On the heights of a product of cyclotomic polynomials

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Abstract.

Let $\Phi_n(z)$ be the n-th cyclotomic polynomial and define $F_N(z) = \prod_{n\leq N} \Phi_n(z)$. In this paper we study the relations between the asymptotic behaviour of some "heights" of F_N and the Prime Number Theorem.

§1 Introduction.

Given a finite subset $S \subset (0,1]$ define its local discrepancy $\rho_S : \mathbf{R} \to \mathbf{R}$ as the periodic function

$$\rho_S(t) = \sum_{\alpha\in S} v(t-\alpha),$$

where $v(t) = t - [t] - 1/2$. It is worth remarking that

$$\begin{aligned}\rho_S(t) &= \sum_{\substack{\alpha\in S\\ \alpha<t}} (t-\alpha-\frac{1}{2}) + \sum_{\substack{\alpha\in S\\ \alpha>t}} (t-\alpha+\frac{1}{2})\\ &= d\,t - \mathrm{Card}(\{\alpha\in S,\ \alpha\leq t\}) + c(S)\end{aligned}$$

for any $t \in [0,1)$, $t \notin S$, where $d = \mathrm{Card}(S)$ is the cardinality of S and $c(S) = \sum_{\alpha\in S} (\frac{1}{2}-\alpha)$.

Let now N be a positive integer and define the N-th Farey set $\Lambda = \Lambda_N$ as the set of rationals a/b with $0 < a \leq b \leq N$ and $(a,b) = 1$. In 1924 Franel [F] and Landau [L] discovered a nice relation between the L^1-norm of the local discrepancy of the Farey set and the Riemann Hypothesis (RH). They proved that the infimum of real λ for which

$$\int_0^1 |\rho_{\Lambda_N}(t)|\, dt \ll N^\lambda$$

is the supremum of the real parts of the zeros of the Riemann zeta-function.

More recently, Niederreiter [N] has remarked that the Prime Number Theorem (PNT) with a suitable remainder term implies

$$N \ll \max_{t\in\mathbf{R}} |\rho_{\Lambda_N}(t)| \ll N.$$

Let us consider the polynomial

$$F_N(z) = \prod_{n\le N} \Phi_n(z),$$

where $\Phi_n(z)$ is the n-th cyclotomic polynomial. Then

$$\log|F_N(e^{i\theta})| = \sum_{\alpha\in\Lambda_N} u(\theta - 2\pi\alpha),$$

where $u(\theta) = \log|1-e^{i\theta}|$. An elementary geometrical argument shows that the conjugate of $u(\theta)$ is $\pi v(\theta/2\pi)$, whence by linearity, the conjugate of $\log|F_N(e^{i\theta})|$ is $\pi\rho_{\Lambda_N}(\theta/2\pi)$. From classical theorems of harmonic analysis, we know that the L_p-norms of two conjugate functions are closely related, so it seems possible to find relations between the RH (or the PNT) and some suitable "heights" of F_N.

Let us define the maximum norm of a polynomial $F \in \mathbf{C}[z]$ as

$$|F| = \max_{|z|=1} |F(z)|.$$

This maximum norm is equivalent to the usual height, i.e. to

$$H(F) = \max|\text{coefficients of } F|,$$

in the sense that $H(F) \le |F| \le (\deg F + 1)H(F)$.

In [A] we also introduce the height

$$\tilde{h}(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|F(e^{i\theta})|\, d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$. For the relations between $\log|F|$ and $\tilde{h}(F)$ we refer to [A]. If F splits as $F(z) = a(z-\alpha_1)\cdots(z-\alpha_d)$ $(a \ne 0)$, its Mahler's measure is

$$M(F) = |a|\prod_{j=1}^{d} \max(|\alpha_j|, 1) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\theta})|\, d\theta\right)$$

(the last equality is easily deduced by Jensen's Formula). If $M(F) = 1$ we have

$$\tilde{h}(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^{+} \frac{1}{|F(e^{i\theta})|}\, d\theta = \frac{1}{4\pi}\int_{-\pi}^{\pi} \big|\log|F(e^{i\theta})|\big|\, d\theta.$$

Let $\mathcal{M}(x) = \sum_{k\le x} \mu(k)$, where $\mu(k)$ is the Möbius function. The following results (see [A], section 5) relate the growth of $\mathcal{M}(x)$ to the behaviour of $\tilde{h}(F_N)$.

Theorem 1.

$$\tilde{h}(F_N) \ge \frac{\mathcal{M}(N)}{4}$$

Theorem 2.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\log|F_N(e^{i\theta})|\right)^2\, d\theta = \frac{\pi^2}{12}\sum_{m=1}^{N}\left(\sum_{k|m}\frac{\mu(k)}{k^2}\right)\left(\sum_{n\le N/m}\frac{\mathcal{M}(N/nm)}{n}\right)^2$$

Let $\lambda \in (0,1)$ and assume $\mathcal{M}(x) \ll x^{\lambda}$. Then it is easy to see that for any $\varepsilon > 0$ we have

$$\sum_{m=1}^{N}\left(\sum_{k|m}\frac{\mu(k)}{k^2}\right)\left(\sum_{n\le N/m}\frac{\mathcal{M}(N/nm)}{n}\right)^2 \ll N^{2\lambda+2\varepsilon}.$$

Moreover, by Hölder's inequality,

$$\tilde{h}(F_N) = \frac{1}{4\pi}\int_{-\pi}^{\pi} |\log|F_N(e^{i\theta})||\, d\theta \le \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\log|F_N(e^{i\theta})|\right)^2\, d\theta\right)^{1/2}.$$

Hence $\mathcal{M}(x) \ll x^{\lambda}$ implies $\tilde{h}(F_N) \ll N^{\lambda+\varepsilon}$. Since the infimum of λ for which $\mathcal{M}(x) \ll x^{\lambda}$ is the supremum of the real parts of the zeros of the Riemann zeta-function, we find the following analogue of Franel and Landau's result:

Corollary 1. *Let $\lambda < 1$. The following two assertions are equivalent:*

1) For any $\varepsilon > 0$ the Riemann zeta function does not vanish for $\Re z \ge \lambda + \varepsilon$;

2) For any $\varepsilon > 0$ we have $\tilde{h}(F_N) \ll N^{\lambda+\varepsilon}$.

Niederreiter's theorem also has an analogue in terms of the usual height of F_N:

Theorem 3. *There exist two constants c_1, $c_2 > 1$ such that*

$$c_1^N \leq |F_N| \leq c_2^N.$$

Moreover, for any $\varepsilon > 0$ we can take $c_1 = e - \varepsilon$ provided that $N \geq N_0(\varepsilon)$.

It would be nice to prove that $\log |F_N| \sim N$ as $N \to +\infty$.

The advantage of having conditional results on the heights of F_N instead of the above assertions on the discrepancy of the Farey set is that the analytic theory of polynomials is fairly rich in results. Some of these can be possibly used to give bounds for the heights of F_N. For example, in Section 4 we prove

Theorem 4. *Let G be a polynomial with complex coefficients and assume that G splits as $G = G_1^{e_1} \cdots G_k^{e_k}$, where $G_1, \ldots, G_k$ are square-free polynomials (not necessarily coprime) of degrees $d_1, \ldots, d_k$ and of Mahler's measure 1, and where $e_1, \ldots, e_k$ are integers (not necessarily positive). Moreover, assume that*

$$\min_{G_j(\alpha)=0} |G_j'(\alpha)| \geq \frac{1}{C_j}, \quad j = 1, \ldots, k \tag{1.1}$$

for some constants $C_j \geq 1/d_j^2$. Then

$$\tilde{h}(G) \leq \frac{3}{2} \sum_{j=1}^{k} |e_j| \log(2 d_j^2 C_j)$$

If we apply Theorem 4 with $G = F_N$ and $k = 1$ we find :

Corollary 2. *Let $\lambda < 1$ and assume that*

$$\log |F_N'(\zeta)| \gg -N^\lambda$$

for every root of unity ζ of order $\leq N$. Then for any $\varepsilon > 0$ the Riemann zeta function does not vanish for $\Re z \geq \lambda + \varepsilon$.

§2 Bounds for $\tilde{h}(F_N)$: proofs of Theorems 1 and 2.

As is remarked in section 1, the conjugate of $\log|F_N(e^{i\theta})|$ is $\pi\rho_{\Lambda_N}(\theta/2\pi)$. Therefore, it is possible to prove Theorem 2 simply by using Franel's original result and the equality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\log|F_N(e^{i\theta})|\right)^2\,dt=\pi\int_0^1\rho_{\Lambda_N}(t)^2\,dt.$$

However, we prefer to give a direct proof. We begin with the Fourier expansion of the periodic function $\log|F_N(e^{i\theta})|$. Since

$$z^n-1=\prod_{d|n}\Phi_d(z),$$

the Möbius Inversion Formula gives

$$\Phi_n(z)=\prod_{d|n}(z^d-1)^{\mu(n/d)}.$$

Hence

$$F_N(z)=\prod_{n=1}^{N}(z^n-1)^{\mathcal{M}(N/n)}. \tag{2.1}$$

Now, using the well-known Fourier expansion

$$\log|e^{i\theta}-1|=-\sum_{\nu=1}^{+\infty}\frac{\cos\nu\theta}{\nu},$$

we obtain

$$\log|F_N(e^{i\theta})|=\sum_{\nu=1}^{+\infty}c_\nu\cos\nu\theta \tag{2.2}$$

where

$$\begin{aligned}c_\nu&=\frac{1}{\pi}\int_{-\pi}^{\pi}\log|F_N(e^{i\theta})|\cdot\cos\nu\theta\,dt=\sum_{n\le N}\mathcal{M}(N/n)\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}\log|e^{in\theta}-1|\cdot\cos\nu\theta\,d\theta\\&=-\frac{1}{\nu}\sum_{\substack{n\le N\\ n|\nu}}n\mathcal{M}(N/n).\end{aligned} \tag{2.3}$$

In particular, $c_1=-\mathcal{M}(N)$; on the other hand

$$|c_1|\le\frac{1}{\pi}\int_{-\pi}^{\pi}\big|\log|F_N(e^{i\theta})|\big|\,d\theta=4\tilde{h}(F_N).$$

Hence $h(F_N) \geq \frac{\mathcal{M}(N)}{4}$, which proves Theorem 1. We now prove Theorem 2. From (2.2), (2.3) and Parseval's formula, we have:

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\log|F_N(e^{i\theta})|\right)^2 \, d\theta = \frac{1}{2}\sum_{\nu=1}^{+\infty} c_\nu^2$$
$$= \frac{1}{2}\sum_{\nu=1}^{+\infty} \sum_{\substack{n_1\leq N\\ n_1|\nu}} \sum_{\substack{n_2\leq N\\ n_2|\nu}} \frac{n_1\mathcal{M}(N/n_1)n_2\mathcal{M}(N/n_2)}{\nu^2}.$$

Putting $d = (n_1, n_2)$, $n_j = dl_j$ $(j = 1, 2)$ and $\nu = dl_1l_2k$, we get

$$I = \frac{1}{2}\sum_{d\leq N} \sum_{\substack{l_1,l_2\leq N/d\\ (l_1,l_2)=1}} \frac{\mathcal{M}(N/dl_1)\mathcal{M}(N/dl_2)}{l_1l_2}\sum_{k=1}^{+\infty}\frac{1}{k^2}$$
$$= \frac{\pi^2}{12}\sum_{d\leq N} \sum_{l_1,l_2\leq N/d} \frac{\mathcal{M}(N/dl_1)\mathcal{M}(N/dl_2)}{l_1l_2}\sum_{\substack{k|l_1\\ k|l_2}}\mu(k).$$

which becomes, with the change of indices $m = kd$ and $dl_j = mn_j$,

$$I = \frac{\pi^2}{12}\sum_{m\leq N}\sum_{k|m}\frac{\mu(k)}{k^2} \sum_{n_1,n_2\leq N/m} \frac{\mathcal{M}(N/mn_1)\mathcal{M}(N/mn_2)}{n_1n_2}$$
$$= \frac{\pi^2}{12}\sum_{m\leq N/m}\left(\sum_{k|m}\frac{\mu(k)}{k^2}\right)\left(\sum_{n\leq N/m}\frac{\mathcal{M}(N/mn)}{n}\right)^2.$$

§3 Bounds for $|F_N|$: proof of Theorem 3.

Let $R > 1$ be a positive real number. From (2.1) and from the maximum principle we obtain

$$\log|F_N| \leq \log\max_{|z|=R}|F_N(z)| \leq \max_{|z|=R}|\log|F_N(z)||$$
$$\leq \sum_{n\leq N}|\mathcal{M}(N/n)|\max_{|z|=R}\big|\log|z^n-1|\big|$$
$$= \sum_{n\leq N}|\mathcal{M}(N/n)|\max\left\{\log\frac{1}{R^n-1}, \log(R^n+1)\right\}.$$

Now choose $R = 1 + 1/N$. Since

$$\log(R^n+1) \leq \log 2 + n\log R \leq \log 2 + \frac{n}{N}$$

and

$$\log \frac{1}{R^n - 1} \le \log \frac{1}{(1+n/N)-1} = \log \frac{N}{n},$$

we have

$$\log |F_N| \ll \sum_{n \le N} |\mathcal{M}(N/n)| \log(N/n).$$

The Prime Number Theorem, in the form $|\mathcal{M}(x)| \ll x/(\log x)^3$, shows that

$$\sum_{n \le N} |\mathcal{M}(N/n)| \log(N/n) \ll N.$$

Therefore, $\log |F_N| \ll N$.

It remains to prove that $|F_N| \ge c_1^N$ for some $c_1 > 1$. To do this, let $G_N = (z-1)^{-1} F_N$. Then (2.1) gives

$$G_N(z) = \sum_{n \le N} \mathcal{M}(N/n) \log \left| \frac{z^n - 1}{z - 1} \right|.$$

Hence

$$\log |F_N'(1)| = \log |G_N(1)| = \sum_{n \le N} \mathcal{M}(N/n) \log n = \sum_{n \le N} \Lambda(n)$$

where $\Lambda(n)$ is the Von Mangoldt function. By the Prime Number Theorem

$$\sum_{n \le N} \Lambda(n) \sim N.$$

Therefore $\log |F_N'(1)| \sim N$. Using Bernstein's inequality $|F'| \le (\deg F)\, |F|$ we obtain

$$\log |F_N| \ge \big(1 + o(1)\big) N.$$

§4 Two Lemmas of Hayman and the proof of Theorem 4.

The following two Lemmas are due to Hayman (see [H], Lemma 2.1 and Lemma 2.2).

Lemma 1. *Let z be a complex number, $C > 0$ and $\varepsilon \in (0, \min\{1, C\}]$. Put*

$$E = \{\theta \in (-\pi, \pi) \; ; \; |e^{i\theta} - z| \le \varepsilon\}.$$

Then

$$\int_E \log \frac{C}{|z - e^{i\theta}|}\, d\theta \leq \pi\varepsilon \left(\log \frac{C}{\varepsilon} + 1 \right).$$

Proof. We can assume $z > 0$. Hence $|e^{i\theta} - z| \geq |\sin\theta|$ and $E \subset [-\theta_0, \theta_0]$, where $\theta_0 \in (0, \frac{\pi}{2}]$ is the smallest root of the equation $\sin\theta_0 = \varepsilon$. Therefore,

$$\int_E \log \frac{C}{|z - e^{i\theta}|}\, d\theta \leq 2\int_0^{\theta_0} \log \frac{C}{\sin\theta}\, d\theta \leq 2\int_0^{\theta_0} \log \frac{\pi C}{2\theta}\, d\theta = 2\theta_0 \left(\log \frac{\pi C}{2\theta_0} + 1 \right). \quad (4.1)$$

Since $\theta_0 \leq \frac{\pi}{2}\varepsilon$ and $2\theta \left(\log \frac{\pi C}{2\theta} + 1 \right)$ is an increasing function for $\theta \leq \pi C/2$, the expression (4.1) is bounded by $\pi\varepsilon \left(\log \frac{C}{\varepsilon} + 1 \right)$.

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Lemma 2. *Let $z_1, \ldots, z_n$ be complex numbers and define*

$$\delta(\theta) = \min_{j=1,\ldots,n} |e^{i\theta} - z_j|.$$

Then for any $C \geq 1/n$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{C}{\delta(\theta)}\, d\theta \leq \frac{3}{2} \log(2nC).$$

Proof.

Let $\varepsilon \in (0, \min\{1, C\}]$ and, for $j = 1, \ldots, n$,

$$E_j = \{\theta \in (-\pi, \pi)\ ;\ |e^{i\theta} - z_j| \leq \varepsilon\}.$$

Then $\theta \in E = \cup_{j \leq n} E_j$ if and only if $\delta(\theta) \leq \varepsilon$. Moreover, for $\theta \in E$,

$$\log^+ \frac{C}{\delta(\theta)} \leq \sum_{j=1}^{n} \log_0 \frac{C}{|e^{i\theta} - z_j|}$$

where

$$\log_0 x = \begin{cases} \log x, & \text{if } x \geq C/\varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, using Lemma 1,

$$\int_E \log^+ \frac{C}{\delta(\theta)}\, d\theta \leq \sum_{j=1}^{n} \int_{E_j} \log \frac{C}{|e^{i\theta} - z_j|}\, d\theta \leq n\pi\varepsilon \left(\log \frac{C}{\varepsilon} + 1 \right).$$

On the other hand,

$$\int_{(-\pi,\pi)\setminus E} \log^+ \frac{C}{\delta(\theta)}\, d\theta \le \int_{-\pi}^{\pi} \log \frac{C}{\varepsilon}\, d\theta \le 2\pi \log \frac{C}{\varepsilon}.$$

Hence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \frac{C}{\delta(\theta)}\, d\theta \le \frac{n\varepsilon}{2}\left(\log \frac{C}{\varepsilon} + 1\right) + \log \frac{C}{\varepsilon}.$$

Choosing $\varepsilon = 1/n$ we obtain:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \frac{C}{\delta(\theta)}\, d\theta \le \frac{3}{2}\log(nC) + \frac{1}{2} \le \frac{3}{2}\log(2nC).$$

∎

Proof of Theorem 4.

We start by the well-known Lagrange interpolation formula: for any square-free polynomial F we have

$$1 = \sum_{F(\alpha)=0} \frac{F(z)}{(z-\alpha)F'(\alpha)}, \quad z \in \mathbf{C},\ F(z) \neq 0.$$

Applying this formula to $G_1, \dots, G_k$ we have, by (1.1),

$$\frac{1}{|G_j(e^{i\theta})|} \le \frac{d_j C_j}{\delta_j(\theta)},$$

where $\delta_j(\theta) = \min_{G_j(\alpha)=0} |e^{i\theta} - \alpha|$. Now, using Lemma 2,

$$\begin{aligned}\tilde{h}(G) &\le \sum_{j=1}^{k} |e_j| \tilde{h}(G_j) = \sum_{j=1}^{k} |e_j| \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \frac{1}{|G_j(e^{i\theta})|}\, d\theta \\ &\le \sum_{j=1}^{k} |e_j| \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \frac{d_j C_j}{\delta_j(\theta)}\, d\theta \le \frac{3}{2}\sum_{j=1}^{k} |e_j| \log(2d_j^2 C_j).\end{aligned}$$

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