\in\Sigma$, i.e. $y_0=0$, we get

$$\begin{aligned}\frac{d}{dr}E(X_0,u,r) &= \frac{2}{r^{n+a-1}}\int_{\partial B_r(X_0)}|y|^a(\partial_r u)^2\mathrm{d}\sigma\\ \frac{d}{dr}H(X_0,u,r) &= \frac{2}{r^{n+a}}\int_{\partial B_r(X_0)}|y|^a u\partial_r u\mathrm{d}\sigma\end{aligned}$$

and consequently

$$\frac{1}{2}\frac{d}{dr}\log N(X_0,u,r)=\frac{\int_{\partial B_r(X_0)}|y|^a(\partial_r u)^2\mathrm{d}\sigma}{\int_{\partial B_r(X_0)}|y|^a u\partial_r u\mathrm{d}\sigma}-\frac{\int_{\partial B_r(X_0)}|y|^a u\partial_r u\mathrm{d}\sigma}{\int_{\partial B_r(X_0)}|y|^a u^2\mathrm{d}\sigma}\geq 0$$

for $r\in(r_1,r_2)$. By the previous differentiation, we have

$$\frac{d}{dr}\log N(X_0,u,r)\geq 0\quad\text{for } r\in(r_1,r_2)$$

and

$$\frac{d}{dr}\log H(X_0,u,r)=\frac{2}{r}N(X_0,u,r). \tag{103}$$

Following the same reasoning in [84], it is quite easy to conclude that the maximum interval is the one with $r_1=0$. □

As a direct consequence of the monotonicity result, we get that the Almgren frequency formula $X \mapsto N(X, u, 0^+)$ on Σ is upper semi-continuous since it is defined as the infimum of continuous function.

A simple consequence of the monotonicity result and (103) is the following comparison property (which, with $r_2 = 2r_1$, is the so called doubling property).

Corollary 3.3.2. *Let $a \in (-1,1)$ and u be L_a-harmonic on B_1. Hence, there given $N = N(X_0, u, 1-|X_0|)$ such that for every $X_0 \in B_1 \cap \Sigma$,*

$$H(X_0, u, r_2) \leq H(X_0, u, r_1)\left(\frac{r_2}{r_1}\right)^{2N}$$

for $0 < r_1 < r_2 < 1 - |X_0|$.

Proof. Fixed $R = 1 - |X_0|$ we have that $N(X_0, u, r) \leq N(X_0, u, R)$ for every $r \in (0, R)$ and integrating (103) between r_1 and r_2, with $0 < r_1 < r_2 \leq R$, we obtain

$$\frac{H(X_0, u, r_2)}{H(X_0, u, r_1)} \leq \left(\frac{r_2}{r_1}\right)^{2N}$$

whit $N = N(X_0, u, R)$. □

In other words, for every $X_0 \in B_1 \cap \Sigma$

$$\frac{1}{R^{n+a}}\int_{\partial B_R(X_0)} |y|^a u^2 \mathrm{d}\sigma \leq \left(\frac{R}{r}\right)^{2N} \frac{1}{r^{n+a}}\int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma$$

with $0 < r < R < 1 - |X_0|$ and $N = N(X_0, u, 1-|X_0|)$, and integrating the previous inequality we get

$$\frac{1}{R^{n+a+1}}\int_{B_R(X_0)} |y|^a u^2 \mathrm{d}X \leq \left(\frac{R}{r}\right)^{2N-1} \frac{1}{r^{n+a+1}}\int_{B_r(X_0)} |y|^a u^2 \mathrm{d}X. \tag{104}$$

In order to justify the analysis of the local behaviour of L_a-harmonic functions, we prove the validity of the strong unique continuation property for the degenerate-singular operator L_a. In general, a function u, is said to *vanish of infinite order* at a point $X_0 \in \Gamma(u)$ if

$$\int_{|X-X_0|<r} u^2 \mathrm{d}X = O(r^k), \quad \text{for every } k \in \mathbb{N},$$

as $r \to 0$. Given an elliptic operator L, L is said to have the *strong unique continuation property* in B_1 if the only solution of $Lu = 0$ in $H^1_{\mathrm{loc}}(B_1)$ which vanishes of infinite order at a point $X_0 \in \Gamma(u)$ is $u = 0$. Moreover, L is said to have the *unique continuation property* in B_1 if the solution of $Lu = 0$ in $H^1_{\mathrm{loc}}(B_1)$ which can vanish in an open subset of B_1 is $u = 0$. (see [48, 49] for more details for the uniformly elliptic case).

Corollary 3.3.3 ([49, Theorem 1.4]). *Let $a \in (-1,1)$ and u be L_a-harmonic in B_1. Then u cannot vanish of infinite order at $X_0 \in \Gamma(u) \cap B_1$ unless $u \equiv 0$ in B_1.*

In [49] the authors stated the proof for analytic nonnegative weights and pointed out the validity for more general, even degenerate, weighted elliptic equations.

The previous result implies that the nodal set $\Gamma(u)$ has empty interior in $\mathbb{R}^{n+1}$. As a consequence of our blow-up analysis, we will prove a posteriori unique continuation property for the restriction of $\Gamma(u)$ on Σ.

Corollary 3.3.4. *Let $a \in (-1,1)$ and u be an L_a-harmonic function on B_1. Then, for every $X_0 \in B_1 \cap \Sigma$ given $R = 1 - |X_0|$ we get*

$$\frac{1}{n+a+1+2N}\int_{\partial B_R(X_0)} |y|^a\, u^2 \mathrm{d}\sigma \leq \int_{B_R(X_0)} |y|^a\, u^2 \mathrm{d}X \leq \frac{1}{n+a+1}\int_{\partial B_R(X_0)} |y|^a\, u^2 \mathrm{d}\sigma,$$

where $N = N(X_0, u, R)$.

Proof. Let $R = 1 - |X_0|$ and $r \in (0, R)$, we get by (103)

$$H(X_0, u, R) = H(X_0, u, r) \exp\left\{2\int_r^R \frac{N(X_0, u, t)}{t} \mathrm{d}t\right\} \geq H(X_0, u, r)$$

or simply

$$\int_{\partial B_r(X_0)} |y|^a\, u^2 \mathrm{d}\sigma \leq H(X_0, u, R) r^{n+a}.$$

Finally, integrating the previous inequality in $(0, R)$ we obtain

$$\begin{aligned}\int_{B_R(X_0)} |y|^a\, u^2 \mathrm{d}X &= \int_0^R \int_{\partial B_r(X_0)} |y|^a\, u^2 \mathrm{d}\sigma \mathrm{d}r \\ &\leq H(X_0, u, R) \int_0^R r^{n+a} \mathrm{d}r = \frac{1}{n+a+1} H(X_0, u, R).\end{aligned}$$

On the other hand, for any $r \in (0, R)$, we have

$$\begin{aligned}H(X_0, u, R) &= H(X_0, u, r) \exp\left\{2\int_r^R \frac{N(X_0, u, t)}{t} \mathrm{d}t\right\} \\ &\geq H(X_0, u, r) \exp\left\{-2N(X_0, u, r) \log r\right\}\end{aligned}$$

and consequently

$$H(X_0, u, r) \geq r^{2N(X_0, u, R)} H(X_0, u, R).$$

Thus, as before

$$\int_{B_R(X_0)} |y|^a u^2 \mathrm{d}X = \int_0^R \int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma \geq \frac{1}{n+a+1+2N(X_0,u,R)} H(X_0,u,R).$$

□

The following result can be viewed as the degenerate-singular counterpart of [55, Theorem 1.6], which gives us a sufficient condition for the presence of the nodal set in the unitary ball.

Proposition 3.3.5. *Let $a \in (-1,1)$ and u be an L_a-harmonic function on B_1. Then, for any $R \in (0,1)$ there exists $N_0 = N_0(R) \ll 1$ such that the following holds:*

1. *if $N(0,u,1) \leq N_0$, then u does not vanish in B_R;*
2. *if $N(0,u,1) > N_0$, then*

$$N\left(X_0, u, \frac{1-R}{2}\right) \leq CN(0,u,1) \quad \text{for any } X_0 \in B_R \cap \Sigma,$$

   where C is a positive constant depending only on n, a and R.

Moreover, the vanishing order, i.e. the Almgren frequency formula, of u at any point of B_R never exceeds $CN(0,u,1)$.

Proof. This prof will follows directly the one in [55, 66]. Moreover, the previous result is known to be true if we restrict our study to the set $B_1 \setminus \Sigma$, by the local uniform ellipticity of the operator L_a outside the characteristic manifold. First, the monotonicity of $r \mapsto N(0,u,r)$ implies that the vanishing order of u at 0 never exceeds $N(0,u,1)$, more precisely

$$\frac{1}{(\lambda R)^{n+1+a}} \int_{B_{\lambda R}} |y|^a u^2 \mathrm{d}X \leq \lambda^{2N(0,u,1)} \frac{1}{R^{n+1+a}} \int_{B_R} |y|^a u^2 \mathrm{d}X$$

for every $R \in (0,1)$ and $\lambda \in (1, 1/R)$. Through this proof we will use the following notation to identify the average of the integrals

$$\fint_{B_r(X_0)} |y|^a u^2 \mathrm{d}X = \frac{1}{r^{n+1+a}} \int_{B_r(X_0)} |y|^a u^2 \mathrm{d}X,$$

and in order to simplify the notations we will use $N = N(0,u,1)$ as the frequency of u in B_1. Under these notations, the previous inequalities become

$$\fint_{\partial B_{\lambda R}} |y|^a u^2 \mathrm{d}\sigma \leq \lambda^{2N} \fint_{\partial B_R} |y|^a u^2 \mathrm{d}\sigma$$
$$\fint_{B_{\lambda R}} |y|^a u^2 \mathrm{d}X \leq \lambda^{-1}\lambda^{2N} \fint_{B_R} |y|^a u^2 \mathrm{d}X$$

where the second one is a consequence of Corollary 3.3.2.
Now, let us prove the claimed result for the case $R = 1/4$ since in the general case it follows by scaling. By definition, we have that $B_{3/4}(X_0) \subset B_1$ and $B_{1/4} \subset B_{1/2}(X_0)$ for any $X_0 \in B_{1/4}$. Hence, we have

$$\fint_{B_{3/4}(X_0)} |y|^a u^2 \mathrm{d}X \leq c(n,a) 4^{2N} \fint_{B_{1/2}(X_0)} |y|^a u^2 \mathrm{d}X$$

for any $X_0 \in B_{1/4}$. Now, let us prove that

$$\fint_{\partial B_{5/8}(X_0)} |y|^a u^2 \mathrm{d}\sigma \leq c(n,a) 4^{2N} \fint_{\partial B_{1/2}(X_0)} |y|^a u^2 \mathrm{d}\sigma. \tag{105}$$

Since by (103) the map $r \mapsto H(X_0, u, r)$ is monotone non decreasing on $(0, 1 - |X_0|)$, and hence

$$\begin{aligned}\int_{B_{3/4}(X_0)} |y|^a u^2 \mathrm{d}X &\geq \int_{B_{3/4}(X_0) \setminus B_{5/8}(X_0)} |y|^a u^2 \mathrm{d}X \\ &= \int_{5/8}^{3/4} r^{n+a} H(X_0, u, r) \mathrm{d}r \\ &\geq C(n,a) H\left(X_0, u, 5/8\right),\end{aligned}$$

and similarly

$$\int_{B_{1/2}(X_0)} |y|^a u^2 \mathrm{d}X = \int_0^{1/2} r^{n+a} H(X_0, u, r) \mathrm{d}r \leq C(n,a) H\left(X_0, u, \frac{1}{2}\right).$$

Finally, integrating (103) between the previous radii, we obtain

$$\log H(X_0, u, r)\Big|_{1/2}^{5/8} = \int_{1/2}^{5/8} \frac{2N(X_0, u, r)}{r} \mathrm{d}r \geq 2C(n) N(X_0, u, 1/2).$$

Combining the previous inequality with the claimed (105), we get

$$c(n,a) N(X_0, u, 1/2) \leq \log\left(c(n,a) 4^{2N(0,u,1)}\right)$$

or equivalently $N(X_0, u, 1/2) \leq c(n,a) N(0, u, 1) + c(n,a)$. Finally, let us consider the second part of the statement. Hence, given $\varepsilon = \varepsilon(n,a)$, sufficiently small, such that $N(0, u, 1) \leq \varepsilon$, let us prove that $u(X_0) \neq 0$ for any $X_0 \in B_{1/4}$. It is not restrictive to assume that $H(0, u, 1) = 1$, which implies by the definition of the Almgren monotonicity formula that

$$\int_{B_1} |y|^a |\nabla u|^2 \mathrm{d}X \leq \varepsilon.$$

By the L_a-harmonicity of u, for every $i = 1, \dots, n$ the derivative $\partial_{x_i} u$ and $\partial_y^a u$ are respectively L_a and L_{-a}-harmonic in B_1. Hence, by [88, Lemma A.2], we get the following interior estimates

$$\sup_{B_{1/2}} |\partial_{x_i} u| \le c(n,a) \left(\frac{1}{r^{n+a+1}} \int_{B_1} |y|^a |\partial_{x_i} u|^2 \, \mathrm{d}X \right)^{1/2} \le c(n,a)\sqrt{\varepsilon}$$

$$\sup_{B_{1/2}} \left|\partial_y^a u\right| \le c(n,-a) \left(\frac{1}{r^{n-a+1}} \int_{B_1} |y|^{-a} \left|\partial_y^a u\right|^2 \mathrm{d}X \right)^{1/2} \le c(n,a)\sqrt{\varepsilon}.$$

By the normalization assumption, we have

$$1 = \int_{\partial B_1} |y|^a u^2 \mathrm{d}\sigma \le c_1(n,a) \int_{\partial B_{1/2}} |y|^a u^2 \mathrm{d}\sigma,$$

and consequently the existence of $X_0 \in \partial B_{1/2}$ such that

$$|u(X_0)|^2 \ge \frac{2}{c_1(n,a)2^{n+a}} \int_{\partial B_1} |y|^a \, \mathrm{d}\sigma.$$

Up to relabeling with $c_1(n,a)$ the previous lower bound, we get that for every $X \in B_{1/2}$ that

$$c_1(n,a) \le |u(X_0)| \le |u(X)| + c(n,a)\sqrt{\varepsilon},$$

which yields $|u(X)| > 0$ of $B_{1/2}$, for $\varepsilon = \varepsilon(n,a)$ sufficiently small. □

Corollary 3.3.6. *Let u be L_a-harmonic on B_1, then for every $X_0 \in \Gamma(u) \cap \Sigma$ we have*

$$N(X_0, u, 0^+) \ge \min\{1, 1-a\}. \tag{106}$$

More precisely

- *if u is symmetric with respect to Σ, we have $N(X_0, u, 0^+) \ge 1$,*
- *if u is antisymmetric with respect to Σ we have $N(X_0, u, 0^+) \ge 1-a$.*

Proof. This result follows by Proposition 3.2.5. More precisely, et $\alpha^* = \min\{1, 1-a\}$ be the coefficient of optimal Hölder regularity for L_a-harmonic function, and suppose by contradiction that (106) is not satisfied.
Since the limit $N(X_0, u, 0^+)$ exists, we obtain the existence of $R > 0$ and $\varepsilon > 0$ such that $N(X_0, u, r) \le \alpha^* - \varepsilon$ for all $0 \le r \le R$. By (103), up to consider a smaller interval of $(0, R)$, we have

$$\frac{d}{dr} \log H(X_0, u, r) = \frac{2}{r} N(X_0, u, r) \le \frac{2}{r}(\alpha^* - \varepsilon).$$

Integrating this inequality between r and R yields

$$\frac{H(X_0,u,R)}{H(X_0,u,r)} \leq \left(\frac{R}{r}\right)^{2(\alpha^*-\varepsilon)}$$

which, together with the fact that u is α^*-Hölder continuous and $u(X_0)=0$, implies

$$C_1 r^{2(\alpha^*-\epsilon)} \leq H(X_0,u,r) \leq C_2 r^{2\alpha^*}.$$

The contradiction follows for small value of $r>0$.

If an addition we suppose that u is symmetric or antisymmetric with respect to Σ, we get respectively that u is Lipschitz continuous or $(1-a)$-Hölder continuous, and the lower bound on the Almgren frequency formula follows immediately. □

In all the first part of this Section, we had supposed that $X_0 \in B_1 \cap \Sigma$, since the degenerate-singular attitude of the operator L_a is constrained to the characteristic manifold Σ. Instead, if $X_0=(x_0,y_0) \in B_1 \setminus \Sigma$, since the operator is uniformly elliptic on $B_R(X_0) \subset \mathbb{R}^{n+1}$, with $R=|y_0|$, the structure and the regularity of the nodal set of u is well known.

At this point, we want to remark how the different scaling of the operator on Σ and on $\mathbb{R}^{n+1}\setminus\Sigma$ affects the Almgren monotonicity formula.
Let $X_0=(x_0,y_0)\in B_1$ and $r>0$ and consider u and L_a-harmonic function in B_1. If we define $u_{X_0,r}(X)=u(X_0+rX)$ we directly see that

$$\frac{|y_0+ry|^a}{r^2}\left(\Delta_X u_{X_0,r}+\frac{ar}{y_0+ry}\partial_y u_{X_0,r}\right)=0 \quad \text{for } X\in\frac{B_1-X_0}{r},$$

and

$$\int_{B_r(X_0)} L_a u \mathrm{d}X = \begin{cases} r^{n-1+a}\displaystyle\int_{B_1}\mathrm{div}\left(|y|^a\nabla u_{X_0,r}\right)\mathrm{d}X & \text{if } X_0\in\Sigma \\ r^{n-1}\displaystyle\int_{B_1}\mathrm{div}\left(|y_0+ry|^a\nabla u_{X_0,r}\right)\mathrm{d}X & \text{if } X_0\notin\Sigma \end{cases}.$$

Inspired by the different scalings of L_a operator, for $X_0\in B_1\setminus\Sigma$, i.e. $y_0\neq 0$, and $r\in(0,|y_0|)$, let us introduce the following functionals

$$\begin{aligned} E(X_0,u,r) =& \frac{1}{r^{n-1}}\int_{B_r(X_0)}|y|^a|\nabla u|^2\mathrm{d}X, \\ H(X_0,u,r) =& \frac{1}{r^n}\int_{\partial B_r(X_0)}|y|^a u^2\mathrm{d}\sigma, \end{aligned}$$

and consequently the Almgren type monotonicity formula

$$N(X_0,u,r) = \frac{E(X_0,u,r)}{H(X_0,u,r)} = \frac{r\displaystyle\int_{B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}X}{\displaystyle\int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma}. \tag{107}$$

As we can see, the expression of the Almgren monotonicity formula is not affected by the position of the point $X_0 \in \mathbb{R}^{n+1}$, i.e if either $X_0 \in B_1 \cap \Sigma$ or $X_0 \in B_1 \setminus \Sigma$. Instead, the rescaling factor in the definitions of $E(X_0,u,r)$ and $H(X_0,u,r)$ are strictly related to the different attitudes of the operator L_a. By [48, 49] we already know the existence of an Almgren type monotonicity formula and the structure/regularity of the nodal set associated to uniformly elliptic operator. For completeness, we give some results on the Almgren type monotonicity result which holds for every $X_0 \in \mathbb{R}^{n+1}$ without using the change of coordinates introduced in [48, 49].

Proposition 3.3.7. *Let $a \in (-1,1)$ and u be an L_a-harmonic function on B_1. Then, for every $X_0 \in B_1 \setminus \Sigma$ there exists $C > 0$ such that $r \mapsto e^{Cr} N(X_0,u,r)$ is absolutely continuous and monotone nondecreasing on $(0,|y_0|)$.*
Hence, there always exists finite the limit

$$N(X_0,u,0^+) = \lim_{r\to 0^+} N(X_0,u,r),$$

which we will call as the Almgren type frequency formula.

Proof. The strategy of the proof is similar to the one for the case $X_0 \in B_1 \cap \Sigma$. By (101) and (102), we already know that passing to the logarithmic derivatives we get from the Cauchy-Schwarz inequality

$$\begin{aligned}\frac{d}{dr}\log N(X_0,u,r) =& \frac{1}{r} + \frac{\displaystyle\frac{d}{dr}\int_{B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}X}{\displaystyle\int_{B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}X} - \frac{\displaystyle\frac{d}{dr}\int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma}{\displaystyle\int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma}\\ \geq& \frac{\displaystyle ay_0 \int_{\partial B_r(X_0)} \frac{|y|^a}{y} u^2 \mathrm{d}\sigma}{\displaystyle r\int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}\sigma} - \frac{\displaystyle ay_0 \int_{B_r(X_0)} \frac{|y|^a}{y} |\nabla u|^2 \,\mathrm{d}\sigma}{\displaystyle r\int_{B_r(X_0)} |y|^a |\nabla u|^2 \,\mathrm{d}\sigma}\end{aligned}$$

for $r \in (r_1, r_2)$. This remainders come out since the Muckenhoupt A_2-weight $\omega(X) = |y|^a$ is homogenous with respect to Σ. Obviously if $a = 0$, by the translation invariance of the Laplacian, we don't need to care anymore about the position of X_0 and also, just substituting $a = 0$, we

obtain the classic Almgren type monotonicity formula of the Laplacian, e.g. [19, 81].
Now, for every $r \in (r_1, r_2)$

$$\frac{d}{dr}\log N(X_0,u,r) \geq \begin{cases} \frac{ay_0}{r}\left(\min_{\partial B_r(X_0)} \frac{1}{y} - \max_{B_r(X_0)} \frac{1}{y}\right) & \text{if } a \cdot y_0 > 0 \\ -\frac{ay_0}{r}\left(\min_{B_r(X_0)} \frac{1}{y} - \max_{\partial B_r(X_0)} \frac{1}{y}\right) & \text{if } a \cdot y_0 < 0 \end{cases}$$

which is equivalent to

$$\frac{d}{dr}\log N(X_0,u,r) \geq -\frac{2\,|ay_0|}{y_0^2 - r^2} \geq -\frac{2\,|ay_0|}{y_0^2 - r_2^2} \quad \text{for } r \in (r_1, r_2),$$

from which we learn that necessary $r_2 < |y_0|$. Consider now

$$H(X_0,u,r) = \frac{1}{r^{n-1}} \int_{\partial B_r(X_0)} |y|^a\, u^2 \mathrm{d}\sigma$$

such that

$$\frac{d}{dr}\log H(X_0,u,r) = \frac{2}{r} N(X_0,u,r) + \frac{a}{r}\left(1 - y_0 \frac{\displaystyle\int_{\partial B_r(X_0)} \frac{|y|^a}{y} u^2 \mathrm{d}\sigma}{\displaystyle\int_{\partial B_r(X_0)} |y|^a\, u^2 \mathrm{d}\sigma}\right).$$

Let us prove the existence of the limit of the Almgren frequency formula as $r \to 0^+$, so suppose by contradiction that $r_1 = \inf\{r > 0 : H(X_0,u,r) > 0 \text{ on } (r, |y_0|)\} > 0$ and consider $r \in (r_1, |y_0|)$. By the previous inequality, we have that there exists a positive constant $C > 0$ such that

$$r \mapsto e^{Cr} N(X_0,u,r)$$

is monotone nondecreasing on $(r_1, |y_0|)$. Then, let $r_1 < r < 2r_1 \leq |y_0|$, since

$$\frac{\displaystyle ay_0 \int_{\partial B_r(X_0)} \frac{|y|^a}{y} u^2 \mathrm{d}\sigma}{\displaystyle r \int_{\partial B_r(X_0)} |y|^a\, u^2 \mathrm{d}\sigma} \geq \begin{cases} \frac{1}{r}\frac{ay_0}{y_0 + 2r_1} & \text{if } a \cdot y_0 > 0 \\ \frac{1}{r}\frac{ay_0}{y_0 - 2r_1} & \text{if } a \cdot y_0 < 0 \end{cases} \tag{108}$$

we have

$$\frac{d}{dr}\log H(X_0,u,r) \leq \frac{2}{r} e^{2Cr_1} N(X_0,u,2r_1) \tag{109}$$

By integrating (109), it follows

$$\frac{H(X_0,u,2r_1)}{H(X_0,u,r)} \leq \left(\frac{2r_1}{r}\right)^{2e^{2Cr_1}N(X_0,u,2r_1)}$$

and since $r \mapsto H(X_0, u, r)$ is continuous, $H(X_0, u, r_1) > 0$ and we seek the contradiction. $\square$

As before, a simple consequence of the monotonicity result and (103) is the following comparison property (which, with $r_2 = 2r_1$, is the so called doubling property).

Corollary 3.3.8. *Let u be an L_a-harmonic function in B_1. For every $X_0 \in B_1 \setminus \Sigma$, there exists $C > 0$ and $R > 0$ such that*

$$H(X_0, u, r_2) \leq H(X_0, u, r_1) \left(\frac{r_2}{r_1}\right)^{2C}$$

for every $0 < r_1 < r_2 < R$.

Proof. Let us consider $R < |y_0|$ and $0 < r_1 < r_2 \leq R$. In order to use the monotonicity of $r \mapsto N(X_0, u, r)$ in this case we need to fix $C, R > 0$ depending on the distance of X_0 from Σ. By (108) we get

$$\frac{d}{dr} \log H(X_0, u, r) \leq \frac{2}{r} e^{2CR} N(X_0, u, R).$$

Now, by integrating the previous inequality we get the claimed result. $\square$

Moreover, since the operator L_a is uniformly elliptic outside Σ, we can apply the same reasoning using the Lipschitz optimal regularity in $\mathbb{R}^n \setminus \Sigma$ and proving

Corollary 3.3.9. *Let u be an L_a-harmonic function in B_1. For every $X_0 \in \Gamma(u) \setminus \Sigma$ we have $N(X_0, u, 0^+) \geq 1$.*

3.4 COMPACTNESS OF BLOW-UP SEQUENCES

All techniques presented in the following Sections involve a local analysis of the solutions, which will be performed via a blow-up procedure. Fix $a \in (-1, 1)$ and u an L_a-harmonic function in B_1. Consider now $X_0 \in \Gamma(u)$ a point on the nodal set of u, then for any $r_k \downarrow 0^+$ we define as the blow-up sequence the collection

$$u_k(X) = \frac{u(X_0 + r_k X)}{\sqrt{H(X_0, u, r_k)}} \quad \text{for } X \in X \in B_{X_0, r_k} = \frac{B_1 - X_0}{r_k},$$

such that $L_a u_k = 0$ and $\|u_k\|_{L^{2,a}(\partial B_1)} = 1$. Through this Chapter we will always apply a blow-up analysis centered in point of the nodal set $\Gamma(u)$ on the characteristic manifold Σ, since as we already remarked the local behaviour of L_a-harmonic function is known outside the characteristic manifold.

In this Section we will prove the convergence of the blow-up sequence and the classification of the blow-up limits starting from the following convergence result.

Theorem 3.4.1. *Let $a \in (-1,1)$ and $\alpha^* = \min\{1, 1-a\}$. Given $X_0 \in \Gamma(u) \cap \Sigma$ and a blow-up sequence u_k centered in X_0 and associated to some $r_k \downarrow 0^+$, there exists $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^n)$ such that, up to a subsequence, $u_k \to p$ in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\alpha \in (0, \alpha^*)$ and strongly in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, the blow-up limit is and entire L_a-harmonic function, i.e.*

$$L_a p = 0 \quad \text{in } \mathbb{R}^{n+1}.$$

In particular, the previous result can be easily improved in the case of L_a-harmonic function purely symmetric with respect to Σ. More precisely, inspired by Proposition 3.2.5, in the first case the convergence holds in $C^{1,\alpha}_{\mathrm{loc}}$ for every $\alpha \in (0,1)$, and this difference relies on the Liouville type theorems introduced in [87].
As in Chapter 1, the proof will be presented in a series of lemmata.

Lemma 3.4.2. *Let $X_0 \in \Gamma(u) \cap \Sigma$. For any given $R > 0$, we have*

$$\|u_k\|_{H^{1,a}(B_R)} \leq C \quad \text{and} \quad \|u_k\|_{L^\infty(\overline{B_R})} \leq C,$$

where $C > 0$ is a constant independent on $k > 0$.

Proof. Let us consider $\rho_k^2 = H(X_0, u, r_k)$, then by definition of the blow-up sequence u_k and Corollary 3.3.2 we get

$$\begin{aligned}
\int_{\partial B_R} |y|^a u_k^2 \mathrm{d}\sigma &= \frac{1}{\rho_k^2} \int_{\partial B_R} |y|^a u^2(X_0 + r_k X) \mathrm{d}\sigma \\
&= \frac{1}{\rho_k^2 r_k^{n+a}} \int_{\partial B_{Rr_k}(X_0)} |y|^a u^2 \mathrm{d}\sigma \\
&= R^{n+a} \frac{H(X_0, u, Rr_k)}{H(X_0, u, r_k)} \\
&\leq R^{n+a} \left(\frac{Rr_k}{r_k}\right)^{2\tilde{C}}
\end{aligned}$$

which gives us $\|u_k\|^2_{L^{2,a}(\partial B_R)} \leq C(R) R^{n+a}$. Similarly

$$\begin{aligned}
\int_{B_R} |y|^a |\nabla u_k|^2 \mathrm{d}\sigma &= N(0, u_k, R) \frac{1}{R} \int_{\partial B_R} |y|^a u_k^2 \mathrm{d}\sigma \\
&\leq C(R) R^{n-1+a} N(X_0, u, Rr_k) \\
&\leq C(R) R^{n-1+a} N(X_0, u, R)
\end{aligned} \tag{110}$$

where in the last inequality we used the monotonicity result of Proposition 3.3.1. Since the map u_k is L_a- harmonic, by [88, Lemma A.2] we get

$$\begin{aligned}\sup_{B_{R/2}} u_k &\leq C(n,s)\left(\frac{1}{R^{n+1+a}}\int_{B_R}|y|^a\, u_k^2 \mathrm{d}X\right)^{1/2}\\ &\leq C(n,s)\left(\frac{H(0,u_k,R)}{n+a+1}\right)^{1/2},\end{aligned}$$

where in the second inequality we used the monotonicity of $r \mapsto H(0,u_k,r)$ in $(0,R)$. Finally, the estimate follows directly from the one the $L^{2,a}(\partial B_R)$-norm. □

So far we have proved the existence of a nontrivial function $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$ such that, up to a subsequence, we have $u_k \rightharpoonup p$ weakly in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $L_a p = 0$ in $\mathcal{D}'(\mathbb{R}^{n+1})$.

The next step is to prove that for $X_0 \in \Gamma(u) \cap \Sigma$ the convergence $u_k \to p$ is indeed strong in $H^{1,a}_{\mathrm{loc}}$ and in $C^{0,\alpha}_{\mathrm{loc}}$ for $\alpha \in (0,\alpha^*)$.

Lemma 3.4.3. *For every $R > 0$, up to a subsequence, $u_k \to p$ strongly in $H^{1,a}(B_R)$.*

Proof. We already know the existence of a blow-up limit $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^n)$, which solves $L_a p = 0$ in $\mathcal{D}'(\mathbb{R}^n)$. Let $\varphi \in C^\infty_c(B_{2R})$ be a cut-off function such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in B_R. By the L_a-harmonicity of u, we get

$$\int_{B_{2R}} |y|^a \varphi \left|\nabla(u_k - p)\right|^2 \mathrm{d}X + \int_{B_{2R}} |y|^a (u_k - p)\langle \nabla(u_k-p), \nabla\varphi\rangle \mathrm{d}X = 0$$

and consequently, we can conclude just by observing that

$$\left|\int_{B_{2R}} |y|^a (u_k - p)\langle \nabla(u_k-p), \nabla\varphi\rangle \mathrm{d}X\right| \leq C \left\|u_k - p\right\|_{L^\infty(B_{2R})} \left\|\nabla u_k\right\|_{L^{2,a}(B_{2R})} \to 0.$$

□

Lemma 3.4.4. *For every $R > 0$ there exists $C > 0$, independent of k, such that*

$$[u_k]_{C^{0,\alpha}(B_R)} = \sup_{X_1,X_2 \in \overline{B}_R} \frac{|u(X_1) - u(X_2)|}{|X_1 - X_2|^\alpha} \leq C$$

for every $\alpha \in (0,\alpha^*)$.

Proof. The proof follows essentially the ideas of the similar results in [83, 84]. Without loss of generality, let $R = 1$ and suppose by contradiction that up to a subsequence

$$L_k = \sup_{X_1, X_2 \in \overline{B}_1} \frac{|\eta(X_1)u_k(X_1) - \eta(X_2)u_k(X_2)|}{|X_1 - X_2|^\alpha} \to \infty$$

where $\eta \in C_c^\infty(B_1)$ is a smooth function such that

$$\begin{cases} \eta(X) = 1, & 0 \le |X| \le 1/2 \\ 0 < \eta(X) \le 1, & 1/2 \le |X| \le 1 \\ \eta(X) = 0, & |X| = 1. \end{cases}$$

Since we may assume that L_k is achieved by $(X_{1,k}, X_{2,k}) \in \overline{B}_1 \times \overline{B}_1$, given $r_k = |X_{1,k} - X_{2,k}|$ we can prove, as $k \to \infty$, that

- $r_k \to 0$
- $\dfrac{\text{dist}(X_{1,k}, \partial B_1)}{r_k} \to \infty, \ \dfrac{\text{dist}(X_{2,k}, \partial B_1)}{r_k} \to \infty.$

Before to continue, let us fix the notations $X_{1,k} = (x_{1,k}, y_{1,k})$ and $X_{2,k} = (x_{2,k}, y_{2,k})$. Now, since by Lemma 3.4.2 the norm $\|u_k\|_{L^\infty(B^+)}$ is uniformly bounded, we have

$$L_k \le \frac{\|u_k\|_{L^\infty(B_1)}}{r_k^\alpha} \left(\eta(X_{1,k}) - \eta(X_{2,k})\right), \tag{111}$$

which immediately implies that $r_k \to 0$. Now, since η is compactly supported in B_1, for every $X \in \overline{B}_1$ we have

$$\eta(X) \le \text{dist}(X, \partial B_1) \text{Lip}(\eta),$$

where obviously $\text{Lip}(\eta)$ denotes the Lipschitz constant of η. Finally, the inequality (111) becomes

$$\frac{\text{dist}(X_{1,k}, \partial B)}{r_k} + \frac{\text{dist}(X_{2,k}, \partial B)}{r_k} \ge \frac{L_k r_k^{\alpha-1}}{\text{Lip}(\eta)\, \|u_k\|_{L^\infty(B_1)}} \to \infty$$

and the result follows by recalling that $\alpha < \alpha^* = \min\{1, 1-a\} \le 1$. As in [83, 84], our proof is based on two different blow-up sequences, indeed we introduce the auxiliary sequences

$$w_k(X) = \eta(P_k) \frac{u_k(P_k + r_k X)}{L_k r_k^\alpha} \quad \text{and} \quad \overline{w}_k(X) = \frac{(\eta u_k)(P_k + r_k X)}{L_k r_k^\alpha}$$

for $X \in B_{P_k,r_k}$ and $P_k = (p_{x,k}, p_{y,k})$ a suitable sequence of points. On one hand, following the same strategy of the blow-up analysis in Chapter 1, $\overline{w}_k$ has an uniform bound on the α-Hölder seminorm, i.e.

$$\sup_{X_1 \neq X_2 \in \overline{B_{P_k,r_k}}} \frac{|\overline{w}_k(X_1) - \overline{w}_k(X_2)|}{|X_1 - X_2|^\alpha} \leq \left| \overline{w}_k\left(\frac{X_1 - P_k}{r_k}\right) - \overline{w}_k\left(\frac{X_2 - P_k}{r_k}\right) \right| = 1,$$

while on the other hand

$$-L_a^k w_{i,k} = 0 \quad \text{in } B_{P_k,r_k}, \quad \text{with } L_a^k = \mathrm{div}\left(\left(y + \frac{p_{y,k}}{r_k}\right)^a \nabla\right). \tag{112}$$

The importance of these two sequences lies in the fact that they have asymptotically equivalent behaviour. Namely, since

$$\begin{aligned} |w_k(X) - \overline{w}_k(X)| \leq & \frac{\|u_k\|_{L^\infty(B_1)}}{r_k^\alpha L_k} |\eta(P_k + r_k X) - \eta(P_k)| \\ \leq & \frac{\mathrm{Lip}(\eta) r_k^{1-\alpha}}{L_k} \|u_k\|_{L^\infty(B_1)} |X| \end{aligned} \tag{113}$$

we get, for any compact $K \subset \mathbb{R}^{n+1}$, that

$$\max_{X \in K \cap B_{P_k,r_k}} |w_k(X) - \overline{w}_k(X)| \longrightarrow 0. \tag{114}$$

Moreover, since $w_k(0) = \overline{w}_k(0)$ we note by (113) that

$$\begin{aligned} |w_k(X) - w_k(0)| \leq & |w_k(X) - \overline{w}_k(X)| + |\overline{w}_k(X) - \overline{w}_k(0)| \\ \leq & C\left(\frac{r_k^{1-\alpha}}{L_k}|X| + |X|^\alpha\right) \end{aligned}$$

and consequently, there exists $C = C(K)$ such that $|w_k(X) - w_k(0)| \leq C$, for every $X \in K$. Let us prove that it is not restrictive to choose $P_k \in \Sigma$ in the definition of $w_k, \overline{w}_k$, since $X_{1,k}, X_{2,k}$ must converge to $B_1 \cap \Sigma$, i.e. there exists $C > 0$ such that, for k sufficiently large,

$$\frac{\mathrm{dist}(X_{1,k}, B_1 \cap \Sigma) + \mathrm{dist}(X_{2,k}, B_1 \cap \Sigma)}{r_k} \leq C.$$

Arguing by contradiction, suppose

$$\frac{\mathrm{dist}(X_{1,k}, B_1 \cap \Sigma) + \mathrm{dist}(X_{2,k}, B_1 \cap \Sigma)}{r_k} \longrightarrow \infty$$

and let us choose $P_k = X_{1,k}$ in the definition of $w_k, \overline{w}_k$ so that $B_{P_k,r_k} \to \mathbb{R}^{n+1}$ and $p_{y,k}^{-1} r_k \to 0^+$. By definition, since $\overline{w}_k$ is a sequence of functions which share the same α-Hölder seminorm and uniformly bounded in every compact $K \subset \mathbb{R}^{n+1}$, by the Ascoli-ArzelÃă theorem, there exists a limit $w \in C(K)$ which, up to a subsequence, is the uniform limit of w_k. By (114), we also find that $w_k \to w$ uniformly con compact sets.
In order to reach a contradiction we can prove that w is a nonconstant globally Hölder harmonic function To this purpose, let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ be a compactly supported smooth function and $\overline{k}$ be sufficiently large so that $\operatorname{supp}\varphi \subset B_{P_k,r_k}$, for all $k \geq \overline{k}$. Fixed $i = 1, \dots, h$, by testing the first equation in (112) with φ we get

$$\int_{\mathbb{R}^{n+1}} \operatorname{div}\left(\left(1 + y\frac{r_k}{p_{y,k}}\right)^a \nabla\varphi\right) w_k \mathrm{d}X = 0.$$

Passing to the uniform limit and observing that

$$\left(1 + y\frac{r_k}{p_{y,k}}\right)^a \to 1 \quad \text{in } C^\infty\left(\operatorname{supp}\varphi\right),$$

we deduce that w is actually harmonic and the contradiction follows naturally by the classical Liouville Theorem, once we have shown that w is globally α - Hölder continuous and not constant. since $P_k = X_{1,k}$ then, up to a subsequence,

$$\frac{X_{2,k} - P_k}{r_k} = \frac{X_{2,k} - X_{1,k}}{|X_{2,k} - X_{1,k}|} \to X_2 \in \partial B_1.$$

Finally, by the equicontinuity and the uniform convergence, we conclude

$$\left|\overline{w}_k\left(\frac{X_1 - P_k}{r_k}\right) - \overline{w}_k\left(\frac{X_2 - P_k}{r_k}\right)\right| = 1 \longrightarrow |\overline{w}(0) - \overline{w}(X_2)| = 1.$$

At this point, the choice $P_k = X_{1,k}$ for every $k \in \mathbb{N}$ guarantees the convergence of the domains $B_{P_k,r_k} \to \mathbb{R}^{n+1}$, while for any compact set $K \subset \mathbb{R}^{n+1}$

$$\max_{X \in K \cap B_{P_k,r_k}} |w_k(X) - \overline{w}_k(X)| \longrightarrow 0.$$

Hence, we are left with two possibilities:

- for any compact set $K \subset \Sigma$ we have $w_k(X) \neq 0$ for every $k \geq k_0$ and $X \in K$;
- there exists a sequence $(X_k)_k \subset \Sigma$ such that $w_k(X_k) = 0$ for every $k \in \mathbb{N}$.

In the first case, if we define $W_k = w_k - w_k(0)$ and $\overline{W}_k = \overline{w}_k - \overline{w}_k(0)$ we obtain that the last sequence is uniformly bounded in $C^{0,\alpha}$ and hence $(W_k)_k$ converges uniformly on compact set to a nonconstant globally α-Hölder continuous L_a-harmonic function W and similarly in the second case the sequence $(w_k)_k$ does converge to a nonconstant globally α-Hölder continuous L_a-harmonic function.
In both cases, the contradiction follows from the Liouville theorem for L_a-harmonic functions since $\alpha \in (0, \alpha^*)$, with $\alpha^* = \min\{1, 1-a\}$. Now, since $\alpha \in (0, \alpha^*)$ with $\alpha^* = \min\{1, 1-a\}$, the contradiction follows immediately from the Liouville theorem for L_a-harmonic functions. □

If instead we consider the general case $X_0 \in \Gamma(u)$ we can prove the following general result

Theorem 3.4.5. *Let u be an L_a-harmonic function in B_1 and $X_0 \in \Gamma(u)$ a point on its nodal set. Given the blow-up sequence u_k centered in X_0 and associated to $r_k \downarrow 0^+$ we have this two cases:*

1. *if $X_0 \in \Sigma$, there exists $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u_k \to p$ in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\alpha \in (0, \alpha^*)$ and strongly in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^n)$. In particular the blow-up limit solves*
$$-L_a p = 0 \quad \text{in} \quad \mathbb{R}^n.$$
2. *if $X_0 \notin \Sigma$, there exists $p \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u_k \to p$ in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\alpha \in (0, 1)$ and strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$. In particular the blow-up limit solves*
$$-\Delta p = 0 \quad \text{in} \quad \mathbb{R}^n.$$

Now we will mention some counterpart of the previous results for the case $X_0 \in \Gamma(u) \setminus \Sigma$.

Lemma 3.4.6. *Let $X_0 \in \Gamma(u) \setminus \Sigma$. For any given $R > 0$, we have $\|u_k\|_{H^1(B_R)} \leq C$ where C is independent on $k > 0$.*

Proof. Since for $X_0 \in \Gamma(u) \setminus \Sigma$ we have
$$\rho_k^2 = H(X_0, u, r_k) = \frac{1}{r_k^{n-1}} \int_{\partial B_{r_k}(X_0)} |y|^a \, u^2 \mathrm{d}\sigma = \int_{\partial B_1} |y_0 + r_k y|^a \, u_k^2 \mathrm{d}\sigma,$$
we get by Corollary 3.3.8 that
$$\begin{aligned} \int_{\partial B_R} |y_0 + r_k y|^a \, u_k^2 \mathrm{d}\sigma &= \frac{1}{\rho_k^2} \int_{\partial B_R} |y_0 + r_k y|^a \, u^2(X_0 + r_k X) \mathrm{d}\sigma \\ &= \frac{1}{\rho_k^2 r_k^{n-1}} \int_{\partial B_{Rr_k}(X_0)} |y|^a \, u^2 \mathrm{d}\sigma \\ &= R^{n-1} \frac{H(X_0, u, Rr_k)}{H(X_0, u, r_k)} \\ &\leq R^{n-1} \left(\frac{Rr_k}{r_k} \right)^{2\tilde{C}}. \end{aligned}$$

Consequently, by the Almgren monotonicity formula, for k so large that $r_k, Rr_k < |y_0|$, we have

$$\int_{B_R} |y_0 + r_k y|^a |\nabla u_k|^2 \, \mathrm{d}X \le C(R) R^{n-2} N(X_0, u, R).$$

Since $X_0 \in \mathbb{R}^n \setminus \Sigma$ and $r_k \downarrow 0^+$ we have

$$\inf_{B_R} |y_0 + r_k y|^a = \inf_{\partial B_R} |y_0 + r_k y|^a \ge |y_0|^a \min\{|1 + r_0|^a, |1 - r_0|^a\}$$

and finally

$$\frac{1}{R^{n-1}} \int_{\partial B_R} u_k^2 \mathrm{d}\sigma + \frac{1}{R^{n-2}} \int_{B_R} |\nabla u_k|^2 \, \mathrm{d}X \le \frac{C'(R)}{|y_0|^a}$$

which gives the claimed bound. □

For completeness we just remarked the uniform bound in H^1 for the blow-up sequence centered outside Σ, but the convergence result for the blow-up sequence centered in $X_0 \notin \Sigma$ is a direct consequence of [81, Theorem 3.3]. Indeed, for every $k > 0$ we have that $\mathrm{div}(|y_0 + r_k y|^a \nabla u_k) = 0$ or equivalently

$$-\Delta u_k = a \frac{r_k}{|y_0 + r_k y|} \partial_y u_k,$$

which implies the existence of a nontrivial function $p \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$ such that, up to a subsequence, we have $u_k \rightharpoonup p$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $-\Delta p = 0$ in $\mathcal{D}'(\mathbb{R}^{n+1})$.

Proposition 3.4.7. *Let $X_0 \in \Gamma(u) \setminus \Sigma$ and p be the blow-up limit of u centered in X_0, as previously defined. Then the following Almgren monotonicty formula*

$$N(Z_0, p, r) = \frac{\dfrac{1}{r^{n-2}} \displaystyle\int_{B_r(Z_0)} |\nabla p|^2 \, \mathrm{d}X}{\dfrac{1}{r^{n-1}} \displaystyle\int_{\partial B_r(Z_0)} p^2 \mathrm{d}\sigma}$$

is well defined for $Z_0 \in \mathbb{R}^{n+1}$ and $r > 0$. In particular the map $r \mapsto N(Z_0, p, r)$ is monotone nondecreasing for every $Z_0 \in \mathbb{R}^{n+1}$ and $r \in (0, +\infty)$.
Moreover, $N(0, p, r) = N(X_0, u, 0^+) =: k$ for every $r > 0$, namely p is k-homogeneous

$$p(X) = |X|^\gamma p\left(\frac{X}{|X|}\right) \text{ for every } X \in \mathbb{R}^{n+1},$$

with $k \in \mathbb{N}, k \ge 1$.

Now we focus our attention on the blow-up limit itself in the challenging case $X_0 \in \Gamma(u) \cap \Sigma$ and the relationship between the value of the Almgren frequency formula and its local behaviour. More precisely, we have

Proposition 3.4.8. *Let $X_0 \in \Gamma(u) \cap \Sigma$ and p be a blow-up limit of u centered in X_0, as previously defined. Then $N(0,p,r) = N(X_0,u,0^+) =: k$ for every $r > 0$ and p is k-homogeneous, i.e.*

$$p(X) = |X|^k p\left(\frac{X}{|X|}\right) \text{ for every } X \in \mathbb{R}^{n+1}.$$

Proof. First of all we prove that $r \mapsto N(0,p,r)$ is constant. Let us observe that $N(0,u_k,r) = N(X_0,u,rr_k)$ and that Theorem 3.4.1 yields that $N(0,p,r) = \lim_k N(0,u_k,r)$. Similarly, for the right hand side we get $\lim_k N(X_0,u,rr_k) = N(X_0,u,0^+)$ by Proposition 3.3.1.

We now compute the derivative of $r \mapsto N(0,p,r)$, in order to prove that p is k-homogeneous, where obviously $k = N(X_0,u,0^+)$ is the Almgren frequency formula. As in the proof of Proposition 3.3.1, we know that

$$\frac{d}{dr}H(0,p,r) = \frac{2}{r^{n+a-1}}\int_{\partial B_r} |y|^a p\partial_r p \mathrm{d}\sigma$$

and by integration by parts that

$$\frac{d}{dr}E(0,p,r) = \frac{1}{r^{n+a-1}}\int_{\partial B_r} |y|^a (\partial_r p)^2 \mathrm{d}\sigma.$$

Hence, this two equalities imply

$$0 = \frac{d}{dr}N(0,p,r) = \frac{2}{r^{2n+2a-2}}\frac{1}{H^2(0,p,r)}\left[\int_{\partial B_r} |y|^a p^2 \mathrm{d}\sigma \int_{\partial B_r} |y|^a |\partial_r p|^2 \mathrm{d}\sigma - \left(\int_{\partial B_r} |y|^a p\partial_r p \mathrm{d}\sigma\right)^2\right]$$

for $r > 0$. This equality yields the existence of $C = C(r) > 0$ such that $\partial_r p = C(r)p$ for every $r > 0$. Using this fact in (103) we get

$$2C(r) = \frac{\int_{\partial B_r} |y|^a p\partial_r p \mathrm{d}\sigma}{\int_{\partial B_r} |y|^a p^2 \mathrm{d}\sigma} = \frac{d}{dr}\log H(0,p,r) = \frac{2}{r}N(0,p,r) = \frac{2}{r}k$$

and thus $C(r) = k/r$ and p is k-homogenous as we claimed. □

In the final part of this Section we classify the possible values of the Almgren frequency formula on the restriction $\Gamma(u) \cap \Sigma$ and consequently the possible blow-up limits, in order to better understand the structure and the stratification of the nodal set of u.

A crucial Corollary of this analysis is that a blow-up limit of u in a point of the nodal set on Σ is either symmetric or antisymmetric with respect Σ: this attitude is due to the fact that near the characteristic manifold the local behaviour of L_a-harmonic function symmetric with respect to Σ is different to the one of antisymmetric L_a-harmonic function, which it is a feature of the degenerate-singular case for $a \in (-1,1)$ and $a \neq 0$.

At this point, we already know that given an L_a-harmonic function u on B_1, for every $X_0 \in \Gamma(u) \cap \Sigma$ and $r_k \downarrow 0^+$ we have, up to a subsequence, that

$$u_k(X) = \frac{u(X_0 + r_k X)}{\sqrt{H(X_0, u, r_k)}} \to p(X),$$

where $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is an nonconstant entire L_a-harmonic function homogenous of order $k \in \mathbb{R}$ with $\|p\|_{L^{2,a}(\partial B_1)} = 1$. In particular, by Proposition 3.4.8 we already know that $k = N(X_0, u, 0^+)$.
Inspired by Proposition 3.2.4, let us consider separately the case when u is symmetric with respect to Σ and the antisymmetric one.

Lemma 3.4.9. *Let $a \in (-1,1)$ and u be an L_a-harmonic function symmetric with respect to Σ. Then, for every $X_0 \in \Gamma(u) \cap \Sigma$, we have*

$$N(X_0, u, 0^+) \in 1 + \mathbb{N}.$$

Proof. Let $X_0 \in \Gamma(u) \cap \Sigma$ and $k = N(X_0, u, 0^+)$ be the Almgren frequency formula in X_0. For every $r_k \to 0^+$ we already know that, up to a subsequence, by Theorem 3.4.1 and Proposition 3.4.8 that

$$u_k(X) = \frac{u(X_0 + r_k X)}{\sqrt{H(X_0, u, r_k)}} \to p(X),$$

where p is an L_a-harmonic k-homogenous function symmetric with respect to Σ.
Since, by Corollary 3.3.6 we already know that $k \geq 1$, let us suppose by contradiction that there exists an homogenous L_a-harmonic function of order $k > 1$ such that $k \notin \mathbb{N}$. Since for every $i = 1, \dots, n$, we have

$$L_a(\partial_{x_i} p) = \partial_{x_i} L_a p = 0,$$

fixed $k = \lfloor k \rfloor$, by Euler's homogeneous function Theorem, we already know that any k-order partial derivative of p with respect to the variables $x_1, \dots, x_n$ must be an homogenous L_a-harmonic of order $\alpha = k - \lfloor k \rfloor \in (0,1)$. The contradiction follows from Proposition 3.3.6, since the homogeneity of an homogenous function is equal to the Almgren frequency formula evaluated in the origin, hence in the symmetric case it must be greater or equal to 1. □

Lemma 3.4.10. *Let $a \in (-1,1)$ and u be an L_a-harmonic function antisymmetric with respect to Σ. Then, for every $X_0 \in \Gamma(u) \cap \Sigma$, we have*

$$N(X_0, u, 0^+) \in 1 - a + \mathbb{N}.$$

Proof. As in the previous Lemma, let $X_0 \in \Gamma(u) \cap \Sigma$ and $k = N(X_0, u, 0^+)$ be the Almgren frequency formula in X_0. For every $r_k \to 0^+$ we already know that, up to a subsequence, we have by Theorem 3.4.1 and Proposition 3.4.8 that $u_k \to p$ where p is an L_a-harmonic k-homogenous function antisymmetric with respect to Σ.
By Proposition 3.2.4, there exists $q \in H^{1,2-a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and L_{2-a}-harmonic function symmetric with respect to Σ, such that $p = qy\,|y|^{-a}$. Since p is k-homogenous, we already know that q must be $(k-1+a)$-homogenous, i.e.

$$q(X) = p(X)y^{-1}\,|y|^a = |X|^{k-1+a}\, p\left(\frac{X}{|X|}\right)\frac{y^{-1}\,|y|^a}{|X|^{-1+a}} = |X|^{k-1+a}\, q\left(\frac{X}{|X|}\right)$$

for every $X \in \mathbb{R}^{n+1}$.
Obviously if $q(0) \neq 0$, then $k = 1 - a$ and q is zero-homogenous, i.e. $q \equiv q(0)$ on $\mathbb{R}^{n+1}$, instead, if $q(0) = 0$, by Lemma 3.4.9 we know that $N(0, q, 0^+) \in 1 + \mathbb{N}$ and consequently $k \in 2 - a + \mathbb{N}$. Similarly, since these two cases correspond to $N(0, q, 0^+) = 0$ and $N(0, q, 0^+) \in 1 + \mathbb{N}$, the final result on k can be formulated as $N(X_0, u, 0^+) \in 1 - a + \mathbb{N}$. □

Proposition 3.4.11. *Let $a \in (-1,1)$ and u be an L_a-harmonic function. Given $X_0 \in \Gamma(u) \cap \Sigma$ and a blow-up sequence u_k centered in X_0 and associated to some $r_k \downarrow 0^+$. Then the blow-up limit $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is either symmetric or antisymmetric with respect to Σ and*

$$N(X_0, u, 0^+) \in \begin{cases} 1 + \mathbb{N}, & \text{if } p \text{ is symmetric,} \\ 1 - a + \mathbb{N}, & \text{if } p \text{ is antisymmetric.} \end{cases}$$

Proof. The proof is a direct consequence of the previous Lemmas. Indeed, let $X_0 \in \Gamma(u) \cap \Sigma$ and $k = N(X_0, u, 0^+)$ be the Almgren frequency formula in X_0. For every $r_k \to 0^+$ we already know that, up to a subsequence, we have by Theorem 3.4.1 and Proposition 3.4.8 that

$$u_k(X) = \frac{u(X_0 + r_k X)}{\sqrt{H(X_0, u, r_k)}} \to p(X),$$

where p is an L_a-harmonic k-homogenous function. By Proposition 3.2.4 and (98), there exist a unique L_a-harmonic function $p_e \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and an L_{2-a}-harmonic function $q_e \in H^{1,2-a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ both symmetric with respect to Σ, such that

$$p(x,y) = p_e(x,y) + q_e(x,y)y\,|y|^{-a}, \quad \text{for every } (x,y) \in \mathbb{R}^{n+1}.$$

Since p is k-homogeneous, we already know by the Definition 3.2.3 that also p_e and $q_e y\,|y|^{-a}$ are k-homogeneous, i.e.

$$N(0, p_e, 0^+) = k = N(0, q_e, 0^+) + 1 - a. \tag{115}$$

If p is purely symmetric or antisymmetric with respect to Σ, the result follows respectively by Lemma 3.4.9 and Lemma 3.4.10. Instead, suppose by contradiction that $p_e \not\equiv 0$ and $q_e \not\equiv 0$, then by (115) the two homogeneity of p_e and q_e can not be simultaneously in $\mathbb{N} + 1$, in contradiction with the previous Lemmas. □

In order to understand the local behaviour of the L_a-harmonic function, we need to construct explicitly the homogenous L_a-harmonic function. As before, we start by classifying the homogeneous solution symmetric with respect to Σ in order to classify all the possible ones.

Lemma 3.4.12. *Let $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ be a nonconstant homogeneous L_a-harmonic function, symmetric with respect to Σ. Then p does not depend on the variable y if and only if it is harmonic in the variable $x_1, \dots, x_n$.*

The proof is trivial and the main consequence is that for every $k \in 1 + \mathbb{N}$ an homogenous harmonic function in the variable $x_1, \dots, x_n$ of order k is an admissible blow-up limit. For this reason, let us concentrate our attention on the case of blow-up limits that depend on the variable y.

Lemma 3.4.13. *[22, Lemma 2.7] Let $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ be an entire L_a-harmonic function symmetric with respect to Σ, such that*

$$|p(X)| \leq C\left(1 + |X|^k\right) \quad \text{in } \mathbb{R}^{n+1},$$

for some $k \in \mathbb{N}$. Then p is a polynomial.

In order to give an explicit formulation of the blow-up limits, at least for $n + 1 = 2$, we remark that if p is a k-homogenous L_a-harmonic function, then for every $i = 1, \dots, n$ the functions $\partial_{x_i} u$ are $(k-1)$-homogeneous L_a-harmonic function and $\partial^2_{yy} u + a y^{-1} \partial_y u$ is a $(k-2)$-homogenous L_a-harmonic function. More precisely,

Lemma 3.4.14. *Let $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ be an L_a-harmonic homogenous polynomial of degree $k \geq 2$ symmetric with respect to Σ. Then p is L_a-harmonic if and only if $\partial_{x_i} p$ and $\partial^2_{yy} p + a y^{-1} \partial_y p$ are L_a-harmonic, for every $i = 1, \dots, n$.*

Proof. Since the derivatives commute, we get

$$\begin{aligned} |y|^{-a} L_a \partial_{x_i} p &= \partial_{x_i}\left(|y|^{-a} L_a p\right), \ \forall i = 1, \dots, n \\ L_a\left(\partial^2_{yy} p + a y^{-1} \partial_y p\right) &= \partial^2_{yy} L_a p + a y^{-1} \partial_y L_a p. \end{aligned} \tag{116}$$

The first implication is obvious since replacing $L_a p = 0$, we get the conditions on the derivatives. Now let us suppose that (116) holds true, since p is an homogenous polynomial of degree k, the function $(x,y) \mapsto |y|^{-a}(L_a p)(x,y)$ is an homogenous polynomial of degree $k-2$ symmetric with respect to Σ. Moreover, since $\partial_{x_i} p$ is L_a-harmonic, for every $i = 1,\dots,n$, from the first conditions in (116) we have $L_a p = q(y)$, with $q(y) = q(-y)$ and

$$\partial^2_{yy} q + a y^{-1} \partial_y q = 0, \tag{117}$$

from the last condition on the derivatives with respect to y.
Since the general solution of (117) is $q(y) = c_1 + c_2 |y|^{-a}$, we immediately get, from the constrained of symmetry, that $q(y) \equiv 0$ and hence that $L_a p = 0$ on $\mathbb{R}^{n+1}$. □

The following Proposition gives a complete picture of the possible entire configurations in $\mathbb{R}^2$. This profiles will be useful in the stratification result of Section 3.7.

Proposition 3.4.15. *Let $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^2)$ be a nonconstant entire L_a-harmonic function symmetric with respect to Σ such that $N(0,p,r) = k$ for every $r > 0$. Suppose that p depends on the variable y, then if $k \in 2\mathbb{N}$ we have*

$$p(x,y) = \frac{(-1)^{\frac{k}{2}}\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{2^k\Gamma\left(1+\frac{k}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}+\frac{k}{2}\right)}\,{}_2F_1\left(-\frac{k}{2},-\frac{k}{2}-\frac{a}{2}+\frac{1}{2},\frac{1}{2},-\frac{x^2}{y^2}\right)y^k, \tag{118}$$

and if $k \in 2\mathbb{N}+1$ we get

$$p(x,y) = -\frac{(-1)^{\frac{k}{2}+\frac{1}{2}}\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{2^{k-1}\Gamma\left(\frac{1}{2}+\frac{k}{2}\right)\Gamma\left(\frac{a}{2}+\frac{k}{2}\right)}\,{}_2F_1\left(\frac{1}{2}-\frac{k}{2},1-\frac{k}{2}-\frac{a}{2},\frac{3}{2},-\frac{x^2}{y^2}\right)xy^{k-1}, \tag{119}$$

where ${}_2F_1$ is the hypergeometric function.

Proof. The proof is by induction and based on the properties related to the derivatives of homogenous L_a-harmonic functions. By Lemma 3.4.13, we already know that every homogenous L_a-harmonic function symmetric with respect to Σ is a polynomial $p(x,y)$ such that, for every $x \in \Sigma$ the map $y \mapsto p(x,y)$ is a polynomial of even degree.
Fix $k = 2m$ with $m \in \mathbb{N}$, consider

$$c(m,a,t) = \frac{(-1)^{m-t}}{2t!}\frac{1}{2^{m-t}(m-t)!}\prod_{i=1}^{m-t}\frac{1}{2i+a-1} = \frac{(-1)^{m-t}\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{2t!(m-t)!2^{2m-2t}\Gamma\left(m-t+\frac{1}{2}+\frac{a}{2}\right)} \tag{120}$$

and consequently

$$p(x,y) = \frac{x^{2m}}{2m!} + \sum_{t=0}^{m-1} c(m,a,t)x^{2t}y^{2m-2t}.$$

which is equivalent to (118). By a direct computation, it is easy to see that $L_a p(x,y) = 0$ for every $(x,y) \in \mathbb{R}^2$. Now, let us prove by induction on $k \geq 2$ that every homogenous L_a-harmonic function is of the form (118). Since the case $k=0$ is trivial, let us take $k=2$. Since p must be of degree 2 and even in the variable y, the polynomial must be like $p(x,y) = a_1x^2 + a_2y^2$ and consequently

$$L_a p = 0 \quad \longleftrightarrow \quad a_2 = -\frac{1}{1+a}a_1,$$

and for $a_1 = 1/2$ we obtain the formula in (120).
Suppose (120) are true for $k \in 2\mathbb{N}$, and consider a L_a-harmonic polynomial p of degree $k+2$, i.e.

$$p(x,y) = a_{m+1}x^{2m+2} + \sum_{t=0}^{m} a_t x^{2t} y^{2m-2t}.$$

Since $\partial^2_x p$ is a L_a-harmonic polynomial of degree k, we must have by the inductive hypothesis

$$a_{m+1}(2m+2)(2m+1) = \frac{1}{2m!}, \quad 2t(2t-1)a_t = c(m,a,t-1) \text{ for } t = 1,\dots,m.$$

which imply, by definition (120), that

$$a_{m+1} = \frac{1}{(2m+2)!}, \quad a_t = \frac{c(m,a,t-1)}{2t(2t-1)} = c(m+1,a,t)$$

for $t = 1,\dots,m$. Finally, let $w = -\partial^2_{yy}p - ay^{-1}\partial_y p$ be a polynomial of degree k. By Lemma 3.4.14 w is L_a-harmonic and, by the inductive hypothesi, we get by linearity that

$$-\partial^2_{yy}(a_0 y^{2m+2}) - ay^{-1}\partial_y(a_0 y^{2m+2}) = c(m,a,0)y^{2m},$$

or in other words that $-(2m+2)(2m+1+a)a_0 = c(m,a,0)$, which implies that

$$a_0 = \frac{c(m,a,0)}{2(m+1)(2m+1+a)} = c(m+1,a,0).$$

We have already proved the formula for the case $k \in 2\mathbb{N}$, while the other one is obtained via an integration respect to the variable x. □

Before to consider the general case $n \geq 3$, we complete the Section with some concrete examples of blow-up profiles in 2-dimensional case. This example, and more generally the class of homogeneous function described by the previous Proposition, will summarize all the possible behaviour of the $(n-2)$-dimensional singular set, as we will see in Section 3.7. For $n \geq 3$, we can not give an explicit formula for the blow- up limits which depend on the variable y, but we can prove that every polynomial in $\mathbb{R}^n$ admits a unique L_a-harmonic extension symmetric with respect to Σ. Since we want to classify the possible blow-up limit of s-harmonic functions on the nodal set, this result suggests that s-harmonic functions can vanish like any polynomial. We will discuss in the following Sections the implication of this classification.

Lemma 3.4.16. *[51, Lemma 5.2] Let $p(x)$ be an homogeneous polynomial of degree d in $\mathbb{R}^n$.Then, there exists a unique polynomial $q(X) = q(x,y)$ of degree d in $\mathbb{R}^{n+1}$ such that*

$$\begin{cases} L_a q = 0 & \text{in } \mathbb{R}^{n+1} \\ q(x,y) = q(x,-y) & \text{in } \mathbb{R}^{n+1} \\ q(x,0) = p(x) & \text{on } \mathbb{R}^n. \end{cases}$$

In particular, it can proved that this extension is obtained by

$$q(x,y) = \sum_{k\geq 0}^{d/2} (-1)^k c_{2k} \Delta^k \frac{x^\alpha}{\alpha!} \frac{y^{2k}}{(2k!)}, \quad c_{2k} = \prod_{i=1}^{k} \frac{2i-1}{2i-2s},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^d$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

Inspired by the previous results, let us introduce the following classes of blow-up limit.

Definition 3.4.17. Given $a \in (-1,1)$ and $k \in \mathbb{R}$, we define the set of all possible blow-up limit of order k, i.e. the set of all L_a-harmonic symmetric polynomials of degree k, as

$$\mathfrak{B}_k^a(\mathbb{R}^{n+1}) = \left\{ p \in H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1}) \;\middle|\; \begin{array}{l} L_a p = 0 \text{ in } \mathbb{R}^{n+1} \\ p(X) = |X|^k p\left(\frac{X}{|X|}\right) \text{ in } \mathbb{R}^{n+1} \end{array} \right\}.$$

Similarly, the set of blow-up limit of order k respectively symmetric or antisymmetric with respect to Σ are defined as

$$\mathfrak{s}\mathfrak{B}_k^a(\mathbb{R}^{n+1}) = \left\{ p \in \mathfrak{B}_k^a(\mathbb{R}^{n+1}) \,|\, p \text{ symmetric with respect to } \Sigma \right\},$$
$$\mathfrak{a}\mathfrak{B}_k^a(\mathbb{R}^{n+1}) = \left\{ p \in \mathfrak{B}_k^a(\mathbb{R}^{n+1}) \,|\, p \text{ antisymmetric with respect to } \Sigma \right\}.$$

By Proposition 3.2.4 and Lemma 3.4.12 we can classify even more the structure of the previous classes emphasizing two subclasses of blow-up limit.

Definition 3.4.18. Given $a \in (-1,1)$ and $k \in \mathbb{R}$, let us define $\mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1}) = \mathfrak{B}^0_k(\mathbb{R}^n_x)$ the set of functions $p \in \mathfrak{B}^a_k(\mathbb{R}^{n+1})$ such that $\Delta_x p = 0$, namely $p(x,y) = p(x)$ in $\mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_y$.

By the previous Section, we already know that for $a \in (-1,1)$ we have $\mathfrak{B}^a_1(\mathbb{R}^{n+1}) = \mathfrak{B}^*_1(\mathbb{R}^{n+1})$ and for $k \geq 2$ we have $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1}) \setminus \mathfrak{B}^*_k(\mathbb{R}^{n+1}) \neq \emptyset$ and it consists of all blow-up limit which depends on the variable y. Finally

Corollary 3.4.19. *For $a \in (-1,1)$, let u be an L_a-harmonic function in B_1 and $X_0 \in \Gamma_k(u)$, for some $k \in 1+\mathbb{N}$ or $k \in 1-a+\mathbb{N}$. Then, every blow-up limit p centered in $X_0 \in \Gamma_k(u)$ is either in $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ or in $\mathfrak{a}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$. Moreover, for every $a \in (-1,1)$ we have*

$$\mathfrak{a}\mathfrak{B}^a_k(\mathbb{R}^{n+1}) = \mathfrak{s}\mathfrak{B}^{2-a}_{k+a-1}(\mathbb{R}^{n+1}) y\,|y|^{-a}.$$

3.5 UNIQUENESS AND CONTINUITY OF TANGENT MAPS AND TANGENT FIELDS

In this Section we start introducing a Weiss type monotonicity formula, which is a fundamental tool well suited for the blow-up analysis at the nodal points $X_0 \in \Gamma(u)$ where $N(X_0,u,0^+) = k$. Starting from this result we will improve our knowledge of the blow-up convergence by proving the existence of a unique no-zero blow-up limit at every point of the nodal set $\Gamma(u)$, which will be called the tangent map φ^{X_0} of u at X_0.
In particular, inspired by the decomposition in (98), we introduce the notion of tangent "*field*" at the nodal point Φ^{X_0}, which take the main role in our blow-up analysis.

Definition 3.5.1. Given u an L_a-harmonic function in B_1, for $k \geq \min\{1, 1-a\}$, we define

$$\Gamma_k(u) := \{X_0 \in \Gamma(u) \colon N(X_0,u,0^+) = k\}.$$

One has to point out that the sets $\Gamma_k(u)$ may be nonempty only for k in a certain set of values. Indeed, by Proposition 3.4.11, we already know that $\Gamma_k(u) \cap \Sigma$ is nonempty if and only if $k \in 1+\mathbb{N}$ or $k \in 1-a+\mathbb{N}$. We remark that all the following results are well known for the case $X_0 \in \Gamma_k(u) \setminus \Sigma$ since the L_a operator is uniformly elliptic outside Σ.

Proposition 3.5.2. *Let u be a nontrivial L_a-harmonic function in B_1. For $X_0 \in \Gamma_k(u) \cap \Sigma$, we introduce the k-Weiss type formula*

$$W_k(X_0,u,r) = \frac{1}{r^{n+a-1+2k}} \int_{B_r(X_0)} |y|^a \, |\nabla u|^2 \,\mathrm{d}X - \frac{k}{r^{n+a+2k}} \int_{\partial B_r(X_0)} |y|^a \, |u|^2 \,\mathrm{d}\sigma.$$

For $r \in (0, 1-|X_0|)$ we have

$$\frac{d}{dr} W_k(X_0,u,r) = \frac{2}{r^{n+a+1+2k}} \int_{\partial B_r(X_0)} |y|^a \left(\langle \nabla u, X - X_0\rangle - ku\right)^2 \mathrm{d}\sigma. \tag{121}$$

which implies that $r \mapsto W_k(X_0, u, r)$ *is monotone nondecreasing in* $(0, 1 - |X_0|)$.
Furthermore, the map $r \mapsto W_k(X_0, u, r)$ *is constant if and only if* u *is homogeneous of degree* k.

Proof. By the definition of the Almgren monotonicity formula, we have

$$W_k(X_0, u, r) = \frac{H(X_0, u, r)}{r^{2k}} \left(N(X_0, u, r) - k\right) \tag{122}$$

which directly implies that

$$\begin{aligned}
\frac{d}{dr} W_k(X_0, u, r) &= \frac{-2k}{r^{2k+1}} \left(E(X_0, u, r) - kH(X_0, u, r)\right) + \frac{1}{r^{2k}} \left(\frac{d}{dr} E(X_0, u, r) - k\frac{d}{dr} H(X_0, u, r)\right) \\
&= \frac{-4k}{r^{2k+1}} E(X_0, u, r) + \frac{2k^2}{r^{2k+1}} H(X_0, u, r) + \frac{1}{r^{2k}} \frac{d}{dr} E(X_0, u, r) \\
&= \frac{2}{r^{n+a+1+2k}} \int_{\partial B_r(X_0)} |y|^a \left(\langle \nabla u, X - X_0 \rangle - ku\right)^2 \mathrm{d}\sigma
\end{aligned}$$

as we previously claimed. □

By a integration by parts, we can rewrite the k-Weiss monotonicity formula as

$$W_k(X_0, u, r) = \frac{1}{r^{n+a+2k}} \int_{\partial B_r(X_0)} |y|^a u \left(\langle \nabla u, X - X_0 \rangle - u\right) \mathrm{d}\sigma.$$

Proposition 3.5.3. *Let* $a \in (-1, 1)$ *and* u *be an* L_a*-harmonic function in* B_1 *and* $X_0 \in \Gamma_k(u) \cap \Sigma$. *For every homogenous* L_a*-harmonic polynomial* $p \in \mathfrak{B}^a_k(\mathbb{R}^{n+1})$, *the map*

$$r \mapsto \frac{H(X_0, u - p_{X_0}, r)}{r^{2k}} = \frac{1}{r^{n+a+2k}} \int_{\partial B_r(X_0)} |y|^a \left(u - p_{X_0}\right)^2 \mathrm{d}\sigma$$

is monotone non decreasing in $(0, 1 - |X_0|)$, *where* $p_{X_0}(X) = p(X - X_0)$.

Through the following Section, we will use the notation $r \mapsto M(X_0, u, p_{X_0}, r)$ for the previous map.

Proof. Since $X_0 \in \Gamma_k(u) \cap \Sigma$ and p is a k-homogenous L_a-harmonic function, we already know that $W_k(X_0, u, r) \geq 0$ and $W_k(X_0, p_{X_0}, r) = 0$ for every $r \in (0, 1 - |X_0|)$. Let $w = u - p_{X_0}$, then

$$\begin{aligned}
W_k(X_0, u, r) &= W_k(X_0, u, r) + W_k(X_0, p_{X_0}, r) \\
&= \frac{1}{r^{n+a-1+2k}} \left(\int_{B_r(X_0)} |y|^a |\nabla w|^2 + 2|y|^a \langle \nabla w, \nabla p \rangle \mathrm{d}X - \frac{k}{r} \int_{\partial B_r(X_0)} |y|^a w^2 + 2|y|^a wp \mathrm{d}\sigma \right) \\
&= W_k(X_0, w, r) + \frac{2}{r^{n+a+2k}} \int_{\partial B_r(X_0)} |y|^a w (\langle \nabla p_{X_0}, X - X_0 \rangle - kp) \mathrm{d}\sigma \\
&= W_k(X_0, u - p_{X_0}, r).
\end{aligned}$$

Hence , by (122), we finally get

$$\begin{aligned}\frac{d}{dr}\frac{H(X_0,u-p_{X_0},r)}{r^{2k}} &= 2\frac{H(X_0,u-p_{X_0},r)}{r^{2k+1}}\left(N(X_0,u-p_{X_0},r)-k\right)\\ &= \frac{2}{r}W_k(X_0,u-p_{X_0},r)\geq 0.\end{aligned}$$

□

Now, we apply the previous monotonicity formulas to study the growth rate of the L_a-harmonic function at the points of the nodal set. In particular, we prove a nondegeneracy and uniqueness result of the blow-up limit, for every points of the nodal set.

Lemma 3.5.4. *Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. Then, for every $X_0 \in \Gamma_k(u)\cap\Sigma$, there exists $C>0$ such that*

$$|u(X)| \leq C\,|X-X_0|^k \quad \text{in } B_{R/2}(X_0).$$

where $R = 1 - \mathrm{dist}(X_0,\partial B_1)$.

Proof. Since whenever $X_0 \in \Gamma_k(u)$ we have $N(X_0,u,r) \geq N(X_0,u,0^+) = k$, then for every $r \in (0,R)$

$$\frac{d}{dr}\log H(X_0,u,r) \geq \frac{2}{r}N(X_0,u,r) \geq \frac{2k}{r}$$

and similarly

$$\log\frac{H(X_0,u,R)}{H(X_0,u,r)} \geq 2k\log\frac{1}{r},$$

which implies $H(X_0,u,r) \leq H(X_0,u,R)r^{2k}$. Now, by [88, Lemma A.2.] and the previous estimate, we get for every $r\in(0,R)$

$$\begin{aligned}\sup_{B_{r/2}} u &\leq C(n,a)\left(\frac{1}{r^{n+1+a}}\int_{B_r}|y|^a\,u^2 \mathrm{d}X\right)^{1/2}\\ &\leq C(n,a)\left(\frac{H(0,u,R)}{n+a+1}\right)^{1/2},\end{aligned}$$

where in the second inequality we used the monotonicity of $r\mapsto H(0,u_k,r)$ in $(0,R)$. □

Lemma 3.5.5 (Nondegeneracy)**.** *Let $a\in(-1,1)$ and u be an L_a-harmonic function in B_1. Then, for every $X_0 \in \Gamma_k(u)\cap\Sigma$ there exists $C>0$ such that*

$$\sup_{\partial B_r(X_0)}|u(X)| \geq Cr^k \quad \text{for } 0<r<R$$

where $R = 1 - \mathrm{dist}(X_0,\partial B_1)$.

Proof. Fix $X_0 \in \Gamma_k(u)$ and suppose by contradiction, given a decreasing sequence $r_j \downarrow 0$, that

$$\lim_{j\to\infty} \frac{H(X_0,u,r_j)^{1/2}}{r_j^k} = \lim_{j\to\infty}\left(\frac{1}{r_j^{n+a+2k}}\int_{\partial B_{r_j}(X_0)} |y|^a\, u^2\, d\sigma\right)^{1/2} = 0.$$

Consider now the blow-up sequence

$$u_j(X) = \frac{u(X_0+r_jX)}{\rho_j} \quad \text{where } \rho_j = H(X_0,u,r_j)^{1/2}$$

constructed starting from r_j and centered in $X_0 \in \Gamma_k(u)$. By Theorem 3.4.1, up to a subsequence $u_j \to p$ uniformly, where p is a nontrivial L_a-harmonic homogenous polynomial of degree k such that $H(0,p,1)=1$.
Let us focus our attention on the functional $M(X_0,u,p_{X_0},r)$ with p_{X_0} as above. By the assumption on the growth of u it follows

$$\begin{aligned}
M(X_0,u,p_{X_0},0^+) &= \lim_{r\to 0}\frac{1}{r^{n+a+2k}}\int_{\partial B_r(X_0)} |y|^a\,(u-p_{X_0})^2\, d\sigma\\
&= \lim_{r\to 0}\int_{\partial B_1} |y|^a\left(\frac{u(X_0+rX)}{r^k} - p(X)\right)^2 d\sigma\\
&= \int_{\partial B_1} |y|^a\, p^2\, d\sigma\\
&= \frac{1}{r^{n+a+2k}}\int_{\partial B_r(X_0)} |y|^a\, p_{X_0}^2\, d\sigma.
\end{aligned}$$

By the monotonicity result of Proposition 3.5.3 on the map $r \mapsto M(X_0,u,p_{X_0},r)$, we obtain

$$\frac{1}{r^{n+a-1+2k}}\int_{\partial B_r(X_0)} |y|^a\,(u-p_{X_0})^2\, d\sigma \geq \frac{1}{r^{n+a-1+2k}}\int_{\partial B_r(X_0)} |y|^a\, p_{X_0}^2\, d\sigma$$

and similarly

$$\int_{\partial B_r(X_0)} |y|^a\,(u^2 - 2up_{X_0})\, d\sigma \geq 0.$$

On the other hand, rescaling the previous inequality and using the blow-up sequence u_j defined as above, we get

$$\int_{\partial B_1} |y|^a\left(H(X_0,u,r_j)u_j^2 - 2H(X_0,u,r_j)^{1/2} r_j^k u_j p\right) d\sigma \geq 0$$

and

$$\int_{\partial B_1} |y|^a\left(\frac{H(X_0,u,r_j)^{1/2}}{r_j^k}u_j^2 - 2u_jp\right) d\sigma \geq 0.$$

The absurd follows passing to the limit for $j \to \infty$, indeed by the previous inequality we get

$$\int_{\partial B_1} |y|^a p^2 \, d\sigma \leq 0$$

in contradiction with $p \not\equiv 0$. □

Theorem 3.5.6 (Uniqueness of the blow-up limit). *Given $a \in (-1,1)$ and u be an L_a-harmonic function in B_1, let us consider $X_0 \in \Gamma_k(u) \cap \Sigma$, i.e. $N(X_0, u, 0^+) = k$. Then there exists a unique nonzero $p \in \mathfrak{B}^a_k(\mathbb{R}^{n+1})$ blow-up limit such that*

$$u_{X_0,r}(X) = \frac{u(X_0 + rX)}{r^k} \longrightarrow p(X). \tag{123}$$

Proof. Up to a subsequence $r_j \to 0^+$, we have that $u_{X_0,r_j} \to p$ in $C^{0,\alpha}_{\text{loc}}$. The existence of such limit follows directly from the previous growth estimate $|u(X)| \leq C\,|X|^k$ and by Lemma 3.5.5 we have p is not identically zero. Now, for any $r > 0$ we have

$$W_k(0, p, r) = \lim_{j\to\infty} W_k(0, u_{X_0,r_j}, r) = \lim_{j\to\infty} W_k(X_0, u, rr_j) = W_k(X_0, u, 0^+) = 0.$$

In particular, Proposition 3.5.2 implies that the L_a-harmonic function p is k-homogeneous and consequently $p \in \mathfrak{B}^a_k(\mathbb{R}^{n+1})$. By Proposition 3.5.3 the limit $M(X_0, u, p_{X_0}, 0^+)$ exists and can be computed by

$$\begin{aligned} M(X_0, u, p_{X_0}, 0^+) &= \lim_{j\to\infty} M(X_0, u, p_{X_0}, r_j) \\ &= \lim_{j\to\infty} M(0, u_{X_0,r_j}, p, 1) \\ &= \lim_{j\to\infty} \int_{\partial B_1} |y|^a (u_{X_0,r_j} - p)^2 \, d\sigma = 0. \end{aligned}$$

Moreover, let us suppose by contradiction that for any other sequence $r_i \to 0^+$ we have that the associated sequence converges to another blow-up limit, i.e. $u_{X_0,r_i} \to q \in \mathfrak{B}^a_k(\mathbb{R}^{n+1}), q \not\equiv p$, then

$$\begin{aligned} 0 = M(X_0, u, p_{X_0}, 0^+) &= \lim_{i\to\infty} M(X_0, u, p_{X_0}, r_i) \\ &= \lim_{i\to\infty} \int_{\partial B_1} |y|^a (u_{r_i} - p)^2 \, d\sigma \\ &= \int_{\partial B_1} |y|^a (q-p)^2 \, d\sigma. \end{aligned}$$

As we claim, since q and p are both homogenous of degree k they must coincide in $\mathbb{R}^n$. □

Inspired by the previous uniqueness and nondegeneracy results, we introduce the notion of tangent map at every point on the nodal set $\Gamma(u)$.

Definition 3.5.7. Given $a \in (-1,1)$, let u be an L_a-harmonic function in B_1 and $X_0 \in \Gamma_k(u) \cap \Sigma$, for $k \geq \min\{1, 1-a\}$. We define as *tangent map* of u at X_0 the unique nonzero map $\varphi^{X_0} \in \mathfrak{B}^a_k(u)$ such that

$$u_{X_0,r}(X) = \frac{u(X_0 + rX)}{r^k} \longrightarrow \varphi^{X_0}(X).$$

Moreover, we define as *normalized* tangent map of u at X_0, the unique nonzero map $p^{X_0} \in \mathfrak{B}^a_k(u)$ normalized with respect to the $L^{2,a}(\partial B_1)$ norm, i.e. the map obtained as

$$u_{X_0,r}(X) = \frac{u(X_0 + rX)}{\sqrt{H(X_0,u,r)}} \longrightarrow p^{X_0}.$$

Exploiting the deep connection between the existence and uniqueness of the tangent map and the Taylor expansion of an L_a-harmonic function, we can find another characterization of the sets $\Gamma_k(u)$.

Corollary 3.5.8. *For $a \in (-1,1)$, let u be an L_a-harmonic function in B_1 and $X_0 \in \Gamma_k(u) \cap \Sigma$, with $k \geq \min\{2, 2-a\}$. Then*

- *if $k \in 2 + \mathbb{N}$, we have $D^\nu u(X_0) = 0$ for every $|\nu| \leq k-1$ and there exists $|\nu_0| = k$ such that $D^{\nu_0} u(X_0) \neq 0$;*
- *if $k \in 2 - a + \mathbb{N}$, we have $D^\nu (uy\,|y|^{-a})(X_0) = 0$ for every $|\nu| \leq k-1$ and there exists $|\nu_0| = k$ such that $D^{\nu_0}(uy\,|y|^{-a})(X_0) \neq 0$.*

Finally, we can prove the validity of the weak unique continuation principle for the restriction of $\Gamma(u)$ on Σ. This result will improve the study of the nodal set of u by showing that its restriction on the characteristic manifold Σ is either with empty interior in Σ or is Σ itself. While in [73] the author proved a similar weak unique continuation property using a boot strap argument based on some regularity estimates for the L_a-operator, in our case we want to emphasize how our blow-up analysis and the classification of the tangent maps allow to study several local property of L_a-harmonic function.

Proposition 3.5.9. *Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. If there exists $X_0 \in B_1 \cap \Sigma$ and $R < 1 - |X_0|$ such that $u = 0$ on $B_R(X_0) \cap \Sigma$, then $u \equiv 0$ on $B_1 \cap \Sigma$.*

Proof. Let $X_0 \in \Gamma(u) \cap \Sigma$ and $R < 1 - |X_0|$. Since we are focusing the attention on the restriction of the nodal set on Σ, by definition of the symmetric part of u with respect to Σ, we can assume

that $u = u_e$ is purely symmetric with respect to Σ.
The idea of the proof is to prove that u is identically zero in the whole ball $B_R(X_0)$ in order to apply the Strong Unique continuation property Corollary 3.3.3, which is actually a stronger result since it does not only concern the trace of u on Σ.
Suppose by contradiction that $u \not\equiv 0$ on $B_R(X_0)$, then

$$H(X_0, u, r) = \frac{1}{r^{n+a}} \int_{\partial B_r(X_0)} |y|^a u^2 \mathrm{d}X > 0$$

for all $r \in (0, R)$. Now, since $X_0 \in \Gamma(u)$, there exists by Theorem 3.5.6 a unique nontrivial tangent map $\varphi^{X_0} \in \mathfrak{B}^a_k(\mathbb{R}^{n+1})$ of u at X_0, where $k = N(X_0, u, 0^+)$. Since u is symmetric with respect to Σ, by Corollary 3.4.9 we know that $\varphi^{X_0} \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$, with $k \in 1 + \mathbb{N}$.
Let us see the points in $B_R(X_0) \cap \Sigma$ as the collection of point $X_0 + r\nu$ for $r < R$ and $\nu \in S^n \cap \Sigma$. By the L^∞_{loc} convergence of the blow-up sequence we get that $\varphi^{X_0}(\nu) = 0$ for all $\nu \in S^n \cap \Sigma$, i.e. $\varphi^{X_0} \equiv 0$ on Σ. Let us prove now that $\varphi^{X_0} \equiv 0$ on $\mathbb{R}^{n+1}$ by induction on the homogeneity $k = N(0, \varphi^{X_0}, 0^+)$.
Let $k = 1$, then up to a rotation $\varphi^{X_0}(x, y) = C\langle X, e_1\rangle = Cx_1$, where $x = (x_1, \cdots, x_n)$ and consequently $C = 0$. Now let us suppose that every k-homogenous L_a-harmonic polynomial symmetric with respect to Σ which is zero on Σ is actually identically zero in $\mathbb{R}^{n+1}$ and consider the case $k + 1$. Given $v_i = \partial_{x_i}\varphi^{X_0} \in H^{1,a}(B_1)$ we have that

$$\begin{cases} L_a v_i = 0 & \text{in } \mathbb{R}^{n+1}, \\ v_i = 0 & \text{on } \Sigma, \\ N(0, v_i, 0^+) \leq k. \end{cases}$$

By the induction hypothesi we have that for every $i = 1, \cdots, n$ $v_i \equiv 0$ on $\mathbb{R}^{n+1}$, i.e. $\partial_{x_i}\varphi^{X_0} \equiv 0$ and consequently φ^{X_0} does not depend on $x \in \Sigma$. The absurd follows immediately since the only L_a-harmonic polynomial in the y-variable is purely antisymmetric and equal, up to a multiplicative constant, to $f(y) = y\,|y|^{-a}$. □

Inspired by the doubling estimate in [73], we get

Proposition 3.5.10. *Let $a \in (-1, 1)$ and u a L_a-harmonic function in B_1. Then $\Gamma(u)$ has empty interior in $\mathbb{R}^{n+1}$ and its restrictions $\Gamma(u) \cap \Sigma$ is either equal to Σ or it has empty interior in Σ itself. More generally,*

$$\Gamma(u) \cap \Sigma = \Gamma(u_e) \cap \Sigma.$$

Proof. Assume by contradiction that there exists $X_0 \in \Gamma(u)$ such that $d = \mathrm{dist}(X_0, \partial\Gamma(u)) < R$, where $R = 1 - |X_0|$. By definition of d, we have $H(X_0, u, r) > 0$ for $r \in (d, d+\varepsilon)$, for some $\varepsilon > 0$. By (103), the map $r \mapsto H(X_0, u, r)$ solves the Cauchy problem

$$\begin{cases} H'(r) = a(r)H(r), & \text{for } r \in (d, d+\varepsilon) \\ H(d) = 0, \end{cases} \tag{124}$$

where $a(r) = 2N(X_0, u, r)/r$, which is continuous at d by the monotonicity result of $r \mapsto N(X_0, u, r)$, i.e. Proposition 3.3.1. Then by uniqueness, $H(r) \equiv 0$ for $r > d$, which contradicts the definition of d and the assumption that u is not identically zero in B_1.
Now, let us consider $\Gamma(u) \cap \Sigma$. By definition of u_e, u_o we easily get

$$\Gamma(u) \cap \Sigma = \Gamma(u_e) \cap \Sigma.$$

Hence, let us suppose that $u \not\equiv u_o$, i.e. $\Gamma(u) \cap \Sigma \subsetneq \Sigma$, and assume as before that there exists $X_0 \in \Gamma(u) \cap \Sigma = \Gamma(u_e) \cap \Sigma$ such that $d = \mathrm{dist}(X_0, \partial\Gamma(u_e) \cap \Sigma) < R$, where $R = 1 - |X_0|$. In other words, the symmetric part u_e of u solves for every $r < d$,

$$\begin{cases} L_a u_e = 0 & \text{on } B_r(X_0) \\ u_e = 0 & \text{on } B_r(X_0) \cap \Sigma \\ \partial^a_y u_e = 0 & \text{on } B_r(X_0) \cap \Sigma, \end{cases}$$

which implies that $u_e \equiv 0$ in $B_d(X_0)$, i.e. $H(X_0, u_e, d) = 0$. As before, by the uniqueness of the Cauchy problem (124), we get that u_e is identically zero in B_1, in contradiction with the assumption $\Gamma(u) \cap \Sigma \subsetneq \Sigma$. □

Looking again to the blow-up sequence, we can establish an auxiliary result concerning the convergence with respect to the Hausdorff distance $d_{\mathcal{H}}$. In particular, we will prove that given the blow-up sequence $(u_{X_0,r})_r$ of u at X_0, then the nodal sets $\Gamma(u_{X_0,r})$ converge to $\Gamma(\varphi^{X_0})$ with respect to the Hausdorff distance. More precisely, given two sets A, B, the Hausdorff distance $d_{\mathcal{H}}(A, B)$ is defined as

$$d_{\mathcal{H}}(A, B) := \max\left\{\sup_{a\in A} \mathrm{dist}(a, B), \sup_{b\in B} \mathrm{dist}(A, b)\right\}.$$

Notice that $d_{\mathcal{H}}(A, B) \leq \varepsilon$ if and only if $A \subseteq N_\varepsilon(B)$ and $B \subseteq N_\varepsilon(A)$, where $N_\varepsilon(\cdot)$ is the closed ε-neighborhood of a set, i.e.

$$N_\varepsilon(A) = \left\{X \in \mathbb{R}^{n+1} : \mathrm{dist}(X, A) \leq \varepsilon\right\}.$$

Proposition 3.5.11. *Let u be an L_a-harmonic function in B_1 and $X_0 \in \Gamma_k(u) \cap \Sigma$. Given, $u_{X_0,r}$ the blow-up sequence at X_0, i.e.*

$$u_{X_0,r}(X) = \frac{u(X_0 + rX)}{r^k} \to \varphi^{X_0}(X).$$

Then $\Gamma(u_{X_0,r}) \cap \Sigma \to \Gamma(\varphi^{X_0}) \cap \Sigma$ with respect to the Hausdorff distance $d_{\mathcal{H}}$ in B_1. More precisely, for every $k \geq \min\{1, 1-a\}$ we have that

$$\Gamma_k(u_{X_0,r}) \cap \Sigma \to \Gamma_k(\varphi^{X_0}) \cap \Sigma$$

with respect to the Hausdorff distance $d_{\mathcal{H}}$ in B_1

Proof. Let $r_i \to 0^+$ and $u_i = u_{X_0,r_i}$ be the blow-up sequence of u at X_0 associated to r_i and $\Gamma_k(u_i)$ be the sequence of nodal sets associated to the blow-up sequence. Through the proof, we will omit the fact that we are just focusing on the restriction of the nodal sets on Σ and we will call $\Gamma_k(\varphi^{X_0})$ as the tangent cone of $\Gamma_k(u)$ at X_0. By Theorem 3.5.6 we already know that φ^{X_0} and $\Gamma(\varphi^{X_0})$ do not depend on the choice of the sequence r_k. By the definition of Hausdorff distance, the claimed result

$$d_{\mathcal{H}}\left(\Gamma_k(u_i) \cap B_1, \Gamma_k(\varphi^{X_0}) \cap B_1\right) \to 0$$

is equivalent to prove that for every $\varepsilon > 0$ there exists $\bar{i} > 0$ such that for every $i \geq \bar{i}$

$$\begin{aligned} \Gamma_k(u_i) \cap B_1 &\subseteq N_\varepsilon\left(\Gamma_k(\varphi^{X_0}) \cap B_1\right) \\ \Gamma_k(\varphi^{X_0}) \cap B_1 &\subseteq N_\varepsilon\left(\Gamma_k(u_i) \cap B_1\right). \end{aligned}$$

Supposing by contradiction that the first inclusion is not true, then there exist $\bar{\varepsilon} > 0$ and a sequence $X_i \in \Gamma_k(u_i) \cap B_1$ such that $\text{dist}\left(X_i, \Gamma_k(\varphi^{X_0}) \cap B_1\right) > \bar{\varepsilon}$. Up to a subsequence, $X_i \to \overline{X} \in \Gamma(\varphi^{X_0}) \cap \overline{B_1}$ by the L^∞_{loc} convergence of $u_i \to \varphi^{X_0}$. Since $X_i \in \Gamma_k(u_i)$ is equivalent to $X_0 + r_i X_i \in \Gamma_k(u)$, given $\Omega \subset\subset B_1$ such that $(X_0 + r_i X_i)_i \subset \Omega$, let us consider

$$\begin{aligned} R_1 &= \min_{p \in \overline{\Omega}} \text{dist}(p, \partial B_1) < 1, \\ \widetilde{C} &= \sup_{p \in \overline{\Omega}} N(p, u, R_1). \end{aligned}$$

Hence, by the monotonicity result Proposition 3.3.1 and Corollary 3.3.2, for $p \in \Omega \cap \Gamma_k(u)$ and $r < R_1$ we get that $N(p, u, r) \geq k$ and

$$N(p, u, r) \leq N(p, u, R_1) \left(\frac{R_1}{r}\right)^{n+a-1+2\tilde{C}} \leq \tilde{C} \frac{1}{r^{n+a-1+2\tilde{C}}}.$$

In particular, from the second inequality we can easily state that for every $\varepsilon > 0$ there exists $\overline{R} = \overline{R}(n, a, \Omega, \varepsilon) > 0$ such that

$$N(p, u, r) \leq k + \varepsilon,$$

for every $p \in \Omega \cap \Gamma_k(u)$ and $r < \overline{R}$.

Now, since for $i > 0$ sufficiently large $N(X_i, u_i, r) \leq N(X_0 + r_i X_i, u, r)$, if we take $p = X_0 + r_i X_i$ in the previous inequality, we get that there exists $\overline{R} = \overline{R}(n, a, X_0) > 0$ sufficiently small, such that for $r < \overline{R}$ we have

$$k \leq N(X_i, u_i, r) \leq k + \min\left(\frac{1}{2}, \frac{1-a}{2}, \frac{|a|}{2}\right).$$

Since $\lim_i N(X_i, u_i, r) = N(\overline{X}, \varphi^{X_0}, r)$ for sufficiently small r, we directly obtain from Proposition 3.4.11 that $N(\overline{X}, \varphi^{X_0}, 0^+) = k$, i.e. $\overline{X} \in \Gamma_k(\varphi^{X_0}) \cap \overline{B_1}$. Finally, the absurd follows immediately since $\Gamma_k(\varphi^{X_0}) \cup \{0\}$ is an homogeneous cone passing through the origin and hence it implies that $\mathrm{dist}(\overline{X}, \Gamma_k(\varphi^{X_0}) \cap B_1) = 0$.

Now let us consider the second inclusion, i.e. for every $\varepsilon > 0$ there exists $\bar{i} > 0$ such that for every $i \geq \bar{i}$

$$\Gamma_k(\varphi^{X_0}) \cap B_1 \subseteq N_\varepsilon\left(\Gamma_k(u_i) \cap B_1\right).$$

Let us start by proving that given $\overline{X} \in \Gamma_k(\varphi^{X_0})$ and $\delta > 0$ such that $B_\delta(\overline{X}) \cap \Gamma(\varphi^{X_0}) = B_\delta(\overline{X}) \cap \Gamma_k(\varphi^{X_0})$ there exiss $\bar{i} > 0$ such that for every $i \geq \bar{i}$ the function u_i must admit a zero of order k in $B_\delta(\overline{X})$, $\Gamma_k(u_i) \cap B_\delta(\overline{X})$. Suppose it is not true, we would have two possibilities: first that $u_i > 0$ in $B_\delta(\overline{X})$ for every $k > 0$ or secondly that every zeros of u_i is not of order k. In the first case, the positivity implies that φ^{X_0} must be an homogeneous L_a-harmonic function nonnegative in $B_\delta(\overline{X})$ with $\varphi^{X_0}(\overline{X}) = 0$, and therefore $\varphi^{X_0} \equiv 0$ in $\mathbb{R}^{n+1}$. In this case the contradiction follows by Lemma 3.5.5 and Theorem 3.5.6.
Secondly, since up to a subsequence there exists a sequence $X_i \in \Gamma_h(u_i) \cap B_\delta(\overline{X})$ for $h \neq k$, by arguing as in the proof of the other inclusion, we can prove that $X_i \to \widetilde{X} \in \Gamma_h(\varphi^{X_0}) \cap B_\delta(\overline{X})$, in contradiction with the definition of $\delta > 0$.
Finally, suppose the existence of $\bar{\varepsilon} > 0$ and $X_i \in \Gamma_k(\varphi^{X_0}) \cap B_1, X_i \to X \in \overline{\Gamma_k(\varphi^{X_0})} \cap \overline{B_1}$, such that $\mathrm{dist}(X_i, \Gamma_k(\varphi^{X_0}) \cap B_1) > \bar{\varepsilon}$. Since $\overline{X} = \{0\}$ is a trivial case, let us focus on the case $\overline{X} \in \Gamma_k(\varphi^{X_0}) \cap \overline{B_1}$. By definition, $\Gamma_k(\varphi^{X_0}) \cup \{0\}$ is an homogenous cone passing through the origin and hence we can take $\overline{X} \in \Gamma_k(\varphi^{X_0}) \cap B_1$ such that $\left|X - \overline{X}\right| \leq \bar{\varepsilon}/4$. Moreover, by the previous paragraph, there exist a sequence $\overline{X}_i \in \Gamma(u_i) \cap B_1$ and $\bar{i} > 0$, such that for $i \geq \bar{i}$ we have $\left|\overline{X}_i - \overline{X}\right| \leq \min\{\delta, \bar{\varepsilon}\}/4$ Hence, we get

$$\mathrm{dist}(X_i, \Gamma_k(\varphi^{X_0}) \cap B_1) \leq \left|X_i - \overline{X}_i\right| \leq |X_i - X| + \left|X - \overline{X}\right| + \left|\overline{X} - \overline{X}_i\right| < \bar{\varepsilon},$$

which leads a contradiction for large $i > 0$. □

The following result will be a fundamental tool in the study of $\Gamma(u) \cap \Sigma$. Indeed, by using the continuation of the tangent map with respect to the $L^{2,a}(\partial B_1)$, we will prove a separation property for the set $\Gamma_k(u) \cap \Sigma$, for $k \geq \min\{2, 2-a\}$.

Theorem 3.5.12 (Continuation of the tangent map on $\Gamma_k(u)$). *Let $X_0 \in \Gamma_k(u) \cap \Sigma$ and φ^{X_0} be the tangent map of u at X_0, such that*

$$u(X) = \varphi^{X_0}(X - X_0) + o(|X - X_0|^k). \tag{125}$$

Thus, the map $X_0 \mapsto \varphi^{X_0}$ from $\Gamma_k(u)$ to $\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ is continuous.Moreover, for any compact set $K \subset \Gamma_k(u) \cap B_1$ there exists a modulus of continuity σ_K such that $\sigma_K(0) = 0$ and

$$\left| u(X) - \varphi^{X_0}(X - X_0) \right| \leq \sigma_K\left(|X - X_0|\right) |X - X_0|^k,$$

for any $X_0 \in K$.

Proof. Since $\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ is a convex subset of a finite-dimensional vector space, namely the space of all k-homogeneous polynomials in $\mathbb{R}^{n+1}$, all the norms on such space are equivalent and hence we can then endow $\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ with the norm of $L^{2,a}(\partial B_1)$.
Fixed $X_0 \in \Gamma(u) \cap \Sigma$, by Theorem 3.5.6 we have the following expansion

$$u(X) = \varphi^{X_0}(X - X_0) + o(|X - X_0|^k).$$

where φ^{X_0} is the unique blow-up limit of u in X_0. Given $\varepsilon > 0$, consider $r_\varepsilon = r_\varepsilon(X_0)$ such that

$$M(X_0, u, \varphi^{X_0}, r_\varepsilon) = \frac{1}{r_\varepsilon^{n+a+2k}} \int_{\partial B_{r_\varepsilon}} |y|^a \left(u(X_0 + X) - \varphi^{X_0}(X)\right)^2 d\sigma < \varepsilon.$$

There exists also $\delta_\varepsilon = \delta_\varepsilon(X_0)$ such that if $X_1 \in \Gamma_k(u) \cap \Sigma$ and $|X_1 - X_0| < \delta_\varepsilon$ then

$$\frac{1}{r_\varepsilon^{n+a+2k}} \int_{\partial B_{r_\varepsilon}} |y|^a \left(u(X_1 + X) - \varphi^{X_0}(X)\right)^2 d\sigma < 2\varepsilon$$

or similarly

$$\int_{\partial B_1} |y|^a \left(\frac{u(X_1 + r_\varepsilon X)}{r_\varepsilon^k} - \varphi^{X_0}(X)\right)^2 d\sigma < 2\varepsilon$$

From Proposition 3.5.3, we have that $M(X_1, u, \varphi^{X_0}, r) < 2\varepsilon$ for $r \in (0, r_\varepsilon)$, which implies

$$\begin{aligned} M(X_1, u, \varphi^{X_0}, 0^+) &= \lim_{r\to 0} M(X_1, u, \varphi^{X_0}, r) \\ &= \lim_{r\to 0} \int_{\partial B_1} |y|^a \left(\frac{u(X_1 + rX)}{r^k} - \varphi^{X_0}(X)\right)^2 d\sigma \\ &= \int_{\partial B_1} |y|^a \left(\varphi^{X_1} - \varphi^{X_0}\right)^2 d\sigma \leq 2\varepsilon. \end{aligned}$$

Now, by the previous computations, for $|X_1 - X_0| < \delta_\varepsilon, 0 < r < r_\varepsilon$ we get

$$\left\| u_{X_1,r} - \varphi^{X_1} \right\|_{L^{2,a}(\partial B_1)} \leq \left\| u_{X_1,r} - \varphi^{X_0} \right\|_{L^{2,a}(\partial B_1)} + \left\| \varphi^{X_0} - \varphi^{X_1} \right\|_{L^{2,a}(\partial B_1)} \leq 2\sqrt{2\varepsilon},$$

where $u_{X_1,r}$ and $u_{X_0,r}$ are the blow-up sequences defined in (123) centered respectively in X_1 and X_0. Now, covering the compact set $K \subset \Gamma_k(u) \cap B_1$ with finitely many balls $B_{\delta_\varepsilon(X_0^i)}(X_0^i)$, for some points $X_0^i \in K, i = 1, \dots, N$, we obtain that the previous inequality is satisfied for all $X_1 \in K$ with $r < r_\varepsilon^K = \min\{r_\varepsilon(X_0^i) : i = 1, \dots, N\}$.
Now, since $u_{X_1,r} - \varphi^{X_1}$ is an L_a-harmonic function in B_1, by [88, Lemma A.2] and (103), we get

$$\begin{aligned} \sup_{B_{1/2}} \left| u_{X_1,r} - \varphi^{X_1} \right| &\leq C(n,a) \left(\int_{B_1} |y|^a \left(u_{X_1,r} - \varphi^{X_1} \right)^2 \mathrm{d}X \right)^{1/2} \\ &\leq 2C(n,a) \sqrt{\frac{2\varepsilon}{n+a+1}} \end{aligned}$$

for all $X_1 \in K, 0 < r < r_\varepsilon^K$, which immediately implies the second part of the Theorem. □

The following definition allows us to study the structure of the restriction $\Gamma(u) \cap \Sigma$. Inspired by Proposition 3.5.10, since $\Gamma(u) \cap \Sigma = \Gamma(u_e) \cap \Sigma$, where u_e is the symmetric part of u with respect to Σ, we characterize the sets $\Gamma_k(u)$ starting from the unique tangent map of u_e. Moreover, since we are dealing with a purely symmetric function, we will see that the structure of the nodal set on Σ is completely defined starting from the blow-up classes $\mathfrak{s}\mathfrak{B}_k^a(\mathbb{R}^{n+1})$ and $\mathfrak{B}_k^*(\mathbb{R}^{n+1})$.

Definition 3.5.13. Given u an L_a-harmonic function on B_1, for $k \geq \min\{1, 1-a\}$ we define on Σ

$$\Gamma_k^*(u) = \left\{ X_0 \in \Gamma_k(u) \cap \Sigma \colon \varphi_e^{X_0} \in \mathfrak{s}\mathfrak{B}_k^*(\mathbb{R}^{n+1}) \right\} \quad \text{and} \quad \Gamma_k^a(u) = \Gamma_k(u) \setminus \Gamma_k^*(u),$$

where $\varphi_e^{X_0} \in \mathfrak{s}\mathfrak{B}_k^a(\mathbb{R}^{n+1})$ is the unique tangent map of u_e at X_0.

In particular $\Gamma_1(u) = \Gamma_1^*(u)$ and for $k \geq 2$ the points in $\Gamma_k^a(u)$ are the ones whose tangent map depends on the variable y.

Corollary 3.5.14. *For every $k \geq 2$ we have that $\overline{\Gamma_k^*(u)} \cap \Gamma_k^a(u) = \emptyset = \Gamma_k^*(u) \cap \overline{\Gamma_k^a(u)}$.*

Proof. The proof of this result is based on the continuation of the tangent map of u on $\Gamma_k(u) \cap \Sigma$ with respect to the norm $L^{2,a}(\partial B_1)$.
First, suppose by contradiction that there exists a sequence $(X_i)_i \subset \Gamma_k^*(u)$ such that $X_i \to X_0 \in \Gamma_k^a(u)$. Let $\varphi^{X_i} = \varphi_e^{X_i}$ and $\varphi^{X_0} = \varphi_e^{X_0}$ be respectively the tangent map of u_e at X_i and X_0, then

by Theorem 3.5.12 we get that $\varphi^{X_i} \to \varphi^{X_0}$ strongly in $L^{2,a}(\partial B_1)$, i.e. for every $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that if $i > N$, then

$$\int_{\partial B_1} |y|^a \left(\varphi^{X_i} - \varphi^{X_0}\right)^2 \mathrm{d}\sigma \leq \varepsilon.$$

Hence, fixed $w_i = \varphi^{X_i} - \varphi^{X_0}$ we get that $L_a w_i = 0$ in B_1 and $\|w_i\|_{L^{2,a}(\partial B_1)} \to 0$. Since w_i is homogenous of degree k, we have

$$\int_{B_1} |y|^a w_i^2 \mathrm{d}X = \frac{1}{n+a+2k+1} \int_{\partial B_1} |y|^a w_i^2 \mathrm{d}\sigma$$
$$\int_{B_1} |y|^a |\nabla w_i|^2 \mathrm{d}X = k \int_{\partial B_1} |y|^a w_i^2 \mathrm{d}\sigma$$

which implies that $w_i \to 0$ strongly in $H^{1,a}(B_1)$. In particular, for every $\phi \in H^{1,a}(B_1)$ we have

$$\int_{B_1} |y|^a \langle \nabla\varphi^{X_0}, \nabla\phi\rangle \mathrm{d}X = \lim_{i\to\infty} \int_{B_1} |y|^a \langle \nabla\varphi^{X_i}, \nabla\phi\rangle \mathrm{d}X$$

If $\phi = \phi(y) \in C^\infty_c((-1,1))$ we get $\nabla\phi = \partial_y\phi\, e_y$ and consequently, since $\varphi^{X_i} \in \mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})$, that

$$\int_{B_1} |y|^a \partial_y\varphi^{X_0}\partial_y\phi \mathrm{d}X = \lim_{i\to\infty} \int_{B_1} |y|^a \partial_y\varphi^{X_i}\partial_y\phi \mathrm{d}X = 0,$$

in contradiction with the fact that $\varphi^{X_0} \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1}) \setminus \mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})$.
Similarly, suppose now there exists a sequence $(X_i)_i \subset \Gamma^a_k(u)$ such that $X_i \to X_0 \in \Gamma^*_k(u)$. As before, let φ^{X_i} and φ^{X_0} be respectively the tangent map of u_e at X_i and X_0, fixed $w_i = \varphi^{X_i} - \varphi^{X_0}$ we get

$$\int_{B_1} |y|^a \phi\Delta_x w_i \mathrm{d}X + \int_{B_1} |y|^a \left(-\partial^2_{yy}\varphi^{X_i} - \frac{a}{y}\partial_y\varphi^{X_i}\right)\phi \mathrm{d}X = k\int_{\partial B_1} |y|^a w_i\phi \mathrm{d}\sigma$$

for every $\phi \in H^{1,a}(B_1)$. The idea now is to reach the contradiction by induction on k, proving that it is impossible that the sequence of L_a-harmonic polynomials in $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1}) \setminus \mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})$ converges strongly in the $L^{2,a}(\partial B_1)$-topology to a function in $\mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})$.
First, for $\phi \in H^{1,a}_0(B_1)$, we have

$$\left|\int_{B_1} |y|^a \left(\partial^2_{yy}\varphi^{X_i} + \frac{a}{y}\partial_y\varphi^{X_i}\right)\phi \mathrm{d}X\right| \leq C(n,k,a)\left(\|\nabla w_i\|_{L^{2,a}(B_1)} + \|w_i\|_{L^{2,a}(B_1)}\right)\|\phi\|_{H^{1,a}(B_1)},$$

which gives us that

$$\psi_i = -\partial^2_{yy}\varphi^{X_i} - \frac{a}{y}\partial_y\varphi^{X_i} \rightharpoonup 0 \text{ in } L^{2,a}(B_1), \tag{126}$$

where ψ_i is a sequence of homogeneous L_a-harmonic polynomial of degree $k-2\geq 0$. Since $X_i \mapsto \psi_i$ is continuous, by Lebesgue's dominated convergence theorem, we get $\|\psi_i\|_{L^{2,a}(B_1)} \to 0$, i.e. $\psi_i \to 0$ strongly in $L^{2,a}(B_1)$.

Hence, let $k = 2$ and $\varphi^{X_i} \in \mathfrak{s}\mathfrak{B}^a_2(\mathbb{R}^{n+1}) \setminus \mathfrak{s}\mathfrak{B}^*_2(\mathbb{R}^{n+1})$ be the sequence that converges to some $\varphi^{X_0} \in \mathfrak{s}\mathfrak{B}^*_2(\mathbb{R}^{n+1})$. As in (126), let us consider the associate sequence ψ_i of L_a-harmonic polynomial of degree $k-2=0$, i.e. a sequence of nonzero constants. Since $\varphi^{X_i} \in \mathfrak{s}\mathfrak{B}^a_2(\mathbb{R}^{n+1}) \setminus \mathfrak{s}\mathfrak{B}^*_2(\mathbb{R}^{n+1})$, by the reasoning in Section 3.4, there exists, up to a multiplicative constant, a unique homogeneous polynomial $u^{X_i} = u^{X_i}(x)$ of degree 2, such that

$$\varphi^{X_i}(x,y) = u^{X_i}(x) - y^2 \quad \text{in } \mathbb{R}^{n+1},$$

where $\Delta_x u^{X_i} = 2(1+a)$ in $\mathbb{R}^n$. In particular, by (126) we get $\psi_i \equiv 2(1+a)$, and the contradiction follows immediately since $a \in (-1,1)$.

Suppose now that we have proved the statement for every $k \leq K$ and let us consider the case $K+1$. By contradiction, let us suppose that $\mathfrak{s}\mathfrak{B}^a_{K+1}(\mathbb{R}^{n+1}) \setminus \mathfrak{s}\mathfrak{B}^*_{K+1}(\mathbb{R}^{n+1})$ is not closed in the $L^{2,a}(\partial B_1)$ topology and $\varphi^{X_i} \to \varphi^{X_0}$ strongly in $L^{2,a}(\partial B_1)$, with $\varphi^{X_0} \in \mathfrak{s}\mathfrak{B}^*_{K+1}(\mathbb{R}^{n+1})$. Thus, we already know that the sequence ψ_i defined by (126) strongly converges to the zero function with respect to the $L^{2,a}(B_1)$ topology. Now, since $(\psi_i)_i$ are $(K-1)$-homogenous, we have that the $L^{2,a}(B_1)$ and $L^{2,a}(\partial B_1)$ topologies are equivalent. Finally, given that $0 < K-1 \leq K$, we have constructed a sequence of $(K-1)$-homogenous L_a-harmonic polynomials ψ_i that converges to the zero function $0 \in \mathfrak{s}\mathfrak{B}^*_{K-1}(\mathbb{R}^{n+1})$, which contradicts the inductive hypothesis. □

In the uniformly elliptic case, the Almgren and Weiss monotonicity formulas allow to prove the uniqueness and non degeneracy of the tangent map and also to construct the generalized Taylor expansion of u at X_0.
In this degenerate-singular setting, since as we already pointed out that the symmetric and antisymmetric cases are complementary, we introduce the notion of *tangential field* of u on its nodal set in order to take care of both this aspect of the solution u.

Definition 3.5.15. Let $a \in (-1,1), u$ be an L_a-harmonic function in B_1 and $X_0 \in \Gamma_k(u) \cap \Sigma$, for some $k \geq \min\{1, 1-a\}$. We define as *tangent field* of u at X_0 the unique nontrivial vector field $\Phi^{X_0} \in (H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1}))^2$ such that

$$\Phi^{X_0} = (\varphi^{X_0}_e, \varphi^{X_0}_o),$$

where $\varphi^{X_0}_e$ and $\varphi^{X_0}_o$ are respectively the tangent map of the symmetric part u_e of u and of the antisymmetric one u_o.

The notion of tangent field will allows us to better understand the regularity of the nodal set $\Gamma(u)$. Indeed, the main weakness of the concept of tangent map in this context is that it take care either of the symmetric part of u or of the even one since they do not share the same optimal regularity and even the same possible vanishing orders. More precisely, by Definition 3.5.7, for every $X_0 \in \Gamma_k(u)$

$$\begin{aligned} u_{X_0,r}(X) &= \frac{u_e(X_0+rX)}{r^k} + \frac{u_o(X_0+rX)}{r^k} \\ &= \frac{u_e^a(X_0+rX)}{r^k} + \frac{u_e^{2-a}(X_0+rX)}{r^{k-1+a}} y\,|y|^{-a} \end{aligned}$$

where both u_e^a and u_e^{2-a} are symmetric with respect to Σ. By Proposition 3.4.11 we already know that the tangent map of u at X_0 is either the tangent map of u_e or the one of u_o.

Definition 3.5.16. Let $a \in (-1,1)$, u be an L_a-harmonic function in B_1 and $X_0 \in \Gamma_k(u)$, for some $k \geq \min\{1, 1-a\}$. We define as Almgren monotonicity formula associated to the tangent field Φ^{X_0} of u at X_0 as the vector

$$N(X, \Phi^{X_0}, r) = \left(N(X, \varphi_e^{X_0}, r), N(X, \varphi_o^{X_0}, r)\right).$$

Obviously, the "*vectorial*" notion of the Almgren frequency formula can be naturally extended to the L_a-harmonic function u as

$$N(X_0, u, r) = (N(X_0, u_e, r), N(X_0, u_o, r)).$$

for every $X_0 \in \Sigma$, but we will avoid this ambiguity on this notion. However, if the function u is symmetric or antisymmetric with respect to Σ, the Almgren monotonicity formula associated to Φ is equal to the one of the tangent map φ^{X_0} of u at X_0 and it does not contain further information on the local behaviour of u at X_0. In general, proving uniqueness result on both the symmetric and antisymmetric part of u with respect to Σ gives the following generalized Taylor expansion

Corollary 3.5.17. *Given $a \in (-1,1)$, let u be an L_a-harmonic function in B_1 and $X_0 \in \Gamma(u) \cap \Sigma$. Then*

$$u(X) = \varphi_e^{X_0}(X - X_0) + \varphi_o^{X_0}(X - X_0) + o(|X - X_0|^k)$$

where $\varphi_e^{X_0} \in \mathfrak{s}\mathfrak{B}_k^a(\mathbb{R}^{n+1})$ and $\varphi_o^{X_0} \in \mathfrak{a}\mathfrak{B}_k^a(\mathbb{R}^{n+1})$ are respectively the tangent maps of u_e and u_o at X_0 and $k = \max\{N(0, \varphi_e^{X_0}, 0^+), N(0, \varphi_o^{X_0}, 0^+)\}$.

Lemma 3.5.18. *Let u be an L_a-harmonic function in B_1 and $\mathcal{R}(u)$ the set*

$$\begin{aligned} \mathcal{R}(u) &= \{X_0 \in \Gamma(u) \colon N(X_0, u_e, 0^+) = 1 \quad \text{or} \quad N(X_0, u_o, 0^+) = 1 - a\} \\ &= \left\{X_0 \in \Gamma(u) \colon N(X_0, \Phi^{X_0}, 0^+) = (1, 1-a)\right\}. \end{aligned}$$

Then $\mathcal{R}(u)\cap\Sigma$ is relatively open in $\Gamma(u)\cap\Sigma$, while for $k\geq 2$ the set $\Gamma_k(u)$ is F_σ, i.e. it is a union of countably many closed sets.

Proof. The first part of the Lemma is a direct consequence of the upper semi-continuity of $X\mapsto N(X,u,0^+)$ on Σ. More precisely, since $\Gamma(u)\cap\Sigma=\Gamma(u_e)\cap\Sigma$, we can restrict our attention on functions symmetric with respect to Σ and hence, we have

$$\mathcal{R}(u)\cap\Sigma=\{X_0\in\Gamma(u)\cap\Sigma\colon N(X_0,u_e,0^+)=1\}$$

Now, by Lemma 3.4.9 we get

$$\{X_0\in\Gamma(u)\colon N(X_0,u_e,0^+)=1\}=\left\{X_0\in\Gamma(u)\colon N(X_0,u_e,0^+)\leq\frac{3}{2}\right\}.$$

Hence, let us focus our attention on the case $\Gamma_k(u_e)\cap\Sigma$, with $k\geq 2$. For $j\in\mathbb{N}$, let us define with E_j the set of points of Σ such that

$$E_j=\left\{X_0\in\Gamma_k(u_e)\cap\Sigma\cap\overline{B_{1-1/j}}\colon\ \frac{1}{j}\rho^k\leq\sup_{|X-X_0|=\rho}|u_e(X)|<j\rho^k,\ 0<\rho<1-|X_0|\right\}.$$

By Lemma 3.5.4 and Lemma 3.5.5 we have that

$$\Gamma_k(u)\cap\Sigma=\bigcup_{j=1}^{\infty}E_j.$$

The result follows immediately once we prove that E_j is a collection of closed sets. Given $X_0\in\overline{E_j}$, since it satisfies

$$\frac{1}{j}\rho^k\leq\sup_{|X-X_0|=\rho}|u_e(X)|<j\rho^k,\tag{127}$$

we need only to show that $X_0\in\Gamma_k(u)\cap\Sigma$, i.e. $N(X_0,u_e,0^+)=k$. Since $X\mapsto N(X,u_e,0^+)$ is upper semi-continuous on Σ, we readily have $N(X_0,u_e,0^+)\geq k$. On the other hand, if $N(X_0,u_e,0^+)=k'>k$, we would have

$$|u_e(X)|\leq C\,|X-X_0|^{k'}\quad\text{in }B_{1-|X_0|}(X_0)\cap\Sigma,$$

which contradicts Lemma 3.5.4 and implies that $X_0\in E_j$. □

An other important consequence of our analysis of the tangent field of u at some nodal point $X_0\in\Gamma(u)\cap\Sigma$ is the following a posteriori result about the "quasi" upper semi-continuity of the Almgren frequency $X\to N(X,u,0^+)$ in the whole $\mathbb{R}^{n+1}$.
Obviously, the restriction of this map on the characteristic manifold Σ and the one on its complementary are both upper semi-continuous, but in general in the whole space $\mathbb{R}^{n+1}$ the upper

semi-continuity is not a immediate consequence of the Almgren monotonicity formula.

This result is based on the decomposition (98) of L_a-harmonic functions and on the regularity result of Proposition 3.2.5 for L_a-harmonic function symmetric with respect to Σ.
Moreover, the following result can be seen as the "*vectorial*" counterpart of the classic one, since it will establish the validity of an upper semi-continuity property for the Almgren frequency in the vectorial sense of Definition 3.5.16. In particular, it allows to relate the notion of vanishing order on Σ to the one on $\mathbb{R}^{n+1} \setminus \Sigma$, which is a fundamental step in order to comprehend the complete topology of the nodal set near Σ.

Proposition 3.5.19. *Let u be an L_a-harmonic function in B_1. Given $(X_i)_i \in \Gamma_k(u) \setminus \Sigma$, with $k \in 1 + \mathbb{N}$ such that $X_i \to X_0 \in \Gamma(u) \cap \Sigma$, then*

$$N(X_i, u, 0^+) \leq \begin{cases} N(X_0, u_e, 0^+), \\ N(X_0, u_o, 0^+) + a. \end{cases}$$

Proof. By Definition 3.2.3 and Proposition 3.2.4, we already know that there exist $f \in H^{1,a}(B_1), g \in H^{1,2-a}(B_1)$ symmetric with respect to Σ and respectively L_a and L_{2-a}-harmonic in B_1, such that

$$u(x, y) = f(x, y) + g(x, y)y\,|y|^{-a} \quad \text{in } B_1, \tag{128}$$

where respectively the first term is the symmetric part u_e of u with respect to Σ and the second one the antisymmetric part u_o.
Through this proof, let us suppose that up to a subsequence $y_i > 0$. Since $(X_i)_i \in \Gamma_k(u) \setminus \Sigma$ and the operator L_a is locally uniformly elliptic on $\mathbb{R}^{n+1} \setminus \Sigma$ we know that $D^\nu u(X_i) = 0$, for any $|\nu| < k$ and there exists $|\nu_0| = k$ such that $D^{\nu_0} u(X_i) \neq 0$. Let us prove the main result by induction on $k \geq 2$. If $k = N(X_i, u, 0^+) = 2$, then for every $j = 1, \cdots, n$ we get from (128) that

$$\begin{aligned} \partial_{x_j} f(x_i, y_i) &= -\partial_{x_j} g(x_i, y_i) y_i^{1-a}, \\ -y_i^a \partial_y f(x_i, y_i) &= (1-a) g(x_i, y_i) + y_i \partial_y g(x_i, y_i), \end{aligned}$$

where the maps $X \mapsto \partial_{x_j} f(X), X \mapsto \partial_y f(X), X \mapsto \partial_{x_j} g(X)$ and $X \mapsto \partial_y g(X)$ are all smooth in B_1 thanks to Proposition 3.2.5. Passing through the limit as $i \to \infty$ we get

$$\partial_{x_j} f(X_0) = 0 \quad \text{and} \quad -\partial_y^a f(X_0) = (1-a) g(X_0).$$

First, since $\partial_y^a f$ is antisymmetric with respect to Σ we get that $g(X_0) = 0$ and consequently $N(X_0, u_o, 0^+) = N(X_0, g, 0^+) + 1 - a \geq 2 - a$. Similarly, if $\varphi^{X_0}(f)$ is the tangent map of f at X_0, we get $\left|\nabla_X(\varphi^{X_0}(f))(X_0)\right| = 0$ and consequently that $N(X_0, f, 0^+) = N(0, \varphi^{X_0}, 0^+) \geq 2$,

as required.
Now, let us prove the inductive step $k-1 \mapsto k$. Let us consider for $j=1,\dots,n$ the collection of symmetric L_a-harmonic functions $v_j = \partial_{x_j} u$ and the antisymmetric L_{-a}-harmonic function $w = \partial^a_y u$. Since $N(X_i,u,0^+)=k$, we get

$$N(X_i,v_j,0^+) = k-1 \quad \text{for } i=1,\dots,n \quad \text{and } N(X_i,w,0^+)=k-1,$$

where we remark that since $X_i \notin \Sigma$ it is the same to consider the order of vanishing of $\partial_y u$ or of the covariant derivative $w=\partial^a_y u$.
By the inductive hypothesi, passing through the limit as $i\to\infty$ we get

$$\begin{cases} N(X_0,v_{j,e},0^+) \geq k-1 \\ N(X_0,v_{j,o},0^+)+a \geq k-1 \end{cases} \text{for } j=1,\dots,n \quad \text{and} \quad \begin{cases} N(X_0,w_e,0^+) \geq k-1 \\ N(X_0,w_o,0^+)-a \geq k-1 \end{cases}.$$

Hence, comparing this result with the notations in (128), since $v_j = \partial_{x_j} f$ and $w=(1-a)g$ on Σ, we get

$$\begin{cases} N(X_0,\partial_{x_j} f,0^+) \geq k-1 \\ N(X_0,\partial_{x_j} g,0^+)+a \geq k-1 \end{cases} \text{for } j=1,\dots,n \quad \text{and} \quad \begin{cases} N(X_0,g,0^+) \geq k-1 \\ N(X_0,\partial^a_y f,0^+)-a \geq k-1 \end{cases},$$

which directly imply that $N(X_0,u_e,0^+)\geq k$ and $N(X_0,u_o,0^+) \geq k-a$, as required. □

Lemma 3.5.20. *Let u be an L_a-harmonic function in B_1 and $\mathcal{R}(u)$ the set*

$$\begin{aligned}\mathcal{R}(u) &= \left\{ X_0 \in \Gamma(u) \;\middle|\; \begin{matrix} N(X_0,u,0^+)=1 & \text{if } X_0\notin\Sigma \\ N(X_0,u_e,0^+)=1 \text{ or } N(X_0,u_o,0^+)=1-a & \text{if } X_0 \in \Sigma \end{matrix} \right\} \\ &= \left\{ X_0 \in \Gamma(u) \;\middle|\; \begin{matrix} N(X_0,u,0^+)=1 & \text{if } X_0\notin\Sigma \\ N(0,\Phi^{X_0},0^+)=(1,1-a) & \text{if } X_0 \in \Sigma \end{matrix} \right\},\end{aligned}$$

is relatively open in $\Gamma(u)$, while for $k\geq \min\{2,2-a\}$ the set $\Gamma_k(u)$ is F_σ, i.e. it is a union of countably many closed sets.

Proof. The first part of the Lemma is a direct consequence of the upper semi-continuity of $X\mapsto N(X,u,r)$ restricted to Σ and to $\mathbb{R}^{n+1}\setminus\Sigma$ and of the Proposition 3.5.19. Hence, let us focus our attention on the case $\Gamma_k(u)$, with $k\geq\min\{2,2-a\}$. For $j\in\mathbb{N}$, let us define with E_j the set of points of Σ such that

$$E_j = \left\{ X_0 \in \Gamma_k(u)\cap\Sigma\cap\overline{B_{1-1/j}} \colon \frac{1}{j}\rho^k \leq \sup_{|X-X_0|=\rho} |u(X)| < j\rho^k,\ 0<\rho<1-|X_0| \right\}.$$

By Lemma 3.5.4 and Lemma 3.5.5 we have that

$$\Gamma_k(u) \cap \Sigma = \bigcup_{j=1}^{\infty} E_j.$$

The result follows immediately once we prove that E_j is a collection of closed sets. Given $X_0 \in \overline{E_j}$, since it satisfies

$$\frac{1}{j}\rho^k \leq \sup_{|X-X_0|=\rho} |u(X)| < j\rho^k,$$

we need only to show that $X_0 \in \Gamma_k(u) \cap \Sigma$, i.e. $N(X_0, u, 0^+) = k$. Since $X \mapsto N(X, u, 0^+)$ is upper semi-continuous on Σ, we readily have $N(X_0, u, 0^+) \geq k$. On the other hand, if $N(X_0, u, 0^+) = k' > k$, we would have

$$|u(X)| \leq C\,|X - X_0|^{k'} \quad \text{in } B_{1-|X_0|}(X_0),$$

which contradicts Lemma 3.5.4 and implies that $X_0 \in E_j$. □

3.6 HAUSDORFF DIMENSION ESTIMATES FOR THE NODAL SET

In this Section we prove different estimates on the Hausdorff dimension of the sets $\Gamma(u)$ and $\Gamma(u) \cap \Sigma$. In the latter, we improve our analysis taking care of the regular and singular part of the restricted nodal set $\Gamma(u) \cap \Sigma$.

Hence, given $a \in (-1, 1)$ and u be an L_a-harmonic function in B_1, let us split the nodal set $\Gamma(u)$ in its regular part

$$\mathcal{R}(u) = \left\{ X_0 \in \Gamma(u) \,\middle|\, \begin{array}{cc} N(X_0, u, 0^+) = 1 & \text{if } X_0 \notin \Sigma \\ N(X_0, u_e, 0^+) = 1 \text{ or } N(X_0, u_o, 0^+) = 1 - a & \text{if } X_0 \in \Sigma \end{array} \right\}, \tag{129}$$

and its singular part

$$\mathcal{S}(u) = \left\{ X_0 \in \Gamma(u) \,\middle|\, \begin{array}{cc} N(X_0, u, 0^+) \geq 2 & \text{if } X_0 \notin \Sigma \\ N(X_0, u_e, 0^+) \geq 2 \text{ and } N(X_0, u_o, 0^+) \geq 2 - a & \text{if } X_0 \in \Sigma \end{array} \right\}. \tag{130}$$

The main idea is to apply a version of the Federer's Reduction Principle due to [78, Appendix A]. More precisely, given a class $\mathcal{F}$ of functions invariant under rescaling and translation and a map $\mathcal{S}$ which associates to each function a subset of $\mathbb{R}^n$, by the Reduction principle we can establishes conditions on F and S which imply that to control the Hausdorff dimension of $\mathcal{S}(u)$ for every $u \in \mathcal{F}$, we just need to control the Hausdorff dimension of $\mathcal{S}(u)$ for elements which are homogeneous of some degree.

Theorem 3.6.1 (Federer's Reduction Principle). *Let $\mathcal{F} \subseteq L^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and define, for any given $u \in \mathcal{F}$, $X_0 \in \mathbb{R}^{n+1}$ and $r > 0$, the rescaled and translated function*

$$u_{X_0,r} := u(X_0 + r\,\cdot).$$

We say that $u_n \to u$ in $\mathcal{F}$ if and only if $u_n \to u$ uniformly on every compact set of $\mathbb{R}^{n+1}$. Moreover, let us assume that $\mathcal{F}$ satisfies the following conditions:

(F1) (Closure under rescaling, translation and normalization) *Given any $|X_0| \leq 1 - r, 0 < r, \rho > 0$ and $u \in \mathcal{F}$, we have that $\rho\, u_{X_0,r} \in \mathcal{F}$.*

(F2) (Existence of a homogeneous blow- up) *Given $|X_0| < 1, r_k \searrow 0$ and $u \in \mathcal{F}$, there exists a sequence $\rho_k \in (0,\infty)$, a real number $\alpha \geq 0$ and a function $\overline{u} \in \mathcal{F}$ α-homogenous such that, if we define $u_k(x) = u(X_0 + r_k x)/\rho_k$ then ,up to a subsequence, we have*

$$u_k \to \overline{u} \quad \textit{in } \mathcal{F}.$$

(F3) (Singular Set hypotheses) *There exists a map $\overline{\mathcal{S}}\colon \mathcal{F} \to \mathcal{C}$, where*

$$\mathcal{C} := \{A \subset \mathbb{R}^{n+1} : A \cap B_1(0) \textit{ is relatively closed in } B_1(0)\}$$

such that

(1) *Given $|X_0| \leq 1 - r, 0 < r < 1$ and $\rho > 0$, it holds*

$$\overline{\mathcal{S}}(\rho\, u_{X_0,r}) = \left(\overline{\mathcal{S}}(u)\right)_{X_0,r} := \frac{\overline{\mathcal{S}}(u) - X_0}{r}.$$

(2) *Given $|X_0| < 1, r_k \searrow 0$ and $u, \overline{u} \in \mathcal{F}$ such that there exists $\rho_k > 0$ satisfying $u_k := \rho_k u_{X_0,r_k} \to \overline{u}$ in $\mathcal{F}$, the following property holds:*

$$\forall \varepsilon > 0,\ \exists k = k(\varepsilon) > 0 \text{ such that for every } k \leq k(\varepsilon)$$
$$\overline{\mathcal{S}}(u_k) \cap B_1(0) \subseteq \{x \in \mathbb{R}^{n+1} : \mathit{dist}(x, \overline{\mathcal{S}}(\overline{u})) < \varepsilon\}.$$

Then, if we define

$$\begin{aligned} d := \max\Big\{&\dim V : V \text{ is a vector subspace of } \mathbb{R}^{n+1} \text{ and there exists } u \in \mathcal{F} \text{ and } \alpha \geq 0 \\ &\text{such that } \overline{\mathcal{S}}(u) \neq \emptyset \text{ and } u_{y,r} = r^\alpha u,\ \forall y \in V,\ r > 0\Big\}, \end{aligned} \tag{131}$$

either $\overline{\mathcal{S}}(u) \cap B_1(0) = \emptyset$ for every $u \in \mathcal{F}$ or else $\dim_{\mathcal{H}}\left(\overline{\mathcal{S}}(u) \cap B_1(0)\right) \leq d$ for every $u \in \mathcal{F}$. Furthermore in the latter case there exists a function $\varphi \in \mathcal{F}$, a d-dimensional subspace $V \subseteq \mathbb{R}^{n+1}$ and a real number $\alpha \geq 0$ such that

$$\varphi_{Y,r} = r^\alpha \varphi \ \text{ for all } \ Y \in V, r > 0 \ \text{ and } \ \overline{\mathcal{S}}(\varphi) \cap B_1(0) = V \cap B_1(0)$$

At last if $d = 0$ then $\overline{\mathcal{S}}(u) \cap B_\rho(0)$ is a fine set for each $u \in \mathcal{F}$ and $0 < \rho < 1$.

We will apply this general result due to Federer in order to construct some estimates on the Hausdorff dimension of the nodal set $\Gamma(u)$ and on its restriction $\Gamma(u) \cap \Sigma$. In the second case, we improve our analysis introducing its regular and singular part on Σ.

Theorem 3.6.2. *Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. Then $\dim_{\mathcal{H}}(\Gamma(u)) \leq n$.*

Proof. A preliminary remark is that we only need to prove the Hausdorff dimensional estimates for the localization of the sets in $K \subset\subset B_1$, since the general statement follows because a countable union of sets with Hausdorff dimension less than or equal to some $n \in \mathbb{R}^+_0$ also has Hausdorff dimension less than or equal to n. Let us consider the class of functions $\mathcal{F}$ defined as

$$\mathcal{F} = \left\{ u \in L^\infty_{\text{loc}}(\mathbb{R}^{n+1}) \setminus \{0\} \colon L_a u = 0 \text{ in } B_r(X_0), \text{ for some } r \in \mathbb{R},\ X_0 \in \mathbb{R}^{n+1} \text{ with } B_r(X_0) \subset B_1 \right\}.$$

By the linearity of the L_a operator, we already know that the closure under rescaling, translation and normalization and assumption $(\mathrm{F}1)$ are all satisfied.
On the other hand, let $|X_0| < 1, r_k \downarrow 0^+$ and $u \in \mathcal{F}$, and choose $\rho_k = \|u(X_0 + r_k \cdot x)\|_{L^{2,a}(\partial B_1)}$. Theorem 3.4.1 and Proposition 3.4.8 yield the existence of a blow-up limit $\varphi^{X_0} \in \mathcal{F}$, i.e. normalized tangent map of u at X_0, such that, up to a subsequence, $u_k \to \varphi^{X_0}$ in $\mathcal{F}$ and φ^{X_0} is a homogeneous function of degree $k = N(X_0, u, 0+) \geq \min\{1, 1-a\}$. Hence also $(\mathrm{F}2)$ holds.

Now, let us consider $\overline{\mathcal{S}} \colon u \mapsto \Gamma(u)$. By the continuity of u, we already know that the set $\Gamma(u) \cap B_1$ is obviously closed in B_1 and it is quite straightforward to check that the two hypotheses in $(\mathrm{F}3)$ are satisfied.
Hence, in order to conclude the analysis, the only thing left to prove is that the integer d in (131) is equal to n. Suppose by contradiction that $d = n+1$, then this would imply the existence of $\varphi \in \mathcal{F}$ with $\overline{\mathcal{S}}(\varphi) = \mathbb{R}^{n+1}$ i.e., $\varphi \equiv 0$ on $\mathbb{R}^{n+1}$, which contradicts the fact the fact the $\Gamma(\varphi)$ has empty interior.
Actually, taking $V = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$ and $\varphi(X) = \langle X, e_n \rangle$, we obtain the claimed estimate on d.

□

Now, we prove a different stratification result for the set $\Gamma(u) \cap \Sigma$, in order to emphasize the different structure of the nodal set with respect of the one of the elliptic case. In particular, with this analysis we want to point out how the different classes of blow-up influence the stratification on the characteristic manifold Σ. Obviously, by Proposition 3.5.10 we already know that

$$\Gamma(u) \cap \Sigma = \Gamma(u_e) \cap \Sigma,$$

and it is either equal to Σ or with empty interior in Σ. Inspired by this fact, since we are dealing with the restriction of the nodal set on the characteristic manifold Σ, we will concentrate our attention on the trace of u on Σ, which is actually equal to the trace of u_e itself.

Theorem 3.6.3. *Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. If $\Gamma(u) \cap \Sigma \neq \Sigma$, then, under the previous notations, we have $\dim_{\mathcal{H}}(\Gamma(u) \cap \Sigma) \leq n-1$ and more precisely*

$$\dim_{\mathcal{H}}(\mathcal{R}(u) \cap \Sigma) = n-1 \quad \textit{and} \quad \dim_{\mathcal{H}}(\mathcal{S}(u) \cap \Sigma) \leq n-1.$$

Proof. Let us consider the class of functions $\mathcal{F}$ defined as

$$\mathcal{F} = \left\{ u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1}) \setminus \{0\} \,\middle|\, \begin{array}{l} L_a u = 0 \text{ in } B_r(X_0), \text{ for some } r \in \mathbb{R},\ X_0 \in \mathbb{R}^{n+1} \\ u \text{ symmetric with respect to } \Sigma \end{array} \right\}.$$

Since the functions in $\mathcal{F}$ are symmetric with respect to Σ and nontrivial, the condition $\Gamma(u) \cap B_r(X_0) \cap \Sigma \neq \Sigma$ is always satisfied.
As before, we already know that the closure under rescaling, translation and normalization and assumption $(\mathrm{F1})$ and $(\mathrm{F2})$ are all satisfied. Moreover, by (129) and (130) we get

$$\begin{aligned} \mathcal{R}(u) \cap \Sigma &= \{X_0 \in \Gamma(u) \cap \Sigma \colon N(X_0, u, 0^+) = 1\}, \\ \mathcal{S}(u) \cap \Sigma &= \bigcup_{k \geq 2} \Gamma_k(u) \cap \Sigma = \bigcup_{k \geq 2} \{X_0 \in \Gamma(u) \cap \Sigma \colon N(X_0, u, 0^+) = k\} \end{aligned}$$

since we are dealing with functions in the class $\mathcal{F}$.
Now, we choose the map $\overline{\mathcal{S}}$ in $(\mathrm{F3})$ according to our needs.

1. Dimensional estimate of $\boldsymbol{\Gamma(u) \cap \Sigma}$
First, let us consider $\overline{\mathcal{S}} \colon u \mapsto \Gamma(u) \cap \Sigma$. By the continuity of u, we already know that the set $\Gamma(u) \cap \Sigma \cap B_1$ is obviously closed in B_1 and it is quite straightforward to check the two hypothesis in $(\mathrm{F3})$. Therefore, in order to conclude the analysis of $\Gamma(u) \cap \Sigma$, the only thing left to prove is that the integer d in (131) is equal to $n-1$.
Suppose by contradiction that $d = n$, this would implies the existence of $\varphi \in \mathcal{F}$ such that $\overline{\mathcal{S}}(\varphi) = \mathbb{R}^n$ i.e., $\varphi \equiv 0$ on Σ. Since φ solves

$$\begin{cases} L_a \varphi = 0 & \text{in } \mathbb{R}^{n+1} \\ \varphi = 0 & \text{on } \Sigma \\ \partial^a_y \varphi = 0 & \text{on } \Sigma, \end{cases}$$

it implies that $\varphi \equiv 0$ on the whole $\mathbb{R}^{n+1}$, which contradicts the fact the $0 \notin \mathcal{F}$. Actually, by taking $V = \mathbb{R}^{n-1} \times (0,0) \subset \Sigma$ and $\varphi(X) = \langle X, e_n \rangle$, we obtain the claimed estimate on d.

2. Dimensional estimate of $\boldsymbol{\mathcal{R}(u) \cap \Sigma}$
Let us consider $\overline{\mathcal{S}} \colon u \mapsto \mathcal{R}(u) \cap \Sigma$. Since we are dealing just with symmetric function with

respect to Σ, by Lemma 3.4.9 we get that necessary $N(X_0, u, 0^+) = 1$ for every $X_0 \in \mathcal{R}(u)$. By the inclusion, we already know that

$$\dim_{\mathcal{H}}(\mathcal{R}(u) \cap \Sigma \cap B_1) \leq n - 1.$$

Finally, we can apply the Reduction principle since (F3) is completely satisfied. More precisely, for $X_0 \in \Sigma \cap B_1, \rho > 0$ and $t > 0$ if $X \in \mathcal{R}(\rho u_{X_0,t}) \cap \Sigma$ then obviously $X_0 + tX \in \mathcal{R}(u) \cap \Sigma$, i.e. $N(X_0 + tX, u, 0^+) = 1$. Secondly, given $u_i, \overline{u} \in \mathcal{F}$ as in (F3), suppose by contradiction that there exists a sequence $X_i \in \Sigma \cap B_1$ and $\overline{\varepsilon} > 0$ such that

$$N(X_i, u_i, 0^+) = 1$$

and $\text{dist}(X_i, \overline{\mathcal{S}}(\overline{u})) \geq \overline{\varepsilon}$. Since, up to a subsequence, $X_i \to \overline{X}$, by the upper semi-continuity of the Almgren frequency formula, we already know that $N(\overline{X}, \overline{u}, 0^+) \geq 1$. Moreover, up to a subsequence, $X_i \to \overline{X} \in \Gamma(\overline{u}) \cap \Sigma \cap \overline{B_1}$ by the L^∞_{loc} convergence of $u_i \to \overline{u}$. The contradiction follows from the same argument of the proof of the second case of Theorem 3.6.2.

More precisely, since $\Gamma(\overline{u}) \cap \Sigma$ is a conical set, i.e. for every $\lambda > 0$ and $\overline{X} \in \Gamma(\overline{u}) \in \Sigma$ we have $\lambda\overline{X} \in \Gamma(\overline{u}) \cap \Sigma$, we deduce that if we can prove $\overline{X} \in \mathcal{R}(\overline{u}) \cap \overline{B_1} \cap \Sigma$ we provide a contradiction, more precisely we get $\text{dist}(\overline{X}, \overline{\mathcal{S}}(\overline{u}) \cap B_1) = 0$. Since there exists $\Omega \subset\subset B_1 \setminus \Sigma$ such that $(X_0 + r_i X_i)_i \subset \Omega$, if we consider

$$R_1 = \min_{p \in \overline{\Omega}} \text{dist}(p, \partial B_1),$$
$$\overline{C} = \sup_{p \in \overline{\Omega}} N(p, u, R_1),$$

we easily get from Corollary 3.3.2 that for $p \in \Omega \cap \mathcal{R}(u)$ and $r < R_1$ we have

$$N(p, u, r) \leq N(p, u, R_1) \left(\frac{R_1}{r}\right)^{n+a-1+2\overline{C}} \leq \overline{C} \frac{1}{r^{n+a-1+2\overline{C}}}.$$

In particular, from the previous inequality we get that there exists $\overline{R} = \overline{R}(n, a, X_0, \varepsilon) > 0$ sufficiently small, such that for $r < \overline{R}$ we have

$$1 \leq N(X_i, u_i, r) \leq 1 + \frac{1}{4}.$$

Since $\lim_i N(X_i, u_i, r) = N(\overline{X}, \overline{u}, r)$ for sufficiently small r, we directly obtain from Lemma 3.4.9 that $N(\overline{X}, \overline{u}, 0^+) = 1$, as we claimed.

As before, let us suppose now that there exist $\varphi \in \mathcal{F}$ and a d-dimensional subspace $V \subset \mathbb{R}^{n+1}$, with $d \leq n-1$, and $k \geq 0$ such that

$$\varphi_{Y,r} = r^k \varphi \ \text{ for all } \ Y \in V, r > 0 \ \text{ and } \ \mathcal{R}(\varphi) \cap \Sigma \cap B_1 = V \cap B_1$$

Since $\varphi \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ is homogenous of degree k with respect to any $Y \in V = \mathcal{R}(\varphi) \cap \Sigma$, namely $N(Y, \varphi, 0^+) = k$, we get that necessary $k = 1$ and that $\mathcal{R}(\varphi) \cap \Sigma$ is d-dimensional. Since every homogenous L_a-harmonic function of order $k = 1$ is one dimensional, i.e. there exists $\nu \in S^{n-1}$ and $C > 0$ such that either

$$\varphi(X) = C\langle X, (\nu, 0)\rangle, \text{ for every } X = (x, y) \in \mathbb{R}^{n+1},$$

we get that $\mathcal{R}(\varphi) \cap \Sigma$ must be $(n-1)$-dimensional, and consequently that

$$\dim_{\mathcal{H}}(\mathcal{R}(u) \cap \Sigma \cap B_1) = n - 1.$$

3. Dimensional estimate of $\mathcal{S}(\mathbf{u}) \cap \mathbf{\Sigma}$

Let us focus on the singular strata

$$\mathcal{S}(u) \cap \Sigma = \bigcup_{k \geq 2} \{X_0 \in \Gamma(u) \cap \Sigma \colon N(X_0, u, 0^+) = k\}$$

Hence, given $\overline{\mathcal{S}} \colon u \mapsto \mathcal{S}(u)$, the map satisfies (F3), since for $X_0 \in \Sigma \cap B_1, \rho > 0$ and $t > 0$, if $X \in \overline{\mathcal{S}}(\rho u_{X_0,t})$ we get

$$N(X, \rho u_{X_0,t}, 0^+) = k \longleftrightarrow N(X_0 + tX, u, 0^+) = k,$$

which is equivalent to $X_0 + tX \in \Gamma_k(u) \subset \mathcal{S}(u)$. Now, given $u_i = \rho_i u_{X_0, r_i}, \overline{u} \in \mathcal{F}$ as in (F3), suppose by contradiction that there exists a sequence $X_i \in B_1$ and $\overline{\varepsilon} > 0$ such that, up to a subsequence, $X_i \to \overline{X}$ and

$$N(X_i, u_i, 0^+) = k \tag{132}$$

and $\text{dist}(X_i, \overline{\mathcal{S}}(\overline{u})) \geq \overline{\varepsilon}$. By the upper semi-continuity of the Almgren frequency formula, we already know that $N(\overline{X}, \overline{u}, 0^+) \geq k$. Since $X_i \in \Gamma_k(u_i)$, there exists $\Omega \subset\subset B_1 \setminus \Sigma$ such that $(X_0 + r_i X_i)_i \subset \Omega$, if we consider

$$\begin{aligned} R_1 &= \min_{p \in \overline{\Omega}} \text{dist}(p, \partial B_1), \\ \overline{C} &= \sup_{p \in \overline{\Omega}} N(p, u, R_1), \end{aligned}$$

we easily get from Corollary 3.3.2 that for $p \in \Omega \cap \Gamma_k(u)$ and $r < R_1$ we have

$$N(p,u,r) \leq N(p,u,R_1)\left(\frac{R_1}{r}\right)^{n+a-1+2\overline{C}} \leq \overline{C}\frac{1}{r^{n+a-1+2\overline{C}}}.$$

In particular, from the previous inequality we get that there exists $\overline{R} = \overline{R}(n,a,X_0,\varepsilon) > 0$ sufficiently small, such that for $r < \overline{R}$ we have

$$k \leq N(X_i,u_i,r) \leq k + \frac{1}{4}.$$

Since $\lim_i N(X_i,u_i,r) = N(\overline{X},\overline{u},r)$ for sufficiently small r, we directly obtain from Lemma 3.4.9 that $N(\overline{X},\overline{u},0^+) = k$, as we claimed.

Since $\mathcal{S}(u) \cap \Sigma \subseteq \Gamma(u) \cap \Sigma$, we already know that

$$\dim_{\mathcal{H}}(\mathcal{S}(u) \cap \Sigma \cap B_1) \leq n-1,$$

which is actually the optimal bound even for the singular set. Indeed, since there exists $\varphi \in \mathcal{F}$, a $(n-1)$-dimensional subspace $V \subset \Sigma$ and $k \geq 0$ such that

$$\varphi_{Y,r} = r^k \varphi \ \text{ for all } \ Y \in V, r > 0 \ \text{ and } \ \mathcal{S}(\varphi) \cap \Sigma \cap B_1 = V \cap B_1.$$

In particular, for every $k \geq 2$, $n \geq 1$ it can be seen by taking $V = \mathbb{R}^{n-1} \times \{0,0\}$ and

$$\varphi(X) = \frac{(-1)^{\frac{k}{2}}\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{2^k\Gamma\left(1+\frac{k}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}+\frac{k}{2}\right)}\,{}_2F_1\left(-\frac{k}{2},-\frac{k}{2}-\frac{a}{2}+\frac{1}{2},\frac{1}{2},-\frac{\langle X,e_n\rangle^2}{\langle X,e_y\rangle^2}\right)\langle X,e_y\rangle^k,$$

as it was previously proved in Section 3.4. □

3.7 REGULARITY OF THE REGULAR AND SINGULAR STRATA

In this Section we show some results about the regularity of the regular and singular strata of the nodal set $\Gamma(u)$. As in Section 3.6, we will consider first the stratification in $\mathbb{R}^{n+1}$ of the whole nodal set $\Gamma(u)$, while in the second case we will focus the attention on the restriction $\Gamma(u) \cap \Sigma$ of the nodal set on the characteristic manifold.

The main idea of this stratification is to classify and then to stratify the nodal set by the spines of the normalized tangent maps, i.e. the largest vector space that leaves the tangent map invariant.

Indeed, we will introduce the subset $\Gamma_k^j(u)$ as the set of points at which every tangent map has at most j independent directions of translation invariance in order to correlate the nodal set of u with the dimension of the set where the tangent map φ^{X_0} vanishes with the same order of u. Moreover, by Theorem 3.6.2 we already know that $\Gamma_k^j(u)$ is well defined for $j \leq n-1$.

More precisely, if $k \geq \min\{2, 2-a\}$ given

$$\Gamma_k(u) = \{X_0 \in \Gamma(u) \colon N(X_0, u, 0^+) = k\}$$

for each $j = 0, \dots, n-1$ let us define

$$\Gamma_k^j(u) = \left\{X_0 \in \Gamma_k(u) \colon \dim \Gamma_k(\varphi^{X_0}) = j\right\},$$

where φ^{X_0} is the unique normalized tangent limit of u at X_0. Obviously, since the uniformly elliptic case is well studied, we focus on the structure of the nodal set $\Gamma(u)$ near Σ.

Before to continue our analysis, let us prove that the concept of dimension in well defined.

Lemma 3.7.1. *Given $a \in (-1,1)$, for every $\varphi \in \mathfrak{B}_k^a(\mathbb{R}^{n+1})$, the singular set $\Gamma_k(\varphi)$ of order $k \geq \min\{2, 2-a\}$ is the largest vector subspace on Σ which leaves φ and $N(\cdot, \varphi, 0^+)$ invariant, i.e.*

$$\Gamma_k(\varphi) = \left\{Z \in \mathbb{R}^{n+1} \colon \varphi(X+Z) = \varphi(X) \text{ for every } X \in \mathbb{R}^{n+1}\right\}.$$

Proof. We can restrict our proof to the case $\varphi \in \mathfrak{s}\mathfrak{B}_k^a(\mathbb{R}^{n+1})$ for $k \geq 2$, since by Corollary 3.4.19 we can easily extend the analysis to the antisymmetric case. Thus, we already know by Corollary 3.5.8 that since $\varphi \in \mathfrak{s}\mathfrak{B}_k^a(\mathbb{R}^{n+1})$ we have

$$\Gamma_k(\varphi) = \left\{X \in \mathbb{R}^{n+1} \colon D^\nu \varphi(X) = 0 \text{ for any } |\nu| \leq k-1\right\}.$$

Obviously $0 \in \Gamma_k(\varphi)$ by the homogeneity of φ and we claim that for every $Z \in \Gamma_k(\varphi)$

$$\varphi(X) = \varphi(X+Z), \quad \text{for all } X \in \mathbb{R}^{n+1},$$

in other words $\Gamma_k(\varphi)$ leaves the map φ invariant. Hence, let $Z \in \Gamma_k(\varphi)$, i.e.

$$D^\nu \varphi(Z) = 0 \quad \text{for any } |\nu| \leq k-1 \tag{133}$$

and write the homogenous polynomial $\varphi \in C^\infty$ as

$$\varphi(X) = \sum_{|\nu|=k} a_\nu X^\nu,$$

where $X^\nu = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots y^{\nu_{n+1}}$ and $a_\nu \in \mathbb{R}$. By (133) we directly get that

$$\varphi(X) = \sum_{|\nu|=k} a_\nu (X-Z)^\nu,$$

which implies the claimed invariance. Since φ is k-homogenous, for every $\lambda > 0$ and $X \in \mathbb{R}^{n+1}$

$$\begin{aligned}\varphi(X) &= \varphi(X-Z)\\ &= (\lambda+1)^k \varphi\left(\frac{X-Z}{\lambda+1}\right)\\ &= (\lambda+1)^k \varphi\left(Z + \frac{X-Z}{\lambda+1}\right)\\ &= \varphi(X+\lambda Z),\end{aligned}$$

therefore, we obtain $D^\nu \varphi(\lambda Z) = 0$ for any $|\nu| \leq k-1$, i.e. $\lambda Z \in \Gamma_k(\varphi)$.
Similarly, noticing that for any $Z, W \in \Gamma_k(\varphi)$ we have $\varphi(Z+W+X) = \varphi(W+X) = \varphi(X)$ for any $X \in \mathbb{R}^{n+1}$, we get $Z + W \in \Gamma_k(\varphi)$. □

Definition 3.7.2. Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. We call d^{X_0} the dimension of $\Gamma_k(u)$ at $X_0 \in \Gamma_k(u)$ as

$$\begin{aligned}d^{X_0} &= \dim \Gamma_k(\varphi^{X_0})\\ &= \dim \left\{\xi \in \mathbb{R}^{n+1} \colon \langle \xi, \nabla_X \varphi^{X_0}(X)\rangle = 0 \text{ for all } X \in \mathbb{R}^{n+1}\right\}.\end{aligned}$$

Following the previous notations we get $\Gamma_k^j(u) = \{X_0 \in \Gamma_k(u) \colon d^{X_0} = j\}$.

Hence, given $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. Then, let us split the nodal set $\Gamma(u)$ in its regular part

$$\mathcal{R}(u) = \left\{ X_0 \in \Gamma(u) \,\middle|\, \begin{array}{cc} N(X_0,u,0^+) = 1 & \text{if } X_0 \notin \Sigma \\ N(X_0,u_e,0^+) = 1 \text{ or } N(X_0,u_o,0^+) = 1-a & \text{if } X_0 \in \Sigma \end{array} \right\},$$

and its singular part

$$\mathcal{S}(u) = \left\{ X_0 \in \Gamma(u) \,\middle|\, \begin{array}{cc} N(X_0,u,0^+) \geq 2 & \text{if } X_0 \notin \Sigma \\ N(X_0,u_e,0^+) \geq 2 \text{ and } N(X_0,u_o,0^+) \geq 2-a & \text{if } X_0 \in \Sigma \end{array} \right\}.$$

As we previously remarked the definition of the regular set $\mathcal{R}(u)$ is well defined in such a way, for every $X_0 \in \Gamma(u) \cap \Sigma$ such that $N(X_0,u_e,0^+) = 1$ or $N(X_0,u_o,0^+) = 1-a$ must exist a sequence of point $(X_i)_i \in \Gamma(u) \setminus \Sigma$ such that $N(X_i,u,0^+) = 1$ and $X_i \to X_0$. The following result gives a generalization in the context of degenerate-singular operator of the concept of regular hypersurface as the set of points where the function vanishes away from its critical set.

Theorem 3.7.3. *Let $a \in (-1,1), a \neq 0$ and u be an L_a-harmonic function in B_1. Then the regular set $\mathcal{R}(u)$ is locally a $C^{k,r}$ hypersurface on $\mathbb{R}^{n+1}$ in the variable $(x, y\,|y|^{-a})$ with*

$$k = \left\lfloor \frac{2}{1-a} \right\rfloor \quad \textit{and} \quad r = \frac{2}{1-a} - \left\lfloor \frac{2}{1-a} \right\rfloor .$$

Moreover, we have that

$$\mathcal{R}(u) = \left\{ X \in \Gamma(u) \colon \; |\nabla_x u(X)|^2 + \left|\partial^a_y u(X)\right|^2 \neq 0 \right\}. \tag{134}$$

Proof. Let us start by proving the characterization of the regular set in terms of the derivatives of the L_a-harmonic function u. By (98), there exist $u_e^a \in H^{1,a}(B_1), u_e^{2-a} \in H^{1,2-a}(B_1)$ respectively L_a and L_{2-a}-harmonic function in B_1, symmetric with respect to Σ, such that

$$u(X) = u_e^a(X) + u_e^{2-a}(X) y\,|y|^{-a} \quad \text{in } B_1.$$

For every $i = 1, \ldots, h$, differentiating the previous equality, we get

$$\partial_{x_i} u(X) = \partial_{x_i} u_e^a(X) + \left(\partial_{x_i} u_e^{2-a}(X)\right) y\,|y|^{-a} \tag{135}$$

$$\partial^a_y u(X) = \left((1-a) u_e^{2-a}(X) + y \partial_y u_e^{2-a}\right) + \partial^a_y u_e^a, \tag{136}$$

where we split the two functions as sum of their symmetric and antisymmetric part. If $X_0 \in \mathcal{R}(u) \setminus \Sigma$ the condition in (134) is obviously satisfied by the local uniformly elliptic regularity outside Σ. Instead, if $X_0 \in \mathcal{R}(u) \cap \Sigma$, if $N(X_0, u_e, 0^+) = 1$ it follows

$$u_e(X) = \varphi_e^{X_0}(\nu^{X_0}) \langle X - X_0, \nu^{X_0} \rangle + o(|X - X_0|),$$

for some $\nu^{X_0} \in S^{n-1} = S^n \cap \Sigma$, and by Theorem 3.5.6 and (135) we get

$$\partial_{x_i} u(X_0) = \partial_{x_i} u_e(X_0) = \varphi_e^{X_0}(\nu^{X_0}) \langle e_i, \nu^{X_0} \rangle,$$

and by the nondegeneracy of the blow-up limit $|\nabla u(X_0)| = \varphi_e^{X_0}(\nu^{X_0}) \neq 0$ (for further details, we remaind to the proof of Theorem 3.7.6). Similarly, taking care of the antisymmetric part, if $N(X_0, u_o, 0^+) = 1 - a$ we get

$$u_o(X) = \varphi_o^{X_0}(e_y) y\,|y|^{-a} + o(|X - X_0|^{1-a}),$$

and consequently $\partial^a_y u(X_0) = \partial^a_y u_o(X_0) = (1-a)\varphi_o^{X_0}(e_y) \neq 0$, as we claimed.
Now, let us consider the other part of the Theorem and let us study the regularity of the regular part $\mathcal{R}(u)$. Since the implicit function theorem implies that the nodal set of a smooth function is a smooth hypersurface away from the critical nodal set, we decide to introduce a suitable change

of variable.
More precisely, let us introduce the change of variable $\Phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that

$$\Phi\colon (x,z) \mapsto \left(x, (1-a)z\,|z|^{\frac{a}{1-a}}\right),$$
$$\Phi^{-1}\colon (x,y) \mapsto \left(x, \frac{y\,|y|^{-a}}{(1-a)^{1-a}}\right),$$

with Jacobian $|J_{\Phi^{-1}}(x,y)| = (1-a)^a\,|y|^{-a}$ and $\Phi(X_0) = X_0$, for every $X_0 \in \Sigma$. By (98) and This change of variable is well known in the literature since it allows to correlate our class of degenerate-singular operator with the class of Baouendi-Grushin Operators (see also [52]). In particular, since $a \in (-1,1)$, we get by simple computations that $\Phi \in C^{k',r'}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$, with

$$k' = \left\lfloor \frac{1}{1-a} \right\rfloor \quad \text{and} \quad r' = \frac{1}{1-a} - \left\lfloor \frac{1}{1-a} \right\rfloor.$$

The previous quantity are well defined since $(1-a)^{-1} > 2$, for every $a \in (-1,1)$ and it blows up as a approaches 1^-.
Now, given $v(x,z) = u(\Phi(x,z))$, we get $\Gamma(u) = \Phi(\Gamma(v))$ and by (98), (135) and (136)

$$\begin{aligned} v(x,z) &= u_e^a(\Phi(x,z)) + u_e^{2-a}(\Phi(x,z))z \\ \partial_{x_i} v(x,z) &= (\partial_{x_i} u)(\Phi(x,z)), \text{ for every } i = 1, \dots, h \\ \partial_z v(x,z) &= (\partial_y^a u)(\Phi(x,z)), \end{aligned}$$

so in particular $|\nabla v(x,z)|^2 = (|\nabla_x u|^2 + \left|\partial_y^a u\right|^2)(\Phi(x,z))$. By (98) and Proposition 3.2.5 we get that given and L_a-harmonic function u in B_1, since $u_e^a(\Phi(x,z)), u_e^{2-a}(\Phi(x,z)) \in C^{k',r'}(B_{1/2})$ we obtain that $v \in C^{k',r'}(B_{1/2})$.Moreover, as we remarked in Section 3.4, since for every $\varphi^{X_0} \in \mathfrak{s}\mathfrak{B}_k^a(\mathbb{R}^{n+1})$ we have that and our change of variables Φ acts only in the y-direction, we get from Proposition 3.2.5 and Theorem 3.5.6 that actually $v \in C^{k,r}(B_{1/2})$ with

$$k = \left\lfloor \frac{2}{1-a} \right\rfloor \geq 1 \quad \text{and} \quad r = \frac{2}{1-a} - \left\lfloor \frac{2}{1-a} \right\rfloor.$$

Now, by the first part of the statement, since $X_0 \in \mathcal{R}(u) \cap \Sigma$ we get by Corollary 3.5.17

$$|\nabla v(X_0)|^2 = |\nabla_x u(X_0)|^2 + \left|\partial_y^a u(X_0)\right|^2 = \varphi_e^{X_0}(\nu^{X_0})^2 + (1-a)^2 \varphi_o^{X_0}(e_y)^2 \neq 0,$$

where $\varphi_e^{X_0}$ and $\varphi_o^{X_0}$ are respectively the tangent map of the symmetric and the antisymmetric part of u with respect to Σ. Since the conclusion follows after an application of the implicit function theorem on the function v and the relation $\Gamma(u) = \Phi(\Gamma(v))$, let us consider three different cases:

(1) $N(X_0, u_e, 0^+) = 1$ and $N(X_0, u_o, 0^+) > 1 - a$, which implies that $\partial_z v(X_0) = 0$ and $\nabla_x v(X_0) = \varphi^{X_0}(\nu^{X_0})\nu^{X_0}$. In this case, up to relabeling the x-variables, by the implicit function theorem we get that there exists $\rho > 0$ and $g \in C^{k,r}(B_\rho(X_0))$ such that $x_1 = g(x) = g(x_2, \dots, x_n, z)$ for every $(x, z) \in \Gamma(v) \cap B_\rho(X_0)$. Going back to the (x, y) variables, we get

$$x_1 = g(x_2, \dots, x_n, y\,|y|^{-a}) \text{ for every } X \in \Gamma(u) \cap B_{\rho/2}(X_0);$$

(2) $N(X_0, u_e, 0^+) > 1$ and $N(X_0, u_o, 0^+) = 1 - a$, in this case since $\partial_{x_i} v(X_0) = 0$ for all $i = 1, \dots, n$ and $\partial_z v(X_0) \neq 0$ we get that there exists $\rho > 0$ and $g \in C^{k,r}(B_\rho(X_0))$ such that $z = g(x) = g(x_1, \dots, x_n)$ for every $(x, z) \in \Gamma(v) \cap B_\rho(X_0)$. Going back to the (x, y) variables, we get

$$y\,|y|^{-a} = g(x) \text{ for every } X \in \Gamma(u) \cap B_{\rho/2}(X_0);$$

(3) $N(X_0, u_e, 0^+) = 1$ and $N(X_0, u_o, 0^+) = 1 - a$, we get that if $a < 0$, by applying the implicit function theorem with respect to the x-variables as in case (1), we get, up to a rotation on Σ, that

$$x_1 = g(x_2, \dots, x_n, y\,|y|^{-a}) \text{ for every } X \in \Gamma(u) \cap B_{\rho/2}(X_0);$$

where in this case $y\,|y|^{-a} \in C^{1,-a}_{\text{loc}}(B_1)$. Otherwise, if $a > 0$ by applying the implicit function theorem on the z-variable as in (2), we get

$$y\,|y|^{-a} = g(x) \text{ for every } X \in \Gamma(u) \cap B_{\rho/2}(X_0),$$

where in the both cases $g \in C^{k,r}(B_\rho(X_0))$.

We remark that the previous records can be changed considering the cases when the minimum between the Almgren frequency of the symmetric and the antisymmetric part of u is achieved by the first or the second one.
Thus, up to consider a smaller radius on the previous cases, the results on $\mathcal{R}(u)$ is a direct consequence of the local one on $\Gamma(u)$ near X_0, since the regular set is relatively open in $\Gamma(u)$ and hence there exists $\rho > 0$ such that $\Gamma(u) \cap B_\rho(X_0) = \mathcal{R}(u) \cap B_\rho(X_0)$. ☐

The previous result explains why the tangent map at a point of the restriction of the nodal set $\Gamma(u) \cap \Sigma$ does not allow to fully understand the geometric picture of the nodal set itself, since we need to take care of both the symmetric and antisymmetric part of u.

Furthermore, we can describe the local behaviour of the regular set $\mathcal{R}(u)$ near the characteristic manifold by using the tangent field Φ^{X_0}, which contains all the geometric information of the regular set. More precisely, as a direct consequence of the previous reports we get

Corollary 3.7.4. *Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. Then the regular part $\mathcal{R}(u)$ of the nodal set intersects the characteristic manifold Σ either orthogonally or tangentially. More precisely, given $X_0 \in \mathcal{R}(u) \cap \Sigma$*

- *if $N(X_0, u, 0^+) = 1$ the direction is orthogonal,*
- *if $N(X_0, u, 0^+) = 1 - a$ the direction is tangential.*

Moreover, independently on $a \in (-1,1)$ and on the value of $N(0, \Phi^{X_0}, 0^+)$, the restriction on Σ of $\mathcal{R}(u)$ is completely described by $\varphi_e^{X_0}$.

Instead, since the structure of the singular set is well known outside of the characteristic manifold Σ, we decided to postpone our analysis and to concentrate our attention to the intersection of the nodal set on Σ.

Hence, in this last part of the Section, we extend the previous analysis focusing on the restriction of the regular and singular set on the characteristic manifold. First, since Lemma 3.7.1 relies on the homogeneity and the regularity of the homogenous polynomial $\varphi \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$, we can reasonably introduce the concept of *dimension restricted to* Σ.

Definition 3.7.5. Given $a \in (-1,1)$, let u be an L_a-harmonic function in B_1. We call $d_\Sigma^{X_0}$ the dimension of $\Gamma_k(u) \cap \Sigma$ at $X_0 \in \Gamma_k(u) \cap \Sigma$ as

$$\begin{aligned} d_\Sigma^{X_0} &= \dim \Gamma_k(\varphi^{X_0}) \cap \Sigma \\ &= \dim \left\{ \xi \in \Sigma \colon \langle \xi, \nabla_x \varphi^{X_0}(x,0) \rangle = 0 \text{ for all } x \in \Sigma \right\}. \end{aligned}$$

Following the previous notations, we define $\Gamma_k^j(u) \cap \Sigma = \{X_0 \in \Gamma_k(u) \cap \Sigma \colon d_\Sigma^{X_0} = j\}$. In the previous Section, we split the restriction on the nodal set on Σ into its regular part

$$\mathcal{R}(u) \cap \Sigma = \{X \in \Gamma(u) \cap \Sigma \colon N(X, u_e, 0^+) = 1\},$$

and its singular part

$$\mathcal{S}(u) \cap \Sigma = \{X \in \Gamma(u) \cap \Sigma \colon N(X, u_e, 0^+) \geq 2\} = \bigcup_{k \geq 2} \Gamma_k(u) \cap \Sigma.$$

Theorem 3.7.6. *Let $a \in (-1,1)$ and u be an L_a-harmonic function in B_1. Then the regular set $\mathcal{R}(u)$ on Σ is locally a smooth hypersurface on Σ and*

$$\mathcal{R}(u) \cap \Sigma = \{X \in \Gamma(u) \cap \Sigma \colon |\nabla_x u_e(X)| \neq 0\}.$$

Proof. By Proposition 3.5.10 we already know that

$$\Gamma(u) \cap \Sigma = \Gamma(u_e) \cap \Sigma,$$

and it is either equal to Σ or with empty interior in Σ. Inspired by this fact, we will concentrate our attention on the trace of u on Σ, which is actually equal to the trace of u_e itself. In order to simplify we will just write u instead of u_e assuming the symmetry with respect to Σ.
Suppose that $\Gamma(u_e) \neq \Sigma$, by Theorem 3.5.6 and our blow-up classification, for every $X_0 \in \mathcal{R}(u) \cap \Sigma$ there exists a linear map $\varphi^{X_0} \in \mathfrak{s}\mathfrak{B}^a_1(\mathbb{R}^{n+1})$ such that

$$u(X) = \varphi^{X_0}(X - X_0) + o(|X - X_0|) = \varphi^{X_0}(\nu^{X_0})\langle X - X_0, \nu^{X_0}\rangle + o(|X - X_0|)$$

for some $\nu^{X_0} \in S^{n-1} = S^n \cap \Sigma$.
Moreover, by Theorem 3.5.12 we know that the map $X_0 \mapsto \varphi^{X_0}(\nu^{X_0})\nu^{X_0}$ is continuous. Passing through its trace on Σ, since $\nu \in \Sigma$ we get

$$u(x, 0) = \varphi^{X_0}(\nu)\langle x - x_0, \nu\rangle + o(|x - x_0|).$$

Since by Proposition 3.2.5 the function $u \in C^\infty(B_{1/2})$, we can use the tangent map in order to compute the directional derivative of u, which will implies the nondegeneracy of the gradient on Σ of u at X_0. More precisely, for every $\xi \in S^{n-1}$

$$\langle \nabla_x u(X_0), \xi\rangle = \frac{d}{dt}u(X_0 + t\xi)\Big|_{t=0} = \lim_{t\to 0}\frac{u(X_0 + t\xi)}{t} = \varphi^{X_0}(\nu^{X_0})\langle \xi, \nu^{X_0}\rangle,$$

and hence $\nabla_x u(X_0) = \varphi^{X_0}(\nu^{X_0})\nu^{X_0}$ which is nonzero by Theorem 3.5.5. Finally, by the implicit function theorem we get the claimed result. □

As we already mentioned, since for $k \geq 2$ we have $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1}) \setminus \mathfrak{B}^*_k(\mathbb{R}^{n+1}) \neq \emptyset$, we decide to introduce the following singular sets

$$\mathcal{S}^*(u) = \bigcup_{k\geq 2}\Gamma^*_k(u) \quad \text{and} \quad \mathcal{S}^a(u) = \bigcup_{k\geq 2}\Gamma^a_k(u),$$

where

$$\Gamma^*_k(u) = \left\{X_0 \in \Gamma_k(u) \cap \Sigma \colon \varphi^{X_0}_e \in \mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})\right\} \text{ and } \Gamma^a_k(u) = (\Gamma_k(u) \cap \Sigma) \setminus \Gamma^*_k(u).$$

The idea is to stratify the singular set taking care of both the dimension $d^{X_0}_\Sigma$ and the different classes of tangent map associated to the sets $\Gamma^*_k(u)$ and $\Gamma^a_k(u)$.

Theorem 3.7.7. *Given $a \in (-1, 1)$, let u be an L_a-harmonic function in B_1. Then for $k \in 2 + \mathbb{N}$ and $j = 0, \cdots, n-1$ the sets $\Gamma^j_k(u) \cap \Sigma$ is contained in a countable union of j-dimensional C^1 manifolds.*

Proof. The proof of this result follows the strategy of [50, Theorem 1.3.8]. Since φ^{X_0} is a polynomial of degree k on Σ, we can write the following

$$\varphi^{X_0}(x,0) = \sum_{|\alpha|=k} \frac{a_\alpha(x_0,0)}{\alpha!}x^\alpha,$$

where the coefficients $X \mapsto a_\alpha(X)$ are continuous on $\Gamma_k(u)\cap\Sigma$ and, since $u(X)=0$ on $\Gamma_k(u)$ it holds

$$\left|\varphi^{X_0}(X-X_0)\right| \leq \sigma\left(|X-X_0|\right)|X-X_0|^k \quad \text{for every } X, X_0 \in K.$$

For any multi-index $|\alpha|\leq k$, let us introduce for any $X\in\Gamma_k(u)$ the collection

$$f_\alpha(X) = \begin{cases} a_\alpha(X) & \text{if } |\alpha| = k \\ 0 & \text{if } |\alpha| < k \end{cases}.$$

Let us prove that the compatibility conditions for the Whitney's extension theorem are fully satisfied in order to guarantee the existence of a function $F\in C^k(\mathbb{R}^{n+1})$ such that

$$\partial^\alpha F = f_\alpha \quad \text{on } E_j,$$

for every $\alpha \leq k$. More precisely, following [92] our claim is that for any $X_0, X\in K$ it holds

$$f_\alpha(X) = \sum_{|\beta|\leq k-|\alpha|} \frac{f_{\alpha+\beta}(X_0)}{\beta!}(X-X_0)^\beta + R_\alpha(X,X_0),$$

with

$$|R_\alpha(X,X_0)| \leq \sigma_\alpha\left(|X-X_0|\right)|X-X_0|^{k-|\alpha|} \tag{137}$$

where $\sigma_\alpha = \sigma_\alpha^K$ is a certain modulus of continuity.
If $|\alpha|=k$, since $R_\alpha(X,X_0)=a_\alpha(X)-a_\alpha(X_0)$, we infer from the continuity of $X\mapsto\varphi^X$ on K that $|R_\alpha(X,X_0)|\leq\sigma_\alpha\left(|X-X_0|\right)$. Instead, for $0\leq|\alpha|<k$ we have

$$R_\alpha(X,X_0) = -\sum_{\substack{\gamma>\alpha\\|\gamma|=k}} \frac{a_\gamma(X_0)}{(\gamma-\alpha)!}(X-X_0)^{\gamma-\alpha} = -\partial^a\varphi^{X_0}(X-X_0). \tag{138}$$

By contradiction, suppose that there is no modulus of continuity σ_α such that (137) is satisfied for $X, X_0\in K$. Then, must exist $\delta>0$ and two sequences $X^i, X_0^i\in K$ with $\rho_i = \left|X^i - X_0^i\right| \searrow 0$ such that

$$\left|\sum_{\substack{\gamma>\alpha\\|\gamma|=k}} \frac{a_\gamma(X_0)}{(\gamma-\alpha)!}(X-X_0)^{\gamma-\alpha}\right| \geq \delta\left|X^i - X_0^i\right|^{k-|\alpha|}.$$

Thus, consider the blow-up sequence associated to the sequences $(X_0^i)_i$ and $(\rho_i)_i$ given by

$$u_i(X) = \frac{u(X_0^i + \rho_i X)}{\rho_i^k}, \quad \xi^i = \frac{X^i - X_0^i}{\rho_i},$$

where it is not restrictive to assume that $X_0^i \to X_0 \in K$ and $\xi^i \to \xi_0 \in \partial B_1$. By Theorem 3.5.12 we get $u^i \to \varphi^{X_0} \in \mathfrak{B}_k^a(\mathbb{R}^{n+1})$ uniformly on compact set and there exist a modulus of continuity such that

$$\left| u_i(X) - \varphi^{X_0^i}(X) \right| \leq \sigma\left(\rho_i \left|X\right|\right) \left|X\right|^k.$$

In particular, since $X_0^i, X^i \in K = E_j$, the inequalities (127) holds true for u^i at 0 and ξ^i. Thus, passing to the limit, we obtain that

$$\frac{1}{j}\rho^k \leq \sup_{|X-\xi_0|=\rho} \left|\varphi^{X_0}(X)\right| < j\rho^k,$$

for $0 < \rho < +\infty$, which implies that $\xi_0 \in \Gamma_k(\varphi^{X_0})$. Finally, since $\partial^\alpha \varphi^{X_0}(\xi_0) = 0$ for $|\alpha| < k$, dividing both the left and the right hand side of (138) by $\rho_i^{k-|\alpha|}$ and passing to the limit, we reach a contradiction since we get

$$\left|\partial^a \varphi^{X_0}(\xi_0)\right| = \left| \sum_{\substack{\gamma > \alpha \\ |\gamma| = k}} \frac{a_\gamma(X_0)}{(\gamma-\alpha)!}(X - X_0)^{\gamma-\alpha} \right| \geq \delta.$$

Finally, under the previous notations, let us consider $X_0 = (x_0, 0) \in \Gamma_k^j(u) \cap E_i$, where E_i is defined in Lemma 3.5.20. Hence, by definition of $d_\Sigma^{X_0}$, there exists $n - d_\Sigma^{X_0}$ linearly independent unit vectors $(\nu_i)_i \subset S^n$, such that

$$\langle \nu_i, \nabla_X \varphi^{X_0} \rangle \neq 0 \text{ on } \Sigma,$$

where $d_\Sigma^{X_0} = j$. Hence, there exist multi-indices α_i or order $|\alpha_i| = k - 1$ such that

$$\partial_{\nu_i} D^{\alpha_i} \varphi^{X_0}(0,0) \neq 0.$$

Since φ^{X_0} is a polynomial of degree k on Σ, we can write the following

$$\varphi^{X_0}(x,0) = \sum_{|\alpha|=k} \frac{a_\alpha(x_0,0)}{\alpha!} x^\alpha,$$

where the coefficients $X \mapsto a_\alpha(X)$ are continuous on $\Gamma_k(u) \cap \Sigma$. Thus, the nondegeneracy condition on φ^{X_0} implies

$$\partial_{\nu_i} D^{\alpha_i} F(x_0, 0) \neq 0, \quad i = 1, \cdots, n - d_\Sigma^{X_0}. \tag{139}$$

Finally, since

$$\Gamma_k^j(u) \cap \Sigma \cap E_i \subset \bigcap_{i=1}^{n-j} \{D^{\alpha_i} F = 0\} \cap \Sigma,$$

in view of the implicit function Theorem, the condition (139) implies that $\Gamma_k^j(u) \cap \Sigma \cap E_i$ is contained in a j-dimensional manifold in a neighborhood of X_0.
The results follows immediately from Lemma 3.5.20 □

We remark that in this particular case of L_a-harmonic function symmetric with respect to Σ, since by the definition of tangent map at a point of the nodal set we have

$$u(x,0) = \sum_{|\alpha|=k} \frac{a_\alpha(x_0,0)}{\alpha!} x^\alpha + o(|x-x_0|^k)$$

and $u \in C^\infty(B_{1/2})$ thanks to Proposition 3.2.5, we get that $D^\alpha u(x_0,0) = 0$ for $|\alpha| = k-1$ and $D^\alpha u(x_0,0) = a_\alpha(x_0,0)$ for $|\alpha| = k$. Thus, the nondegeneracy condition on φ^{X_0} implies

$$\partial_{\nu_i} D^{\alpha_i} u(x_0,0) \neq 0, \quad i = 1,\cdots,n-d_\Sigma^{X_0}. \tag{140}$$

Hence, we can obtain conclusion just looking at the strata $\{D^{\alpha_i} u = 0\}$, with $i = 1,\ldots,n-j$. Instead, the previous proof is more general and it will be applied to a more general class of degenerate-singular operators in Section 3.8.

The following is the main Theorem of this stratification analysis, in particular it allows to emphasize the degenerate-singular attitude of the operator L_a near the characteristic manifold Σ by showing the presence of a $(n-1)$-dimensional singular stratum for $a \in (-1,1)$ with $a \neq 0$.

Theorem 3.7.8. *Given $a \in (-1,1)$, let u be L_a-harmonic in B_1. Then there holds*

$$\mathcal{S}(u) \cap \Sigma = \mathcal{S}^*(u) \cup \mathcal{S}^a(u)$$

where $\mathcal{S}^*(u)$ is contained in a countable union of $(n-2)$-dimensional C^1 manifolds and $\mathcal{S}^a(u)$ is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. Moreover

$$\mathcal{S}^*(u) = \bigcup_{j=0}^{n-2} \mathcal{S}_j^*(u) \quad \text{and} \quad \mathcal{S}^a(u) = \bigcup_{j=0}^{n-1} \mathcal{S}_j^a(u),$$

where both $\mathcal{S}_j^*(u)$ and $\mathcal{S}_j^a(u)$ are contained in a countable union of j-dimensional C^1 manifolds.

Proof. The proof can be seen as an improvement of Proposition 3.7.7 since it consists on applying the previous strategy for the dimension and the regularity of the set $\Gamma_k^j(u)$ taking care on the case when the tangent map belongs to $\mathfrak{sB}^*_k(\mathbb{R}^{n+1})$ or not. Indeed, this two cases influence the upper bound on the dimension $d^{X_0}_\Sigma$ and consequently the dimension of the singular strata. Hence, let us set

$$\mathcal{S}^*(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^*_j(u) = \bigcup_{j=0}^{n-2}\bigcup_{k\geq 2}\{X\in\Gamma^*_k(u)\colon d^{X_0}_\Sigma = j\},$$
$$\mathcal{S}^a(u) = \bigcup_{j=0}^{n-1} \mathcal{S}^a_j(u) = \bigcup_{j=0}^{n-1}\bigcup_{k\geq 2}\{X\in\Gamma^a_k(u)\colon d^{X_0}_\Sigma = j\}.$$

Since for every $k\geq 2$ the functions $\varphi\in\mathfrak{sB}^*_k(\mathbb{R}^{n+1})$ are homogeneous polynomial harmonic in Σ, we have that dim $(\mathcal{S}(\varphi)\cap\Sigma)\leq n-2$, and consequently $d^{X_0}_\Sigma\leq n-2$ for every $X_0\in\Gamma^*_k(u)$. Similarly, following Proposition 3.4.15 and the remarks in the proof of Theorem 3.6.3, since for every $k\geq 2$ there exists $\varphi\in\mathfrak{sB}^a_k(\mathbb{R}^{n+1})\setminus\mathfrak{sB}^*_k(\mathbb{R}^{n+1})$ such that dim $(\mathcal{S}(u)\cap\Sigma) = n-1$ we get that for $X_0\in\Gamma^a_k(u)$ it holds $d^{X_0}_\Sigma\leq n-1$.
Now, by applying the same argument in the proof of Proposition 3.7.7, if we set

$$\mathcal{S}^*_j(u) = \bigcup_{k\geq 2}\left\{X\in\Gamma^*_k(u)\colon d^{X_0}_\Sigma = j\right\}\quad \text{for } j=0,\cdots,n-2$$
$$\mathcal{S}^a_j(u) = \bigcup_{k\geq 2}\left\{X\in\Gamma^a_k(u)\colon d^{X_0}_\Sigma = j\right\}\quad \text{for } j=0,\cdots,n-1,$$

we get that $\mathcal{S}^*_j(u)$ and $\mathcal{S}^a_j(u)$ are contained in j-dimensional C^1 manifold. □

Furthermore, by Proposition 3.4.15 we get that for any $X_0\in\mathcal{S}^a_{n-1}(u)$ the leading polynomial of u at X_0, i.e. the first term of the Taylor expansion of u at X_0, is an homogenous polynomial of two variables of the form (118) or (119), up to a rotation on Σ.

3.8 FRACTIONAL POWER OF ELLIPTIC OPERATOR IN DIVERGENCE FORM

In this Section, we find an application to the previous analysis relating, via the extension technique, the study of the restriction of the nodal set on the characteristic manifold Σ to the local properties of solutions of fractional power of elliptic differential equations in divergence form. Initially we start focusing the attention on the case of the fractional Laplacians $(-\Delta)^s$ and then we discuss the monotonicity formula and its consequences for solutions of fractional elliptic differential equations of the second order with Lipschitz leading coefficients.

Let $s\in(0,1)$ and $u\colon B_1\subset\mathbb{R}^n\to\mathbb{R}$ be a nontrivial s-harmonic function in B_1, that is

$$(-\Delta)^s u(x) = 0 \quad \text{in } B_1. \tag{141}$$

Here we define the s-Laplacian

$$(-\Delta)^s u(x) = C(n,s)\ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y \ ,$$

where

$$C(n,s) = \frac{2^{2s} s\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(1-s)} \in \left(0, 4\Gamma\left(\frac{n}{2}+1\right)\right] \ . \tag{142}$$

In general, the s-Laplacian can be defined in various ways, which we review now. First, in order to better understand these definitions, we introduce the spaces

$$\tilde{H}^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \colon \ |\xi|^s (\mathcal{F}u)(\xi) \in L^2(\mathbb{R}^n) \right\},$$

where $s \in (0,1)$ and $\mathcal{F}$ denotes the Fourier transform. In the literature, the spaces $\tilde{H}^s(\mathbb{R}^n)$ are called Bessel spaces and in particular they can be equivalently defined as a Sobolev-Slobodeckij spaces. More precisely, fixed $\Omega \subseteq \mathbb{R}^n$ an open set, for every fractional exponent $s \in (0,1)$ we define $H^s(\Omega)$ as the set of all functions u defined on Ω with a finite norm

$$\|u\|_{H^s(\Omega)} = \left(\int_\Omega |u|^2 \,\mathrm{d}x + \frac{C(n,s)}{2} \int_\Omega \int_\Omega \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}} \mathrm{d}x\mathrm{d}z \right)^{1/2} ,$$

where the term

$$[u]_{H^s(\Omega)} = \left(\frac{C(n,s)}{2} \int_\Omega \int_\Omega \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}} \mathrm{d}x\mathrm{d}z \right)^{1/2} \tag{143}$$

is the so-called Gagliardo seminorm of u in $H^s(\Omega)$. It can be proved that $\tilde{H}^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ and in particular, for every $u \in H^s(\mathbb{R}^n)$ we get

$$[u]^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \,\mathrm{d}\xi = \left\| (-\Delta)^{s/2} u \right\|^2_{L^2(\mathbb{R}^n)} .$$

Note that one can also define the fractional Laplacian acting on spaces of functions with weaker regularity.
More precisely, following [77], let $\mathcal{S}$ be the Schwartz space of rapidly decreasing smooth functions in R^n and $\mathcal{S}^s(\mathbb{R}^n)$ be the space of smooth function u such that $(1+|x|^{n+2s})D^k f(x)$ is bounded in $\mathbb{R}^n$, for every $k \geq 0$, endowed with the topology given by the family of seminorms

$$[f]_k = \sup_{x \in \mathbb{R}^n} \left(1 + |x|^{n+2s}\right) D^k f(x).$$

Under this notations, the fractiona Laplacian of $f \in \mathcal{S}$ is well defined in $(-\Delta)^s f \in \mathcal{S}_s$ and, by duality, this allows to define the fractional Laplacian for functions in the space

$$\begin{aligned}\mathcal{L}^1_s(\mathbb{R}^n) &= \left\{u \in L^1_{loc}(\mathbb{R}^n)\colon \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}}\mathrm{d}x < +\infty\right\}\\ &= L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'_s(\mathbb{R}^n),\end{aligned}$$

where $\mathcal{S}'_s(\mathbb{R}^n)$ stands for the dual of $\mathcal{S}_s(\mathbb{R}^n)$. We remark that necessary a function in $\mathcal{L}^1_s(\mathbb{R}^n)$ needs to keep an algebraic growth of power strictly smaller than $2s$, in order to make the above expression meaningful, as we pointed out in Chapter 1 and Chapter 2.

In order to study the local behaviour of u, let us look at the extension technique popularized by Caffarelli and Silvestre (see [23]), characterizing the fractional Laplacian in $\mathbb{R}^n$ as the Dirichlet-to-Neumann map for a variable v depending on one more space dimension. Namely for every $u \in H^s(\mathbb{R}^n)$, let us consider $v \in H^{1,a}(\mathbb{R}^{n+1}_+)$ satisfying

$$\begin{cases}\mathrm{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+,\\ v(x,0) = u(x) & \text{in } \Sigma\ .\end{cases} \tag{144}$$

with $a = 1-2s \in (-1,1)$. Such an extension exists unique and is given by the formula

$$v(x,y) = \gamma(n,s)\int_{\mathbb{R}^n} \frac{y^{2s}u(x)}{(|x-\eta|^2+y^2)^{n/2+s}}\mathrm{d}\eta \qquad \text{where } \gamma(n,s)^{-1} =\colon \int_{\mathbb{R}^n}\frac{1}{(|\eta|^2+1)^{n/2+s}}\mathrm{d}\eta\,,$$

where the nonlocal operator $(-\Delta)^s$ translates into the Dirichlet-to-Neumann opeartor type

$$(-\Delta)^s\colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n), \quad u \longmapsto -\frac{C(n,s)}{\gamma(n,s)}\lim_{y\to 0^+} y^{1-2s}\partial_y v(x,y),$$

with $C(n,s)$ the normalization constant deeply studied in Chapter 2. By [70], it is known that the space $H^s(\mathbb{R}^n)$ coincides with the trace on $\partial\mathbb{R}^{n+1}$ of the weighted Sobolev space $H^{1,a}(\mathbb{R}^{n+1}_+)$ and in general

$$[u]^2_{H^s(\mathbb{R}^n)} = \frac{C(n,s)}{\gamma(n,s)}\int_{\mathbb{R}^{n+1}} |y|^{1-2s}\,|\nabla v|^2\,\mathrm{d}X,$$

where v is the L_a-harmonic extension of u defined by (144). Since in the context of the extension problem the equation (141) translates in the homogeneous Neumann condition

$$\partial^a_y v(x,0) = -\frac{C(n,s)}{\gamma(n,s)}\lim_{y\to 0^+} y^{1-2s}\partial_y v(x,y) = 0 \quad \text{on } B_1 \subset \Sigma,$$

by applying the even reflection through Σ, we can study the structure of the nodal set of s-harmonic function in $\mathbb{R}^n$ as the restriction of the nodal set $\Gamma(v)$ on the characteristic manifold Σ of the solution

$$\begin{cases} L_a v = 0 & \text{in } B_1^+ \\ v(x,-y) = v(x,y) & \text{in } B_1^+ \\ v(x,0) = u(x) & \text{in } B_1 \end{cases} \tag{145}$$

where $a = 1-2s \in (-1,1)$ and B_1^+ is the unitary $(n+1)$-dimensional ball in $\mathbb{R}^{n+1}$.
Moreover, by [70] is it well known that the class of trace on $B_1^+ \cap \Sigma = B_1$ of function L_a-harmonic in B_1^+ is equal to the space $H^s(B_1)$.

Through this Section we will always identify as v the L_a-harmonic extension of u in $\mathbb{R}^{n+1}$ symmetric with respect to Σ and with $B_r(x_0)^+$ the ball in $\mathbb{R}^{n+1}_+$ of radius $r>0$ and centered in the point $X_0 = (x_0, 0) \in \Sigma$ in the characteristic manifold associated to Σ.
The following results are a direct consequence of the ones obtained for purely symmetric L_a-harmonic function. For this reason the proof of the majority of them is skipped when the result is obtained just passing through the L_a-harmonic extension.

Proposition 3.8.1. *Given $s \in (0,1)$, let u be s-harmonic in B_1. Then, there for every $x_0 \in B_1$,*

$$\frac{1}{R^n}\int_{B_R(x_0)} u^2 \mathrm{d}x \le C(n,s)\left(\frac{R}{r}\right)^{2N-1}\frac{1}{r^n}\int_{B_r(x_0)} u^2 \mathrm{d}x, \tag{146}$$

for $0<r<R<1-|x_0|$ and $N = N(X_0, v, 1-|X_0|)$, with v the L_a-harmonic extension of u.

Proof. Let $v \in H^{1,a}(B_1)$ be the L_a-harmonic extension of u in $\mathbb{R}^{n+1}$, symmetric with respect to Σ. The idea of this proof is to "move" the doubling condition on $\mathbb{R}^{n+1}$ to the characteristic manifold Σ. In [73] the author used a similar strategy to prove a so called "bulk doubling property". In our case we improve the proof by using our blow up analysis developed in Section 3.4 and applying the correct factor of scaling in order to pass from a doubling condition in the dimension $n+a+1$ to the one on Σ.
Let $X_0 \in B_1 \cap \Sigma$, and v the L_a-harmonic extension symmetric with respect to Σ. Integrating the inequality in Corollary 3.3.2, see (104), we obtain that

$$\int_{B_R^+(X_0)} |y|^a v^2 \mathrm{d}X \le \left(\frac{R}{r}\right)^{2C+n+a}\int_{B_r^+(X_0)} |y|^a v^2 \mathrm{d}X$$

for every $0 < r < R < 1-|X_0|$, with $N = N(X_0, v, 1-|X_0|)$. By the interpolation estimate in [73], we get

$$\begin{aligned}\frac{1}{R^n}\int_{B_R(X_0)} u^2 \mathrm{d}x &\leq C(n,a)\left(\frac{1}{R^{n+a+1}}\int_{B_R^+(X_0)} |y|^a v^2 \mathrm{d}X + \frac{1}{R^{n+a-1}}\int_{B_R^+(X_0)} |y|^a |\nabla v|^2 \mathrm{d}X\right)\\ &\leq C(n,a)\left(\frac{1}{R^{n+a+1}}\int_{B_R^+(X_0)} |y|^a v^2 \mathrm{d}X + \frac{1}{R^{n+a+1}}\int_{B_{2R}^+(X_0)} |y|^a v^2 \mathrm{d}X\right)\\ &\leq C(n,a)2^{2N+n+a+1}\frac{1}{R^{n+a}}\int_{B_R^+(X_0)} |y|^a v^2 \mathrm{d}X\end{aligned}$$

where in the second inequality we used the Caccioppoli estimate (99) and in the last one the doubling condition. Since it yields the desired lower bound for the left hand side of the doubling condition on Σ, we left to prove the upper bound. Let us prove by contradiction the existence of $C > 0$ and a radius $0 < \overline{r} < R$ such that

$$\int_{\partial B_r^+(X_0)} |y|^a v^2 \mathrm{d}X \leq C(n,a) r^{a+1}\int_{\partial B_r(X_0)} u^2 \mathrm{d}x \quad \text{for all } 0 < r \leq \overline{r}, \tag{147}$$

which will finally implies (146) after a simple integration.
Hence, suppose there exists a sequence $r_k \searrow 0^+$ such that

$$\|v\|_{L^{2,a}(\partial B_{r_k}^+(X_0))} \geq k r_k^{\beta/2} \|u\|_{L^2(\partial B_{r_k}(X_0))}, \tag{148}$$

with $\beta = a+1$. Then let us consider the blow-up sequence of u centered at X_0 associated to $(r_k)_k$

$$v_k(X) = \frac{v(X_0 + r_k X)}{\rho_k} \quad \text{with } \rho_k^2 = \frac{1}{r_k^{n+a}}\int_{\partial B_{r_k}^+(X_0)} |y|^a v^2 \mathrm{d}X = H(X_0, v, r_k).$$

By definition we have $\|v_k\|_{L^{2,a}(\partial B_1^+)} = 1$, and by Lemma 3.4.2 the sequence $(v_k)_k$ is uniformly bounded in $H^{1,a}(B_R^+)$ and $L^\infty(\overline{B_R})$, for every $R > 0$. In particular, by (148), we get

$$\|u_k\|_{L^2(\partial B_1)} = \|v_k\|_{L^2(\partial B_1)} = \frac{r_k^{-\frac{n-1}{2}}\|v\|_{L^2(\partial B_{r_k}(X_0))}}{r_k^{-\frac{n+a}{2}}\|v\|_{L^{2,a}(\partial B_{r_k}^+(X_0))}} \leq k^{-1} r_k^{\frac{a+1-\beta}{2}} = k^{-1}.$$

Thus, up to a subsequence, by Theorem 3.4.1 the blow-up sequence $(v_k)_k$ strongly converge in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, for every $\alpha \in (0,1)$ to some homogeneous blow-up limit $\overline{v} \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ such that $\overline{v} = 0$ on B_1, $\|\overline{v}\|_{L^{2,a}(\partial B_1^+)} = 1$ and it satisfies

$$\begin{cases} L_a \overline{v} = 0 & \text{in } \mathbb{R}^{n+1}\\ \partial_y^a \overline{v} = 0 & \text{in } \Sigma. \end{cases}$$

Hence, by Proposition 3.5.10 we get that $\overline{v} \equiv 0$ in contradiction with $\|\overline{v}\|_{L^{2,a}(B_1^+)} = 1$. □

In order to justify the analysis of the local behaviour of s-harmonic functions, it is necessary to ensure the validity of the strong unique continuation property. It is known by [45] that an s-harmonic function in B_1 enjoys the *strong unique continuation property* in B_1, i.e. the only solutions which vanishes of infinite order at a point $X_0 \in \Gamma(u)$ is $u \equiv 0$. Similarly, an s-harmonic function in B_1 is said to satisfies the *unique continuation property* in B_1 if the only solution of $(-\Delta)^s u = 0$ in $H^s_{\mathrm{loc}}(B_1)$ which can vanish in an open subset of B_1 is $u \equiv 0$. Indeed, as a direct consequence of Proposition 3.5.10 we prove

Corollary 3.8.2. *Let $s \in (0,1)$ and u be s-harmonic in B_1. Then the nodal set $\Gamma(u)$ has either empty interior in B_1 or $u \equiv 0$.*

Hence, it is reasonable to define the notion of vanishing order of u at $x_0 \in \Gamma(u)$. More precisely, the strong unique continuation property guarantees the existence of $k \in \mathbb{R}$ such that

$$\limsup_{r\to 0^+} \frac{1}{r^{n+2k}} \int_{B_r(x_0)} u^2 \mathrm{d}x > 0.$$

In order to correlate the notion of vanishing order of s-harmonic functions with the one for their L_a-harmonic extension, let us introduce the following common definition.

Definition 3.8.3. Given $s \in (0,1)$, let u be an s-harmonic function in B_1 and $x_0 \in \Gamma(u)$. The vanishing order of u in x_0 is defined as the number $\mathcal{O}(u,x_0) \in \mathbb{R}$ such that

$$\limsup_{r\to 0^+} \frac{1}{r^{n-1+2k}} \int_{\partial B_r(x_0)} u^2 \mathrm{d}x = \begin{cases} 0 & \text{if } k < \mathcal{O}(u,x_0) \\ +\infty & \text{if } k > \mathcal{O}(u,x_0). \end{cases}$$

In particular, from Lemma 3.4.9 and Proposition 3.8.1 we get

Corollary 3.8.4. *Let $s \in (0,1)$ and $a = 1-2s \in (-1,1)$. Given u an s-harmonic function in B_1, then the vanishing order $\mathcal{O}(u,x_0)$ of u in $x_0 \in \Gamma(u)$ satisfy*

$$\mathcal{O}(u,x_0) = N(X_0, v, 0^+) = \lim_{r\to 0^+} \frac{r \displaystyle\int_{B_r^+(X_0)} |y|^a |\nabla v|^2 \mathrm{d}X}{\displaystyle\int_{\partial B_r^+(X_0)} |y|^a v^2 \mathrm{d}\sigma},$$

where v is the unique L_a-harmonic extension of u symmetric with respect to Σ and $X_0 = (x_0, 0)$.

Hence, for $k \in 1 + \mathbb{N}$, we define the subsets

$$\Gamma_k(u) := \{x_0 \in \Gamma(u) \colon \mathcal{O}(u,x_0) = k\},$$

which is coherent with the Definition for the L_a-harmonic case. Indeed, inspired by the results in Section 3.4, we can prove a convergence result for the blow-up sequence associated to $x_0 \in \Gamma(u)$

to some blow-up limit $\varphi \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$.

Before to prove the main convergence result, let us introduce two different classes of tangent maps strictly related to the ones introduced in Definition 3.4.17 and Definition 3.4.18. In particular, we will see that the structure of the nodal set is completely defined starting from these blow-up classes.

Definition 3.8.5. Given $s \in (0,1)$ and $k \in 1+\mathbb{N}$, we define the set of all possible blow-up limit of order k, i.e. the set of the traces of all L_a-harmonic polynomial of degree k symmetric with respect to Σ, as

$$\mathfrak{B}^s_k(\mathbb{R}^n) = \left\{\varphi \in H^s_{\mathrm{loc}}(\mathbb{R}^n) : \text{ the } L_a\text{-extension of } \varphi \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})\right\}.$$

Moreover, by Lemma 3.4.12 and Lemma 3.4.16, the space $\mathfrak{B}^s_k(\mathbb{R}^n)$ is the set of all possible homogenous polynomial of order k in $\mathbb{R}^n$, which is, by the results in [70], the space of traces on Σ of $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^n)$. Similarly, if we define with $\mathfrak{B}^*_k(\mathbb{R}^n)$ the set of function $\varphi \in \mathfrak{B}^s_k(\mathbb{R}^n)$ such that $\Delta\varphi = 0$ in $\mathbb{R}^n$, namely the collection of homogeneous harmonic polynomial of order k, it holds that $\mathfrak{B}^*_k(\mathbb{R}^n)$ coincides with the set of traces of blow-up limits in $\mathfrak{s}\mathfrak{B}^*_a(\mathbb{R}^{n+1})$.

The following result is a direct application of Theorem 3.4.1, Lemma 3.5.5, Theorem 3.5.6 and Theorem 3.5.12 on the L_a-harmonic extension of u symmetric with respect to Σ and it ensure the existence of a unique non trivial tangent map at every point of the nodal set of u.

Proposition 3.8.6. *Given $s \in (0,1)$, let u be an s-harmonic function in B_1 and $x_0 \in \Gamma_k(u)$. Then there exists a unique k-homogenous polynomial $\varphi^{x_0} \in \mathfrak{B}^s_k(\mathbb{R}^n)$ such that*

$$u_{x_0,r}(x) = \frac{u(x_0 + r_k x)}{r^k} \longrightarrow \varphi^{x_0}(x),$$

where the blow-up sequence $(u_{x_0,r})_r$ converges strongly in $H^s_{\mathrm{loc}}(\mathbb{R}^n)$ and in $C^{1,\alpha}_{\mathrm{loc}}(B_1)$, for every $\alpha \in (0,1)$. Moreover, the unique tangent map φ^{x_0} is nontrivial and it satisfies the following generalized Taylor expansion

$$u(x) = \varphi^{x_0}(x - x_0) + o(|x - x_0|^k),$$

where the map $x_0 \mapsto \varphi^{x_0}$ from $\Gamma_k(u)$ to the space $\mathfrak{B}^s_k(\mathbb{R}^n)$ is continuous.

Thus, let

$$\begin{aligned}\mathcal{R}(u) &= \{x_0 \in \Gamma(u) \colon \mathcal{O}(u, x_0) = 1\},\\ \mathcal{S}(u) &= \bigcup_{k\geq 2} \Gamma_k(u) = \bigcup_{k\geq 2} \{x_0 \in \Gamma(u) \colon \mathcal{O}(u, x_0) = k\},\end{aligned}$$

be respectively the *regular* and *singular* part of $\Gamma(u)$. Moreover, by Corollary 3.5.8 we can find a different characterization of the singular strata $\Gamma_k(u)$ for $k \geq 2$, i.e.

$$\Gamma_k(u) = \left\{ x_0 \in \Gamma_k(u) \;\middle|\; \begin{array}{ll} D^\nu u(x_0) = 0 & \text{for every } |\nu| \leq k-1 \\ D^{\nu_0} u(x_0) \neq 0 & \text{for some } |\nu_0| = k \end{array} \right\}.$$

The following are the main theorems related to the regularity and the geometric structure of the nodal set: while in the first result we focus the attention on the regular part of the nodal set, proving a result similar to its local counterpart (see [56, 66]), in the ones related to the singular strata we highlight the presence of a singular subset $\mathcal{S}^s(u)$ strictly related to the nonlocal attitude of the fractional Laplacian.

Theorem 3.8.7. *Given $s \in (0,1)$, let u be s-harmonic in B_1. Then the regular set $\mathcal{R}(u)$ is relatively open in $\Gamma(u)$ and is locally a smooth hypersurface on $\mathbb{R}^n$. Moreover*

$$\mathcal{R}(u) = \{x \in \Gamma(u) \colon\ |\nabla u(x)| \neq 0\}.$$

Proof. By Corollary 3.8.2, let us suppose that $u \not\equiv 0$ in B_1 and hence $\Gamma(u)$ has empty interior. Given v the unique L_a-harmonic extension of u symmetric with respect to Σ, well defined in $H^{1,a}(B_1^+)$ by (145), it is obvious to infer that

$$\mathcal{R}(v) \cap \Sigma = \mathcal{R}(u).$$

Moreover, by Lemma 3.5.20 we already know that $\mathcal{R}(u)$ is relatively open in $\Gamma(u)$ and by the application of the Federer reduction principle in Theorem 3.6.3 we get

$$\dim_{\mathcal{H}}(\mathcal{R}(u)) = n-1.$$

Now, by Theorem 3.8.6 and our blow-up classification, for every $x_0 \in \mathcal{R}(u) \cap \Sigma$ there exists a linear map $\varphi^{x_0} \in \mathfrak{B}_1^s(\mathbb{R}^n)$ such that

$$u(x) = \varphi^{x_0}(x-x_0) + o(|x-x_0|) = \varphi^{x_0}(\nu^{x_0})\langle x-x_0, \nu^{x_0}\rangle + o(|x-x_0|)$$

for some $\nu^{x_0} \in S^{n-1}$.

Moreover, still by Theorem 3.8.6 we know that the map $x_0 \mapsto \varphi^{x_0}(\nu^{x_0})\nu^{x_0}$ is continuous. By Proposition 3.2.5, since $u \in C^\infty(B_{1/2})$ we can use the tangent map in order to compute the directional derivative of u, which will ensures the nondegeneracy of the gradient of u at x_0. More precisely, for every $\xi \in S^{n-1}$

$$\langle \nabla u(x_0), \xi \rangle = \frac{d}{dt} u(x_0 + t\xi)\Big|_{t=0} = \lim_{t\to 0} \frac{u(x_0+t\xi)}{t} = \varphi^{x_0}(\nu^{x_0})\langle \xi, \nu^{x_0}\rangle,$$

and hence $\nabla u(x_0) = \varphi^{x_0}(\nu^{x_0})\nu^{x_0}$ which is nonzero by the nondegeneracy of the tangent map. Finally, by the implicit function theorem we get the claimed result. □

As in Section 3.7, initially we will prove a stratification result for the singular set $\mathcal{S}(u)$. The main idea of this stratification is to stratify the nodal set by the spines of the normalized tangent maps. Indeed, we will introduce the subset $\Gamma_k^j(u)$ as the set of points at which every tangent map has at most j independent directions of translation invariance in order to correlate the nodal set of u with the dimension of the set where the tangent map φ^{X_0} vanishes with the same order of u.

We remark that these result are a direct consequence of Theorem 3.7.7 and Theorem 3.7.8, nevertheless, for the sake of completeness, we present some technical details.
From Definition 3.7.5, given $s \in (0,1)$ we call d^{x_0} the dimension of $\Gamma_k(u)$ at $x_0 \in \Gamma_k(u)$ as

$$d^{x_0} = \dim \{\xi \in \mathbb{R}^n \colon \langle \xi, \nabla\varphi^{x_0}(x)\rangle = 0 \text{ for all } x \in \mathbb{R}^n\}.$$

Now, fixed $k \geq 2$, for each $j = 0, \dots, n-1$ let us define

$$\Gamma_k^j(u) = \{x_0 \in \Gamma_k(u) \colon \dim \Gamma_k(\varphi^{x_0}) = j\},$$

where φ^{x_0} is the unique tangent limit of u at x_0. As we already mentioned, since for $k \geq 2$ we have $\mathfrak{B}_k^s(\mathbb{R}^n) \setminus \mathfrak{B}_k^*(\mathbb{R}^n) \neq \emptyset$, we decide to introduce the following singular sets

$$\mathcal{S}^*(u) = \bigcup_{k\geq 2} \Gamma_k^*(u) \quad \text{and} \quad \mathcal{S}^s(u) = \bigcup_{k\geq 2} \Gamma_k^s(u),$$

where

$$\Gamma_k^*(u) = \{x_0 \in \Gamma_k(u) \colon \varphi^{x_0} \in \mathfrak{B}_k^*(\mathbb{R}^n)\} \quad \text{and} \quad \Gamma_k^s(u) = \Gamma_k(u) \setminus \Gamma_k^*(u).$$

The idea is to stratify the singular set taking care of both the dimension d^{x_0} and the different classes of tangent map associated to the sets $\Gamma_k^*(u)$ and $\Gamma_k^s(u)$.

Theorem 3.8.8. *Given $s \in (0,1)$ let u be s-harmonic in B_1. Then it holds*

$$\mathcal{S}(u) = \mathcal{S}^*(u) \cup \mathcal{S}^s(u)$$

where $\mathcal{S}^*(u)$ is contained in a countable union of $(n-2)$-dimensional C^1 manifolds and $\mathcal{S}^s(u)$ is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. Moreover

$$\mathcal{S}^*(u) = \bigcup_{j=0}^{n-2} \mathcal{S}_j^*(u) \quad \textit{and} \quad \mathcal{S}^s(u) = \bigcup_{j=0}^{n-1} \mathcal{S}_j^s(u),$$

where both $\mathcal{S}_j^*(u)$ and $\mathcal{S}_j^s(u)$ are contained in a countable union of j-dimensional C^1 manifolds.

Proof. The proof is based on a combination of Theorem 3.7.7 Theorem 3.7.8. Since for every $k \geq 2$ the functions $\varphi \in \mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})$ are homogeneous polynomial harmonic in Σ, we have that $\dim(\mathcal{S}(\varphi) \cap \Sigma) \leq n-2$, and consequently $d^{X_0}_\Sigma \leq n-2$ for every $X_0 \in \Gamma^*_k(u)$.
Similarly, following Proposition 3.4.15 and the remarks in the proof of Theorem 3.6.3, since for every $k \geq 2$ there exists $\varphi \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1}) \setminus \mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})$ such that $\dim(\mathcal{S}(u) \cap \Sigma) = n-1$ we get that for $X_0 \in \Gamma^a_k(u)$ it holds $d^{X_0}_\Sigma \leq n-1$.
Now, by applying the same argument in the proof of Theorem 3.7.8, if we set

$$\mathcal{S}^*_j(u) = \bigcup_{k\geq 2} \{x \in \Gamma^*_k(u) \colon d^{x_0} = j\} \quad \text{for } j = 0, \cdots, n-2$$
$$\mathcal{S}^s_j(u) = \bigcup_{k\geq 2} \{x \in \Gamma^s_k(u) \colon d^{x_0} = j\} \quad \text{for } j = 0, \cdots, n-1,$$

we get the claimed result. □

Furthermore, by Proposition 3.4.15 we get that for any $x_0 \in \mathcal{S}^s_{n-1}(u)$ the leading polynomial of u at x_0 is a monomial of degree k with $k \in 2 + \mathbb{N}$ depending only on one variable of $\mathbb{R}^n$.

In order to show the optimality of the result, we will now present an explicit example of s-harmonic function in $B_1 = (-1,1) \subset \mathbb{R}$ with vanishing order $k \geq 2$. More precisely, the following construction allows to exhibit an s-harmonic in $B_1 \subset \mathbb{R}^n$ with $\Gamma(u) = \mathcal{S}^s_{n-1}(u)$.
Fixed $s \in (0,1)$, let $B_1 = (-1,1) \subset \mathbb{R}$ be the unitary ball in the real line and $f \in \mathcal{L}^1_s(\mathbb{R}) \cap C(\mathbb{R})$ an admissible function. By the classical potential theory is it known that the unique solution of

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1 \\ u = f & \text{in } \mathbb{R} \setminus B_1 \end{cases}$$

an be computed explicitly as

$$u(x) = \int_{\mathbb{R}\setminus B_1} P(x,y) f(y) \mathrm{d}y = \frac{\Gamma(1/2)\sin\pi s}{\pi^{3/2}} \left(1 - |x|^2\right)^s \int_{\mathbb{R}\setminus B_1} \frac{1}{(|y|^2-1)^s} \frac{f(y)}{|x-y|} \mathrm{d}y.$$

We remark that several results and reference about the Poisson kernel can be found in the classical book of Landkof [63].
Now, given $f \in \mathcal{L}^1_s(\mathbb{R}) \cap C(\mathbb{R})$, let us consider $f_e, f_o \in \mathcal{L}^1_s(\mathbb{R}) \cap C(\mathbb{R})$ respectively the even and odd part of f uniquely defined as

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Under this notations, we get for $x \in (-1,1)$

$$u(x) = \frac{2\Gamma(1/2)\sin\pi s}{\pi^{3/2}} \left(1 - |x|^2\right)^s \left[\int_1^{+\infty} \frac{f_e(y) y}{(|y|^2-1)^s (y^2 - x^2)} \mathrm{d}y + x \int_1^{+\infty} \frac{f_o(y)}{(|y|^2-1)^s (y^2 - x^2)} \mathrm{d}y\right].$$

Since for every $y \in \mathbb{R} \setminus B_1$ we have $|y| > |x|$, using the series expression

$$\frac{1}{y^2 - x^2} = \frac{1}{y^2} \sum_{n=0}^{\infty} \frac{x^{2n}}{y^{2n}},$$

we obtain

$$u(x) = \frac{2\Gamma(1/2)\sin \pi s}{\pi^{3/2}} \left(1 - |x|^2\right)^s \left[\sum_{n=0}^{\infty} A_{2n}(f)x^{2n} + \sum_{n=0}^{\infty} A_{2n+1}(f)x^{2n+1}\right],$$

where for every $n \in \mathbb{N}$

$$A_{2n}(f) = \int_1^{+\infty} \frac{f_e(y)}{y(|y|^2 - 1)^s y^{2n}} \mathrm{d}y \quad \text{and} \quad A_{2n+1}(f) = \int_1^{+\infty} \frac{f_o(y)}{y(|y|^2 - 1)^s y^{2n+1}} \mathrm{d}y.$$

In particular, if we consider $f(x) = (|x|^2 - 1)^s g(x^{-1})$ we get by a simple change of variables

$$A_{2n}(f) = \int_0^1 \frac{g_e(y)}{y} y^{2n} \mathrm{d}y \quad \text{and} \quad A_{2n+1}(f) = \int_0^1 g_o(y) y^{2n} \mathrm{d}y.$$

Hence, for every fixed order of vanishing $k \in 2 + \mathbb{N}$, there exists a polynomial function $g(x)$ such that $A_i(f) = 0$, for every $i \leq k - 1$. We remark that all these coefficients can be computed explicitly. Moreover, this construction it implies that for every vanishing order $k \in 2 + \mathbb{N}$ there exists an s-harmonic function in $(-1, 1)$ which vanishes in zero with order k, which shows the purely nonlocal attitude of the singular set of s-harmonic functions.

In this part we will generalize the previous result to a more general class of fractional power of divergence form operator following the change of variables first introduced in [4] and deeply popularized in the works [48, 49] . Inspired by works, we consider solutions of homogeneous linear elliptic differential equations of the second order with Lipschitz leading coefficients and no lower order terms. We remark that in general the regularity assumption on the coefficient is optimal thanks to the counterexample of [68].

Let $A(x) = (a_{ij}(x))$ be a symmetric $n \times n$ matrix-valued function in $\overline{B}_1$ satisfying the following assumptions:

1. there exists $\lambda \in (0, 1)$ such that

$$\lambda |\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1} |\xi|^2 \quad \text{for any } x \in \overline{B}_1 \text{ and } \xi \in \mathbb{R}^n;$$

2. there exists $\Gamma > 0$ such that for any $1 \leq i, j \leq n$

$$|a_{ij}(x) - a_{ij}(z)| \leq \gamma |x - z| \quad \text{for any } x, z \in B_1.$$

Hence, consider now the uniformly elliptic operator

$$Lu = \text{div}\left(A(x)\nabla u(x)\right) = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u\right) = 0 \quad \text{in } B_1. \tag{149}$$

By [80], we already know the existence of characterization for the fractional powers of second order partial differential operators in some suitable class.

Proposition 3.8.9. *Let $s \in (0,1)$ and $u \in \mathcal{L}^1_s(\mathbb{R}^n)$. Given $a = 1-2s \in (-1,1)$, a solution of the extension problem*

$$\begin{cases} Lv + \frac{a}{y}\partial_y v + \partial^2_{yy} v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ v(x,0) = u(x) & \text{in } \mathbb{R}^n; \end{cases} \tag{150}$$

is given by

$$v(x,y) = \frac{1}{\Gamma(s)}\int_0^\infty (e^{tL}(-L)^s f)(x) e^{-\frac{y^2}{4t}} \frac{\mathrm{d}t}{t^{1-s}},$$

and

$$(-L)^s u(x) = -\frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)} \lim_{y\to 0^+} y^a \partial_y u(x,y).$$

A similar extension can be constructed in the context of fractional powers $(-\Delta_M)^s$ of the Laplace-Beltrami operator on a Riemannian manifold M and to conformal fractional Laplacian on conformally compact Einstein manifolds and asymptotically hyperbolic manifold, thanks to the extension technique developed in [27] and the asymptotic expansion of their geodesic boundary defining function.
In this Section, we just consider the case of divergence form operator L in order to show how to deal with the limit case of Lipschitz coefficients. Therefore, this analysis will extend the results also to the case of Laplace-Beltrami with Lipschitz metric.

As we did for the fractional Laplacian, in order to study the local behaviour of solution of fractional elliptic equation associated to operator L in divergence form, let $s \in (0,1)$ and u be a solution of the extended problem (150) associated to L, even with respect to the y-direction, i.e. such that

$$\begin{cases} \text{div}_{x,y}(|y|^a \overline{A}(x)\nabla_{x,y} u) = 0, & \text{in } \mathbb{R}^{n+1} \\ u(x,y) = u(x,-y), & \text{in } \mathbb{R}^{n+1}. \end{cases} \tag{151}$$

where $\overline{A}(x)$ is a symmetric $(n+1)\times(n+1)$ matrix-valued function in B_1 such that

$$\overline{A}(x) = \left(\begin{array}{c|c} A(x) & 0 \\ \hline 0 & 1 \end{array}\right). \tag{152}$$

Inspired by Definition 3.2.1 we define the natural generalization of notion of L_a-harmonicity in the context of divergence form operator L with Lipschitz leading coefficient.

Definition 3.8.10. Let $a \in (-1,1)$, we say $u \in H^{1,a}(B_1)$ is L_a^A-harmonic in B_1 if for every $\varphi \in C_c^\infty(B_1)$ we have

$$\int_{B_1} |y|^a \langle \overline{A}(x)\nabla u, \nabla\varphi\rangle \mathrm{d}X = 0,$$

where $\overline{A}(x)$ is the symmetric $(n+1)\times(n+1)$ matrix-valued function defined in (152).

Through this Section we will state all the result in the contxt of L_a^A-harmonic function in B_1 symmetric with respect to Σ, since the nodal set of the fractional powers $(-L)^s$ is completely defined as the restriction of the nodal set of L_a^A-harmonic function symmetric with respect to Σ, as we did in the previous part of the Section.
Obviously, in order to better understand the behaviour of general degenerate operator with Lipschitz leading coefficient, one could consider general L_a^A-harmonic solution and apply the ideas and the decomposition of the previous Sections.

In order to develop a blow-up analysis, let us construct a monotonicity formula base on a geometrical reduction introduce in [4] and deeply used in the local case [48, 49]. Hence, for $n \geq 3$, define a Lipschitz metric $\overline{g} = \overline{g}_{ij}(x,y)\mathrm{d}x_i \otimes \mathrm{d}x_j + \overline{g}_{yy}(x,y)\mathrm{d}y \otimes \mathrm{d}y$ on B_1 by setting

$$\overline{g}_{ij} = \overline{a}^{ij}\left(\det \overline{A}\right)^{\frac{1}{n-1}} = \begin{cases} a^{ij}\,|A|^{\frac{1}{n-1}}, & \text{if } 1 \leq i,j \leq n \\ |A|^{\frac{1}{n-1}}, & \text{otherwise} \end{cases} \tag{153}$$

where $\overline{a}^{i,j}$ and $a^{i,j}$ denote respectively the entries of $\overline{A}^{-1}$ and A^{-1}. Letting similarly $\overline{g}^{ij}$ be the entries of the inverse metric of $\overline{g}$, consider

$$\begin{aligned} r(x,y)^2 &= \overline{g}_{ij}(0)x_ix_j + \overline{g}_{yy}(0)y^2 \\ &= |A|^{\frac{1}{n-1}}\left(a^{ij}(0)x_ix_j + y^2\right) \end{aligned}$$

and

$$\begin{aligned} \eta(x,y) &= \frac{1}{r^2(x,y)}\left(\overline{g}^{kl}(x)\overline{g}_{ik}(0)\overline{g}_{jl}(0)x_ix_j + \overline{g}^{yy}(x)\overline{g}_{yy}(0)\overline{g}_{yy}(0)y^2\right) \\ &= \frac{a_{kl}(x)a^{ik}(0)a^{jl}(0)x_ix_j + y^2}{a^{ij}(0)x_ix_j + y^2}. \end{aligned}$$

We can easily verify that η is a positive Lipschitz function in B_1, whose Lipschitz constant depends on n, λ, Γ but not on $a \in (-1,1)$.
Next, we introduce a new metric tensor $g = g_{ij}(x,y)\mathrm{d}x_i \otimes \mathrm{d}x_j + g_{yy}(x,y)\mathrm{d}y \otimes \mathrm{d}y$ in B_1 by

defining $g = \eta(x,y)\overline{g}$. In the intrinsic geodesic polar coordinates with pole at zero of the Riemannian manifold (B_1, g_{ij}), the metric tensor takes the form

$$g = \mathrm{d}r \otimes \mathrm{d}r + r^2 b_{ij}(r,\theta)\mathrm{d}\theta_i \otimes \mathrm{d}\theta_j,$$

where

$$b_{ij}(0,0) = \delta_{ij}, \quad |\partial_r b_{ij}(r,\theta)| \leq \Lambda(n,\lambda,\Gamma), \text{ for } 1 \leq i,j \leq n. \tag{154}$$

Moreover, if we denote $|g| = |\det g|$ we get

$$\sqrt{|g|} = \eta^{\frac{n+1}{2}}\sqrt{|\overline{g}|} = \eta^{\frac{n+1}{2}}|A|^{\frac{1}{n-1}}. \tag{155}$$

Here we denote by $\nabla_g u$ and $\mathrm{div}_g X$ respectively the intrinsic gradient of a function u and the intrinsic divergence of a vector field X on B_1 in the metric g, i.e.

$$\nabla_g u = g^{ij}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}, \quad \mathrm{div}_g X = \frac{1}{\sqrt{|g|}}\left(\sum_{i=1}^n \frac{\partial}{\partial x_i}(\sqrt{|g|}X_i) + \frac{\partial}{\partial y}(\sqrt{|g|}X_y)\right).$$

Finally, in this new metric we rewrite the divergence form equation in (151) as

$$\mathrm{div}_g\left(|y|^a \mu \nabla_g u\right) = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial y}\left[\left(1 - \sqrt{|g|}g^{yy}\mu\right)u\frac{\partial}{\partial y}|y|^a\right]$$

where $\mu = \mu(x,y)$ is a positive Lipschitz function given by

$$\mu(x,y) = \eta(x,y)^{-\frac{n-1}{2}}$$

bounded in $\overline{B}_1$ and such that, in polar coordinates, it satisfies

$$\mu(0,0) = 1, \quad \left|\frac{\partial}{\partial r}\mu(r,\theta)\right| \leq \Lambda(n,\lambda,\Gamma). \tag{156}$$

By (153), (155) and the definition of μ, for every $(x,y) \in B_1$

$$\sqrt{|g|}g^{yy}\mu = \eta^{\frac{n+1}{2}}|A|^{\frac{1}{n-1}}|A|^{-\frac{1}{n-1}}\eta^{-1}\eta^{-\frac{n-1}{2}} = 1.$$

To proceed, given $u \in H^{1,a}(B_1, \mathrm{d}V_g)$ a solution of

$$\mathrm{div}_g\left(|y|^a \mu \nabla_g u\right) = 0 \quad \text{in } B_1 \tag{157}$$

symmetric with respect to Σ, let us define for any $r \in (0,1)$

$$E_g(u,r) = \frac{1}{r^{n+a-1}}\int_{B_g(r)} |y|^a \mu |\nabla_g u|^2 \mathrm{d}V_g$$

$$H_g(u,r) = \frac{1}{r^{n+a}}\int_{\partial B_g(r)} |y|^a \mu u^2 \mathrm{d}V_{\partial B_r}$$

where here $B_g(r)$ represents the geodesic ball in the metric g of radius r centered at the origin. We remark that by the polar decomposition of g, $B_g(r)$ coincides with the usual Euclidian ball.

In [52] the authors introduced a new monotonicity formula for a class of generalized Baouendi-Grushin operators. Since it is well known the existence of a connection between this two families of degenerate elliptic operator, their result gives an analogue counterpart in the context of our weighted degenerate operator, firstly introduced in the pioneering papers [43, 44]. More precisely, let us introduce the change of variable $\Phi\colon \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+$ such that

$$(x,z) = \Phi(x,y) = \left(x, \frac{y^{1-a}}{(1-a)^{1-a}}\right),$$

with inverse $\Phi^{-1}(x,z) = \left(x, (1-a)z^{\frac{1}{1-a}}\right)$. Now, given a function $u(x,z)$ defined for $(x,z) \in \mathbb{R}^{n+1}_+$, we define a function $\tilde{u}(x,y)$ with $(x,y) \in \mathbb{R}^{n+1}_+$ as $\tilde{u}(x,y) = u(\Phi(x,y))$. A simple computations gives

$$L\tilde{u}(x,y) + \partial_{yy}\tilde{u}(x,y) + \frac{a}{y}\partial_y \tilde{u}(x,y) = z^{-\frac{2a}{1-a}}\left[\partial_{zz}u(x,z) + z^{\frac{2a}{1-a}} Lu(x,z)\right].$$

As we can see, the operator within square brackets in the right-hand side of the previous equation is a special case of the family of operators in $\mathbb{R}^n_x \times \mathbb{R}^1_z$ known as generalized Baouendi-Grushin operator.
Nevertheless, our problem does not satifsy the hyphotesis of the remarkable result obtained in [52] and consequently we need to construct a new monotonicity formula, which "does" extend the class of generalized Baouendi-Grushin operator for which a unique continuation principle holds true.

Under the previous notations, for $r \in (0,1)$, we define the Almgren type monotonicity formula as

$$N_g(u,r) = \frac{E_g(u,r)}{H_g(u,r)} = \frac{r\displaystyle\int_{B_g(r)} |y|^a \mu \left|\nabla_g u\right|^2 \mathrm{d}V_g}{\displaystyle\int_{\partial B_g(r)} |y|^a \mu u^2 \mathrm{d}V_{\partial B_r}}.$$

Theorem 3.8.11. *Let $a \in (-1,1)$ and u be a solution of (157) symmetric with respect to Σ. Then there exist a constant $C > 0$ such that the map $r \mapsto e^{Cr}N_g(u,r)$ is absolutely continuous and monotone nondecreasing on $(0,1)$. Hence, there always exists finite the limit*

$$N_g(u,0^+) = \lim_{r\to 0^+} N_g(u,r),$$

which we will call as the Almgren frequency formula.

Proof. By assumption, both $r \mapsto E_g(u,r)$ and $r \mapsto H_g(u,r)$ are locally absolutely continuous function on $(0,1)$, that is both their derivative are $L^1_{\text{loc}}(0,1)$. First, passing to the logarithmic derivatives, the monotonicity of $r \mapsto N_g(u,r)$ is a direct consequence of the claim

$$\frac{d}{dr}\log N(X_0,u,r) = \frac{1}{r} + \frac{\displaystyle\frac{d}{dr}\int_{B_g(r)} |y|^a \mu |\nabla u|^2 \,\mathrm{d}V_g}{\displaystyle\int_{B_g(r)} |y|^a \mu |\nabla u|^2 \,\mathrm{d}V_g} - \frac{\displaystyle\frac{d}{dr}\int_{\partial B_g(r)} |y|^a \mu u^2 \mathrm{d}V_{\partial B_r}}{\displaystyle\int_{\partial B_g(r)} |y|^a \mu u^2 \mathrm{d}V_{\partial B_r}} \geq 0$$

for $r \in (0,1)$. First, by setting $b(r,\theta) = |\det b_{ij}(r,\theta)|$, we get $\sqrt{g(r,\theta)} = r^n\sqrt{b(r,\theta)}$ and we can rewrite the denominator $H_g(u,r)$ of the Almgren monotonicity formula as

$$H_g(u,r) = \int_{\partial B_g(1)} |\theta_n|^a \mu(r,\theta) u^2(r,\theta)\sqrt{b(r,\theta)}\mathrm{d}\theta,$$

where θ_n is the spherical coordinate associated to the y-direction. By differentiating respect to $r \in (0,1$, we obtain

$$\frac{d}{dr}H_g(u,r) = \frac{2}{r^{n+a}}\int_{\partial B_g(r)} |y|^a \mu u \partial_\rho u \mathrm{d}V_{\partial B_r} + \frac{1}{r^{n+a}}\int_{\partial B_g(r)} \frac{|y|^a}{\sqrt{b}}\frac{\partial}{\partial\rho}\left(\mu\sqrt{b}\right) u^2 \mathrm{d}V_{\partial B_r}$$

where $\partial_\rho u$ denotes the radial differentiation $\partial_\rho u = \langle \nabla_g u, X/\rho\rangle$ for $X \in \mathbb{R}^{n+1}$. Finally, by (154) and (156) we get

$$\frac{d}{dr}H_g(u,r) = \frac{2}{r^{n+a}}\int_{\partial B_g(r)} |y|^a \mu u \partial_\rho u \mathrm{d}V_{\partial B_r} + O(1)H_g(u,r), \tag{158}$$

with $O(1)$ a function bounded in absolute value by a constant $C = C(n,\Lambda)$. On the other hand, the divergence theorem gives

$$\int_{B_g(r)} |y|^a \mu |\nabla u|^2 \,\mathrm{d}V_g = -\int_{B_g(r)} u \mathrm{div}_g \left(|y|^a \mu \nabla u\right) \mathrm{d}V_g + \int_{\partial B_g(r)} |y|^a \mu u \partial_\rho u \mathrm{d}V_{\partial B_r}$$

and hence we can rewrite (158) as

$$\frac{d}{dr}H_g(u,r) = \frac{2}{r}E_g(u,r) + O(1)H_g(u,r). \tag{159}$$

We now focus on the derivative of $r \mapsto E_g(u,r)$, following the idea of the radial deformation in [48, 49]: for $0 < r, \Delta r < 1/2$ fixed, we define $w_t \colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$w_t(\rho) = \begin{cases} t, & \text{if } \rho \leq r \\ 1, & \text{if } \rho \geq r + \Delta r \\ t\dfrac{r+\Delta r - \rho}{\Delta r} + \dfrac{\rho - r}{\Delta r}, & \text{if } r \leq \rho \leq r + \Delta r. \end{cases}$$

Now, for $0 < t < 1 + \Delta r/(r + \Delta r)$, we define the bi-Lipschitz map $l_t \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as

$$l_t(X) = w_t(\rho(X))X,$$

with $\rho(X) = \mathrm{dist}_g(0, X)$, and consequently the radial deformation u_t of u as

$$u^t(X) = u(l_t^{-1}(X)) \in H^{1,a}(B_1, \mathrm{d}V_g).$$

By definition we have $u^t(Z) = u(X)$, with $Z = l_t(X)$. Since u is a solution of (157), given the functional $I(t) = E_g(u^t, 1)$ we have

$$\frac{d}{dt} I(t)\Big|_{t=1} = 0. \tag{160}$$

In order to ease the notations, through the following computations we will simply use B_r instead of $B_g(r)$. Inspired by the definition of $w(t)$, let us set

$$\begin{aligned} I(t) &= \int_{B_{rt}} |y|^a \mu \left|\nabla u^t\right|^2 \mathrm{d}V_{B_r} + \int_{B_{r+\Delta r}\setminus B_{rt}} |y|^a \mu \left|\nabla u^t\right|^2 \mathrm{d}V_{B_r} + \int_{B_1\setminus B_{r+\Delta r}} |y|^a \mu \left|\nabla u^t\right|^2 \mathrm{d}V_{B_r} \\ &= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

It is easy to see that

$$I_3(t) = \int_{B_1\setminus B_{r+\Delta r}} |y|^a \mu \left|\nabla u^t\right|^2 \mathrm{d}V_{B_r} = \int_{B_1\setminus B_{r+\Delta r}} |y|^a \mu \left|\nabla u\right|^2 \mathrm{d}V_{B_r}$$

and consequently that $I_3(t)$ does not give contribution to the derivative of $I(t)$. Next, we have

$$\begin{aligned} I_1(t) &= \int_{B_{rt}} |y|^a \mu \left|\nabla u^t\right|^2 \mathrm{d}V_{B_r} \\ &= \int_0^r \int_{\partial B_1} t^a \left|(\rho, \theta_n)\right|^a \mu(t\rho, \theta) \partial_\rho u^2(\rho, \theta) \frac{\sqrt{g(t\rho, \theta)}}{t} \mathrm{d}\theta \mathrm{d}\rho \\ &\quad + \int_0^r \int_{\partial B_1} t^a \left|(\rho, \theta_n)\right|^a \mu(t\rho, \theta) b^{ij}(t\rho, \theta) \partial_{\theta_i} u(\rho, \theta) \partial_{\theta_j} u(\rho, \theta) t \sqrt{g(t\rho, \theta)} \mathrm{d}\theta \mathrm{d}\rho, \end{aligned}$$

where obviously b^{ij} are the entries of the inverse of $(b_{ij})_{ij}$ associated to the metric g. By (156), we get

$$\left|\frac{\partial}{\partial t} \mu(t\rho, \theta)\right| \leq \Lambda(n, \lambda, \Gamma)\rho.$$

Furthermore, we can rewrite

$$\begin{cases} \sqrt{g(t\rho, \theta)} = t^n \rho^n \sqrt{b(t\rho, \theta)} \\ b^{ij}(t\rho, \theta) \sqrt{g(t\rho, \theta)} = t^{n-2} \rho^{n-2} \left[\delta_{ij} + \varepsilon_{ij}(t\rho, \theta)\right] \end{cases} \tag{161}$$

for some $(\varepsilon_{ij}(t\rho,\theta))_{ij}$. Since that (154), we have

$$\left|\frac{\partial}{\partial t}\sqrt{b(t\rho,\theta)}\right| \leq C(n,\Lambda)\rho, \quad \left|\frac{\partial}{\partial t}\sqrt{\varepsilon_{ij}(t\rho,\theta)}\right| \leq C(n,\Lambda)\rho$$

which gives

$$\begin{aligned} I_1(t) = t^{n+a-1}&\left[\int_0^r\int_{\partial B_1}|(\rho,\theta_n)|^a\,\mu(t\rho,\theta)\partial_\rho u^2(\rho,\theta)\rho^n\sqrt{b(t\rho,\theta)}\mathrm{d}\theta\mathrm{d}\rho\right. \\ &\left.+\int_0^r\int_{\partial B_1}|(\rho,\theta_n)|^a\,\mu(t\rho,\theta)\rho^{n-2}(\delta_{ij}+\varepsilon(t\rho,\theta))\partial_{\theta_i}u(\rho,\theta)\partial_{\theta_j}u(\rho,\theta)\mathrm{d}\theta\mathrm{d}\rho\right], \end{aligned}$$

and consequently

$$\left.\frac{d}{dt}I_1(t)\right|_{t=1} = (n+a-1)\int_{B_r}|y|^a\,\mu\,|\nabla_g u|^2\,\mathrm{d}V_g + O(r)\int_{B_r}|y|^a\,\mu\,|\nabla_g u|^2\,\mathrm{d}V_g, \tag{162}$$

with $O(r)$ a function of (r,θ) whose absolute value is bounded by $C(n,\Lambda)r$.

Finally, in order to estimate the second term of $I(t)$, we need to introduce the following notations. Hence, given $X\in B_{r+\Delta r}\setminus B_r$ and $Z=l_t(X)\in B_{r+\Delta r}\setminus B_{rt}$ let us consider their expression in the intrinsic geodesic polar coordinates associated to g, namely $X=(\rho,\theta)$ and $Z=(\gamma_t(\rho),\theta)$, where

$$\gamma_t(\rho) = \mathrm{dist}_g(Z,0) = w_t(X)\rho = \rho\left[t\frac{r+\Delta r-\rho}{\Delta r}+\frac{\rho-r}{\Delta r}\right].$$

and

$$\frac{\partial}{\partial\rho}\gamma_t(\rho) = t\frac{r+\Delta r-2\rho}{\Delta r}+\frac{2\rho-r}{\Delta r}.$$

Then, still using the polar coordinates, we have

$$\begin{aligned} \left|\nabla_g u^t(Z)\right|^2 &= \left.\left|\partial_s u^t(s,\theta)\right|^2+\frac{1}{s^2}b^{ij}(s,\theta)\partial_{\theta_i}u^t(s,\theta)\partial_{\theta_j}u^t(s,\theta)\right|_{s=\gamma_t(\rho)} \\ &= |\partial_\rho u(\rho,\theta)|^2\left(\left.\frac{\partial}{\partial s}\gamma_t^{-1}(s)\right|_{s=\gamma_t(\rho)}\right)^2+\frac{1}{\gamma_t(\rho)^2}b^{ij}(\gamma_t(\rho),\theta)\partial_{\theta_i}u(\rho,\theta)\partial_{\theta_j}u(\rho,\theta), \\ &= |\partial_\rho u(\rho,\theta)|^2\,h_t(\rho)^2+\frac{1}{\gamma_t(\rho)^2}b^{ij}(\gamma_t(\rho),\theta)\partial_{\theta_i}u(\rho,\theta)\partial_{\theta_j}u(\rho,\theta), \end{aligned}$$

and similarly the volume element is given by

$$\mathrm{d}V_{B_r}(Z) = \gamma_t(\rho)^n\sqrt{g(\gamma_t(\rho),\theta)}\frac{\partial}{\partial\rho}\gamma_t(\rho)\mathrm{d}\rho\mathrm{d}\theta.$$

By the previous computations and the expansions in (161), we get

$$\begin{aligned}
I_2(t) &= \int_{B_{r+\Delta r}\setminus B_{rt}} |y|^a \,\mu \left|\nabla u^t\right|^2 \mathrm{d}V_{B_r}(Z) \\
&= \int_r^{r+\Delta r}\int_{\partial B_1} s^n \,|(s,\theta_n)|^a \,h_t(\rho)\mu(s,\theta)\partial_\rho u^2(\rho,\theta)\sqrt{b(s,\theta)}\Big|_{s=\gamma_t(\rho)} \mathrm{d}\theta\mathrm{d}\rho \\
&\quad + \int_r^{r+\Delta r}\int_{\partial B_1} s^{n-2}\,|(s,\theta_n)|^a \,\frac{\partial}{\partial\rho}\gamma_t(\rho)\mu(s,\theta)(\delta_{ij}+\varepsilon(s,\theta))\partial_{\theta_i}u(\rho,\theta)\partial_{\theta_j}u(\rho,\theta)\Big|_{s=\gamma_t(\rho)} \mathrm{d}\theta\mathrm{d}\rho.
\end{aligned}$$

Since

$$h_t(\rho) = \frac{\Delta r + t\rho - \rho}{t(r+\Delta r-\rho)+\rho - r}, \qquad \frac{\partial}{\partial t}h_t(\rho)\Big|_{t=1} = -\frac{\Delta r + r - 2\rho}{\Delta r},$$

we can conclude

$$\begin{aligned}
\frac{d}{dt}I_2(t)\Big|_{t=1} &= \int_{B_{r+\Delta r}\setminus B_r} |y|^a\,\mu\left[(n+a+O(\rho))\,\frac{r+\Delta r-\rho}{\Delta r} - \frac{r+\Delta r-2\rho}{\Delta r}\right](\partial_\rho u)^2 \mathrm{d}V_{B_r} \\
&\quad + \int_{B_{r+\Delta r}\setminus B_r} |y|^a\,\mu\left[(n+a-2+O(\rho))\,\frac{r+\Delta r-\rho}{\Delta r} + \frac{r+\Delta r-2\rho}{\Delta r}\right]\left(|\nabla_g u|^2 - (\partial_\rho u)^2\right)\mathrm{d}V_{B_r}.
\end{aligned}$$

Finally, by letting $\Delta r \to 0^+$ we get

$$\frac{d}{dt}I_2(t)\Big|_{t=1} = 2r\int_{\partial B_r} |y|^a\,\mu(\partial_\rho u)^2 \mathrm{d}V_{\partial B_r} - r\int_{\partial B_r}|y|^a\,\mu\,|\nabla_g u|^2\,\mathrm{d}V_{\partial B_r}. \tag{163}$$

From (160), (162) and (163), we obtain

$$r\frac{d}{dr}\int_{B_r}|y|^a\,\mu\,|\nabla_g u|^2\,\mathrm{d}V_{\partial B_r} - (n+a-1+O(r))\int_{B_r}|y|^a\,\mu\,(\nabla_g u)^2\,\mathrm{d}V_{B_r} = 2r\int_{\partial B_r}|y|^a\,\mu(\partial_\rho u)^2\mathrm{d}V_{\partial B_r},$$

which implies with (159) that

$$\frac{d}{dr}\log N(X_0,u,r) = O(1) + \frac{2\displaystyle\int_{\partial B_r}|y|^a\,\mu(\partial_\rho u)^2\mathrm{d}V_{\partial B_r}}{\displaystyle\int_{\partial B_r}|y|^a\,\mu u\partial_\rho u\mathrm{d}V_{\partial B_r}} - \frac{2\displaystyle\int_{\partial B_r}|y|^a\,\mu u\partial_\rho u\mathrm{d}V_{\partial B_r}}{\displaystyle\int_{\partial B_g(r)}|y|^a\,\mu u^2\mathrm{d}V_{\partial B_r}} \geq -C(n,\Lambda),$$

where the inequality is a consequence of Schwarz's inequality. It follows immediately that the map $r \mapsto \exp(C(n,\Lambda)r)N_g(u,r)$ is a monotone nondecreasing function on $r \in (0,1)$ as required. □

Returning to the formulation of the problem in the euclidian metric, for $X_0 \in \Sigma$ and $r \in (0, 1-|X_0|)$ we set

$$E(X_0,u,r) = \frac{1}{r^{n+a-1}} \int_{B_r(X_0)} |y|^a \langle \overline{A}(X)\nabla u, \nabla u\rangle \mathrm{d}X$$
$$H(X_0,u,r) = \frac{1}{r^{n+a}} \int_{\partial B_r(X_0)} |y|^a \mu_0 u^2 \mathrm{d}\sigma,$$

and consequently

$$N(X_0,u,r) = \frac{E(X_0,u,r)}{H(X_0,u,r)},$$

with μ_0 is a positive Lipschitz function bounded in B_1 satisfying (154) with Λ depending only on n, λ and Γ.

Corollary 3.8.12. *Let $a \in (-1,1)$ and u be a solution of (151) in B_1 symmetric with respect to Σ. Then there exist a constant $C > 0$ such that for every $X_0 \in B_1 \cap \Sigma$ the map*

$$r \mapsto e^{Cr} N(X_0,u,r)$$

is absolutely continuous and monotone nondecreasing on $(0, 1-|X_0|)$.
Hence, there exists finite the Almgren frequency formula *defined as*

$$N(X_0,u,0^+) = \lim_{r\to 0^+} N(X_0,u,r) = \inf_{r>0} N(X_0,u,r).$$

Now, we can finally apply the previous analysis to the general case $(-L)^s$, by proving the validity of a doubling condition, a compactness result for blow-up sequences and a general Theorem on the structure of the nodal set itself.

Proposition 3.8.13. *Let $a \in (-1,1)$ and u be a solution of (151) in B_1. Hence, there exists a constant $C = C(n,\Lambda)$ such that, for every $X_0 \in B_1 \cap \Sigma$,*

$$H(X_0,u,r_2) \leq CH(X_0,u,r_1)\left(\frac{r_2}{r_1}\right)^{2\tilde{C}}$$

for $0 < r_1 < r_2 < 1-|X_0|$, where $\tilde{C} = N(X_0,u,R)e^{C(n,\Lambda)R}$.

Proof. Fixed $R = 1-|X_0|$, by Corollary 3.8.12 we have that $N(X_0,u,r) \leq e^{CR} N(X_0,u,R)$ for every $r \in (0,R)$. By (159) we get

$$\begin{aligned} \frac{d}{dr}\log H(X_0,u,r) &= \frac{2}{r}N(X_0,u,r) + O(1) \\ &\leq \frac{2}{r}N(X_0,u,R)e^{C(n,\Lambda)R} + C(n,\Lambda), \end{aligned}$$

for every $0 < r < R$. Now we integrate between $0 < r_1 < r_2 < R$, obtaining

$$\log \frac{H(X_0, u, r_2)}{H(X_0, u, r_1)} \leq 2N(X_0, u, R)e^{C(n,\Lambda)R} \log \frac{r_2}{r_1} + C(n, \Lambda)(r_2 - r_1)$$

and finally

$$\frac{H(X_0, u, r_2)}{H(X_0, u, r_1)} \leq e^{C(n,\Lambda)R} \left(\frac{r_2}{r_1}\right)^{2\tilde{C}}$$

with $\tilde{C} = N(X_0, u, R)e^{C(n,\Lambda)R}$. □

Moreover, since we are dealing with the extension L_a^A of operator uniformly elliptic in divergence form with Lipschitz coefficent, we can easily extend Corollary 3.3.6 to our new class of operator following the technique developed in [85]. Indeed, since the lower bound on the Almgren frequency formula is based on the Hölder regularity of L_a^A-harmonic function, we easily get

Corollary 3.8.14. *Let u be L_a^A-harmonic on B_1, then for every $X_0 \in \Gamma(u) \cap \Sigma$ we have*

$$N(X_0, u, 0^+) \geq \min\{1, 1 - a\}. \tag{164}$$

More precisely

- *if u is symmetric with respect to Σ, we have $N(X_0, u, 0^+) \geq 1$,*
- *if u is antisymmetric with respect to Σ we have $N(X_0, u, 0^+) \geq 1 - a$.*

In particular, since in this Section we are focusing on the symmetric case, we directly get $N(X_0, u, 0^+) \geq 1$, for any $X_0 \in \Gamma(u) \cap \Sigma$. All techniques presented in this manuscript involve a local analysis of the solutions, which will be performed via a blow-up procedure. The following result are a generalization of the ones in Section 3.4. Fixed $a \in (-1, 1)$ and u an L_a^A-harmonic function in B_1, for every $X_0 = (x_0, 0) \in \Gamma(u) \cap \Sigma$ and $r_k \downarrow 0^+$ we define as the blow-up sequence the collection

$$u_k(X) = \frac{u(X_0 + r_k X)}{\sqrt{H(X_0, u, r_k)}} \quad \text{for } X \in X \in B_{X_0, r_k} = \frac{B_1 - X_0}{r_k},$$

such that $L_a^{A_k} u_k = 0$ and $\|u_k\|_{L^{2,a}(\partial B_1)} = 1$, where

$$L_a^{A_k} = \mathrm{div}_{x,y}\left(|y|^a \overline{A}_k(x) \nabla_{x,y}\right), \quad \text{with } \overline{A}_k(x) = \overline{A}(x_0 + r_k x),$$

for every $X \in B_{X_0, r_k}$.

Proposition 3.8.15. *Let $a \in (-1,1)$. Given $X_0 \in \Gamma(u) \cap \Sigma$ and a blow-up sequence u_k centered in X_0 and associated to some $r_k \downarrow 0^+$, there exists $p \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^n)$ such that, up to a subsequence, $u_k \to p$ in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\alpha \in (0,1)$ and strongly in $H^{1,a}_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, the blow-up limit is and entire solution of following elliptic equation with constant coefficient*

$$div_{x,y}\left(|y|^a \overline{A}(x_0)\nabla_{x,y} p\right) = 0 \quad \text{in } \mathbb{R}^{n+1}.$$

The proof of this result is a straightforward adaption of the one of Theorem 3.4.1. In particular, since the coefficient of $\overline{A}$ are Lipschitz continuous and uniformly elliptic, all the computations of the blow-up argument follows the line of the local counterpart in [56, 66, 49, 48, 81, 4].

Moreover, since for every $X_0 \in \Gamma(u) \cap \Sigma$ the blow- up limit satisfies a degenerate-singular equation with constant coefficients, it is not restrictive to suppose that $\overline{A}(x_0) = \mathrm{Id}$, since by trivial transformation we can rewrite the equation in a canonical form.
Therefore, all the results on the structure of the singular strata, proved in the previous part of the Section for the nodal set of s-harmonic functions, remain valid for the nodal set of fractional power of divergence form operator with Lipschitz leading coefficients. Indeed, as we already pointed out, in the proof of Theorem 3.7.7 and Theorem 3.7.8 we never used Proposition 3.2.5 in order to attain the result on the structure of the singular strata on Σ. The crucial idea is that the Whitney extension allows to study the structure of the nodal set just by using the generalized Taylor expansion (3.5.17) for symmetric function without the high-order differentiability of the function itself. In this way the results can be easily generalized to our class of operators.

Proposition 3.8.16. *Given $s \in (0,1)$, let u be a solution of*

$$(-L)^s u = 0 \quad \text{in } B_1,$$

with L a uniformly elliptic operator with Lipschitz coefficient defined as (149). Then the nodal set $\Gamma(u)$ splits into its regular and singular part

$$\mathcal{R}(u) = \{x \in \Gamma(u) \colon |\nabla u(x)| \neq 0\} \quad \text{and} \quad \mathcal{S}(u) = \{x \in \Gamma(u) \colon |\nabla u(x)| = 0\}.$$

Moreover, if $u \in C^1(B_{1/2})$ on one hand $\mathcal{R}(u)$ is locally a smooth hypersurface and on the other one it holds

$$\mathcal{S}(u) = \mathcal{S}^*(u) \cup \mathcal{S}^s(u)$$

where $\mathcal{S}^*(u)$ is contained in a countable union of $(n-2)$-dimensional C^1 manifolds and $\mathcal{S}^s(u)$ is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. Moreover

$$\mathcal{S}^*(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^*_j(u) \quad \text{and} \quad \mathcal{S}^s(u) = \bigcup_{j=0}^{n-1} \mathcal{S}^s_j(u),$$

where both $\mathcal{S}^*_j(u)$ and $\mathcal{S}^s_j(u)$ are contained in a countable union of j-dimensional C^1 manifolds.

3.9 MEASURE ESTIMATES OF NODAL SETS OF s-HARMONIC FUNCTIONS

In this last Section, we estimate the measure of the nodal set $\Gamma(u)$ in the case of s-harmonic functions. This result can be seen as the nonlocal counterpart of a conjecture that Lin proposed in [66]. Indeed, following his strategy, we give an explicit estimate on the $(n-1)$-Hausdorff measure of the nodal set in terms of the Almgren monotonicity formula of its L_a-extension.

As we already did in the previous Section, since the local structure of the nodal set $\Gamma(u)$ can be described using the results on the restriction of the nodal set of L_a-harmonic function on the characteristic manifold Σ, we will follow the notations previously introduced. More precisely, through this Section we will denote with $v \in H^{1,a}(B_1^+)$ the restriction on the unitary ball in $\mathbb{R}^{n+1}_+$ of the L_a-harmonic extension, defined by (144), symmetric with respect to Σ (see (145)).

Since the fractional Laplacian $(-\Delta)^s$ admits a representation formula, we directly have that the analyticity assumption, which is fundamental in order to apply a strategy developed in [66], is fully satisfied on every compact set $K \subset\subset B_1$. Moreover, by Proposition 3.8.1 we already provide a quantitative doubling condition for s-harmonic functions strictly correlated to the one in the extended space $\mathbb{R}^{n+1}$.

In order to achieve the estimate on the Hausdorff measure of the nodal set $\Gamma(u)$ we use the following lemma relating the growth of a complex analytic function with the number of its zeros introduced in [41].

Lemma 3.9.1. *Let $f\colon B_1 \subset \mathbb{C} \to \mathbb{C}$ be an analytic function such that*

$$|f(0)| = 1 \quad \textit{and} \quad \sup_{B_1} |f| \leq 2^N,$$

for some positive constant N. Then for any $r \in (0,1)$

$$\#\{z \in B_r \colon f(z) = 0\} \leq cN$$

and

$$\#\{z \in B_{1/2} \colon f(z) = 0\} \leq N,$$

where C is a positive constant depending only on the radius r.

Before to state the main result on the measure of the nodal sets $\Gamma(u)$ in terms of the Almgren monotonicity formula of the L_a-harmonic extension, let us start with an example in the setting of tangent maps $\mathfrak{B}^s_k(\mathbb{R}^n)$ that emphasizes how the measure of the nodal set is strictly related to the class of tangent maps that we are considering. More precisely, the classes $\mathfrak{B}^*_k(\mathbb{R}^n)$ and

$\mathfrak{B}^s_k(\mathbb{R}^n) \setminus \mathfrak{B}^*_k(\mathbb{R}^n)$ strictly affect the local measure of the nodal set.
First, it is not restrictive to assume that $\varphi \in \mathfrak{B}^s_k(\mathbb{R}^2)$ for some $k \in 1+\mathbb{N}$. Hence, consider the case $n=2$ with the notation $(x,z) \in \mathbb{R}^2$. Since every $\varphi \in \mathfrak{B}^*_k(\mathbb{R}^2)$ is harmonic in $\mathbb{R}^2$, it is known that

$$\mathcal{H}^1\left(\Gamma(\varphi) \cap B_1\right) = 2k.$$

Instead, for $\varphi \in \mathfrak{B}^s_k(\mathbb{R}^2) \setminus \mathfrak{B}^*_k(\mathbb{R}^2)$, with $k \geq 2$, the previous bound turn to be not optimal. More precisely, given the constant $k' = \#\{t \in \mathbb{R} \colon \varphi(t,1) = 0\}$, we get

$$\mathcal{H}^1\left(\Gamma(\varphi) \cap B_1\right) = 2k',$$

where, by the Fundamental Theorem of Algebra, it is obvious to see that $0 \leq k' \leq k$.

In general, we prove the following result which is based on an argument first introduced in [66] in the context of solution of second order elliptic equation with analytic coefficient.
More recently, in [6] the author constructs a similar estimate in a more general context connecting the Hausdorff measure of the nodal set of smooth functions with their finite vanishing order, which can be also applied to our case. Unfortunately, the remarkable difference between the case $\mathfrak{B}^*_k(\mathbb{R}^n)$ and $\mathfrak{B}^s_k(\mathbb{R}^n) \setminus \mathfrak{B}^*_k(\mathbb{R}^n)$ (or similarly $\mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})$ and $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1}) \setminus \mathfrak{s}\mathfrak{B}^*_k(\mathbb{R}^{n+1})$) implies the not optimality of the result of Bär in our setting.

Theorem 3.9.2. *Given $s \in (0,1)$, let u be an s-harmonic function in B_1 and $0 \in \Gamma(u)$. Then*

$$\mathcal{H}^{n-1}\left(\Gamma(u) \cap B_{\frac{1}{2}}\right) \leq C(n,s)N,$$

where $N = N(0,v,1)$ is the frequency of the L_a-harmonic extension v in B_1^+ defined by

$$N = \frac{\displaystyle\int_{B_1^+} |y|^a \left|\nabla v\right|^2 \mathrm{d}X}{\displaystyle\int_{\partial B_1^+} |y|^a v^2 \mathrm{d}\sigma}.$$

Proof. Let $(B_R(p_i))_i$ be a finite cover of $B_{1/2}$ with $R < 1/8$ and $p_i \in B_{1/2}$. Moreover, up to a normalization, it is not restrictive to assume that

$$\fint_{B_1} u^2 \mathrm{d}x = 1.$$

By Proposition 3.3.5 and Proposition 3.8.1, for every $p_i \in B_{1/2}$ we have

$$\fint_{B_r(p_i)} u^2 \mathrm{d}x \geq 4^{-C(n,s)N} \fint_{B_{2r}(p_i)} u^2 \mathrm{d}x,$$

with $0 < r < 1/4$ and $N = N(0, v, 1)$. Moreover, using the normalization hypothesi, we get

$$\fint_{B_R(p_i)} u^2 \mathrm{d}x \geq 4^{-C(n,s)N}.$$

Given $p_1, \dots, p_j \in B_{1/2}$ the collection of points associated to the covering, let us consider $(x_{p_i})_i \in B_R(p_i)$ such that

$$|u(x_{p_i})| \geq 2^{-C(n,s)N}, \quad \text{for any } i = 1, \dots, j.$$

In order to apply Lemma 3.9.1, for $i = 1, \dots, j$ consider the collection of analytic functions of one complex variable defined as

$$f_i(w, z) = u(x_{p_i} + 4Rzw), \quad \text{for } w \in S^{n-1}, z \in B_1^{\mathbb{C}}$$

Then, by construction, we have

$$|f_i(w, 0)| \geq 2^{-C(n,s)N} \quad \text{and} \quad |f_i(w, z)| \leq C,$$

for some positive dimensional constant $C > 0$. Since, by Lemma 3.9.1 we have

$$\begin{aligned} N_i(w) &= \# \left\{ x \in B_{2R}(x_{p_i}) \colon u(x) = 0 \quad \text{for } (x - x_{p_j}) \parallel w \right\} \\ &\leq \# \left\{ z \in B_{1/2}^{\mathbb{C}} \colon f_i(w, z) = 0 \right\} \\ &\leq c(n, s, N) N, \end{aligned}$$

for every $i = 1, \cdots, j$, by the integral geometric formula in [46, Theorem 3.2.27], we finally obtain

$$\mathcal{H}^{n-1}\left(\Gamma(u) \cap B_{1/2}\right) \leq \sum_{i=1}^{j} \mathcal{H}^{n-1}\left(\Gamma(v) \cap B_R(p_i)\right) \leq c(n, s, N) \sum_{i=1}^{j} \int_{S^{n-1}} N_i(w) \mathrm{d}w \leq C(n, s, N) N$$

where in the second inequality we used $B_R(p_i) \subset B_{2R}(x_{p_i})$ for every $i = 1, \dots, j$. □

In the end, since our estimate on the Hausdorff measure is deeply based on the existence of an L_a-harmonic extension of u and on the validity of an Almgren's type monotonicity result, we expect to improve Theorem **??** exploiting the connection between the Dirichlet energy associated to the L_a-extension and the Gagliardo seminorm introduced in (143). This improvement would show a purely nonlocal version of the result in [66].

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