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Subtraction of higher-order infrared singularities with a universal analytic algorithm in massless QCD

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Subtraction of higher-order infrared singularities with a universal analytic algorithm in massless QCD

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A chi è parte del cammino,
a chi ancora vorrebbe esserlo

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Abstract

The future of high-energy particle physics will be soon characterised by a rapid increase in experimental data precision as it approaches the upcoming high-luminosity phase of the Large Hadron Collider. To maximise the potential for uncovering new physics phenomena, it is crucial to establish a solid theoretical framework capable of providing highly-accurate predictions, thus facilitating the identification of any unforeseen deviations from the Standard Model background. Since strong interactions dominate physical processes at high-energy particle colliders, perturbative corrections in Quantum Chromodynamics (QCD) play a central role in this endeavour. In this thesis, we introduce a novel subtraction scheme designed for the systematic and universal cancellation of higher-order infrared singularities in massless QCD, paving the way for a general analytic solution to the next-to-next-to-leading-order (NNLO) infrared problem. Built upon the Local Analytic Sector Subtraction framework, we first present the construction of a next-to-leading-order (NLO) subtraction scheme, covering both initial- and final-state radiation. We explain key features of the method, including an optimisation procedure to improve numerical stability without adding analytic complexity, followed by a comprehensive numerical validation. In the second part of this work, we extend this strategy to construct analytic expressions implementing a fully-local infrared subtraction at NNLO, applicable to generic coloured massless final states. We rigorously verify the cancellation of all explicit infrared poles and analytically evaluate finite contributions using ordinary polylogarithms up to weight three. The resulting subtraction formula can be readily implemented in any numerical framework containing the relevant matrix elements up to NNLO.

List of publications

The content of this thesis is based on the following publications:

- G. Bertolotti, P. Torrielli, S. Uccirati and M. Zaro, *Local analytic sector subtraction for initial- and final-state radiation at NLO in massless QCD*, *JHEP* **12** (2022) 042 [2209.09123];
- G. Bertolotti, L. Magnea, G. Pelliccioli, A. Ratti, C. Signorile-Signorile, P. Torrielli et al., *NNLO subtraction for any massless final state: a complete analytic expression*, *JHEP* **07** (2023) 140 [2212.11190].

Contents

Introduction

Walking through a vintage market in the streets of Turin, I came across a forgotten copy of *Le Scienze*, the Italian edition of *Scientific American*, dated back to 1973. Within the pages of this magazine, an educational article by A. M. Litke and R. Wilson [3, 4] was reported, addressing the following topic:

ELECTRON-POSITRON COLLISIONS

When the electron and its antiparticle collide, they can annihilate each other and give rise to radiation or other particles. Such processes are currently studied with opposed beams of electrons and positrons

Herein, the authors discuss the technology employed in high-energy particle colliders, and explore the advantages that these experiments offer as unique and controlled environments that enable researchers to test theoretical predictions in physical domains that would otherwise be inaccessible through any other means. The enthusiasm surrounding particle discoveries and the vibrant pursuit of a unified explanation of the observed phenomena can be clearly perceived, especially in the research area devoted to the production of strongly-interacting nuclear particles called hadrons. At that time, a comprehensive predictive theory for this class of events was still in its early stages of development.

As we fast forward to the present day, high-energy physics has made remarkable strides in our understanding of the fundamental laws of Nature over the past 50 years. One of the monumental achievements has been the completion of the Standard Model (SM) of particle physics, the quantum field theory describing elementary constituents of matter and their interactions [5]. This milestone has been the result of several decades of collaborative theoretical and experimental efforts, which culminated with the discovery of the last missing piece, the Higgs boson, in July 2012 at the Large Hadron Collider (LHC) [6, 7].

While the Standard Model has been remarkably successful in describing the behaviour of known particles and forces, demonstrating an impressive agreement between theoretical predictions and experimental data, there are clear indications implying that this is not the final answer. The limitations it presents in explaining certain astrophysical observations, such as the existence of dark matter, or in addressing fundamental questions like the origin of neutrino masses and the hierarchy problem, have motivated the development of numerous innovative theories beyond the Standard Model (BSM), which aim to offer a more exhaustive understanding of these phenomena.

Eleven years after the discovery of the Higgs boson, ongoing experimental searches for hints of physics beyond the Standard Model at modern energy-frontier colliders have not yet uncovered any clear indication of new particles or interactions. As a result, various proposed BSM scenarios have been ruled out. This implies that if new physics does indeed exist within the currently accessible energy ranges, its effects would likely be extremely tiny, possibly requiring a large volume of experimental data to become evident. To detect these effects, which are expected to manifest as small deviations from SM predictions, it results imperative to enhance our control over Standard Model background processes.

To maximise the potential for discoveries on the experimental front, a significant upgrade is scheduled for the LHC, expected to take place around 2029. This upcoming phase, known as High Luminosity LHC (HL-LHC), will boost the integrated luminosity of this hadron collider by a factor of 10 compared to its previous design, resulting in a substantial reduction in statistical uncertainties due to the vast amount of collected data. Meanwhile, the field of particle physics is actively exploring another promising avenue of research: the development of future lepton colliders [8–14]. Thanks to the *cleaner* environment resulting from the collision of elementary particles, these experiments enable highly precise measurements and offer a significant reduction in background noise when compared to the more intricate hadron interactions. Long-term projects for future hadron colliders [15] are also under consideration.

In light of this forthcoming experimental progress, the availability of highly-accurate theoretical predictions covering a wide spectrum of scattering processes and relevant collider observables becomes of paramount importance. This precision is definitely crucial for enabling meaningful comparisons with ever-more precise experimental data, thereby ensuring their reliable interpretation. Within the context of high-energy colliders, where physical processes are primarily dominated by strong interactions, Quantum Chromodynamics (QCD) assumes a central role. In this theoretical framework, physical observables are evaluated as perturbative expansions in the strong coupling constant α_{S}, and missing higher-order corrections in the corresponding computations stands out as one of the primary source of theoretical uncertainties. While leading-order (LO) calculations offer little more than rough estimates for physical observables, next-to-leading-order (NLO) corrections are essential for assessing scale dependence and obtaining reliable results. However, calculations at this perturbative order frequently struggle to achieve the level of precision required by current data. Consequently, predictions at the next-to-next-to-leading order (NNLO) in the strong coupling are rapidly becoming the standard for meeting the few percent-level precision demanded by collider measurements.

The computation of higher-order QCD corrections for fully-differential observables relies on the existence of effective and automated methods for handling infrared (IR) singularities that manifest in quantum field theories beyond the Born approximation. These singularities, which stem from long-distance interactions such as soft or collinear emissions of massless particles, impact intermediate stages of the calculation and must be systematically removed to recover a finite and physically meaningful result. During the 1990s, an improved understanding of the universal infrared behaviour of scattering amplitudes paved the way for the development of general frameworks aimed at addressing the singularity problem at NLO in perturbation theory [16–21]. These frameworks, or rather *subtraction schemes*, played a pivotal role in the accuracy revolution that significantly contributed to the high-energy physics programmes at the LHC and other experiments.

The scenario takes a significant turn at NNLO, where tackling the infrared-subtraction problem becomes an extremely demanding task, driven by a sharp increase in technical complexity. Beyond the NLO regime, calculations of QCD perturbative corrections are hindered by two primary sources of complexity: a rapid proliferation of overlapping singular regions and the intricate interplay between virtual poles and phase-space singularities. Consequently, endeavours to attain the same degree of universality and computational efficiency achieved at NLO have spanned nearly two decades. The availability of two-loop amplitudes and the technology for producing differential NNLO calculations, established by a number of methods belonging to the large spectrum of proposed algorithms [22–57], have led to the successful computation of all relevant $2 \to 2$ processes at the LHC, thereby extending the current theoretical frontier towards NNLO predictions for $2 \to 3$ reactions [58–67]. Furthermore, recent progress has streamlined the calculation of N^{3}LO predictions for benchmark processes (for state-of-the-art reviews, see Refs. [68, 69]). Although the fundamental mechanisms underlying the cancellation of infrared singularities are conceptually straightforward and well-understood, the concrete technical implementation of such approaches typically entails significant computational intricacy. As a result, a universal solution to the infrared-subtraction problem beyond NLO is still missing.

This thesis presents the development of a novel subtraction scheme for the systematic and universal treatment of infrared singularities in fully-differential predictions at higher orders in perturbative QCD. The ambitious goal of this programme is to establish the groundwork for a fully-general, analytic solution to the infrared problem at NNLO.

The structure of this thesis is outlined as follows. Chapter 1 sets the stage by delineating the issue this manuscript aims to solve. We start by introducing Quantum Chromodynamics and its fundamental properties, we then delve into the computation of physical predictions for hadronic processes, with a particular emphasis on the origin of infrared and collinear singularities arising in QCD perturbative calculations. An overview of existing methods for removing infrared divergences at NLO and NNLO accuracy is provided, highlighting their accomplishments and limitations. We conclude this Chapter by presenting our proposed strategy for the formulation of a general, analytic, and local solution to the infrared-subtraction problem.

In this thesis, we focus on developing and validating our algorithm in two distinct scenarios. First, we apply our strategy to formulate a subtraction scheme at NLO accuracy in massless QCD, capable of handling both initial- and final-state radiation. Chapter 2 provides a thorough account of the construction of this scheme. This step plays a fundamental role in assessing the algorithm's performance, mitigating potential instabilities, and devising optimisations that can be readily extended to higher-order implementations.

With this experience in hand, in Chapter 3 we address the construction of a novel subtraction scheme at NNLO precision. A detailed description of all ingredients is presented, as well as a guided procedure showcasing their intricate interplay. The outcome of this extensive effort is a fully-analytic and universal formula which achieves the cancellation of NNLO infrared singularities in processes involving an arbitrary number of colourful (as well as colourless) final-state particles in massless QCD. On the phenomenological side, this result paves the way for a deeper analysis of theory-data comparisons in current (and future) e^+e^- colliders, enabling for example the calculation of the NNLO-accurate cross section for four-jet production.

Chapter 4 is finally dedicated to the validation of our algorithms. We present both integrated and differential results obtained employing our NLO subtraction formula, which have been implemented in MadNkLO, an automated Python framework built upon MadGraph5_aMC@NLO. While ongoing efforts are directed towards the low-level implementation and optimisation of MadNkLO, particularly in the view of computationally-demanding NNLO computations, a detailed example illustrating how the cancellation of NNLO infrared divergences is achieved within our algorithm, is presented in a non-trivial case study.

The construction of these subtraction schemes, as described in Chapter 2 and Chapter 3, rely on a considerable number of lengthy expressions, which we provide in a series of Appendices: Appendix A offers a quick reference for the general notations used throughout this thesis, Appendix B contains all the relevant NLO material, and Appendix C collects all building blocks entering our NNLO subtraction formula.

In the Conclusions, we provide a summary of the outcomes of this thesis work and explore potential directions for future developments and applications of our algorithm.

Chapter 1

Collider predictions facing infrared singularities: a brief overview

This first Chapter offers a brief overview aimed at introducing the reader to the fundamental concepts necessary for a comprehensive understanding of this thesis work.

To achieve this, we will first present the theoretical framework of Quantum Chromodynamics (Section 1.1) and provide insights into how calculations are performed within this context. We will then place particular emphasis on predictions for hadron colliders, delving into the issue of infrared singularities that affect them (Section 1.2). To further assist the reader in situating our work within the broader theoretical landscape of collider phenomenology, we will provide an overview of the current state-of-the-art algorithms developed and employed for the cancellation of infrared singularities at higher orders (Section 1.3). This will include an introduction to the strategy underlying our own proposed subtraction method.

For those interested in delving deeper into these topics, we recommend referring to standard textbooks of quantum field theory, such as Refs. [70–72], and specific lecture notes [73, 74], which offer more extensive coverage of the subject matter.

1.1 Quantum Chromodynamics

Quantum Chromodynamics (QCD) is the gauge field theory describing the strong interactions among coloured quarks and gluons. In Section 1.1.1 we provide an overview of its fundamental properties. Since the interactions of strongly interacting particles play a dominant role in high-energy collider experiments, our attention turns to perturbation theory. Within this framework, the calculation of scattering amplitudes is organised in terms of Feynman diagrams. When going beyond the leading-order representation, this approach involves the evaluation of closed loops of partons, which are plagued by ultraviolet divergences stemming from the integration over arbitrarily-large loop momenta. These infinities can be reabsorbed into a redefinition of the parameters of the Lagrangian through the renormalisation procedure discussed in Section 1.1.2. As a result, the coupling constant α_{S} acquires a scale-dependent behaviour, and in Section 1.1.3, we delve into the running of α_{S} and its important implications.

1.1.1 Basics

The dynamics of quarks and gluons is encoded in the QCD Lagrangian density, given by

$$\mathcal{L} = \sum_f \bar{\psi}_{f,i}\big(i\gamma^\mu\partial_\mu - m_f\big)\psi_{f,i} + \sum_f \bar{\psi}_{f,i}\big(-g_{\mathrm{S}}\gamma^\mu t^a_{ij}A^a_\mu\big)\psi_{f,j} - \frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu}_a\,, \tag{1.1}$$

where summation over repeated indices is understood. The first contribution to Eq. (1.1) represents the kinetic propagation term for the fermion fields. Specifically, $\psi_{f,i}$ is the quark field, of flavour f and mass m_f, defined to transform under the triplet (fundamental) representation of the SU(3) gauge group (the corresponding color index i runs from 1 to $N_c = 3$). The sum over f runs over the six quark flavours included in the Standard Model, while γ^μ are the Dirac γ-matrices connecting the spinor representation of the quark fields to the vector representation of the Lorentz group. The second quantity in Eq. (1.1) is the fermion-gauge boson interaction vertex that couples the quark fields to the gluon field A^a_μ by means of the SU(3) generators t^a_{ij}, hermitian and traceless matrices in the fundamental representation. An explicit expression for these generators is given by the eight 3×3 Gell-mann matrices [72], defined as $t^a_{ij} \equiv \lambda^2_{ij}/2$, with the following standard normalisation

$$\mathrm{Tr}\left[t^a_{ij}t^b_{ij}\right] = T_R\,\delta^{ab} = \frac{1}{2}\,\delta^{ab}\,. \tag{1.2}$$

The coupling constant of the strong interaction is g_{S}, which is also commonly expressed as $\alpha_{\mathrm{S}} = g_{\mathrm{S}}^2/4\pi$, where α_{S} is the parameter on which the perturbative expansion of QCD is based. The last term of Eq. (1.1) contains the contraction of field strength tensors $F^a_{\mu\nu}$, defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_{\mathrm{S}} f_{abc} A^b_\mu A^c_\nu\,, \tag{1.3}$$

where A^a_μ represents the gluon field, the gauge field of SU(3), in the adjoint representation (with color index a running from 1 to $N_c^2 - 1 = 8$), and f_{abc} are the real and anti-symmetric structure constants of SU(3), appearing in the commutation relation defining the group algebra:

$$\left[t^a, t^b\right] = i f_{abc}\, t^c\,. \tag{1.4}$$

The expansion of the contraction between tensors in Eq. (1.3) reveals a kinetic term for gluons and self-interaction contributions among gauge bosons, in the form of triple-gluon vertex (proportional to g_{S}) and four-gluon vertex (proportional to g_{S}^2), arising from the fact that gluons are force mediator which carry colour charge themselves, differently from the QED electrically-neutral mediator, the photon.

The structure of the QCD Lagrangian in Eq. (1.1) is constrained by the requirement of invariance under SU(3) local gauge transformations, i.e. independent redefinitions of quark and gluon fields at every point in space and time have not to change the physics content of the theory. A generic element of this group, the unitary matrix U, can be written in exponential form in terms of the generators t^a of the Lie-algebra and the real phase parameters of the local gauge transformation $\omega_a \equiv \omega_a(x)$, as

$$U \;=\; e^{i\omega_a t^a}\,. \tag{1.5}$$

One can verify that with the following transformation properties of quark fields, gluon fields, and strength tensor, namely

$$\begin{aligned} \psi_{f,i} &\to U_{ij}\,\psi_{f,j}\,, \\ t^a A^a_\mu &\to U\,t^a A^a_\mu\,U^{-1} + \frac{i}{g_s}\,(\partial_\mu U)\,U^{-1}\,, \\ t^a F^a_{\mu\nu} &\to U\,t^a F^a_{\mu\nu}\,U^{-1}\,, \end{aligned} \tag{1.6}$$

the QCD Lagrangian density is left unchanged, thus proving its gauge invariance.

In order to perform perturbative calculations, the classical Lagrangian (1.1) has to be supplemented by a *gauge-fixing* (GF) term, without which the propagator for the gluon field could not be defined. A possible class of gauge choices, denoted as

$$\mathcal{L}_{\rm GF} \;=\; -\frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2\,, \tag{1.7}$$

are the *covariant* gauges with gauge parameter ξ. In non-abelian gauge theories as QCD, the covariant gauge-fixing term (1.7) requires the inclusion of a ghost Lagrangian, as

$$\mathcal{L}_{\rm ghost} \;=\; -\bar{c}^a\partial^\mu\left(\partial_\mu\delta_{ab} + g_{\rm s} f_{abc}A^c_\mu\right)c^b\,. \tag{1.8}$$

The *Faddev-Popov* ghosts, denoted by c^a, are anti-commuting scalar fields in the adjoint representation which interact with the gluon fields, and remove unphysical degrees of freedom which would otherwise appear in covariant gauges.

For future convenience, we introduce here the colour-charge vector $\mathbf{T}_i$, associated with the emission of a gluon from a coloured particle i, acting on the colour space as $\mathbf{T}_i \equiv T^a_i\,|a\rangle$. The components of this vector are $n\times n$ matrices whose dimension depends on the nature of the particle i: if i is a gluon, $T^a_{cb} \equiv i f_{cab}$ with $n = 8$, while if i is a quark, $n = 3$ and $T^a_{\alpha\beta} \equiv t^a_{\alpha\beta}$ (in the case of emitting antiquark one has $T^a_{\alpha\beta} \equiv \bar{t}^a_{\alpha\beta} = -t^a_{\beta\alpha}$). Applying colour-algebra relations, one can derive that the scalar products obey

$$\mathbf{T}_i \cdot \mathbf{T}_j \;=\; \mathbf{T}_j \cdot \mathbf{T}_i\,, \qquad \text{if} \quad i \neq j\,, \tag{1.9}$$

which indicates that the respective matrices act on different spaces; while, for $i = j$, one obtains

$$\mathbf{T}_i \cdot \mathbf{T}_i \;=\; (\mathbf{T}_i)^2 \;=\; C_{f_i}\,, \tag{1.10}$$

where C_{f_i} is the Casimir operator taking value $C_{f_i} = C_F$ if i is a quark (antiquark), and $C_{f_i} = C_A$ if i is a gluon. Specifically,

$$\begin{aligned} \sum_a t^a_{ik} t^a_{kj} &= C_F\, \delta_{ij} \qquad \text{with} \qquad C_F \;=\; \frac{N_c^2-1}{2N_c} \xrightarrow{N_c=3} \frac{4}{3}\,, \\ \sum_{a,b} f^{abc} f^{abd} &= C_A\, \delta^{cd} \qquad \text{with} \qquad C_A \;=\; N_c \xrightarrow{N_c=3} 3\,. \end{aligned} \tag{1.11}$$

The colour factor C_F is associated with the gluon emission from a quark, while the colour factor C_A is associated with the gluon emission from a gluon. The factor T_R introduced in Eq. (1.2) is connected to the process of a gluon splitting into a $q\bar{q}$ pair. We refer the reader to Ref. [18] for more details on the colour-charge algebra.

1.1.2 Ultraviolet renormalisation

Starting from the Lagrangian density given by Eq. (1.1), along with Eqs. (1.7)-(1.8), one can deduce the Feynman rules of QCD (as detailed for instance in Ref. [71]). These rules provide analytical expressions for computing scattering amplitudes $\mathcal{A}$ in perturbation theory, which are subsequently used in the calculation of physical observables.

However, going beyond the tree-level approximation, the evaluation of loop corrections to scattering amplitudes gives rise to divergences of ultraviolet (UV) origin, as a consequence of the integration over virtual loop momenta which can become arbitrarily large, since not constrained by any physical condition. To restore the predictive power of the theory, we first need to introduce a regulator to isolate and quantify the singularities, so as to work with finite quantities in the intermediate stages of the calculations. *Dimensional regularisation* [75] is by far the most commonly used approach, as it preserves the Lorentz and gauge invariance of the theory. Basically it consists in shifting the number of space-time dimensions from 4 to $d = 4 - 2\epsilon$, and taking the limit $\epsilon \to 0$ just at the end of the calculation. In this framework, UV divergences manifest as isolated ϵ poles, schematically as

$$g_{\rm S}^2 \int \frac{d^4k}{(2\pi)^4} \cdots \to g_{\rm S}^2\, \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \cdots \;=\; g_{\rm S}^2\, \mu^{2\epsilon} \left[\frac{c_{-1}}{\epsilon} + c_0 + \mathcal{O}(\epsilon)\right]. \tag{1.12}$$

Notice that the introduction of an arbitrary mass scale μ is necessary to maintain the correct dimensionality of the integral. Since physical observables must be independent of ϵ, we need to reabsorb the effects of UV degrees of freedom in a redefinition of the unobservable parameters of the theory, as masses, fields and couplings of the original (*bare*) Lagrangian. This procedure is known as *renormalisation*, and ultimately leads to the systematic cancellation of UV divergences order by order in perturbation theory. The definition of a renormalised Lagrangian capable of providing finite predictions involves the introduction of a multiplicative constant Z for each bare parameter (denoted with the label *bare*), explicitly as

$$\psi_i^{\rm bare} \,=\, Z_\psi^{1/2}\,\psi_i\,, \quad A_\mu^{\rm bare} \,=\, Z_A^{1/2}\,A_\mu\,, \quad c^{\rm bare} \,=\, Z_c^{1/2}\,c\,, \quad \alpha_{\rm S}^{\rm bare} \,=\, Z_{\alpha_{\rm S}}\,\alpha_{\rm S}\,, \tag{1.13}$$

which divergent parts are fixed by the requirement of matching the UV divergences. On the other hand, the inclusion of finite constants within the Z factors is completely arbitrary. Different prescriptions for the finite parts result in different *renormalisation schemes*: the solely subtraction of the pole content corresponds to the Minimal Subtraction (MS) scheme, whereas the addition of a finite term given by the universal factor $\ln(4\pi) - \gamma_E$ defines the Modified Minimal Subtraction ($\overline{\rm MS}$) scheme.

The renormalisation procedure introduces a new arbitrary mass scale, known as *renormalisation scale* μ_R, which generally defines a separation between the energy accessible by experiments and the effects of UV physics. The regularisation scale μ in Eq. (1.12) and the renormalisation scale μ_R are usually chosen to coincide[1].

1.1.3 The running coupling constant

In the context of perturbative QCD, predictions for physical observables are formulated in terms of the renormalised strong coupling $\alpha_{\rm S} = \alpha_{\rm S}(\mu^2)$, whose value depends on the renormalisation scale μ at which $\alpha_{\rm S}$ is evaluated. This *running* of the coupling constant is governed by a renormalisation group equation,

$$\mu^2 \frac{d\alpha_{\rm S}(\mu^2)}{d\mu^2} \,=\, \beta(\alpha_{\rm S}(\mu^2))\,, \tag{1.14}$$

where the μ-dependence is encoded in the QCD β-function. This function can be expressed as a power series in $\alpha_{\rm S}(\mu^2)$, as

$$\beta(\alpha_{\rm S}) \,=\, -\,\alpha_{\rm S}\sum_{n=0}^{\infty} b_n\,\alpha_{\rm S}^{n+1} \,=\, -\left(b_0\alpha_{\rm S}^2 + b_1\alpha_{\rm S}^3 + b_2\alpha_{\rm S}^4 + \cdots\right)\,, \tag{1.15}$$

[1] In this thesis we set $\mu_R \equiv \mu$, and we adopt μ to identify the renormalisation scale.

where in particular

$$b_0 = \frac{11C_A - 4T_R N_f}{12\pi}, \qquad b_1 = \frac{17C_A^2 - N_f T_R(10C_A + 6C_F)}{24\pi^2}, \tag{1.16}$$

are the one-loop (b_0) and the two-loop (b_1) coefficients[2]. Here N_f is the number of active light flavours, defined as the quark flavours with mass smaller than the scale μ_R. Beyond two loops, the b_n coefficients become scheme-dependent, and they are currently known up to five loops [76–78].

The solution to the renormalisation group equation in Eq. (1.14) can be obtained iteratively. Retaining the first order in the β expansion, the analytic solution for the running of α_S reads

$$\alpha_S(\mu^2) = \frac{\alpha_S(\mu_0^2)}{1 + \alpha_S(\mu_0^2)\, b_0 \ln\frac{\mu^2}{\mu_0^2}}, \tag{1.17}$$

where μ_0 is an initial scale at which $\alpha_S(\mu_0^2)$ is known or measured. We observe that the slope of $\alpha_S(\mu^2)$ is controlled by the sign of the β-function: for a sufficiently small number of active flavours N_f, as it is the case for QCD, the coefficient b_0 is positive and the strength of the coupling constant decreases with increasing energies. This behaviour, known as *asymptotic freedom* [79, 80], is a fundamental feature of QCD. The small value of α_S allows for reliable perturbative calculations in the high-energy regime, where the resulting predictions are expected to converge with the inclusion of just a few expansion terms. Eq. (1.17) can be equivalently written in terms of the non-perturbative scale Λ, or Λ_{QCD}, as

$$\alpha_S(\mu^2) = \frac{1}{b_0 \ln\frac{\mu^2}{\Lambda^2}}, \tag{1.18}$$

at which the strong coupling diverges. The value of Λ_{QCD} is not precisely defined as it depends on the definition of α_S, which in turn is determined by the scale μ_0, the renormalisation scheme, and the order considered for the β-function within the renormalisation group equation. This scale, which order of magnitude is around 200 MeV, is indicative of the energy domain where non-perturbative effects become relevant, QCD is strongly coupled, and perturbation theory ceases to be applicable. This property is consistent with the phenomenon known as *confinement*, according to which colour-charged quarks and gluons are constrained to form colourless observable hadrons. Ultimately, our ability of making predictions for processes in hadronic colliders relies on *factorisation theorems*, which play a crucial role in separating the impact of non-perturbative dynamics from the high-energy domain, where employing perturbation theory is allowed. We will discuss this point in more detail in Section 1.2.2.

[2] The definition of b_0 in Eq. (1.16) and β_0 in Appendix A are linked by the relation $b_0 = \beta_0/4\pi$.

1.2 Hadronic processes in perturbative QCD

The first theoretical approach for calculating cross sections in generic hadronic collisions was built upon the *parton model* [81, 82], a QFT-based framework grounded on the observation that, in high-energy scattering processes, hadrons (or more specifically, protons) can be effectively treated as loosely-bound collections of point-like *partons*, namely massless quarks and gluons. In hadron-initiated reactions, these partons are the actual particles that partecipate in the hard, short-distance interaction, carrying a fraction x of the longitudinal momentum of the respective parent hadron. The probability of extracting a parton of flavour i with energy fraction x is encoded in a *parton distribution function* (PDF), $f_i(x)$. Since these distribution functions parametrise our knowledge of the internal dynamics of hadrons, of intrinsic non-perturbative nature, they cannot be computed using perturbation theory. Conversely, they must be determined from experimental data through global fits. Due to their universality, once obtained they can be employed to describe various processes[3]. The cross section for a scattering process initiated by two hadrons with momenta p_1 and p_2 can be written as

$$\sigma(p_1,p_2) = \sum_{a,b} \int_0^1 dx_1 dx_2 \, f_a(x_1) f_b(x_2) \, \hat{\sigma}_{ab}(x_1 p_1, x_2 p_2) + \mathcal{O}\big((\Lambda_{\rm QCD}/Q)^p\big) \,, \tag{1.19}$$

where the sum runs over all flavours of the incoming partons a and b carrying momenta $p_a = x_1 p_1$ and $p_b = x_2 p_2$, respectively, while Q is the relevant hard scale of the process[4]. The quantity $\hat{\sigma}_{ab}$ is the *partonic cross section*, encoding the hard, short-distance dynamics of the colliding partons. Given a partonic centre-of-mass energy $s = q^2$, with $q^\mu = p_a^\mu + p_b^\mu$, it may be specifically evaluated as

$$\hat{\sigma}_{ab}(p_a,p_b) \; = \; \frac{1}{2s} \int d\Phi_m \, |\mathcal{M}_{ab\to X_m}|^2 \,, \tag{1.20}$$

where $2s$ is the flux factor, X_m indicates a generic m-particle final state, and $d\Phi_m$ represents the corresponding differential phase-space measure, explicitly reading

$$d\Phi_m \equiv (2\pi)^d \delta^{(d)}\Big(q - \sum_{i=1}^m p_i\Big) \left[\prod_{i=1}^m \frac{d^d p_i}{(2\pi)^{d-1}} \, \delta(p_i^2 - m_i^2) \, \theta(p_i^0) \right] \,, \tag{1.21}$$

[3]In this PDF definition, we are neglecting transverse momentum components of the colliding partons that would have originated from hadron-internal motions. An alternative formulation, known as Transverse Momentum Dependent (TMD) PDFs, has been developed to include non-perturbative information on intrinsic transverse momentum and polarization in parton distributions, which are essential in QCD predictions of multi-scale, exclusive collider observables (see e.g. [83] for a review on this topic).

[4]Eq. (1.19) is easily generalised to describe lepton-hadron and lepton-lepton scattering processes by setting the hadronic PDFs to unity. Actually, electron and muon PDFs [84–86] have also been developed in view of a direct application in future lepton colliders.

with Lorentz-invariant factors enforcing momentum conservation and positive-energy mass-shell condition. The matrix element $\mathcal{M}$ gives the transition probability for obtaining a final state X_m from the scattering of the two incoming states a and b. The expression for the matrix elements squared in Eq. (1.20) includes appropriate sums/averages over colour and spin states of the involved particles. Within the framework of perturbative QCD, it can be expressed as a perturbative expansion in the small coupling constant, as

$$\mathcal{M}_{ab\to X_m} = \sum_{i=0}^{\infty} \alpha_{\mathrm{S}}^i \, \mathcal{M}_{ab\to X_m}^{(i)} , \tag{1.22}$$

where the sum index i indicates the increasing loop order. As a result, the partonic cross section can be computed order-by-order in perturbation theory as

$$\hat{\sigma}_{ab} = \sum_{i=0}^{\infty} \hat{\sigma}_{ab}^{(i)} = \hat{\sigma}_{ab}^{(0)} + \hat{\sigma}_{ab}^{(1)} + \hat{\sigma}_{ab}^{(2)} + \dots , \tag{1.23}$$

where the first term in Eq. (1.23) is known as the leading-order (LO) correction, the second term as the next-to-leading-order (NLO) correction, and so forth. When considering a LO process of order α_{S}^k, each component in the expansion incorporates an extra power of the strong coupling, according to $\alpha_{\mathrm{S}}^{k+i}$. The size of subsequent contributions is expected to steadily decrease at higher orders, and ultimately improve the comparison with experimental results. The last quantity in Eq. (1.19) accounts for non-perturbative power corrections that go beyond this perturbative scattering picture. These contributions are expected to scale as $(\Lambda_{\mathrm{QCD}}/Q)^p$, where the exponent $p \geqslant 1$ depends on the specific observable under consideration. For instance, recent studies have been dedicated to quantifying the effects of linear ($p = 1$) power corrections in collider observables [87–91].

Nowadays, the parton model is recognised as providing merely an approximation of the lowest-order perturbative description of collider observables in QCD. As we will see, this approach does not survive when QCD radiative corrections are taken into account. In Section 1.2.1, we will analyse the *infrared divergent* behaviour of next-to-leading order corrections, specifically arising in the evaluation of real-emission and virtual one-loop contributions. This will be crucial in determining which observables are ultimately insensitive to these singularities, thus reliably calculable in perturbative QCD. We will find a class of divergences, originating from initial-state collinear emissions, which do not cancel in the sum of the separately divergent NLO corrections, effectively making Eq. (1.19) no more applicable. The solution to this problem will be given in Section 1.2.2, where we employ a *collinear factorisation* procedure for reabsorbing these divergences into the PDFs, in the spirit of the UV renormalisation performed in Section 1.1.2. As a consequence, PDFs will acquire a dependence on the factorisation scale μ_F, which in turn implies the presence of a renormalisation group equation. This latter is known as the DGLAP evolution equation, and it will be discussed in Section 1.2.3.

1.2.1 Origin of infrared and collinear singularities

Two different categories of events contribute to the NLO correction in the perturbative expansion of a total cross section (1.23): the real emission of an extra gluon in the final state of the corresponding tree-level process, and the one-loop virtual correction (interfered with the Born amplitude), where a virtual gluon is exchanged between two partonic legs. When attempting the evaluation of such $\mathcal{O}(\alpha_{\rm S})$ corrections, we inevitably face the presence of *infrared* (IR) divergences, which appear to hinder the computation of meaningful predictions beyond the LO approximation. In the following we will identify the origin of these divergences and assess their impact on perturbative calculations.

We start by analysing the behaviour of the real-radiation correction in a simple scenario, as the tree-level $q\bar{q}$ pair production in e^+e^- annihilation. The cross section for the real-emission process $e^+e^- \to q\bar{q} + g$ is given by

$$\sigma_R = \frac{1}{2s} \int d\Phi_{q\bar{q}g} \, |\mathcal{M}_{q\bar{q}g}|^2 \,. \tag{1.24}$$

We are interested in the hadronic side of the real amplitude $\mathcal{M}_{q\bar{q}g}$, whose relevant diagrams are illustrated in Figure 1.1.

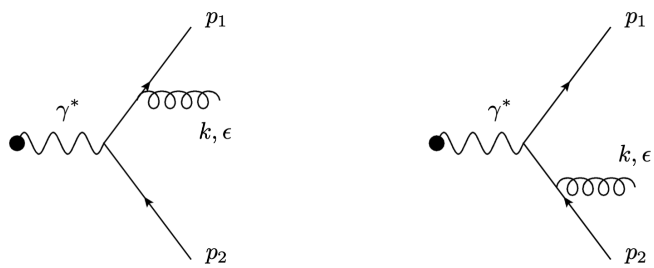

Figure 1.1. Gluon emission off a final-state quark (left) and a final-state antiquark (right).

The analytic expression for these contributions explicitly reads

$$\mathcal{M}^{\mu}_{q\bar{q}g} = \bar{u}(p_1)\, ig_{\rm S}\not{\epsilon}t^a \, \frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2} \, ie_q\gamma^\mu \, v(p_2) + \bar{u}(p_1)\, ie_q\gamma^\mu \, \frac{-i(\not{p}_2 + \not{k})}{(p_2+k)^2} \, ig_{\rm S}\not{\epsilon}t^a \, v(p_2) \,, \tag{1.25}$$

where the two terms in the sum correspond to the gluon emission from a quark and the gluon emission from an antiquark, respectively. Here $\bar{u}(p_1)$ and $v(p_2)$ are the spinors of the massless final-state quark and antiquark, ϵ is the gluon polarisation vector, and e_q identifies the quark electric charge. In the expression above, the colour indices of quarks are omitted, but they will be later reintroduced in the matrix element squared. Additionally, $\not{b} = b_\mu\gamma^\mu$. Beginning with Eq. (1.25) and working in the soft gluon approximation, which assumes that the gluon energy is significantly smaller than the total available energy, the calculations simplify. This yields a squared real matrix element summed over colour and polarisations, which finally reads

$$|\mathcal{M}_{q\bar{q}g}|^2 \xrightarrow{soft} |\mathcal{M}_{q\bar{q}}|^2 \, g_{\rm S}^2 \, C_F \, \frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \, . \tag{1.26}$$

The quantity $\mathcal{M}_{q\bar{q}}$ is the scattering amplitude for the underlying Born process $e^+e^- \to q\bar{q}$, whereas the remaining terms constitute the *eikonal* factor, a universal structure depending on colour charges and kinematic invariants that factorises from scattering amplitudes in soft limits. Next, our attention turns to the real phase-space measure, which takes on the factorised form

$$d\Phi_{q\bar{q}g} \xrightarrow{soft} d\Phi_{q\bar{q}} \, \frac{d^3\vec{k}}{2E_g(2\pi)^3} \; = \; d\Phi_{q\bar{q}} \, \frac{E_g dE_g \, d\cos\theta \, d\phi}{2(2\pi)^3} \, , \tag{1.27}$$

described by the polar (θ) and azimuthal (ϕ) angles of the gluon with respect to the direction of the emitting quark. In the centre-of-mass frame, where quarks are in a back-to-back configuration, the combination of Eq. (1.26) and Eq. (1.27) leads to

$$\sigma_R^{soft} \; = \; \sigma_B \, \frac{2\alpha_{\rm S} C_F}{\pi} \int \frac{dE_g}{E_g} \, \frac{d\cos\theta}{(1-\cos\theta)(1+\cos\theta)} \, . \tag{1.28}$$

This resulting cross section, albeit obtained in the soft gluon approximation, contains all the interesting features. Specifically, the integral in Eq. (1.28) is affected by non-integrable divergences originating in two distinct limits. First, it diverges when the energy of the emitted gluon E_g becomes vanishingly small, which is referred to as the *soft singularity*. Second, it also diverges when the angle θ between the gluon and quark (or antiquark) momenta approaches zero (or π), resulting in a nearly collinear configuration, thus known as *collinear singularity*[5].

Despite the simplicity of the chosen example, the presence of soft and collinear infrared divergences is a common characteristic of QCD calculations. These infinities are associated with long-range interactions that occur over time scales significantly longer than those of the hard scattering event. Therefore, predictions for total cross sections should, in principle, remain unaffected by them. This is indeed the case when we account for the one-loop virtual correction as well, whose contribution also exhibits the same kind of non-integrable singularities that emerge (with a negative sign) as the loop momentum explores infrared kinematic regions. Consequently, in the sum of the real and virtual corrections these divergences cancel out, ultimately enabling the calculation of finite predictions. The Bloch-Nordsieck theorem [92], initially formulated within the framework of QED, and its later generalisation to QCD by Kinoshita, Lee, and Nauenberg (KLN) [93, 94], ensure

[5]Note that when dealing with massive quarks, the mass acts as a regulator for collinear singularities, specifically as $(p+k)^2 = 2E_qE_g(1-\beta\cos\theta)$, with $\beta = \sqrt{(1-m^2/E_q^2)}$, while soft divergences still remain.

that this cancellation takes place when summing over all initial and final degenerate states, namely those states that appear physically indistinguishable with respect to a specific detector resolution. This theorem is particularly relevant for a specific class of observables referred to as *infrared and collinear* (IRC) safe, which possess the property of being essentially insensitive to the effects of soft and collinear long-distance dynamics. As a result, they can be reliably calculated within the framework of perturbative QCD. Concretely, this property implies that when we evaluate a given observable $\mathcal{O}_{n+1}$ with an $(n+1)$-body kinematics, its behaviour smoothly approaches that of the corresponding $\mathcal{O}_n$ observable, schematically as

$$\begin{aligned} \mathcal{O}_{n+1}(k_1, \cdots, k_i, \cdots, k_{n+1}) &\xrightarrow{k_i \to 0} \mathcal{O}_n(k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_{n+1}), \\ \mathcal{O}_{n+1}(k_1, \cdots, k_i, k_j, \cdots, k_{n+1}) &\xrightarrow{k_i || k_j} \mathcal{O}_n(k_1, \cdots, k_i + k_j, \cdots, k_{n+1}), \end{aligned} \tag{1.29}$$

where, on the right-hand side, $\mathcal{O}_n$ is evaluated with n-body kinematics obtained either by removing a soft gluon momentum, or by replacing the collinear particles k_i, k_j with their combined momenta.

The KLN theorem, however, does not apply to hadronic collisions, as the non-perturbative nature of hadrons prevents the summation over initial degenerate states. This leads to the presence of initial-state collinear singularities that cannot be canceled, thereby affecting the parton model description of hadron-initiated cross sections, as shown in Eq. (1.19). We can make this issue evident when considering a generic process involving an initial-state hadron, and focusing on the combination of the real-emission correction, given by the radiation of a final-state gluon off an incoming quark p, as depicted in the first panel of Figure 1.2, and the respective one-loop virtual contribution, as in the second graph in Figure 1.2.

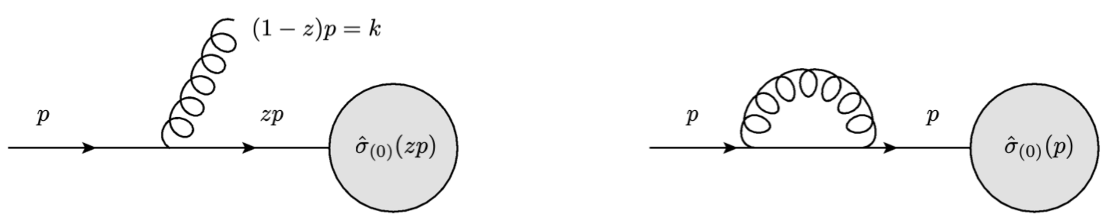

Figure 1.2. Gluon emission off an incoming quark (left) and one-loop virtual correction (right).

The real contribution to the partonic cross section can be expressed as

$$\hat{\sigma}_{R,(1)} = \frac{1}{2s} \int d\Phi(p;k)\, |\mathcal{M}_{(1)}|^2 , \tag{1.30}$$

where the phase-space measure is understood to depend on the incoming momentum p, and to be integrated over the outgoing momentum k of the emitted gluon. It is convenient to examine the divergent behaviour of the real matrix element $\mathcal{M}_{(1)}$ in the specific configuration where the quark and gluon become collinear, denoted as $p \,||\, k$. To this end, we employ the so-called Sudakov parametrisation to decompose the gluon momentum along the relevant directions, as

$$k^\mu \;=\; (1-z)\, p^\mu + k_\perp^\mu - \frac{1}{1-z}\,\frac{k_\perp^2}{2p\!\cdot\! p_r}\, p_r^\mu\,, \qquad (p+k)^2 \;=\; -\frac{k_\perp^2}{1-z}\,, \tag{1.31}$$

where p_r is an arbitrary massless on-shell vector necessary to define the gluon transverse momentum $k_\perp$ (with $k_\perp^2 < 0$), satisfying $k_\perp\!\cdot p = k_\perp\!\cdot p_r = 0$. The variable z represents the longitudinal momentum fraction, defined as $z = (p-k)\!\cdot p_r / p \cdot p_r$. In this framework, the radiative phase-space measure is given by

$$d\Phi(p;k) \xrightarrow{coll.} d\Phi(zp)\, \frac{dk_\perp^2}{16\pi^2}\, \frac{dz}{(1-z)}\,, \tag{1.32}$$

while, after some manipulations (for explicit calculations, see e.g. Ref. [73]), the matrix element squared in the limit of small emission angles takes the following form

$$|\mathcal{M}_{(1)}|^2 \xrightarrow{coll.} \frac{2g_{\rm S}^2}{(p_1+k)^2}\, \frac{1}{z} \left(\frac{1+z^2}{1-z}\right) |\mathcal{M}_{(0)}(zp)|^2\,, \tag{1.33}$$

where the notation $\mathcal{M}_{(0)}(zp)$ indicates that the Born-level amplitude should be calculated using zp as the four-momentum associated with the incoming quark. Combining all the ingredients in Eq. (1.32) and Eq. (1.33), one obtains

$$\hat\sigma_{R,\,(1)} \;=\; \frac{\alpha_{\rm S} C_F}{2\pi} \int \hat\sigma_{(0)}(zp) \left(\frac{1+z^2}{1-z}\right) \frac{dk_\perp^2}{k_\perp^2}\, dz\,. \tag{1.34}$$

When one combines this result with the contribution from the virtual one-loop correction, the final outcome can be expressed as follows:

$$\hat\sigma_{R,\,(1)} + \hat\sigma_{V,\,(1)} \;=\; \frac{\alpha_{\rm S} C_F}{2\pi} \int \Big[\hat\sigma_{(0)}(zp) - \hat\sigma_{(0)}(p)\Big] \left(\frac{1+z^2}{1-z}\right) \frac{dk_\perp^2}{k_\perp^2}\, dz\,. \tag{1.35}$$

In this equation, the soft singularity at $z \to 1$ cancels out between the real and virtual terms, as the content of the square brackets vanishes. On the other hand, the collinear singularity associated with the limit of small $k_\perp$ does not cancel. Specifically, the integration over the transverse momentum is not bounded from below, and as a result, the integral exhibits a logarithmic divergence as $k_\perp$ approaches zero.

1.2.2 Collinear factorisation

The problem of initial-state collinear singularities spoiling the interaction picture proposed by the parton model can be solved through a *collinear factorisation* procedure.

This approach involves quantifying the divergent contributions that affect the partonic cross section and reabsorbing them into a redefinition of universal (*bare*) objects as the parton distribution functions. We will now illustrate how this process works in a simple scenario, and then extend the concept to create an improved parton-model formulation for describing hadronic interactions within perturbative QCD.

Let us consider a scattering process involving one initial-state hadron with momentum p, initiated at the hard-interaction level by a quark carrying momentum $\hat{p} = yp$. The naïve prediction for the total cross section at next-to-leading-order can be written as

$$\sigma(p) \;=\; \int dy\, f_q^{(0)}(y)\,\hat{\sigma}(yp) \;=\; \int dy\, f_q^{(0)}(y)\left(\hat{\sigma}_{(0)}(yp) + \hat{\sigma}_{(1)}(yp)\right), \tag{1.36}$$

where $f_q^{(0)}$ represents the *bare* quark PDF, which solely depends on the longitudinal fraction y, $\hat{\sigma}_{(0)}$ is the leading-order partonic cross section, and $\hat{\sigma}_{(1)}$ collects the contribution from the corresponding radiative corrections, schematically depicted in Figure 1.2. As previously discussed, the computation of these corrections, as given in Eq. (1.35), displays a logarithmic singularity arising from the integration over the transverse momentum. To regularise this divergence, we introduce an infrared cutoff λ, which represents the unknown dependence on low-scale dynamics[6]. Integrating over $k_\perp$ now leads to

$$\hat{\sigma}_{(1)}(\hat{p}) \;=\; \frac{\alpha_S}{2\pi}\int_{\lambda^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\int_0^1 dz\, P_{qq}(z)\,\hat{\sigma}_{(0)}(z\hat{p}) \;=\; \frac{\alpha_S}{2\pi}\,\log\frac{Q^2}{\lambda^2}\int_0^1 dz\, P_{qq}(z)\,\hat{\sigma}_{(0)}(z\hat{p}), \tag{1.37}$$

where Q defines a characteristic hard scale of the process, and P_{qq} is the regularised $q \to qg$ splitting function, given by

$$P_{qq}(z) \;=\; C_F\left(\frac{1+z^2}{1-z}\right)_+ . \tag{1.38}$$

The *plus distribution* introduced above provides a method to regularise an integral that would otherwise diverge at $z = 1$, and it is defined as

$$\int_0^1 dx\,\big[g(x)\big]_+\, f(x) \;=\; \int_0^1 dx\, g(x)\,\big(f(x) - f(1)\big)\,. \tag{1.39}$$

The complete expression for the partonic cross section can then be written as

$$\hat{\sigma}(\hat{p}) \;=\; \hat{\sigma}_{(0)}(\hat{p}) + \hat{\sigma}_{(1)}(\hat{p}) \;=\; \int_0^1 dz\,\Gamma(z,Q^2)\,\hat{\sigma}_{(0)}(z\hat{p})\,, \tag{1.40}$$

[6]Despite the introduced cutoff, there are various methods for regularisation. Dimensional regularisation, already adopted for dealing with ultraviolet divergences, can also serve as an alternative technique in this context.

with the distribution Γ defined as

$$\Gamma(z, Q^2) \;=\; \delta(1-z) + \frac{\alpha_s}{2\pi}\,\log\frac{Q^2}{\lambda^2}\,P_{qq}(z)\,. \tag{1.41}$$

In Eq. (1.40) one may notice a certain analogy with the parton model formula, suggesting that even partons have a substructure that depends on the scale at which they are probed. By substituting Eq. (1.40) back into the hadronic cross-section formula in Eq. (1.36) and using the identity $\int dx\,\delta(x-zy) = 1$, we can absorb the contribution from gluon emission into the bare parton density, essentially returning to a parton model-like formula. Specifically, one has

$$\begin{aligned}\sigma(p) &= \int_0^1 dx \int_0^1 dy \int_0^1 dz\, f_q^{(0)}(y)\,\Gamma(z,Q^2)\,\hat\sigma_{(0)}(xp)\,\delta(x-zy)\\ &= \int_0^1 dx\, f_q(x,Q^2)\,\hat\sigma_{(0)}(xp)\,,\end{aligned} \tag{1.42}$$

with

$$f_q(x,Q^2) \;=\; \int_0^1 dy \int_0^1 dz\, f_q^{(0)}(y)\,\Gamma\big(z,Q^2\big)\,\delta(x-zy)\,. \tag{1.43}$$

We can then apply a similar procedure to reabsorb the singular collinear behaviour of the partonic cross section into the universal PDF. To do this, we first isolate the divergence by introducing in the integration over the transverse momentum in Eq. (1.37) an intermediate scale μ_F, known as *factorisation scale*, thus separating the singular contributions originating from any emissions characterised by $k_\perp \lesssim \mu_F^2$ from the finite part of the radiative correction. At the integrated level, this amounts to divide the logarithm resulting in Eq. (1.41) into two parts, as

$$\log\frac{Q^2}{\lambda^2} \;=\; \log\frac{Q^2}{\mu_F^2} + \log\frac{\mu_F^2}{\lambda^2}\,, \tag{1.44}$$

and consequently rewrite Eq. (1.40) as

$$\hat\sigma(\hat p) \;=\; \int_0^1 dz\,\left[\Gamma(z,\mu_F^2) + \frac{\alpha_s}{2\pi}\,\log\frac{Q^2}{\mu_F^2}\,P_{qq}(z)\right]\hat\sigma_{(0)}(z\hat p)\,, \tag{1.45}$$

where the initial-state collinear divergence has been subtracted from the last term in the right-hand side and now embedded in $\Gamma(z,\mu_F^2)$. Therefore, by redefining the bare parton density $f_q^{(0)}$ as[7]

$$f_q(x,\mu_F^2) = \int_0^1 dy \int_0^1 dz\, f_q^{(0)}(y)\, \Gamma(z,\mu_F^2)\, \delta(x-zy)$$
$$= f_q^{(0)}(x) + \frac{\alpha_S}{2\pi} \log\frac{\mu_F^2}{\lambda^2} \int_x^1 \frac{dy}{y}\, f_q^{(0)}(y)\, P_{qq}\left(\frac{x}{y}\right), \tag{1.46}$$

the hadronic cross section can be finally written in the following factorised form:

$$\sigma(p) = \int_0^1 dx\, f_q(x,\mu_F^2)\, \hat{\sigma}(xp,\mu_F^2)\,. \tag{1.47}$$

The product of the modified parton distribution with the finite *short-distance* cross section $\hat{\sigma}(xp,\mu_F^2)$, this latter defined as

$$\hat{\sigma}(\hat{p},\mu_F^2) = \hat{\sigma}_{(0)}(\hat{p}) + \frac{\alpha_S}{2\pi} \log\frac{Q^2}{\mu_F^2} \int dz\, P_{qq}(z)\, \hat{\sigma}_{(0)}(z\hat{p})\,, \tag{1.48}$$

can be demonstrated to reproduce Eq. (1.36), up to $\mathcal{O}(\alpha_S^2)$ terms.

The generalisation of the factorisation discussed above to the context of hadron-hadron collisions results in the QCD-*improved* parton model formula, reading

$$\sigma(p_1,p_2) = \sum_{a,b} \int_0^1 dx_1 dx_2\, f_a(x_1,\mu_F^2) f_b(x_2,\mu_F^2)\, \hat{\sigma}_{ab}(x_1p_1,x_2p_2,\mu_F^2) + \mathcal{O}\big((\Lambda_{\mathrm{QCD}}/Q)^p\big)\,. \tag{1.49}$$

The main difference with the analogous formula in Eq. (1.19) lies in the explicit dependence on the factorisation scale μ_F. In this context, the modified parton distributions presented in Eq. (1.46) are consistently extended to include any kind of partonic collinear emissions, as

$$f_a(x,\mu_F^2) = f_a^{(0)}(x) + \frac{\alpha_S}{2\pi} \log\frac{\mu_F^2}{\lambda^2} \sum_c \int_x^1 \frac{dy}{y}\, f_c^{(0)}(y)\, P_{ac}\left(\frac{x}{y}\right). \tag{1.50}$$

Here, the summation involves all initial-state particles labeled with flavour c that have the potential to undergo a collinear splitting, resulting in the emergence of an incoming particle a that actively participates in the hard process, represented as $a \leftarrow c$.

The non-cancellation of initial-state collinear singularities serves as a strong example of the crucial role played by general factorisation theorems [96] in our ability to provide theoretical descriptions of collider processes. These theorems enable a clear separation between the short-distance, high-energy physics governing parton interactions and the long-distance, low-energy behaviour of QCD degrees of freedom.

[7] It is possible to reabsorb arbitrary finite contributions into the PDF definition. Different choices lead to different *factorisation schemes*. The PDFs are usually defined in the $\overline{\mathrm{MS}}$ scheme, already introduced in Sec. 1.1.2.

1.2.3 The DGLAP evolution equation

The (physical) parton distribution functions $f_a(x, \mu_F^2)$ in Eq. (1.50) are universal non-perturbative quantities whose value and x-dependence have to be extracted from experimental data. However, their additional dependence of the factorisation scale μ_F can be predicted within perturbative QCD. Starting from the fundamental requirement that fixed-order physical predictions should, up to higher-order contributions, remain independent of the arbitrary choice of the factorisation scale, we can derive a renormalisation group equation that characterises the scale evolution of the PDFs. By taking the derivative of Eq. (1.50), we obtain the following general expression

$$\begin{aligned} \mu_F^2 \frac{\partial f_a(x,\mu_F^2)}{\partial \mu_F^2} &= \frac{\alpha_S(\mu_F^2)}{2\pi} \sum_c \int_x^1 \frac{dy}{y}\, f_c(y,\mu_F^2)\, P_{ac}\left(\frac{x}{y}\right) \\ &= \frac{\alpha_S(\mu_F^2)}{2\pi} \sum_c \Big[P_{ac} \otimes f_c\Big](x,\mu_F^2)\,, \end{aligned} \tag{1.51}$$

which is known as the DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) [97–99] evolution equation. Note that the symbol $\otimes$ represents convolutions over momentum fractions, and the value of the strong coupling is calculated at the scale μ_F. Here $P_{ac}(x, \mu_F^2)$ are the regularised all-order Altarelli-Parisi kernels, which can be perturbatively expanded as

$$P_{ac}(x,\mu_F^2) = \sum_{n=0}^{\infty} \left(\frac{\alpha_S(\mu_F^2)}{2\pi}\right)^n P_{ac}^{(n)}(x)\,. \tag{1.52}$$

At leading order, they are given by[8]

$$\begin{aligned} P_{qq}^{(0)}(x) &= C_F \left(\frac{1+z^2}{1-z}\right)_+\,, \\ P_{qg}^{(0)}(x) &= T_R\Big(x^2 + (1-x)^2\Big)\,, \\ P_{gq}^{(0)}(x) &= C_F\, \frac{1+(1-x)^2}{x}\,, \\ P_{gg}^{(0)}(x) &= 2\,C_A \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \delta(1-x)\,\frac{\beta_0}{2}\,, \end{aligned} \tag{1.53}$$

with $\beta_0 = (11\,C_A - 4\,T_R\,N_f)/3$. The next-to-leading splitting functions $P_{ij}^{(1)}(x)$ have been computed in Refs. [100, 101], while comprehensive results for the next-to-next-to-leading coefficients $P_{ij}^{(2)}(x)$ can be found in Refs. [102, 103]. Recent studies are actively

[8]For a generic initial-state branching $c \to a + b$, with b being a final-state particle, the LO splitting functions listed in Appendix B.1 are defined using subscripts that indicate the particle flavours resulting from the splitting, such as ab. This is in contrast with the notation used in Eq. (1.53), where the parent particle and the particle entering the hard scattering process are identified, as ac.

contributing to expanding our understanding of four-loop splitting functions, with some partial information already available, as presented in Refs. [104–110].

1.3 Cancellation of infrared singularities at higher orders

The calculation of higher-order QCD corrections to collider observables is of utmost importance to produce theoretical predictions accurate enough to enable a meaningful comparison with the increasingly-precise data obtained in current (and future) energy-frontier programmes. These days, achieving NNLO accuracy in the strong coupling is becoming the standard requirement for fixed-order predictions for hard scattering processes at the Large Hadron Collider (LHC). To compute differential distributions beyond the leading order, it is essential to employ an efficient and automated method for handling infrared singularities. As shown in Sections 1.2.1 and 1.2.2, these singularities must either cancel between virtual corrections and the phase-space integrals of unresolved final-state radiation, or be universally factorised in the case of collisions involving hadrons in the initial state. From the theoretical viewpoint, the origins and characteristics of the infrared behaviour of perturbative QCD corrections are well understood (see Ref. [111] for a recent review). When dimensional regularisation is employed, IR singularities in virtual corrections manifest as explicit ϵ poles that are known to factorise from scattering amplitudes in terms of universal functions [112–123]. The anomalous dimensions required for this infrared factorisation are fully known up to three loops [124, 125]. Real-radiation matrix elements have also been shown to factorise in soft and collinear limits, and the corresponding real-radiation splitting kernels have been computed at order α_{s}^2 [126–131], with extensive information also available at α_{s}^3 [132–145].

Despite the existence of general theorems applicable in perturbation theory [92–96], which guarantee the cancellation (or factorisation) of infrared divergences when considering infrared-safe observables, the practical implementation of procedures to efficiently remove these singularities remains a non-trivial task. Indeed, as we explore high-multiplicity processes and consider typical collider observables, the complexity of phase-space constraints rapidly grows to a point where analytic integration becomes impractical. Consequently, the use of numerical tools is essential to handle this complexity effectively. However, since virtual and real contributions are separately divergent quantities in $d = 4$ space-time dimensions, numerical Monte Carlo techniques cannot be directly employed to handle these integrations. Hence, a strategy is necessary to explicitly extract the poles in the d-dimensional integration over the radiative degrees of freedom (once completely decoupled from the resolved phase space), cancel them against the poles originating from virtual corrections, and thereby enable the numerical computation of fully-differential predictions for infrared-safe observables.

At NLO, this problem was first approached with *phase-space slicing* methods [146, 147]. Given a differential real correction $d\sigma_{R,\,d}$ with a d-dimensional phase-space measure, the strategy can be outlined as follows:

$$\int d\sigma_{R,\,d} \;=\; \int_0^\delta \Big[d\sigma_R^{\text{approx}}\Big]_{d,\,\text{poles}} + \int_\delta^1 \Big[d\sigma_R\Big]_{4,\,\text{fin}} + \mathcal{O}(\delta)\,. \tag{1.54}$$

The basic idea consists in isolating the phase-space regions where the real emission is singular by introducing a cutoff scale δ, associated with a proper IR-sensitive observable. For these singular regions, approximate expressions of the relevant matrix elements, denoted here as $d\sigma_R^{\text{approx}}$, are introduced. The integration is then carried out analytically up to the slicing parameter, thus exposing the poles, while the leftover terms are finite quantities that can be numerically evaluated in $d = 4$ dimensions. The slicing parameter δ must be chosen sufficiently small to ensure that the final result becomes ultimately δ-independent. Although conceptually simple, this method requires careful control over the residual dependence on the slicing parameter, as it has the potential to introduce significant power corrections spoiling the convergence of the calculation.

An alternative strategy that avoids the extra dependence on a slicing parameter is the *subtraction* method. This technique involves defining local counterterms that reproduce the singular behaviour of the real matrix element across all regions of phase space. By subtracting these counterterms from the full real-radiation matrix elements and then adding back their exact integrals, one schematically obtains

$$\int d\sigma_{R,\,d} \;=\; \int \Big[d\sigma_R - d\sigma_{\text{ct}}\Big]_{4,\,\text{fin}} + \int \Big[d\sigma_{\text{ct}}\Big]_{d,\,\text{poles}}. \tag{1.55}$$

This yields a finite, numerically-integrable subtracted real contribution, identified by the first square brackets in the r.h.s., while the d-dimensional analytic integration of the counterterms exposes the implicit ϵ poles. In contrast to slicing schemes, subtraction is a local and exact procedure that is not affected by issues related to cutoffs and power corrections. On the other hand, constructing effective subtraction terms can be a challenging task: indeed, they have to be designed so as to mimic infrared singularities overlapping in the phase space, while, at the same time, their structure has to remain simple enough to allow for analytic integration.

Starting from the mid-90s, the automated cancellation of infrared singularities in NLO predictions has been achieved and generalised by two pioneering subtraction schemes, which we review in Section 1.3.1. The situation is however rather different at NNLO accuracy, where efforts to achieve the same level of universality and efficiency as was attained at NLO have been ongoing for nearly two decades. In Section 1.3.2, we provide a brief overview of the various approaches that have been proposed and pursued during this time, encompassing both slicing and subtraction methods. These algorithms cover a wide spectrum, ranging from primarily numerical to predominantly analytic tools, and they can be applied to distinct classes of processes of high phenomenological significance. Recent developments have also enabled the computation of N^3LO predictions for LHC benchmark processes. The construction of such methods, however, has proven to be extremely challenging due to a substantial increase in technical complexity, arising either at the level of the analytic integration of counterterms, or at the level of numerical implementation. As a consequence, a comprehensive solution to the infrared-subtraction problem beyond NLO remains elusive. Thus, aware of the fact that there is still room for improvement in the universality, versatility and efficiency of existing algorithms, we have developed a novel approach to the cancellation of infrared singularities called *Local Analytic Sector Subtraction*, aiming at a solution of the NNLO QCD subtraction problem for generic processes. We present the framework and the specific features of this subtraction algorithm, which will constitute the central core of this thesis work, in Section 1.3.3.

We reserve a special mention to a different category of algorithms that aim at addressing the infrared-singularity problem by combining the real and virtual corrections in an alternative way with respect to the subtraction approach presented so far. These recently-proposed techniques [52–57, 148] are based on the Loop-Tree Duality (LTD) theorem [149–152], which allows for a fully local cancellation of IR singularities directly at the integrand level. Several ongoing endeavours are dedicated to automating and extending these methods, especially towards the application to higher-order calculations (for a recent review, see Ref. [153]).

1.3.1 Subtraction schemes at NLO

In this Section we briefly present the relevant features of the three main subtraction algorithms that handle the cancellation of IR divergences at NLO accuracy in full generality.

The first process-independent algorithm to appear was the Frixione-Kunszt-Signer (FKS) subtraction [16, 17]. This method relies on the introduction of *sectors*, namely partition functions designed to disentangle the structure of overlapping singularities within the radiative phase space. This partition crucially reduces the number of infrared divergences that must be simultaneously addressed, effectively suppressing all but one collinear and one soft singularity. Sectors can be treated independently, and each of them features an adapted phase-space parametrisation. Local counterterms, identified by δ-functions and plus distributions, can be analytically integrated thanks to the use of *sum rules*, which eliminate the explicit dependence of sector functions within their definitions.

A different approach to tackle the infrared problem at NLO is the Catani-Seymour (CS) dipole subtraction [18, 19]. In this scheme, local counterterms mimic the IRC singular behaviour of the real-emission matrix element across the entire phase space, eliminating the need for partition functions. These subtraction terms are constructed as sums of universal functions, or *dipoles*, which interpolate between soft and collinear singularities. Within each dipole, the Born-level kinematics are determined through momentum mappings involving only three partons, namely the *emitted*, the *emitter* and the *spectator* (this latter referred to as *local* recoiler). While dipoles can be rather complex objects, the simple phase-space factorisation and parametrisation achieved through these mappings make the analytical integration of the local counterterms a straightforward task.

A more recent, viable alternative to the previous approaches is the Nagy-Soper subtraction [20, 21], firstly introduced in the formulation of an improved parton shower [154–156]. This method embraces the philosophy of Catani-Seymour method in constructing local counterterms, but it crucially reduces the substantial number of mappings introduced in dipole functions by a factor of n, thus improving convergence, where n represents the massless final states in a high multiplicity process. This advantage however comes at the price of more complex expressions for the subtraction terms, which require the use of semi-numerical methods for the corresponding integrations.

Nowadays, some of these methods have been developed in full generality, and versions of the corresponding algorithms have been incorporated into several fast and efficient multi-purpose NLO event generators [157–169]. These implementations provide robust solutions to the subtraction problem at the NLO accuracy level.

1.3.2 Landscape of available algorithms at NNLO (and beyond)

As we move beyond NLO, the handling of infrared singularities becomes significantly more challenging, both conceptually and practically. Calculations of perturbative corrections at NNLO (and beyond) are substantially complicated due to the proliferation of overlapping divergent regions in phase space and the inclusion of mixed real-virtual contributions. This complexity makes it evident that extending mature NLO techniques to higher orders is unfeasible without introducing new tools to address these overlaps. Nevertheless, the knowledge and experience gained at NLO have triggered the development of several innovative strategies over the last few decades, each of which characterised by its own range of applicability, degree of universality, and computational efficiency. Collectively, these strategies have paved the way for the possibility of performing calculations at NNLO and beyond.

The idea underlying the singularity-cancellation mechanism introduced by the Catani-Seymour dipole subtraction at NLO has inspired two major generalisations at NNLO. One of these is called *antenna* subtraction [23–27]. In this method, counterterms are defined using *antenna functions*, which are essentially ratios of physical spin-averaged matrix elements obtained exploiting the universal factorisation properties of colour-ordered amplitudes [170, 171], which naturally incorporate the information about the relevant infrared singular regions. These universal functions are almost local, except for angular correlations, which are removed by averaging over azimuthal angles. This non-locality is cured by increasing the number of numerical evaluations. Although antennae exhibit complex and large expressions, analytic integration for all counterterms can be achieved through integration-by-parts (IBP) identities [172, 173] and the reduction to a small set of master integrals. The antenna subtraction method is applicable to both hadronic initial and final states and is implemented in the private parton-level event generator NNLOJET [174], providing efficient predictions for jet production processes at NNLO. Ongoing studies aim at extending the construction of antenna functions to higher-order calculations [175–179]. In addition, a reformulation of the antenna subtraction approach taming former limitations while facilitating the application to high-multiplicity processes has been recently proposed [28].

The *CoLoRFulNNLO* method [29–38] is another approach rooted in the dipole subtraction philosophy. In this framework, local subtraction terms are defined using universal singular kernels, and the associated momentum mappings involve the momenta of all outgoing particles. The integration of counterterms is performed analytically for the infrared poles, while numerical methods are employed for evaluating the finite parts. This subtraction scheme has been fully worked out for processes involving hadronic final states, and has been partially extended to the treatment of initial-state radiation.

The FKS subtraction scheme has in turn stimulated the development of various algorithms that recognise the phase-space partition as an effective strategy for separating overlapping singularities, thus enabling the implementation of less intricate subtraction mechanisms. The *sector-improved residue* subtraction [39–42] is a fully numerical framework that combines a phase-space partition with the sector decomposition technique [180]. Counterterms are generated through iterated subtractions, and unlike some other schemes, it does not introduce kinematic mappings. Instead, it relies on specific phase-space parametrisations that facilitate effective numerical cancellation. Counterterm integrals are first analytically decomposed in their ϵ-expansion in dimensional regularisation, and the resulting coefficients, sector functions included, are integrated numerically. Subtraction is then made on the fly, on a process-by-process basis. The sector-improved residue subtraction applies to initial- and final-state radiation. Currently implemented in the non-public code STRIPPER, the numerical efficiency of this procedure is expected to outperform schemes based on Catani-Seymour subtraction, because of the presence of sectors.

Expanding upon the ideas of the sector-improved residue subtraction method, the *nested soft-collinear* subtraction [43] optimises the algorithm by reducing sector redundancy, resulting in a more physically-transparent approach. This method employs color coherence to iteratively extract soft singularities from the double-real process, and subsequently regulates collinear singularities by partitioning the angular phase-space into sectors. It is a fully local scheme that provides analytic expressions for integrated subtraction counterterms. As of now, its applications have been limited to processes involving only two external color-charged particles at the tree level. However, ongoing efforts are actively focusing on extending the method to processes with an arbitrary number of coloured partons, starting from the production of N-gluon final states in $q\bar{q}$ annihilation [44, 181].

While local subtraction schemes tend to outperform slicing methods in terms of precision and numerical efficiency at NLO accuracy, slicing methods regain competitiveness at NNLO. This resurgence is attributed to factors such as the enhanced understanding of the analytic structure of matrix elements and the increased availability of computational power. The q_T-subtraction method [45] is a non-local subtraction implemented as a slicing procedure that achieves the cancellation of infrared singularities in colour-singlet production. Given the transverse momentum (q_T) of the generic colour-singlet system V, the NNLO infrared divergences associated with $q_T \neq 0$ configurations correspond to the NLO singularities in the process with a final state involving V + jets. These NLO singularities can be regularised employing any available subtraction scheme. The remaining NNLO divergences due to the $q_T \to 0$ limit are treated with an additional counterterm, constructed by exploiting the universal behaviour of real emissions in the transverse-momentum distribution, which is known via resummation techniques. Neglecting the subtracted contribution below the cutoff, a residual dependence on this parameter remains in the form of power-suppressed contributions whose size determines the efficiency of the computation. Procedures have been recently developed to account for linear power corrections, thus improving numerical convergence and reducing systematic uncertainties [182, 183]. This formalism has been extended to processes with a pair of massive coloured particles [46, 184, 185], and alternative slicing variables for jet processes are being explored [186]. NNLO predictions obtained via q_T-subtraction are collected in the public framework MATRIX [187].

Another algorithm that incorporates a phase-space slicing procedure is the subtraction method based on the N-jettiness event-shape variable [47–49], which fully captures the singularity structure of QCD amplitudes for processes involving final-state jets. In this framework, soft-collinear effective theory (SCET) [188–192] is employed to derive the behaviour of the infrared singular contributions of jet cross sections in the limit as the N-jettiness parameter (τ_N) approaches zero. The N-jettiness subtraction method has been implemented, and results for various processes are available in the publicly-accessible parton-level Monte Carlo program called MCFM [193–196]. In addition to the methods briefly described in this discussion[9], a variety of additional strategies have been explored, see e.g. Refs. [22, 50, 51].

The availability of two-loop amplitudes and the technology for producing differential NNLO calculations, established by some of the methods belonging to the broad spectrum of available schemes, resulted in the successful computation of all relevant $2 \to 1$ and $2 \to 2$ processes at LHC. This achievement has pushed the frontier towards NNLO predictions

[9]For a more detailed review on IR-subtraction methods, see Ref. [197].

for $2 \to 3$ collider processes, for which many results started to appear [58–67]. Some of the previously cited algorithms have also been extended and applied to the calculation of a selection of simple $2 \to 1$ benchmark processes at N^{3}LO accuracy, involving only coloured singlets in the final state. We refer the reader to Refs. [68, 69] for thorough reports on state-of-the-art QCD predictions. It is worth emphasising that as of today, obtaining generic NNLO calculations is still not a straightforward task. In fact, these latter are typically computationally demanding and can be performed either using private codes, or with a limited number of publicly-accessible computer programmes as Refs. [187, 195], which usually offer predefined sets of available processes. Recently, there have been developments in the form of the HighTEA web platform [198], which aims to provide easy access to the analysis of NNLO predictions for a specific collection of precomputed events, obtained using Monte Carlo methods at a fixed level of statistical precision.

1.3.3 Local Analytic Sector Subtraction: the framework

The vast number of approaches proposed to address the local cancellation of infrared singularities at NNLO, some of which were briefly introduced in Section 1.3.2 and reviewed in Refs. [68, 197], provides a clear picture of the intricacy of the problem at hand. The considerable complexity arising in the development of such techniques explains why research groups are still actively working on possible optimisations to streamline the structure and/or broaden the applicability of existing schemes. The absence of a completely comprehensive and satisfactory solution to the NNLO problem, especially when compared to the well-established NLO strategies, fuels further investigation into this area. Specifically, the trade-off between the complete locality of subtraction terms and their analytical integrability, which appears to be a defining characteristic of existing local methods, serves as a promising starting point for a fundamental re-examination of the subtraction mechanism.

The in-depth analysis of such a criticality has resulted in the development of a novel subtraction scheme, which ambitiously aims to lay the foundations for a fully general analytic solution to the cancellation of NNLO infrared singularities. We call this framework *Local Analytic Sector Subtraction* [1, 2, 199]. The fundamental idea driving this method is to address the NNLO infrared problem by making full use of the freedom available to define local counterterms, in order to ultimately identify the simplest possible structure that optimises the process of subtracting infrared singularities at every step of the calculation. In practical terms, to achieve this simplicity we construct local counterterms by drawing upon key concepts that have been successfully applied at NLO accuracy. A fundamental ingredient of our method is the implementation of a phase-space partition, in the spirit of FKS subtraction [16]. Each sector within this partition is carefully designed to isolate a minimal subset of soft and collinear singularities, allowing in turn for the construction of a minimal local counterterm that has to reproduce the behaviour of the real-radiation amplitude squared only in those sector-relevant singular limits. These sector functions must further obey a set of *sum rules*, which facilitate the recombination of different sectors and their removal from local counterterms. As a result, this simplifies the subsequent process of analytical integration. Another crucial ingredient of our scheme is a flexible family of momentum mappings, akin to those introduced by Catani-Seymour subtraction [18]. These kinematic mappings can be tailored to the specific counterterms or even be adapted to different contributions within a given counterterm. The obtained phase-space factorisation and corresponding parametrisation, which adjust to the various singular kernels, significantly enhance the simplicity of the required analytical integrations [200]. The final outcome of such a programme is a completely general and analytic formula that can be implemented within any existing numerical framework without any additional work, thus enabling the production of NNLO phenomenological results. Its applicability to multi-particle processes will be primarily constrained by computational resources and the availability of multi-loop matrix elements (see for instance Ref. [201]). The central core of this thesis will be dedicated to implementing this strategy in practice, specifically providing a comprehensive step-by-step explanation of the process that culminates in the construction of a fully analytic subtraction formula for the treatment of infrared singularities at NNLO accuracy.

The implementation of this method is accompanied by a second line of investigation, which delves into more formal aspects of subtraction, specifically focusing on factorisation [202, 203]. This line of research aims to explore the connection between the structure of local counterterms for real radiation and the structural simplification resulting from the factorisation of gauge-theory amplitudes, with the hope of improving the construction of minimal and process-independent counterterms, with a specific emphasis on the organisation of strongly-ordered singular limits, which become relevant for the first time at NNLO.

Chapter 2

Local Analytic Sector Subtraction at NLO

In the mind of the reader who learned in Section 1.3.1 about the existence of well-established algorithms which have efficiently removed next-to-leading-order infrared singularities since the '90s, the necessity and practical significance of developing another subtraction scheme at this perturbative order may be questioned. In our case, testing the strategy proposed in Section 1.3.3 by implementing a general fully-fledged subtraction in a simpler NLO playground represents an instrumental step towards the construction of a universal cancellation procedure for the more involved and demanding NNLO scenario. Indeed, this offers a valuable opportunity to address potential criticalities, fine-tune the methodology, and assess the numerical performance of our scheme, away from higher-order complexity.

We therefore present in this Chapter the details of the construction of a general analytic formula for the cancellation of NLO infrared singularities, developed within the framework of Local Analytic Sector Subtraction. This formulation applies to processes featuring initial- and final-state massless QCD radiation, thus covering all kinds of particle colliders.

The outline is as follows. We provide an overview of the architecture of our method in Section 2.1, introducing the relevant notations and the ingredients required for an infrared subtraction at NLO, for massless initial and final states. We devote Section 2.2 to the detailed construction of our *local* counterterm K: we put at work the strategy outlined in Section 1.3.3, analysing its pros and cons; we then turn to the validation of our approach by testing the locality of the subtraction, while implementing various optimisations. In Section 2.3, we perform the integration of the designed counterterm over the single unresolved radiation parametrised according to tailored momentum mappings, and recast the outcomes into Born-level kinematic quantities. Corresponding results are collected in Appendix B.3. Finally, Section 2.4 shows the explicit cancellation of virtual and collinear-factorisation poles. We then present the finite remainders of our computations, which result in a very compact and completely analytic NLO formula.

2.1 Generalities

Let us consider a generic scattering reaction that at Born level features n massless coloured partons (as well as an arbitrary number of massless or massive colourless particles) in the final state, with up to two massless coloured partons in the initial state. We denote with $\mathcal{A}_n(k_i)$ the relevant scattering amplitude, which can be expanded in perturbation theory as

$$\mathcal{A}_n(k_i) \; = \; \mathcal{A}_n^{(0)}(k_i) \; + \; \mathcal{A}_n^{(1)}(k_i) \; + \; \mathcal{A}_n^{(2)}(k_i) \; + \; \dots \, , \tag{2.1}$$

where $\mathcal{A}_n^{(k)}$ denotes the k-loop correction, and includes the appropriate power of the strong coupling constant. In this notation, we define $i = 1, \dots, n$ for final-state particles, and $i = a, b$ for initial-state particles. For such a process, we consider a generic IRC-safe observable X, and we write the corresponding differential distribution as

$$\frac{d\sigma}{dX} \; = \; \frac{d\sigma_{\rm LO}}{dX} \; + \; \frac{d\sigma_{\rm NLO}}{dX} \; + \; \frac{d\sigma_{\rm NNLO}}{dX} \; + \; \dots \, . \tag{2.2}$$

The formulation of the Born contribution in terms of Eq. (2.1) allows to express the LO coefficient of Eq. (2.2) as

$$\frac{d\sigma_{\rm LO}}{dX} \; = \; \int d\Phi_n \, B \, \delta_n(X) \, , \qquad \text{with} \quad B \; = \; \big| \mathcal{A}_n^{(0)} \big|^2 \, . \tag{2.3}$$

By also specifying the real emission, and ($\overline{\rm MS}$-renormalised) virtual contributions, as

$$R \; = \; \Big| \mathcal{A}_{n+1}^{(0)} \Big|^2 \, , \qquad V \; = \; 2 \, \mathbf{Re} \left[\mathcal{A}_n^{(0)*} \, \mathcal{A}_n^{(1)} \right] \, , \tag{2.4}$$

we write the standard expression for the NLO term as the combination

$$\frac{d\sigma_{\rm NLO}}{dX} \; = \; \lim_{d \to 4} \left[\int d\Phi_n \, V \, \delta_n(X) + \int d\Phi_{n+1} \, R \, \delta_{n+1}(X) + \int d\Phi_n^{x\hat{x}} \, C(x, \hat{x}) \, \delta_n(X) \right] , \tag{2.5}$$

where $\delta_m(X) \equiv \delta(X - X_m)$, X_m standing for the observable computed with m-body kinematics, $d\Phi_m = d\Phi_m(k_a, k_b)$ denotes the Lorentz-invariant phase-space measure for m massless final-state particles, including suitable polarisation sums/averages and flux factors; the convolution phase space $d\Phi_n^{x\hat{x}}$, defined as

$$\int d\Phi_n^{x\hat{x}} \; \equiv \; \int_0^1 \frac{dx}{x} \int_0^1 \frac{d\hat{x}}{\hat{x}} \int d\Phi_n(x k_a, \hat{x} k_b) \, , \tag{2.6}$$

shows a dependence on rescaled initial-state partonic momenta $x k_a$ and $\hat{x} k_b$, where $0 \leqslant x, \hat{x} \leqslant 1$. The PDF collinear counterterm $C(x, \hat{x})$, encoding the full μ_F dependence of the partonic cross section, is defined in $\overline{\text{MS}}$ scheme as

$$C(x,\hat{x}) \;=\; \frac{\alpha_{\rm S}}{2\pi}\,\frac{1}{\epsilon}\,\frac{(e^{\gamma_E})^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon}\Big[\bar{P}_a(x)\,\delta(1-\hat{x})+\bar{P}_b(\hat{x})\,\delta(1-x)\Big]\,B(xk_a,\hat{x}k_b)\,, \quad (2.7)$$

where $\bar{P}_i(x)$ represent the lowest-order four-dimensional regularised Altarelli-Parisi splitting kernels (for their explicit expressions, see Appendix B.1).

While the finiteness of the comprehensive NLO correction in Eq. (2.5) is ensured by the KLN theorem [93, 94] supplemented with PDF renormalisation, as well as by the IRC-safety of X, the n-body and $(n+1)$-body contributions are manifestly divergent when considered separately, thus preventing a straightforward numerical evaluation. In dimensional regularisation, where amplitudes are evaluated in $d = 4-2\epsilon$ space-time dimensions, such divergences appear at NLO as double and single $1/\epsilon$ poles in the expression of V. On the other hand, the real contribution R, which is finite for $\epsilon \to 0$, features IRC phase-space singularities which translate into double and single $1/\epsilon$ poles upon integration over the radiative phase space.

The procedure of infrared subtraction enables the explicit cancellation of such poles, while avoiding analytic integration of the full real-radiation amplitudes. This is accomplished by adding and subtracting to Eq. (2.5) a counterterm cross section

$$\begin{aligned}\frac{d\sigma_{\rm NLO}}{dX}\bigg|_{\rm ct} &\equiv \int d\Phi_{n+1}\,K\,\delta_n(X)\\ &\equiv \int d\Phi_n\,I\,\delta_n(X)+\int d\Phi_n^{x\hat{x}}\,J(x,\hat{x})\,\delta_n(X)\,. \end{aligned} \quad (2.8)$$

The *local* counterterm K is designed so as to reproduce the singular IR behaviour of the real-radiation matrix element R *locally* in phase space, and at the same time, it is expected to be simple enough to be analytically integrated in the phase space of the unresolved radiation (once a parametrisation of the radiative phase space is in place). The outcome of this integration can be recast into the sum of an $(x,\hat{x})$-independent contribution I and an $(x,\hat{x})$-dependent contribution J, which exhibit the same ϵ - pole content (with opposite signs) as V and $C(x,\hat{x})$, respectively. It is now possible to rewrite Eq. (2.5) identically as

$$\frac{d\sigma_{\rm NLO}}{dX} \;=\; \int d\Phi_n V_{\rm sub}(X)\,+\int d\Phi_{n+1}\,R_{\rm sub}(X)+\int d\Phi_n^{x\hat{x}}\,C_{\rm sub}(X)\,, \quad (2.9)$$

with

$$V_{\rm sub}(X) \;=\; \big(V+I\big)\,\delta_n(X)\,, \quad (2.10)$$

$$R_{\rm sub}(X) \;=\; R\,\delta_{n+1}(X)-K\,\delta_n(X)\,, \quad (2.11)$$

$$C_{\rm sub}(X) \;=\; C(x,\hat{x})\,\delta_n(X)+J(x,\hat{x})\,\delta_n(X)\,. \quad (2.12)$$

Each contribution in Eqs. (2.10)-(2.12) is now separately finite in $d = 4$ dimensions, and therefore well-suited for numerical evaluations of the corresponding phase-space integrals. In particular, the subtracted real matrix element $R_{\rm sub}(X)$ is free from phase-space singularities by construction, $V_{\rm sub}(X)$ is finite as $\epsilon \to 0$ as a consequence of the KLN theorem, and $C_{\rm sub}(X)$ contains just $(x, \hat{x})$-dependent non-singular remainders. Notice once again that the IRC safety of the observable X is necessary for the cancellation, which requires that $\delta_{n+1}(X)$ turns smoothly into $\delta_n(X)$ in all unresolved limits.

It is worth emphasising that the points discussed above equally apply to lepton-hadron collisions, up to the formal substitutions

$$\int d\Phi_n^{x\hat{x}} \;\to\; \int d\Phi_n^{x} \;\equiv\; \int_0^1 \frac{dx}{x} \int d\Phi_n(xk_a)\,,$$
$$C(x,\hat{x}) \;\to\; C(x) \;\equiv\; \frac{\alpha_{\rm S}}{2\pi}\,\frac{1}{\epsilon}\,\frac{(e^{\gamma_E})^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon} \bar{P}_a(x)\,B(xk_a)\,, \tag{2.13}$$

which, as a consequence, necessitate the definition of a single-argument counterterm $J(x)$ instead of $J(x, \hat{x})$. In lepton-lepton collisions, the structure further simplifies because of the absence of x-variable dependencies (the collinear term of Eq. (2.5) vanishes), and the integration of the counterterm K reduces to

$$\int d\Phi_{n+1}\, K\, \delta_n(X) \;=\; \int d\Phi_n\, I\, \delta_n(X)\,. \tag{2.14}$$

2.2 The subtracted real contribution $R_{\rm sub}$

In this Section we approach the step-by-step construction of the *local* counterterm K within the context of Local Analytic Sector Subtraction, such as to enable the definition of an integrable real correction $R_{\rm sub}$. As outlined in Section 1.3.3, the strategy we propose is based on the introduction of a unitary phase-space partition (Sec. 2.2.1), which helps to disentangle the overlapping structure of singularities, and consequently to sketch a first minimal *candidate* counterterm (Sec. 2.2.2). However, this formulation suffers from the presence of Born-level matrix elements potentially evaluated with unphysical n-body kinematics. This issue can be solved through appropriate phase-space mappings (Sec. 2.2.3). At this stage, we are finally able to design a proper *local* counterterm (Sec. 2.2.4), and thanks to a detailed analysis testing the actual locality of the subtraction, we can claim the construction of a real-radiation correction $R_{\rm sub}$ which is finite and integrable in the whole phase space. Lastly, we explore two different types of optimisations for the *local* counterterm K, with the aim of laying the foundations for an efficient implementation of our subtraction algorithm in a numerical framework. The first optimisation introduces a *symmetrised* phase-space partition (Sec. 2.2.5), reducing the overall number of sectors and limits that need to be evaluated in a NLO computation. The second suggestion modifies singular kernels with *damping factors* (Sec. 2.2.6), such as to constrain the counterterm contribution away from the singular phase-space regions, thus preventing potential instabilities in the cancellation between R and K.

2.2.1 Sector functions

First, we define *projection operators* $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ that extract from the real-radiation squared matrix element R its singular behaviour in soft and collinear limits. In practice, it is necessary to pick specific phase-space variables in order to perform the projection, and we opt for Lorentz-invariant quantities. More precisely, we introduce the dimensionless variables

$$e_i \equiv \frac{s_{qi}}{s}\,, \qquad w_{ij} \equiv \frac{s\, s_{ij}}{s_{qi}s_{qj}} = \frac{1-\cos\theta_{ij}}{2}\,, \tag{2.15}$$

associated with the energy of the parton i and the angle between i and j particles in the centre-of-mass frame, respectively; moreover, s is the centre-of-mass energy with partonic centre-of-mass four momentum $q^\mu = (\sqrt{s}, \vec{0}\,)$, $s_{ij} = 2k_i \cdot k_j$ and $s_{q\ell} = 2q \cdot k_\ell$. We proceed by defining $\mathbf{S}_i$ as extracting the leading power in e_i, and $\mathbf{C}_{ij} = \mathbf{C}_{ji}$ as extracting the leading power in w_{ij}. It is straightforward to verify that, with this definition, the two operators commute when acting on the squared matrix element, $\mathbf{S}_i\,\mathbf{C}_{ij}\,R = \mathbf{C}_{ij}\,\mathbf{S}_i\,R$.

Local Analytic Sector Subtraction builds upon the well known idea [16, 17] of dividing the radiative phase space into regions, each of which tied to the IRC singularities stemming from an identified set of partons (two at NLO). This can be achieved by introducing a unitary phase-space partition,

$$\sum_i \sum_{j\neq i} \mathcal{W}_{ij} = 1\,, \tag{2.16}$$

defined by *sector functions*, $\mathcal{W}_{ij}$, namely a set of kinematical weights smoothly dampening all radiative singularities but those due to particle i becoming soft, or becoming collinear to a second particle j, as

$$\mathbf{S}_i\,\mathcal{W}_{ab} = 0\,, \qquad \forall\, i \neq a\,, \tag{2.17}$$

$$\mathbf{C}_{ij}\,\mathcal{W}_{ab} = 0\,, \qquad \forall\, ab \notin \{ij,\, ji\}\,. \tag{2.18}$$

We formulate our sector functions in terms of Lorentz invariants. Specifically, we define[1]

$$\mathcal{W}_{ij} \equiv \frac{\sigma_{ij}}{\sum_{k\neq l}\sigma_{kl}}\,, \qquad \sigma_{ij} \equiv \frac{\theta_{i\in\mathrm{F}}}{e_i w_{ij}}\,, \tag{2.19}$$

[1]Eq. (2.19) generalises the definition of sector functions firstly proposed in [199] to processes involving partonic initial states.

where the symbol $\theta_{\mathcal{C}}$ is 1 or 0 if condition $\mathcal{C}$ is or is not fulfilled, so that $\theta_{a\in \mathrm{F}}$ ($\theta_{a\in \mathrm{I}}$) enforces parton a to belong to the final (initial) state. These sector functions have the further defining property of satisfying the following *sum rules*, namely

$$\mathbf{S}_i \sum_{k\neq i} \mathcal{W}_{ik} \; = \; \theta_{i\in \mathrm{F}} \, , \qquad \mathbf{C}_{ij} \big(\mathcal{W}_{ij} + \mathcal{W}_{ji} \big) \; = \; 1 - \theta_{i\in \mathrm{I}} \, \theta_{j\in \mathrm{I}} \, , \qquad \mathbf{S}_i \, \mathbf{C}_{ij} \, \mathcal{W}_{ij} \; = \; \theta_{i\in \mathrm{F}} \, , \tag{2.20}$$

which express that the sum over all sectors that share a given soft or collinear singularity still forms a partition of unity. Eq. (2.20) not only guarantees that, upon summing over sectors, the full soft and collinear singularities will be recovered, but it also allows to eliminate sector functions upon suitable combination of particle labels, which will prove crucial in view of analytic counterterm integration.

2.2.2 Candidate local counterterm

Considering now one partition at a time, we can readily identify a combination which is by construction integrable in the radiative phase space: this is achieved by collecting the singular limits which are relevant within each real contribution $R\,\mathcal{W}_{ij}$, and subsequently subtracting them from it. Indeed, in sector (ij)

$$(1 - \mathbf{S}_i) \, (1 - \mathbf{C}_{ij}) \; R \, \mathcal{W}_{ij} \; = \; R \, \mathcal{W}_{ij} - \mathbf{L}^{(\mathbf{1})}_{ij} \, R \, \mathcal{W}_{ij} \; \to \; \text{integrable} \, , \tag{2.21}$$

where we defined the set of projectors

$$\mathbf{L}^{(\mathbf{1})}_{ij} \equiv \mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \, \mathbf{C}_{ij} \tag{2.22}$$

as the incoherent sum of soft and collinear limits, corrected by the $-\,\mathbf{S}_i \, \mathbf{C}_{ij}$ term which removes the double-counting of soft-collinear configurations. We stress here that the operators $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ are defined to act on all elements located to their right: therefore, when denoting a generic singular limit as $\mathbf{L}$, the relation $\mathbf{L} \, R \, \mathcal{W}_{ij} \; \equiv \; (\mathbf{L} \, R) \, (\mathbf{L} \, \mathcal{W}_{ij})$ is understood. Performing the sum over sectors, we get to the expression for our *candidate* local counterterm, namely

$$\sum_i \sum_{j\neq i} \mathbf{L}^{(\mathbf{1})}_{ij} R \, \mathcal{W}_{ij} \; \equiv \; \sum_i \sum_{j\neq i} \Big[\mathbf{S}_i + \mathbf{C}_{ij} \big(1 - \mathbf{S}_i \big) \Big] R \, \mathcal{W}_{ij} \, , \tag{2.23}$$

which satisfies the requirement of reproducing the singular behaviour of the real matrix element in all soft and collinear regions of phase space. In particular, when such soft and collinear projection operators act on sector functions as defined in Eq. (2.19), one obtains

$$\begin{aligned} \mathbf{S}_i\,\mathcal{W}_{ij} &= \theta_{i\in\mathrm{F}}\,\frac{1/w_{ij}}{\sum\limits_{l\neq i} 1/w_{il}}\,, \\ \mathbf{C}_{ij}\,\mathcal{W}_{ij} &= \theta_{i\in\mathrm{F}}\Big(\theta_{j\in\mathrm{F}}\frac{e_j}{e_i+e_j}+\theta_{j\in\mathrm{I}}\Big)\,, \\ \mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij} &= \theta_{i\in\mathrm{F}}\,, \end{aligned} \tag{2.24}$$

which clearly strictly depend on the specific definition introduced for $\mathcal{W}_{ij}$. Conversely, the singular parts of the QCD matrix elements for real emission can be singled out in a general way by using the factorisation properties of soft and collinear radiation. Despite the well-known structure of such universal process-independent singular kernels, we report below the soft and collinear factorised expression for R written in terms of Lorentz invariants and in a manifestly flavour-symmetric notation, that we will later adopt for defining our NLO counterterm.

Soft limit

The real matrix element squared can be expressed in the soft limit $\mathbf{S}_i$ as

$$\mathbf{S}_i\,R = -\,\mathcal{N}_1\sum_{c\neq i}\sum_{d\neq i,c}\,\mathcal{E}^{(i)}_{cd}\,B_{cd}\big(\{k\}_{\not i}\big)\,, \tag{2.25}$$

where the *eikonal* kernel,

$$\mathcal{E}^{(i)}_{cd} \;=\; \theta_{i\in\mathrm{F}}\,\delta_{f_i g}\,\frac{s_{cd}}{s_{ic}\,s_{id}}\,, \tag{2.26}$$

is non-vanishing only if the final-state parton i, with flavour f_i, is a gluon. Note that no constraints on particle (initial or final) position have been placed on the sums running over c and d in Eq. (2.25). The soft kinematics $\{k\}_{\not i}$ represents the set of real-radiation momenta after removal of soft momentum k_i. The colour-correlated Born matrix element is schematically defined as

$$B_{cd} \;=\; \mathcal{A}^{(0)*}_n\,(\mathbf{T}_c\cdot\mathbf{T}_d)\,\mathcal{A}^{(0)}_n\,, \tag{2.27}$$

where $\mathcal{A}_n$ is understood as a ket in colour space [18] which undergoes non-trivial transformations under the action of the SU(N_c) generators $\mathbf{T}_a$. Lastly, the coefficient $\mathcal{N}_1$ reads

$$\mathcal{N}_1 \;=\; 8\pi\alpha_{\mathrm{S}}\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon}. \tag{2.28}$$

Collinear limit

The kinematics of a pair of final-state particles i and j subjected to a collinear splitting $[ij] \to i + j$ in the $\mathbf{C}_{ij}$ limit can be described with the introduction of a Sudakov parametrisation of momenta, as

$$k_i^\mu \;=\; z_i\,\bar{k}_{[ij]}^\mu + \widetilde{k}_{\rm F}^\mu - \frac{1}{z_i}\frac{\widetilde{k}_{\rm F}^2}{s_{[ij]r}}k_r^\mu\,, \qquad k_j^\mu \;=\; z_j\,\bar{k}_{[ij]}^\mu - \widetilde{k}_{\rm F}^\mu - \frac{1}{z_j}\frac{\widetilde{k}_{\rm F}^2}{s_{[ij]r}}k_r^\mu\,, \tag{2.29}$$

$$k_{[ij]}^\mu \equiv k_i^\mu + k_j^\mu\,, \qquad s_{[ij]r} \equiv s_{ir} + s_{jr}\,, \qquad \bar{k}_{[ij]}^\mu = k_{[ij]}^\mu - \frac{s_{ij}}{s_{[ij]r}}\,k_r^\mu\,, \quad r = r_{ij}\,,$$

where massless vector $\bar{k}_{[ij]}^\mu$ identifies the collinear direction, while k_r^μ represents a light-like reference vector whose prescription $r = r_{ij}$ enforces r to be any particle different from i, j, chosen according to the rule defined in Eq. (A.13) (in this case it means that the same r must be chosen for the pair ij and for the pair ji). The z_i variable represents the longitudinal momentum fraction of the k_i momenta, and $\widetilde{k}_{\rm F}^\mu$ the transverse momentum of parton i with respect to the collinear direction, which respectively read

$$z_i \;=\; \frac{s_{ir}}{s_{[ij]r}}\,, \qquad \widetilde{k}_{\rm F}^\mu \;=\; k_i^\mu - z_i\,k_{[ij]}^\mu - (1-2z_i)\frac{s_{ij}}{s_{[ij]r}}k_r^\mu\,, \tag{2.30}$$

satisfying the conditions $z_i + z_j = 1$, $\widetilde{k}_{\rm F}\cdot\bar{k}_{[ij]} = \widetilde{k}_{\rm F}\cdot k_r = 0$. In the alternative case in which the collinear configuration involves an outgoing parton i and an incoming parton j, described with the splitting $j \to [ij] + i$, the final-state momentum k_i^μ is parametrised in terms of its transverse momentum $\widetilde{k}_{\rm I}^\mu$ and the longitudinal momentum fraction x_i, as

$$k_i^\mu \;=\; x_i\;k_j^\mu + \widetilde{k}_{\rm I}^\mu - \frac{1}{x_i}\frac{\widetilde{k}_{\rm I}^2}{s_{jr}}k_r^\mu\,, \tag{2.31}$$

where

$$x_i \;=\; \frac{s_{ir}}{s_{jr}}\,, \qquad \widetilde{k}_{\rm I}^\mu \;=\; k_i^\mu - x_i\,k_j^\mu - \frac{s_{ij}}{s_{jr}}\,k_r^\mu\,, \tag{2.32}$$

satisfying $x_{[ij]} + x_i = 1$, $\widetilde{k}_{\rm I}\cdot k_r = \widetilde{k}_{\rm I}\cdot k_j = 0$. The collinear direction is identified by

$$\bar{k}_{[ij]}^\mu \;=\; x_{[ij]}\,k_j^\mu - \widetilde{k}_{\rm I}^\mu - \frac{1}{x_{[ij]}}\frac{\widetilde{k}_{\rm I}^2}{s_{jr}}k_r^\mu\,. \tag{2.33}$$

The universal un-regularised (d-dimensional) Altarelli-Parisi splitting kernels [97–99] are matrices in spin space which encode the collinear behaviour of R, and can be compactly written as

$$P^{\mu\nu}_{ab(r),\star}(\xi) \,=\, P_{ab(r)}(\xi)\left(-g^{\mu\nu}\right) + Q_{ab(r),\star}(\xi)\left[-g^{\mu\nu} + (d-2)\,\frac{\widetilde{k}^{\mu}_{\star}\,\widetilde{k}^{\nu}_{\star}}{\widetilde{k}^{2}_{\star}}\right], \tag{2.34}$$

where ξ represents the longitudinal momentum fraction of splitting parton a, and the dependence on $\star = \mathrm{I,F}$ will be specified in a moment[2]. We also make the dependence of such kernels on the reference vector r explicit in the subscript $ab(r)$, as it enters the definition of the longitudinal fraction ξ (see Eqs. (2.30) and (2.32)). In a manifestly flavour-symmetric notation, the spin- averaged components $P_{ab(r)}(\xi)$ of Eq. (2.34) read

$$\begin{aligned} P_{ab(r)}(\xi) = {} & \delta_{f_a g}\delta_{f_b g}\, 2\, C_A \left[\frac{\xi}{1-\xi} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right] + \delta_{\{f_a f_b\}\{q\bar{q}\}}\, T_R \left[1 - \frac{2\,\xi(1-\xi)}{1-\epsilon}\right] \\ & + \delta_{f_a\{q,\bar{q}\}}\delta_{f_b g}\, C_F \left[2\frac{\xi}{1-\xi} + (1-\epsilon)(1-\xi)\right] + \delta_{f_a g}\delta_{f_b\{q,\bar{q}\}}\, C_F \left[2\frac{1-\xi}{\xi} + (1-\epsilon)\,\xi\right], \end{aligned} \tag{2.35}$$

where we introduced flavour delta functions as

$$\delta_{f_a\{q,\bar{q}\}} \equiv \delta_{f_a q} + \delta_{f_a \bar{q}}\,, \qquad \delta_{\{f_a f_b\}\{q\bar{q}\}} \equiv \delta_{f_a q}\,\delta_{f_b \bar{q}} + \delta_{f_a \bar{q}}\,\delta_{f_b q}\,, \tag{2.36}$$

which, respectively, specify the cases in which a particle a is a quark or anti-quark, or the particles ab are a quark/anti-quark pair of the same flavour. According to QCD helicity conservation, the collinear azimuthal kernels $Q_{ab(r),\star}(\xi)$ are non-zero only when the virtual parton participating in the splitting is a gluon: the expression for $Q_{ab(r),\star}(\xi)$ thus depends on whether the virtual gluon is the outgoing splitting parent ($\star = \mathrm{F}$),

$$Q_{ab(r),\mathrm{F}}(\xi) \,=\, -\,\delta_{f_a g}\delta_{f_b g}\, 2\, C_A\, \xi(1-\xi) + \delta_{\{f_a f_b\}\{q\bar{q}\}}\, T_R\, \frac{2\,\xi(1-\xi)}{1-\epsilon}\,, \tag{2.37}$$

or the incoming splitting sibling ($\star = \mathrm{I}$),

$$Q_{ab(r),\mathrm{I}}(\xi) \,=\, -\,\delta_{f_a g}\delta_{f_b g}\, 2\, C_A\, \frac{1-\xi}{\xi} - \delta_{f_a g}\delta_{f_b\{q,\bar{q}\}}\, 2\, C_F\, \frac{1-\xi}{\xi}\,. \tag{2.38}$$

This notation is indeed reminiscent of the fact that at NLO the two cases are relevant to final- and initial-state splittings, respectively.

[2] Note that the formulation of the Altarelli-Parisi kernels in Eq. (2.34) is analogous to the one reported in Eq. (C.8) when restricted to final-state particle splittings.

In terms of such kernels, the collinear $\mathbf{C}_{ij}$ limit of the real matrix element can be written as

$$\mathbf{C}_{ij}\, R = \frac{\mathcal{N}_1}{s_{ij}} \Bigg[\theta_{i\in\mathrm{F}}\, \theta_{j\in\mathrm{F}}\, P^{\mu\nu}_{ij(r),\mathrm{F}}(z_i)\, B_{\mu\nu}\big(\{k\}_{\not i \not j}, k_{[ij]}\big) + \theta_{i\in\mathrm{F}}\, \theta_{j\in\mathrm{I}}\, \frac{P^{\mu\nu}_{[ij]i(r),\mathrm{I}}(x_{[ij]})}{x_{[ij]}}\, B_{\mu\nu}\big(\{k\}_{\not i \not j}, x_{[ij]} k_j\big) + \theta_{j\in\mathrm{F}}\, \theta_{i\in\mathrm{I}}\, \frac{P^{\mu\nu}_{[ji]j(r),\mathrm{I}}(x_{[ji]})}{x_{[ji]}}\, B_{\mu\nu}\big(\{k\}_{\not i \not j}, x_{[ji]} k_i\big) \Bigg], \tag{2.39}$$

where $B_{\mu\nu}$ is the spin-correlated Born amplitude obtained by stripping the gluon polarisation vectors from the matrix element and from its complex conjugate, while $(\{k\}_{\not a \not b}, k_c)$ is the real-radiative kinematics with k_a and k_b removed and replaced by k_c. Note that the factorised structure in Eq. (2.39) shows an overall symmetry in $i \leftrightarrow j$ index exchange. In particular, the first two lines of Eq. (2.39) can be pictorially represented in the left and right panels of Figure 2.1, respectively, while the third line is obtained from the second upon $i \leftrightarrow j$ exchange.

Figure 2.1. Final-state (left) splitting and initial-state (right) splitting.

Soft-collinear limit

In the soft-collinear $\mathbf{S}_i\,\mathbf{C}_{ij}$ limit, an outgoing gluon i exhibits both soft and collinear behaviour with respect to an initial- or final-state parton j. The corresponding kernel is

$$\mathbf{S}_i\,\mathbf{C}_{ij}\, R \;=\; \mathbf{C}_{ij}\,\mathbf{S}_i\, R \;=\; \mathcal{N}_1\, 2\, C_{f_j}\, \mathcal{E}^{(i)}_{jr}\, B(\{k\}_{\not i})\,, \tag{2.40}$$

where $C_{f_j} = C_A\, \delta_{f_j g} + C_F\, \delta_{f_j \{q,\bar{q}\}}$ is the $\mathrm{SU}(N_c)$ Casimir operator associated to flavour f_j; moreover, in this case $f_j = f_{[ij]}$ since the δ_{f_i} function within the eikonal kernel (2.26) forces i to be a gluon.

For later convenience, we define the hard-collinear version of the Altarelli-Parisi kernels in Eq. (2.35), which are obtained by removing their respective soft limits: for a final-state splitting, both collinear siblings i and j can induce a soft singularity, thus

$$\begin{aligned} P^{\rm hc}_{ij(r),{\rm F}}(z_i) &\equiv \big(1 - \mathbf{S}_i - \mathbf{S}_j\big)\, P_{ij}(z_i) \\ &= \delta_{f_i g}\delta_{f_j g}\, 2\, C_A\, z_i\, z_j + \delta_{\{f_i f_j\}\{q\bar q\}}\, T_R \left(1 - \frac{2\, z_i z_j}{1-\epsilon}\right) \\ &\quad + \delta_{f_i\{q,\bar q\}}\delta_{f_j g}\, C_F (1-\epsilon)\, z_j + \delta_{f_i g}\delta_{f_j\{q,\bar q\}}\, C_F (1-\epsilon)\, z_i\,, \end{aligned} \tag{2.41}$$

while, in an initial-state splitting, just the outgoing sibling i can potentially be soft, so

$$\begin{aligned} P^{\rm hc}_{[ij]i(r),{\rm I}}(x_{[ij]}) &\equiv x_{[ij]}\big(1 - \mathbf{S}_i\big)\, \frac{P_{[ij]i(r)}(x_{[ij]})}{x_{[ij]}} \\ &= \delta_{f_{[ij]}g}\delta_{f_i g}\, 2 C_A \left[\frac{x_i}{x_{[ij]}} + x_{[ij]} x_i\right] + \delta_{\{f_{[ij]} f_i\}\{q\bar q\}}\, T_R \left[1 - \frac{2\, x_{[ij]} x_i}{1-\epsilon}\right] \\ &\quad + \delta_{f_{[ij]}\{q,\bar q\}}\delta_{f_i g}\, C_F (1-\epsilon)\, x_i + \delta_{f_{[ij]}g}\delta_{f_i\{q,\bar q\}}\, C_F \left[2\frac{x_i}{x_{[ij]}} + (1-\epsilon)\, x_{[ij]}\right]. \end{aligned} \tag{2.42}$$

In analogy with Eq. (2.34), we introduce the compact notation[3]

$$P^{\mu\nu,\,{\rm hc}}_{ab(r),\star}(\xi) = P^{\rm hc}_{ab(r),\star}(\xi)\,\big(-g^{\mu\nu}\big) + Q_{ab(r),\star}(\xi) \left[-g^{\mu\nu} + (d-2)\,\frac{\widetilde{k}^\mu_\star\, \widetilde{k}^\nu_\star}{\widetilde{k}^2_\star}\right]. \tag{2.43}$$

2.2.3 Phase-space mappings

Although the *candidate* counterterm in Eq. (2.23) locally reproduces all phase-space singularities of the real matrix element, it cannot yet be used directly in Eq. (2.9): in fact, the Born matrix elements which factorise in the soft and collinear limits result evaluated with n-body kinematics that either do not satisfy momentum conservation (in the soft case, $\{k\}_{\not i}$ in Eq. (2.25)), or feature an off-shell leg (in the collinear case, $(\{k\}_{\not a\not b}, k_c)$ in Eq. (2.39)) outside the relevant singular region of phase space.

Conversely, it is essential for the Born matrix elements appearing in counterterms to have a physical (i.e. on-shell and momentum conserving) n-body kinematics for all choices of the $n+1$ radiative momenta, and not only for specific singular configurations. For this purpose, we must introduce a set of *mappings* that relate the $(n+1)$-particle momenta $\{k\}$ to the n-particle momenta $\{\bar k\}$, preserving at the same time the soft and collinear limits at leading power. A convenient way of achieving this is through the adoption of the Catani-Seymour mappings [18], which generally involve a triplet of massless momenta k_a, k_b, and k_c (the *emitted*, *emitter*, and *recoiler* parton, respectively) and map them onto a dipole of Born-level momenta $\bar k_b^{(abc)}$ and $\bar k_c^{(abc)}$[4]. Based on the position of the chosen triplet of real momenta, we employ distinct mapping prescriptions:

[3]Eq. (2.43) coincides with Eq. (C.11) when restricted to final-state particle splittings.

[4]This prescription does not apply to the initial-initial case in the third panel of Figure 2.2, where all final states are shifted by the defined mapping (see Eq. (2.51) below).

- For three final-state momenta k_a, k_b, k_c (all different), as in leftmost configuration of Figure 2.2, we construct the n-tuple

$$\{\bar{k}\}^{(abc)} \; = \; \Big\{ \{k\}_{\not a \not b \not c}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \Big\} \,, \tag{2.44}$$

with

$$\bar{k}_b^{(abc)} \; = \; k_a + k_b - \frac{y}{1-y}\, k_c \,, \qquad \bar{k}_c^{(abc)} \; = \; \frac{1}{1-y}\, k_c \,, \tag{2.45}$$

and all other momenta left unchanged $\bar{k}_i^{(abc)} = k_i$, $i \neq a, b, c$ (for i running from 1 to $n+1$), where we defined the kinematic variables y and z as

$$y \; = \; \frac{s_{ab}}{s_{abc}} \,, \qquad z \; = \; \frac{s_{ac}}{s_{ac} + s_{bc}} \,, \tag{2.46}$$

such that $0 \leqslant y, z \leqslant 1$.

- For two different final-state momenta k_a, k_b and an initial-state momentum k_c, as in the central panel of Figure (2.2), we construct the n-tuple

$$\{\bar{k}\}^{(abc)} \; = \; \Big\{ \{k\}_{\not a \not b \not c}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \Big\} \,, \tag{2.47}$$

with

$$\bar{k}_b^{(abc)} \; = \; k_a + k_b - (1-x)\, k_c \,, \qquad \bar{k}_c^{(abc)} \; = \; x\, k_c \,, \tag{2.48}$$

and all other momenta left unchanged ($\bar{k}_i^{(abc)} = k_i$, $i \neq a, b, c$); here we introduced the kinematic variables x and z as

$$x \; = \; \frac{s_{ac} + s_{bc} - s_{ab}}{s_{ac} + s_{bc}} \,, \qquad z \; = \; \frac{s_{ac}}{s_{ac} + s_{bc}} \,, \tag{2.49}$$

such that $0 \leqslant x, z \leqslant 1$.

- For one final-state momentum k_a and two different initial-state momenta k_b, k_c (last configuration in Figure 2.2), we construct the n-tuple

$$\{\bar{k}\}^{(abc)} \; = \; \Big\{ \{k\}_{\not a \not b \not f}, \bar{k}_b^{(abc)}, \bar{k}_f^{(abc)} \Big\} \,, \tag{2.50}$$

where $k_f = \{k_j\}_{j \in \mathrm{F}, j \neq a}$ stands for the collection of all final-state momenta different from k_a; in this setting,

$$\bar{k}_b^{(abc)} \; = \; x\, k_b \,, \qquad \bar{k}_f^{(abc)} \; = \; k_f - \frac{2k_f \cdot (K + \bar{K})}{(K + \bar{K})^2}\, (K + \bar{K}) + \frac{2k_f \cdot K}{K^2}\, \bar{K} \,, \quad \forall f \neq a \,, \tag{2.51}$$

where the momentum k_c is left unchanged ($\bar{k}_c^{(abc)} = k_c$), and

$$K = k_b + k_c - k_a \,, \qquad \bar{K} = \bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} \,. \tag{2.52}$$

The kinematic variables adopted in this case, satisfying $0 \leqslant x, v \leqslant 1$, are

$$x = \frac{s_{bc} - s_{ab} - s_{ac}}{s_{bc}} \,, \qquad v = \frac{s_{ab}}{s_{ab} + s_{ac}} \,. \tag{2.53}$$

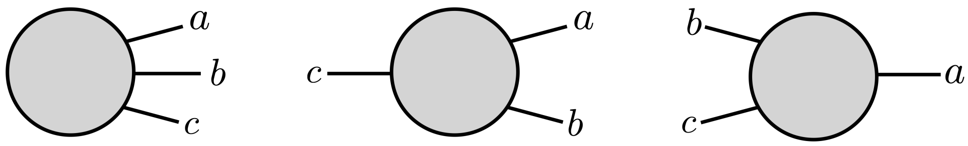

Figure 2.2. Final-final (left), final-initial (middle), and initial-initial (right) dipoles.

All kinematic mappings listed above satisfy the on-shell requirement,

$$\left(\bar{k}_m^{(abc)}\right)^2 = 0 \,, \quad m = 1, \ldots, n \,, \tag{2.54}$$

as well as the momentum-conservation condition, according to

$$\begin{aligned}
\theta_{a\in\mathrm{F}}\,\theta_{b\in\mathrm{F}}\,\theta_{c\in\mathrm{F}} : \qquad & \bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c \,, \\
\theta_{a\in\mathrm{F}}\,\theta_{b\in\mathrm{F}}\,\theta_{c\in\mathrm{I}} : \qquad & \bar{k}_b^{(abc)} - \bar{k}_c^{(abc)} = k_a + k_b - k_c \,, \\
\theta_{a\in\mathrm{F}}\,\theta_{b\in\mathrm{I}}\,\theta_{c\in\mathrm{I}} : \qquad & \sum_{\substack{i\in\mathrm{F}\\ i\neq a}} \bar{k}_i^{(abc)} - \bar{k}_b^{(abc)} - \bar{k}_c^{(abc)} = \sum_{\substack{i\in\mathrm{F}\\ i\neq a}} k_i + k_a - k_b - k_c \,.
\end{aligned} \tag{2.55}$$

One easily verifies that the sets of momenta reported in the left- and right-hand sides of Eq. (2.55) respectively coincide when k_a vanishes (i.e. it goes soft), and when k_a becomes collinear to k_b.

In turn, each of these mapping operations lead to the factorisation of the $(n+1)$-body phase space $d\Phi_{n+1}$ into a remapped n-body phase space $d\Phi_n$ times a single-radiative measure $d\Phi_{\rm rad}$: this factorisation can be exact when just final-state particles are involved, or it may feature an additional convolution with respect to the variable x defining the boost applied to the initial-state momentum, as in Eqs. (2.48) and (2.51). In both cases, the resulting phase-space factorisations enable the independent analytic integration of the radiative degrees of freedom at fixed underlying Born kinematics. More details will be given when approaching the counterterm integration in Section 2.3.

2.2.4 Local counterterm with improved limits

Finally, we can promote the *candidate* counterterm (2.23) to a *local* counterterm K: all we need to do is to make use of the factorised expressions for soft and collinear limits of R, and evaluate the corresponding Born-level squared matrix elements with well-defined mapped n-body kinematics. To achieve this, we introduce *improved versions* of our limit operators $\mathbf{S}_i$ and $\mathbf{C}_{ij}$, which are defined at NLO to project on leading-power soft and collinear limits (as was the case for their bare formulation), while simultaneously applying the selected phase-space mappings. Furthermore, the freedom left in this procedure of defining the action of the improved singular limits allows also for the potential introduction of modifications in the kernel structures, which may turn out to be instrumental in minimising the structural complexity of the counterterm, and possibly curing undesirable spurious effects.

In practice, there is considerable flexibility in how to associate mappings to singular kernels, as long as they do not compromise the locality of the subtraction (more details on this specific requirement will be given shortly): in particular, the choice of the mapping dipoles can be adapted to the identity of the partons involved in the singular configuration. In the soft limit, each eikonal kernel $\mathcal{E}^{(i)}_{cd}$ leads reasonably to the choice (icd) or (idc), where the momentum of the first particle in the triplet (i.e. i) is the one vanishing in the soft limit; on the other hand, the most natural mapping for collinear limits involves the splitting partons and the recoiler, as $(abc) = (ijr)$ or $(abc) = (irj)$. Denoting with a bar the *improved limits* $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$ which convey in their action the kinematic mappings according to the aforementioned prescription, we thus define the soft counterterm to be

$$\overline{\mathbf{S}}_i\, R = -\,2\,\mathcal{N}_1 \sum_{c\neq i} \sum_{\substack{d\neq i \\ d<c}} \mathcal{E}^{(i)}_{cd} \Big[\big(\theta_{c\in \mathrm{I}}\,\theta_{d\in \mathrm{I}} + \theta_{c\in \mathrm{F}}\,\theta_{d\in \mathrm{I}} + \theta_{c\in \mathrm{F}}\,\theta_{d\in \mathrm{F}}\big)\, \bar{B}^{(icd)}_{cd} + \theta_{c\in \mathrm{I}}\,\theta_{d\in \mathrm{F}}\, \bar{B}^{(idc)}_{cd} \Big], \quad (2.56)$$

where the colour-correlated Born matrix elements $\bar{B}^{(abc)}_{\dots} \equiv B_{\dots}(\{\bar{k}\}^{(abc)})$ are evaluated with mapped momenta, and are therefore also denoted with a bar and with a label identifying the specific mapping to be employed. As for collinear and soft-collinear kernels, we define

$$\begin{aligned}
\overline{\mathbf{C}}_{ij}\, R = \frac{\mathcal{N}_1}{s_{ij}} \Bigg[& \theta_{i\in \mathrm{F}}\,\theta_{j\in \mathrm{F}}\, P^{\mu\nu}_{ij(r),\mathrm{F}}(z)\, \bar{B}^{(ijr)}_{\mu\nu} \\
& + \theta_{i\in \mathrm{F}}\,\theta_{j\in \mathrm{I}}\, \frac{P^{\mu\nu}_{[ij]i(r),\mathrm{I}}(x)}{x} \Big(\theta_{r\in \mathrm{F}}\, \bar{B}^{(irj)}_{\mu\nu} + \theta_{r\in \mathrm{I}}\, \bar{B}^{(ijr)}_{\mu\nu} \Big) \\
& + \theta_{j\in \mathrm{F}}\,\theta_{i\in \mathrm{I}}\, \frac{P^{\mu\nu}_{[ji]j(r),\mathrm{I}}(x)}{x} \Big(\theta_{r\in \mathrm{F}}\, \bar{B}^{(jri)}_{\mu\nu} + \theta_{r\in \mathrm{I}}\, \bar{B}^{(jir)}_{\mu\nu} \Big) \Bigg], \qquad (2.57)
\end{aligned}$$

$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R = \mathcal{N}_1\, 2\, C_{f_j}\, \mathcal{E}^{(i)}_{jr} \Bigg[\theta_{j\in \mathrm{F}}\, \bar{B}^{(ijr)} + \theta_{j\in \mathrm{I}} \Big(\frac{\theta_{r\in \mathrm{F}}}{1-z}\, \bar{B}^{(irj)} + \theta_{r\in \mathrm{I}}\, (1-v)\, \bar{B}^{(ijr)} \Big) \Bigg], \quad (2.58)$$

where the rule $r = r_{ij}$ applies for both expressions. Note that the soft-collinear counterterm in Eq. (2.58) features extra kinematical factors multiplying the singular kernels, which are written in terms of the variable z (defined as in Eq. (2.49) for the $\theta_{j\in \rm I}\theta_{r\in \rm F}$ term) and the variable v (defined as in Eq. (2.53) for the $\theta_{j\in \rm I}\theta_{r\in \rm I}$ term)[5], which serve the purpose of reconstructing the hard-collinear kernels of Eqs. (2.41) and (2.42) in the following compact form,

$$
\begin{aligned}
\overline{\mathbf{HC}}_{ij}\, R &\equiv (1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j)\, \overline{\mathbf{C}}_{ij}\, R \\
&= \frac{\mathcal{N}_1}{s_{ij}} \Bigg[\theta_{i\in \rm F}\, \theta_{j\in \rm F}\, P^{\mu\nu,\,\rm hc}_{ij(r),\rm F}(z)\, \bar{B}^{(ijr)}_{\mu\nu} \\
&\qquad + \theta_{i\in \rm F}\, \theta_{j\in \rm I}\, \frac{P^{\mu\nu,\,\rm hc}_{[ij]i(r),\rm I}(x)}{x} \left(\theta_{r\in \rm F}\, \bar{B}^{(irj)}_{\mu\nu} + \theta_{r\in \rm I}\, \bar{B}^{(ijr)}_{\mu\nu} \right) \\
&\qquad + \theta_{j\in \rm F}\, \theta_{i\in \rm I}\, \frac{P^{\mu\nu,\,\rm hc}_{[ji]j(r),\rm I}(x)}{x} \left(\theta_{r\in \rm F}\, \bar{B}^{(jri)}_{\mu\nu} + \theta_{r\in \rm I}\, \bar{B}^{(jir)}_{\mu\nu} \right) \Bigg] ,
\end{aligned}
\tag{2.59}
$$

where we have defined $\overline{\mathbf{S}}_j\, \overline{\mathbf{C}}_{ij} \equiv \overline{\mathbf{S}}_j\, \overline{\mathbf{C}}_{ji}$.

At this stage, the definition in our *local* counterterm still lacks one final ingredient to be complete, which is the specification of how the improved projectors, denoted collectively with $\overline{\mathbf{L}} = \overline{\mathbf{S}}_i,\, \overline{\mathbf{C}}_{ij},\, \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}$, operate on sector functions. The simplest and straightforward option is to set

$$
\overline{\mathbf{L}}\, \mathcal{W}_{ab} \;\equiv\; \mathbf{L}\, \mathcal{W}_{ab}\,, \tag{2.60}
$$

essentially leaving unchanged the action of the improved limits $\overline{\mathbf{L}}$ with respect to the un-improved operators $\mathbf{L}$ (see Eq. (2.24) for explicit expressions). However, there is nothing that actually prevents us from redefining their structure in order to meet certain conditions.

The combination of all the ingredients discussed so far finally leads to define the sought *local* counterterm K as

$$
K \equiv \sum_i \sum_{j\neq i} K_{ij}\,, \qquad K_{ij} \;\equiv\; \Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij} \Big]\, R\, \mathcal{W}_{ij}\,, \tag{2.61}
$$

where again $\overline{\mathbf{L}}\, R\, \mathcal{W}_{ij} \equiv (\overline{\mathbf{L}}\, R)\, (\overline{\mathbf{L}}\, \mathcal{W}_{ij})$. As mentioned earlier, the entire construction and consequent validity of the counterterm K is crucially subjected to the stringent requirement that the improved operators must preserve the correct soft and collinear limits of R in order to guarantee the locality of the subtraction procedure. In practise, this condition,

[5] We emphasise that the definitions of the z, x, and v variables in the previous equations are mapping-dependent: for instance, one should correctly interpret the notation $f(x)(\theta_{r\in \rm F}\, \bar{B}^{(irj)}_{\mu\nu} + \theta_{r\in \rm I}\, \bar{B}^{(ijr)}_{\mu\nu})$ to mean $\theta_{r\in \rm F}\, f(x^{(irj)})\, \bar{B}^{(irj)}_{\mu\nu} + \theta_{r\in \rm I}\, f(x^{(ijr)})\, \bar{B}^{(ijr)}_{\mu\nu}$, and similarly for the other terms.

explicitly as

$$R\,\mathcal{W}_{ij} - K_{ij} \;=\; R\,\mathcal{W}_{ij} - \big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\big)\,R\,\mathcal{W}_{ij} \;\to\; \text{integrable}\,, \tag{2.62}$$

translates into the verification of a set of *consistency relations*. Specifically, for Eq. (2.62) to be true, it must be checked that the leading divergences cancel under the *primary*[6] limits $\mathbf{S}_i$ and $\mathbf{C}_{ij}$, as

$$\begin{aligned} \mathbf{S}_i\Big[R\,\mathcal{W}_{ij} - \big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\big)\,R\,\mathcal{W}_{ij}\Big] \;&\to\; \text{integrable}\,,\\ \mathbf{C}_{ij}\Big[R\,\mathcal{W}_{ij} - \big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\big)\,R\,\mathcal{W}_{ij}\Big] \;&\to\; \text{integrable}\,, \end{aligned} \tag{2.63}$$

leading to

$$\begin{aligned} \mathbf{S}_i\,\overline{\mathbf{S}}_i\,R\,\mathcal{W}_{ij} \;&=\; \mathbf{S}_i\,R\,\mathcal{W}_{ij}\,, & \mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ij}\,R\,\mathcal{W}_{ij} \;&=\; \mathbf{C}_{ij}\,R\,\mathcal{W}_{ij}\,,\\ \mathbf{S}_i\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R\,\mathcal{W}_{ij} \;&=\; \mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,R\,\mathcal{W}_{ij}\,, & \mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R\,\mathcal{W}_{ij} \;&=\; \mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ij}\,R\,\mathcal{W}_{ij}\,. \end{aligned} \tag{2.64}$$

Under the assumption (2.60) for sector functions, these expressions easily reduce to

$$\begin{aligned} \mathbf{S}_i\,\overline{\mathbf{S}}_i\,R \;&=\; \mathbf{S}_i\,R\,, & \mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ij}\,R \;&=\; \mathbf{C}_{ij}\,R\,,\\ \mathbf{S}_i\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R \;&=\; \mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,R\,, & \mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R \;&=\; \mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,R\,. \end{aligned} \tag{2.65}$$

Once these relations are satisfied, they provide evidence of the correctness of our mapping-adaptation procedure and further redefinitions of singular structures, ultimately ensuring the cancellation of phase-space singularities. It can be checked (see Appendix B.2) that the consistency relations in Eq. (2.65) are verified by the definitions of the soft and collinear counterterms in Eqs. (2.56)-(2.59).

Instabilities within counterterms

However, the freedom we are exploiting to rework singular kernel definitions through improved limits can prove to be a double-edged sword: in fact, despite the local cancellation being checked by consistency relations, the quantity K_{ij} defined in Eq. (2.61) contains a subtlety, which must be analysed with care. Let's consider, for example, the final-state[7] DGLAP kernels $P^{\mu\nu}_{ij(r),\mathrm{F}}$ reported in Eq. (2.34), which are written in terms of the invariants

$$z_i \;=\; \frac{s_{ir}}{s_{ir}+s_{jr}}\,, \qquad z_j \;=\; \frac{s_{jr}}{s_{ir}+s_{jr}}\,, \tag{2.66}$$

[6] The adjective *primary* pertains to the singular limits selected by sector functions, to be distinguished from the *auxiliary* limits, which we will elaborate on in the next Section.

[7] The following discussion holds equally for initial-state Altarelli-Parisi splitting kernels.

as opposed to the energy fractions $e_i/(e_i+e_j)$, $e_j/(e_i+e_j)$. This is a useful choice in view of analytical integration, and a legitimate one since x_i and x_j reduce to e_i and e_j in the collinear limit $\mathbf{C}_{ij}$. This choice, however, introduces *spurious singularities* in the collinear limits $\mathbf{C}_{ir}$ and $\mathbf{C}_{jr}$, which are generated by the denominators of the Altarelli-Parisi kernels, and are not present in $R\,\mathcal{W}_{ij}$. As a result, the combination $(1-\overline{\mathbf{S}}_i)(1-\overline{\mathbf{C}}_{ij})R\,\mathcal{W}_{ij}$ is not integrable in those limits. Nevertheless, the problem can be solved by using our freedom to define the action of the improved operators $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$ on the sector functions $\mathcal{W}_{ij}$, whose structure has been so far left unchanged within the local counterterm. These new definitions we introduce as $(r=r_{ij})$

$$
\begin{aligned}
\overline{\mathbf{S}}_i\,\mathcal{W}_{ij} &= \theta_{i\in\mathrm{F}}\,\frac{1/w_{ij}}{\sum\limits_{l\neq i}1/w_{il}}\,,\\
\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ij} &= \theta_{i\in\mathrm{F}}\left(\theta_{j\in\mathrm{F}}\frac{e_j\,w_{jr}}{e_i\,w_{ir}+e_j\,w_{jr}}+\theta_{j\in\mathrm{I}}\right),\\
\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ij} &= \theta_{i\in\mathrm{F}}\,,
\end{aligned}
\tag{2.67}
$$

in fact represent a modification of Eq. (2.24). The presence of the angular factors w_{ir} and w_{jr} vanishing in the $\mathbf{C}_{ir}$ and $\mathbf{C}_{jr}$ limits respectively, allows to satisfy the following *auxiliary consistency relations*

$$
\begin{aligned}
\mathbf{C}_{ir}\left\{1\,,\;\overline{\mathbf{S}}_i\,,\;\overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i\right)\right\}R\,\mathcal{W}_{ij} &\;\rightarrow\; \text{integrable}\,,\\
\mathbf{C}_{jr}\left\{1\,,\;\overline{\mathbf{S}}_i\,,\;\overline{\mathbf{C}}_{ij}\,,\;\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\right\}R\,\mathcal{W}_{ij} &\;\rightarrow\; \text{integrable}\,,
\end{aligned}
\tag{2.68}
$$

on top of the standard ones, corresponding to Eq. (2.65), which now need to be written explicitly including also sector functions. More compactly, one has

$$
\begin{aligned}
\mathbf{S}_i\left\{\left(1-\overline{\mathbf{S}}_i\right),\;\overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i\right)\right\}R\,\mathcal{W}_{ij} &\;\rightarrow\; \text{integrable}\,,\\
\mathbf{C}_{ij}\left\{\left(1-\overline{\mathbf{C}}_{ij}\right),\;\overline{\mathbf{S}}_i\left(1-\overline{\mathbf{C}}_{ij}\right)\right\}R\,\mathcal{W}_{ij} &\;\rightarrow\; \text{integrable}\,.
\end{aligned}
\tag{2.69}
$$

Recall that in Eq. (2.68) the index r labels the reference vector used to define the collinear kernel $P^{\mu\nu}_{ab(r),\star}$: in fact, all collinear projection operators $\mathbf{C}_{ab}$ should properly be labelled with the index r, which in general we omit for brevity. Notice also that our definition of improved limits of sector functions, Eq. (2.67), is not symmetric under $i\leftrightarrow j$ exchange. As a consequence, the two lines of Eq. (2.68) are not identical: in the first line, only the combination $\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)$ gives an integrable result in the ir collinear limit, when acting on $R\,\mathcal{W}_{ij}$ (which is sufficient for our purposes), while in the second line $\overline{\mathbf{C}}_{ij}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}$ give separately integrable contributions in the jr collinear limit.

With the definitions provided in Eqs. (2.56)-(2.59) supplemented by Eq. (2.67), we can build the *local counterterm* K, as presented in Eq. (2.61), which is sufficient to construct a fully functional subtraction algorithm at NLO. Finally, the subtracted real matrix element squared is given by

$$R_{\rm sub}(X) \;=\; \sum_{i,j\neq i} R^{\rm sub}_{ij}(X)\,, \quad R^{\rm sub}_{ij}(X) \;=\; R\,\mathcal{W}_{ij}\,\delta_{n+1}(X) - K_{ij}\,\delta_n(X)\,. \tag{2.70}$$

In the previous Sections, we have presented what can be considered the *basic implementation* of our counterterm. However, we are of the opinion that there is good potential to introduce optimisations in the various stages of the formulation.

2.2.5 Room for optimisation: symmetrised phase-space partition

We acknowledge that sector functions $\mathcal{W}_{ij}$ are a valuable tool to identify the improved limits to be defined, and the consistency relations they must satisfy, but we are also aware of the fact that the stability of numerical integrations generally improves when sectors involving the same parametrisations are combined (specifically, in our case, sector functions sharing a collinear singularity would be parametrised in the same way in a numerical code). To pursue this idea, we introduce *symmetrised sector functions* as

$$\mathcal{Z}_{ij} \;=\; \mathcal{W}_{ij} + \mathcal{W}_{ji}\,, \tag{2.71}$$

whose corresponding improved limits read

$$\begin{aligned} &\overline{\mathbf{S}}_i\,\mathcal{Z}_{ij} \;=\; \theta_{i\in{\rm F}}\,\frac{1/w_{ij}}{\sum\limits_{l\neq i} 1/w_{il}}\,, \qquad \overline{\mathbf{S}}_j\,\mathcal{Z}_{ij} \;=\; \theta_{j\in{\rm F}}\,\frac{1/w_{ij}}{\sum\limits_{l\neq j} 1/w_{jl}}\,, \\ &\overline{\mathbf{C}}_{ij}\,\mathcal{Z}_{ij} \;=\; 1-\theta_{i\in{\rm I}}\,\theta_{j\in{\rm I}}\,, \qquad \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\mathcal{Z}_{ij} \;=\; \theta_{i\in{\rm F}}\,, \qquad \overline{\mathbf{S}}_j\,\overline{\mathbf{C}}_{ij}\,\mathcal{Z}_{ij} \;=\; \theta_{j\in{\rm F}}\,. \end{aligned} \tag{2.72}$$

This symmetrised phase-space partition reduces the overall number of sectors and singular limits to be evaluated, thus simplifying the scheme (to some extent) and enhancing its numerical efficiency. In fact, the counterterm K, with symmetrised sector functions, can be written as

$$K \;=\; \sum_{i,j>i} K_{\{ij\}}\,, \qquad K_{\{ij\}} \;=\; \big(\overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{HC}}_{ij}\big)R\,\mathcal{Z}_{ij}\,, \tag{2.73}$$

recalling that $\overline{\mathbf{HC}}_{ij} \;=\; \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j)$. The subtracted real contribution can now be written as

$$R_{\rm sub}(X) \;=\; \sum_{i,j>i} R^{\rm sub}_{\{ij\}}(X)\,, \qquad R^{\rm sub}_{\{ij\}}(X) \;=\; R\,\mathcal{Z}_{ij}\,\delta_{n+1}(X) - K_{\{ij\}}\,\delta_n(X)\,. \tag{2.74}$$

A third expression for the NLO counterterm[8], is obtained by summing over all sectors. Using sum rules in Eq. (2.20), one can then write

$$R_{\rm sub}(X) \;=\; R\,\delta_{n+1}(X) - K\,\delta_n(X)\,, \tag{2.75}$$

with

$$K \;=\; \sum_i \overline{\mathbf{S}}_i\, R + \sum_{i,j>i} \overline{\mathbf{HC}}_{ij}\, R\,. \tag{2.76}$$

Here K results purely defined as a collection of universal soft and collinear NLO kernels. This last formulation is particularly well-suited for counterterm integration: not only this avoids analytically integrating over the (arbitrarily complicated) sector functions, but it also eliminates the necessity to recompute the integrated counterterms I and J (see Eq. (2.8)) upon redefinition of the sectors themselves, provided the sum rules in Eq. (2.20) are preserved. Conversely, the expression for $R_{\rm sub}(X)$ in Eq. (2.74), with symmetrised sector functions, is to be preferred for the numerical implementation, since it allows to parallelise the contribution of different sectors, and to independently optimise their numerical evaluation.

2.2.6 Room for optimisation: damping factors

Since the subtraction procedure is necessary only in the infrared corners of the phase space, one has the flexibility to adjust the counterterm contribution in the non-singular regions, thereby reducing potential numerical instabilities. This is customarily achieved in the literature by introducing parameters (such as the α parameter in CS [204], and the δ and $\xi_{\rm cut}$ parameters in FKS [16]) that impose a hard boundary to the phase space allowed for counterterms. The improved numerical stability of this procedure generally comes at the cost of a more cumbersome analytic counterterm integration, which may become untenable at NNLO.

What we propose, instead, is to multiply the local counterterms in Eqs. (2.56)-(2.58) with smooth *damping factors* (as opposed to hard step functions) in order to gradually suppress their contribution away from the singular regions. Although there is some flexibility in constructing such damping factors, provided the validity of Eqs. (2.68) and (2.69) is not spoiled, it is highly convenient to define them as powers, with tunable exponents, of the kinematic invariants proper of the chosen phase-space parametrisation. This approach allows controlled inclusion of subleading power terms in the normal variables used to write the IRC kernels. As a result, the presence of damping factors does not impact the complexity of the analytic integrations, which is crucial for exporting this optimisation to

[8] We already provided the counterterm definition (2.61) in terms of $\mathcal{W}_{ij}$ functions, and the expression (2.73) in terms of $\mathcal{Z}_{ij}$ functions.

higher perturbative orders. The explicit dependence of (the finite part of) the integrated counterterms upon the damping parameters, namely the tunable exponents mentioned above, must cancel against an analogous dependence in the local counterterms, which is known to offer a powerful handle to verify the numerical implementation of the subtraction method.

We start by including damping factors in the soft counterterm, Eq. (2.56):

$$\overline{\mathbf{S}}_i \, R = -\,2\,\mathcal{N}_1 \sum_{c\neq i} \sum_{\substack{d\neq i\\ d<c}} \mathcal{E}^{(i)}_{cd} \Big\{ \theta_{c\in \mathrm{F}}\,(1-z)^{\alpha} \Big[\theta_{d\in \mathrm{F}}\,(1-y)^{\alpha} + \theta_{d\in \mathrm{I}}\, x^{\alpha} \Big] \bar{B}^{(icd)}_{cd} + \theta_{c\in \mathrm{I}}\, x^{\alpha} \Big[\theta_{d\in \mathrm{F}}\,(1-z)^{\alpha}\, \bar{B}^{(idc)}_{cd} + \theta_{d\in \mathrm{I}}\, \bar{B}^{(icd)}_{cd} \Big] \Big\}\,, \tag{2.77}$$

where $\alpha \geqslant 0$, and the kinematic variables x, y, z are those associated to the (icd) or (idc) phase-space mappings, i.e. they are different for each term in the eikonal double sum. In detail, they are defined as in Eq. (2.46), Eq. (2.49), Eq. (2.53) for $(cd) = \mathrm{FF}, \mathrm{FI/IF}, \mathrm{II}$, respectively. The case with no damping, Eq. (2.56), is simply obtained by setting $\alpha = 0$.

As far as collinear and soft-collinear contributions are concerned, we modify Eq. (2.57) and Eq. (2.58) as

$$\overline{\mathbf{C}}_{ij} \, R = \frac{\mathcal{N}_1}{s_{ij}} \Bigg\{ \theta_{i\in \mathrm{F}}\,\theta_{j\in \mathrm{F}}\, P^{\mu\nu}_{ij(r),\mathrm{F}}(z) \Big[\theta_{r\in \mathrm{F}}\,(1-y)^{\beta} + \theta_{r\in \mathrm{I}}\, x^{\beta} \Big] \bar{B}^{(ijr)}_{\mu\nu} + \theta_{i\in \mathrm{F}}\,\theta_{j\in \mathrm{I}}\, \frac{P^{\mu\nu}_{[ij]i(r),\mathrm{I}}(x)}{x} \Big[\theta_{r\in \mathrm{F}}\,(1-z)^{\gamma}\, \bar{B}^{(irj)}_{\mu\nu} + \theta_{r\in \mathrm{I}}\,(1-v)^{\gamma}\, \bar{B}^{(ijr)}_{\mu\nu} \Big] + \theta_{j\in \mathrm{F}}\,\theta_{i\in \mathrm{I}}\, \frac{P^{\mu\nu}_{[ji]j(r),\mathrm{I}}(x)}{x} \Big[\theta_{r\in \mathrm{F}}\,(1-z)^{\gamma}\, \bar{B}^{(jri)}_{\mu\nu} + \theta_{r\in \mathrm{I}}\,(1-v)^{\gamma}\, \bar{B}^{(jir)}_{\mu\nu} \Big] \Bigg\}, \tag{2.78}$$

$$\overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R = \mathcal{N}_1 \, 2\, C_{f_j}\, \mathcal{E}^{(i)}_{jr} \Big\{ \theta_{j\in \mathrm{F}}\,(1-z)^{\alpha} \Big[\theta_{r\in \mathrm{F}}\,(1-y)^{\beta} + \theta_{r\in \mathrm{I}}\, x^{\beta} \Big] \bar{B}^{(ijr)} + \theta_{j\in \mathrm{I}}\, x^{\alpha} \Big[\theta_{r\in \mathrm{F}}\,(1-z)^{\gamma-1} \bar{B}^{(irj)} + \theta_{r\in \mathrm{I}}\,(1-v)^{\gamma+1} \bar{B}^{(ijr)} \Big] \Big\}\,, \tag{2.79}$$

where α is the same exponent appearing in the damped soft counterterm, Eq. (2.77), while $\beta, \gamma \geqslant 0$ are relevant for final- and initial-state collinear splitting, respectively. The kinematic variables defining the damping factors depend on the mapping appearing in the relevant Born matrix element, similar to the soft case. The un-damped limits can be retrieved by setting $\alpha = \beta = \gamma = 0$.

By following the same steps outlined in Appendix B.2, it can be checked that the damped counterterm definitions in Eqs. (2.77)-(2.79) correctly satisfy the consistency relations in Eqs. (2.68) and (2.69). Furthermore, it will be shown in Section 2.4 that, as expected, the ϵ poles of the integrated counterterms do not exhibit any dependence on the arbitrary parameters α, β, γ, which thus appear only in the finite part $\mathcal{O}(\epsilon^0)$. While in Section 4.1, we will provide a first numerical validation of such damped local counterterms at both integrated and differential level.

We point out that the structure of the local counterterm K and of its sector components K_{ij} and $K_{\{ij\}}$, as given in Eqs. (2.61, 2.73, 2.76), is not affected by the presence of damping factors and remains formally valid for any value of α, β, γ. The damped $\overline{\mathbf{HC}}_{ij}\,R$ counterterms can still be expressed in terms of the hard-collinear kernels $P^{\rm hc}_{ij(r),\star}$ as

$$\begin{aligned}\overline{\mathbf{HC}}_{ij}\,R &\equiv (1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j)\,\overline{\mathbf{C}}_{ij}\,R \\ &= \theta_{i\in{\rm F}}\,\theta_{j\in{\rm F}}\,\overline{\mathbf{HC}}_{ij,{\rm F}}\,R+\theta_{i\in{\rm F}}\,\theta_{j\in{\rm I}}\,\overline{\mathbf{HC}}_{ij,{\rm I}}\,R+\theta_{j\in{\rm F}}\,\theta_{i\in{\rm I}}\,\overline{\mathbf{HC}}_{ji,{\rm I}}\,R\,,\end{aligned} \tag{2.80}$$

with

$$\begin{aligned}\overline{\mathbf{HC}}_{ij,{\rm F}}\,R &\equiv \mathcal{N}_1\Big[\theta_{r\in{\rm F}}\,(1-y)^\beta+\theta_{r\in{\rm I}}\,x^\beta\Big] \\ &\quad\left[\frac{P^{\mu\nu,{\rm hc}}_{ij(r),{\rm F}}(z)}{s_{ij}}\bar{B}^{(ijr)}_{\mu\nu}+2\Big[C_{f_j}\mathcal{E}^{(i)}_{jr}\big(1-(1-z)^\alpha\big)+C_{f_i}\mathcal{E}^{(j)}_{ir}(1-z^\alpha)\Big]\bar{B}^{(ijr)}\right], \\ \overline{\mathbf{HC}}_{ij,{\rm I}}\,R &\equiv \mathcal{N}_1\left[\theta_{r\in{\rm F}}\,(1-z)^\gamma\left(\frac{P^{\mu\nu,{\rm hc}}_{[ij]i(r),{\rm I}}(x)}{x\,s_{ij}}\bar{B}^{(irj)}_{\mu\nu}+2C_{f_j}\,\mathcal{E}^{(i)}_{jr}\frac{1-x^\alpha}{1-z}\bar{B}^{(irj)}\right)\right. \\ &\quad\left.+\,\theta_{r\in{\rm I}}\,(1-v)^\gamma\left(\frac{P^{\mu\nu,{\rm hc}}_{[ij]i(r),{\rm I}}(x)}{x\,s_{ij}}\bar{B}^{(ijr)}_{\mu\nu}+2C_{f_j}\mathcal{E}^{(i)}_{jr}(1-x^\alpha)(1-v)\bar{B}^{(ijr)}\right)\right],\end{aligned} \tag{2.81}$$

which will be integrated in the next Section.

2.3 Integration of the real-radiation counterterm

In order to analytically integrate the counterterm K, it is convenient to start from Eq. (2.76), relying on the kernel definitions in Eqs. (2.77) and (2.80). The counterterm expression is summed over sectors, as its integral must reproduce the poles of the virtual matrix element, which is not partitioned. For future convenience, we split K into the corresponding soft, final-state hard-collinear and initial-state hard-collinear contributions

$$K\;\equiv\;K_{\rm s}\;+\;K_{\,\rm hc,F}\;+\;K_{\,\rm hc,I}\,, \tag{2.82}$$

defined as

$$K_{\rm s}\equiv\sum_i\overline{\mathbf{S}}_i\,R\,, \tag{2.83}$$

$$K_{\,\rm hc,F}\equiv\sum_i\sum_{j<i}\theta_{i\in{\rm F}}\,\theta_{j\in{\rm F}}\,\overline{\mathbf{HC}}_{ij,{\rm F}}\,R\,, \tag{2.84}$$

$$K_{\,\rm hc,I}\equiv\sum_i\sum_{j<i}\Big[\theta_{i\in{\rm F}}\,\theta_{j\in{\rm I}}\,\overline{\mathbf{HC}}_{ij,{\rm I}}\,R+\theta_{j\in{\rm F}}\,\theta_{i\in{\rm I}}\,\overline{\mathbf{HC}}_{ji,{\rm I}}\,R\Big]\,. \tag{2.85}$$

Phase-space parametrisations

We start by introducing precise definitions of the phase-space measures used for integration. As detailed in Section 2.2.3, we have three possible mapping prescriptions to parametrise the unresolved phase space. We examine them in turn.

The first mapping $\{\bar{k}\}^{(abc)}$ with final-state momenta k_a, k_b, k_c (all different), presented in Eqs. (2.44) and (2.45), induces the exact factorisation

$$\int d\Phi_{n+1} = \frac{\varsigma_{n+1}}{\varsigma_n} \int d\Phi_n^{(abc)} \int d\Phi_{\rm rad}^{(abc)} , \tag{2.86}$$

where we explicitly extracted the ratio of the relevant symmetry factors ς_{n+1} and ς_n, and

$$d\Phi_n^{(abc)} \equiv d\Phi_n\left(\{\bar{k}\}^{(abc)}\right) , \qquad d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_{\rm rad}\left(\bar{s}_{bc}^{(abc)}; y, z, \phi\right) . \tag{2.87}$$

The radiative measure of integration is

$$\int d\Phi_{\rm rad}^{(abc)} = N(\epsilon) \left(\bar{s}_{bc}^{(abc)}\right)^{1-\epsilon} \int_0^\pi d\phi \,(\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[y(1-y)^2 z(1-z)\right]^{-\epsilon} (1-y) , \tag{2.88}$$

where the expression of the invariants composed by k_a, k_b, k_c, written in terms of the integration variables, are

$$s_{ab} = y\, \bar{s}_{bc}^{(abc)} , \qquad s_{ac} = z(1-y)\, \bar{s}_{bc}^{(abc)} , \qquad s_{bc} = (1-z)(1-y)\, \bar{s}_{bc}^{(abc)} , \tag{2.89}$$

such that $s_{abc} = s_{ab} + s_{ac} + s_{bc} = \bar{s}_{bc}^{(abc)}$. Finally, we define

$$N(\epsilon) \equiv \frac{(4\pi)^{\epsilon-2}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)} . \tag{2.90}$$

The second mapping $\{\bar{k}\}^{(abc)}$ with final-state momenta k_a, k_b (all different) and an initial-state momentum k_c, as in Eqs. (2.47) and (2.48), leads to the following phase-space *convolution*,

$$\int d\Phi_{n+1}(k_c) = \frac{\varsigma_{n+1}}{\varsigma_n} \int\!\!\int d\Phi_n^{(abc)}(xk_c)\, d\Phi_{\rm rad}^{(abc)} , \tag{2.91}$$

with

$$d\Phi_n^{(abc)}(xk_c) \equiv d\Phi_n\left(\{\bar{k}\}^{(abc)}\right) , \qquad d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_{\rm rad}\left(\bar{s}_{bc}^{(abc)}; x, z, \phi\right) . \tag{2.92}$$

Indeed, in this case, achieving exact factorisation is not possible due to the residual dependence on the variable x associated to the rescaled momentum of the initial-state parton, over which we are not integrating. As a function of the reference invariant $\bar{s}_{bc}^{(abc)} \equiv$

$2\bar{k}_b^{(abc)} \cdot k_c = 2\bar{k}_b^{(abc)} \cdot \bar{k}_c^{(abc)}/x$, the relevant dot products read

$$s_{ab} = (1-x)\,\bar{s}_{bc}^{(abc)}, \qquad s_{ac} = z\,\bar{s}_{bc}^{(abc)}, \qquad s_{bc} = (1-z)\,\bar{s}_{bc}^{(abc)}. \tag{2.93}$$

The single unresolved phase space in terms of the kinematic variables results in

$$\int d\Phi_{\rm rad}^{(abc)} = N(\epsilon)\left(\bar{s}_{bc}^{(abc)}\right)^{1-\epsilon}\int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\int_0^1 dx\int_0^1 dz\,[(1-x)\,z(1-z)]^{-\epsilon}. \tag{2.94}$$

The third mapping configuration $\{\bar{k}\}^{(abc)}$ with final-state momentum k_a, and two different initial-state momenta k_b, k_c, as reported in Eqs. (2.50) and (2.51), conveys a similar *convolution*,

$$\int d\Phi_{n+1}(k_b,k_c) = \frac{\varsigma_{n+1}}{\varsigma_n}\int\int d\Phi_n^{(abc)}(xk_b,k_c)\,d\Phi_{\rm rad}^{(abc)}, \tag{2.95}$$

where

$$d\Phi_n^{(abc)}(xk_b,k_c) \equiv d\Phi_n\left(\{\bar{k}\}^{(abc)}\right), \qquad d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_{\rm rad}\left(\bar{s}_{bc}^{(abc)};x,v,\phi\right), \tag{2.96}$$

leading to the explicit expression

$$\int d\Phi_{\rm rad}^{(abc)} = N(\epsilon)\left(\bar{s}_{bc}^{(abc)}\right)^{1-\epsilon}\int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\int_0^1 dx\int_0^1 dv\,[(1-x)^2\,v(1-v)]^{-\epsilon}\,(1-x). \tag{2.97}$$

With respect to the invariant $\bar{s}_{bc}^{(abc)} \equiv 2k_b\cdot k_c = 2\bar{k}_b^{(abc)}\cdot\bar{k}_c^{(abc)}/x$, we express the dipole invariants as

$$s_{ab} = (1-x)\,v\,\bar{s}_{bc}^{(abc)}, \qquad s_{ac} = (1-x)\,(1-v)\,\bar{s}_{bc}^{(abc)}, \qquad s_{bc} = \bar{s}_{bc}^{(abc)}. \tag{2.98}$$

Integration of soft and collinear counterterms

We start with the integration of the soft counterterm $K_{\rm s}$ in Eq. (2.83), which leads to

$$\begin{aligned}
\int d\Phi_{n+1}\,\overline{\mathbf{S}}_i\,R \;=& \qquad\qquad (2.99)\\
=& -2\,\mathcal{N}_1\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{c\neq i}\sum_{\substack{d\neq i\\ d<c}}\Bigg[\theta_{c\in{\rm F}}\,\theta_{d\in{\rm F}}\int d\Phi_n^{(icd)}\int d\Phi_{\rm rad}^{(icd)}\,(1-y)^\alpha(1-z)^\alpha\,\mathcal{E}_{cd}^{(i)}\,\bar B_{cd}^{(icd)}\\
&+\theta_{c\in{\rm F}}\,\theta_{d\in{\rm I}}\int\!\!\int d\Phi_n^{(icd)}(xk_d)\,d\Phi_{\rm rad}^{(icd)}\,x^\alpha\,(1-z)^\alpha\,\mathcal{E}_{cd}^{(i)}\,\bar B_{cd}^{(icd)}\\
&+\theta_{d\in{\rm F}}\,\theta_{c\in{\rm I}}\int\!\!\int d\Phi_n^{(idc)}(xk_c)\,d\Phi_{\rm rad}^{(idc)}\,x^\alpha\,(1-z)^\alpha\,\mathcal{E}_{cd}^{(i)}\,\bar B_{cd}^{(idc)}\\
&+\theta_{c\in{\rm I}}\,\theta_{d\in{\rm I}}\int\!\!\int d\Phi_n^{(icd)}(xk_c,k_d)\,d\Phi_{\rm rad}^{(icd)}\,x^\alpha\,\mathcal{E}_{cd}^{(i)}\,\bar B_{cd}^{(icd)}\Bigg]\\
\equiv& -2\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{c\neq i}\sum_{\substack{d\neq i\\ d<c}}\Bigg\{\theta_{c\in{\rm F}}\,\theta_{d\in{\rm F}}\int d\Phi_n^{(icd)}\,I_{\rm s,FF}^{icd}\,\bar B_{cd}^{(icd)}\\
&+\theta_{c\in{\rm F}}\,\theta_{d\in{\rm I}}\bigg[\int d\Phi_n^{(icd)}(k_d)\,I_{\rm s,FI}^{icd}+\int_0^1\frac{dx}{x}\int d\Phi_n^{(icd)}(xk_d)\,J_{\rm s,FI}^{icd}(x)\bigg]\bar B_{cd}^{(icd)}\\
&+\theta_{d\in{\rm F}}\,\theta_{c\in{\rm I}}\bigg[\int d\Phi_n^{(idc)}(k_c)\,I_{\rm s,FI}^{idc}+\int_0^1\frac{dx}{x}\int d\Phi_n^{(idc)}(xk_c)\,J_{\rm s,FI}^{idc}(x)\bigg]\bar B_{cd}^{(idc)}\\
&+\theta_{c\in{\rm I}}\,\theta_{d\in{\rm I}}\bigg[\int d\Phi_n^{(icd)}(k_c,k_d)\,I_{\rm s,II}^{icd}+\int_0^1\frac{dx}{x}\int d\Phi_n^{(icd)}(xk_c,k_d)\,J_{\rm s,II}^{icd}(x)\bigg]\bar B_{cd}^{(icd)}\Bigg\}.
\end{aligned}$$

The integrals $I^{iab}_{\rm s,\star\star}$ and $J^{iab}_{\rm s,\star\star}(x)$ are reported in Appendix B.3.1, where the latter (former) collect x-(in)dependent contributions.

Moving to the hard-collinear counterterms $K_{{\rm hc},\star}$ in Eqs. (2.84)-(2.85), we notice that the azimuthal contribution multiplying $Q_{ab(r),\star}$ in the collinear kernels vanishes upon integration (see Appendix C.4). Therefore, only unpolarised Altarelli-Parisi kernels need to be integrated. For a final-state j, relevant to $K_{\rm hc,F}$, the result is as follows:

$$\begin{aligned}
\int d\Phi_{n+1}\,\overline{\mathbf{HC}}_{ij,{\rm F}}\,R \;=&\\
=&\,\mathcal{N}_1\,\frac{\varsigma_{n+1}}{\varsigma_n}\bigg[\theta_{r\in{\rm F}}\int d\Phi_n^{(ijr)}\int d\Phi_{\rm rad}^{(ijr)}\,(1-y)^\beta+\theta_{r\in{\rm I}}\int\!\!\int d\Phi_n^{(ijr)}(xk_r)\,d\Phi_{\rm rad}^{(ijr)}\,x^\beta\bigg]\\
&\left[\frac{P^{\rm hc}_{ij,{\rm F}}(z)}{s_{ij}}+2\Big[C_{f_j}\,\mathcal{I}_{jr}^{(i)}\big(1-(1-z)^\alpha\big)+C_{f_i}\,\mathcal{I}_{ir}^{(j)}\,(1-z^\alpha)\Big]\right]\bar B^{(ijr)}\\
\equiv&\,\frac{\varsigma_{n+1}}{\varsigma_n}\bigg[\theta_{r\in{\rm F}}\int d\Phi_n^{(ijr)}\Big(I_{\rm hc,FF}^{ijr}+I_{\rm sc,FF}^{ijr}+I_{\rm sc,FF}^{jir}\Big)\\
&+\theta_{r\in{\rm I}}\int d\Phi_n^{(ijr)}(k_r)\Big(I_{\rm hc,FI}^{ijr}+I_{\rm sc,FI}^{ijr}+I_{\rm sc,FI}^{jir}\Big)\\
&+\theta_{r\in{\rm I}}\int_0^1\frac{dx}{x}\int d\Phi_n^{(ijr)}(xk_r)\,\Big(J_{\rm hc,FI}^{ijr}(x)+J_{\rm sc,FI}^{ijr}(x)+J_{\rm sc,FI}^{jir}(x)\Big)\bigg]\,\bar B^{(ijr)}\,. \qquad (2.100)
\end{aligned}$$

The contributions proportional to $\theta_{r\in\mathrm{F}}$ or $\theta_{r\in\mathrm{I}}$ correspond to different prescriptions for the position of the recoiler particle. Similarly, integrating the constituents of $K_{\,\mathrm{hc,I}}$ yields

$$
\begin{aligned}
&\int d\Phi_{n+1}\,\overline{\mathbf{HC}}_{ij,\mathrm{I}}\,R\;= \qquad (2.101)\\
&= \mathcal{N}_1\,\frac{\varsigma_{n+1}}{\varsigma_n}\Bigg[\theta_{r\in\mathrm{F}}\iint d\Phi_n^{(irj)}(xk_j)\,d\Phi_{\mathrm{rad}}^{(irj)}(1-z)^{\gamma}\left(\frac{P^{\mathrm{hc}}_{[ij]i,\mathrm{I}}(x)}{x\,s_{ij}}+2C_{f_j}\mathcal{I}^{(i)}_{jr}\,\frac{1-x^{\alpha}}{1-z}\right)\bar{B}^{(irj)}\\
&\qquad +\theta_{r\in\mathrm{I}}\iint d\Phi_n^{(ijr)}(xk_j,k_r)\,d\Phi_{\mathrm{rad}}^{(ijr)}(1-v)^{\gamma}\left(\frac{P^{\mathrm{hc}}_{[ij]i,\mathrm{I}}(x)}{x\,s_{ij}}+2C_{f_j}\mathcal{I}^{(i)}_{jr}(1-x^{\alpha})(1-v)\right)\bar{B}^{(ijr)}\Bigg]\\
&\equiv \frac{\varsigma_{n+1}}{\varsigma_n}\Bigg\{\theta_{r\in\mathrm{F}}\Bigg[\int_0^1\frac{dx}{x}\int d\Phi_n^{(irj)}(xk_j)\Big(J^{irj}_{\mathrm{hc,IF}}(x)+J^{irj}_{\mathrm{sc,IF}}(x)\Big)+\int d\Phi_n^{(irj)}(k_j)\,I^{irj}_{\mathrm{sc,IF}}\Bigg]\bar{B}^{(irj)}\\
&\qquad +\theta_{r\in\mathrm{I}}\Bigg[\int_0^1\frac{dx}{x}\int d\Phi_n^{(ijr)}(xk_j,k_r)\Big(J^{ijr}_{\mathrm{hc,II}}(x)+J^{ijr}_{\mathrm{sc,II}}(x)\Big)+\int d\Phi_n^{(ijr)}(k_j,k_r)\,I^{ijr}_{\mathrm{sc,II}}\Bigg]\bar{B}^{(ijr)}\Bigg\}.
\end{aligned}
$$

All integrals $I^{iab}_{\mathrm{hc/sc},\star\star}$ and $J^{iab}_{\mathrm{hc/sc},\star\star}(x)$ appearing in the previous equations are collected in Appendix B.3.2.

Notice that, thanks to the procedure we implemented to construct our *local* counterterm, which combines a unitary phase-space partition with a smart mapping adaptation, all necessary integrations turn out to be surprisingly straightforward, and the resulting integrals involve nothing more complex than logarithms of kinematic invariants (see Appendix B.3).

Rearranging the outcomes

To obtain the final integrated counterterms I and J, two last steps are required. First, all various Born-level parametrisations are identified, as the corresponding phase spaces have identical support. This process entails the following relabelings:

$$
\{\bar{k}\}^{(abc)}\;\to\;\{k\}\,,\qquad d\Phi_n^{(abc)}\;\to\;d\Phi_n\,,\qquad \bar{B}^{(abc)}_{\dots}\;\to\;B_{\dots}\,. \qquad (2.102)
$$

Next, sums over $(n+1)$-body labels must be converted into Born-level sums. When removing a final-state gluon i, which is relevant to the soft case, one has

$$
\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{i\in\mathrm{F}}\delta_{f_i g}\;=\;1\,; \qquad (2.103)
$$

when two final-state particles i and j are replaced by the *parent* particle p, the sums over i and j can be reorganized as a sum over p according to

$$\begin{aligned}
\frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i\in\mathrm{F}} \sum_{\substack{j\in\mathrm{F}\\ j<i}} \delta_{\{f_i f_j\}\{q\bar{q}\}} &= N_f \sum_{p\in\mathrm{F}} \delta_{f_p g} \,, \\
\frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i\in\mathrm{F}} \sum_{\substack{j\in\mathrm{F}\\ j<i}} \big(\delta_{f_i\{q,\bar{q}\}}\delta_{f_j g} + \delta_{f_j\{q,\bar{q}\}}\delta_{f_i g}\big) &= \sum_{p\in\mathrm{F}} \delta_{f_p\{q,\bar{q}\}} \,, \\
\frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i\in\mathrm{F}} \sum_{\substack{j\in\mathrm{F}\\ j<i}} \delta_{f_i g}\, \delta_{f_j g} &= \frac{1}{2} \sum_{p\in\mathrm{F}} \delta_{f_p g} \,,
\end{aligned} \tag{2.104}$$

where N_f denotes the number of light active flavours; in the case of a final-state particle i and an initial-state particle j replaced by the resulting initial-state particle a, the relevant relations are

$$\begin{aligned}
\frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i\in\mathrm{F}} \sum_{j\in\mathrm{I}} \delta_{\{f_{[ij]} f_i\}\{q\bar{q}\}} &= \sum_{a\in\mathrm{I}} \delta_{f_a g} \,, \\
\frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i\in\mathrm{F}} \sum_{j\in\mathrm{I}} \delta_{f_{[ij]}\{q,\bar{q}\}}\, \delta_{f_i g} &= \sum_{a\in\mathrm{I}} \delta_{f_a\{q,\bar{q}\}} \,, \\
\frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i\in\mathrm{F}} \sum_{j\in\mathrm{I}} \delta_{f_{[ij]} g}\, \delta_{f_i\{q,\bar{q}\}} &= \sum_{a\in\mathrm{I}} \delta_{f_a\{q,\bar{q}\}} \,, \\
\frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i\in\mathrm{F}} \sum_{j\in\mathrm{I}} \delta_{f_{[ij]} g}\, \delta_{f_i g} &= \sum_{a\in\mathrm{I}} \delta_{f_a g} \,.
\end{aligned} \tag{2.105}$$

After such a procedure, all integrals mentioned above are naturally written in terms of Born-level quantities. For $\star = \mathrm{F}, \mathrm{I}$, one has

$$\begin{aligned}
I^{abc}_{\mathrm{s},\star\star} &\;\to\; I_{\mathrm{s},\star\star}\,(s_{bc}) \,, &\qquad J^{abc}_{\mathrm{s},\star\star}(x) &\;\to\; J_{\mathrm{s},\star\star}\,(s_{bc}, x) \,, \\
I^{abc}_{\mathrm{sc},\star\star} &\;\to\; 2\,C_{f_b}\, I_{\mathrm{sc},\star\star}\,(s_{bc}) \,, &\qquad J^{abc}_{\mathrm{sc},\star\star}(x) &\;\to\; 2\,C_{f_b}\, J_{\mathrm{sc},\star\star}\,(s_{bc}, x) \,,
\end{aligned} \tag{2.106}$$

$$\begin{aligned}
I^{abc}_{\mathrm{hc,F}\star} &\to \delta_{f_b g} \left[\frac{1}{2}\, I^{(2\mathrm{g})}_{\mathrm{hc,F}\star}\,(s_{bc}) + N_f\, I^{(0\mathrm{g})}_{\mathrm{hc,F}\star}\,(s_{bc}) \right] + \delta_{f_b\{q,\bar{q}\}}\, I^{(1\mathrm{g})}_{\mathrm{hc,F}\star}\,(s_{bc}) \,, \\
J^{abc}_{\mathrm{hc,F}\star}(x) &\to \delta_{f_b g} \left[\frac{1}{2}\, J^{(2\mathrm{g})}_{\mathrm{hc,F}\star}\,(s_{bc}, x) + N_f\, J^{(0\mathrm{g})}_{\mathrm{hc,F}\star}\,(s_{bc}, x) \right] + \delta_{f_b\{q,\bar{q}\}}\, J^{(1\mathrm{g})}_{\mathrm{hc,F}\star}\,(s_{bc}, x) \,, \\
J^{abc}_{\mathrm{hc,I}\star}(x) &\to \delta_{f_b g} \left[J^{(2\mathrm{g})}_{\mathrm{hc,I}\star}\,(s_{bc}, x) + J^{(0\mathrm{g})}_{\mathrm{hc,I}\star}\,(s_{bc}, x) \right] + \delta_{f_b\{q,\bar{q}\}}\, J^{(1\mathrm{g})}_{\mathrm{hc,I}\star}\,(s_{bc}, x) \,,
\end{aligned} \tag{2.107}$$

where, on the right-hand sides, b and c are Born-level labels. The quantities $I_{\mathrm{s/sc/hc},\star\star}(s)$ and $J_{\mathrm{s/sc/hc},\star\star}(s,x)$ appearing in the above identifications are collected in Appendices B.3.1 and B.3.2.

2.4 The subtracted virtual $V_{\rm sub}$ and collinear $C_{\rm sub}$ contributions

We are now at the stage of verifying that the integrated counterterm correctly reproduces all virtual ϵ poles, thus providing a valid local subtraction formula for generic NLO processes without massive colourful particles. We separately analyse the three distinct cases of 0, 1, 2 initial-state QCD partons, relevant to lepton-lepton, lepton-hadron, and hadron-hadron collisions, respectively. For these three process categories, we label the counterterm K as $K_{\rm F}$, $K_{\rm IF}$, and $K_{\rm IIF}$, respectively.

To properly define the hard-collinear contributions of these counterterms, it is essential to establish a rule for assigning the recoiler particle, denoted as r, to each splitting pair i and j. Our approach is to associate r with an initial-state particle whenever possible; otherwise, we assign it to a final-state particle. In the following discussion, we emphasise the assignment of r using theta factors like $\theta_{r\in\star}$, which have no purpose other than enabling a more straightforward interpretation of the equations.

Final-state radiation

We define the counterterm for leptonic processes as

$$K_{\rm F} \,=\, K_{\rm s} + \theta_{r\in{\rm F}}\, K_{\,{\rm hc,F}}\,, \tag{2.108}$$

where $K_{\rm s}$ and $K_{\,{\rm hc,F}}$ are defined in Eqs. (2.83) and (2.84). The notation underlines the fact that the emitting dipole jr appearing in the hard-collinear kernels is bound to belong to the final state.

The integration over the radiative phase space in Eq. (2.94), up to $\mathcal{O}(\epsilon)$, yields

$$I_{\rm F} \,=\, I_{\rm poles} + I_{\rm fin,F}\,, \tag{2.109}$$

where[9]

$$I_{\rm poles} \,=\, \frac{\alpha_{\rm S}}{2\pi}\bigg[\frac{1}{\epsilon^2}\sum_j C_{f_j}\,B + \frac{1}{\epsilon}\Big(\sum_j \gamma_j\,B + \sum_{c,d\neq c}{\rm L}_{cd}\,B_{cd}\Big)\bigg]\,, \tag{2.110}$$

$$\begin{aligned} I_{\rm fin,F} \,=\, \frac{\alpha_{\rm S}}{2\pi}\bigg\{&\Big[\sum_{k\in{\rm F}}\phi_k - \sum_j \gamma_j^{\rm hc}\,{\rm L}_{jr}\Big]B + \sum_{c,d\neq c}{\rm L}_{cd}\Big(2-\frac{1}{2}{\rm L}_{cd}\Big)B_{cd} \\ &+ 2\,A_2(\alpha)\Big[\sum_j C_{f_j}\,{\rm L}_{jr}\,B + \sum_{c,d\neq c}{\rm L}_{cd}\,B_{cd}\Big] + \sum_{k\in{\rm F}}\gamma_k^{\rm hc}\,A_2(\beta)\,B \\ &+\Big[A_2(\alpha)\Big(A_2(\alpha)-2\,A_2(\beta)\Big) - A_3(\alpha)\Big]\sum_j C_{f_j}\,B\bigg\}\,. \end{aligned} \tag{2.111}$$

[9]The expressions in Eqs. (2.110, 2.111) include sums running on final-state labels only, $\sum_{k\in{\rm F}}$, as well as on final- and initial-state labels, such as $\sum_j$ and $\sum_{c,d\neq c}$. Although in leptonic collisions the distinction is immaterial, as $C_{f_a} = \gamma_a = 0$ for initial-state particles, such a notation enables the direct unmodified use of Eq. (2.109) for hadronic collisions as well.

We have introduced some short-hand notation for logarithms, denoted as $\mathrm{L}_{ab} = \ln(s_{ab}/\mu^2)$, and for anomalous dimensions,

$$\begin{aligned}\gamma_a &= \frac{3}{2} C_F\, \delta_{f_a\{q,\bar q\}} + \frac{1}{2}\beta_0\, \delta_{f_a g}\,, \qquad \gamma_a^{\mathrm{hc}} \;=\; \gamma_a - 2\, C_{f_a}\,, \\ \phi_a &= \frac{13}{3} C_F\, \delta_{f_a\{q,\bar q\}} + \frac{4}{3}\beta_0\, \delta_{f_a g} + \left(\frac{2}{3} - \frac{7}{2}\zeta_2\right) C_{f_a}\,, \end{aligned} \tag{2.112}$$

where $C_A = N_c$, $C_F = (N_c^2-1)/(2N_c)$, $T_R = 1/2$, and $\beta_0 = \big(11\, C_A - 4\, T_R\, N_f\big)/3$ is the first coefficient of the QCD beta function. The functions $A_n(x)$ are specified in Appendix B.3.

The poles collected in Eq. (2.110) are correctly independent of the damping parameters α and β, and can be checked to exactly match those of virtual origin, see for instance [116], thus verifying the cancellation of singularities in Eq. (2.11). As a consequence, we can write the subtracted virtual contribution in an integrable form, as

$$V_{\mathrm{sub}}(X) \;=\; \big(V_{\mathrm{fin}} + I_{\mathrm{fin,F}}\big)\, \delta_n(X)\,, \tag{2.113}$$

where V_{fin} stands for the finite remainder of the one-loop correction. As for the finite contribution in Eq. (2.111), the second and third lines collect the full dependence upon the damping parameters, and cancel out as $\alpha = \beta = 0$.

Initial-state radiation: one initial-state QCD parton

The relevant local counterterm for a reaction with one incoming QCD parton is

$$K_{\mathrm{IF}} \;=\; K_{\mathrm{s}} + \theta_{r\in \mathrm{I}}\, K_{\,\mathrm{hc,F}} + \theta_{r\in \mathrm{F}}\, K_{\,\mathrm{hc,I}}\,, \tag{2.114}$$

where the singular kernels are listed in Eqs. (2.83)-(2.85). In $K_{\,\mathrm{hc,I}}$, a final-state recoiler is assigned since the only initial-state coloured parton is identified with j, the initial-state splitting particle. As for $K_{\,\mathrm{hc,F}}$, assigning a final-state recoiler is only possible if the process features at least one massless colourful parton in the final state at Born level, in addition to the final-state emitter j. On the other hand, identifying the recoiler with the initial-state colourful parton is always allowed.

The integration over the radiative phase space up to $\mathcal{O}(\epsilon)$ gives

$$\int d\Phi_{n+1}\, K_{\mathrm{IF}} \;=\; \int d\Phi_n(k_a)\, \Big(I_{\mathrm{F}} + I_{\mathrm{fin,I}}\Big) + \int_0^1 \frac{dx}{x} \int d\Phi_n(xk_a)\, J_{\mathrm{I}}(x)\,, \tag{2.115}$$

where I_{F} is the same as in Eq. (2.109), while $I_{\mathrm{fin,I}}$ is a purely finite contribution, which can be expressed as

$$I_{\mathrm{fin,I}} \;=\; \frac{\alpha_{\mathrm{s}}}{2\pi}\, 2\, C_{f_a} \left[1 + \frac{\zeta_2}{4} - A_2(\alpha)\Big(A_1(\gamma) - A_2(\beta) - 1\Big) + A_3(\alpha)\right] B\,, \tag{2.116}$$

where a labels the initial-state coloured parton. The x-independent integral on the right-hand side of Eq. (2.115) once again successfully reproduces the general pole structure of the virtual contribution. The remaining integral $J_{\rm I}(x)$, reading

$$\begin{aligned} J_{\rm I}(x) = \frac{\alpha_{\rm S}}{2\pi} \Bigg\{ & -\Big(\frac{1}{\epsilon} - {\rm L}_{ar}\Big)\bar{P}_a(x) + P^{(1)}_{a,{\rm fin}}(x) - \left(\frac{x^{1+\beta}}{1-x}\right)_+ \sum_{k\in{\rm F}} \Big(\gamma_k^{\rm hc} - 2\,C_{f_k}\,A_2(\alpha)\Big) \\ & + 2\,C_{f_a}\Bigg[\left(\frac{x\ln(1-x)}{1-x}\right)_+ - \left(\frac{x}{1-x}\right)_+ A_1(\gamma) \\ & \quad + \left(\frac{x^{1+\alpha}}{1-x}\right)_+ \Big(A_1(\gamma) - A_2(\alpha) - 1 - {\rm L}_{ar}\Big)\Bigg]\Bigg\}\,B \\ & - \frac{\alpha_{\rm S}}{2\pi}\left(\frac{x^{1+\alpha}}{1-x}\right)_+ \sum_{k\in{\rm F}} 2\,{\rm L}_{ak}\,B_{ak}\,, \end{aligned} \tag{2.117}$$

with $P^{(1)}_{a,{\rm fin}}(x)$ defined in Appendix B.1, is instrumental to tame the single pole stemming from collinear factorisation, as contained in Eq. (2.13). It is straightforward to check that the sum $C_{\rm sub}(X) = \big(C(x) + J_{\rm I}(x)\big)\,\delta_n(X)$ is finite in $d = 4$, and exhibits a leftover logarithmic dependence upon the factorisation scale μ_F, in the form

$$C(x) + J_{\rm I}(x) \supset -\frac{\alpha_{\rm S}}{2\pi}\,\ln\mu_F^2\,\bar{P}_a(x)\,B\,, \tag{2.118}$$

which cancels the $\mathcal{O}(\alpha_{\rm S})$ DGLAP μ_F dependence from the PDF.

Initial-state radiation: two initial-state QCD partons

The local counterterm for a process involving two incoming colourful partons is

$$K_{\rm IIF} = K_{\rm s} + \theta_{r\in{\rm I}}\Big(K_{\rm hc,F} + K_{\rm hc,I}\Big)\,, \tag{2.119}$$

where the selection of an initial recoiler r is dictated by the general availability, for this class of processes, of an extra initial-state QCD parton regardless of the position of the emitter j.

Counterterm integration up to $\mathcal{O}(\epsilon)$ gives

$$\int d\Phi_{n+1}\,K_{\rm IIF} = \int d\Phi_n(k_a,k_b)\Big(I_{\rm F} + I_{\rm fin,II}\Big) + \int_0^1\frac{dx}{x}\int_0^1\frac{d\hat{x}}{\hat{x}}\int d\Phi_n(xk_a,\hat{x}k_b)\,J_{\rm II}(x,\hat{x})\,. \tag{2.120}$$

As above, $I_{\rm F}$ refers to Eq. (2.109), reproducing the general virtual-pole content. The remaining x-independent contributions are collected in

$$\begin{aligned} I_{\rm fin,II} = \frac{\alpha_{\rm S}}{2\pi}\Bigg\{ & \Big[2 + \frac{\zeta_2}{2} + 3\,A_3(\alpha) - A_2(\alpha)\Big(2\,A_1(\gamma) - 2\,A_2(\beta) + A_2(\alpha)\Big)\Big]\Big(C_{f_a} + C_{f_b}\Big)\,B \\ & + 4\Big(\zeta_2 - 1 + A_3(\alpha)\Big)\,B_{ab}\Bigg\}\,, \end{aligned} \tag{2.121}$$

with a, b labelling the two incoming coloured partons.

The contribution $J_{\rm II}(x,\hat{x}) \equiv J_{a,\rm II}(x)\,\delta(1-\hat{x}) + J_{b,\rm II}(\hat{x})\,\delta(1-x)$ accounts separately for the configurations in which the initial-state colourful parton a or b, respectively, enters the Born-level amplitude with rescaled momentum. Since none of our mappings features a simultaneous rescaling of both initial-state momenta, the simultaneous dependence on both x and $\hat{x}$ is trivial in $J_{\rm II}(x,\hat{x})$. Explicitly, for $i=a,b$, one has

$$\begin{aligned}
J_{i,\rm II}(x) = \frac{\alpha_{\rm S}}{2\pi}\bigg\{ &-\Big(\frac{1}{\epsilon}-{\rm L}_{ir}\Big)\bar{P}_i(x) + P^{(2)}_{i,\rm fin}(x) - \bigg(\frac{x^{1+\beta}}{1-x}\bigg)_+ \sum_{k\in \rm F}\Big(\gamma_k^{\rm hc} - 2\,C_{f_k}\,A_2(\alpha)\Big) \\
&+ 2\,C_{f_i}\bigg[\,2\,\bigg(\frac{x\ln(1-x)}{1-x}\bigg)_+ - \bigg(\frac{x^{1+\alpha}\ln(1-x)}{1-x}\bigg)_+ - \bigg(\frac{x}{1-x}\bigg)_+ A_1(\gamma) \\
&\qquad + \bigg(\frac{x^{1+\alpha}}{1-x}\bigg)_+ \Big(A_1(\gamma) - A_2(\alpha) - 1 - {\rm L}_{ab}\Big)\bigg]\bigg\}\,B \\
&- \frac{\alpha_{\rm S}}{2\pi}\,2\bigg[\bigg(\frac{x^{1+\alpha}\ln(1-x)}{1-x}\bigg)_+ + \bigg(\frac{x^{1+\alpha}}{1-x}\bigg)_+ \Big(A_2(\alpha) + 1 + {\rm L}_{ab}\Big)\bigg]\,B_{ab} \\
&- \frac{\alpha_{\rm S}}{2\pi}\bigg(\frac{x^{1+\alpha}}{1-x}\bigg)_+ \sum_{k\in \rm F} 2\,{\rm L}_{ik}\,B_{ik}\,.
\end{aligned} \tag{2.122}$$

The same considerations regarding collinear-pole cancellation and μ_F dependence apply as in the case of single initial-state QCD parton, therefore concluding the proof of ϵ - pole cancellation by means of the Local Analytic Sector Subtraction procedure.

Chapter 3

Local Analytic Sector Subtraction at NNLO

The implementation of our general analytic formula to address the NLO QCD subtraction problem, as detailed in Chapter 2, has provided valuable insights into the underlying mechanisms of our approach. In particular, this effort has demonstrated that optimising the counterterm structure throughout all stages of the calculation, by systematically leveraging all available degrees of freedom, offers substantial advantages in terms of simplifying the required integrations. The discussion in Section 2.3 and the explicit results presented in Appendix B.3 underscore the effectiveness of this strategy.

The achieved computational simplicity is an encouraging result that strongly motivates the extension of this method to address the subtraction problem beyond NLO. Exporting this simplicity to higher perturbative orders is highly desirable, especially considering that practical implementations of algorithms producing state-of-the-art predictions often encounter substantial computational complexity. This complexity has, so far, hindered the community endevours to reach the same degree of universality and efficiency as was accomplished at NLO (see Section 1.3.2).

In this Chapter we present the extension of the subtraction procedure developed within the framework of Local Analytic Sector Subtraction to address the treatment of NNLO infrared singularities. The final outcome of this algorithm is a completely analytic subtraction formula, which provides the NNLO contribution to the differential distribution for any infrared-safe observable built out of massless coloured final states (along with an arbitrary number of massive or massless colourless final-state particles). It only requires as input the relevant matrix elements, which include the double-virtual correction to the Born-level process, the one-loop correction to the single-radiation process, and the tree-level expression for the double-real-emission contribution.

This Chapter is structured as follows. We start by introducing the framework of our algorithm in Section 3.1, which expands upon the discussion outlined in Section 2.1 by introducing the essential ingredients for a NNLO subtraction. In Section 3.2, we implement the strategy proposed in Section 1.3.3 to construct three distinct local counterterms, labelled as the *single-unresolved*, *uniform double-unresolved* and *strongly-ordered double-unresolved* subtraction terms, specifically designed to reproduce all double-unresolved phase-space singularities. The resulting outcome is a subtracted double-real contribution that is integrable over the entire radiative phase space. At this level, the multiplicity of singular configurations and of their overlaps will lead to long and intricate expressions: therefore, detailed formulas for NNLO soft and collinear kernels, and for the relevant improved limits, will be presented in the Appendices C.1 and C.2. Section 3.3 organises the integration procedure for all counterterms associated with double-real radiation, expressing the necessary integrals in terms of a small set of constituent integrals, which are collected in Appendix C.5. These integrals have all been computed analytically [200], requiring only standard techniques. Section 3.4 presents the subtracted real-virtual correction, providing an explicit expression for the real-virtual counterterm. By combining together the real-virtual correction with its local counterterm, and the integrals of the single-unresolved and the strongly-ordered counterterms, we build an expression that is both free of infrared poles and integrable in the radiative phase space. Next, we discuss in Section 3.5 the integration of the real-virtual counterterm, which again can be organised in terms of simple integrals. Lastly, Section 3.6 introduces the subtracted double-virtual contribution, which is free of infrared poles. This finally completes our subtraction programme for generic massless QCD final states.

3.1 Generalities

The NNLO contribution to the differential cross section in Eq. (2.2) when QCD radiation is limited to the final state can be written as

$$\frac{d\sigma_{\rm NNLO}}{dX} = \lim_{d\to 4}\left[\int d\Phi_n\, VV\,\delta_n(X) + \int d\Phi_{n+1}\, RV\,\delta_{n+1}(X) + \int d\Phi_{n+2}\, RR\,\delta_{n+2}(X)\right], \quad (3.1)$$

where

$$RR = \left|\mathcal{A}^{(0)}_{n+2}\right|^2,$$

$$RV = 2\,\mathbf{Re}\left[\mathcal{A}^{(0)\dagger}_{n+1}\,\mathcal{A}^{(1)}_{n+1}\right], \qquad VV = \left|\mathcal{A}^{(1)}_{n}\right|^2 + 2\,\mathbf{Re}\left[\mathcal{A}^{(0)\dagger}_{n+1}\,\mathcal{A}^{(2)}_{n+1}\right]. \quad (3.2)$$

In this case, the $\overline{\rm MS}$-renormalised double-virtual contribution VV exhibits IR poles up to ϵ^{-4}, the double-real RR contains up to four phase-space singularities, and the $\overline{\rm MS}$-renormalised real-virtual term RV has poles up to ϵ^{-2} and up to two phase-space singularities. In order to rewrite Eq. (3.1) as a sum of finite contributions, we introduce *four* local counterterms, denoted as $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$, $K^{(\mathbf{12})}$ and $K^{(\mathbf{RV})}$. The counterterm $K^{(\mathbf{1})}$ is designed to reproduce all phase-space singularities of RR where a single particle becomes unresolved, while $K^{(\mathbf{2})}$ addresses situations where two particles become unresolved at the same rate. The two sets of singularities overlap, and $K^{(\mathbf{12})}$ is responsible for subtracting the double-counted overlap region. Finally, $K^{(\mathbf{RV})}$ is defined to subtract the phase-space singularities arising from the single-real radiation in RV.

In order to integrate these counterterms, we need to introduce phase-space parametrisations factorising single and double radiation. In this case we will require the factorisations

$$d\Phi_{n+2} \;=\; \frac{\varsigma_{n+2}}{\varsigma_{n+1}}\, d\Phi_{n+1}\, d\Phi_{\rm rad}\,,$$

$$d\Phi_{n+2} \;=\; \frac{\varsigma_{n+2}}{\varsigma_{n}}\, d\Phi_{n}\, d\Phi_{\rm rad,2}\,, \qquad d\Phi_{n+1} \;=\; \frac{\varsigma_{n+1}}{\varsigma_{n}}\, d\Phi_{n}\, d\Phi_{\rm rad}\,. \tag{3.3}$$

Once a parametrisation yielding Eq. (3.3) is in place, one can define *integrated counterterms* as

$$I^{(\mathbf{1})} \;\equiv\; \int d\Phi_{\rm rad}\, K^{(\mathbf{1})}\,, \qquad I^{(\mathbf{2})} \;\equiv\; \int d\Phi_{\rm rad,2}\, K^{(\mathbf{2})}\,,$$

$$I^{(\mathbf{12})} \;\equiv\; \int d\Phi_{\rm rad}\, K^{(\mathbf{12})}\,, \qquad I^{(\mathbf{RV})} \;\equiv\; \int d\Phi_{\rm rad}\, K^{(\mathbf{RV})}\,. \tag{3.4}$$

We are now ready to present the master formula for our subtraction at NNLO: in practice, we aim to construct an expression of the form

$$\frac{d\sigma_{\rm NNLO}}{dX} \;=\; \int d\Phi_n\, VV_{\rm sub}(X) + \int d\Phi_{n+1}\, RV_{\rm sub}(X) + \int d\Phi_{n+2}\, RR_{\rm sub}(X)\,, \tag{3.5}$$

where each one of the three contributions is finite in ϵ and is free from phase-space singularities.

Using the previously introduced local counterterms, and their integrals over the radiative degrees of freedom, the subtracted matrix elements $VV_{\rm sub}$, $RV_{\rm sub}$ and $RR_{\rm sub}$ can be expressed as

$$VV_{\rm sub}(X) \equiv \big(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\big)\, \delta_n(X)\,, \tag{3.6}$$

$$RV_{\rm sub}(X) \equiv \big(RV + I^{(\mathbf{1})}\big)\, \delta_{n+1}(X) \;-\; \big(K^{(\mathbf{RV})} + I^{(\mathbf{12})}\big)\, \delta_n(X)\,, \tag{3.7}$$

$$RR_{\rm sub}(X) \equiv RR\, \delta_{n+2}(X) - K^{(\mathbf{1})}\, \delta_{n+1}(X) - \big(K^{(\mathbf{2})} - K^{(\mathbf{12})}\big)\, \delta_n(X)\,. \tag{3.8}$$

Eqs. (3.5) and (3.6)-(3.8) provide an identical rewriting of Eq. (3.1), and their logic is as follows:

- in Eq. (3.8), $RR_{\rm sub}(X)$ term must be integrated in the full phase space Φ_{n+2}, and it is built out of tree-level quantities[1], therefore has no explicit IR poles. It is also free from phase-space singularities, since single-unresolved contributions are subtracted

[1] We have implicitly assumed that the underlying Born reaction is associated with tree-level diagrams; however, in case of loop-induced processes, all arguments and techniques presented in this Chapter carry over.

by $K^{(\mathbf{1})}$, double-unresolved contributions are subtracted by $K^{(\mathbf{2})}$, and their double-counted overlap is reinstated by adding back $K^{(\mathbf{12})}$.

- in Eq. (3.7), RV must be integrated in Φ_{n+1}, and is affected by both explicit IR poles and phase-space singularities. The IR poles arising from the loop integration in RV are removed by the integral $I^{(\mathbf{1})}$, by virtue of general cancellation theorems, making the first parenthesis finite. However, both terms are singular in the phase space of the radiated particle. By construction, the phase-space singularities of $I^{(\mathbf{1})}$ are cancelled by $I^{(\mathbf{12})}$, while $K^{(\mathbf{RV})}$ is designed to cancel the phase-space singularities of RV. Nevertheless, this does not guarantee that explicit IR poles will cancel in the second parenthesis. Anyway, one can fine-tune the definition of $K^{(\mathbf{RV})}$, by including explicit IR poles not affecting the phase-space singularity cancellation with RV, in order to make the second parenthesis finite as well. At this point, Eq. (3.7) is both finite and integrable.
- The complete cancellation of real and virtual singularities in Eq. (3.7) and Eq. (3.8) guarantees then, as a consequence of the KLN theorem, that Eq. (3.6), which is to be integrated over the Born-level phase space Φ_n, will be free of IR poles.

3.2 The subtracted double-real contribution $RR_{\rm sub}$

In this Section we provide a detailed construction of the subtracted matrix element squared for double-real radiation, $RR_{\rm sub}$. As noted in Eq. (3.8), this will require the definition of three distinct local counterterms. This task represents the most intricate part of the NNLO-subtraction programme from a combinatorial viewpoint, due to the large number of overlapping singular limits affecting double-real radiation. In analogy to Section 2.2, we will proceed as follows: first, in Section 3.2.1, we will identify the relevant singular limits, which can be single- or double-unresolved; next, we will introduce a set of sector functions, smoothly partitioning the $(n+2)$-particle phase space so as to minimise the number of singular configurations to be considered in any given sector (Sec. 3.2.2). These sectors will naturally be grouped into three different *topologies*, corresponding to the specific structure of the limits relevant to each sector. In Section 3.2.3, we will identify specific combinations of limits that yield integrable contributions in each topology, in the spirit of Eq. (2.21); we will then construct a family of phase-space mappings in order to properly factorise the double-radiative phase space in all relevant configurations (Sec. 3.2.4). Finally, in Section 3.2.5, we will introduce improved limits appropriate for each topology, discuss the required consistency relations, and then use those improved limits to write an expression for the subtracted double-real contribution $RR_{\rm sub}$. As was the case at NLO for single-real radiation, it is possible to improve upon the resulting expression for $RR_{\rm sub}$ by introducing symmetrised sector functions in order to optimise the subsequent numerical integration

(Sec. 3.2.6). We note that the construction presented in the following sections differs slightly in some technical choices from the one given in Ref. [199]: we will stress the differences as we go along.

3.2.1 Singular limits

Double-real squared matrix elements are characterised by a variety of overlapping singular limits. It is important, from the outset, to select a complete set of limits, in order to study (and then subtract) their overlaps, preventing double counting. Clearly, single-unresolved soft and collinear limits are relevant also for double radiation, so our list must include the limits $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ introduced in Section 2.2.1. Next, we collect all possible double-unresolved limits. Importantly, when two particles become unresolved, one needs to distinguish *uniform* limits, where two particles become unresolved at the same rate, and *strongly-ordered* limits, where one particle becomes unresolved at a higher rate with respect to the second one. Obviously, this distinction becomes significant starting at NNLO. Our set of fundamental uniform limits consists of four independent configurations. First, two particles i and j can become soft at the same rate, a limit which we denote by $\mathbf{S}_{ij}$; second, a single hard particle can branch into three collinear ones, i, j and k, a limit which we denote by $\mathbf{C}_{ijk}$; third, two hard partons can independently branch into two collinear pairs, which we denote by $\mathbf{C}_{ijkl}$, with (i, j) and (k, l) labelling the two independent pairs; finally, a particle i can become soft while another pair of particles, j and k, become collinear at the same rate[2], which we denote by $\mathbf{SC}_{ijk}$. In these four limits, the double-real-radiation squared matrix element factorises, with the universal relevant kernels derived and presented in Ref. [129]. Given these uniform limits, the strongly-ordered ones can be reached by acting iteratively: for example, the strongly-ordered double-soft limit, with particle i becoming soft faster than particle j, can be reached by computing $\mathbf{S}_i\,\mathbf{S}_{ij}$,while the strongly-ordered double-collinear limit, with particles i and j becoming collinear faster than the third particle k, will be given by the combination $\mathbf{C}_{ij}\,\mathbf{C}_{ijk}$. All singular configurations can be reached in this way.

In order to proceed, we need to characterise the limits in terms of phase-space variables. As was the case at NLO, we choose to define the limits in terms of Mandelstam invariants, and we pay attention to the fact that all limits must commute when applied to the double-real radiation squared matrix element. Using the variables e_i and w_{ij} given in Eq. (2.15), the definitions of the independent single- and double-unresolved limits are specified in Table 3.1. Importantly, our choice of independent limits is related to how we choose to define sector functions, which will be tuned so that only a minimal pre-defined set of the chosen limits will contribute in each sector.

[2] In Ref. [199], two strongly-ordered soft-collinear limits were considered, instead of the uniform one chosen here.

$\mathbf{S}_i$	$e_i \to 0$ (soft configuration of parton i)
$\mathbf{C}_{ij}$	$w_{ij} \to 0$ (collinear configuration of partons (i,j))
$\mathbf{S}_{ij}$	$e_i, e_j \to 0$ and $e_i/e_j \to$ constant (uniform double-soft configuration of partons (i,j))
$\mathbf{C}_{ijk}$	$w_{ij}, w_{ik}, w_{jk} \to 0$ and $w_{ij}/w_{ik}, w_{ij}/w_{jk}, w_{ik}/w_{jk} \to$ constant (uniform double-collinear configuration of partons (i,j,k))
$\mathbf{C}_{ijkl}$	$w_{ij}, w_{kl} \to 0$ and $w_{ij}/w_{kl} \to$ constant (uniform double-collinear configuration of partons (i,j) and (k,l))
$\mathbf{SC}_{ijk}$	$e_i, w_{jk} \to 0$ and $e_i/w_{jk} \to$ constant (uniform soft and collinear configuration for partons i and (j,k))

Table 3.1. Definitions of the single-unresolved singular limits $\mathbf{S}_i$, $\mathbf{C}_{ij}$ and of our set of basic independent double-unresolved singular limits $\mathbf{S}_{ij}$, $\mathbf{C}_{ijk}$, $\mathbf{C}_{ijkl}$, $\mathbf{SC}_{ijk}$.

3.2.2 Sector functions and topologies

We now introduce a smooth unitary partition of the double-real-radiation phase space, in the spirit of Ref. [16]. Since at most four particles can be involved in singular infrared limits at NNLO, we label the sector functions with four indices, and denote them by $\mathcal{W}_{ijkl}$. We reserve the first two indices to label the single-unresolved configurations assigned to the chosen sector. In particular, we will design the sector $(ijkl)$ to be non-zero in the limits $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ (thus we take $j \neq i$). We then need to distinguish sectors involving only three distinct particles from sectors involving four distinct particles. In sectors where only three particles are involved, the double-unresolved limit $\mathbf{C}_{ijk}$ will be relevant; furthermore, a second particle (besides i) may become soft in these configurations, and it can be particle j *or* particle k. Correspondingly, we will have distinct sector functions $\mathcal{W}_{ijjk}$ and $\mathcal{W}_{ijkj}$, where we take the third index to indicate the second particle that can become soft. Similarly, if the four indices are all different, we take $\mathcal{W}_{ijkl}$ to identify the sector where particles i and k can become soft, while the possible collinear pairs are (i,j) and (k,l). Notice that in all cases the last three indices j, k and l are distinct from i, and $k \neq l$. We will refer to the three allowed combinations of sector indices, $(ijjk)$, $(ijkj)$ and $(ijkl)$ as *topologies*, and we will denote them collectively by $\tau \equiv abcd \in \{ijjk, ijkj, ijkl\}$.

It is now necessary to introduce a precise definition of NNLO sector functions, which will enable us to list all the fundamental limits contributing to each topology. In analogy with NLO case (see Eq. (2.19)), we define NNLO sector functions as ratios of the type

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sigma}, \qquad \sigma = \sum_{a,b\neq a} \sum_{\substack{c\neq a \\ d\neq a,c}} \sigma_{abcd}, \tag{3.9}$$

so that

$$\sum_{a,b\neq a} \sum_{\substack{c\neq a \\ d\neq a,c}} \mathcal{W}_{abcd} = 1. \tag{3.10}$$

Such a partition allows us to rewrite the double-real squared matrix element RR as

$$RR = \sum_{i,\,j\neq i}\sum_{k\neq i}\sum_{l\neq i,k} RR\,\mathcal{W}_{ijkl} = \sum_{i,\,j\neq i}\sum_{k\neq i,j}\left[RR\,\mathcal{W}_{ijjk} + RR\,\mathcal{W}_{ijkj} + \sum_{l\neq i,j,k} RR\,\mathcal{W}_{ijkl}\right]. \quad (3.11)$$

In particular, our choice for the functions σ_{abcd}[3] is given by

$$\sigma_{abcd} = \frac{1}{(e_a\,w_{ab})^{\alpha}}\frac{1}{(e_c+\delta_{bc}\,e_a)\,w_{cd}}\,, \qquad \alpha > 1\,. \quad (3.12)$$

Having specified Eq. (3.12), we can list which of the fundamental limits discussed in Section 3.2.1 affect each topology. One finds that the combination $RR\,\mathcal{W}_\tau$ is singular in the limits listed below.

$$\begin{aligned}
RR\,\mathcal{W}_{ijjk} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ij}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,; \\
RR\,\mathcal{W}_{ijkj} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,, \quad \mathbf{SC}_{kij}\,; \\
RR\,\mathcal{W}_{ijkl} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijkl}\,, \quad \mathbf{SC}_{ikl}\,, \quad \mathbf{SC}_{kij}\,.
\end{aligned} \quad (3.13)$$

In analogy with the NLO sum-rule requirements in Eq. (2.20), also NNLO sector functions which share a given singular configuration must form a unitary partition. This is a crucial feature in order to minimise the complexity of the counterterm structure in view of analytic integration. The choice of the functions σ_{abcd} in Eq. (3.12) guarantees that the required partial sums reduce to unity. As an example, we report the sum rules for the double-unresolved limits in Table 3.1, which read

$$\mathbf{S}_{ik}\left(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\right) = 1\,, \quad (3.14)$$

$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\pi(ijk)}\left(\mathcal{W}_{abbc}+\mathcal{W}_{abcb}\right) = 1\,, \quad (3.15)$$

$$\mathbf{C}_{ijkl}\sum_{\substack{ab\,\in\,\pi(ij)\\ cd\,\in\,\pi(kl)}}\left(\mathcal{W}_{abcd}+\mathcal{W}_{cdab}\right) = 1\,, \quad (3.16)$$

$$\mathbf{SC}_{ijk}\left(\sum_{\substack{d\neq i\\ ab\,\in\,\pi(jk)}}\mathcal{W}_{idab} + \sum_{\substack{d\neq i,a\\ ab\,\in\,\pi(jk)}}\mathcal{W}_{abid}\right) = 1\,, \quad (3.17)$$

where by $\pi(ij)$ and $\pi(ijk)$ we denote the sets $\{ij, ji\}$ and $\{ijk, ikj, jik, jki, kij, kji\}$, respectively.

In order for the double-real contribution to properly combine with the real-virtual

[3] This choice corresponds to setting $\alpha = \beta$ in the NNLO sector functions introduced in Ref. [199].

correction, we further require NNLO sector functions to factorise into NLO-like sector functions under the action of single-unresolved limits. As discussed in Ref. [199], and below in Section 3.4, this ensures the local cancellation of integrated phase-space singularities with the poles of the real-virtual correction, sector by sector in the single-radiative phase space: indeed RV needs to be partitioned with NLO-like sector functions, since it involves a single-real radiation. As an easy check, one may verify that the sector functions for the topology $(ijjk)$ satisfy

$$\begin{aligned} \mathbf{S}_i \, \mathcal{W}_{ijjk} &= \mathcal{W}_{jk} \, \mathbf{S}_i \, \mathcal{W}_{ij}^{(\alpha)} \, , \\ \mathbf{C}_{ij} \, \mathcal{W}_{ijjk} &= \mathcal{W}_{[ij]k} \, \mathbf{C}_{ij} \, \mathcal{W}_{ij}^{(\alpha)} \, , \\ \mathbf{S}_i \, \mathbf{C}_{ij} \, \mathcal{W}_{ijjk} &= \mathcal{W}_{jk} \, \mathbf{S}_i \, \mathbf{C}_{ij} \, \mathcal{W}_{ij}^{(\alpha)} \, , \end{aligned} \tag{3.18}$$

where $\mathcal{W}_{[ij]k}$ is the NLO sector function defined in the $(n+1)$-particle phase space including the parent parton $[ij]$ of the collinear pair (i,j), and we introduced the NLO-like, α-dependent sector functions

$$\mathcal{W}_{ij}^{(\alpha)} \equiv \frac{\sigma_{ij}^{(\alpha)}}{\sum_{k \neq l} \sigma_{kl}^{(\alpha)}} \, , \qquad \sigma_{ij}^{(\alpha)} \equiv \frac{1}{(e_i \, w_{ij})^{\alpha}} \, , \qquad \alpha > 1 \, , \tag{3.19}$$

so that ordinary NLO sector functions are given by $\mathcal{W}_{ij} = \mathcal{W}_{ij}^{(1)}$. Similar relations hold for the other two topologies.

3.2.3 Candidate local counterterms

As listed in Eq. (3.13), a limited number of products of IR projectors is sufficient to collect all singular configurations of the double-real squared matrix element in each topology. Since the action of the relevant limits on both RR and on the sector functions does not depend on the order they are applied, the following combinations are by construction integrable in the whole phase space:

$$\begin{aligned} (1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})(1-\mathbf{SC}_{ijk}) \, RR \, \mathcal{W}_{ijjk} &\to \text{integrable,} \\ (1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})(1-\mathbf{SC}_{ijk})(1-\mathbf{SC}_{kij}) \, RR \, \mathcal{W}_{ijkj} &\to \text{integrable,} \\ (1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})(1-\mathbf{SC}_{ikl})(1-\mathbf{SC}_{kij}) \, RR \, \mathcal{W}_{ijkl} &\to \text{integrable.} \end{aligned} \tag{3.20}$$

Note that, in analogy to the definition used for NLO projection operators, if we take $\mathbf{L}$ to be any one of the singular limits in Table 3.1, the relation $\mathbf{L} \, RR \, \mathcal{W}_{abcd} \equiv (\mathbf{L} \, RR) \, (\mathbf{L} \, \mathcal{W}_{abcd})$ is understood for all topologies.

Applying directly Eq. (3.20) to construct local counterterms would be quite cumbersome, as the three lines generate a total of 160 terms. Fortunately, the resulting combinations of limits are not all independent, and several non-trivial relations can be obtained exploiting the symmetries of the limits under exchanges of indices, as well as the actual definitions of the various limits involved as projection operators on singular terms of RR. Consider for example, in four-particle sector $\mathcal{W}_{ijkl}$, the projection $(1-\mathbf{S}_{ik})\,RR\,\mathcal{W}_{ijkl}$. This will contain only terms in RR that are not singular in sector $(ijkl)$ when the uniform soft limit is taken for particles i and k. As a consequence, if further projections involving *both* the i and k soft limits are taken, the result will be integrable. We conclude, for example, that

$$\mathbf{SC}_{ikl}\,\mathbf{SC}_{kij}(1-\mathbf{S}_{ik})\,RR\,\mathcal{W}_{ijkl} \rightarrow \text{integrable}\,. \tag{3.21}$$

Working in this way, topology by topology, we can write a set of finite relations, which help us remove redundant configurations contributing to Eq. (3.20). They read

$$\begin{aligned}
\mathbf{C}_{ij}\,\mathbf{SC}_{ijk}(1-\mathbf{S}_i)(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\rightarrow \text{integrable}\,,\\
\mathbf{S}_i\,\mathbf{SC}_{kij}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\rightarrow \text{integrable}\,,\\
\mathbf{C}_{ij}\,\mathbf{SC}_{ijk}(1-\mathbf{S}_i)(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\rightarrow \text{integrable}\,,\\
\mathbf{C}_{ij}\,\mathbf{S}_{ik}(1-\mathbf{S}_i)(1-\mathbf{SC}_{kij})(1-\mathbf{C}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\rightarrow \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\mathbf{SC}_{kij}(1-\mathbf{S}_{ik})\,RR\,\mathcal{W}_{ijkj} &\rightarrow \text{integrable}\,,\\
\mathbf{S}_i\,\mathbf{SC}_{kij}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\,RR\,\mathcal{W}_{ijkl} &\rightarrow \text{integrable}\,,\\
\mathbf{S}_i\,\mathbf{C}_{ijkl}(1-\mathbf{S}_{ik})(1-\mathbf{SC}_{ikl})\,RR\,\mathcal{W}_{ijkl} &\rightarrow \text{integrable}\,,\\
\mathbf{C}_{ij}\,\mathbf{SC}_{ikl}(1-\mathbf{S}_i)(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\,RR\,\mathcal{W}_{ijkl} &\rightarrow \text{integrable}\,,\\
\mathbf{C}_{ij}\,\mathbf{S}_{ik}(1-\mathbf{S}_i)(1-\mathbf{SC}_{kij})(1-\mathbf{C}_{ijkl})\,RR\,\mathcal{W}_{ijkl} &\rightarrow \text{integrable}\,,\\
\mathbf{SC}_{ikl}\,\mathbf{SC}_{kij}(1-\mathbf{S}_{ik})\,RR\,\mathcal{W}_{ijkl} &\rightarrow \text{integrable}\,,\\
\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}(1-\mathbf{SC}_{ikl})\,RR\,\mathcal{W}_{ijkl} &\rightarrow \text{integrable}\,,\\
\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}(1-\mathbf{SC}_{kij})\,RR\,\mathcal{W}_{ijkl} &\rightarrow \text{integrable}\,.
\end{aligned} \tag{3.22}$$

These relations considerably simplifies Eq. (3.20), leading to the integrable expression

$$RR\,\mathcal{W}_\tau - \Big(\mathbf{L}^{(\mathbf{1})}_{ij} + \mathbf{L}^{(\mathbf{2})}_{\tau} - \mathbf{L}^{(\mathbf{12})}_{\tau}\Big)\,RR\,\mathcal{W}_\tau \;\rightarrow\; \text{integrable}\,, \tag{3.23}$$

which is the NNLO equivalent of Eq. (2.21) for double-real radiation[4]. In Eq. (3.23) we distinguished, for each topology τ, the single-unresolved limit $\mathbf{L}^{(\mathbf{1})}_{ij}$, the uniform double-unresolved limit $\mathbf{L}^{(\mathbf{2})}_{\tau}$, and the strongly-ordered double-unresolved limit $\mathbf{L}^{(\mathbf{12})}_{\tau}$. Their explicit expressions for each topology, in terms of the projectors discussed in Section 3.2.1,

[4]Note that there is no ambiguity in the notation: we denote by (ij) the first two indices of the sector, which are common to all three topologies.

are

$$
\begin{aligned}
\mathbf{L}^{(\mathbf{1})}_{ij} &= \mathbf{S}_i + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\ , \\
\mathbf{L}^{(\mathbf{2})}_{ijjk} &= \mathbf{S}_{ij} + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij}) + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij})\,(1-\mathbf{SC}_{ijk})\ , \\
\mathbf{L}^{(\mathbf{2})}_{ijkj} &= \mathbf{S}_{ik} + (\mathbf{SC}_{ijk} + \mathbf{SC}_{kij})\,(1-\mathbf{S}_{ik}) + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ik})\,(1-\mathbf{SC}_{ijk}-\mathbf{SC}_{kij})\ , \\
\mathbf{L}^{(\mathbf{2})}_{ijkl} &= \mathbf{S}_{ik} + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})\,(1-\mathbf{S}_{ik}) + \mathbf{C}_{ijkl}\,(1+\mathbf{S}_{ik}-\mathbf{SC}_{ikl}-\mathbf{SC}_{kij})\ , \\
\mathbf{L}^{(\mathbf{12})}_{ijjk} &= \mathbf{S}_i\Big[\mathbf{S}_{ij} + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij}) + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij})\,(1-\mathbf{SC}_{ijk})\Big] \\
&\qquad + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij})\Big]\,, \\
\mathbf{L}^{(\mathbf{12})}_{ijkj} &= \mathbf{S}_i\Big[\mathbf{S}_{ik} + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ik}) + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ik})\,(1-\mathbf{SC}_{ijk})\Big] \\
&\qquad + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\Big[\mathbf{SC}_{kij} + \mathbf{C}_{ijk}\,(1-\mathbf{SC}_{kij})\Big]\,, \\
\mathbf{L}^{(\mathbf{12})}_{ijkl} &= \mathbf{S}_i\Big[\mathbf{S}_{ik} + \mathbf{SC}_{ikl}\,(1-\mathbf{S}_{ik})\Big] + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\Big[\mathbf{SC}_{kij} + \mathbf{C}_{ijkl}\,(1-\mathbf{SC}_{kij})\Big]\,.
\end{aligned}
\tag{3.24}
$$

The projection operators appearing in Eq. (3.24) are organised so as to display, in order, the soft ($\mathbf{S}$), the uniform soft and collinear ($\mathbf{SC}$) and the collinear ($\mathbf{C}$) singular contributions. Upon summing over sectors, Eq. (3.23) and Eq. (3.24) build up the equivalent at NNLO of Eq. (2.21) and Eq. (2.23), for double-real radiation: indeed, applying the limits defined in Eq. (3.24) on RR and on the sector functions gives the starting point to determine the form of the counterterms for each sector, since the limits contain all phase-space singularities of RR in a given sector, without double counting. In order to promote them to actual counterterms, it is now necessary to introduce phase-space mappings, allowing to properly factorise the $(n+2)$-body phase space into an $(n+1)$-body phase space times a single-radiation phase space for $\mathbf{L}^{(\mathbf{1})}$ and $\mathbf{L}^{(\mathbf{12})}$, and into an n-body phase space times a double-radiation phase space for $\mathbf{L}^{(\mathbf{2})}$, as shown in Eq. (3.3). We now turn to the discussion of these mappings.

3.2.4 Phase-space mappings

There is considerable freedom in defining phase-space mappings for double-real radiation (see for example [205]). We have chosen to use nested Catani-Seymour final-state mappings, which involve a minimal set of the $(n+2)$ momenta, and are built in terms of Mandelstam invariants. This choice simplifies both the factorised expression for the $(n+2)$-body phase space and the dependence of the counterterms on the integration variables of the radiative phase spaces. In this framework, the mappings to factorise the $(n+2)$-body phase space into an $(n+1)$-body phase space times a single-radiation phase space, required for $\mathbf{L}^{(\mathbf{1})}$ and $\mathbf{L}^{(\mathbf{12})}$, can be constructed using the same procedure as followed at NLO. This leads us to Eq. (2.45) and Eq. (2.54), with i running from 1 to $n+2$, and m from 1 to $n+1$.

For the construction of an on-shell, momentum conserving n-tuple of massless momenta in the $(n+2)$-particle phase space, necessary for $\mathbf{L^{(2)}}$, we report in the following three of the possible choices.

- For six final-state massless momenta k_a, k_b, k_c, k_d, k_e, k_f (all different), we construct the n-tuple (without k_a and k_b)

$$\{\bar{k}\}^{(acd,bef)} = \left\{\{k\}_{\not a\not b\not c\not d\not e\not f},\, \bar{k}_c^{(acd,bef)}, \bar{k}_d^{(acd,bef)}, \bar{k}_e^{(acd,bef)}, \bar{k}_f^{(acd,bef)}\right\}\,, \tag{3.25}$$

with

$$\begin{aligned} \bar{k}_c^{(acd,bef)} &= k_a + k_c - \frac{s_{ac}}{s_{[ac]d}}\, k_d\,, & \bar{k}_d^{(acd,bef)} &= \frac{s_{acd}}{s_{[ac]d}}\, k_d\,, \\ \bar{k}_e^{(acd,bef)} &= k_b + k_e - \frac{s_{be}}{s_{[be]f}}\, k_f\,, & \bar{k}_f^{(acd,bef)} &= \frac{s_{bef}}{s_{[be]f}}\, k_f\,, \end{aligned} \tag{3.26}$$

while all other momenta are left unchanged ($\bar{k}_n^{(acd,bef)} = k_n\,,\ n \neq a,b,c,d,e,f$). Here and in the following $s_{[ab]c} = s_{ac} + s_{bc}$.

- For five final-state massless momenta k_a, k_b, k_c, k_d, k_e (all different), we construct the n-tuple (without k_a and k_b)

$$\{\bar{k}\}^{(acd,bed)} = \left\{\{k\}_{\not a\not b\not c\not d\not e},\, \bar{k}_c^{(acd,bed)}, \bar{k}_d^{(acd,bed)}, \bar{k}_e^{(acd,bed)}\right\}\,, \tag{3.27}$$

with

$$\begin{aligned} \bar{k}_c^{(acd,bed)} &= k_a + k_c - \frac{s_{ac}}{s_{[ac]d}}\, k_d\,, & \bar{k}_d^{(acd,bed)} &= \left(1 + \frac{s_{ac}}{s_{[ac]d}} + \frac{s_{be}}{s_{[be]d}}\right) k_d\,, \\ \bar{k}_e^{(acd,bed)} &= k_b + k_e - \frac{s_{be}}{s_{[be]d}}\, k_d\,, \end{aligned} \tag{3.28}$$

while all other momenta are left unchanged ($\bar{k}_n^{(acd,bed)} = k_n\,,\ n \neq a,b,c,d,e$).

- For four final-state massless momenta k_a, k_b, k_c, k_d (all different), we construct the n-tuple (without k_a and k_b)

$$\{\bar{k}\}^{(acd,bcd)} = \{\bar{k}\}^{(abc,bcd)} = \{\bar{k}\}^{(abcd)} = \left\{\{k\}_{\not a\not b\not c\not d},\, \bar{k}_c^{(abcd)}, \bar{k}_d^{(abcd)}\right\}\,, \tag{3.29}$$

with

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad}+s_{bd}+s_{cd}}\, k_d\,, \qquad \bar{k}_d^{(abcd)} = \frac{s_{abcd}}{s_{ad}+s_{bd}+s_{cd}}\, k_d\,, \tag{3.30}$$

while all other momenta are left unchanged ($\bar{k}_n^{(abcd)} = k_n\,,\ n \neq a,b,c,d$).

Other possibilities for the construction of nested mappings, like the combinations (abc, deb) or (abc, dcb), are achieved by iteratively applying NLO final-state mappings (see Eq. (2.45)). With these tools, we are now fully equipped to construct improved infrared projectors, with a proper factorised structure, and we can use them to define our local counterterms.

3.2.5 Local counterterms with improved limits

To write explicitly the counterterms we introduce *improved* versions of the projection operators in Table 3.1

$$\overline{\mathbf{S}}_a, \quad \overline{\mathbf{C}}_{ab}, \quad \overline{\mathbf{S}}_{ab}, \quad \overline{\mathbf{C}}_{abc}, \quad \overline{\mathbf{C}}_{abcd}, \quad \overline{\mathbf{SC}}_{abc} \, .$$

Similarly to the logic already discussed in Section 2.2.4, these new limits should be interpreted as operators which, not only extract the corresponding singular limit on the objects they act on, but also convey a specific mapping of momenta, to be defined a case-by-case basis. They may be further refined (for example by tuning their action on sector functions) in order to ensure the local cancellation of singularities after the implementation of phase-space mappings.

Once given the definitions of the improved limits (to be discussed below), we can formulate the expression for $RR_{\rm sub}$ in the following way. First, we define the improved version of the various $\mathbf{L}$ operators which correspond to the un-improved limits in Eq. (3.24), denoting the improved operators by $\overline{\mathbf{L}}$. Next, for each topology $\tau = ijjk, ijkj, ijkl$, we define our *local* counterterms as

$$K_\tau^{(\mathbf{1})} = \overline{\mathbf{L}}_{ij}^{(\mathbf{1})} RR\, \mathcal{W}_\tau \,, \qquad K_\tau^{(\mathbf{2})} = \overline{\mathbf{L}}_{\tau}^{(\mathbf{2})} RR\, \mathcal{W}_\tau \,, \qquad K_\tau^{(\mathbf{12})} = \overline{\mathbf{L}}_{\tau}^{(\mathbf{12})} RR\, \mathcal{W}_\tau \,. \tag{3.31}$$

Explicit expressions for these counterterms are reported in Eqs. (C.138), (C.151) and (C.165), topology by topology. The subtracted double-real squared matrix element for arbitrary topology τ can then be written as

$$RR_\tau^{\rm sub}(X) = RR\, \mathcal{W}_\tau\, \delta_{n+2}(X) - K_\tau^{(\mathbf{1})}\, \delta_{n+1}(X) - \Big(K_\tau^{(\mathbf{2})} - K_\tau^{(\mathbf{12})}\Big)\delta_n(X) \,. \tag{3.32}$$

Summing now the contributions from all sectors, we finally build the complete $RR_{\rm sub}(X)$ of Eq. (3.5) reading

$$RR_{\rm sub}(X) = \sum_{i,\, j\neq i} \sum_{k\neq i,j} \bigg[RR_{ijjk}^{\rm sub}(X) + RR_{ijkj}^{\rm sub}(X) + \sum_{l\neq i,j,k} RR_{ijkl}^{\rm sub}(X) \bigg] \,. \tag{3.33}$$

The structure of Eq. (3.8) with no sector functions is then recovered by using Eq. (3.11), and by defining

$$\begin{aligned} K^{(\mathbf{1})} &= \sum_{i,\, j\neq i}\sum_{k\neq i,j}\bigg[K^{(\mathbf{1})}_{ijjk} + K^{(\mathbf{1})}_{ijkj} + \sum_{l\neq i,j,k} K^{(\mathbf{1})}_{ijkl}\bigg]\,, \\ K^{(\mathbf{2})} &= \sum_{i,\, j\neq i}\sum_{k\neq i,j}\bigg[K^{(\mathbf{2})}_{ijjk} + K^{(\mathbf{2})}_{ijkj} + \sum_{l\neq i,j,k} K^{(\mathbf{2})}_{ijkl}\bigg]\,, \\ K^{(\mathbf{12})} &= \sum_{i,\, j\neq i}\sum_{k\neq i,j}\bigg[K^{(\mathbf{12})}_{ijjk} + K^{(\mathbf{12})}_{ijkj} + \sum_{l\neq i,j,k} K^{(\mathbf{12})}_{ijkl}\bigg]\,. \end{aligned} \tag{3.34}$$

We stress that the definitions of the counterterms in Eq. (3.31) are actually complete only after specifying both the action of improved limits on the double-real matrix element, $\overline{\mathbf{L}}\,RR$, as well as the action on sector functions, $\overline{\mathbf{L}}\,\mathcal{W}_\tau$. We report in Appendix C.2 the detailed descriptions of all improved limits, which are written in terms of the soft and collinear kernels listed in Appendix C.1, multiplying appropriate versions of the Born-level matrix element squared, expressed in terms of mapped momenta.

To provide the readers with an insight into the kind of expressions that emerge from this procedure, we present here two representative examples. First, consider the uniform double-unresolved double-soft improved limit $\overline{\mathbf{S}}_{ik}$ $(i \neq k)$, which can be written as

$$\begin{aligned} \overline{\mathbf{S}}_{ik}\,RR \equiv \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\Bigg\{\,\mathcal{E}^{(i)}_{cd}\sum_{e\neq i,k,c,d}\Bigg[\sum_{f\neq i,k,c,d,e}\mathcal{E}^{(k)}_{ef}\,\bar{B}^{(icd,kef)}_{cdef} + 4\,\mathcal{E}^{(k)}_{ed}\,\bar{B}^{(icd,ked)}_{cded}\Bigg] \\ + 2\,\mathcal{E}^{(i)}_{cd}\mathcal{E}^{(k)}_{cd}\,\bar{B}^{(icd,kcd)}_{cdcd} + \mathcal{E}^{(ik)}_{cd}\,\bar{B}^{(ikcd)}_{cd}\Bigg\}\,, \end{aligned} \tag{3.35}$$

where the NLO eikonal kernel $\mathcal{E}^{(i)}_{cd}$ and the NNLO eikonal kernel $\mathcal{E}^{(ik)}_{cd}$ are reported in Eqs. (C.3) and (C.5), and for the colour-correlated Born terms we employed six-, five- and four-particle mappings, according to the numbers of particles involved. Note in particular that all eikonal dipoles are mapped differently, which is essential to ease the respective analytic integration. This concept is discussed in Ref. [200] and in Section 3.3 below.

On the other hand, we can express the strongly-ordered double-unresolved double-soft improved limit $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}$ $(i \neq k)$ as follows:

$$\begin{aligned} \overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR \equiv \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\Bigg\{\,\mathcal{E}^{(i)}_{cd}\Bigg[\sum_{e\neq i,k,c,d}\Bigg(\sum_{f\neq i,k,c,d,e}\bar{\mathcal{E}}^{(k)(icd)}_{ef}\,\bar{B}^{(icd,kef)}_{cdef} + 2\,\bar{\mathcal{E}}^{(k)(icd)}_{ed}\,\bar{B}^{(icd,ked)}_{cded}\Bigg) \\ + 2\sum_{e\neq i,k,c,d}\bar{\mathcal{E}}^{(k)(idc)}_{ed}\,\bar{B}^{(idc,ked)}_{cded} + 2\,\bar{\mathcal{E}}^{(k)(icd)}_{cd}\Big(\bar{B}^{(icd,kcd)}_{cdcd} + C_A\,\bar{B}^{(icd,kcd)}_{cd}\Big)\Bigg] \\ - 2\,C_A\Big[\mathcal{E}^{(i)}_{kc}\,\bar{\mathcal{E}}^{(k)(ick)}_{cd}\,\bar{B}^{(ick,kcd)}_{cd} + \mathcal{E}^{(i)}_{kd}\,\bar{\mathcal{E}}^{(k)(ikd)}_{cd}\,\bar{B}^{(ikd,kcd)}_{cd}\Big]\Bigg\}\,. \end{aligned} \tag{3.36}$$

Despite the more intricate combinatorics, the complexity of the kernels has diminished with respect to Eq. (3.35), as might be expected: note indeed that the expression solely features NLO eikonal factors. Unlike the previous example, here we used mapped momenta also in the eikonal kernels corresponding to the least-unresolved particle k.

It is important to stress that, while there appears to be considerable flexibility in the definitions of the improved limits, there are also stringent constraints that must be satisfied. Specifically, the improved limits $\overline{\mathbf{L}}\, RR$ must preserve the same symmetries under index exchange which are carried by the corresponding unimproved countertparts, so that the improved collections $\overline{\mathbf{L}}^{(\mathbf{1})}, \overline{\mathbf{L}}^{(\mathbf{2})}, \overline{\mathbf{L}}^{(\mathbf{12})}$ are still consistent with Eq. (3.24), whose content relies on the validity of the integrable relations listed in Eq. (3.22). Within the limitations of this requirement, there is still a residual freedom to adjust how the improved limits act on both RR and sector functions, compared to the outcomes obtained with their bare unimproved version. However, it is essential to ensure that this procedure does preserve the locality of the cancellation of singularities, or, analogously, the finiteness of $RR^{\rm sub}_{ijjk}$, $RR^{\rm sub}_{ijkj}$ and $RR^{\rm sub}_{ijkl}$, in Eq. (3.33). To this end, we proved the consistency of the improved limits listed in Appendix C.2 by analytically verifying that, for any topology τ, the corresponding $RR^{\rm sub}_{\tau}$ is in fact integrable in all singular limits of that tolopogy. Concretely, this process involved the analytical verification of the following *consistency relations*:

$$
\begin{aligned}
\big\{\mathbf{S}_i,\ \mathbf{C}_{ij},\ \mathbf{S}_{ij},\ \mathbf{C}_{ijk},\ \mathbf{SC}_{ijk}\big\}\, RR^{\rm sub}_{ijjk} &\to \text{integrable}\,,\\
\big\{\mathbf{S}_i,\ \mathbf{C}_{ij},\ \mathbf{S}_{ik},\ \mathbf{C}_{ijk},\ \mathbf{SC}_{ijk},\ \mathbf{SC}_{kij}\big\}\, RR^{\rm sub}_{ijkj} &\to \text{integrable}\,,\\
\big\{\mathbf{S}_i,\ \mathbf{C}_{ij},\ \mathbf{S}_{ik},\ \mathbf{C}_{ijkl},\ \mathbf{SC}_{ikl},\ \mathbf{SC}_{kij}\big\}\, RR^{\rm sub}_{ijkl} &\to \text{integrable}\,.
\end{aligned}
\tag{3.37}
$$

As was the case at NLO (see Section 2.2.4), also NNLO-relevant collinear kernels of Appendix C.1 display *spurious* collinear singularities involving the reference momentum k_r, which are not always screened by the respective sector functions when evaluated in those specific limits. Once the action of the improved limits on such sector functions has been tuned by the introduction of angular factors (explicitly in Appendix C.2.2), we analytically verified that also the following relations hold

$$
\begin{aligned}
\big\{\mathbf{C}_{ir},\ \mathbf{C}_{jr},\ \mathbf{C}_{ijr}\big\}\, RR^{\rm sub}_{ijjk} &\to \text{integrable}\,,\\
\big\{\mathbf{C}_{ir},\ \mathbf{C}_{kr},\ \mathbf{C}_{ikr}\big\}\, RR^{\rm sub}_{ijkj} &\to \text{integrable}\,,\\
\big\{\mathbf{C}_{ir},\ \mathbf{C}_{kr}\big\}\, RR^{\rm sub}_{ijkl} &\to \text{integrable}\,.
\end{aligned}
\tag{3.38}
$$

To complete this analysis, we also consider it necessary to verify that $RR_\tau^{\rm sub}$ is integrable in what we refer to as *secondary* limits, such as

$$\begin{aligned} &\mathbf{S}_j \quad \text{for topology } \tau = ijjk, && \mathbf{S}_k \quad \text{for topology } \tau = ijkj, ijkl, \\ &\mathbf{C}_{jk} \quad \text{for topology } \tau = ijjk, ijkj, && \mathbf{C}_{kl} \quad \text{for topology } \tau = ijkl, \end{aligned} \tag{3.39}$$

where the word *secondary* identifies those singular limits that are suppressed by the full sector function in $RR\mathcal{W}_\tau$, but that can still lead to divergent terms when acting on counterterms, where sector functions are taken in specific limits, thus no longer complete. The singular limits appearing in Eq. (3.39) are also a consequence of the definition of $\mathcal{W}_\tau$. For the interested reader, we have collected in Appendix C.3 a detailed catalogue of all the consistency relations outlined in Eqs. (3.37), (3.38), and (3.39). These relations are broken down into lists of minimal conditions among counterterms, which provide valuable insights into the singularity-cancellation mechanism at the core of our scheme.

Having passed these tests, we can claim that the *local* counterterms presented in Eq. (3.31), assembled according to Eqs. (3.32)-(3.33), and constructed with the improved limits listed in Appendix C.2, provide a fully local subtraction of phase-space singularities for the double-real-emission contribution to the cross section, and Eq. (3.33) is indeed integrable in the $(n+2)$-particle phase space. We now go on to investigate a different (optimised) construction for $RR_{\rm sub}$ based on symmetrised sector functions, similarly to what was done in Section 2.2.5 at NLO.

3.2.6 Local counterterms with symmetrised sector functions

The partition of the $(n+2)$-particle phase space by means of the sector functions $\mathcal{W}_{abcd}$ that we introduced in Section 3.2.2 is not the only possible way forward. Analogously to what has been discussed at NLO (see Section 2.2.5), this sector structure can be adapted to meet certain symmetry conditions that reduce the actual number of sectors: in particular, since sectors sharing the same double-collinear singularities would naturally be parametrised in the same way in a numerical implementation, grouping such sectors in a single contribution is expected to improve numerical stability. Exploiting the symmetries of the improved limit $\overline{\mathbf{C}}_{ijk}$, we thus collect the 6 permutations of i, j, k in sectors $\mathcal{W}_{ijjk}, \mathcal{W}_{ijkj}$ introducing the *symmetrised sector functions*

$$\begin{aligned} \mathcal{Z}_{ijk} = {} & \mathcal{W}_{ijjk} + \mathcal{W}_{ikkj} + \mathcal{W}_{jiik} + \mathcal{W}_{jkki} + \mathcal{W}_{kiij} + \mathcal{W}_{kjji} \\ & + \mathcal{W}_{ijkj} + \mathcal{W}_{ikjk} + \mathcal{W}_{jiki} + \mathcal{W}_{jkik} + \mathcal{W}_{kiji} + \mathcal{W}_{kjij} \, . \end{aligned} \tag{3.40}$$

Similarly, within four-particle sectors $\mathcal{W}_{ijkl}$, we can leverage the symmetries of the improved limit $\overline{\mathbf{C}}_{ijkl}$ to sum up the 8 permutations $ijkl$, $ijlk$, $jikl$, $jilk$, $klij$, $klji$, $lkij$, $lkji$, and define

$$\mathcal{Z}_{ijkl} \;=\; \mathcal{W}_{ijkl}+\mathcal{W}_{ijlk}+\mathcal{W}_{jikl}+\mathcal{W}_{jilk}+\mathcal{W}_{klij}+\mathcal{W}_{klji}+\mathcal{W}_{lkij}+\mathcal{W}_{lkji}\,. \tag{3.41}$$

We also find it useful to introduce some notation for the NLO-type symmetric sector functions, based on $\mathcal{W}_{ij}^{(\alpha)}$ definition in Eq. (3.19), reading

$$\mathcal{Z}_{ij}^{(\alpha)} \;\equiv\; \mathcal{W}_{ij}^{(\alpha)}+\mathcal{W}_{ji}^{(\alpha)}\,, \qquad \mathcal{Z}_{ij} \;\equiv\; \mathcal{Z}_{ij}^{(1)}\,, \tag{3.42}$$

and the corresponding soft limit,

$$\mathcal{Z}_{\mathrm{s},ij}^{(\alpha)} \;\equiv\; \mathbf{S}_i\,\mathcal{Z}_{ij}^{(\alpha)} \;=\; \mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha)} \;=\; \frac{\frac{1}{w_{ij}^{\alpha}}}{\sum\limits_{l\neq i}\frac{1}{w_{il}^{\alpha}}}\,, \qquad \mathcal{Z}_{\mathrm{s},ij} \;\equiv\; \mathcal{Z}_{\mathrm{s},ij}^{(1)}\,. \tag{3.43}$$

By employing $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$ functions and consequently reducing the number of sectors, the expression of the counterterms further simplifies. In fact, deriving the action of the generic improved limit $\overline{\mathbf{L}}$ on the new sector functions (which can be directly obtained from the $\overline{\mathbf{L}}\,\mathcal{W}_{abcd}$ definitions in Appendix C.2.2), we verify that, thanks to their symmetries, any improved limit involving either the operator $\overline{\mathbf{C}}_{ijk}$, or the operator $\overline{\mathbf{C}}_{ijkl}$, reduces $\mathcal{Z}_{ijk}$, or $\mathcal{Z}_{ijkl}$, to unity, according to

$$\overline{\mathbf{C}}_{ijk}\big(\ldots\big)\,RR\,\mathcal{Z}_{ijk} \;=\; \overline{\mathbf{C}}_{ijk}\big(\ldots\big)\,RR\,, \qquad \overline{\mathbf{C}}_{ijkl}\big(\ldots\big)\,RR\,\mathcal{Z}_{ijkl} \;=\; \overline{\mathbf{C}}_{ijkl}\big(\ldots\big)\,RR\,, \tag{3.44}$$

where the ellipsis denotes a generic sequence of improved limits.

In analogy with Eq. (3.31), we now define our local counterterms with symmetrised sector functions by

$$K_{\{\sigma\}}^{(\mathbf{1})} = \overline{\mathbf{L}}_{\{\sigma\}}^{(\mathbf{1})}\,RR\,\mathcal{Z}_{\sigma}\,, \qquad K_{\{\sigma\}}^{(\mathbf{2})} = \overline{\mathbf{L}}_{\{\sigma\}}^{(\mathbf{2})}\,RR\,\mathcal{Z}_{\sigma}\,, \qquad K_{\{\sigma\}}^{(\mathbf{12})} = \overline{\mathbf{L}}_{\{\sigma\}}^{(\mathbf{12})}\,RR\,\mathcal{Z}_{\sigma}\,, \tag{3.45}$$

where $\sigma \in \{ijk, ijkl\}$ denotes the symmetrised topologies, and the limits $\overline{\mathbf{L}}_{\{\sigma\}}$ are symmetrised versions of the limits in Eq. (3.24), to be presented below. The subtracted double-real contribution for a given symmetrised sector, in analogy with Eq. (3.32), is then given by

$$RR_{\{\sigma\}}^{\mathrm{sub}}(X) \equiv RR\;\mathcal{Z}_{\sigma}\,\delta_{n+2}(X) - K_{\{\sigma\}}^{(\mathbf{1})}\,\delta_{n+1}(X) - \Big(K_{\{\sigma\}}^{(\mathbf{2})} - K_{\{\sigma\}}^{(\mathbf{12})}\Big)\,\delta_n(X)\,, \tag{3.46}$$

and finally the full expression for $RR_{\mathrm{sub}}(X)$ of Eq. (3.5) is obtained by summing the contributions from the symmetrised sectors $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$. It reads

$$RR_{\mathrm{sub}}(X) = \sum_{i,\,j>i}\Bigg[\sum_{k>j} RR_{\{ijk\}}^{\mathrm{sub}}(X) + \sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq i,j\\l>k}} RR_{\{ijkl\}}^{\mathrm{sub}}(X)\Bigg]\,. \tag{3.47}$$

This expression can be written in the form of Eq. (3.8) by specifing the complete counterterms $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$ in terms of symmetrised sector functions, as

$$
\begin{aligned}
K^{(\mathbf{1})} &= \sum_{i,\,j>i} \Bigg[\sum_{k>j} K^{(\mathbf{1})}_{\{ijk\}} + \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq i,j\\ l>k}} K^{(\mathbf{1})}_{\{ijkl\}} \Bigg] , \\
K^{(\mathbf{2})} &= \sum_{i,\,j>i} \Bigg[\sum_{k>j} K^{(\mathbf{2})}_{\{ijk\}} + \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq i,j\\ l>k}} K^{(\mathbf{2})}_{\{ijkl\}} \Bigg] , \\
K^{(\mathbf{12})} &= \sum_{i,\,j>i} \Bigg[\sum_{k>j} K^{(\mathbf{12})}_{\{ijk\}} + \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq i,j\\ l>k}} K^{(\mathbf{12})}_{\{ijkl\}} \Bigg] . \qquad (3.48)
\end{aligned}
$$

The collections of improved limits required to compute the symmetrised counterterms defined in Eq. (3.45) can be derived from the limits designed for the $\mathcal{W}_{abcd}$ sector functions, which were presented in Eq. (3.24) before improvement. The symmetrisation must be done carefully, in order not to overcount singular configurations. We adopt the following procedure. First, we expand all products in Eq. (3.24), and we express the corresponding lists of *improved* limits as flat sums running over the respective sets of relevant singular projectors. For example, we write

$$
\begin{aligned}
\overline{\mathbf{L}}^{(\mathbf{1})}_{ab} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{1})}_{ab}} \bar\ell\,, \qquad \text{where} \quad \mathcal{L}^{(\mathbf{1})}_{ab} = \Big\{ \overline{\mathbf{S}}_{a},\, \overline{\mathbf{C}}_{ab},\, -\overline{\mathbf{S}}_{a}\,\overline{\mathbf{C}}_{ab} \Big\} , \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{abbc} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{2})}_{abbc}} \bar\ell\,, \qquad \text{where} \quad \mathcal{L}^{(\mathbf{2})}_{abbc} = \Big\{ \overline{\mathbf{S}}_{ab},\, \overline{\mathbf{SC}}_{abc},\, -\overline{\mathbf{SC}}_{abc}\,\overline{\mathbf{S}}_{ab},\, \overline{\mathbf{C}}_{abc},\, -\overline{\mathbf{S}}_{ab}\,\overline{\mathbf{C}}_{abc}, \\
&\qquad\qquad\qquad\qquad\qquad -\overline{\mathbf{SC}}_{abc}\,\overline{\mathbf{C}}_{abc},\, \overline{\mathbf{SC}}_{abc}\,\overline{\mathbf{S}}_{ab}\,\overline{\mathbf{C}}_{abc} \Big\} , \qquad (3.49)
\end{aligned}
$$

and similarly for the remaining limits given in Eq. (3.24). Next, we introduce the index sets

$$
\begin{aligned}
\alpha &= \{ij, ji, ik, ki, jk, kj\}\,, & \beta &= \{ij, ji, kl, lk\}\,, \\
\gamma_1 &= \{ijjk, ikkj, jkki, jiik, kiij, kjji\}\,, & \gamma_2 &= \{ijkj, ikjk, jkik, jiki, kiji, kjij\}\,, \\
\delta &= \{ijkl, ijlk, jikl, jilk, klij, klji, lkij, lkji\}\,, & & \qquad (3.50)
\end{aligned}
$$

which enumerate the permutations that will need to be summed in order to perform the required symmetrisations. The collections of limits $\overline{\mathbf{L}}^{(\mathbf{1})}_{\{\sigma\}}$, $\overline{\mathbf{L}}^{(\mathbf{2})}_{\{\sigma\}}$ and $\overline{\mathbf{L}}^{(\mathbf{12})}_{\{\sigma\}}$ can now be defined by sums running over unions of the sets $\mathcal{L}$. Specifically, we define

$$
\begin{aligned}
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijk\}} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{1})}_\alpha} \bar\ell\,, &\qquad \text{where}\quad \mathcal{L}^{(\mathbf{1})}_\alpha &= \bigcup_{ab\,\in\,\alpha} \mathcal{L}^{(\mathbf{1})}_{ab}\,, \\
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijkl\}} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{1})}_\beta} \bar\ell\,, &\qquad \text{where}\quad \mathcal{L}^{(\mathbf{1})}_\beta &= \bigcup_{ab\,\in\,\beta} \mathcal{L}^{(\mathbf{1})}_{ab}\,, \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{\{ijk\}} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{2})}_\gamma} \bar\ell\,, &\qquad \text{where}\quad \mathcal{L}^{(\mathbf{2})}_\gamma &= \left[\bigcup_{abbc\,\in\,\gamma_1} \mathcal{L}^{(\mathbf{2})}_{abbc}\right] \cup \left[\bigcup_{abcb\,\in\,\gamma_2} \mathcal{L}^{(\mathbf{2})}_{abcb}\right], \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{\{ijkl\}} &= \sum_{\bar\ell\,\in\,\mathcal{L}^{(\mathbf{2})}_\delta} \bar\ell\,, &\qquad \text{where}\quad \mathcal{L}^{(\mathbf{2})}_\delta &= \bigcup_{abcd\,\in\,\delta} \mathcal{L}^{(\mathbf{2})}_{abcd}\,.
\end{aligned}
\tag{3.51}
$$

Similarly, the strongly-ordered double-unresolved limits $\overline{\mathbf{L}}^{(\mathbf{12})}_{\{\sigma\}}$ are obtained by analogous sums, where for $\sigma = ijk$ the sum runs over the collection $\mathcal{L}^{(\mathbf{12})}_\gamma$, and, for $\sigma = ijkl$, the sum runs over the collection $\mathcal{L}^{(\mathbf{12})}_\delta$, defined as in the last two lines of Eq. (3.51), with the replacement $(\mathbf{2}) \to (\mathbf{12})$. While assembling the set unions introduced in Eq. (3.51), one must be careful in counting only once all limits that coincide by symmetry: thus, for example, one should use the fact that $\overline{\mathbf{C}}_{ij} = \overline{\mathbf{C}}_{ji}$, and $\overline{\mathbf{SC}}_{ijk} = \overline{\mathbf{SC}}_{ikj}$. To further illustrate the procedure, we make explicit the first line of Eq. (3.51), which becomes

$$
\begin{aligned}
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijk\}} &= \overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{ij} + \overline{\mathbf{C}}_{ik} + \overline{\mathbf{C}}_{jk} \\
&\quad - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_j\,\overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ik} - \overline{\mathbf{S}}_k\,\overline{\mathbf{C}}_{ik} - \overline{\mathbf{S}}_j\,\overline{\mathbf{C}}_{jk} - \overline{\mathbf{S}}_k\,\overline{\mathbf{C}}_{jk} \\
&= \overline{\mathbf{S}}_i + \overline{\mathbf{S}}_j + \overline{\mathbf{S}}_k + \overline{\mathbf{HC}}_{ij} + \overline{\mathbf{HC}}_{ik} + \overline{\mathbf{HC}}_{jk}\,,
\end{aligned}
\tag{3.52}
$$

properly including all relevant singular regions without double counting.

The explicit results for the sums in Eq. (3.51) appear rather cumbersome at first sight, but in fact they result in relatively compact expressions when the limits are evaluated. Indeed, thanks to the symmetry properties of $\mathcal{Z}_{ijk}$ and $\mathcal{Z}_{ijkl}$, it is possible to merge subsets of singular limits which factor identical combinations of symmetrised sector functions. One finds then that only certain combinations of singular limits survive in the result. In detail, all single-unresolved collections $\overline{\mathbf{L}}^{(\mathbf{1})}_{\{\sigma\}}$ can be written explicitly as sums of single-soft limits $\overline{\mathbf{S}}_a$ plus hard-collinear combinations $\overline{\mathbf{HC}}_{ab}$, defined in Eq. (C.48). Furthermore, it is useful to introduce a soft-subtracted version of the uniform double-unresolved limit $\overline{\mathbf{SC}}_{abc}$, which is given by

$$
\overline{\mathbf{SHC}}_{abc} \equiv \overline{\mathbf{SC}}_{abc}\left(1 - \overline{\mathbf{S}}_{ab} - \overline{\mathbf{S}}_{ac}\right)\,. \tag{3.53}
$$

This combination appears only when attached to either the $\overline{\mathbf{S}}_a$ or $\overline{\mathbf{C}}_{abc}$ limits: indeed, in any other case, the operators $\overline{\mathbf{SC}}_{abc}$ and $\overline{\mathbf{S}}_{ab}\,\overline{\mathbf{SC}}_{abc}$ do not share the same sector functions in the limit. Similarly, considering the double-unresolved improved collinear limit $\overline{\mathbf{C}}_{abc}$, we can distinguish three useful combinations, defined by

$$\begin{aligned}
\overline{\mathbf{HC}}_{abc} &\equiv \overline{\mathbf{C}}_{abc}\left(1-\overline{\mathbf{S}}_{ab}-\overline{\mathbf{S}}_{bc}-\overline{\mathbf{S}}_{ac}\right) , \\
\overline{\mathbf{HC}}_{abc}^{\,(\mathbf{s})} &\equiv \overline{\mathbf{C}}_{abc}\left(1-\overline{\mathbf{S}}_{ab}-\overline{\mathbf{S}}_{ac}\right)\left(1-\overline{\mathbf{SC}}_{abc}\right) , \\
\overline{\mathbf{HC}}_{abc}^{\,(\mathbf{c})} &\equiv \overline{\mathbf{C}}_{abc}\left(1-\overline{\mathbf{S}}_{ab}-\overline{\mathbf{SC}}_{cab}\right) ,
\end{aligned} \tag{3.54}$$

which reflect three different possible strategies for removing soft singularities from the triple collinear kernel. The superscripts ($\mathbf{s}$) and ($\mathbf{c}$) in the second and third line of Eq. (3.54) refer to the fact that the ($\mathbf{s}$) combination can appear only in association with a single-soft limit $\overline{\mathbf{S}}_d$ (with $d \in \{a,b,c\}$), while the ($\mathbf{c}$) combination can appear only in association with single hard-collinear limits $\overline{\mathbf{HC}}_{de}$, with $de \in \{ab, ac, bc\}$. Finally, for the four-particle double-collinear improved limit $\overline{\mathbf{C}}_{ijkl}$ we introduce

$$\begin{aligned}
\overline{\mathbf{HC}}_{abcd} &\equiv \overline{\mathbf{C}}_{abcd}\left(1+\overline{\mathbf{S}}_{ac}+\overline{\mathbf{S}}_{bc}+\overline{\mathbf{S}}_{ad}+\overline{\mathbf{S}}_{bd}-\overline{\mathbf{SC}}_{acd}-\overline{\mathbf{SC}}_{bcd}-\overline{\mathbf{SC}}_{cab}-\overline{\mathbf{SC}}_{dab}\right) , \\
\overline{\mathbf{HC}}_{abcd}^{\,(\mathbf{c})} &\equiv \overline{\mathbf{C}}_{abcd}\left(1-\overline{\mathbf{SC}}_{cab}-\overline{\mathbf{SC}}_{dab}\right) ,
\end{aligned} \tag{3.55}$$

where again the notation ($\mathbf{c}$) refers to the fact that the combined limit in the second line of Eq. (3.55) can only appear in association with the hard-collinear single-unresolved limits $\overline{\mathbf{HC}}_{ab}$ and $\overline{\mathbf{HC}}_{cd}$.

With these preliminary definitions in place, we can formulate explicit expressions for the symmetrised improved limits defined in Eq. (3.51). They are as follows:

$$\begin{aligned}
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijk\}} &= \overline{\mathbf{S}}_i+\overline{\mathbf{S}}_j+\overline{\mathbf{S}}_k+\overline{\mathbf{HC}}_{ij}+\overline{\mathbf{HC}}_{jk}+\overline{\mathbf{HC}}_{ik} , \\
\overline{\mathbf{L}}^{(\mathbf{1})}_{\{ijkl\}} &= \overline{\mathbf{S}}_i+\overline{\mathbf{S}}_j+\overline{\mathbf{S}}_k+\overline{\mathbf{S}}_l+\overline{\mathbf{HC}}_{ij}+\overline{\mathbf{HC}}_{kl} , \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{\{ijk\}} &= \overline{\mathbf{S}}_{ij}+\overline{\mathbf{S}}_{jk}+\overline{\mathbf{S}}_{ik}+\overline{\mathbf{SC}}_{ijk}(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik})+\overline{\mathbf{SC}}_{jik}(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{jk})+\overline{\mathbf{SC}}_{kij}(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}) \\
&\quad +\overline{\mathbf{HC}}_{ijk}-\overline{\mathbf{C}}_{ijk}(\,\overline{\mathbf{SHC}}_{ijk}+\overline{\mathbf{SHC}}_{jik}+\overline{\mathbf{SHC}}_{kij}\,) , \\
\overline{\mathbf{L}}^{(\mathbf{2})}_{\{ijkl\}} &= \overline{\mathbf{S}}_{ik}+\overline{\mathbf{S}}_{jk}+\overline{\mathbf{S}}_{il}+\overline{\mathbf{S}}_{jl}+\overline{\mathbf{SC}}_{ikl}\left(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{il}\right)+\overline{\mathbf{SC}}_{jkl}\left(1-\overline{\mathbf{S}}_{jk}-\overline{\mathbf{S}}_{jl}\right) \\
&\quad +\overline{\mathbf{SC}}_{kij}\left(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\right)+\overline{\mathbf{SC}}_{lij}\left(1-\overline{\mathbf{S}}_{il}-\overline{\mathbf{S}}_{jl}\right)+\overline{\mathbf{HC}}_{ijkl} , \\
\overline{\mathbf{L}}^{(\mathbf{12})}_{\{ijk\}} &= \overline{\mathbf{S}}_i\left(\overline{\mathbf{S}}_{ij}+\overline{\mathbf{S}}_{ik}+\overline{\mathbf{SHC}}_{ijk}\right)+\overline{\mathbf{S}}_j\left(\overline{\mathbf{S}}_{ij}+\overline{\mathbf{S}}_{jk}+\overline{\mathbf{SHC}}_{jik}\right)+\overline{\mathbf{S}}_k\left(\overline{\mathbf{S}}_{ik}+\overline{\mathbf{S}}_{jk}+\overline{\mathbf{SHC}}_{kij}\right) \\
&\quad +\left(\overline{\mathbf{S}}_i+\overline{\mathbf{S}}_j+\overline{\mathbf{S}}_k\right)\overline{\mathbf{HC}}^{\,(\mathbf{s})}_{ijk}+\overline{\mathbf{HC}}_{ij}\left(\overline{\mathbf{S}}_{ij}+\overline{\mathbf{SC}}_{kij}+\overline{\mathbf{HC}}^{\,(\mathbf{c})}_{ijk}\right) \\
&\quad +\overline{\mathbf{HC}}_{jk}\left(\overline{\mathbf{S}}_{jk}+\overline{\mathbf{SC}}_{ijk}+\overline{\mathbf{HC}}^{\,(\mathbf{c})}_{ijk}\right)+\overline{\mathbf{HC}}_{ik}\left(\overline{\mathbf{S}}_{ik}+\overline{\mathbf{SC}}_{jik}+\overline{\mathbf{HC}}^{\,(\mathbf{c})}_{ijk}\right) , \qquad (3.56) \\
\overline{\mathbf{L}}^{(\mathbf{12})}_{\{ijkl\}} &= \overline{\mathbf{S}}_i\left(\overline{\mathbf{S}}_{ik}+\overline{\mathbf{S}}_{il}\right)+\overline{\mathbf{S}}_j\left(\overline{\mathbf{S}}_{jk}+\overline{\mathbf{S}}_{jl}\right)+\overline{\mathbf{S}}_k\left(\overline{\mathbf{S}}_{ik}+\overline{\mathbf{S}}_{jk}\right)+\overline{\mathbf{S}}_l\left(\overline{\mathbf{S}}_{il}+\overline{\mathbf{S}}_{jl}\right) \\
&\quad +\overline{\mathbf{S}}_i\,\overline{\mathbf{SHC}}_{ikl}+\overline{\mathbf{S}}_j\,\overline{\mathbf{SHC}}_{jkl}+\overline{\mathbf{S}}_k\,\overline{\mathbf{SHC}}_{kij}+\overline{\mathbf{S}}_l\,\overline{\mathbf{SHC}}_{lij} \\
&\quad +\overline{\mathbf{HC}}_{ij}\left(\overline{\mathbf{SC}}_{kij}+\overline{\mathbf{SC}}_{lij}\right)+\overline{\mathbf{HC}}_{kl}\left(\overline{\mathbf{SC}}_{ikl}+\overline{\mathbf{SC}}_{jkl}\right)+\left(\overline{\mathbf{HC}}_{ij}+\overline{\mathbf{HC}}_{kl}\right)\overline{\mathbf{HC}}^{\,(\mathbf{c})}_{ijkl} .
\end{aligned}$$

The actions of all these improved limits on RR and on the symmetrised sector functions $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$ are reported in Appendix C.2.

Comparing the collections of singular projectors relevant to $\mathcal{W}_{abcd}$ sector functions in Eq. (3.24) with the ones reported in Eq. (3.56) for the symmetrised case, it is immediate to notice that the number of different non-trivial singular limits contributing to a given sector changes, depending on the type of partition we introduce. In particular, this number increases for our choice of $\mathcal{Z}_{ijk}$ and $\mathcal{Z}_{ijkl}$. Despite this, though, the ordered sums in Eq. (3.47), building up the relevant integrable contributions, lead to a significantly more compact final expression (in terms of the number of different objects one needs to define and evaluate). This is a feature that will crucially translate into a gain in computational time and resources in the numerical implementation of the method.

3.3 Integration of the double-real-radiation counterterms

In the previous Section we constructed $RR_{\rm sub}$ of Eq. (3.8), a combination which is integrable everywhere in the double-radiative phase space, by subtracting the local counterterms $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$ (given in Eq. (3.34), or equivalently in Eq. (3.48)) from the double-real squared matrix element RR. These counterterms must now be added back, after integrating out one or two emissions, yielding the integrated counterterms $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$, $I^{(\mathbf{12})}$. The integration procedure at this perturbative order involves rather intricate combinatorics, and generates lengthy expressions in the intermediate stages. However, all integrals that need to be computed are remarkably simple, and in almost all cases exhibit trivial (logarithmic) dependence on the Mandelstam invariants [200].

We will begin by introducing the relevant phase-space factorisations and parameterisations, derived from the nested Catani-Seymour mappings introduced in Section 3.2.4. Then, we will report the integration of the counterterms $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$, $K^{(\mathbf{12})}$, specifying how each singular contribution is treated. The resulting expressions can be simplified, by relabelling the momenta and rewriting the flavour sums of the original $(n+2)$-body phase space. This finally enable us to recombine the contributions carrying different mappings, resulting in relatively compact collections of integrals for $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$, $I^{(\mathbf{12})}$. At this stage, the results can be directly employed in the subtraction formula, Eq. (3.5).

It is natural to define $I^{(\mathbf{1})}$ as the integral of $K^{(\mathbf{1})}$ in the single-unresolved radiation, and $I^{(\mathbf{2})}$ as the integral of $K^{(\mathbf{2})}$ in both unresolved emissions. For the strongly-ordered counterterm $K^{(\mathbf{12})}$ both possibilities are in principle viable. In our framework, we define $I^{(\mathbf{12})}$ as the integral of $K^{(\mathbf{12})}$ in a single radiation[5], corresponding to the 'most unresolved' radiated particle, as explicitly noted in Eq. (3.4). As a consequence, before performing the integrations, we rewrite both $K^{(\mathbf{1})}$ and $K^{(\mathbf{12})}$ by summing up the sector functions

[5]We note that in the context of the `CoLoRFul` approach to subtraction [206, 207], the strongly-ordered counterterm is integrated directly in both unresolved radiations.

related to the most unresolved radiation (the ones carrying the suffix α), while keeping the sector functions for the second (least-unresolved) radiation untouched. Note however that these remaining sector functions carry $(n+1)$-body mapped kinematics. In this way, it will be possible to combine directly the integrated counterterms $I^{\,(\mathbf{1})}$ and $I^{\,(\mathbf{12})}$ with the real-virtual contribution RV, and with the real-virtual counterterm $K^{(\mathbf{RV})}$, in Eq. (3.7), sector by sector in Φ_{n+1}. For the sake of simplicity, in the following all integrations are described using the representations of $K^{\,(\mathbf{1})}$, $K^{\,(\mathbf{2})}$ and $K^{\,(\mathbf{12})}$ in terms of symmetrised sector functions, as provided in Eq. (3.48), but we will also present the resulting expressions for $I^{\,(\mathbf{1})}$, $I^{\,(\mathbf{2})}$ and $I^{\,(\mathbf{12})}$ in terms of the $\mathcal{W}$ sector functions.

Phase-space parametrisations

We start by giving precise definitions for the measures of integration in the radiative phase spaces $d\Phi_{\rm rad}$ and $d\Phi_{\rm rad,2}$, according to Eq. (3.3), now focusing on the dependence on the chosen mappings (discussed in Section 3.2.4), and making specific choices of integration variables.

The single-unresolved counterterm $K^{\,(\mathbf{1})}$ contains just single mappings of the type $\{\bar{k}\}^{(acd)}$ (a, c, d all different) and is going to be integrated in the corresponding single-radiation phase space. Conversely, $K^{\,(\mathbf{12})}$ and $K^{\,(\mathbf{2})}$ are built by means of iterated mappings of the type $\{\bar{k}\}^{(acd,bef)}$ (a, c, d all different and b, e, f all different). However, while $K^{\,(\mathbf{12})}$ has to be integrated just in the phase space of the single radiation corresponding to the first mapping, $K^{\,(\mathbf{2})}$ must be integrated in the whole double-radiation phase space.

We start specifying the first term in Eq. (3.3), needed for the integration of $K^{\,(\mathbf{1})}$ and $K^{\,(\mathbf{12})}$. We write

$$\int d\Phi_{n+2}(\{k\}) \;=\; \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi_{n+1}^{(acd)} \int d\Phi_{\rm rad}^{(acd)} \,, \tag{3.57}$$

where we defined

$$d\Phi_{n+1}^{(acd)} \;\equiv\; d\Phi_{n+1}(\{\bar{k}\}^{(acd)})\,. \tag{3.58}$$

The explicit expression for the radiative measure has already been reported in Eq. (2.88). The invariants composed by the momenta k_a, k_c, k_d are related to the integration variables y and z by

$$s_{ac} \;=\; y\,\bar{s}_{cd}^{(acd)}\,, \qquad s_{ad} \;=\; z(1-y)\,\bar{s}_{cd}^{(acd)}\,, \qquad s_{cd} \;=\; (1-z)(1-y)\,\bar{s}_{cd}^{(acd)}\,, \tag{3.59}$$

so that $s_{acd} \;=\; s_{ac} + s_{ad} + s_{cd} = \bar{s}_{cd}^{(acd)}$.

To parametrise the double-radiative phase space for $K^{\,(\mathbf{2})}$ integration (as in the second entry of Eq. (3.3)), we employ double mappings of three different types, as discussed in Section 3.2.4. We examine them in turn.

The six-particle mapping $\{\bar{k}\}^{(acd,bef)}$ (a, b, c, d, e, f all different) presented in Eqs. (3.25) and (3.26) induces the factorisation

$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n} \int d\Phi_n^{(acd,bef)} \int d\Phi_{\rm rad,2}^{(acd,bef)} , \quad d\Phi_n^{(acd,bef)} \equiv d\Phi_n(\{\bar{k}\}^{(acd,bef)}), \tag{3.60}$$

and the resulting radiative measure of integration is

$$\begin{aligned} \int d\Phi_{\rm rad,2}^{(acd,bef)} &= N^2(\epsilon) \left(\bar{s}_{cd}^{(acd,bef)} \, \bar{s}_{ef}^{(acd,bef)} \right)^{1-\epsilon} \int_0^\pi d\phi' \, (\sin\phi')^{-2\epsilon} \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi \, (\sin\phi)^{-2\epsilon} \\ &\times \int_0^1 dy \int_0^1 dz \left[y'(1-y')^2 \, z'(1-z') \, y(1-y)^2 \, z(1-z) \right]^{-\epsilon} (1-y')(1-y), \end{aligned} \tag{3.61}$$

with $N(\epsilon)$ as in Eq. (2.90). The expressions for relevant invariants in terms of the integration variables are

$$\begin{aligned} s_{ac} &= y' \, \bar{s}_{cd}^{(acd,bef)} , & s_{be} &= y \, \bar{s}_{ef}^{(acd,bef)} , \\ s_{ad} &= z' \, (1-y') \, \bar{s}_{cd}^{(acd,bef)} , & s_{bf} &= z \, (1-y) \, \bar{s}_{ef}^{(acd,bef)} , \\ s_{cd} &= (1-z')(1-y') \, \bar{s}_{cd}^{(acd,bef)} , & s_{ef} &= (1-z)(1-y) \, \bar{s}_{ef}^{(acd,bef)} , \end{aligned} \tag{3.62}$$

so that

$$\begin{aligned} s_{acd} &= s_{ac} + s_{ad} + s_{cd} = \bar{s}_{cd}^{(acd,bef)} = \bar{s}_{cd}^{(acd)} , \\ s_{bef} &= s_{be} + s_{bf} + s_{ef} = \bar{s}_{ef}^{(acd,bef)} = \bar{s}_{ef}^{(bef)} . \end{aligned} \tag{3.63}$$

The five-particle mapping $\{\bar{k}\}^{(acd,bed)}$ (a, b, c, d, e all different) presented in Eqs. (3.27) and (3.28) induces the factorisation

$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n} \int d\Phi_n^{(acd,bed)} \int d\Phi_{\rm rad,2}^{(acd,bed)} , \qquad d\Phi_n^{(acd,bed)} \equiv d\Phi_n(\{\bar{k}\}^{(acd,bed)}), \tag{3.64}$$

and we write

$$\begin{aligned} \int d\Phi_{\rm rad,2}^{(acd,bed)} &= N^2(\epsilon) \left(\bar{s}_{cd}^{(acd,bed)} \, \bar{s}_{ed}^{(acd,bed)} \right)^{1-\epsilon} \int_0^\pi d\phi' \, (\sin\phi')^{-2\epsilon} \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi \, (\sin\phi)^{-2\epsilon} \int_0^1 dy \\ &\times \int_0^1 dz \left[y'(1-y')^2 \, z'(1-z') \, y(1-y)^3 \, z(1-z) \right]^{-\epsilon} (1-y')(1-y)^2, \end{aligned} \tag{3.65}$$

with

$$\begin{aligned} s_{ac} &= y' \, (1-y) \, \bar{s}_{cd}^{(acd,bed)} , & s_{ad} &= z' \, (1-y')(1-y) \, \bar{s}_{cd}^{(acd,bed)} , \\ s_{be} &= y \, \bar{s}_{ed}^{(acd,bed)} , & s_{bd} &= (1-y') \, z \, (1-y) \, \bar{s}_{ed}^{(acd,bed)} , \\ s_{cd} &= (1-y')(1-z')(1-y) \, \bar{s}_{cd}^{(acd,bed)} , & s_{ed} &= (1-y')(1-z)(1-y) \, \bar{s}_{ed}^{(acd,bed)} , \end{aligned} \tag{3.66}$$

so that the invariant $s_{abcde} = s_{ab} + s_{ac} + s_{ad} + s_{ae} + s_{bc} + s_{bd} + s_{be} + s_{cd} + s_{ce} + s_{de}$ is equal to $\bar{s}_{cde}^{(acd,bed)} = \bar{s}_{cd}^{(acd,bed)} + \bar{s}_{ce}^{(acd,bed)} + \bar{s}_{de}^{(acd,bed)}$.

Finally, we have the four-particle mapping, $\{\bar{k}\}^{(acd,bcd)} = \{\bar{k}\}^{(abcd)}$, $(a, b, c, d$ all different), presented in Eqs. (3.29) and (3.30). This is the most intricate mapping, inducing the factorisation

$$\int d\Phi_{n+2}(\{k\}) = \frac{\varsigma_{n+2}}{\varsigma_n} \int d\Phi_n^{(abcd)} \int d\Phi_{\mathrm{rad},2}^{(abcd)}, \qquad d\Phi_n^{(abcd)} \equiv d\Phi_n(\{\bar{k}\}^{(abcd)}), \tag{3.67}$$

where we write

$$\begin{aligned} \int d\Phi_{\mathrm{rad},2}^{(abcd)} &= 2^{-2\epsilon} N^2(\epsilon) \left(\bar{s}_{cd}^{(abcd)}\right)^{2-2\epsilon} \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi \, (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \\ &\times \Big[w'(1-w')\Big]^{-1/2-\epsilon} \Big[y'(1-y')^2 \, z'(1-z') \, y^2(1-y)^2 \, z(1-z)\Big]^{-\epsilon} (1-y') \, y \, (1-y) , \end{aligned} \tag{3.68}$$

with

$$\begin{aligned} s_{ab} &= y' \, y \, \bar{s}_{cd}^{(abcd)}, \qquad s_{ac} = z'(1-y') \, y \, \bar{s}_{cd}^{(abcd)}, \qquad s_{bc} = (1-y')(1-z') \, y \, \bar{s}_{cd}^{(abcd)}, \\ s_{cd} &= (1-y')(1-y)(1-z) \, \bar{s}_{cd}^{(abcd)}, \\ s_{ad} &= (1-y) \Big[y'(1-z')(1-z) + z'z - 2(1-2w')\sqrt{y'z'(1-z')z(1-z)} \Big] \bar{s}_{cd}^{(abcd)}, \\ s_{bd} &= (1-y) \Big[y'z'(1-z) + (1-z')z + 2(1-2w')\sqrt{y'z'(1-z')z(1-z)} \Big] \bar{s}_{cd}^{(abcd)}, \end{aligned} \tag{3.69}$$

so that $s_{abcd} = s_{ab} + s_{ac} + s_{ad} + s_{bc} + s_{bd} + s_{cd} = \bar{s}_{cd}^{(abcd)}$.

Integration of $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$

We now have all the ingredients to perform the required integrations. Our task is simplified by the fact that the integrals of the azimuthal parts of the collinear kernels vanish, as proved in Appendix C.4. All remaining integrals are then computed following the techniques explained in [200].

The integration of the single-unresolved counterterm $K^{(\mathbf{1})}$ involves

$$\int d\Phi_{n+2} \, K^{(\mathbf{1})} = \int d\Phi_{n+2} \Bigg\{ \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{S}}_i \, RR \, \bar{\mathcal{Z}}_{jk} + \sum_{i,\,j>i} \sum_{k\neq i} \sum_{\substack{l\neq i \\ l>k}} \overline{\mathbf{HC}}_{ij} \, RR \, \bar{\mathcal{Z}}_{kl} \Bigg\} . \tag{3.70}$$

The integrand on the right-hand side has been obtained from $K^{(\mathbf{1})}$ of Eq. (3.48) by summing up the NLO sector functions with label α of Eqs. (C.129)-(C.130). As explained in Appendix C.2, the mapped sector functions $\bar{\mathcal{Z}}_{ij}$ are understood to carry the same mapping as the matrix elements they multiply. Given that Eq. (3.70) will have to be combined with the real-virtual contribution RV within Eq. (3.7), it becomes necessary to express the integral in Eq. (3.70) as a sum of terms where the integration over the single-particle radiative phase space has been performed, a specific parametrisation for the $(n+1)$-particle phase space has been identified, and the full single-real-radiation squared matrix element R has been factored, and computed in the chosen variables. The outcomes for the individual components of the two terms in Eq. (3.70) take the form

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_i\,RR\,\bar{\mathcal{Z}}_{jk} = -\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{c\neq i}\sum_{d\neq i,c}\int d\Phi^{(icd)}_{n+1}\,J_s^{icd}\,\bar{R}^{(icd)}_{cd}\,\bar{\mathcal{Z}}^{(icd)}_{jk}\,, \tag{3.71}$$

$$\int d\Phi_{n+2}\,\overline{\mathbf{HC}}_{ij}\,RR\,\bar{\mathcal{Z}}_{kl} = \frac{\varsigma_{n+2}}{\varsigma_{n+1}}\int d\Phi^{(ijr)}_{n+1}\,J_{\rm hc}^{ijr}\,\bar{R}^{(ijr)}\,\bar{\mathcal{Z}}^{(ijr)}_{kl}\,, \qquad r = r_{ijkl}\,, \tag{3.72}$$

where the measure of integration $d\Phi^{(acd)}_{n+1}$ was defined in Eq. (3.58). The integration over the appropriate $d\Phi_{\rm rad}$ has been performed, yielding the integrals J_s^{icd} and $J_{\rm hc}^{ijr}$, whose explicit expressions are given in Eq. (C.183) and in Eq. (C.189), respectively. The reference particle $r = r_{ijkl} \neq i,j,k,l$[6] in Eq. (3.72) has been chosen following the rule of Eq. (A.13), which reflects the prescription made for $\overline{\mathbf{HC}}_{ij}\,RR$ in Eq. (C.48). Notice that this choice causes a dependence of the integrated kernel $J_{\rm hc}^{ijr}$ on the indices k and l of $\bar{\mathcal{Z}}^{(ijr)}_{kl}$, thus preventing the complete summation of sector functions in the second line of Eq. (3.7).

We now turn to the integration of $K^{(\mathbf{2})}$, which constitutes the most involved part of the calculation. In this case, since $I^{(\mathbf{2})}$ lives in Φ_n, as part of double-virtual correction in Eq. (3.6), we start from $K^{(\mathbf{2})}$ in Eq. (3.48) and perform the complete sum over sector functions by means of their sum rules (see for example Eqs. (3.14)-(3.17)), thus removing any dependence from the latter. This gives

$$\begin{aligned}\int d\Phi_{n+2}\,K^{(\mathbf{2})} = \int d\Phi_{n+2}\bigg[&\sum_{i,j>i}\overline{\mathbf{S}}_{ij}\,RR + \sum_{i,j\neq i}\sum_{\substack{k\neq i\\k>j}}\overline{\mathbf{SHC}}_{ijk}\big(1-\overline{\mathbf{C}}_{ijk}\big)\,RR\\ &+\sum_{i,j>i}\sum_{k>j}\overline{\mathbf{HC}}_{ijk}RR + \sum_{i,j>i}\sum_{\substack{k\neq j\\k>i}}\sum_{\substack{l\neq j\\l>k}}\overline{\mathbf{HC}}_{ijkl}\,RR\bigg]\,.\end{aligned} \tag{3.73}$$

Each of the four terms in Eq. (3.73) must be written as a sum of contributions, where the double-radiation kernels have been integrated over the parametrised radiative phase space, and one is left with a Born-level factor, expressed in terms of mapped momenta. To guide the eye of the reader through the following rather intricate expressions, we emphasise that, for each one of the limits involved, the results are of the form

$$\int d\Phi_{n+2}\,\overline{\mathbf{L}}^{(\mathbf{2})}_{\dots}\,RR = \text{constant}\sum_{\{\mu\}}\int d\Phi^{(\mu)}_n\,J^{\mu}_{\rm limit}\,\bar{B}^{(\mu)}_{\rm colour}\,, \tag{3.74}$$

where the overall constant is related to multiplicities, the sum is over the set $\{\mu\}$ of

[6] Notice that setting $r = r_{ijkl}$ implies the need for at least five massless partons in Φ_{n+2}, namely three massless final-state partons at Born level. A solution for the case of two massless final-state partons in the Born phase space requires minor technical modifications.

mappings that have been employed, the Born factor may have different colour correlations, and J will always denote the results of the integration of the kernels appropriate to the limit being taken[7]. The relevant J's will be listed in Appendix C.5. Beginning with the integrated double-soft limit in Eq. (3.73), we find the explicit expression

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR = \frac{1}{2}\,\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\sum_{e\neq i,j,c,d}\Bigg[\sum_{f\neq i,j,c,d,e}\int d\Phi_n^{(icd,jef)}\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcdef}\,\bar{B}_{cdef}^{(icd,jef)} + 4\int d\Phi_n^{(icd,jed)}\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcde}\,\bar{B}_{cded}^{(icd,jed)}\Bigg] + \int d\Phi_n^{(ijcd)}\Big[2\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcd}\,\bar{B}_{cdcd}^{(ijcd)} + J_{\mathrm{ss}}^{ijcd}\,\bar{B}_{cd}^{(ijcd)}\Big]\Bigg\}, \tag{3.75}$$

where we collected colour correlations involving four, three and two partons, and each term has been mapped differently, to simplify the corresponding integration. The integrals relevant for double-soft radiation are presented in Eq. (C.185). We now turn to the second term in Eq. (3.73), and we find (with $r = r_{ijk}$)

$$\begin{aligned}\int d\Phi_{n+2}\,\overline{\mathbf{SHC}}_{ijk}\,\big(1-\overline{\mathbf{C}}_{ijk}\big)\,RR = \\ -\frac{\varsigma_{n+2}}{\varsigma_n}\Bigg\{\sum_{\substack{c\neq i,j,k,r\\ d\neq i,j,k,r,c}}\int d\Phi_n^{(jkr,icd)}\,J_{\mathrm{s}\otimes\mathrm{hc}}^{jkricd}\,\bar{B}_{cd}^{(jkr,icd)} + 2\sum_{c\neq i,j,k,r}\int d\Phi_n^{(jkr,icr)}\,J_{\mathrm{s}\otimes\mathrm{hc}}^{jkricr}\,\bar{B}_{cr}^{(jkr,icr)} \\ + \Bigg[\sum_{c\neq i,j,k}\int d\Phi_n^{(krj,icj)}\,J_{\mathrm{s}\otimes\mathrm{hc}}^{krjic}\Big(\rho_{jk}^{(C)}\,\bar{B}_{[jk]c}^{(krj,icj)} + \tilde{f}_{jk}^{\,q\bar{q}}\,\bar{\mathcal{B}}_{[jk]c}^{(krj,icj)}\Big) \\ + C_{f_{[jk]}}\,\rho_{jk}^{(C)}\int d\Phi_n^{(krj,ijr)}\,J_{\mathrm{s}\otimes\mathrm{hc}}^{krjir}\,\bar{B}^{(krj,irj)} + (j\leftrightarrow k)\Bigg]\Bigg\},\end{aligned} \tag{3.76}$$

where $[jk]$ represents the parent particle of the pair (j,k), the factors $\rho_{jk}^{(C)}$, involving combinations of Casimir eigenvalues, are defined in Eq. (A.7), the flavour factors such as $\tilde{f}_{jk}^{\,q\bar{q}}$ are presented in Eq. (A.3), and $\mathcal{B}_{cd}$ is a colour projection of the Born contribution involving the symmetric tensor d_{ABC}, defined in Eq. (A.5); furthermore, the phase-space integrals $J_{\mathrm{s}\otimes\mathrm{hc}}$ are presented in Eq. (C.196). The remaining contributions to Eq. (3.73) are purely hard-collinear. For the integral of the emission of a cluster of three hard-collinear particles we find

$$\int d\Phi_{n+2}\,\overline{\mathbf{HC}}_{ijk}RR = \frac{\varsigma_{n+2}}{\varsigma_n}\int d\Phi_n^{(ijkr)}\,J_{\mathrm{hcc}}^{ijkr}\,\bar{B}^{(ijkr)}\,, \qquad r = r_{ijk}\,, \tag{3.77}$$

[7] Note that, since the limit $\overline{\mathbf{L}}$ is expressed as a sum of terms that can be mapped differently, several J's will contribute to each $\overline{\mathbf{L}}$.

while for the emission of two distinct pairs of hard-collinear particles the integral reads

$$\int d\Phi_{n+2}\,\overline{\mathbf{HC}}_{ijkl}\,RR = \frac{\varsigma_{n+2}}{\varsigma_n}\int d\Phi_n^{(ijr,klr)}\,J_{\mathrm{hc}\otimes\mathrm{hc}}^{ijklr}\,\bar{B}^{(ijr,klr)}\,, \qquad r = r_{ijkl}\,, \tag{3.78}$$

where the integrals J_{hcc} and $J_{\mathrm{hc}\otimes\mathrm{hc}}$ are reported in Eq. (C.192) and in Eq. (C.194), respectively.

We finally turn to the integration of the strongly-ordered counterterm $K^{(\mathbf{12})}$. As announced, we integrate $K^{(\mathbf{12})}$ only in the phase space of the most unresolved radiation, so the integrals involved will be the same that appeared in the case of $K^{(\mathbf{1})}$. Starting from the expression for $K^{(\mathbf{12})}$ in Eq. (3.48), we then sum up the NLO sector functions with label α of Eqs. (C.133)-(C.134), and we get

$$\begin{aligned}\int d\Phi_{n+2}\,K^{(\mathbf{12})} = \int d\Phi_{n+2}\Bigg\{ &\sum_{i,j\neq i}\overline{\mathbf{S}}_i\Bigg[\sum_{k\neq i,j}\overline{\mathbf{S}}_{ij}\,RR\,\bar{\mathcal{Z}}_{\mathrm{s},jk} + \sum_{\substack{k\neq i\\ k>j}}\big(\overline{\mathbf{SHC}}_{ijk} + \overline{\mathbf{HC}}_{ijk}^{(\mathbf{s})}\big)\,RR\Bigg] \\ &+ \sum_{i,\,j>i}\sum_{k\neq i,j}\overline{\mathbf{HC}}_{ij}\Bigg[\overline{\mathbf{S}}_{ij}\,RR\,\bar{\mathcal{Z}}_{\mathrm{s},jk} + \sum_{l\neq i,k}\overline{\mathbf{SC}}_{kij}\,RR\,\bar{\mathcal{Z}}_{\mathrm{s},kl} \\ &\qquad + \overline{\mathbf{HC}}_{ijk}^{(\mathbf{c})}\,RR + \sum_{\substack{l\neq i,j\\ l>k}}\overline{\mathbf{HC}}_{ijkl}^{(\mathbf{c})}\,RR\Bigg]\Bigg\}\,,\end{aligned} \tag{3.79}$$

where again the mapped sector functions $\bar{\mathcal{Z}}_{\mathrm{s},ab}$ carry the same mapping as the matrix elements they multiply. No other sector functions appear in Eq. (3.79), since the use of symmetrised sector functions has allowed to perform the corresponding sector sums in the collinear contributions, thus replacing sector functions by unity. Once again, to highlight the general structure of the expressions listed below, we anticipate that they are all of the form

$$\int d\Phi_{n+2}\,\overline{\mathbf{L}}_{\dots}^{(\mathbf{12})}\,RR = \text{constant}\sum_{\{\mu_1,\mu_2\}}\int d\Phi_{n+1}^{(\mu_1)}\;J_{\mathrm{limit}}^{\mu_1}\;\bar{\mathcal{K}}_{\mu_2}^{(\mu_1)}\;\bar{B}_{\mathrm{colour}}^{(\mu_1,\mu_2)}\,. \tag{3.80}$$

In this case, the only integrals required for the most unresolved radiation will again be J_s^{ilm} and J_{hc}^{ijr}, reported in Eq. (C.183) and in Eq. (C.189) respectively. The contribution $\bar{\mathcal{K}}$ identifies either a soft or a collinear kernel associated with the second (least-unresolved) radiation, which carries mapping (μ_1), i.e. the first one in the nested mapping (μ_1, μ_2) of the Born matrix elements. Following in the order the content of Eq. (3.79), the integrated strongly-ordered double-soft limit is given by

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij}\,RR\,\bar{\mathcal{Z}}_{\mathrm{s},jk} \;= \tag{3.81}$$
$$\mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\int d\Phi_{n+1}^{(icd)}\,J_s^{icd}\Bigg[\sum_{e\neq i,j,c,d}\Bigg(\frac{1}{2}\sum_{f\neq i,j,c,d,e}\bar{\mathcal{E}}_{ef}^{(j)(icd)}\,\bar{B}_{cdef}^{(icd,jef)}+\bar{\mathcal{E}}_{ed}^{(j)(icd)}\,\bar{B}_{cded}^{(icd,jed)}\Bigg)$$
$$+\,\bar{\mathcal{E}}_{cd}^{(j)(icd)}\Big(\bar{B}_{cdcd}^{(icd,jcd)}+C_A\bar{B}_{cd}^{(icd,jcd)}\Big)\Bigg]\bar{\mathcal{Z}}_{\mathrm{s},\,jk}^{(icd)}$$
$$+\int d\Phi_{n+1}^{(idc)}\,J_s^{idc}\sum_{e\neq i,j,c,d}\bar{\mathcal{E}}_{ed}^{(j)(idc)}\,\bar{B}_{cded}^{(idc,jed)}\,\bar{\mathcal{Z}}_{\mathrm{s},\,jk}^{(idc)}$$
$$-\,C_A\int d\Phi_{n+1}^{(icj)}\,J_s^{icj}\,\bar{\mathcal{E}}_{cd}^{(j)(icj)}\,\bar{B}_{cd}^{(icj,jcd)}\,\bar{\mathcal{Z}}_{\mathrm{s},\,jk}^{(icj)}$$
$$-\,C_A\int d\Phi_{n+1}^{(ijd)}\,J_s^{ijd}\,\bar{\mathcal{E}}_{cd}^{(j)(ijd)}\,\bar{B}_{cd}^{(ijd,jcd)}\,\bar{\mathcal{Z}}_{\mathrm{s},\,jk}^{(ijd)}\Bigg\}\,,$$

and it is entirely expressed in terms of the simple NLO eikonal kernels given in Eq. (C.3). Next, we perform the integral (with $r = r_{ijk}$)

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{SHC}}_{ijk}\,RR \;= \tag{3.82}$$
$$-\,\mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{c\neq i,j,k}\Bigg\{\sum_{d\neq i,j,k,c}\int d\Phi_{n+1}^{(icd)}\,J_s^{icd}\,\frac{\bar{P}_{jk}^{(icd)\mathrm{hc},\mu\nu}}{\bar{s}_{jk}^{(icd)}}\,\bar{B}_{\mu\nu,cd}^{(icd,jkr)}$$
$$+\Bigg[\int d\Phi_{n+1}^{(ijc)}\,J_s^{ijc}\,\frac{\bar{P}_{jk(r)}^{(ijc)\mathrm{hc},\mu\nu}}{2\bar{s}_{jk}^{(ijc)}}\Big(\rho_{jk}^{(C)}\,\bar{B}_{\mu\nu,[jk]c}^{(ijc,krj)}+\tilde{f}_{jk}^{\,q\bar{q}}\,\bar{\mathcal{B}}_{\mu\nu,[jk]c}^{(ijc,krj)}\Big)+(j\leftrightarrow k)\Bigg]$$
$$+\Bigg[\int d\Phi_{n+1}^{(icj)}\,J_s^{icj}\,\frac{\bar{P}_{jk(r)}^{(icj)\mathrm{hc},\mu\nu}}{2\bar{s}_{jk}^{(icj)}}\Big(\rho_{jk}^{(C)}\,\bar{B}_{\mu\nu,[jk]c}^{(icj,krj)}+\tilde{f}_{jk}^{\,q\bar{q}}\,\bar{\mathcal{B}}_{\mu\nu,[jk]c}^{(icj,krj)}\Big)+(j\leftrightarrow k)\Bigg]\Bigg\}\,,$$

where the hard-collinear kernels are given in Eq. (C.11). We now turn to limits involving triple-collinear configurations. First we need

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{HC}}_{ijk}^{(\mathbf{s})}\,RR \;= \tag{3.83}$$
$$\mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\,\frac{C_{f_{[jk]}}}{2}\Bigg\{\Bigg[\rho_{jk}^{(C)}\int d\Phi_{n+1}^{(ijr)}\,J_s^{ijr}\,\frac{\bar{P}_{jk(r)}^{(ijr)\mathrm{hc},\mu\nu}}{\bar{s}_{jk}^{(ijr)}}\Big(\bar{B}_{\mu\nu}^{(ijr,jkr)}-\bar{B}_{\mu\nu}^{(ijr,krj)}\Big)+(j\leftrightarrow k)\Bigg]$$
$$+\Bigg[\rho_{jk}^{(C)}\int d\Phi_{n+1}^{(irj)}\,J_s^{irj}\,\frac{\bar{P}_{jk(r)}^{(irj)\mathrm{hc},\mu\nu}}{\bar{s}_{jk}^{(irj)}}\Big(\bar{B}_{\mu\nu}^{(irj,jkr)}-\bar{B}_{\mu\nu}^{(irj,krj)}\Big)+(j\leftrightarrow k)\Bigg]$$
$$-\,\rho_{[jk]}^{(C)}\Bigg[\int d\Phi_{n+1}^{(ijk)}\,J_s^{ijk}\,\frac{\bar{P}_{jk(r)}^{(ijk)\mathrm{hc},\mu\nu}}{\bar{s}_{jk}^{(ijk)}}\,\bar{B}_{\mu\nu}^{(ijk,jkr)}+(j\leftrightarrow k)\Bigg]\Bigg\}\,,\qquad r=r_{ijk}\,.$$

Next we consider (again with $r = r_{ijk}$)

$$\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{S}}_{ij}\, RR\, \bar{\mathcal{Z}}_{\mathrm{s},\,jk} \;=\; -\,\mathcal{N}_1 \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}} \sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \bar{\mathcal{E}}^{(j)(ijr)}_{cd}\, \bar{B}^{(ijr,jcd)}_{cd}\, \bar{\mathcal{Z}}^{(ijr)}_{\mathrm{s},\,jk}\,, \tag{3.84}$$

where the choice of r different from i, j, k, analogously to the integral of $\overline{\mathbf{HC}}_{ij}\, RR$ in Eq. (3.72), causes a dependence of the integrated kernel J^{ijr}_{hc} on the index k of the sector function $\bar{\mathcal{Z}}^{(ijr)}_{\mathrm{s},\,jk}$. Next we have ($r = r_{ijkl}$, $r' = r_{ijk}$)

$$\begin{aligned}\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{SC}}_{kij}\, RR\, \bar{\mathcal{Z}}_{\mathrm{s},\,kl} \;=\; -\,\mathcal{N}_1 \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}} \Bigg[\sum_{\substack{c\neq i,j,k,r'\\ d\neq i,j,k,r',c}} \bar{\mathcal{E}}^{(k)(ijr)}_{cd}\, \bar{B}^{(ijr,kcd)}_{cd} \\ +\, 2 \sum_{c\neq i,j,k,r'} \bar{\mathcal{E}}^{(k)(ijr)}_{cr'}\, \bar{B}^{(ijr,kcr')}_{cr'} + 2 \sum_{c\neq i,j,k} \bar{\mathcal{E}}^{(k)(ijr)}_{jc}\, \bar{B}^{(ijr,kcj)}_{jc} \Bigg]\, \bar{\mathcal{Z}}^{(ijr)}_{\mathrm{s},\,kl}\,.\end{aligned} \tag{3.85}$$

Finally we move to strongly-ordered hard-collinear limits. First, with a collinear cluster of three particles we find (with $r = r_{ijk}$)

$$\begin{aligned}\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{HC}}^{(\mathbf{c})}_{ijk}\, RR = \mathcal{N}_1 \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \Bigg\{ \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}}\, \frac{\bar{P}^{(ijr)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\, \bar{B}^{(ijr,jkr)}_{\mu\nu} \\ -\, 2\, C_{f_{[ij]}} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}}\, \bar{\mathcal{E}}^{(k)(ijr)}_{jr} \left(\bar{B}^{(ijr,krj)} - \bar{B}^{(ijr,kjr)} \right) \Bigg\}\,.\end{aligned} \tag{3.86}$$

Then, with two independent pairs of collinear particles, we obtain

$$\int d\Phi_{n+2}\, \overline{\mathbf{HC}}_{ij}\, \overline{\mathbf{HC}}^{(\mathbf{c})}_{ijkl}\, RR = \mathcal{N}_1 \frac{\varsigma_{n+2}}{\varsigma_{n+1}} \int d\Phi^{(ijr)}_{n+1}\, J^{ijr}_{\mathrm{hc}}\, \frac{\bar{P}^{(ijr)\mathrm{hc},\mu\nu}_{kl(r)}}{\bar{s}^{(ijr)}_{kl}}\, \bar{B}^{(ijr,klr)}_{\mu\nu}\,, \qquad r = r_{ijkl}\,. \tag{3.87}$$

This concludes the list of all required integrals for double-real radiation.

Relabelling of momenta and flavour sums

Our next step will be to collect the results from different contributions and combine them by relabelling mapped momenta. More precisely, in all $(n+1)$-body phase spaces $d\Phi^{(abc)}_{n+1}$ appearing in the integrals of $K^{(\mathbf{1})}$ and $K^{(\mathbf{12})}$, we rename the sets of mapped momenta $\{\bar{k}^{(abc)}\}_{n+1}$ as a unique set of $(n+1)$ momenta $\{k\}_{n+1}$. With this new labelling, the indices of the mapped momenta refer directly to the particles of the unique $(n+1)$-body phase space, and the reference to the first mapping can be simply removed. The relabelling thus leads to

$$d\Phi_{n+1}^{(abc)} \to d\Phi_{n+1}\,, \qquad \bar{\mathcal{Z}}_{\dots}^{(abc)} \to \mathcal{Z}_{\dots}\,, \qquad \bar{R}_{\dots}^{(abc)} \to R_{\dots}\,, \qquad \bar{B}_{\dots}^{(abc,def)} \to \bar{B}_{\dots}^{(def)}\,,$$
$$\bar{s}_{ij}^{(abc)} \to s_{ij}\,, \qquad \bar{P}_{ij(r)}^{(abc)\mathrm{hc},\mu\nu} \to P_{ij(r)}^{\mathrm{hc},\mu\nu}\,, \qquad \bar{\mathcal{E}}_{lm}^{(i)(abc)} \to \mathcal{E}_{lm}^{(i)}\,. \tag{3.88}$$

Similarly, in the n-body phase spaces $d\Phi_n^{(abc,def)}$ appearing in the integral of $K^{(\mathbf{2})}$, the sets of mapped momenta $\{\bar{k}^{(abc,def)}\}_n$ are renamed as a unique set of n momenta $\{k\}_n$, which in practice means performing the substitutions

$$d\Phi_n^{(abc,def)} \;\to\; d\Phi_n\,, \qquad \bar{B}_{\dots}^{(abc,def)} \;\to\; B_{\dots} \qquad \bar{s}_{ij}^{(abc,def)} \;\to\; s_{ij}\,. \tag{3.89}$$

In particular, in the integral of $\overline{\mathbf{SHC}}_{ijk}(1-\overline{\mathbf{C}}_{ijk})RR$ in Eq. (3.76), which involves a collinear splitting of partons j and k, the momenta $\bar{k}_k^{(jkr,icd)}$, $\bar{k}_k^{(jkr,icr)}$, $\bar{k}_j^{(krj,icj)}$ and $\bar{k}_j^{(krj,ijr)}$ are all renamed as k_p, where p is the parent particle of j and k.

At this stage, in all integrated counterterms, the only recollection of the particles of the original $(n+2)$-body phase space is confined to the flavour factors f_i^q, $f_i^{\bar{q}}$, f_i^g. These can be summed up, and, if needed, translated into flavour factors for the particles of the $(n+1)$-body and n-body phase spaces. We now provide the rules to carry out these sums.

Let us begin with the simple case in which only one particle is integrated out, which is the case for $K^{(\mathbf{1})}$ and $K^{(\mathbf{12})}$. In this context, the following rules come into play.

- When going from an $(n+2)$-body phase space to an $(n+1)$-body phase space by discarding particle i, which happens when particle i is a soft gluon, the sum over flavour factors satisfies[8]

$$\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_i f_i^g = 1\,. \tag{3.90}$$

  For example, if all $(n+2)$ particles are gluons, one has $\varsigma_{n+2} = 1/(n+2)!$ and $\varsigma_{n+1} = 1/(n+1)!$, and the sum yields the missing factor of $n+2$.

- When going from an $(n+2)$-body phase space to an $(n+1)$-body phase space by replacing two particles i, j with their parent particle p, which happens when i and j form a collinear pair, the sum over the flavour factors of particles i, j can be written

[8] Eq. (3.90) is equal to Eq. (2.103), up to the appropriate symmetry factors.

as a sum over flavour factors for particle p according to the rules[9]

$$\begin{aligned}
\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_{i,j>i} f_{ij}^{q\bar{q}} &= N_f \sum_p f_p^g \,, \\
\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_{i,j>i} (f_{ij}^{gq} + f_{ij}^{g\bar{q}}) &= \sum_p (f_p^q + f_p^{\bar{q}}) \,, \\
\frac{\varsigma_{n+2}}{\varsigma_{n+1}} \sum_{i,j>i} f_{ij}^{gg} &= \frac{1}{2} \sum_p f_p^g \,.
\end{aligned} \tag{3.91}$$

As an example, consider the production of n gluons and a collinear $q\bar{q}$ pair. In this case the first line of Eq. (3.91) applies, and one must take into account the fact that quark flavours must be summed, since the quark pair is integrated out. One then has $\varsigma_{n+2} = N_f/n!$ and $\varsigma_{n+1} = 1/(n+1)!$, since the new final state involves $(n+1)$ gluons. For the same reason, the *r.h.s.* yields $N_f(n+1)$.

Not surprisingly, the flavour sum rules for the integrated $K^{(\mathbf{2})}$ are both more varied and more intricate. This complexity arises from integrating out two particles, either by removing them (when they are soft), or by replacing them with their (grand)parent particles when they form collinear sets. We consider the various cases in turn.

- When going from an $(n+2)$-body phase space to an n-body phase space by discarding two particles i, j, the sum over particles i, j satisfies

$$\frac{\varsigma_{n+2}}{\varsigma_n} \sum_i \sum_{j>i} f_{ij}^{gg} = \frac{1}{2} \,, \qquad \frac{\varsigma_{n+2}}{\varsigma_n} \sum_i \sum_{j>i} f_{ij}^{q\bar{q}} = N_f \,. \tag{3.92}$$

  As before, the first equality is easily verified when all $(n+2)$ particles are gluons, as is the second one when the final state consists of n gluons and a quark-antiquark pair.

- When going from an $(n+2)$-body phase space to an n-body phase space by replacing two particles j, k with their parent particle p, and by discarding particle i, the sum over particles i, j, k can be written as a sum over p according to the following rules,

$$\begin{aligned}
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ k>j}} f_i^g f_{jk}^{q\bar{q}} &= N_f \sum_p f_p^g \,, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ k>j}} f_i^g (f_{jk}^{gq} + f_{jk}^{g\bar{q}}) &= \sum_p (f_p^q + f_p^{\bar{q}}) \,, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ k>j}} f_i^g f_{jk}^{gg} &= \frac{1}{2} \sum_p f_p^g \,,
\end{aligned} \tag{3.93}$$

  where it is important to pay attention to the range of the various sums.

[9] The sums in Eq. (3.91) are equal to ones in Eq. (2.104), up to the appropriate symmetry factors.

- When going from an $(n+2)$-body phase space to an n-body phase space by replacing three particles i, j, k with their grandparent particle p, the sum over particles i, j, k can be replaced by a sum over p, as

$$
\begin{aligned}
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{k>j} \left(f_{ijk}^{q\bar{q}q'} + f_{ijk}^{q\bar{q}\bar{q}'}\right) &= N_f \sum_p \left(f_p^q + f_p^{\bar{q}}\right), \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{k>j} \left(f_{ijk}^{q\bar{q}q} + f_{ijk}^{q\bar{q}\bar{q}}\right) &= \frac{1}{2} \sum_p \left(f_p^q + f_p^{\bar{q}}\right), \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{k>j} f_{ijk}^{q\bar{q}g} &= N_f \sum_p f_p^g, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{k>j} \left(f_{ijk}^{ggq} + f_{ijk}^{gg\bar{q}}\right) &= \frac{1}{2} \sum_p \left(f_p^q + f_p^{\bar{q}}\right), \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{k>j} f_{ijk}^{ggg} &= \frac{1}{6} \sum_p f_p^g,
\end{aligned}
\tag{3.94}
$$

  where one easily recognises in the five lines the five possible partonic channels involving the production of a cluster of three collinear particles: in the first line, the final quark-antiquark pair can have any flavour (including that of the grandparent (anti)quark, which is the same as that of the final (anti)quark q'), while in the second line all three (anti)quarks have the same flavour.

- The most elaborated channel for flavour sums arises when going from an $(n+2)$-body phase space to an n-body phase space by replacing two pairs of particles i, j and k, l with their parent particles, p and t, respectively. In this case, the sum over particles i, j, k, l can be replaced by a sum over p and t according to the following rules:

$$
\begin{aligned}
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} f_{ij}^{q\bar{q}} f_{kl}^{q'\bar{q}'} &= \frac{N_f^2}{2} \sum_{p,t\neq p} f_{pt}^{gg}, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} \left[f_{ij}^{q\bar{q}}\left(f_{kl}^{gq'} + f_{kl}^{g\bar{q}'}\right) + \left(f_{ij}^{gq'} + f_{ij}^{g\bar{q}'}\right) f_{kl}^{q\bar{q}}\right] &= \frac{N_f}{2} \sum_{p,t\neq p} \left(f_{pt}^{gq} + f_{pt}^{g\bar{q}}\right), \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} \left(f_{ij}^{q\bar{q}} f_{kl}^{gg} + f_{ij}^{gg} f_{kl}^{q\bar{q}}\right) &= \frac{N_f}{2} \sum_{p,t\neq p} f_{pt}^{gg}, \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} \left(f_{ij}^{gq} + f_{ij}^{g\bar{q}}\right)\left(f_{kl}^{gq'} + f_{kl}^{g\bar{q}'}\right) &= \frac{1}{2} \sum_{p,t\neq p} \left(f_p^q + f_p^{\bar{q}}\right)\left(f_t^{q'} + f_t^{\bar{q}'}\right), \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} \left[\left(f_{ij}^{gq} + f_{ij}^{g\bar{q}}\right) f_{kl}^{gg} + f_{ij}^{gg}\left(f_{kl}^{gq} + f_{kl}^{g\bar{q}}\right)\right] &= \frac{1}{4} \sum_{p,t\neq p} \left(f_{pt}^{gq} + f_{pt}^{g\bar{q}}\right), \\
\frac{\varsigma_{n+2}}{\varsigma_n} \sum_{i,\,j>i} \sum_{\substack{k\neq j\\ k>i}} \sum_{\substack{l\neq j\\ l>k}} f_{ij}^{gg} f_{kl}^{gg} &= \frac{1}{8} \sum_{p,t\neq p} f_{pt}^{gg}.
\end{aligned}
\tag{3.95}
$$

We emphasise that the flavour sum rules listed in this Section apply for any final-state multiplicity and flavour structure. We now have all the tools to assemble the complete integrated counterterms, which will be naturally organised according to the flavour structures of the $(n+1)$-particle and of the n-particle phase spaces, as needed.

Assembling the complete integrated counterterms

After summing all contributions that were differently mapped, relabelling their momenta, and making use of the flavour rules, the resulting integrated counterterms no longer retain any trace of the original $(n+2)$-body phase space, and we can actually get full results for $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$, $I^{(\mathbf{12})}$, as defined in Eq. (3.4).

The simplest case is the integral of the single-unresolved counterterm $I^{(\mathbf{1})}$, which reads

$$\begin{aligned} I^{(\mathbf{1})} &= \sum_{i,j\neq i} I^{(\mathbf{1})}_{ij}\, \mathcal{W}_{ij} = \sum_{i,j>i} I^{(\mathbf{1})}_{ij}\, \mathcal{Z}_{ij}\,, \\ I^{(\mathbf{1})}_{ij} &= -\sum_{c,d\neq c} J_s(s_{cd})\, R_{cd} + \sum_k J^k_{\mathrm{hc}}(s_{kr})\, R\,, \qquad r = r_{ijk}\,. \end{aligned} \tag{3.96}$$

Here R is the full squared matrix element for single-real radiation, defined in Eq. (2.4), and R_{cd} is its colour-correlated counterpart, defined in Eq. (A.6). The single-soft integral J_s is given in Eq. (C.184), and the collinear integral J^k_{hc} is reported in Eq. (C.191). Because of the rule $r = r_{ijk}$, a dependence of $J^k_{\mathrm{hc}}(s_{kr})$ on i and j is left, excluding the possibility to sum over sectors in the hard-collinear part of $I^{(\mathbf{1})}$.

Moving to the integral of the double-unresolved counterterm, $I^{(\mathbf{2})}$, we assemble the corresponding contributions according to

$$I^{(\mathbf{2})} = I^{(\mathbf{2})}_{\mathrm{SS}} + I^{(\mathbf{2})}_{\mathrm{SHC}} + I^{(\mathbf{2})}_{\mathrm{HCC}} + I^{(\mathbf{2})}_{\mathrm{HCHC}}\,, \tag{3.97}$$

distinguishing double-soft, soft-times-hard-collinear and double-hard-collinear contributions, the last of which may involve three or four Born-level particles. For $I^{(\mathbf{2})}_{\mathrm{SS}}$ we get contributions containing Born-level colour correlations involving four, three and two particles, and we write

$$\begin{aligned} I^{(\mathbf{2})}_{\mathrm{SS}} = \frac{1}{4} \sum_{c,d\neq c} \Bigg\{ \sum_{e\neq c,d} \Bigg[\sum_{f\neq c,d,e} J^{(4)}_{\mathrm{s}\otimes\mathrm{s}}(s_{cd}, s_{ef}) B_{cdef} + 4\, J^{(3)}_{\mathrm{s}\otimes\mathrm{s}}(s_{cd}, s_{ed}) B_{cded} \Bigg] \\ + 2\, J^{(2)}_{\mathrm{s}\otimes\mathrm{s}}(s_{cd}) B_{cdcd} + 2 \Big[2\, N_f\, T_R\, J^{(\mathrm{q\bar{q}})}_{\mathrm{ss}}(s_{cd}) - C_A\, J^{(\mathrm{gg})}_{\mathrm{ss}}(s_{cd}) \Big] B_{cd} \Bigg\}\,, \end{aligned} \tag{3.98}$$

where the constituent integrals are given in Eq. (C.186). The soft-times-hard-collinear contribution yields

$$I^{(\mathbf{2})}_{\rm SHC} = -\sum_k \bigg\{ J^{\,k}_{\rm hc}(s_{kr}) \sum_{c,d\neq c} J_{\rm s}(s_{cd}) B_{cd} + J^{\,k}_{\rm shc}(s_{kr}) B + J^{\,k,{\rm A}}_{\rm shc}(s_{kr}) B_{kr} + \sum_{c\neq k,r} \Big[J^{\,k,{\rm B}}_{\rm shc}(s_{kr}, s_{kc}) B_{kc} + J^{\,k,{\rm B}}_{\rm shc}(s_{kr}, s_{cr}) B_{cr} \Big] \bigg\} , \qquad r = r_k , \tag{3.99}$$

where the rule $r = r_k$, as defined in Eq. (A.13), prevents r from being equal to k. In Eq. (3.99) we have introduced the following soft-times-hard-collinear integrals

$$\begin{aligned}
J^{\,k}_{\rm shc}(s) &= (f^q_k + f^{\bar q}_k) \Big\{ 2\, C_F\, J^{gqg}_{\rm s\otimes hc}(s) + C_A \Big[J^{ggq}_{\rm s\otimes hc}(s) - J^{gqg}_{\rm s\otimes hc}(s) \Big] \Big\} \\
&\quad + f^g_k\, C_A \Big[2\, N_f\, J^{gqq}_{\rm s\otimes hc}(s) + J^{ggg}_{\rm s\otimes hc}(s) \Big] , \\
J^{\,k,{\rm A}}_{\rm shc}(s) &= (f^q_k + f^{\bar q}_k) \Big\{ 2\, J^{gqg}_{\rm s\otimes hc}(s) + \frac{C_A}{C_F} \Big[J^{ggq}_{\rm s\otimes hc}(s) - J^{gqg}_{\rm s\otimes hc}(s) \Big] - 2\, J^{4(2{\rm g})}_{\rm s\otimes hc}(s,s) \Big\} \\
&\quad + f^g_k \Big\{ 2\, N_f \Big[J^{gqq}_{\rm s\otimes hc}(s) - J^{4(1{\rm g})}_{\rm s\otimes hc}(s,s) \Big] + J^{ggg}_{\rm s\otimes hc}(s) - J^{4(3{\rm g})}_{\rm s\otimes hc}(s,s) \Big\} , \\
J^{\,k,{\rm B}}_{\rm shc}(s,s') &= (f^q_k + f^{\bar q}_k) \Big[2\, J^{3(2{\rm g})}_{\rm s\otimes hc}(s,s') - 2\, J^{4(2{\rm g})}_{\rm s\otimes hc}(s,s') \Big] \\
&\quad + f^g_k \Big\{ 2\, N_f \Big[J^{3(1{\rm g})}_{\rm s\otimes hc}(s,s') - J^{4(1{\rm g})}_{\rm s\otimes hc}(s,s') \Big] + J^{3(3{\rm g})}_{\rm s\otimes hc}(s,s') - J^{4(3{\rm g})}_{\rm s\otimes hc}(s,s') \Big\} ,
\end{aligned} \tag{3.100}$$

whose constituent integrals can be found in Eq. (C.197). Next, we turn to the double-hard-collinear integral involving three Born-level particles, which reads

$$I^{(\mathbf{2})}_{\rm HCC} = \sum_k \bigg\{ (f^q_k + f^{\bar q}_k) \Big[N_f\, J^{(0{\rm g})}_{\rm hcc}(s_{kr}) + \frac{1}{2}\, J^{(0{\rm g},{\rm id})}_{\rm hcc}(s_{kr}) + \frac{1}{2}\, J^{(2{\rm g})}_{\rm hcc}(s_{kr}) \Big] + f^g_k \Big[N_f\, J^{(1{\rm g})}_{\rm hcc}(s_{kr}) + \frac{1}{6}\, J^{(3{\rm g})}_{\rm hcc}(s_{kr}) \Big] \bigg\} B , \qquad r = r_k ,$$

where the relevant constituent integrals are given in Eq. (C.193). Finally, we come to the double-hard-collinear integral involving four Born-level particles, for which we obtain

$$\begin{aligned}
I^{(\mathbf{2})}_{\rm HCHC} = \frac{1}{2} \sum_{j,l\neq j} \bigg\{ & (f^q_j + f^{\bar q}_j)(f^{q'}_l + f^{\bar q'}_l) J^{\rm qgqg}_{\rm hc\otimes hc}\big(s_{jr} s_{lr}\big) \\
& + (f^{gq}_{jl} + f^{g\bar q}_{jl}) \Big[N_f\, J^{\rm qqqg}_{\rm hc\otimes hc}\big(s_{jr} s_{lr}\big) + \frac{1}{2}\, J^{\rm qggg}_{\rm hc\otimes hc}\big(s_{jr} s_{lr}\big) \Big] \\
& + f^{gg}_{jl} \Big[N_f^2 J^{\rm qqqq}_{\rm hc\otimes hc}\big(s_{jr} s_{lr}\big) + N_f J^{\rm qqgg}_{\rm hc\otimes hc}\big(s_{jr} s_{lr}\big) + \frac{1}{4} J^{\rm gggg}_{\rm hc\otimes hc}\big(s_{jr} s_{lr}\big) \Big] \bigg\} B , \quad r = r_{jl} ,
\end{aligned} \tag{3.101}$$

where the constituent integrals are given in Eq. (C.195).

Similarly to $I^{\,(\mathbf{1})}$, we provide expressions for the integral of the strongly-ordered counterterm, $I^{\,(\mathbf{12})}$, with both unsymmetrised and symmetrised sector functions, so as to make it straightforward to prove that $I^{(\mathbf{12})}$ compensates sector by sector the phase-space singularities of $I^{(\mathbf{1})}$. Starting with the expression involving the original sector functions $\mathcal{W}_{ij}$, we write

$$I^{(\mathbf{12})} = \sum_{i,j\neq i} I^{(\mathbf{12})}_{ij}\,, \qquad I^{(\mathbf{12})}_{ij} = I^{(\mathbf{12})}_{\mathrm{S},ij}\,\mathcal{W}_{\mathrm{s},ij} + I^{(\mathbf{12})}_{\mathrm{C},ij} - I^{(\mathbf{12})}_{\mathrm{SC},ij}\,, \tag{3.102}$$

where the soft limit of sector functions $\mathcal{W}_{\mathrm{s},ij}$ is given in Eq. (C.41). The soft integral $I^{(\mathbf{12})}_{\mathrm{S},ij}$ can again be organised in terms of quadruple, triple and simple Born-level colour correlations, which in this case will be multiplied by eikonal kernels for the second radiation, and NLO-type soft and hard-collinear integrals. We find ($r = r_{ik}$, $r' = r_{ij}$, $r'' = r_{ijk}$)

$$\begin{aligned}
I^{(\mathbf{12})}_{\mathrm{S},ij} = \mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \bigg\{ \frac{1}{2} \sum_{\substack{e\neq i\\ f\neq i,e}} J_s(s_{ef})\,\bar{B}^{(icd)}_{cdef} + \sum_{e\neq i,d} J_s(s_{de}) \left(\bar{B}^{(icd)}_{cded} - \bar{B}^{(idc)}_{cded}\right) \\
- C_A \Big[J_s(s_{ic}) + J_s(s_{id}) - J_s(s_{cd})\Big]\bar{B}^{(icd)}_{cd} - J^{i}_{\mathrm{hc}}(s_{ir'})\bar{B}^{(icd)}_{cd} \bigg\} \\
- \mathcal{N}_1 \sum_{k\neq i} J^{k}_{\mathrm{hc}}(s_{kr''}) \bigg[\sum_{\substack{c\neq i,k,r\\ d\neq i,c,k,r}} \mathcal{E}^{(i)}_{cd}\,\bar{B}^{(icd)}_{cd} + 2 \sum_{c\neq i,k,r} \mathcal{E}^{(i)}_{cr}\,\bar{B}^{(icr)}_{cr} + 2 \sum_{c\neq i,k} \mathcal{E}^{(i)}_{kc}\,\bar{B}^{(ick)}_{kc} \bigg]\,,
\end{aligned} \tag{3.103}$$

where the component integrals are given in Eq. (C.184) and in Eq. (C.191). We notice that the expression contains three different reference particles r, r' and r'', all built according to the rule in Eq. (A.13). In particular, $r' = r_{ij}$ and $r'' = r_{ijk}$ introduce a dependence in $I^{(\mathbf{12})}_{\mathrm{S},ij}$ on the particle j of the soft sector function $\mathcal{W}_{\mathrm{s},ij}$. The collinear integral $I^{(\mathbf{12})}_{\mathrm{C},ij}$ in Eq. (3.102) is formulated in terms of spin-correlated Born-level squared matrix elements, which in this case are multiplied by LO collinear kernels for the least-unresolved collinear splitting, and times suitable combinations of the same constituent integrals as in Eq. (3.103). We find (with $r = r_{ij}$, $r' = r_{ijk}$)

$$\begin{aligned}
I^{(\mathbf{12})}_{\mathrm{C},ij} = -\mathcal{N}_1 \frac{P^{\mu\nu}_{ij(r)}}{s_{ij}} \bigg\{ \sum_{c\neq i,j} \sum_{d\neq i,j,c} J_s(s_{cd})\,\bar{B}^{(ijr)}_{\mu\nu,cd} + C_{f_{[ij]}}\,\rho^{(C)}_{[ij]}\,J_s(s_{ij})\,\bar{B}^{(ijr)}_{\mu\nu} \\
+ \bigg[\sum_{c\neq i,j} J_s(s_{ic}) \left(\rho^{(C)}_{ij}\,\bar{B}^{(jri)}_{\mu\nu,[ij]c} + \tilde{f}^{\,q\bar{q}}_{ij}\,\bar{\mathcal{B}}^{(jri)}_{\mu\nu,[ij]c}\right) \\
+ C_{f_{[ij]}}\,\rho^{(C)}_{ij}\,J_s(s_{ir}) \left(\bar{B}^{(jri)}_{\mu\nu} - \bar{B}^{(ijr)}_{\mu\nu}\right) + (i \leftrightarrow j)\bigg]\bigg\} \mathcal{W}_{\mathrm{c},ij(r)} \\
+ \mathcal{N}_1 \frac{P^{\mu\nu}_{ij(r)}}{s_{ij}} \Big[J^{i}_{\mathrm{hc}}(s_{ir}) + J^{j}_{\mathrm{hc}}(s_{jr})\Big]\bar{B}^{(ijr)}_{\mu\nu}\,\mathcal{W}_{\mathrm{c},ij(r)} + \mathcal{N}_1 \sum_{k\neq i,j} \frac{P^{\mu\nu}_{ij(r')}}{s_{ij}}\,J^{k}_{\mathrm{hc}}(s_{kr'})\,\bar{B}^{(ijr')}_{\mu\nu}\,\mathcal{W}_{\mathrm{c},ij(r')}\,,
\end{aligned} \tag{3.104}$$

where the collinear limit of sector functions $\mathcal{W}_{\mathrm{c},ij}$ is given in Eq. (C.42), and again two reference particles have to be introduced. Finally, the soft-collinear integral has a similar structure and reads (with $r = r_{ij}$, $r' = r_{ijk}$)

$$
\begin{aligned}
I^{(\mathbf{12})}_{\mathrm{SC},ij} = & -2\,\mathcal{N}_1\,\mathcal{E}^{(i)}_{jr}\bigg\{ C_{f_j}\sum_{c\neq i,j}\sum_{d\neq i,j,c} J_s(s_{cd})\,\bar{B}^{(ijr)}_{cd} - C_{f_j}C_A\,J_s(s_{ij})\,\bar{B}^{(ijr)} \\
& + C_A\bigg[\sum_{c\neq i,j} J_s(s_{ic})\,\bar{B}^{(jri)}_{[ij]c} + C_{f_j}\,J_s(s_{ir})\Big(\bar{B}^{(jri)} - \bar{B}^{(ijr)}\Big)\bigg] \\
& + (2C_{f_j} - C_A)\bigg[\sum_{c\neq i,j} J_s(s_{jc})\,\bar{B}^{(irj)}_{[ij]c} + C_{f_j}\,J_s(s_{jr})\Big(\bar{B}^{(irj)} - \bar{B}^{(ijr)}\Big)\bigg]\bigg\} \\
& + 2\,\mathcal{N}_1 C_{f_j}\mathcal{E}^{(i)}_{jr}\Big[J^i_{\mathrm{hc}}(s_{ir})\,\bar{B}^{(ijr)} + J^j_{\mathrm{hc}}(s_{jr})\,\bar{B}^{(irj)}\Big] + 2\,\mathcal{N}_1 C_{f_j}\mathcal{E}^{(i)}_{jr'}\sum_{k\neq i,j} J^k_{\mathrm{hc}}(s_{kr'})\,\bar{B}^{(ijr')}.
\end{aligned}
\tag{3.105}
$$

As already noted, a more compact expression for $I^{(\mathbf{12})}$ can be obtained using symmetrised sector functions. We can equivalently write

$$
I^{(\mathbf{12})} = \sum_{i,j>i} I^{(\mathbf{12})}_{\{ij\}}\,, \qquad\qquad I^{(\mathbf{12})}_{\{ij\}} = I^{(\mathbf{12})}_{\mathrm{S},ij}\,\mathcal{Z}_{\mathrm{s},ij} + I^{(\mathbf{12})}_{\mathrm{S},ji}\,\mathcal{Z}_{\mathrm{s},ji} + I^{(\mathbf{12})}_{\mathrm{HC},ij}\,, \tag{3.106}
$$

where the soft contributions were already introduced in Eq. (3.103), while the hard-collinear contribution $I^{(\mathbf{12})}_{\mathrm{HC},ij}$ reads ($r = r_{ij}$, $r' = r_{ijk}$)

$$
\begin{aligned}
I^{(\mathbf{12})}_{\mathrm{HC},ij} = & \; I^{(\mathbf{12})}_{\mathrm{C},ij} + I^{(\mathbf{12})}_{\mathrm{C},ji} - I^{(\mathbf{12})}_{\mathrm{SC},ij} - I^{(\mathbf{12})}_{\mathrm{SC},ji} \\
= & -\mathcal{N}_1\,\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\bigg\{\sum_{c\neq i,j}\sum_{d\neq i,j,c} J_s(s_{cd})\,\bar{B}^{(ijr)}_{\mu\nu,cd} + C_{f_{[ij]}}\,\rho^{(C)}_{[ij]}\,J_s(s_{ij})\,\bar{B}^{(ijr)}_{\mu\nu} \\
& + \bigg[\sum_{c\neq i,j} J_s(s_{ic})\,\Big(\rho^{(C)}_{ij}\,\bar{B}^{(jri)}_{\mu\nu,[ij]c} + \tilde{f}^{\,q\bar{q}}_{ij}\,\bar{\mathcal{B}}^{(jri)}_{\mu\nu,[ij]c}\Big) \\
& + C_{f_{[ij]}}\,\rho^{(C)}_{ij}\,J_s(s_{ir})\,\Big(\bar{B}^{(jri)}_{\mu\nu} - \bar{B}^{(ijr)}_{\mu\nu}\Big) + (i\leftrightarrow j)\bigg]\bigg\} \\
& + \mathcal{N}_1\,\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\Big[J^i_{\mathrm{hc}}(s_{ir}) + J^j_{\mathrm{hc}}(s_{jr})\Big]\bar{B}^{(ijr)}_{\mu\nu} + \mathcal{N}_1\sum_{k\neq i,j}\frac{P^{\mathrm{hc},\mu\nu}_{ij(r')}}{s_{ij}}\,J^k_{\mathrm{hc}}(s_{kr'})\,\bar{B}^{(ijr')}_{\mu\nu} \\
& - 2\,\mathcal{N}_1\bigg[C_{f_i}\mathcal{E}^{(j)}_{ir}\,J^i_{\mathrm{hc}}(s_{ir})\,\Big(\bar{B}^{(jri)} - \bar{B}^{(jir)}\Big) + C_{f_j}\mathcal{E}^{(i)}_{jr}\,J^j_{\mathrm{hc}}(s_{jr})\,\Big(\bar{B}^{(irj)} - \bar{B}^{(ijr)}\Big)\bigg].
\end{aligned}
\tag{3.107}
$$

This concludes the list of integrated counterterms for double-real radiation. We now turn to the treatment of real-virtual contributions.

3.4 The subtracted real-virtual contribution $RV_{\rm sub}$

Let us review what we have accomplished up to this point. After subtracting the appropriate combination of the local counterterms $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$ from the double-real squared matrix element RR, and after adding back the corresponding integrated counterterms, $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$ and $I^{(\mathbf{12})}$, we can write a partially subtracted expression for the differential distribution in Eq. (3.1). It reads

$$\begin{aligned}\frac{d\sigma_{\rm NNLO}}{dX} &= \int d\Phi_n \Big[VV + I^{(\mathbf{2})}\Big]\,\delta_n(X) \\ &+ \int d\Phi_{n+1}\Big[\,\big(RV + I^{(\mathbf{1})}\big)\,\delta_{n+1}(X) - I^{(\mathbf{12})}\,\delta_n(X)\Big] \\ &+ \int d\Phi_{n+2}\, RR_{\rm sub}(X)\,. \end{aligned} \tag{3.108}$$

Notice that no approximations have been made in reaching Eq. (3.108), since all local terms that were subtracted from Eq. (3.1) have been added back exactly in their integrated form. At this stage, $RR_{\rm sub}$, given in Eq. (3.33) or in Eq. (3.47), is free of phase-space singularities in Φ_{n+2}, and (evidently) does not contain explicit poles in ϵ. Therefore, it can be directly integrated in four dimensions, as desired.

We can then safely turn our attention to the second line of Eq. (3.108), namely the real-virtual correction. As discussed in Section 3.1, this contribution is affected by the presence of phase-space singularities, due to the extra single-unresolved radiative emission, as well as explicit poles in ϵ, originating from its one-loop nature. These characteristics inevitably make the devising of a strategy to remove those singularities, a non-trivial task. We will proceed by steps. We will start by assessing the role played by the insertion of the integrals $I^{(\mathbf{1})}$ and $I^{(\mathbf{12})}$ in the cancellation pattern (Section 3.4.1). We will however soon realise that these contributions are not sufficient to reach our goal, as they too bring with them further singularities that must be taken care of, in addition to those of RV. It will be therefore necessary to introduce a fourth counterterm, $K^{(\mathbf{RV})}$, specifically designed to make the full RV correction comprehensively free from ϵ poles, and integrable in the $(n+1)$-body phase space (Section 3.4.2). Once verified the finiteness of the second line of Eq. (3.108), in Section 3.4.3 we will present the formulation of the real-virtual counterterm in terms of symmetrised sector functions.

3.4.1 Integrated contributions

By introducing the integrated counterterms $I^{(\mathbf{1})}$ and $I^{(\mathbf{12})}$ as defined in Eq. (3.96) and Eq. (3.102), we can confirm that the second line of Eq. (3.108) verifies two crucial properties that follow from general cancellation theorems and from the definitions provided up to this point. Specifically, the combination of these ingredients leads to

$$\begin{aligned} &(1) \qquad \big(RV + I^{(\mathbf{1})}\big)\,\delta_{n+1}(X) \;\to\; \text{finite}\,, \\ &(2) \qquad I^{(\mathbf{1})}\,\delta_{n+1}(X) - I^{(\mathbf{12})}\,\delta_n(X) \;\to\; \text{integrable}\,. \end{aligned} \tag{3.109}$$

The first property derives from the KLN theorem: indeed, $I^{(\mathbf{1})}$ is the integral over the most unresolved radiation of RR, and its IR poles must compensate the virtual poles arising when one of the two unresolved particles becomes virtual, while the other one remains unaffected. These are precisely the poles of RV. To check this, which provides a test of the results obtained so far, it is sufficient to perform the ϵ expansion of $I^{(\mathbf{1})}$, as given in Eq. (3.96), writing

$$I^{(\mathbf{1})} = I^{(\mathbf{1})}_{\rm poles} + I^{(\mathbf{1})}_{\rm fin} + \mathcal{O}(\epsilon)\,. \tag{3.110}$$

Performing the sum over sectors in $I^{(\mathbf{1})}_{\rm poles}$, we get

$$I^{(\mathbf{1})}_{\rm poles} = \frac{\alpha_s}{2\pi}\left[\frac{1}{\epsilon^2}\,\Sigma_C\,R + \frac{1}{\epsilon}\left(\Sigma_\gamma\,R + \sum_{c,d\neq c} L_{cd}\,R_{cd}\right)\right] = -\,RV_{\rm poles}\,, \tag{3.111}$$

while keeping the complete dependence on sector functions in $I^{(\mathbf{1})}_{\rm fin}$, we find

$$I^{(\mathbf{1})}_{\rm fin} = \sum_{i,j\neq i} I^{(\mathbf{1})}_{{\rm fin},ij}\,\mathcal{W}_{ij} = \sum_{i,j>i} I^{(\mathbf{1})}_{{\rm fin},ij}\,\mathcal{Z}_{ij}\,, \tag{3.112}$$

$$I^{(\mathbf{1})}_{{\rm fin},ij} = \frac{\alpha_s}{2\pi}\left[\left(\Sigma_\phi - \sum_k \gamma_k^{\rm hc}\,L_{kr}\right)R + \sum_{c,d\neq c} L_{cd}\left(2-\frac{1}{2}\,L_{cd}\right)R_{cd}\right], \qquad r = r_{ijk}\,.$$

In Eqs. (3.111)-(3.112), $L_{ab} = \log(s_{ab}/\mu^2)$, and the numerical coefficients are given in Eqs. (A.7)-(A.10). One easily verifies that $I^{(\mathbf{1})}_{\rm poles}$ matches the explicit poles of the real-virtual matrix element $RV_{\rm poles}$, which have the well-known universal NLO structure (see for example [116, 146]), upon replacing the n-point amplitude with the $(n+1)$-point amplitude.

The second property in Eq. (3.109) guarantees the cancellation of phase-space singularities arising from to the real-radiation matrix elements squared R present in $I^{(\mathbf{1})}$. In order to prove it, we start from the decompositions of the integrated counterterms in terms of the sector fuctions $\mathcal{W}_{ij}$, provided in Eqs. (3.96)-(3.102), which combination reads

$$I^{(\mathbf{1})}\delta_{n+1}(X) - I^{(\mathbf{12})}\delta_n(X) = \sum_{i,j\neq i}\left\{I^{(\mathbf{1})}_{ij}\mathcal{W}_{ij}\,\delta_{n+1}(X) - \left[I^{(\mathbf{12})}_{{\rm S},ij}\,\mathcal{W}_{{\rm s},ij} + I^{(\mathbf{12})}_{{\rm C},ij} - I^{(\mathbf{12})}_{{\rm SC},ij}\right]\delta_n(X)\right\}. \tag{3.113}$$

The NLO sector functions $\mathcal{W}_{ij}$ and $\mathcal{W}_{{\rm s},ij}$ are defined in Eq. (2.19) and Eq. (C.41) respectively. The local subtraction of phase-space singularities in Eq. (3.113) is thus expected to occur at the level of single sectors $\mathcal{W}_{ij}$, owing to the relations

$$\mathbf{S}_i \left[I^{(\mathbf{1})}_{ij} \, \mathcal{W}_{ij} - I^{(\mathbf{12})}_{\mathrm{S},ij} \, \mathcal{W}_{\mathrm{s},ij} \right] \to \text{integrable} \, , \qquad \mathbf{S}_i \left[I^{(\mathbf{12})}_{\mathrm{C},ij} - I^{(\mathbf{12})}_{\mathrm{SC},ij} \right] \to \text{integrable} \, ,$$
$$\mathbf{C}_{ij} \left[I^{(\mathbf{1})}_{ij} \, \mathcal{W}_{ij} - I^{(\mathbf{12})}_{\mathrm{C},ij} \right] \to \text{integrable} \, , \quad \mathbf{C}_{ij} \left[I^{(\mathbf{12})}_{\mathrm{S},ij} \, \mathcal{W}_{\mathrm{s},ij} - I^{(\mathbf{12})}_{\mathrm{SC},ij} \right] \to \text{integrable} \, . \quad (3.114)$$

For concreteness, consider the first relation. Under soft limit, the $(n+1)$-particle matrix element in $I^{(\mathbf{1})}_{ij}$ returns a sum of products of eikonal factors and n-particle Born-level, colour-correlated matrix elements, and its sector function $\mathcal{W}_{ij}$ becomes equal to $\mathcal{W}_{\mathrm{s},ij}$. At the same time, when the operator $\mathbf{S}_i$ acts on $I^{(\mathbf{12})}_{\mathrm{S},ij}$, it effectively removes the phase-space mappings, so that Eq. (3.103) tends to the $\mathbf{S}_i$ limit of the sum of soft and collinear integrals in Eq. (3.96), up to the overall sign. Similar steps show the validity of the other relations in Eq. (3.114).

3.4.2 Definition of the local counterterm

At this stage, we have established that the sum $\big(RV + I^{\,(\mathbf{1})}\big)\, \delta_{n+1}(X)$ is free of explicit poles, while the combination $I^{\,(\mathbf{1})}\, \delta_{n+1}(X) - I^{\,(\mathbf{12})}\, \delta_n(X)$ is integrable in Φ_{n+1}. Nonetheless, the former expression still contains phase-space singularities arising from the real-virtual correction RV, whereas the latter still exhibits explicit poles in ϵ, specifically from $I^{\,(\mathbf{12})}$.

In order to build a fully subtracted real-virtual matrix element RV_{sub}, it is necessary to define a real-virtual counterterm $K^{(\mathbf{RV})}_{ij}$. This counterterm, sector by sector, must satisfy the two further properties:

$$\begin{aligned} &(3) \qquad K^{(\mathbf{RV})}_{ij} + I^{(\mathbf{12})}_{ij} \;\to\; \text{finite} \, , \\ &(4) \qquad RV \, \mathcal{W}_{ij} \, \delta_{n+1}(X) - K^{(\mathbf{RV})}_{ij} \, \delta_n(X) \;\to\; \text{integrable} \, . \end{aligned} \qquad (3.115)$$

Once these conditions are met, the subtracted real-virtual contribution to the cross section, defined in Eq. (3.7), is manifestly finite and integrable in Φ_{n+1}. To explicitly prove the relations in Eq. (3.115), we reformulate RV_{sub} as a sum over sectors, obtaining

$$RV_{\mathrm{sub}}(X) \;=\; \sum_{i, j \neq i} \left[\Big(RV + I^{(\mathbf{1})}_{ij} \Big) \mathcal{W}_{ij} \, \delta_{n+1}(X) - \Big(K^{(\mathbf{RV})}_{ij} + I^{(\mathbf{12})}_{ij} \Big) \, \delta_n(X) \right] . \quad (3.116)$$

As sector functions $\mathcal{W}_{ij}$ selects only single soft and collinear singular limits associated with a specific pair of partons ij, the second condition in Eq. (3.115) effectively reduces to verify that

$$RV \, \mathcal{W}_{ij} \, \delta_{n+1}(X) - K^{(\mathbf{RV})}_{ij} \, \delta_n(X) \;\to\; \text{integrable in the limits } \mathbf{S}_i, \, \mathbf{C}_{ij} \, . \qquad (3.117)$$

In order to find a suitable expression for $K^{(\mathbf{RV})}_{ij}$, we adopt a strategy akin to the one used in the NLO scenario. We start by defining soft and collinear *improved* limits, $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$, for the real-virtual squared matrix element. As a crucial requirement, such limits must reproduce the behaviour of RV in all singular regions of phase space. This is verified by checking a set of *consistency relations*[10], specifically

$$\begin{aligned} \mathbf{S}_i \Big\{ \big(1 - \overline{\mathbf{S}}_i\big) \,,\, \overline{\mathbf{C}}_{ij} \big(1 - \overline{\mathbf{S}}_i\big) \Big\} \, RV \, \mathcal{W}_{ij} \;&\to\; \text{integrable} \,, \\ \mathbf{C}_{ij} \Big\{ \big(1 - \overline{\mathbf{C}}_{ij}\big) \,,\, \overline{\mathbf{S}}_i \big(1 - \overline{\mathbf{C}}_{ij}\big) \Big\} \, RV \, \mathcal{W}_{ij} \;&\to\; \text{integrable} \,. \end{aligned} \tag{3.118}$$

As previously mentioned, these improved limits are devised to act on the real-virtual matrix element not only by extracting its leading behaviour in the singular phase-space regions, they also associate specific kinematic mappings to each counterterm contribution in order to provide well-behaved (i.e. on-shell and momentum conserving) Born-level kinematics, in the whole phase space. Moreover, such mappings can be selected to simplify as much as possible the analytic integration over the corresponding radiation phase space. Following the discussion presented at NLO, and the choices made in Ref. [200], we introduce

$$\begin{aligned} \overline{\mathbf{S}}_i \, RV \, \mathcal{W}_{ij} \equiv& \\ & - \mathcal{N}_1 \sum_{\substack{c \neq i \\ d \neq i,c}} \bigg[\mathcal{E}^{(i)}_{cd} \, \bar{V}^{(icd)}_{cd} - \frac{\alpha_{\rm S}}{2\pi} \bigg(\tilde{\mathcal{E}}^{(i)}_{cd} + \mathcal{E}^{(i)}_{cd} \, \frac{\beta_0}{2\epsilon} \bigg) \bar{B}^{(icd)}_{cd} + \alpha_{\rm S} \sum_{e \neq i,c,d} \tilde{\mathcal{E}}^{(i)}_{cde} \, \bar{B}^{(icd)}_{cde} \bigg] \mathcal{W}_{{\rm s},ij} \,, \\ \overline{\mathbf{C}}_{ij} \, RV \, \mathcal{W}_{ij} \equiv& \, \frac{\mathcal{N}_1}{s_{ij}} \bigg[P^{\mu\nu}_{ij(r)} \, \bar{V}^{(ijr)}_{\mu\nu} + \frac{\alpha_{\rm S}}{2\pi} \bigg(\tilde{P}^{\mu\nu}_{ij(r)} - P^{\mu\nu}_{ij(r)} \frac{\beta_0}{2\epsilon} \bigg) \bar{B}^{(ijr)}_{\mu\nu} \bigg] \mathcal{W}_{{\rm c},ij} \,, \\ \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, RV \, \mathcal{W}_{ij} \equiv& \, 2 \mathcal{N}_1 \, C_{f_j} \bigg[\mathcal{E}^{(i)}_{jr} \bar{V}^{(ijr)} - \frac{\alpha_{\rm S}}{2\pi} \bigg(\tilde{\mathcal{E}}^{(i)}_{jr} + \mathcal{E}^{(i)}_{jr} \, \frac{\beta_0}{2\epsilon} \bigg) \bar{B}^{(ijr)} \bigg] \,, \qquad r = r_{ij} \,. \end{aligned} \tag{3.119}$$

The kernels $\mathcal{E}^{(i)}_{cd}$ and $P^{\mu\nu}_{ij(r)}$ are the eikonal and collinear kernels from tree-level factorisation, introduced already at NLO, while $\tilde{\mathcal{E}}^{(i)}_{cd}$, $\tilde{\mathcal{E}}^{(i)}_{cde}$ and $\tilde{P}^{\mu\nu}_{ij(r)}$ are the genuine real-virtual soft and collinear kernels [128, 131]. Explicit expressions can be found in Eqs. (C.3), (C.8), (C.6), (C.25), respectively.

Since the combination $(1 - \overline{\mathbf{S}}_i)(1 - \overline{\mathbf{C}}_{ij}) \, RV \, \mathcal{W}_{ij}$ is integrable everywhere in Φ_{n+1}, one would expect to define the counterterm $K^{(\mathbf{RV})}_{ij}$ simply as an NLO-like collection of improved limits, as

$$K^{(\mathbf{RV})}_{ij,\,\rm naive} \;=\; \Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} \big(1 - \overline{\mathbf{S}}_i\big) \Big] \, RV \, \mathcal{W}_{ij} \,, \tag{3.120}$$

employing the definitions provided in Eqs. (3.119). Although such a choice preserves the minimal structure of the real-virtual counterterm, and automatically fulfils condition (4)

[10]Note that these consistency relations are analogous to those introduced at the NLO level in Eq. (2.69) for dealing with the real-emission correction, consistent with the fact that, in both cases, the phase-space singularities to be cured arise from a single-unresolved radiation.

of Eq. (3.115), surprisingly, explicit computations show that it spoils condition (3) of Eq. (3.115). Let us explain the reasons behind this violation.

The pole content of $K^{(\mathbf{RV})}_{ij,\,\mathrm{naive}}$ is designed to match the poles of RV that are accompanied by phase-space singularities, as a necessary requirement to verify condition (4) of Eq. (3.115). On the other hand, $I^{(\mathbf{12})}_{ij}$ is the result of integrating the strongly-ordered counterterm $K^{(\mathbf{12})}_{ij}$ over the phase space of the most unresolved radiation: in fact, it collects precisely terms that exhibit phase-space singularities in the remaining radiation (in the form of singular kernels multiplied by mapped Born-level matrix elements), as well as poles that should match their virtual counterpart, given by RV. Hence, it would be natural to expect that the poles of Eq. (3.120) cancel those of $I^{(\mathbf{12})}_{ij}$. However, unforeseen subtleties, stemming from the specific phase-space mappings adopted in the improved limits entering $K^{(\mathbf{RV})}_{ij,\,\mathrm{naive}}$, or $I^{(\mathbf{12})}_{ij}$ in their integrated form, prevent this from occurring. The first class of mismatches we find concerns single ϵ-pole terms that appear multiplied by kinematics-dependent coefficients within the contributions under investigation. Specifically, we notice that the residues of the single poles in $I^{(\mathbf{12})}_{ij}$ (refer to Eqs. (3.103)-(3.105)) are proportional to logarithms of Lorentz invariants constructed with *unmapped* momenta, *i.e.* with $(n+1)$-body kinematics; on the contrary, the residues of the single poles in the ϵ expansions of Eq. (3.119) can also depend on logarithms of *mapped* invariants, obtained via momentum mappings from the $(n+1)$- to the n-particle phase space. This is the case, for instance, for the virtual component of the soft limit $\overline{\mathbf{S}}_i\, RV$ in Eq. (3.119): the pole content of $\bar{V}^{(icd)}_{cd}$ includes terms of the type,

$$\Big[\overline{\mathbf{S}}_i\, RV\Big]_{\mathrm{poles}} \supset \frac{\alpha_{\mathrm{S}}}{2\pi}\,\mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd}\,\frac{1}{\epsilon}\left(\frac{1}{2}\sum_{\substack{e\neq i,c\\ f\neq i,c,e}} \log\frac{\bar{s}^{(icd)}_{ef}}{\mu^2}\,\bar{B}^{(icd)}_{cdef} + \sum_{e\neq i,d}\log\frac{\bar{s}^{(icd)}_{de}}{\mu^2}\,\bar{B}^{(icd)}_{cded}\right), \quad (3.121)$$

which cannot appear in the soft part of $I^{(\mathbf{12})}_{ij}$, where we find instead

$$\Big[I^{(\mathbf{12})}_{\mathrm{S},ij}\Big]_{\mathrm{poles}} \supset -\frac{\alpha_s}{2\pi}\,\mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd}\,\frac{1}{\epsilon}\left(\frac{1}{2}\sum_{\substack{e\neq i,c\\ d\neq i,c,e}} \log\frac{s_{ef}}{\mu^2}\,\bar{B}^{(icd)}_{cdef} + \sum_{e\neq i,d}\log\frac{s_{de}}{\mu^2}\,\bar{B}^{(icd)}_{cded}\right). \quad (3.122)$$

The second category of mismatches emerges exclusively in the integrated counterterm $I^{(\mathbf{12})}_{ij}$, and basically consists in differences of terms that would vanish if not for the different mappings appearing in the Born-level matrix elements associated with such contributions. Here is an example of such a discrepancy arising in Eq. (3.103):

$$\Big[I^{(\mathbf{12})}_{\mathrm{S},ij}\Big]_{\mathrm{poles}} \supset \frac{\alpha_s}{2\pi}\,2\,\mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd}\left(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\right) C_{f_c}\left(\bar{B}^{(icd)}_{cd} - \bar{B}^{(idc)}_{cd}\right). \quad (3.123)$$

More intricate inconsistencies occur in the collinear sector, where the kinematics of the poles of $I^{(\mathbf{12})}_{ij}$ does not align with that of $K^{(\mathbf{RV})}_{ij,\,\mathrm{naive}}$ outside the collinear region, regardless of mappings.

The fact that all discrepancies in the single pole in ϵ disappear in the singular regions of phase space, as they must, gives us the possibility to refine the definition of $K^{(\mathbf{RV})}_{ij,\,\mathrm{naive}}$, by adding back precisely the mismatched terms, thus obtaining the desired cancellation of the $I^{(\mathbf{12})}_{ij}$ poles, without introducing new phase-space singularities. Schematically, we define

$$K^{(\mathbf{RV})}_{ij} \;\equiv\; K^{(\mathbf{RV})}_{ij,\,\mathrm{naive}} + \Delta_{ij} \;=\; \Big[\,\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}\left(1 - \overline{\mathbf{S}}_i\right)\Big] RV\,\mathcal{W}_{ij} + \Delta_{ij}\,. \tag{3.124}$$

The extra term Δ_{ij} appearing in Eq. (3.124) is required not to spoil condition (4) of Eq. (3.115), and therefore cannot have any phase-space singularity in the limits $\mathbf{S}_i$ and $\mathbf{C}_{ij}$. Therefore, we impose the conditions

$$\mathbf{S}_i\,\Delta_{ij} \;\to\; \text{integrable}\,, \qquad \mathbf{C}_{ij}\,\Delta_{ij} \;\to\; \text{integrable}\,. \tag{3.125}$$

At the same time, Δ_{ij} has the crucial role of matching the explicit ϵ poles of $I^{(\mathbf{12})}_{ij}$, implying the finiteness of the combination $K^{(\mathbf{RV})}_{ij} + I^{(\mathbf{12})}_{ij}$, in agreement with condition (3) of Eq. (3.115). We introduce Δ_{ij} starting from a decomposition into soft, collinear and soft-collinear components, following the structure outlined for $I^{(\mathbf{12})}_{ij}$ in Eq. (3.102). Using this decomposition,

$$\Delta_{ij} \;\equiv\; \Delta_{\mathrm{S},i}\,\mathcal{W}_{\mathrm{s},ij} + \Delta_{\mathrm{C},ij} - \Delta_{\mathrm{SC},ij}\,, \tag{3.126}$$

the properties Eq. (3.125) can be better detailed, and read

$$\begin{aligned} \mathbf{S}_i\,\Delta_{\mathrm{S},i}\,\mathcal{W}_{\mathrm{s},ij} &\to \text{integrable}\,, & \mathbf{S}_i\left(\Delta_{\mathrm{C},ij} - \Delta_{\mathrm{SC},ij}\right) &\to \text{integrable}\,, \\ \mathbf{C}_{ij}\,\Delta_{\mathrm{C},ij} &\to \text{integrable}\,, & \mathbf{C}_{ij}\left(\Delta_{\mathrm{S},i}\,\mathcal{W}_{\mathrm{s},ij} - \Delta_{\mathrm{SC},ij}\right) &\to \text{integrable}\,. \end{aligned} \tag{3.127}$$

Furthermore, we can enforce the desired cancellation between $K^{(\mathbf{RV})}_{ij}$ and $I^{(\mathbf{12})}_{ij}$ for each component, specifically by requiring that

$$\begin{aligned} \Big[\overline{\mathbf{S}}_i\,RV\,\mathcal{W}_{ij} + \Big(\Delta_{\mathrm{S},i} + I^{(\mathbf{12})}_{\mathrm{S},ij}\Big)\,\mathcal{W}_{\mathrm{s},ij}\Big]_{\mathrm{poles}} &= 0\,, \\ \Big[\overline{\mathbf{C}}_{ij}\,RV\,\mathcal{W}_{ij} + \Big(\Delta_{\mathrm{C},ij} + I^{(\mathbf{12})}_{\mathrm{C},ij}\Big)\Big]_{\mathrm{poles}} &= 0\,, \\ \Big[\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,RV\,\mathcal{W}_{ij} + \Big(\Delta_{\mathrm{SC},ij} + I^{(\mathbf{12})}_{\mathrm{SC},ij}\Big)\Big]_{\mathrm{poles}} &= 0\,. \end{aligned} \tag{3.128}$$

Since the pole parts of both $I^{(\mathbf{12})}_{ij}$ and $K^{(\mathbf{RV})}_{ij,\,\mathrm{naive}}$ are explicitly known, the necessary compensating terms are easily determined. An expression for the three components of Δ_{ij} can be constructed in a minimal way by considering all and only the single poles of $I^{(\mathbf{12})}_{ij}$

with mismatching kinematics. Given that they consist in differences of logarithms, or differences of Born matrix elements (which vanish in the soft or collinear limit), we chose to promote the differences of logarithms to ratios of scales, raised to a power vanishing with ϵ. This non-minimal structure simplifies subsequent integrations, and it only affects finite parts, without introducing new phase-space singularities. Beginning with the soft term $\Delta_{\mathrm{S},i}$, we define

$$\begin{aligned}\Delta_{\mathrm{S},i} = & -\frac{\alpha_s}{2\pi}\mathcal{N}_1\sum_{\substack{c\neq i\\ d\neq i,c}}\mathcal{E}^{(i)}_{cd}\Bigg\{\frac{1}{2\epsilon^2}\sum_{\substack{e\neq i,c\\ f\neq i,c,e}}\Bigg[\bigg(\frac{s_{ef}}{\bar{s}^{(icd)}_{ef}}\bigg)^{-\epsilon}-1\Bigg]\bar{B}^{(icd)}_{efcd}+\frac{1}{\epsilon^2}\sum_{e\neq i,d}\Bigg[\bigg(\frac{s_{ed}}{\bar{s}^{(icd)}_{ed}}\bigg)^{-\epsilon}-1\Bigg]\bar{B}^{(icd)}_{edcd}\\ & \qquad +\Bigg[\bigg(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\bigg)2\,C_{f_c}+\frac{\gamma_c^{\mathrm{hc}}}{\epsilon}\Bigg]\Big(\bar{B}^{(icd)}_{cd}-\bar{B}^{(idc)}_{cd}\Big)\Bigg\}\\ & -\frac{\alpha_{\mathrm{S}}}{2\pi}\mathcal{N}_1\sum_{\substack{k\neq i\\ c\neq i,k,r}}\mathcal{E}^{(i)}_{cr}\,\frac{\gamma_k^{\mathrm{hc}}}{\epsilon}\Big(\bar{B}^{(irc)}_{cr}-\bar{B}^{(icr)}_{cr}\Big)\,, \qquad r=r_{ik}\,. \end{aligned} \tag{3.129}$$

Thanks to the fact that in the soft limit the mapped momenta coincide with the unmapped ones, the first condition in Eq. (3.127) is fulfilled in a trivial way. The first relation in Eq. (3.128) is less evident, but can be proven by simply performing the ϵ expansion of $\overline{\mathbf{S}}_i\,RV$, $\Delta_{\mathrm{S},i}$ and $I^{(\mathbf{12})}_{\mathrm{S},ij}$. For the collinear component, we define ($r=r_{ij}$, $r'=r_{ijk}$)

$$\begin{aligned}\Delta_{\mathrm{C},ij} = & \frac{\alpha_{\mathrm{S}}}{2\pi}\mathcal{N}_1\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{1}{\epsilon^2}\sum_{c\neq i,j}\Bigg\{\sum_{d\neq i,j,c}\Bigg[\bigg(\frac{s_{cd}}{\bar{s}^{(ijr)}_{cd}}\bigg)^{-\epsilon}-1\Bigg]\bar{B}^{(ijr)}_{\mu\nu,cd}+2\Bigg[1-\bigg(\frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}}\bigg)^{-\epsilon}\Bigg]\bar{B}^{(ijr)}_{\mu\nu,[ij]c}\\ & \qquad +\Bigg\{\rho^{(C)}_{ij}\Bigg[\bigg(\frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}}\bigg)^{-\epsilon}-\bigg(\frac{s_{ir}\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}\,s_{ic}}\bigg)^{-\epsilon}\Bigg]\bar{B}^{(jri)}_{\mu\nu,[ij]c}\\ & \qquad +\tilde{f}^{\,q\bar{q}}_{ij}\Bigg[\bigg(\frac{\bar{s}^{(jri)}_{ic}}{\mu^2}\bigg)^{-\epsilon}-\bigg(\frac{\bar{s}^{(jri)}_{ic}}{s_{ic}}\bigg)^{-\epsilon}\Bigg]\bar{\mathcal{B}}^{(jri)}_{\mu\nu,[ij]c}+(i\leftrightarrow j)\Bigg\}\Bigg\}\,\mathcal{W}_{\mathrm{c},ij(r)}\\ & +\frac{\alpha_{\mathrm{S}}}{2\pi}\mathcal{N}_1\sum_{k\neq i,j}\bigg(\frac{\gamma_k^{\mathrm{hc}}}{\epsilon}+\phi_k^{\mathrm{hc}}\bigg)\Bigg[\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{B}^{(ijr)}_{\mu\nu}\,\mathcal{W}_{\mathrm{c},ij(r)}-\frac{P^{\mu\nu}_{ij(r')}}{s_{ij}}\,\bar{B}^{(ijr')}_{\mu\nu}\,\mathcal{W}_{\mathrm{c},ij(r')}\Bigg]\,, \end{aligned} \tag{3.130}$$

where $\rho^{(C)}_{ij}$, $\tilde{f}^{\,q\bar{q}}_{ij}$, γ_k^{hc}, ϕ_k^{hc} and $\bar{\mathcal{B}}$ are defined in Appendix A, and $\mathcal{W}_{\mathrm{c},ij(r)}$ is given in Eq. (C.42). The third condition in Eq. (3.127) can be verified by considering that, in the collinear limit $\mathbf{C}_{ij}$, we have

$$\bar{k}^{(ijr)}_j,\bar{k}^{(jri)}_i\xrightarrow{\mathbf{C}_{ij}}k_{[ij]},\qquad \bar{k}^{(ijr)}_r,\bar{k}^{(jri)}_r\xrightarrow{\mathbf{C}_{ij}}k_r,\qquad \bar{k}^{(ijr)}_c,\bar{k}^{(jri)}_c\xrightarrow{\mathbf{C}_{ij}}k_c. \tag{3.131}$$

Again, the second relation in Eq. (3.128) can be proven upon expansion in ϵ. Finally, for the soft-collinear component we introduce ($r = r_{ij}$, $r' = r_{ijk}$)

$$\begin{aligned}
\Delta_{\mathrm{SC},ij} \;=&\; \\
&\frac{\alpha_s}{2\pi}\, 2\,\mathcal{N}_1\, C_{f_j}\, \mathcal{E}^{(i)}_{jr} \frac{1}{\epsilon^2} \sum_{c\neq i,j} \Bigg\{ \sum_{d\neq i,j,c} \Bigg[\bigg(\frac{s_{cd}}{\bar{s}^{(ijr)}_{cd}} \bigg)^{-\epsilon} - 1 \Bigg] \bar{B}^{(ijr)}_{cd} + 2 \Bigg[\bigg(\frac{s_{jr}}{s_{[ij]r}} \bigg)^{-\epsilon} - \bigg(\frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}} \bigg)^{-\epsilon} \Bigg] \bar{B}^{(ijr)}_{[ij]c} \\
&\qquad + \frac{C_A}{C_{f_j}} \Bigg[\bigg(\frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}} \bigg)^{-\epsilon} - \bigg(\frac{s_{ir}\,\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}\, s_{ic}} \bigg)^{-\epsilon} \Bigg] \bar{B}^{(jri)}_{[ij]c} \\
&\qquad + \frac{2\,C_{f_j} - C_A}{C_{f_j}} \Bigg[\bigg(\frac{\bar{s}^{(irj)}_{jc}}{\bar{s}^{(irj)}_{jr}} \bigg)^{-\epsilon} - \bigg(\frac{s_{jr}\,\bar{s}^{(irj)}_{jc}}{\bar{s}^{(irj)}_{jr}\, s_{jc}} \bigg)^{-\epsilon} \Bigg] \bar{B}^{(irj)}_{[ij]c} \Bigg\} \\
&+ \frac{\alpha_s}{2\pi}\, 2\,\mathcal{N}_1\, C_{f_j} \Bigg[J^{j}_{\mathrm{hc}}(s_{jr})\, \mathcal{E}^{(i)}_{jr} \Big(\bar{B}^{(ijr)} - \bar{B}^{(irj)} \Big) \\
&\qquad + \sum_{k\neq i,j} \bigg(\frac{\gamma^{\mathrm{hc}}_k}{\epsilon} + \phi^{\mathrm{hc}}_k \bigg) \Big(\mathcal{E}^{(i)}_{jr}\, \bar{B}^{(ijr)} - \mathcal{E}^{(i)}_{jr'}\, \bar{B}^{(ijr')} \Big) \Bigg]. \qquad (3.132)
\end{aligned}$$

By employing the latter definition, we can demonstrate the validity of the second and fourth relation in Eq. (3.127) by exploiting the colour algebra of the colour-connected matrix elements. We can also prove the cancellation of the ϵ poles in the third line of Eq. (3.128). The explicit expression of the components of Δ_{ij} in Eq. (3.126) completes the list of definitions required to implement the subtracted real-virtual squared matrix element $RV_{\rm sub}$. Because of its finiteness in $d = 4$, we can now rephrase Eq. (3.116) as

$$RV_{\rm sub}(X) \;=\; \sum_{i,j\neq i} \Bigg[\Big(RV_{\rm fin} + I^{(\mathbf{1})}_{\mathrm{fin},ij} \Big) \mathcal{W}_{ij}\, \delta_{n+1}(X) - \Big(K^{(\mathbf{RV})}_{\mathrm{fin},ij} + I^{(\mathbf{12})}_{\mathrm{fin},ij} \Big)\, \delta_n(X) \Bigg], \quad (3.133)$$

where the subscript emphasises that, at this stage, all the explicit poles have already been cancelled. The finite component $I^{(\mathbf{1})}_{\mathrm{fin},ij}$ is given in Eq. (3.112), while $I^{(\mathbf{12})}_{\mathrm{fin},ij}$ can easily be derived from Eqs. (3.103)-(3.105). Finally, we obtain the finite contribution $K^{(\mathbf{RV})}_{\mathrm{fin},ij}$ by computing the expansion in powers of ϵ of the sum of Eqs. (3.119) and (3.129)-(3.132). We refrain from giving here the explicit expression for the quantities in Eq. (3.133), as we will derive a more compact result for $RV_{\rm sub}(X)$ in terms of symmetrised sector functions in the next Section.

3.4.3 $RV_{\rm sub}$ with symmetrised sector functions

In analogy to the procedure applied at NLO in Eq. (2.73), and later generalised to $RR_{\rm sub}$ in Section 3.2.6, we rewrite the real-virtual counterterm $K^{(\mathbf{RV})}$ in terms of the symmetrised sector functions, which is defined as

$$K^{(\mathbf{RV})}_{\{ij\}} = K^{(\mathbf{RV})}_{ij} + K^{(\mathbf{RV})}_{ji} \,, \qquad\qquad K^{(\mathbf{RV})} = \sum_{i,j\neq i} K^{(\mathbf{RV})}_{ij} = \sum_{i,j>i} K^{(\mathbf{RV})}_{\{ij\}} \,. \tag{3.134}$$

Starting from Eq. (3.133), it is not difficult to obtain the corresponding formulation for $\mathcal{Z}_{ij}$ functions, as

$$RV_{\rm sub}(X) \;=\; \sum_{i,j>i} \bigg\{ \Big[RV_{\rm fin} + I^{(\mathbf{1})}_{{\rm fin},ij} \Big] \mathcal{Z}_{ij}\, \delta_{n+1}(X) - \Big[K^{(\mathbf{RV})}_{{\rm fin},\{ij\}} + I^{(\mathbf{12})}_{{\rm fin},\{ij\}} \Big]\, \delta_n(X) \bigg\} \,, \tag{3.135}$$

with $I^{(\mathbf{1})}_{{\rm fin},ij}$ given in Eq. (3.112). The remaining finite contributions collected in the right-most square brackets in Eq. (3.135) can be organised in terms of soft and hard-collinear components, leading to the expression

$$K^{(\mathbf{RV})}_{{\rm fin},\{ij\}} + I^{(\mathbf{12})}_{{\rm fin},\{ij\}} \;=\; K^{(\mathbf{RV+12})}_{{\rm S},ij}\, \mathcal{Z}_{{\rm s},ij} + K^{(\mathbf{RV+12})}_{{\rm S},ji}\, \mathcal{Z}_{{\rm s},ji} + K^{(\mathbf{RV+12})}_{{\rm HC},ij} \,, \tag{3.136}$$

where the soft limit of the symmetrised sector functions, $\mathcal{Z}_{{\rm s},ij}$, is defined in Eq. (3.43). The finite soft counterterm in the first contribution to the right-hand side is obtained through the combination

$$K^{(\mathbf{RV+12})}_{{\rm S},ij} \;=\; \overline{\mathbf{S}}_i\, RV + \Delta_{{\rm S},i} + I^{(\mathbf{12})}_{{\rm S},ij} \,, \tag{3.137}$$

derived from the definitions in Eqs. (3.119), (3.129) and (3.103), dropping the explicit poles. The result is extremely compact, and, except for the process-dependent finite part of the single-virtual squared matrix element, it displays only simple logarithmic dependence on the kinematics. We find ($r = r_{ik}$, $r' = r_{ij}$, $r'' = r_{ijk}$)

$$\begin{aligned}
K^{(\mathbf{RV+12})}_{{\rm S},ij} = 4\,\alpha_{\rm S}^2 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \Bigg\{ & \sum_{\substack{e\neq i\\ f\neq i,e}} \bigg(L_{ef} - \frac{1}{4} L_{ef}^2 \bigg)\, \bar{B}^{(icd)}_{cdef} + 2 \sum_{e\neq i,d} \bigg(L_{ed} - \frac{1}{4} L_{ed}^2 \bigg) \Big(\bar{B}^{(icd)}_{cded} - \bar{B}^{(idc)}_{cded} \Big) \\
& + \sum_{e\neq i,d} \ln^2 \frac{\bar{s}^{(icd)}_{de}}{s_{de}}\, \bar{B}^{(icd)}_{cded} - \frac{1}{2} \ln^2 \frac{\bar{s}^{(icd)}_{cd}}{s_{cd}}\, \bar{B}^{(icd)}_{cdcd} - 2\pi \sum_{e\neq i,c,d} \ln \frac{s_{id} s_{ie}}{\mu^2\, s_{de}}\, \bar{B}^{(icd)}_{cde} \\
& + \Bigg[\bigg(6 - \frac{7}{2}\,\zeta_2 \bigg) \big(\Sigma_C + 2 C_{f_d} - 2 C_{f_c} \big) + \sum_k \phi^{\rm hc}_k - \sum_{k\neq i} \gamma^{\rm hc}_k\, L_{kr''} - \gamma^{\rm hc}_i\, L_{ir'} \\
& \quad + C_A \bigg(6 - \zeta_2 - \ln \frac{s_{ic}}{s_{cd}} \ln \frac{s_{id}}{s_{cd}} - 2\, \ln \frac{s_{ic} s_{id}}{\mu^2 s_{cd}} \bigg) \Bigg]\, \bar{B}^{(icd)}_{cd} \Bigg\} \\
+ 4\,\alpha_{\rm S}^2 \sum_{k\neq i} \big(\phi^{\rm hc}_k - \gamma^{\rm hc}_k L_{kr''} \big) & \Bigg[\sum_{c\neq i,k} \mathcal{E}^{(i)}_{kc} \Big(\bar{B}^{(ick)}_{kc} - \bar{B}^{(ikc)}_{kc} \Big) + \sum_{c\neq i,k,r} \mathcal{E}^{(i)}_{cr} \Big(\bar{B}^{(icr)}_{cr} - \bar{B}^{(irc)}_{cr} \Big) \Bigg] \\
+ 8\pi\, \alpha_{\rm S} \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd}\, \bar{V}^{(icd)}_{{\rm fin},cd} \,, &
\end{aligned} \tag{3.138}$$

where $\bar{V}^{(icd)}_{\mathrm{fin},cd}$ is the finite part of the colour-correlated, single-virtual squared matrix element, expressed in the n-body mapped kinematics. We notice that the presence of the reference particle $r' = r_{ij}$ introduces a dependence on the particle j of the soft sector function $\mathcal{Z}_{\mathrm{s},ij}$ which multiplies $K^{(\mathbf{RV+12})}_{\mathrm{S},ij}$, as was the case for $I^{(\mathbf{12})}_{\mathrm{S},ij}$.

To conclude this section, we report the expression for the finite hard-collinear counterterm introduced in the right-hand side of Eq. (3.136), which is the result of summing Eqs. (3.119), (3.130), (3.104), (3.132), and (3.105), as

$$K^{(\mathbf{RV+12})}_{\mathrm{HC},ij} \;=\; \overline{\mathbf{HC}}_{ij}\, RV + \Delta_{\mathrm{HC},ij} + I^{(\mathbf{12})}_{\mathrm{C},ij} - I^{(\mathbf{12})}_{\mathrm{SC},ij} - I^{(\mathbf{12})}_{\mathrm{SC},ji}\,, \tag{3.139}$$

where we introduced the shorthand notation

$$\overline{\mathbf{HC}}_{ij}\, RV \equiv \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j)\, RV\,, \qquad \Delta_{\mathrm{HC},ij} \equiv \Delta_{\mathrm{C},ij} + \Delta_{\mathrm{C},ji} - \Delta_{\mathrm{SC},ij} - \Delta_{\mathrm{SC},ji}\,. \tag{3.140}$$

Explicitly we find (with $r = r_{ij}$, $r' = r_{ijk}$)

$$\begin{aligned}
K^{(\mathbf{RV+12})}_{\mathrm{HC},ij} =\; & 4\,\alpha_{\mathrm{S}}^2\,\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\Bigg\{\sum_{c\neq i,j}\Bigg[\ln^2\frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}}\,\bar{B}^{(ijr)}_{\mu\nu,[ij]c} - \frac{1}{2}\sum_{d\neq i,j,c}\big(4L_{cd}-L^2_{cd}\big)\,\bar{B}^{(ijr)}_{\mu\nu,cd}\Bigg] \\
& - \sum_{c\neq i,j,r}\Bigg[\ln^2\frac{\bar{s}^{(ijr)}_{cr}}{s_{cr}}\,\bar{B}^{(ijr)}_{\mu\nu,cr} + \frac{\rho^{(C)}_{ij}}{2}\,\mathcal{L}_{ijcr}\,\bar{B}^{(jri)}_{\mu\nu,[ij]c} + \frac{\rho^{(C)}_{ji}}{2}\,\mathcal{L}_{jicr}\,\bar{B}^{(irj)}_{\mu\nu,[ij]c}\Bigg] \\
& - \frac{1}{2}\sum_{c\neq i,j}\tilde{f}^{\,q\bar{q}}_{ij}\,\Big(\tilde{\mathcal{L}}_{ijcr}\,\bar{\mathcal{B}}^{(jri)}_{\mu\nu,[ij]c} - \tilde{\mathcal{L}}_{jicr}\,\bar{\mathcal{B}}^{(irj)}_{\mu\nu,[ij]c}\Big) \\
& - \Bigg[\bigg(6-\frac{7}{2}\,\zeta_2\bigg)\Big(\Sigma_C - C_{f_{[ij]}}\rho^{(C)}_{[ij]}\Big) + C_{f_{[ij]}}\frac{\rho^{(C)}_{[ij]}}{2}\,\big(4L_{ij}-L^2_{ij}\big) \\
& - C_{f_{[ij]}}\frac{\rho^{(C)}_{ij}}{2}\,\big(4L_{ir}-L^2_{ir}\big) - C_{f_{[ij]}}\frac{\rho^{(C)}_{ji}}{2}\,\big(4L_{jr}-L^2_{jr}\big) + \Sigma^{\mathrm{hc}}_{\phi}\Bigg]\bar{B}^{(ijr)}_{\mu\nu}\Bigg\} \\
& - 4\,\alpha_{\mathrm{S}}^2\Bigg[2\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\,C_{f_{[ij]}}\ln^2\frac{s_{jr}}{s_{[ij]r}}\,\bar{B}^{(ijr)} + (i\leftrightarrow j)\Bigg] \\
& + 4\,\alpha_{\mathrm{S}}^2\Bigg[\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\Big(\gamma^{\mathrm{hc}}_i L_{ir} + \gamma^{\mathrm{hc}}_j L_{jr}\Big)\,\bar{B}^{(ijr)}_{\mu\nu} + \sum_{k\neq i,j}\frac{P^{\mathrm{hc},\mu\nu}_{ij(r')}}{s_{ij}}\,\gamma^{\mathrm{hc}}_k\,L_{kr'}\,\bar{B}^{(ijr')}_{\mu\nu}\Bigg] \\
& - 4\,\alpha_{\mathrm{S}}^2\,\frac{\tilde{P}^{\mathrm{hc},\mu\nu}_{\mathrm{fin},ij(r)}}{s_{ij}}\,\bar{B}^{(ijr)}_{\mu\nu} - 8\pi\,\alpha_{\mathrm{S}}\,\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{V}^{(ijr)}_{\mathrm{fin},\mu\nu}\,,
\end{aligned} \tag{3.141}$$

where we introduced the shorthand notation

$$\mathcal{L}_{ijcr} \;=\; 2\ln\frac{s_{ic}}{s_{ir}}\Bigg[2 - L_{ic} + \ln\frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}}\Bigg]\,, \qquad \tilde{\mathcal{L}}_{ijcr} \;=\; 2L_{ic}\Bigg[2 - L_{ic} + \ln\frac{\bar{s}^{(jri)}_{ic}}{\mu^2}\Bigg]\,. \tag{3.142}$$

Notice that even in Eq. (3.141), the kinematic dependence occurs only in terms of simple logarithms.

3.5 Integration of the real-virtual counterterm

In Eqs. (3.124), (3.134) we have defined the counterterm $K^{(\mathbf{RV})}$, that enabled us to build the subtracted real-virtual squared matrix element $RV_{\rm sub}$, integrable in the whole $(n+1)$-body phase space, and free of poles in ϵ. The $K^{(\mathbf{RV})}$ counterterm needs to be integrated in $d = 4-2\epsilon$ dimensions in the radiation phase space, and then the result must be added back, according to the subtraction structure given in Eqs. (3.5)-(3.7). To compute the integrated counterterm, $I^{(\mathbf{RV})}$, as defined in Eq. (3.4), we first sum over all sectors $\mathcal{W}_{ij}$, so that sector functions drop out of the calculation due to the sum rules they satisfy (refer to those in (2.20)). We then perform the integration over the radiative phase space with the measure $d\Phi_{\rm rad}^{(acd)}$, naturally induced by the mapping (acd), according to

$$\int d\Phi_{n+1}(\{k\}) \;=\; \frac{\varsigma_{n+1}}{\varsigma_n}\int d\Phi_n^{(acd)}\int d\Phi_{\rm rad}^{(acd)}\,, \qquad d\Phi_n^{(acd)} \;\equiv\; d\Phi_n(\{\bar{k}\}^{(acd)})\,, \tag{3.143}$$

where $d\Phi_{\rm rad}^{(acd)}$ is defined in Eq. (2.88). The integration of $K^{(\mathbf{RV})}$ is carried out following the methods described in Ref. [200], and using the fact that the spin-correlated contributions proportional to the kernels $Q^{\mu\nu}_{ij(r)}$ and $\tilde{Q}^{\mu\nu}_{ij(r)}$ vanish upon integration, as discussed in Appendix C.4. The formal expression for the integration of $K^{(\mathbf{RV})}$ can be written as

$$\int d\Phi_{n+1}\,K^{(\mathbf{RV})} \;=\; \int d\Phi_{n+1}\left[\sum_i\Big(\overline{\mathbf{S}}_i\,RV+\Delta_{{\rm S},i}\Big)+\sum_{i,\,j>i}\Big(\overline{\mathbf{HC}}_{ij}\,RV+\Delta_{{\rm HC},ij}\Big)\right], \tag{3.144}$$

where the integrands are defined in Eqs. (3.119) and (3.129)-(3.132), and we used the shorthand notations introduced in Eq. (3.140). Before integrating, we can further simplify the expressions for $\Delta_{{\rm S},i}$ and $\Delta_{{\rm C},ij}$, given in (3.129)-(3.130). In fact, since $\bar{s}^{(icd)}_{ef} = s_{ef}$ for $e,f \neq i,c,d$, and $\bar{s}^{(ijr)}_{cd} = s_{cd}$ for $c,d \neq i,j,r$, one finds that

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{e\neq i,c\\ f\neq i,c,e}}\left[\left(\frac{s_{ef}}{\bar{s}^{(icd)}_{ef}}\right)^{-\epsilon}-1\right]\bar{B}^{(icd)}_{efcd}+\sum_{e\neq i,d}\left[\left(\frac{s_{ed}}{\bar{s}^{(icd)}_{ed}}\right)^{-\epsilon}-1\right]\bar{B}^{(icd)}_{edcd} \;= \\
&\qquad = \; 2\sum_{e\neq i,c,d}\left[\left(\frac{s_{ed}}{\bar{s}^{(icd)}_{ed}}\right)^{-\epsilon}-1\right]\bar{B}^{(icd)}_{edcd}+\left[\left(\frac{s_{cd}}{\bar{s}^{(icd)}_{cd}}\right)^{-\epsilon}-1\right]\bar{B}^{(icd)}_{cdcd}\,,
\end{aligned} \tag{3.145}$$

as well as

$$\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\left[\left(\frac{s_{cd}}{\bar{s}^{(ijr)}_{cd}}\right)^{-\epsilon}-1\right]\bar{B}^{(ijr)}_{\mu\nu,cd} \;=\; 2\sum_{c\neq i,j,r}\left[\left(\frac{s_{cr}}{\bar{s}^{(ijr)}_{cr}}\right)^{-\epsilon}-1\right]\bar{B}^{(ijr)}_{\mu\nu,cr}\,. \tag{3.146}$$

After integration, the soft contributions to Eq. (3.144) read

$$\int d\Phi_{n+1}\,\overline{\mathbf{S}}_i\,RV = -\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{\substack{c\neq i\\ d\neq i,c}}\int d\Phi_n^{(icd)}\bigg[\,J_{\rm s}^{icd}\,\bar V_{cd}^{(icd)} - \frac{\alpha_{\rm S}}{2\pi}\left(\tilde J_{\rm s}^{icd}+J_{\rm s}^{icd}\frac{\beta_0}{2\epsilon}\right)\bar B_{cd}^{(icd)} + \alpha_{\rm S}\sum_{e\neq i,c,d}\tilde J_{\rm s}^{icde}\,\bar B_{cde}^{(icd)}\bigg], \tag{3.147}$$

while ($r = r_{ik}$)

$$\begin{aligned}\int d\Phi_{n+1}\,\Delta_{{\rm S},i} = &-\frac{\alpha_s}{2\pi}\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{\substack{c\neq i\\ d\neq i,c}}\Bigg\{\int d\Phi_n^{(icd)}\bigg[\sum_{e\neq i,c,d}J_{\Delta_{\rm s}}^{icd(e)}\,\bar B_{edcd}^{(icd)} + J_{\Delta_{\rm s}}^{icd}\,\bar B_{cdcd}^{(icd)}\bigg]\\ &+\left[2\,C_{f_c}\left(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\right)+\frac{\gamma_c^{\rm hc}}{\epsilon}\right]\left[\int d\Phi_n^{(icd)}J_{\rm s}^{icd}\bar B_{cd}^{(icd)} - \int d\Phi_n^{(idc)}J_{\rm s}^{idc}\bar B_{cd}^{(idc)}\right]\Bigg\}\\ &-\frac{\alpha_{\rm S}}{2\pi}\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{\substack{k\neq i\\ c\neq i,k,r}}\frac{\gamma_k^{\rm hc}}{\epsilon}\left[\int d\Phi_n^{(irc)}J_{\rm s}^{irc}\,\bar B_{cr}^{(irc)} - \int d\Phi_n^{(icr)}J_{\rm s}^{icr}\,\bar B_{cr}^{(icr)}\right].\end{aligned} \tag{3.148}$$

Explicit expressions for the constituent integrals $\tilde J_s^{icd}$, $\tilde J_{\rm s}^{icde}$, $J_{\Delta_{\rm s}}^{icd(e)}$ and $J_{\Delta_{\rm s}}^{icd}$ are given in Eq. (C.187), while the NLO integral $J_{\rm s}^{icd}$ is given in Eq. (C.183). We notice that the soft integrated real-virtual counterterm in Eq. (3.147) receives contributions from the triple-colour-correlated squared matrix element $\bar B_{cde}$. However, the pole content of such term vanishes upon performing the appropriate colour sums (see Ref. [200] for further details). This cancellation represents a strong test for the method: it is protected by the fact that no singular contributions proportional to colour tripoles can arise from double-virtual nor from double-real corrections. On the other hand, integrating the tripole contribution to the soft real-virtual kernel requires the non-trivial procedure described in Ref. [200], which is necessary in order to verify the pole cancellation, and to compute the finite remainder. To complete the discussion we also report the integrated hard-collinear component, reading ($r = r_{ij}$)

$$\int d\Phi_{n+1}\,\overline{\mathbf{HC}}_{ij}\,RV = \frac{\varsigma_{n+1}}{\varsigma_n}\int d\Phi_n^{(ijr)}\left[J_{\rm hc}^{ijr}\,\bar V^{(ijr)} + \frac{\alpha_{\rm S}}{2\pi}\left(\tilde J_{\rm hc}^{ijr} - J_{\rm hc}^{ijr}\frac{\beta_0}{2\epsilon}\right)\bar B^{(ijr)}\right], \tag{3.149}$$

while the compensating hard-collinear term integrates to ($r = r_{ij}$, $r' = r_{ijk}$)

$$\int d\Phi_{n+1}\,\Delta_{\mathrm{HC},ij} = \frac{\alpha_{\mathrm{S}}}{2\pi}\frac{\varsigma_{n+1}}{\varsigma_n}\Bigg\{\int d\Phi_n^{(ijr)}\Bigg[\sum_{c\neq i,j,r} J_{\Delta\mathrm{hc}}^{ijr}\,\bar{B}_{cr}^{(ijr)} + \sum_{c\neq i,j} J_{\Delta\mathrm{hc}}^{ijrc}\,\bar{B}_{[ij]c}^{(ijr)}\Bigg] \tag{3.150}$$
$$+ \sum_{c\neq i,j,r}\Bigg[\int d\Phi_n^{(jri)}\,J_{\Delta\mathrm{hc}}^{jri,c}\,\bar{B}_{[ij]c}^{(jri)} + \int d\Phi_n^{(irj)}\,J_{\Delta\mathrm{hc}}^{irj,c}\,\bar{B}_{[ij]c}^{(irj)}\Bigg]$$
$$+ \sum_{k\neq i,j}\left(\frac{\gamma_k^{\mathrm{hc}}}{\epsilon} + \phi_k^{\mathrm{hc}}\right)\Bigg[\int d\Phi_n^{(ijr)} J_{\mathrm{hc}}^{ijr}\,\bar{B}^{(ijr)} - \int d\Phi_n^{(ijr')} J_{\mathrm{hc}}^{ijr'}\,\bar{B}^{(ijr')}\Bigg]$$
$$+ \tilde{f}_{ij}^{q\bar{q}}\sum_{c\neq i,j}\Bigg[\int d\Phi_n^{(jri)}\,\tilde{J}_{\Delta\mathrm{hc}}^{jri,c}\,\bar{\mathcal{B}}_{[ij]c}^{(jri)} - \int d\Phi_n^{(irj)}\,\tilde{J}_{\Delta\mathrm{hc}}^{irj,c}\,\bar{\mathcal{B}}_{[ij]c}^{(irj)}\Bigg]\Bigg\}\,.$$

Explicit expressions for the hard-collinear constituent integrals $\tilde{J}_{\mathrm{hc}}^{ijr}$, $J_{\Delta\mathrm{hc}}^{ijr}$, $J_{\Delta\mathrm{hc}}^{ijrc}$, $J_{\Delta\mathrm{hc}}^{jri,c}$, and $\tilde{J}_{\Delta\mathrm{hc}}^{jri,c}$ are given in Eq. (C.198), while the NLO hard-collinear integral J_{hc}^{ijr} can be found in Eq. (C.189).

Having computed all relevant integrals, we now recombine them, following a procedure analogous to the one described at the end of Section 2.3. We rename the sets of mapped momenta $\{\bar{k}^{(abc)}\}_n$ to the same set of Born-level momenta $\{k\}_n$ by means of the replacements

$$d\Phi_n^{(abc)} \to d\Phi_n\,, \qquad \bar{B}_{\dots}^{(abc)} \to B_{\dots}\,, \qquad \bar{\mathcal{B}}_{\dots}^{(abc)} \to \bar{\mathcal{B}}_{\dots}\,, \qquad \bar{s}_{lm}^{(abc)} \to s_{lm}\,, \tag{3.151}$$

where the ellipsis in the Born-level matrix element stands for a generic colour correlation. In particular, in the integral of $\Delta_{\mathrm{HC},ij}$ in Eq. (3.150), all momenta $\bar{k}_j^{(ijr)}$, $\bar{k}_i^{(jri)}$, $\bar{k}_j^{(irj)}$, and $\bar{k}_j^{(ijr')}$ are renamed as k_p, where p is the label of the parent particle splitting into i and j. As a consequence of this renaming, the integrals involving $\bar{\mathcal{B}}_{[ij]c}$ can be recombined, and do not contribute to the integrated counterterm. Indeed,

$$\int d\Phi_n^{(jri)}\,\tilde{J}_{\Delta\mathrm{hc}}^{jri,c}\,\bar{\mathcal{B}}_{[ij]c}^{(jri)} - \int d\Phi_n^{(irj)}\,\tilde{J}_{\Delta\mathrm{hc}}^{irj,c}\,\bar{\mathcal{B}}_{[ij]c}^{(irj)} = \tag{3.152}$$
$$= \int d\Phi_n^{(jri)}\,\tilde{J}_{\Delta\mathrm{hc}}^{c}\Big(\bar{s}_{ir}^{(jri)},\bar{s}_{ic}^{(jri)}\Big)\,\bar{\mathcal{B}}_{[ij]c}^{(jri)} - \int d\Phi_n^{(irj)}\,\tilde{J}_{\Delta\mathrm{hc}}^{c}\Big(\bar{s}_{jr}^{(irj)},\bar{s}_{jc}^{(irj)}\Big)\,\bar{\mathcal{B}}_{[ij]c}^{(irj)}$$
$$\to \int d\Phi_n\,\tilde{J}_{\Delta\mathrm{hc}}^{c}\Big(s_{pr},s_{pc}\Big)\,\mathcal{B}_{pc} - \int d\Phi_n\,\tilde{J}_{\Delta\mathrm{hc}}^{c}\Big(s_{pr},s_{pc}\Big)\,\mathcal{B}_{pc} = 0\,.$$

The dependence on the $(n+1)$-body phase-space particles is now limited to the flavour factors f_i^q, $f_i^{\bar{q}}$ and f_i^g, which can be translated into flavour factors for the n-body-phase-space particles, as was done in Section 2.3. In particular, when going from an $(n+1)$-body phase space to an n-body phase space the relations in Eq. (2.103) and Eq. (2.104) apply. After performing the flavour sums, no residual dependence on the original $(n+1)$-body phase space remains. Simplifying the colour correlations where possible, we finally get

$$
\begin{aligned}
I^{(\mathbf{RV})} = & - \sum_{c,d\neq c} \bigg[J_{\rm s}(s_{cd})\, V_{cd} + J_{\rm sRV}(s_{cd})\, B_{cd} + J^{(2)}_{\rm sRV}(s_{cd})\, B_{cdcd} + \sum_{e\neq c,d} J^{cde}_{\rm sRV}\, B_{cde} \bigg] \\
& + \sum_j \Bigg\{ J^j_{\rm hc}(s_{jr})\, V + J^j_{\rm hcRV}(s_{jr})\, B + J^{j,{\rm A}}_{\rm hcRV}(s_{jr})\, B_{jr} \\
& \quad + \sum_{c\neq j,r} \bigg[J^{j,{\rm B}}_{\rm hcRV}(s_{jc})\, B_{jc} + J^{j,{\rm C}}_{\rm hcRV}(s_{jr})\, B_{cr} \bigg] \\
& \quad + \frac{\alpha_{\rm S}}{2\pi} \sum_{k\neq j} \left(\frac{\gamma^{\rm hc}_k}{\epsilon} + \phi^{\rm hc}_k \right) \Big[J^j_{\rm hc}(s_{jr}) - J^j_{\rm hc}(s_{jr'}) \Big] \Bigg\}\,,
\end{aligned} \tag{3.153}
$$

where we introduced the following combinations of constituent integrals:

$$
J_{\rm sRV}(s) = -\frac{\alpha_{\rm S}}{2\pi} \bigg[C_A\, \tilde{J}_{\rm s}(s) + \frac{\beta_0}{2\epsilon}\, J_{\rm s}(s) + 2\, C_{f_d}\, J^{(3)}_{\Delta{\rm s}}(s) \bigg]\,, \tag{3.154}
$$

$$
J^{(2)}_{\rm sRV}(s) = \frac{\alpha_{\rm S}}{2\pi} \Big[J^{(2)}_{\Delta{\rm s}}(s) - J^{(3)}_{\Delta{\rm s}}(s) \Big]\,, \tag{3.155}
$$

$$
J^{cde}_{\rm sRV} = -\frac{\alpha_{\rm S}^2}{2\pi} \bigg[\frac{1}{2} \ln\frac{s_{ce}}{s_{de}}\, \ln^2\frac{s_{cd}}{\mu^2} + \frac{1}{6} \ln^3\frac{s_{ce}}{s_{de}} + {\rm Li}_3\bigg(-\frac{s_{ce}}{s_{de}} \bigg) + \mathcal{O}(\epsilon) \bigg]\,, \tag{3.156}
$$

$$
\begin{aligned}
J^j_{\rm hcRV}(s) = \frac{\alpha_{\rm S}}{2\pi} \Bigg\{ & \big(f^q_j + f^{\bar q}_j\big) \bigg[\tilde{J}^{(1{\rm g})}_{\rm hc}(s) - \frac{\beta_0}{2\epsilon}\, J^{(1{\rm g})}_{\rm hc}(s) \\
& \qquad - C_F\, J^{(1{\rm g})}_{\Delta{\rm hc,A}}(s) - C_F\, J^{\rm qg}_{\Delta{\rm hc,A}}(s) - C_F\, J^{\rm gq}_{\Delta{\rm hc,A}}(s) \bigg] \\
& + f^g_j \bigg[\frac{1}{2} \bigg(\tilde{J}^{(2{\rm g})}_{\rm hc}(s) - \frac{\beta_0}{2\epsilon}\, J^{(2{\rm g})}_{\rm hc}(s) - C_A\, J^{(2{\rm g})}_{\Delta{\rm hc,A}}(s) - 2\, C_A\, J^{\rm gg}_{\Delta{\rm hc,A}}(s) \bigg) \\
& \quad + N_f \bigg(\tilde{J}^{(0{\rm g})}_{\rm hc}(s) - \frac{\beta_0}{2\epsilon}\, J^{(0{\rm g})}_{\rm hc}(s) - C_A\, J^{(0{\rm g})}_{\Delta{\rm hc,A}}(s) - 2\, C_A\, J^{\rm qq}_{\Delta{\rm hc,A}}(s) \bigg) \bigg] \Bigg\}\,,
\end{aligned} \tag{3.157}
$$

$$
\begin{aligned}
J^{j,{\rm A}}_{\rm hcRV}(s) = \frac{\alpha_{\rm S}}{2\pi} \Bigg\{ & \big(f^q_j + f^{\bar q}_j\big) \Big(J^{(1{\rm g})}_{\Delta{\rm hc,B}}(s) - J^{\rm qg}_{\Delta{\rm hc,A}}(s) - J^{\rm gq}_{\Delta{\rm hc,A}}(s) \Big) \\
& + f^g_j \bigg[\frac{1}{2} \Big(J^{(2{\rm g})}_{\Delta{\rm hc,B}}(s) - 2\, J^{\rm gg}_{\Delta{\rm hc,A}}(s) \Big) + N_f \Big(J^{(0{\rm g})}_{\Delta{\rm hc,B}}(s) - 2\, J^{\rm qq}_{\Delta{\rm hc,A}}(s) \Big) \bigg] \Bigg\}\,,
\end{aligned} \tag{3.158}
$$

$$
\begin{aligned}
J^{j,{\rm B}}_{\rm hcRV}(s) = \frac{\alpha_{\rm S}}{2\pi} \Bigg\{ & \big(f^q_j + f^{\bar q}_j\big) \Big(J^{(1{\rm g})}_{\Delta{\rm hc,B}}(s) + J^{\rm qg}_{\Delta{\rm hc,B}}(s) + J^{\rm gq}_{\Delta{\rm hc,B}}(s) \Big) \\
& + f^g_j \bigg[\frac{1}{2} \Big(J^{(2{\rm g})}_{\Delta{\rm hc,B}}(s) + 2\, J^{\rm gg}_{\Delta{\rm hc,B}}(s) \Big) + N_f \Big(J^{(0{\rm g})}_{\Delta{\rm hc,B}}(s) + 2\, J^{\rm qq}_{\Delta{\rm hc,B}}(s) \Big) \bigg] \Bigg\}\,,
\end{aligned} \tag{3.159}
$$

$$J_{\rm hcRV}^{j,{\rm C}}(s) = \frac{\alpha_{\rm S}}{2\pi} \left\{ \left(f_j^q + f_j^{\bar{q}}\right) J_{\Delta{\rm hc}}^{(1{\rm g})}(s) + f_j^g \left[\frac{1}{2} J_{\Delta{\rm hc}}^{(2{\rm g})}(s) + N_f \, J_{\Delta{\rm hc}}^{(0{\rm g})}(s) \right] \right\} . \tag{3.160}$$

All new constituent integrals appearing in the above results are listed in Appendix C.5: the soft integrals are presented in Eq. (C.188), the hard-collinear integrals in Eq. (C.199), and the integrals arising from the compensating Δ_{ij} terms in Eqs. (C.200)-(C.202). We note once again that all integrals involved are single-scale, and thus involve only simple logarithms. Interestingly, the only exception is Eq. (3.156), with a uniform-weight-three function featuring three scales and a single trilogarithm: this integral arises as a finite remainder of the non-trivial integration of the tripole term.

The integrated counterterm $I^{(\mathbf{RV})}$ given in Eq. (3.153), which features Born-level kinematics, contains explicit poles in ϵ, that must be combined with those of the integrated counterterm $I^{(\mathbf{2})}$, and must, together, cancel the singularities of the double-virtual squared matrix element.

3.6 The subtracted double-virtual contribution $VV_{\rm sub}$

Finally, we turn our attention to the first line in Eq. (3.6), which we rewrite here as

$$VV_{\rm sub}(X) = \Big(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \Big) \, \delta_n(X) \, . \tag{3.161}$$

It is our task to show that the equation above is free of ϵ poles. To verify this, we first explicitly derive the ϵ poles of the double-virtual correction VV (Section 3.6.1), and then we provide the complete ϵ expansion of $I^{(\mathbf{2})} + I^{(\mathbf{RV})}$, including $\mathcal{O}(\epsilon^0)$ terms, obtained by combining Eq. (3.97) and Eq. (3.153) (Section 3.6.2).

3.6.1 The pole part of the double-virtual matrix element VV

All infrared poles of gauge-theory scattering amplitudes can be expressed in a factorised form through the formula [116, 117, 120, 121, 123]

$$\mathcal{A}\left(\frac{k_i}{\mu}, \alpha_s(\mu), \epsilon\right) = \mathbf{Z}\left(\frac{k_i}{\mu}, \alpha_s(\mu), \epsilon\right) \mathcal{H}\left(\frac{k_i}{\mu}, \alpha_s(\mu), \epsilon\right) , \tag{3.162}$$

where $\mathcal{H}$ is finite as $\epsilon \to 0$, and $\mathbf{Z}$ is a colour operator with a universal form, to be discussed below. The infrared operator $\mathbf{Z}$ obeys a (matrix) renormalisation-group equation, which can be solved in exponential form, with a trivial initial condition, in terms of an anomalous-dimension matrix $\mathbf{\Gamma}$. One may write

$$\mathbf{Z}\left(\frac{k_i}{\mu}, \alpha_s(\mu), \epsilon\right) = \mathcal{P} \exp\left[\int_0^\mu \frac{d\lambda}{\lambda} \, \mathbf{\Gamma}\left(\frac{k_i}{\lambda}, \alpha_s(\lambda), \epsilon\right) \right] , \tag{3.163}$$

where the integral converges at $\lambda = 0$ in dimensional regularisation thanks to the behaviour of the β function in $d = 4 - 2\epsilon$, for $\epsilon < 0$ ($d > 4$). Indeed, in dimensional regularisation one has

$$\mu \frac{d\alpha_s}{d\mu} \equiv \beta\left(\epsilon, \alpha_s\right) = -2\epsilon\, \alpha_s - \frac{\alpha_s^2}{2\pi}\beta_0 + \mathcal{O}\left(\alpha_s^3\right) , \tag{3.164}$$

whose solution implies [114] that the d-dimensional running coupling $\alpha_s(\mu, \epsilon)$ vanishes at $\mu = 0$ for $\epsilon < 0$, so that the corresponding initial condition is $\mathbf{Z}(\mu = 0) = \mathbf{1}$, leading to Eq. (3.163). For the purposes of NNLO subtraction (and thus at two loops for virtual amplitudes), $\mathbf{\Gamma}$ is given by the dipole formula [120, 121]

$$\mathbf{\Gamma}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda), \epsilon\right) = \frac{1}{2}\widehat{\gamma}_K\left(\alpha_s(\lambda, \epsilon)\right)\sum_{i,j>i}\ln\left(\frac{s_{ij}\, e^{\mathrm{i}\pi\sigma_{ij}}}{\lambda^2}\right)\mathbf{T}_i \cdot \mathbf{T}_j + \sum_i \gamma_i\left(\alpha_s(\lambda, \epsilon)\right) . \tag{3.165}$$

In Eq. (3.165), the phases σ_{ij} are given by $\sigma_{ij} = +1$ if partons i and j are either both in the initial state or both in the final state, while $\sigma_{ij} = 0$ otherwise. For our present final-state application, we can thus henceforth replace all phase factors using $e^{\mathrm{i}\pi\sigma_{ij}} = -1$, with the understanding that the logarithm is taken above the cut.

The anomalous dimensions appearing in Eq. (3.165) are the cusp anomalous dimension $\widehat{\gamma}_K\left(\alpha_s\right)$ and the collinear anomalous dimensions $\gamma_i\left(\alpha_s\right)$. More precisely, in the derivation of Eq. (3.165) it has been assumed that the (light-like) cusp anomalous dimension $\gamma_K^{(r)}(\alpha_s)$, in colour representation r, obeys 'Casimir scaling', *i.e.* it can be written as

$$\gamma_K^{(r)}(\alpha_s) = C_r\, \widehat{\gamma}_K(\alpha_s) , \tag{3.166}$$

where C_r is the quadratic Casimir eigenvalue for colour representation r, while $\widehat{\gamma}_K(\alpha_s)$ is a universal (representation-independent) function. This assumption is known to fail at four loops [208, 209]. The collinear anomalous dimensions $\gamma_i(\alpha_s)$ are related to the anomalous dimensions of quark and gluon fields, and can be derived from essentially colour-singlet calculations such as those of form factors.

One important consequence of the dipole formula is that the scale integration in Eq. (3.163) can be performed without affecting the colour structure (which is scale-independent): one may therefore omit the path-ordering in Eq. (3.163), simplifying considerably the necessary calculations. Expanding the various ingredients perturbatively as

$$\widehat{\gamma}_K(\alpha_s) = \sum_{n=1}^{\infty}\widehat{\gamma}_K^{(n)}\left(\frac{\alpha_s}{2\pi}\right)^n, \quad \gamma_i(\alpha_s) = \sum_{n=1}^{\infty}\gamma_i^{(n)}\left(\frac{\alpha_s}{2\pi}\right)^n, \quad \mathbf{\Gamma}(\alpha_s) = \sum_{n=1}^{\infty}\mathbf{\Gamma}^{(n)}\left(\frac{\alpha_s}{2\pi}\right)^n, \tag{3.167}$$

one gets at NLO

$$\mathbf{\Gamma}^{(1)} = \frac{1}{4}\,\widehat{\gamma}_K^{(1)}\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j \,+\, \sum_i \gamma_i^{(1)} - \frac{1}{4}\,\widehat{\gamma}_K^{(1)}\ln\left(\frac{\mu^2}{\lambda^2}\right)\sum_i C_{f_i}\,, \tag{3.168}$$

and consequently

$$\mathbf{Z}^{(1)}\left(\frac{p_i}{\mu},\epsilon\right) = -\frac{1}{\epsilon^2}\frac{\widehat{\gamma}_K^{(1)}}{8}\Sigma_C - \frac{1}{\epsilon}\left(\frac{\widehat{\gamma}_K^{(1)}}{8}\sum_{i,j\neq i}L_{ij}\,\mathbf{T}_i\cdot\mathbf{T}_j + \frac{1}{2}\Sigma_\gamma\right) + \mathrm{i}\pi\,\frac{\gamma_K^{(1)}}{8\epsilon}\,\Sigma_C\,, \tag{3.169}$$

where $L_{ij} = \ln(s_{ij}/\mu^2)$. Eq. (3.169) is in agreement with [116, 121], with the one-loop anomalous-dimension coefficients given by

$$\widehat{\gamma}_K^{(1)} = 4\,, \quad \gamma_i^{(1)} \equiv \gamma_i = \frac{3}{2}C_F\,(f_i^q + f_i^{\bar{q}}) + \frac{1}{2}\beta_0\, f_i^g\,, \quad \Sigma_C = \sum_i C_{f_i}\,, \quad \Sigma_\gamma = \sum_i \gamma_i\,, \tag{3.170}$$

where we noted that in the text we have sometimes used the notation γ_i for the one-loop coefficient denoted here by $\gamma_i^{(1)}$. Expanding the anomalous dimensions to two loops and performing the relevant integrals, the NNLO result for the $\mathbf{Z}$ factor is

$$\begin{aligned}
\mathbf{Z}^{(2)} =\; & \frac{1}{\epsilon^4}\,\frac{\left(\widehat{\gamma}_K^{(1)}\right)^2}{128}\Sigma_C^2 \\
& + \frac{1}{\epsilon^3}\,\frac{\widehat{\gamma}_K^{(1)}}{64}\Sigma_C\left[3\beta_0 \,+\, 4\Sigma_\gamma \,+\, \widehat{\gamma}_K^{(1)}\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j\right] \\
& + \frac{1}{\epsilon^2}\,\frac{1}{8}\left[\frac{\beta_0\,\widehat{\gamma}_K^{(1)}}{4}\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j \,+\, \beta_0\Sigma_\gamma \,-\, \frac{\widehat{\gamma}_K^{(2)}}{4}\,\Sigma_C\right. \\
& \qquad + \Sigma_\gamma^2 \,+\, \frac{\widehat{\gamma}_K^{(1)}}{2}\,\Sigma_\gamma\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j \\
& \qquad \left. + \frac{\left(\widehat{\gamma}_K^{(1)}\right)^2}{16}\sum_{\substack{i,j\neq i\\ k,l\neq k}}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\ln\left(\frac{-s_{kl}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j\,\mathbf{T}_k\cdot\mathbf{T}_l\right] \\
& - \frac{1}{\epsilon}\,\frac{1}{4}\left[\frac{\widehat{\gamma}_K^{(2)}}{4}\sum_{i,j\neq i}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j \,+\, \Sigma_\gamma^{(2)}\right],
\end{aligned} \tag{3.171}$$

which agrees with [121], with the anomalous dimension coefficients given in Eq. (A.12), and where we defined $\Sigma_\gamma^{(2)} = \sum_i \gamma_i^{(2)}$. Having deduced the $\mathbf{Z}$ elements up to the needed order, we can now interfere the double-virtual amplitude with the Born, and extract the poles. The perturbative expansion of (3.162) yields

$$\begin{aligned}
\mathcal{A}^{(0)} &= \mathcal{H}^{(0)} \,, \\
\mathcal{A}^{(1)} &= \frac{\alpha_s}{2\pi} \Big[\mathcal{H}^{(1)} + \mathbf{Z}^{(1)} \mathcal{H}^{(0)} \Big] \equiv \frac{\alpha_s}{2\pi} \, A^{(1)} \,, \\
\mathcal{A}^{(2)} &= \left(\frac{\alpha_s}{2\pi} \right)^2 \Big[\mathcal{H}^{(2)} + \mathbf{Z}^{(1)} \mathcal{H}^{(1)} + \mathbf{Z}^{(2)} \mathcal{H}^{(0)} \Big] \equiv \left(\frac{\alpha_s}{2\pi} \right)^2 A^{(2)} \,,
\end{aligned} \tag{3.172}$$

implying

$$\begin{aligned}
|\mathcal{A}|^2 = \left| \mathcal{H}^{(0)} \right|^2 &+ \frac{\alpha_s}{2\pi} \, 2 \, \mathbf{Re} \Big[\big(\mathcal{H}^{(0)} \big)^\dagger \mathcal{H}^{(1)} + \big(\mathcal{H}^{(0)} \big)^\dagger \mathbf{Z}^{(1)} \mathcal{H}^{(0)} \Big] \\
&+ \left(\frac{\alpha_s}{2\pi} \right)^2 \Big[2 \, \mathbf{Re} \Big(\big(\mathcal{H}^{(0)} \big)^\dagger \mathcal{H}^{(2)} + \big(\mathcal{H}^{(0)} \big)^\dagger \mathbf{Z}^{(1)} \mathcal{H}^{(1)} + \big(\mathcal{H}^{(0)} \big)^\dagger \mathbf{Z}^{(2)} \mathcal{H}^{(0)} \Big) \\
&\qquad + \left| \mathcal{H}^{(1)} \right|^2 + \big(\mathcal{H}^{(0)} \big)^\dagger \big(\mathbf{Z}^{(1)} \big)^\dagger \mathbf{Z}^{(1)} \mathcal{H}^{(0)} + 2 \, \mathbf{Re} \Big(\big(\mathcal{H}^{(1)} \big)^\dagger \mathbf{Z}^{(1)} \mathcal{H}^{(0)} \Big) \Big] + \mathcal{O}(\alpha_s^3) \,.
\end{aligned} \tag{3.173}$$

We are interested in the divergent contributions to Eq. (3.173) at $\mathcal{O}(\alpha_s^2)$: we extract them in turn. First, the direct contribution of the two-loop $\mathbf{Z}$ matrix is given by

$$\begin{aligned}
2 \, \mathbf{Re} \Big(\big(\mathcal{H}^{(0)} \big)^\dagger \mathbf{Z}^{(2)} \mathcal{H}^{(0)} \Big) &= \mathcal{H}^{(0)\dagger} \big(\mathbf{Z}^{(2)} + \mathbf{Z}^{(2)\dagger} \big) \mathcal{H}^{(0)} \\
&= \frac{1}{\epsilon^4} \frac{1}{4} \Sigma_C^2 \, B + \frac{1}{\epsilon^3} \frac{1}{2} \Sigma_C \left[\left(\frac{3}{4} \beta_0 + \Sigma_\gamma \right) B + \sum_{i, j \neq i} L_{ij} \, B_{ij} \right] \\
&+ \frac{1}{\epsilon^2} \frac{1}{4} \Bigg[\left(\beta_0 \Sigma_\gamma - \frac{\widehat{\gamma}_K^{(2)}}{4} \Sigma_C + \Sigma_\gamma^2 \right) B + \left(\beta_0 + 2 \, \Sigma_\gamma \right) \sum_{i, j \neq i} L_{ij} \, B_{ij} \\
&\qquad + \frac{1}{2} \sum_{\substack{i, j \neq i \\ k, l \neq k}} \left(L_{ij} \, L_{kl} - \pi^2 \right) B_{ijkl} \Bigg] \\
&- \frac{1}{\epsilon} \frac{1}{8} \left[4 \, \Sigma_\gamma^{(2)} \, B + \widehat{\gamma}_K^{(2)} \sum_{i, j \neq i} L_{ij} \, B_{ij} \right] ,
\end{aligned} \tag{3.174}$$

where again $L_{ij} = \ln(s_{ij}/\mu^2)$, and the colour-correlated Born amplitudes B_{ij} and B_{ijkl} are defined in Eq. (A.5). The square of the one-loop $\mathbf{Z}$ matrix contributes

$$\begin{aligned}
\mathcal{H}^{(0)\dagger} \mathbf{Z}^{(1)\dagger} \mathbf{Z}^{(1)} \mathcal{H}^{(0)} &= \frac{1}{\epsilon^4} \frac{1}{4} \, \Sigma_C^2 B + \frac{1}{\epsilon^3} \frac{1}{2} \, \Sigma_C \left[\Sigma_\gamma \, B + \sum_{i, j \neq i} L_{ij} \, B_{ij} \right] \\
&+ \frac{1}{\epsilon^2} \frac{1}{4} \left[\Sigma_\gamma^2 \, B + 2 \, \Sigma_\gamma \sum_{i, j \neq i} L_{ij} \, B_{ij} + \frac{1}{2} \sum_{\substack{i, j \neq i \\ k, l \neq k}} \left(L_{ij} \, L_{kl} + \pi^2 \right) B_{ijkl} \right] .
\end{aligned} \tag{3.175}$$

Note that in Eq. (3.174) and in Eq. (3.175), for simplicity, we already substituted $\widehat{\gamma}_K^{(1)} = 4$.

Finally, terms involving the product of the one-loop hard part and the one-loop $\mathbf{Z}$ matrix give

$$2\,\mathbf{Re}\left(\mathcal{H}^{(0)^\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(1)} + \mathcal{H}^{(1)^\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(0)}\right) = \mathcal{H}^{(0)^\dagger}\left(\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)^\dagger}\right)\mathcal{H}^{(1)} + \mathcal{H}^{(1)^\dagger}\left(\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)^\dagger}\right)\mathcal{H}^{(0)} . \qquad (3.176)$$

In order to make use in practice of Eq. (3.176), it is useful to rewrite $\mathcal{H}^{(1)}$ in terms of the full virtual amplitude $A^{(1)}$, using

$$\mathcal{H}^{(1)} = A^{(1)} - \mathbf{Z}^{(1)}\mathcal{H}^{(0)} . \qquad (3.177)$$

Eq. (3.176) then becomes

$$2\,\mathbf{Re}\left(\mathcal{H}^{(0)^\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(1)} + \mathcal{H}^{(1)^\dagger}\mathbf{Z}^{(1)}\mathcal{H}^{(0)}\right) = \mathcal{H}^{(0)^\dagger}\left(\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)^\dagger}\right)A^{(1)} + A^{(1)^\dagger}\left(\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)^\dagger}\right)\mathcal{H}^{(0)} - \mathcal{H}^{(0)^\dagger}\left(\mathbf{Z}^{(1)^2} + 2\,\mathbf{Z}^{(1)^\dagger}\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)^{\dagger 2}}\right)\mathcal{H}^{(0)}. \qquad (3.178)$$

The term on the second line of Eq. (3.178) is easily computed using Eq. (3.169) and yields

$$-\,\mathcal{H}^{(0)^\dagger}\left(\mathbf{Z}^{(1)^2} + 2\,\mathbf{Z}^{(1)^\dagger}\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)^{\dagger 2}}\right)\mathcal{H}^{(0)} = -\,\frac{1}{\epsilon^4}\,\Sigma_C^2 B - \frac{1}{\epsilon^3} 2\,\Sigma_C\left[\Sigma_\gamma\, B + \sum_{i,j\neq i} L_{ij}\, B_{ij}\right] - \frac{1}{\epsilon^2}\left[\Sigma_\gamma^2 B + 2\,\Sigma_\gamma \sum_{i,j\neq i} L_{ij}\, B_{ij} + \frac{1}{2}\sum_{\substack{i,j\neq i\\ k,l\neq k}} L_{ij}\, L_{kl}\, B_{ijkl}\right] . \qquad (3.179)$$

The first two terms on the *r.h.s.* of Eq. (3.178) can be expressed in terms of the one-loop virtual correction to the cross section. One finds

$$\begin{aligned}
&\frac{\alpha_s}{2\pi}\left[\mathcal{H}^{(0)^\dagger}\left(\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)^\dagger}\right)A^{(1)} + A^{(1)^\dagger}\left(\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)^\dagger}\right)\mathcal{H}^{(0)}\right] \\
&\quad = \mathcal{H}^{(0)^\dagger}\left[-\frac{1}{\epsilon^2}\frac{\widehat{\gamma}_K^{(1)}}{4}\Sigma_C - \frac{1}{\epsilon}\left(\frac{\widehat{\gamma}_K^{(1)}}{4}\sum_{i,j\neq i} L_{ij}\,\mathbf{T}_i\cdot\mathbf{T}_j + \Sigma_\gamma\right)\right]\mathcal{A}^{(1)} \;+\; \text{h. c.} \\
&\quad = -\frac{1}{\epsilon^2}\,\Sigma_C\, V - \frac{1}{\epsilon}\Sigma_\gamma\, V - \frac{1}{\epsilon}\sum_{i,j\neq i} L_{ij}\, V_{ij} \,, \qquad (3.180)
\end{aligned}$$

where the colour-correlated virtual correction V_{ij} is defined in Eq. (A.6). Combining Eq. (3.174) with Eq. (3.175) and Eq. (3.180), we get a complete and explicit expression for the pole part of the double-virtual contribution to the cross section,

$$
\begin{aligned}
VV_{\rm poles} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ &- \frac{1}{\epsilon^4}\,\frac{1}{2}\,\Sigma_C^2\,B + \frac{1}{\epsilon^3}\,\Sigma_C\left[\left(\frac{3}{8}\,\beta_0\,-\,\Sigma_\gamma\right)B - \sum_{i,j\neq i} L_{ij}\,B_{ij}\right] \\
&+ \frac{1}{\epsilon^2}\,\frac{1}{4}\Bigg[\left(\beta_0\Sigma_\gamma\,-\,\frac{\widehat{\gamma}_K^{(2)}}{4}\Sigma_C\,-\,2\,\Sigma_\gamma^2\right)B \\
&\qquad + \Big(\beta_0 - 4\,\Sigma_\gamma\Big)\sum_{i,j\neq i} L_{ij}\,B_{ij} - \sum_{\substack{i,j\neq i \\ k,l\neq k}} L_{ij}\,L_{kl}\,B_{ijkl}\Bigg] \\
&- \frac{1}{\epsilon}\,\frac{1}{8}\left[4\,\Sigma_\gamma^{(2)}\,B\,+\,\widehat{\gamma}_K^{(2)}\sum_{i,j\neq i} L_{ij}\,B_{ij}\right]\Bigg\} \\
- \frac{\alpha_s}{2\pi}&\left[\frac{1}{\epsilon^2}\,\Sigma_C\,V + \frac{1}{\epsilon}\Sigma_\gamma\,V + \frac{1}{\epsilon}\sum_{i,j\neq i} L_{ij}\,V_{ij}\right].
\end{aligned}
\tag{3.181}
$$

Eq. (3.181) can now be combined with the integrals of the double-radiative and the real-virtual counterterms to form the subtracted double-virtual contribution to the cross section, $VV_{\rm sub}$, given in Eq. (3.161).

3.6.2 Integrated counterterms for double-virtual poles

The expressions for the relevant integrated counterterms, $I^{\,(\mathbf{2})}$ and $I^{(\mathbf{RV})}$, were given in Eq. (3.97) and in Eq. (3.153), respectively. All we have to do now is expand these expressions in powers of ϵ, including terms up to $\mathcal{O}(\epsilon^0)$. We define

$$
I^{\,(\mathbf{2})} + I^{(\mathbf{RV})} \;\equiv\; I_{\rm poles}^{(\mathbf{2+RV})} + I_{\rm fin}^{(\mathbf{2+RV})} + \mathcal{O}(\epsilon)\,. \tag{3.182}
$$

As expected, the pole part $I_{\rm poles}^{(\mathbf{2+RV})}$ exactly cancels Eq. (3.181):

$$
I_{\rm poles}^{(\mathbf{2+RV})} \;=\; -\,VV_{\rm poles}\,. \tag{3.183}
$$

We note in particular that *it is not necessary* to compute NLO virtual corrections up to $\mathcal{O}(\epsilon^2)$, since the last term in Eq. (3.181), containing virtual corrections times explicit poles up to ϵ^{-2}, is exactly reproduced by $I_{\rm poles}^{(\mathbf{2+RV})}$, so that $\mathcal{O}(\epsilon)$ contributions to NLO corrections never appear in our subtraction formula[11]. This was anticipated in Ref. [210] and emerges clearly in our approach thanks to the factorisation properties of the one-loop amplitude, and the minimal scheme we adopt for the factorisation of virtual corrections. The finite part derived from the sum of the integrated counterterms in Eq. (3.182) can

[11] This understands the technical capability by a two-loop provider to turn off the $\mathcal{O}(\epsilon)$ NLO virtual contribution in the computation of VV. Were this is not the case, the evaluation of $I^{\,(\mathbf{2})}$ as well would have to be performed with such a contribution turned on.

be written as ($r = r_j$, $r' = r_{jl}$)

$$
\begin{aligned}
I^{(\mathbf{2+RV})}_{\rm fin} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \Bigg[I^{(0)} + \sum_j I^{(1)}_j \, L_{jr} + \sum_j I^{(2)}_j \, L^2_{jr} + \frac{1}{2} \sum_{j,l\neq j} \gamma_j^{\rm hc} \, \gamma_l^{\rm hc} \, L_{jr'} L_{lr'} \Bigg] B \\
& + \sum_j \Big[I^{(0)}_{jr} + I^{(1)}_{jr} \, L_{jr} \Big] B_{jr} - 2 \left(1 - \zeta_2\right) \sum_{j,c\neq j,r} \gamma_j^{\rm hc} \left(2 - L_{cr}\right) B_{cr} \\
& + \sum_{c,d\neq c} L_{cd} \Bigg[I^{(0)}_{cd} + I^{(1)}_{cd} \, L_{cd} + \frac{\beta_0}{12} \, L^2_{cd} - \frac{1}{2} \big(4 - L_{cd}\big) \sum_j \gamma_j^{\rm hc} \, L_{jr} \Bigg] B_{cd} \\
& + \sum_{c,d\neq c} \Bigg[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2\big(1 - \zeta_3\big)\, L_{cd} \Bigg] B_{cdcd} \\
& + \big(1 - \zeta_2\big) \sum_{\substack{c,d\neq c \\ e\neq d}} L_{cd} \, L_{ed} \, B_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}} L_{cd} \, L_{ef} \Bigg[1 - \frac{1}{2} \, L_{cd} \bigg(1 - \frac{1}{8} L_{ef} \bigg) \Bigg] B_{cdef} \\
& + \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \Bigg[\ln \frac{s_{ce}}{s_{de}} \, L^2_{cd} + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2 \, {\rm Li}_3 \bigg(- \frac{s_{ce}}{s_{de}} \bigg) \Bigg] B_{cde} \Bigg\} \\
+ \frac{\alpha_s}{2\pi} \Bigg[& \bigg(\Sigma_\phi - \sum_j \gamma_j^{\rm hc} \, L_{jr} \bigg) V_{\rm fin} + \sum_{c,d\neq c} L_{cd} \bigg(2 - \frac{1}{2} L_{cd} \bigg) V^{\rm fin}_{cd} \Bigg] ,
\end{aligned}
\tag{3.184}
$$

where $V_{\rm fin}$ and $V^{\rm fin}_{cd}$ are the $\mathcal{O}(\epsilon^0)$ terms in the virtual and colour-correlated virtual contributions, which are obtained from the full virtual contributions V and V_{cd} by subtracting the IR poles given explicitly by Eq. (3.169). We emphasise that the kinematic dependence of Eq. (3.184) is only through simple logarithms of kinematic invariants, with the single exception of the trilogarithm multiplying the tripole Born-level colour correlation B_{cde} on the one-but-last line of Eq. (3.184). All the integral coefficients appearing in Eq. (3.184) are pure numbers, and they are collected in the following expressions:

$$
\begin{aligned}
I^{(0)} = \; & \\
& N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4 \right] + N_g N_q C_F \left[C_A \left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4 \right) + \beta_0 \left(\frac{77}{12} - \frac{53}{12}\zeta_2 \right) \right] \\
& + N_g^2 \left[C_A^2 \left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{1}{8}\zeta_2 \right) + C_A \beta_0 \left(- \frac{1}{9} - \frac{11}{3}\zeta_2 \right) \right] \\
& + N_q C_F \Bigg[C_F \left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4 \right) + C_A \left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4 \right) \\
& \qquad + \beta_0 \left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3 \right) \Bigg] \\
& + N_g \Bigg[C_F C_A \left(- \frac{737}{48} + 11\zeta_3 \right) + C_F \beta_0 \left(\frac{67}{16} - 3\zeta_3 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{3}{8}\zeta_2 \right) \\
& \qquad + C_A^2 \left(- \frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4 \right) + C_A \beta_0 \left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3 \right) \Bigg] ,
\end{aligned}
\tag{3.185}
$$

$$
\begin{aligned}
I_j^{(1)} &= (f_j^q + f_j^{\bar q})\, C_F \bigg[N_q C_F \bigg(\frac{5}{2} - \frac{7}{4}\zeta_2\bigg) + N_g C_A \bigg(\frac{1}{3} - \frac{7}{4}\zeta_2\bigg) + \frac{2}{3} N_g \beta_0 \\
&\qquad\qquad + C_F \bigg(- \frac{3}{8} - 4\zeta_2 + 2\zeta_3\bigg) + C_A \bigg(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\bigg) + \beta_0 \bigg(\frac{1}{24} + \zeta_2\bigg)\bigg] \\
&\quad + f_j^g \bigg[N_q C_F C_A \big(10 - 7\zeta_2\big) - N_q C_F \beta_0 \bigg(\frac{5}{2} - \frac{7}{4}\zeta_2\bigg) + N_g C_A^2 \bigg(\frac{4}{3} - 7\zeta_2\bigg) + N_g C_A \beta_0 \bigg(\frac{7}{3} + \frac{7}{4}\zeta_2\bigg) \\
&\qquad - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4} C_F C_A - \frac{3}{4} C_F \beta_0 + C_A^2 \bigg(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\bigg) - C_A \beta_0 \bigg(\frac{2}{3} - \frac{5}{2}\zeta_2\bigg)\bigg] , \\
I_j^{(2)} &= \frac{1}{8}\big(15\, C_A - 7\, \beta_0\big) C_{f_j} - \frac{1}{4}\big(5\, C_A - 2\, \beta_0\big)\gamma_j + \frac{1}{8}\big(16\, \zeta_2 - 15\big) C_{f_j}^2 , \\
I_{jr}^{(0)} &= \big(- 1 + 3\zeta_2 - 2\zeta_3\big) C_A - \frac{1}{2}\big(13 + 10\zeta_2 + 2\zeta_3\big) C_{f_j} + \big(5 + 2\zeta_3\big)\gamma_j , \\
I_{jr}^{(1)} &= \big(1 - \zeta_2\big) C_A + \frac{1}{2}\big(4 + 7\zeta_2\big) C_{f_j} - \big(2 + \zeta_2\big)\gamma_j , \\
I_{cd}^{(0)} &= \bigg(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\bigg) C_A + \frac{31}{9}\, \beta_0 + 2\, \Sigma_\phi + 8\, \big(1 - \zeta_2\big)\, C_{f_d} , \\
I_{cd}^{(1)} &= -\bigg(\frac{1}{3} - \frac{1}{2}\zeta_2\bigg) C_A - \frac{11}{12}\, \beta_0 - \frac{1}{2}\, \Sigma_\phi .
\end{aligned} \tag{3.186}
$$

We emphasise that, as expected, the pole part $I^{(\mathbf{2+RV})}_{\rm poles}$ does not depend on the reference momenta r, r'; conversely, the dependence on r, r' arising in the finite part $I^{(\mathbf{2+RV})}_{\rm fin}$ is essential for removing the corresponding explicit dependence present in the counterterms $K^{(\mathbf{2})}$ and $K^{(\mathbf{RV})}$.

Chapter 4

Numerical implementation and validation of the scheme

With the completion of the Local Analytic Sector Subtraction programme for the cancellation of NNLO infrared singularities in massless QCD final states, as presented in Chapter 3, we are now ready to approach the fundamental task of implementing and validating this algorithm at the numerical level. The universal and entirely analytic nature of our subtraction procedure makes it naturally well-suited to be incorporated in a general automated Monte Carlo event generator, which provides an optimal environment to fully exploit the potential of our scheme. The development of such an event generator, coupled with the extension of the algorithm to the treatment of initial-state radiation, would result in a cutting-edge tool capable of producing fully-differential NNLO predictions. This would be highly valuable for the wider phenomenological community, especially considering the current absence of a fully automated and publicly available code yielding perturbative corrections beyond NLO, as discussed in Section 1.3.2.

Reaching this ambitious goal requires the successful finalisation of several intermediate yet equally significant steps. In this Chapter we report on the progress and the current status of the implementation of our subtraction procedure within MadNkLO, a Python-based framework designed to automate the generation and handling of local subtraction terms at higher orders in perturbation theory, in the spirit of the well-established MadGraph5_aMC@NLO package [166, 211], on which it builds. Motivated by the logical approach that guided the analytic construction of the scheme, we first start by implementing the NLO subtraction formula developed in Chapter 2, and subsequently assess its performances in both integrated and differential calculations. We report the results in Section 4.1. Then, in Section 4.2, we provide an update on the current progress of the NNLO implementation. Additionally, we offer an analytic demonstration showcasing how the cancellation of phase-space singularities is achieved within a non-trivial process at NNLO accuracy.

4.1 Testing the NLO subtraction scheme

In this Section we present numerical results obtained through the application of the Local Analytic Sector Subtraction algorithm[1] to the computation of NLO cross sections for realistic scattering processes. As previously mentioned, we choose to work in the MadNkLO framework [212–215], which provides a flexible high-level platform suitable for deploying meta-codes that implement generic subtraction schemes for IRC divergences at higher orders. MadNkLO builds on the MadGraph5_aMC@NLO environment [166, 211], relying on the latter for the generation of tree-level and one-loop matrix elements.[2] Specifically, once the user specifies the scattering process and the desired perturbative order (e.g. NLO or NNLO in QCD, and possibly mixed QCD-EW corrections), MadNkLO identifies all the building blocks required for the corresponding computations, i.e. the matrix elements and the local counterterms necessary for handling singular limits appropriately. It is a developer's task to implement in this framework those ingredients which are specific to a given subtraction scheme, such as the expression of the local and integrated counterterms, momentum mappings, possibly sector functions, as well as functions that generate code in low-level programming languages. In the following, we present numerical results that validate our scheme both at the local and at the integrated level, in Section 4.1.1 and Section 4.1.2, respectively. In Section 4.1.3, we also analyse differential cross sections and verify the effectiveness of the damping factors introduced in Section 2.2.6. The interested reader can find details on the technical implementation of our subtraction scheme in MadNkLO in Appendix E of Ref. [1].

4.1.1 Cancellation of IRC singularities

In this Section we showcase how the numerical cancellation of IRC singularities is achieved for a selection of processes and of singular configurations at NLO accuracy. To carry out this demonstration, we use the built-in testing routine provided by MadNkLO, which allows us to examine the behavior of matrix elements and local counterterms in singular phase-space regions. In detail, we evaluate the $(n+1)$-body matrix element and the relevant counterterms in a randomly-chosen phase-space point, then we progressively deform it in order to approach a specific singular configuration (soft or collinear). The closeness to the singular configuration is controlled by a scaling parameter, λ. For the purpose of this Section, the reader should bear in mind that $\lambda \sim E_i^2$ ($\lambda \sim \theta_{ij}^2$) in the soft $\mathbf{S}_i$ (collinear $\mathbf{C}_{ij}$) limit (more details on the implementation of such scaling variable can be found in Section 6.1.1 of Ref. [212], and Appendix A of Ref. [213]).

[1] Specifically, we implement the version of the NLO subtraction scheme that incorporates symmetrised sector functions and damping factors.

[2] We remind the reader that one-loop matrix elements in MadGraph5_aMC@NLO are generated by the MadLoop module [216].

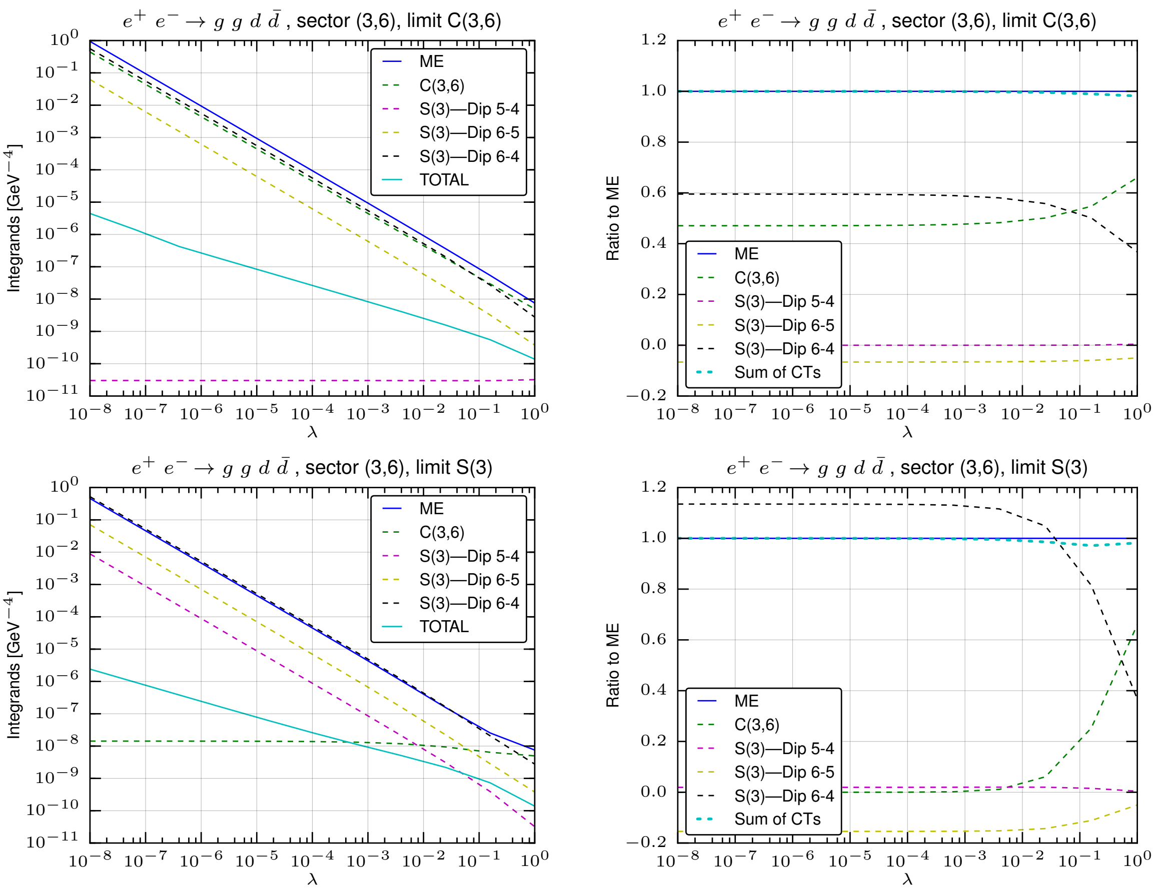

Figure 4.1. The singular behaviour of the real-emission matrix element and counterterms for the process $e^+e^- \to gg d\bar{d}$, in the sector identified by particles 3, 6. Top row: collinear configuration C(3,6); bottom row: soft configuration S(3).

We start by showing in Figure 4.1 the real channel $e^+e^- \to gg d\bar{d}$ of the annihilation process $e^+e^- \to jjj$, and consider the sector identified by the first gluon and the $\bar{d}$ quark (labelled as 3, 6 in the particle list) both in case they become collinear (top row), and in case the gluon becomes soft (bottom row). Several quantities are displayed: the solid blue line represents the exact $(n+1)$-body matrix element, dubbed **ME**; thin dashed lines of different colours indicate the collinear counterterms $\mathrm{C}(x,y)$, which include soft-collinear contributions, and the soft counterterms $\mathrm{S}(z)$, split according to the different eikonal (or radiating dipole) contributions **Dip a-b**; the subtracted matrix element, labelled with **TOTAL**, is marked with a solid teal line, while the sum of all counterterms (**Sum of CTs**) is displayed with a thicker dashed line. Contributions are shown either in absolute value (left panels) or divided by the matrix element (right panels). Both sets of panels help conveying the message that the local cancellation of singularities has been achieved. In the left-hand plots, the λ^{-1} slope of the real matrix element and of the counterterms is apparent, reducing to a $\lambda^{-1/2}$ behaviour for the subtracted result, which in turn becomes regular once combined with the phase-space measure. In the right-hand plots one can

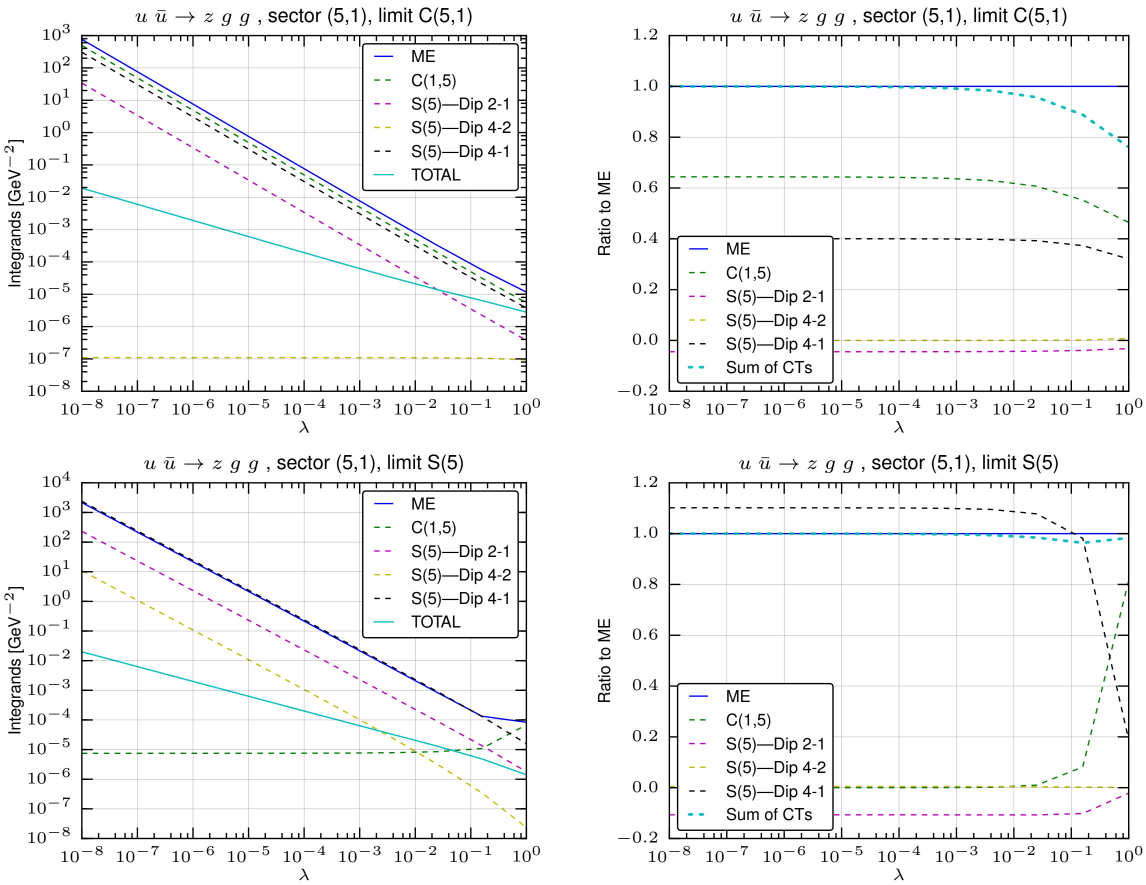

Figure 4.2. The singular behaviour of the real-emission matrix element and counterterms for the process $u\bar{u} \to Zgg$, in the sector identified by particles 1, 5. Top row: collinear configuration C(1,5); bottom row: soft configuration S(5).

appreciate how the various counterterms combine in such a way that their sum matches the matrix element in the relevant singular limit.

Turning to processes initiated by coloured particles, we consider in Figure 4.2 the real correction $u\bar{u} \to Zgg$ to the reaction $pp \to Zj$. We show the C(1,5) and S(5) configurations (i.e. those for which the last gluon (5) is collinear to the incoming u quark (1), or soft), in the sector identified by particles 1, 5.

Analogously, in Figure 4.3, we consider the real channel $dd \to ggdd$ of the three-level process $pp \to jjj$, in the C(1,3) and S(3) configurations, in the sector identified by particles 1, 3. Such a process has as many as 11 counterterms in this configuration (1 collinear and 10 soft dipoles), thus the displayed integrable scaling of the subtracted matrix element provides a highly non-trivial test of the correctness of the local subtraction mechanism.

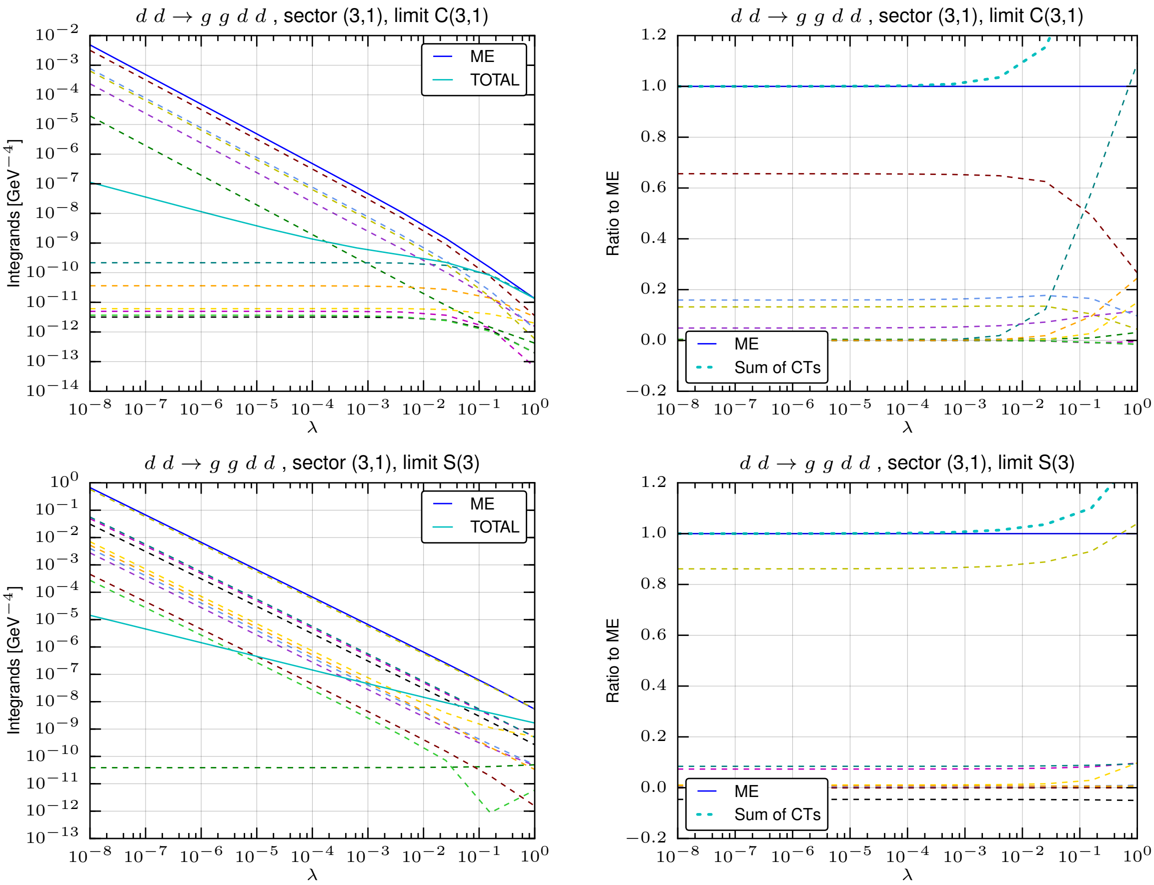

Figure 4.3. The singular behaviour of the real-emission matrix element and counterterms for the process $dd \to ggdd$, in the sector identified by particles 1, 3. Top row: collinear configuration C(1,3); bottom row: soft configuration S(3).

4.1.2 Integrated results

We now turn to the numerical validation of our approach at the level of integrated cross sections for a selection of processes at NLO, comparing our results against those obtained with MadGraph5_aMC@NLO. The two main current limitations of our MadNkLO-based framework are the absence of a low-level code implementation, and of optimised phase-space integration routines. In fact, the integration is steered by a code written in Python, using Vegas3 [217, 218] as integrator. Such a behaviour somewhat limits the complexity of the processes that can be run within a reasonable amount of time and computing resources; still, the processes we consider in the following cover all radiation topologies and both leptonic and hadronic collisions, hence we reckon them a sufficient subset for validation purposes.

The numerical setup we use is the following: processes at lepton colliders are run at a centre-of-mass energy of 1 TeV. Hadronic processes are instead run at the LHC RunII energy of 13 TeV. In the latter case, the PDF4LHC15_nlo_30 PDFs are employed [219], via the LHAPDF interface [220]. The fine-structure and Fermi constants have the values

$$\alpha = 1/132.507, \qquad G_f = 1.16639 \cdot 10^{-5}\,\text{GeV}^{-2}, \tag{4.1}$$

while the particle masses are given by:[3]

$$m_Z = 91.188\,\text{GeV}, \qquad m_W = 80.419\,\text{GeV}, \qquad m_b = 4.7\,\text{GeV}, \qquad m_t = 173\,\text{GeV}. \tag{4.2}$$

Renormalisation and factorisation scales are kept fixed to $\mu = \mu_F = m_Z$.
Whenever light partons are present in the final state at the Born level, they are clustered into jets with the anti-k_t algorithm [221], as implemented in FastJet [222], with radius parameter $R = 0.4$. Jets are then required to satisfy the following kinematic cuts:

$$p_T(j) > 20\,\text{GeV}, \qquad |\eta(j)| < 5. \tag{4.3}$$

The processes we consider are

$$e^+e^- \to jj, \tag{4.4}$$
$$e^+e^- \to jjj, \tag{4.5}$$
$$pp \to Z, \tag{4.6}$$
$$pp \to Zj, \tag{4.7}$$
$$pp \to W^+W^-j. \tag{4.8}$$

For these processes, we have computed the LO cross section and the corresponding NLO correction, which are quoted in Table 4.1. In this case, no damping factors are applied. Results from MadGraph5_aMC@NLO (dubbed aMC in the table) and MadNkLO are in general very well compatible, the largest deviations being of the order of the combined integration error, which is at or below the per-mille level.

Process	aMC LO	MadNkLO LO	aMC NLO corr.	MadNkLO NLO corr.
$e^+e^- \to jj$	0.53209(6)	0.53208(6)	0.019991(7)	0.019991(10)
$e^+e^- \to jjj$	0.4739(3)	0.4740(3)	-0.1461(1)	-0.1463(6)
$pp \to Z$	46361(3)	46362(3)	6810.9(8)	6810.8(4)
$pp \to Zj$	11270(7)	11258(5)	3770(6)	3776(17)
$pp \to W^+W^-j$	42.42(1)	42.39(2)	10.68(5)	10.53(13)

Table 4.1. Validation table with predictions for LO cross sections and NLO corrections. Numbers are in pb. Integration errors, on the last digit(s), are shown in parentheses.

[3]In our model m_W is derived from α, G_f and m_Z; also, the presence of a non-zero value for m_b is formally inconsistent with the employed PDF set, however this is of no relevance as far as validation is concerned.

We also consider the case of non-zero values for the damping parameters α, β, γ presented in Section 2.2.6. For simplicity, we set the three parameters to a common value, ranging from 0 to 2. Results for the NLO corrections are shown in Table 4.2, together with their breakdown into n-body and $(n+1)$-body contributions (the former including virtual corrections and integrated counterterms, the latter including subtracted real emissions). While the n- and $(n+1)$-body terms, if consider separately, show a very significant dependence upon the unphysical damping parameters, their sum remains stable, as expected. Results with the three different damping choices are totally compatible within the respective integration errors, and, in turn, with the MadGraph5_aMC@NLO results.

Process	MadNkLO $\alpha=\beta=\gamma=0$	MadNkLO $\alpha=\beta=\gamma=1$	MadNkLO $\alpha=\beta=\gamma=2$
$e^+e^- \to jj$			
V+I	0.02664732(9)	0.01998531(7)	0.00666183(2)
R-K	-0.00666(1)	0.000004(6)	0.013329(6)
NLO corr.	0.019991(10)	0.019985(6)	0.019991(6)
$pp \to Z$			
V+I+C+J	3981.5(4)	-3472.7(4)	-9163.2(5)
R-K	2829.3(2)	10284.3(4)	15974.1(6)
NLO corr.	6810.8(4)	6811.6(6)	6810.9(8)
$pp \to Zj$			
V+I+C+J	7172(2)	5246(2)	3624(2)
R-K	-3395(17)	-1469(25)	156(22)
NLO corr.	3776(17)	3777(25)	3780(22)

Table 4.2. Validation table with predictions for the NLO corrections, broken down between n and $n+1$ contributions, when different damping factors (α, β, γ) are considered. Numbers are in pb. The integration error, on the last digit(s), is shown in parentheses.

4.1.3 Validation at differential level

Finally, we validate the correctness of the damping factors at the differential level in the simple case of $e^+e^- \to \gamma^* \to jj$, at centre-of-mass energy $\sqrt{s} = 100$ GeV, with $\mu = 35$ GeV. The plots in Figure 4.4 show differential cross sections with respect to transverse momentum and (absolute value of) pseudo-rapidity of the two hardest jets in the events (clustered with the k_t algorithm [223, 224]), which are NLO-accurate observables receiving contribution from subtraction counterterms across the whole spectrum. A comparison is provided between predictions obtained with MadGraph5_aMC@NLO and an in-house implementation of Local Analytic Sector Subtraction, limited to the above-mentioned process. Various combinations of parameters α and β, ranging from 0 to 3, are chosen, in order to cover different damping possibilities (γ is irrelevant for final-state radiation).

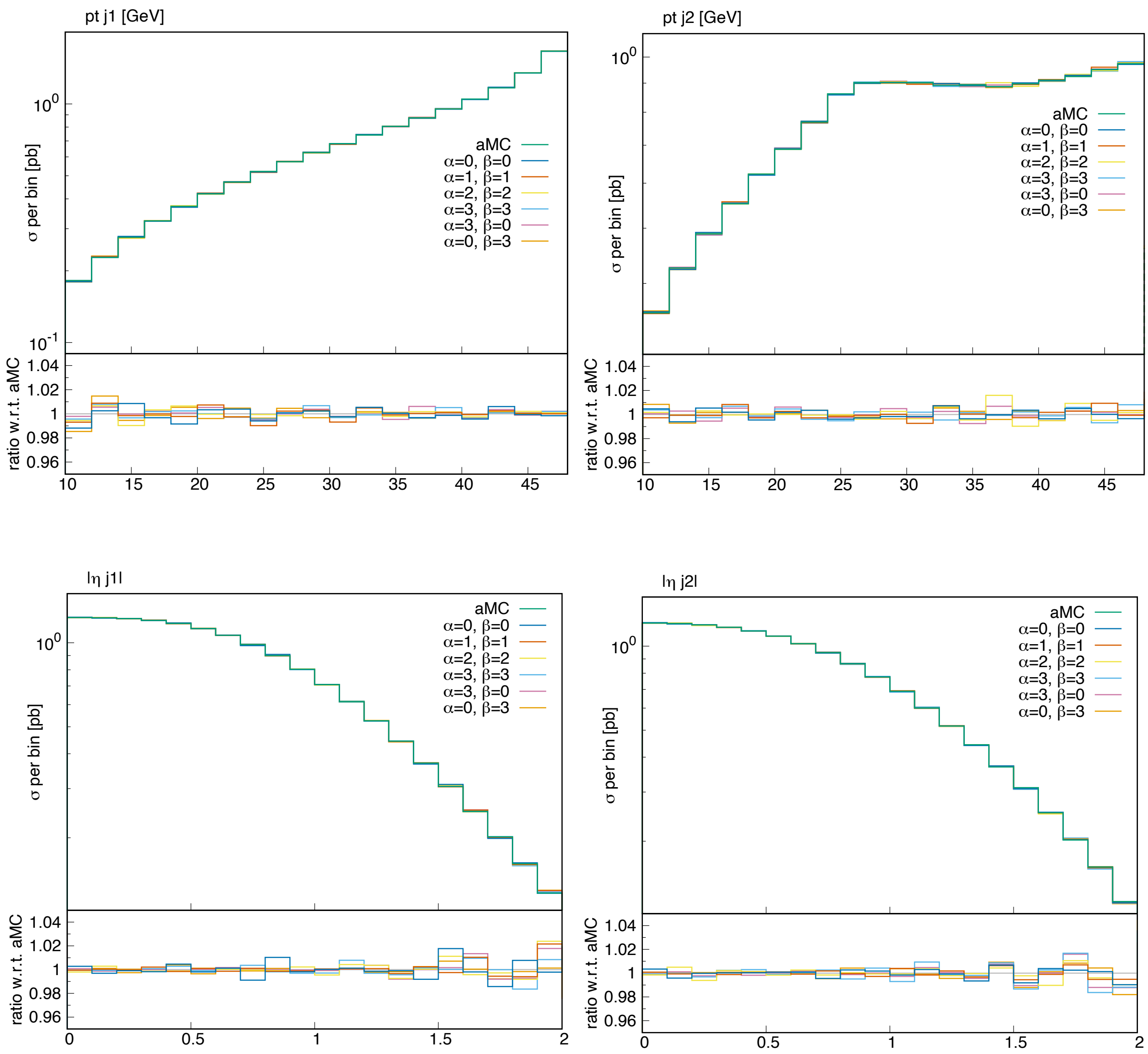

Figure 4.4. Transverse momenta and pseudo-rapidities for the two hardest jets in $e^+e^- \to \gamma^* \to jj$ at NLO, comparing MadGraph5_aMC@NLO (aMC in the labels) and Local Analytic Sector Subtraction.

As evident from Figure 4.4, predictions generated using our NLO subtraction algorithm for all chosen damping profiles are in excellent agreement with those obtained with MadGraph5_aMC@NLO within the numerical accuracy used for the runs. A systematic study of the performance of different damping choices at the differential level in more complex processes and setups would certainly be valuable. Such an analysis is, however, beyond the scope of this first scheme validation, and thus postponed to future work.

4.2 Testing the NNLO subtraction scheme

While the MadNkLO framework is already equipped with all the necessary structures and routines for handling the components required for a NNLO subtraction, the highly-flexible

Python code is not optimised for efficiently performing extensive NNLO phenomenological computations within a reasonable runtime. In order to approach these computationally-expensive calculations, we must first address the current limitations of our code, pointed out in Section 4.1.2. On-going efforts are dedicated to the construction of a Fortran-level implementation and optimisation of the code, so as to allow for a faster evaluation of the integrands. Additionally, we are incorporating mature phase-space routines that leverage the sector structure and kinematic mappings of our algorithm, which will further enhance the code's performance. Once fully developed, MadNkLO will become a versatile automated platform, providing immediate access to the generation of state-of-the-art specialised results for both theoretical and experimental communities, with a significant impact on the scientific programmes of the LHC and future colliders.

In the absence, as yet, of a numerical tool to test our NNLO subtraction scheme, in Section 4.2.1 we provide an analytical, explicit example that illustrates the cancellation of phase-space singularities for a selected double-real channel of the process $e^+e^- \to q\bar{q}g$.

4.2.1 Cancellation of IR singularities: a case study

In this Section we work out in detail the cancellation of IR singularities for the process $e^+e^- \to q(1)\bar{q}(2)g([345])$ at NNLO, focusing on the double-real-emission channel where an extra quark-antiquark pair is emitted, namely $e^+e^- \to q(1)\bar{q}(2)g(3)q'(4)\bar{q}'(5)$ (with q and q' being different quark flavours).

We pick the sector $\mathcal{W}_{4353}$ as a test case. As such sector function selects $\overline{\mathbf{C}}_{43}$ as single-unresolved limit, the only possible underlying single-real-emission channel to be considered is $e^+e^- \to q(1)\bar{q}(2)\; q'([34])\bar{q}'(5)$. Moreover, the only double-unresolved configurations allowed are $\overline{\mathbf{S}}_{45}$ and $\overline{\mathbf{C}}_{435}$. In Fig. 4.5 we show a sample NNLO double-real-emission Feynman diagram contributing to this sector (left), together with its underlying NLO single-real-emission diagram (middle), as well as the LO Born diagram (right).

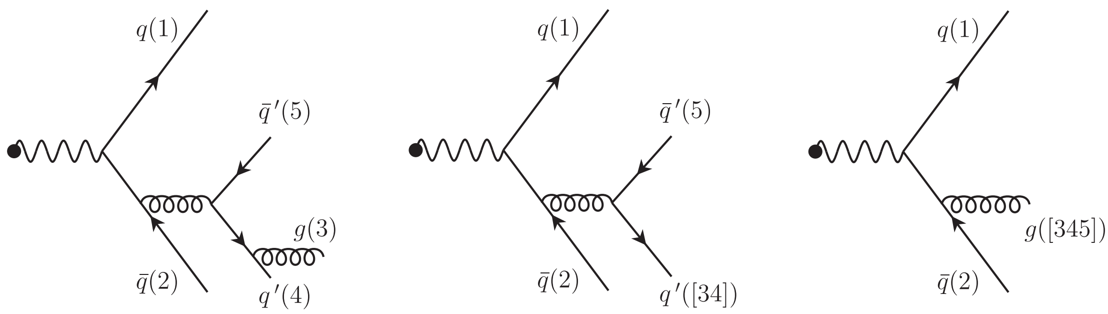

Figure 4.5. Sample Feynman diagrams for three-jet production in leptonic collisions: NNLO double-real emission (left), NLO single-real emission (middle), Born process (right).

The improved limits on RR relevant for $\mathcal{W}_{4353}$ sector are given by (see Appendix C.1)

$$\overline{\mathbf{C}}_{43}\, RR \equiv \frac{\mathcal{N}_1}{s_{43}}\, P_{43(r)}\, \bar{R}^{(43r)}\,, \qquad \overline{\mathbf{C}}_{435}\, RR \,\equiv\, \frac{\mathcal{N}_1^2}{s_{435}^2}\, P^{\mu\nu}_{435(r)}\, \bar{B}^{(435r)}_{\mu\nu}\,,$$
$$\overline{\mathbf{S}}_{45}\, RR \equiv \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq 4,5\\ d\neq 4,5,c}} \mathcal{E}^{(45)}_{cd}\, \bar{B}^{(45cd)}_{cd}\,, \qquad \overline{\mathbf{S}}_{45}\, \overline{\mathbf{C}}_{435}\, RR \,\equiv\, -\mathcal{N}_1^2\, C_{f_3}\, \mathcal{E}^{(45)}_{3r}\, \bar{B}^{(453r)}\,,$$
$$\overline{\mathbf{C}}_{43}\, \overline{\mathbf{C}}_{435}\, RR \,\equiv\, \mathcal{N}_1^2\, \frac{P_{43(r)}}{s_{43}}\, \frac{\bar{P}^{(43r)\mu\nu}_{[34]5(r)}}{\bar{s}^{(43r)}_{[34]5}}\, \bar{B}^{(43r,[34]5r)}_{\mu\nu}\,. \tag{4.9}$$

The expression for the relevant kernels $\overline{\mathbf{S}}_{45}\, RR$ and $\overline{\mathbf{C}}_{43}\, \overline{\mathbf{C}}_{435}\, RR$, according to Eqs. (C.49) and (C.76), is rather simple in this configuration. Since partons 4 and 5 are quarks, the eikonal kernels $\mathcal{E}^{(4)}_{ab}$ and $\mathcal{E}^{(5)}_{ab}$ vanish; moreover, since the parent parton [34] of the 34 pair is a quark, there is no azimuthal dependence in the collinear splitting kernels, hence $Q^{\mu\nu}_{43(r)} = 0$, and $P^{\mu\nu}_{43(r)} \bar{R}^{(43r)}_{\mu\nu} = P_{43(r)} \bar{R}^{(43r)}$. In the case we are considering, the reference index r (used for collinear limits) could be either $r = 1$ or $r = 2$, without any distinction. The subtracted double-real contribution in this sector is thus

$$RR^{\rm sub}_{4353} \,=\, RR\,\mathcal{W}_{4353} - K^{(\mathbf{1})}_{4353} - \Big(K^{(\mathbf{2})}_{4353} - K^{(\mathbf{12})}_{4353}\Big)\,, \tag{4.10}$$

where the NNLO counterterms read

$$K^{(\mathbf{1})}_{4353} \equiv \overline{\mathbf{C}}_{43}\, RR\,\mathcal{W}_{4353} \,=\, \mathcal{N}_1\, \frac{P_{43(r)}}{s_{43}}\, \bar{R}^{(43r)}\, \overline{\mathcal{W}}^{(43r)}_{5[34]}\, \mathcal{W}^{(\alpha)}_{{\rm c},43(r)}\,, \tag{4.11}$$
$$\begin{aligned} K^{(\mathbf{2})}_{4353} \equiv{}& \Big[\overline{\mathbf{S}}_{45} + \overline{\mathbf{C}}_{435}\,\big(1 - \overline{\mathbf{S}}_{45}\big)\Big]\, RR\,\mathcal{W}_{4353} \\ ={}& \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq 4,5\\ d\neq 4,5,c}} \mathcal{E}^{(45)}_{cd}\, \bar{B}^{(45cd)}_{cd}\, \big(\overline{\mathbf{S}}_{45}\, \mathcal{W}_{4353}\big) + \frac{\mathcal{N}_1^2}{s_{435}^2}\, P^{\mu\nu}_{435(r)}\, \bar{B}^{(435r)}_{\mu\nu} \big(\overline{\mathbf{C}}_{435}\, \mathcal{W}_{4353}\big) \\ &+ \mathcal{N}_1^2\, C_{f_3}\, \mathcal{E}^{(45)}_{3r}\, \bar{B}^{(453r)}\, \big(\overline{\mathbf{S}}_{45}\, \overline{\mathbf{C}}_{435}\, \mathcal{W}_{4353}\big)\,, \end{aligned}$$
$$K^{(\mathbf{12})}_{4353} \equiv \overline{\mathbf{C}}_{43}\, \overline{\mathbf{C}}_{435}\, RR\,\mathcal{W}_{4353} \,=\, \mathcal{N}_1^2\, \frac{P_{43(r)}}{s_{43}}\, \frac{\bar{P}^{(43r)\mu\nu}_{[34]5(r)}}{\bar{s}^{(43r)}_{[34]5}}\, \bar{B}^{(43r,[34]5r)}_{\mu\nu}\, \overline{\mathcal{W}}^{(43r)}_{{\rm c},5[34](r)}\, \mathcal{W}^{(\alpha)}_{{\rm c},43(r)}\,.$$

Looking at Eqs. (3.37)-(3.39), we can identify the consistency relations that must be satisfied in order to prove the integrability of the subtracted double-real contribution $RR^{\rm sub}_{4353}$. Those relations are given by

- *primary* limits: $\big\{\mathbf{C}_{43},\ \mathbf{S}_{45},\ \mathbf{C}_{435}\big\}\, RR^{\rm sub}_{4353} \,\to\,$ integrable ;
- *auxiliary* limits: $\big\{\mathbf{C}_{4r},\ \mathbf{C}_{5r},\ \mathbf{C}_{45r}\big\}\, RR^{\rm sub}_{4353} \,\to\,$ integrable ;
- *secondary* limits: $\mathbf{C}_{35}\, RR^{\rm sub}_{4353} \,\to\,$ integrable .

Let us begin by showing the cancellation of the singular behaviour in the $\mathbf{C}_{43}$ limit. We note that particle momenta in this limit obey $\mathbf{C}_{43}\{k\} = \mathbf{C}_{43}\{\bar{k}\}^{(43r)} = \{k\}_{\not{4}\not{3},[34]}$, which implies that the limit taken on sector functions gives

$$\mathbf{C}_{43}\, \overline{\mathcal{W}}^{(43r)}_{5[34]} = \mathcal{W}_{5[34]}\,, \qquad \mathbf{C}_{43}\, \mathcal{W}^{(\alpha)}_{\mathrm{c},43(r)} = \mathbf{C}_{43}\, \mathcal{W}^{(\alpha)}_{43}\,. \tag{4.12}$$

This, together with the relation $\mathbf{C}_{43}\mathcal{W}_{4353} = \mathbf{C}_{43}\mathcal{W}_{5[34]}\mathcal{W}^{(\alpha)}_{43}$, and with the known $\mathbf{C}_{43}$ limit of RR, is sufficient to show that

$$\mathbf{C}_{43}\left[1 - \overline{\mathbf{C}}_{43}\right] RR\, \mathcal{W}_{4353} \to \text{integrable}\,. \tag{4.13}$$

Next, for the remaining contributions, we need to show that

$$\mathbf{C}_{43}\left[K^{(\mathbf{2})}_{4353} - K^{(\mathbf{12})}_{4353}\right] \to \text{integrable}\,. \tag{4.14}$$

To this end, let us note that the double-soft kernel factorised in $\overline{\mathbf{S}}_{45}\, RR$ is not singular in the $\mathbf{C}_{43}$ limit, since the denominator of $\mathcal{E}^{(45)}_{cd}$ (see Eq. (4.9)) features the sum $s_{[34]5} \equiv s_{34} + s_{35}$. The same is true for the $\overline{\mathbf{C}}_{435}\, \overline{\mathbf{S}}_{45}\, RR$ limit, which is constructed with the same double-soft kernel. As a consequence, checking the requirement in Eq. (4.14) reduces to verifying that

$$\mathbf{C}_{43}\, \overline{\mathbf{C}}_{435}\left[1 - \overline{\mathbf{C}}_{43}\right] RR\, \mathcal{W}_{4353} \to \text{integrable}\,. \tag{4.15}$$

As far as sector functions are concerned, it is straightforward to check that

$$\begin{aligned} \mathbf{C}_{43}\, \overline{\mathbf{C}}_{435}\, \mathcal{W}_{4353} &= \mathbf{C}_{43}\, \overline{\mathbf{C}}_{43}\, \overline{\mathbf{C}}_{435}\, \mathcal{W}_{4353} \\ &= \mathbf{C}_{43}\, \overline{\mathcal{W}}^{(43r)}_{\mathrm{c},5[34](r)}\, \mathcal{W}^{(\alpha)}_{\mathrm{c},43(r)} \;=\; \mathbf{C}_{[34]5}\, \mathcal{W}_{5[34]}\, \mathbf{C}_{43}\, \mathcal{W}^{(\alpha)}_{43}\,. \end{aligned} \tag{4.16}$$

The mapped kinematics is such that the identity $\{\bar{k}\}^{(435r)} = \{\bar{k}\}^{(43r,[34]5r)}$ holds also far from the collinear limit. Finally, the double-collinear kernel can be written as

$$\frac{P^{\mu\nu}_{435(r)}}{s^2_{345}} = \frac{1}{s^2_{345}}\left[-P^{(\mathrm{1g})}_{453(r)}\, g^{\mu\nu} + \sum_{a=3,4,5} Q^{(\mathrm{1g}),a}_{453(r)}\, d^{\mu\nu}_a\right]. \tag{4.17}$$

It is easy to show that in the collinear limit $\mathbf{C}_{43}$ the non-abelian contribution to this kernel is non-singular[4], while the abelian part, owing to the relations $x_3 = s_{3r}/s_{[34]r}$ and

[4] To this end we use the following equivalence relations in the $\mathbf{C}_{43}$ limit ($\tilde{k}^\mu_5 = -\tilde{k}^\mu_3 - \tilde{k}^\mu_4$):

$$\mathbf{C}_{43}\,\frac{s_{35}}{s_{45}} = \mathbf{C}_{43}\,\frac{s_{3r}}{s_{4r}} = \mathbf{C}_{43}\,\frac{z_3}{z_4}\,, \qquad \mathbf{C}_{43}\,\tilde{k}^\mu_3 = -\,\mathbf{C}_{43}\,\frac{z_3}{z_{34}}\,\tilde{k}^\mu_5\,, \qquad \mathbf{C}_{43}\,\tilde{k}^\mu_4 = -\,\mathbf{C}_{43}\,\frac{z_4}{z_{34}}\,\tilde{k}^\mu_5\,.$$

$x_4 = 1 - x_3$, becomes

$$\begin{aligned}
\mathbf{C}_{43}\,\frac{P^{\mu\nu}_{435(r)}}{s_{345}^2} &= \mathbf{C}_{43}\,\frac{C_F\,T_R}{s_{[34]5}^2}\,\frac{2x_4 + (1-\epsilon)x_3^2}{x_3}\,\frac{s_{[34]5}}{s_{43}}\left[-\,g^{\mu\nu} + 4\,z_5(1-z_5)\frac{\tilde{k}_5^\mu \tilde{k}_5^\nu}{\tilde{k}_5^2}\right] \\
&= \mathbf{C}_{43}\,\frac{P_{43(r)}}{s_{43}}\,\frac{P^{\mu\nu}_{[34]5(r)}}{s_{[34]5}} \;=\; \mathbf{C}_{43}\,\frac{P_{43(r)}}{s_{43}}\,\frac{\bar{P}^{(43r)\mu\nu}_{[34]5(r)}}{\bar{s}^{(43r)}_{[34]5}}\,, \qquad (4.18)
\end{aligned}$$

which shows that singular terms in the $\mathbf{C}_{43}$ limit cancel in Eq. (4.15).

Moving on to the double-soft $\mathbf{S}_{45}$ limit, we note that its action on kinematics is such that $\mathbf{S}_{45}\,\{k\} = \mathbf{S}_{45}\,\{\bar{k}\}^{(45cd)} = \{k\}_{\not{4}\not{5}}$. This consideration, together with the relation $\mathbf{S}_{45}\,\mathcal{W}_{4353} = \mathbf{S}_{45}\,\overline{\mathbf{S}}_{45}\,\mathcal{W}_{4353}$, and the known double-soft limit of the double-real matrix element, immediately implies that

$$\mathbf{S}_{45}\left[1 - \overline{\mathbf{S}}_{45}\right] RR\,\mathcal{W}_{4353} \to \text{integrable}\,. \qquad (4.19)$$

On the other hand, the single-unresolved kernel $K^{(\mathbf{1})}_{4353}$ is non-singular in the $\mathbf{S}_{45}$ double-soft limit, as it does not feature any $1/s_{45}$ enhancement, so $\mathbf{S}_{45}\,K^{(\mathbf{1})}_{4353} = 0$. The same holds for the strongly-ordered collinear kernel $\overline{\mathbf{C}}_{43}\,\overline{\mathbf{C}}_{435}\,RR$, whence $\mathbf{S}_{45}\,K^{(\mathbf{12})}_{4353} = 0$. We thus are left with the requirement

$$\mathbf{S}_{45}\,\overline{\mathbf{C}}_{435}\,\big(1 - \overline{\mathbf{S}}_{45}\big)RR\,\mathcal{W}_{4353} \to \text{integrable}\,. \qquad (4.20)$$

As far as sector functions are concerned, it is straightforward to verify that $\mathbf{S}_{45}\,\overline{\mathbf{C}}_{435}\,\mathcal{W}_{4353} = \mathbf{S}_{45}\,\overline{\mathbf{S}}_{45}\,\overline{\mathbf{C}}_{435}\,\mathcal{W}_{4353}$, using Eq. (C.96) and Eq. (C.98). As to matrix elements, the relevant kernel is

$$\overline{\mathbf{C}}_{435}\,\big(1 - \overline{\mathbf{S}}_{45}\big)RR = \frac{\mathcal{N}_1^2}{s_{435}^2}\,P^{\mu\nu}_{435(r)}\,\bar{B}^{(435r)}_{\mu\nu} + \mathcal{N}_1^2\,C_{f_3}\mathcal{E}^{(45)}_{3r}\,\bar{B}^{(453r)}\,. \qquad (4.21)$$

Here, using the kinematic condition $\mathbf{S}_{45}\,\{\bar{k}\}^{(435r)} = \mathbf{S}_{45}\,\{\bar{k}\}^{(453r)}$, one can show that the second term on the right-hand side precisely removes from the first one all double-soft enhancements proportional to $1/s_{45}$, as was the case for unimproved limits. Thus, we verify that

$$\mathbf{S}_{45}\,\overline{\mathbf{C}}_{435}\,\big(1 - \overline{\mathbf{S}}_{45}\big)RR \to \text{integrable.} \qquad (4.22)$$

Next, we consider the behaviour of the counterterm $K^{(\mathbf{2})}_{4353}$ under the double-collinear limit $\mathbf{C}_{435}$. First, we notice that

$$\mathbf{C}_{435}\,\{k\} = \mathbf{C}_{435}\,\{\bar{k}\}^{(435r)} = \mathbf{C}_{435}\,\{\bar{k}\}^{(453r)} = \{k\}_{\not{3}\not{4}\not{5}[345]}\,, \qquad (4.23)$$

and that, using Eq. (3.12) and Eq. (C.43), one has $\mathbf{C}_{435}\,\overline{\mathbf{C}}_{435}\,\mathcal{W}_{4353} = \mathbf{C}_{435}\,\mathcal{W}_{4353}$. These relations, together with the known $\mathbf{C}_{435}$ limit of RR, are sufficient to show that

$$\mathbf{C}_{435}\,\overline{\mathbf{C}}_{435}\,RR\,\mathcal{W}_{4353} = \mathbf{C}_{435}\,RR\,\mathcal{W}_{4353}\,. \tag{4.24}$$

As for the remaining part of $K^{(\mathbf{2})}_{4353}$, we have

$$\mathbf{C}_{435}\,\overline{\mathbf{S}}_{45}\,\mathcal{W}_{4353} \;=\; \mathbf{C}_{435}\,\overline{\mathbf{S}}_{45}\,\overline{\mathbf{C}}_{435}\,\mathcal{W}_{4353} \;=\; \frac{\sigma_{4353}}{\sigma_{4353}+\sigma_{5343}+\sigma_{4553}+\sigma_{5443}}\,, \tag{4.25}$$

as can be deduced from the definitions in Eq. (C.43) and Eq. (C.98). The action of the double-soft kernel on the matrix element, on the other hand, gives

$$\mathbf{C}_{435}\sum_{\substack{c\neq 4,5\\ d\neq 4,5,c}}\mathcal{E}^{(45)}_{cd}\,\bar{B}^{(45cd)}_{cd} \;=\; 2\,\mathcal{E}^{(45)}_{3r}\sum_{d\neq 4,5,3}\mathbf{C}_{435}\,\bar{B}^{(453d)}_{3d} \;=\; 2\,\mathcal{E}^{(45)}_{3r}\sum_{d\neq 4,5,3}\mathbf{C}_{435}\,\bar{B}^{(453r)}_{3d}\,, \tag{4.26}$$

where, in the last step, we used $\mathbf{C}_{435}\,\{\bar{k}\}^{(453d)} = \mathbf{C}_{435}\,\{\bar{k}\}^{(453r)}$. By performing the sum over colours, Eq. (4.26) becomes

$$2\,\mathcal{E}^{(45)}_{3r}\sum_{d\neq 4,5,3}\mathbf{C}_{435}\,\bar{B}^{(453r)}_{3d} \;=\; -2\,C_{f_3}\,\mathcal{E}^{(45)}_{3r}\mathbf{C}_{435}\,\bar{B}^{(453r)}\,, \tag{4.27}$$

which matches (with opposite sign) the kernel in $\overline{\mathbf{C}}_{435}\,\overline{\mathbf{S}}_{45}\,RR$, finally showing that

$$\mathbf{C}_{435}\,\overline{\mathbf{S}}_{45}\Big[1-\overline{\mathbf{C}}_{435}\Big]\,RR\,\mathcal{W}_{4353} \to \text{integrable}\,. \tag{4.28}$$

In order to complete the proof of the cancellation of singular contributions in the $\mathbf{C}_{435}$ limit, it is finally necessary to show that

$$\mathbf{C}_{435}\Big[K^{(\mathbf{1})}_{4353} - K^{(\mathbf{12})}_{4353}\Big] \to \text{integrable}\,. \tag{4.29}$$

The sector functions appearing in $K^{(\mathbf{1})}_{4353}$ and $K^{(\mathbf{12})}_{4353}$ approach the same value under the double-collinear limit, since $\mathbf{C}_{435}\,\overline{\mathcal{W}}_{\mathrm{c},5[34](r)} = \mathbf{C}_{435}\,\overline{\mathcal{W}}_{5[34]}$. As for the kernels, one just needs to check that

$$\mathbf{C}_{435}\,\bar{R}^{(43r)} \;=\; \mathbf{C}_{435}\,\frac{\bar{P}^{(43r)\mu\nu}_{[34]5(r)}}{\bar{s}^{(43r)}_{[34]5}}\,\bar{B}^{(43r,[34]5r)}_{\mu\nu}\,, \tag{4.30}$$

which is indeed the case, since the $\mathbf{C}_{435}$ double-collinear limit acts on the mapped kinematics as a single-collinear limit between parton 5 and the parent parton [34].

After proving the local cancellation of all phase-space singularities in sector $\mathcal{W}_{4353}$, we proceed by showing that the functional form chosen for the sector functions is also capable of eliminating *spurious* singularities, arising from collinear kernels, as detailed in Eq. (3.38). To this end, we consider the $\mathbf{C}_{45r}$ limit (in fact, neither the $\mathbf{C}_{4r}$ nor $\mathbf{C}_{5r}$ limits generate any spurious singularities in this specific case). We introduce a common scaling parameter λ for the invariants vanishing in this limit, s_{45}, s_{4r} and s_{5r}. The $K^{(\mathbf{1})}_{4353}$ counterterm is non-singular in this limit, since it does not feature any isolated $1/s_{4r}$ denominator, so we focus on $K^{(\mathbf{2})}_{4353}$ and $K^{(\mathbf{12})}_{4353}$. The kernel $\overline{\mathbf{S}}_{45}\, RR$ diverges as λ^{-2} in the limit, due to denominators of type $1/s_{45}^2$, or $1/s_{45}s_{[45]r}$, however the corresponding sector function $\overline{\mathbf{S}}_{45}\, \mathcal{W}_{4353}$ scales as λ^2, thus compensating the singularity. Analogously, the counterterm $\overline{\mathbf{C}}_{43}\, \overline{\mathbf{C}}_{435}\, RR\, \mathcal{W}_{4353}$ is non-singular in the $\mathbf{C}_{45r}$ limit. As for the remaining counterterms, we have

$$\begin{aligned}
\mathbf{C}_{45r}\, \overline{\mathbf{C}}_{435}\, RR \;&=\; \mathbf{C}_{45r}\, \mathcal{N}_1^2\, 2\, C_A T_R \left[\frac{1}{s_{45}^2} \left(\frac{s_{43}s_{5r} - s_{4r}s_{53}}{s_{[45]3}s_{[45]r}} \right)^2 - \frac{s_{3r}}{s_{45}s_{[45]3}s_{[45]r}} \right] \;\sim\; \lambda^{-2} \\
&=\; \mathbf{C}_{45r}\, \overline{\mathbf{S}}_{45}\, \overline{\mathbf{C}}_{435}\, RR\,, \\
\mathbf{C}_{45r}\, \overline{\mathbf{C}}_{435}\, \mathcal{W}_{4353} \;&=\; \mathbf{C}_{45r}\, \frac{\hat{\sigma}_{4353}}{\hat{\sigma}_{4553} + \hat{\sigma}_{5443}} \;\sim\; \lambda^{\alpha} \\
&=\; \mathbf{C}_{45r}\, \overline{\mathbf{S}}_{45}\, \overline{\mathbf{C}}_{435}\, \mathcal{W}_{4353}\,,
\end{aligned} \tag{4.31}$$

where the dependence on the parameter α emerges from the definition of the sector function, see Eq. (3.12). In this case, both $\overline{\mathbf{C}}_{435}\, RR\, \mathcal{W}_{4353}$ and $\overline{\mathbf{S}}_{45}\, \overline{\mathbf{C}}_{435}\, RR\, \mathcal{W}_{4353}$ display at most an integrable $\lambda^{\alpha-2}$ singularity, which is ultimately due to the chosen sector. However, even in a sector in which the two contributions are separately non-integrable (for instance in $\mathcal{W}_{4553}$ or $\mathcal{W}_{5443}$), the fact that both kernels and sector functions tend to become identical prevents a singular scaling of the double-hard collinear contribution $\overline{\mathbf{C}}_{435}(1 - \overline{\mathbf{S}}_{45})RR\, \mathcal{W}_{4353}$.

Finally, we analyse the secondary limit $\mathbf{C}_{35}$. Since the counterterms $K^{(\mathbf{1})}_{4353}$, $K^{(\mathbf{12})}_{4353}$, and the double-soft and soft-collinear contributions in $K^{(\mathbf{2})}_{4353}$, are non-singular under this limit, we are left to verify that

$$\begin{aligned}
\mathbf{C}_{35}\, RR\, \mathcal{W}_{4353} &\to \text{integrable}\,, \\
\mathbf{C}_{35}\, \overline{\mathbf{C}}_{435}\, RR\, \mathcal{W}_{4353} &\to \text{integrable}\,.
\end{aligned} \tag{4.32}$$

Denoting with λ the scaling variable associated with the vanishing invariant s_{35}, it can be shown that both the double-unresolved singular kernel in $\overline{\mathbf{C}}_{435}\, RR$, and the collinear kernel resulting from

$$\mathbf{C}_{35}\, RR \;=\; \mathcal{N}_1\, \frac{P_{35(r)}}{s_{35}}\, R^{([35])}\,, \tag{4.33}$$

display a λ^{-1} singular behaviour. At the same time, the corresponding sector functions evaluated in the $\mathbf{C}_{35}$ limit vanish with a $\lambda^{\alpha-1}$ rate. As a result, both contributions in Eq. (4.32) display an overall $\lambda^{\alpha-2}$ scaling, leading to an integrable singularity.

This completes our analysis of this example, showing that the subtracted double-real emission contribution under consideration is completely free of phase-space singularities.

Conclusions & Perspective

In this thesis, we addressed the challenging problem of handling infrared singularities in higher-order perturbative calculations in massless QCD. In Chapter 1, we introduced an innovative strategy for the formulation of a novel subtraction scheme, known as *Local Analytic Sector Subtraction*, which ambitiously aims to lay the foundations for systematic and universal solutions to this long-standing issue. Inspired by successful NLO schemes, our procedure seeks to optimise the counterterm structure across all stages of the calculation by systematically leveraging every available degree of freedom, significantly simplifying the required analytic integrations.

We initially applied this strategy to develop a general subtraction scheme, capable of addressing unresolved radiation in both initial and final states within the NLO framework. In Chapter 2, we provided a detailed construction of our local counterterm, coupled with a comprehensive analysis of its advantages and limitations. To improve numerical stability, we introduced an optimisation procedure to smoothly mitigate the contribution of subtraction terms in the non-singular regions of phase space while preserving the method's fundamental properties, specifically the simplicity of the involved analytical integrations. This resulting computational simplicity stands as a highly desirable characteristic as we look towards extending our approach to higher perturbative orders.

With these promising results in hand, in Chapter 3 we addressed the extension of our subtraction procedure to handle NNLO infrared singularities. Herein, we provided an exhaustive description of all essential ingredients contributing to the formulation of this scheme, accompanied by a step-by-step explanation of their intricate combination. The outcome of this substantial effort culminated in a fully analytic and universal formula which achieves the cancellation of NNLO infrared singularities for a broad spectrum of processes, involving an arbitrary number of colourful as well as colourless final-state particles in massless QCD. The cancellation of all phase-space singularities in double-real and real-virtual contributions has been proven by the verification of all relevant consistency checks. Furthermore, all counterterms were analytically integrated through standard techniques, exhibiting singular contributions characterised by single-scale integrals with trivial logarithmic dependence on Born-level kinematic invariants. All explicit ϵ poles originating from the singular part of the double-virtual contribution to the cross section have also been shown to cancel once properly combined with the phase-space counterterm integrations. We also achieved the analytic integration of all finite remainders, which manifest a similar level of simplicity, with the exception of a single contribution (proportional to a colour tripole) introducing a weight-three polylogarithm depending on two physical scales. This analytic formula, representing a significant novelty in our field, can be readily implemented in any numerical framework equipped with the relevant matrix elements.

Finally, we dedicated Chapter 4 of this thesis to the validation of our subtraction algorithms. We reported on the progress towards their numerical implementation within MadNkLO, a Python-based framework designed to automate the generation and handling of local subtraction terms at higher orders in perturbation theory. In this context, we presented numerical results obtained by applying the NLO subtraction scheme to the computation of cross sections for realistic scattering processes. We discussed the performance of the method both at the integrated and differential level. While efforts are underway to achieve an efficient implementation of the NNLO subtraction formula within our numerical framework, we provided a detailed example illustrating the analytical cancellation of phase-space singularities in a non-trivial scattering process.

The results obtained in this thesis have direct implications in state-of-the-art phenomenological studies. A clear research avenue is, for example, the analysis of theory-data comparisons in current (and future) e^+e^- colliders: in this context, our subtraction formula can be readily applied to the computation of NNLO-accurate predictions, extending to quantities such as the cross section for four-jet production, as well as energy-energy correlations in hadronic final states.

The future steps naturally following after these achievements are clearly defined. Among the high-priority tasks is the numerical implementation and testing of the NNLO subtraction algorithm within an automated Monte Carlo event generator. As we highlighted in Chapter 3, work is under way to overcome the current limitations of MadNkLO, specifically focusing on a systematic optimisation of the numerical software to reduce the time and CPU resources needed for the production of computationally-demanding NNLO phenomenological results. Without a doubt, a crucial goal at NNLO is to extend the treatment of unresolved radiation to non-trivial initial states, particularly in view of relevant LHC applications. This generalisation is anticipated to involve no new major technical obstacles: as observed at NLO, it will require the introduction of new classes of mappings and a consistent implementation of collinear factorisation, but all of these developments are expected to be relatively straightforward. Importantly, also new phase-space integrals are expected to be of the same level of complexity as those presented in the final-state scenario, suggesting that a completely analytic result is within reach. Work is in progress also on this front. Looking ahead, another important step for generalisation is the inclusion of massive QCD particles in the final state. This task will be simplified by the fact that the number and type of singular limits associated with massive coloured particles are limited, as collinear limits for real radiation are non-singular in this case. Since our approach is combinatorially intensive, this simplicity is expected to be of great advantage.

On the other hand, massive particles will require adjustments in phase-space mappings, and will likely involve new classes of integrals, with a more intricate scale dependence. We are, nonetheless, confident that a complete analytic expression can be derived also in that case.

Finally, we are of the opinion that, despite the simplicity of our analytic results, there is further room for optimisation. Specifically, we believe that extending the damping factors, initially introduced at NLO, to NNLO accuracy could significantly enhance the numerical efficiency of our algorithm.

In summary, we believe that our results mark a significant step towards establishing a fully general, local, analytic, and efficient NNLO-subtraction formalism.

Appendix A

General notation

We denote by s the squared centre-of-mass energy and by $q^\mu = (\sqrt{s}, \vec{0}\,)$ the centre-of-mass four-momentum. Given two final-state momenta k_i^μ and k_j^μ, we define

$$s_{qi} \;=\; 2\,q\cdot k_i\,, \quad s_{ij} \;=\; 2\,k_i\cdot k_j\,, \quad L_{ij} \;=\; \ln\frac{s_{ij}}{\mu^2}\,, \quad e_i \;=\; \frac{s_{qi}}{s}\,, \quad w_{ij} \;=\; \frac{s\,s_{ij}}{s_{qi}\,s_{qj}}\,. \tag{A.1}$$

In addition, given four final-state momenta k_a^μ, k_b^μ, k_c^μ and k_d^μ, we define

$$\begin{aligned} s_{abc} \;&=\; s_{ab}+s_{bc}+s_{ac}\,, \qquad s_{[ab]c} \;=\; s_{ac}+s_{bc}\,, \qquad k_{[ab]}^\mu \;=\; k_a^\mu+k_b^\mu\,, \\ s_{abcd} \;&=\; s_{ab}+s_{ac}+s_{ad}+s_{bc}+s_{bd}+s_{cd}\,, \qquad s_{[abc]d} \;=\; s_{ad}+s_{bd}+s_{cd}\,. \end{aligned} \tag{A.2}$$

For the sake of compactness, we define the following flavour structures:

$$f_i^q = \begin{cases} 1 & \text{if } i \text{ is a quark} \\ 0 & \text{if } i \text{ is not a quark} \end{cases} \qquad f_i^{\bar q} = \begin{cases} 1 & \text{if } i \text{ is an antiquark} \\ 0 & \text{if } i \text{ is not an antiquark} \end{cases}$$

$$f_i^g = \begin{cases} 1 & \text{if } i \text{ is a gluon} \\ 0 & \text{if } i \text{ is not a gluon} \end{cases}$$

$$f_{ij}^{q\bar q} = f_i^q f_j^{\bar q} + f_i^{\bar q} f_j^q\,, \qquad f_{ij}^{gg} = f_i^g f_j^g\,, \qquad f_{ijk}^{ggg} = f_i^g f_j^g f_k^g\,, \qquad \tilde f_{ij}^{\,q\bar q} = f_i^q f_j^{\bar q} - f_i^{\bar q} f_j^q\,, \tag{A.3}$$

which are special cases of the general rule

$$f_{i_1\dots i_n}^{f_1\dots f_n} = \sum_{\substack{g_1,\dots,g_n= \\ P(f_1,\dots,f_n)}} f_{i_1}^{g_1}\cdots f_{i_n}^{g_n}\,, \qquad \tilde f_{i_1\dots i_n}^{f_1\dots f_n} = \sum_{\substack{g_1,\dots,g_n= \\ P(f_1,\dots,f_n)}} \text{sign}(P)\, f_{i_1}^{g_1}\cdots f_{i_n}^{g_n}\,, \tag{A.4}$$

where $P(f_1, \dots, f_n)$ is a generic permutation of indices $f_1, \dots, f_n$.
We introduce a compact notation for Born-level colour correlations:

$$\begin{aligned} B_{cd} \;&\equiv\; \mathcal{A}_n^{(0)\dagger}\, \mathbf{T}_c\cdot\mathbf{T}_d\, \mathcal{A}_n^{(0)}\,, \qquad B_{cdef} \;\equiv\; \mathcal{A}_n^{(0)\dagger}\, \{\mathbf{T}_c\cdot\mathbf{T}_d, \mathbf{T}_e\cdot\mathbf{T}_f\}\, \mathcal{A}_n^{(0)}\,, \\ \mathcal{B}_{cd} \;&\equiv\; f_c^g\, \mathcal{A}_n^{(0)\dagger}\, \mathcal{T}_c\cdot\mathbf{T}_d\, \mathcal{A}_n^{(0)}\,, \qquad (\mathcal{T}_A)_{BC} \;=\; d_{ABC}\,. \end{aligned} \tag{A.5}$$

Analogously, the colour-correlated real and virtual matrix elements are defined as

$$V_{cd} \equiv 2\,\mathbf{Re}\left[\mathcal{A}_n^{(1)\dagger}\,\mathbf{T}_c\cdot\mathbf{T}_d\,\mathcal{A}_n^{(0)}\right], \qquad R_{cd} \equiv \mathcal{A}_{n+1}^{(0)\dagger}\,\mathbf{T}_c\cdot\mathbf{T}_d\,\mathcal{A}_{n+1}^{(0)}, \tag{A.6}$$

which are of relative order α_s with respect to the corresponding Born-level terms.
We define the following combinations of Casimir operators,

$$\rho_{ab}^{(C)} = \frac{C_{f_{[ab]}} + C_{f_a} - C_{f_b}}{C_{f_{[ab]}}}, \qquad \rho_{[ab]}^{(C)} = \frac{C_{f_{[ab]}} - C_{f_a} - C_{f_b}}{C_{f_{[ab]}}}, \qquad \Sigma_C = \sum_a C_{f_a}, \tag{A.7}$$

and

$$\gamma_a = \frac{3}{2}\,C_F\,(f_a^q + f_a^{\bar q}) + \frac{1}{2}\,\beta_0\,f_a^g, \qquad \Sigma_\gamma = \sum_a \gamma_a, \qquad \gamma_a^{\rm hc} = \gamma_a - 2C_{f_a}, \tag{A.8}$$

$$\phi_a = \frac{13}{3}\,C_F\,(f_a^q + f_a^{\bar q}) + \frac{4}{3}\,\beta_0\,f_a^g + \left(\frac{2}{3} - \frac{7}{2}\zeta_2\right)C_{f_a}, \qquad \Sigma_\phi = \sum_a \phi_a, \tag{A.9}$$

$$\phi_a^{\rm hc} = \frac{13}{3}\,C_F\,(f_a^q + f_a^{\bar q}) + \frac{4}{3}\,\beta_0\,f_a^g - \frac{16}{3}\,C_{f_a}, \qquad \Sigma_\phi^{\rm hc} = \sum_a \phi_a^{\rm hc}, \tag{A.10}$$

where the sums run over all final-state QCD partons and

$$\beta_0 = \frac{11 C_A - 4\,T_R\,N_f}{3}. \tag{A.11}$$

The two-loop anomalous dimensions are given by

$$\begin{aligned}
\widehat{\gamma}_K^{(2)} &= 4\left\{\left(\frac{67}{18} - \zeta_2\right)C_A - \frac{10}{9}T_R\,N_f\right\} = \left(\frac{8}{3} - 4\zeta_2\right)C_A + \frac{10}{3}\,\beta_0, \\
\gamma_i^{(2)} &= (f_i^q + f_i^{\bar q})\,C_F\left[3\left(\frac{1}{8} - \zeta_2 + 2\zeta_3\right)C_F + \left(\frac{41}{36} - \frac{13}{2}\zeta_3\right)C_A + \left(\frac{65}{72} + \frac{3}{4}\zeta_2\right)\beta_0\right] \\
&\quad + f_i^g\left\{C_A\left[-\frac{11}{4}\,C_F + \left(-\frac{1}{9} - \frac{1}{2}\zeta_3\right)C_A\right] + \beta_0\left[\frac{3}{4}\,C_F + \left(\frac{16}{9} - \frac{1}{4}\zeta_2\right)C_A\right]\right\}.
\end{aligned} \tag{A.12}$$

As for the labelling of particles we introduce the notation

$$r_{i_1\ldots i_n} = R_n(i_1,\ldots,i_n) \neq i_1,\ldots,i_n, \tag{A.13}$$

to indicate a generic particle label different from $i_1,\ldots,i_n$, defined following a specific rule R_n. Such a rule is arbitrary to some extent, and could for instance assign $r_{i_1\ldots i_n}$ as the smallest label different from all $i_1,\ldots,i_n$, or the largest, and so on. A crucial feature, however, is that R_n must be symmetric under permutations of all indices $i_1,\ldots,i_n$, and must be the same for all $r_{i_1\ldots i_n}$ with the same n. As a consequence, the notation $r_{i_1\ldots i_n}$ always refers to the rule $R_n(i_1,\ldots,i_n)$, which is a symmetric function of its indices $i_1,\ldots,i_n$, and just depends on n.

Appendix B

NLO Appendices

B.1 Altarelli-Parisi splitting kernels

We collect here the expression for the regularised Altarelli-Parisi collinear kernels appearing in the lowest-order DGLAP [97–99] evolution equations.

$$\begin{aligned}\bar{P}_a(x)\, B \equiv \delta_{f_a g}\Big[\bar{P}_{gg}(x)\, B^{(g)} + \bar{P}_{q\bar{q}}(x)\left(B^{(q)} + B^{(\bar{q})}\right)\Big] \\ + \delta_{f_a \{q,\bar{q}\}}\Big[\bar{P}_{gq}(x)\, B^{(g)} + \bar{P}_{qg}(x)\, B^{(f_a)}\Big]\,,\end{aligned} \tag{B.1}$$

where

$$\bar{P}_{gg}(x) = 2\, C_A \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \delta(1-x)\,\frac{\beta_0}{2}\,, \tag{B.2}$$

$$\bar{P}_{q\bar{q}}(x) = T_R\left[x^2 + (1-x)^2\right]\,, \quad \bar{P}_{qg}(x) = C_F\left(\frac{1+x^2}{1-x}\right)_+\,, \quad \bar{P}_{gq}(x) = C_F\,\frac{1+(1-x)^2}{x}\,,$$

$C_A = N_c$, $C_F = (N_c^2-1)/(2N_c)$, $T_R = 1/2$, $\beta_0 = (11\, C_A - 4\, T_R\, N_f)/3$, and $B^{(f_i)}$ denotes the Born contribution initiated by a parton of flavour f_i, stemming from the splitting of parton a.

We also collect here finite terms arising from the integration of initial-state collinear counterterms, see Section 2.4, which are related to the Altarelli-Parisi kernels:

$$\begin{aligned}P^{(\lambda)}_{a,\mathrm{fin}}(x)\, B \equiv \delta_{f_a g}\Big[p^{(\lambda)}_{gg}(x)\, B^{(g)} + p^{(\lambda)}_{q\bar{q}}(x)\left(B^{(q)} + B^{(\bar{q})}\right)\Big] \\ + \delta_{f_a \{q,\bar{q}\}}\Big[p^{(\lambda)}_{gq}(x)\, B^{(g)} + p^{(\lambda)}_{qg}(x)\, B^{(f_a)}\Big]\,,\end{aligned} \tag{B.3}$$

where $\lambda = 1, 2$,

$$\begin{aligned}p^{(\lambda)}_{gg}(x) &= 2\, C_A \left(\frac{1-x}{x} + x(1-x)\right)\Big[\lambda\, \ln(1-x) - A_1(\gamma)\Big]\,, \\ p^{(\lambda)}_{q\bar{q}}(x) &= T_R\left(x^2 + (1-x)^2\right)\Big[\lambda\, \ln(1-x) - A_1(\gamma)\Big] + T_R\, 2x\,(1-x)\,,\end{aligned}$$

$$
\begin{aligned}
p^{(\lambda)}_{qg}(x) &= C_F\,(1-x)\Big[\lambda\,\ln(1-x) + 1 - A_1(\gamma)\Big]\,, \\
p^{(\lambda)}_{gq}(x) &= C_F\,\frac{1+(1-x)^2}{x}\Big[\lambda\,\ln(1-x) - A_1(\gamma)\Big] + C_F\,x\,,
\end{aligned}
\tag{B.4}
$$

and $A_1(\gamma)$ is defined in Appendix B.3.

B.2 NLO consistency relations

In this Appendix we explicitly verify the relations in Eqs. (2.65) for initial- and final-state radiation, ensuring the locality of the subtraction procedure.

- $\mathbf{S}_i\,\overline{\mathbf{S}}_i\,R \;=\; \mathbf{S}_i\,R$

  This relation is trivially verified since the mapped kinematics in Eq. (2.56) reduce to $\mathbf{S}_i\,\{\bar{k}\}^{(icd)} = \mathbf{S}_i\,\{\bar{k}\}^{(idc)} = \{k\}_{\not{i}}$ for vanishing i, hence

$$
\mathbf{S}_i\,\overline{\mathbf{S}}_i\,R = -\mathcal{N}_1 \sum_{c\neq i}\sum_{d\neq i,c} \mathcal{E}^{(i)}_{cd}\;B_{cd}(\{k\}_{\not{i}}) \;=\; \mathbf{S}_i\,R\,,
\tag{B.5}
$$

  which coincides with Eq. (2.25).

- $\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ij}\,R \;=\; \mathbf{C}_{ij}\,R$

  The key ingredients for this consistency are the limits

$$
\mathbf{C}_{ij}\,x\,\theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{I}} \;=\; x_{[ij]}\,\theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{I}}\,,
\tag{B.6}
$$

  as well as

$$
\begin{aligned}
\mathbf{C}_{ij}\,\{\bar{k}\}^{(ijr)}\,\theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{F}} &= \big(\{k\}_{\not{i}\not{j}}, k_{[ij]}\big)\,\theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{F}}\,, \\
\mathbf{C}_{ij}\,\{\bar{k}\}^{(ijr)}\,\theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{I}}\,\theta_{r\in\mathrm{I}} &= \mathbf{C}_{ij}\,\{\bar{k}\}^{(irj)}\,\theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{I}}\,\theta_{r\in\mathrm{F}} \\
&= \big(\{k\}_{\not{i}\not{j}}, x_{[ij]}\,k_j\big)\,\theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{I}}\,,
\end{aligned}
\tag{B.7}
$$

  from which one immediately deduces

$$
\begin{aligned}
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ij}\,R = \frac{\mathcal{N}_1}{s_{ij}}\Bigg[&\theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{F}}\,P^{\mu\nu}_{ij(r),\mathrm{F}}(z_i)\,B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}}, k_{[ij]}\big) \\
&+ \theta_{i\in\mathrm{F}}\,\theta_{j\in\mathrm{I}}\,\frac{P^{\mu\nu}_{[ij]i(r),\mathrm{I}}\big(x_{[ij]}\big)}{x_{[ij]}}\,B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}}, x_{[ij]}k_j\big) \\
&+ \theta_{j\in\mathrm{F}}\,\theta_{i\in\mathrm{I}}\,\frac{P^{\mu\nu}_{[ji]j(r),\mathrm{I}}\big(x_{[ji]}\big)}{x_{[ji]}}\,B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}}, x_{[ji]}k_i\big)\Bigg] \;=\; \mathbf{C}_{ij}\,R\,,
\end{aligned}
\tag{B.8}
$$

  exactly as Eq. (2.39).

- $\mathbf{S}_i\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R \;=\; \mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,R$

  This relation is a consequence of the fact that

$$\mathbf{S}_i\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R \;=\; \mathcal{N}_1\,\delta_{f_i g}\,2\,C_{f_j}\,\frac{s_{jr}}{s_{ij}\left(s_{ir}+\theta_{j\in\mathrm{I}}\,\theta_{r\in\mathrm{I}}\,s_{ij}\right)}\,B\big(\{k\}_{\not i}\big) \;=\; \mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,R\,, \tag{B.9}$$

  having explicitly employed the soft behaviour of the Altarelli-Parisi kernels.

- $\mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R \;=\; \mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,R$

  This final relation is instead slightly subtler. The explicit action of the collinear limit on the soft counterterm reads

$$\begin{aligned}
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,R \;=&\; -2\,\mathcal{N}_1\,\mathcal{E}^{(i)}_{jr}\,\mathbf{C}_{ij}\Bigg\{\sum_{\substack{c\neq i\\ c<j}}\Big[\big(\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}+\theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{I}}+\theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{F}}\big)\bar{B}^{(ijc)}_{jc}+\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{F}}\,\bar{B}^{(icj)}_{jc}\Big]\\
&+\sum_{\substack{c\neq i\\ c>j}}\Big[\big(\theta_{c\in\mathrm{I}}\,\theta_{j\in\mathrm{I}}+\theta_{c\in\mathrm{F}}\,\theta_{j\in\mathrm{I}}+\theta_{c\in\mathrm{F}}\,\theta_{j\in\mathrm{F}}\big)\bar{B}^{(icj)}_{jc}+\theta_{c\in\mathrm{I}}\,\theta_{j\in\mathrm{F}}\,\bar{B}^{(ijc)}_{jc}\Big]\Bigg\}\\
=&\; -2\,\mathcal{N}_1\,\mathcal{E}^{(i)}_{jr}\,\mathbf{C}_{ij}\Bigg\{\sum_{\substack{c\neq i\\ c<j}}\Big[\big(\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}+\theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{F}}\big)\,\bar{B}^{(ijc)}_{jc}\Big]\\
&+\sum_{\substack{c\neq i\\ c>j}}\Big[\big(\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}+\theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{F}}\big)\,\bar{B}^{(icj)}_{jc}\Big]\\
&+\sum_{c\neq i,j}\Big[\theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{I}}\,\bar{B}^{(ijc)}_{jc}+\theta_{j\in\mathrm{I}}\,\theta_{k\in\mathrm{F}}\,\bar{B}^{(icj)}_{jc}\Big]\Bigg\}\,,
\end{aligned}\tag{B.10}$$

  aware of the fact that the eikonal kernel $\mathbf{C}_{ij}\,\mathcal{E}^{(i)}_{jc} = \mathcal{E}^{(i)}_{jr}$ is independent of c, thus it can be taken out of the sum. The action of $\mathbf{C}_{ij}$ on the mapped Born kinematics reads

$$\begin{aligned}
\mathbf{C}_{ij}\,\theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{F}}\,\bar{B}^{(ijc)}_{jc} &= \mathbf{C}_{ij}\,\theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{F}}\,\bar{B}^{(icj)}_{jc} = \theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{F}}\,B_{jc}\big(\{k\}_{\not i\not j},k_{[ij]}\big)\,,\\
\mathbf{C}_{ij}\,\theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{I}}\,\bar{B}^{(ijc)}_{jc} &= \theta_{j\in\mathrm{F}}\,\theta_{c\in\mathrm{I}}\,B_{jc}\big(\{k\}_{\not i\not j},k_{[ij]}\big)\,,\\
\mathbf{C}_{ij}\,\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{F}}\,\bar{B}^{(icj)}_{jc} &= \theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{F}}\,B_{jc}\big(\{k\}_{\not i\not j},x_{[ij]}k_j\big)\,,\\
\mathbf{C}_{ij}\,\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}\,\bar{B}^{(ijc)}_{jc} &= \mathbf{C}_{ij}\,\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}\,\bar{B}^{(icj)}_{jc} = \theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}\,B_{jc}\big(\{k\}_{\not i\not j},x_{[ij]}k_j\big)\,,
\end{aligned}\tag{B.11}$$

  where the latter non-trivial equality is proven in Appendix B.2.1. At this point, one can recast Eq. (B.10) in

$$\mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,R \;=\; -2\,\mathcal{N}_1\,\mathcal{E}^{(i)}_{jr}\sum_{c\neq i,j}\Big[\theta_{j\in\mathrm{I}}\,B_{jc}\big(\{k\}_{\not i\not j},x_{[ij]}k_j\big)+\theta_{j\in\mathrm{F}}\,B_{jc}\big(\{k\}_{\not i\not j},k_{[ij]}\big)\Big]\,. \tag{B.12}$$

Upon enforcing colour conservation, $\sum_{c\neq j}\mathbf{T}_c = -\mathbf{T}_j$, this becomes

$$\mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,R \;=\; 2\mathcal{N}_1\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\left[\theta_{j\in\mathrm{I}}\,B\big(\{k\}_{\not i\not j},x_{[ij]}k_j\big)+\theta_{j\in\mathrm{F}}\,B\big(\{k\}_{\not i\not j},k_{[ij]}\big)\right]. \quad \text{(B.13)}$$

Recalling that $\mathbf{C}_{ij}\,z^{(irj)}\,\theta_{j\in\mathrm{I}}\,\theta_{r\in\mathrm{F}} \;=\; \mathbf{C}_{ij}\,v^{(ijr)}\,\theta_{j\in\mathrm{I}}\,\theta_{r\in\mathrm{I}} \;=\; 0$, it is straightforward to verify that the expression in Eq. (B.10) matches the result of $\mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R$ for all choices of mapping, showing the consistency relation.

B.2.1 Collinear limits on mappings with two initial-state partons

We show the last line of Eqs. (B.11), namely that, under the collinear $\mathbf{C}_{ij}$ limit, both $\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}\,\bar{B}^{(ijc)}_{jc}$ and $\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}\,\bar{B}^{(icj)}_{jc}$ tend to $\theta_{j\in\mathrm{I}}\,\theta_{c\in\mathrm{I}}\,B_{jc}\big(\{k\}_{\not i\not j},x_{[ij]}k_j\big)$. The proof relies on the fact that, although the two sets of momenta do not match in the limit, the colour- (as opposed to spin-) connected Born squared amplitudes depend on kinematics only through Mandelstam invariants, which do coincide in the $\mathbf{C}_{ij}$ limit, as shown below.

Considering particles j and c in the initial state, while particles i and f in the final state, we analyse the $\mathbf{C}_{ij}$ limit for the mappings (ijc) and (icj).

- Mapping (ijc)

$$\begin{aligned}\bar{k}_j &= x\,k_j\,,\\ \bar{k}_c &= k_c\,,\\ \bar{k}_f &= k_f-\frac{2k_f\cdot\big(K+\bar{K}_{(1)}\big)}{\big(K+\bar{K}_{(1)}\big)^2}\big(K+\bar{K}_{(1)}\big)+\frac{2k_f\cdot K}{K^2}\,\bar{K}_{(1)}\,,\end{aligned} \quad \text{(B.14)}$$

with

$$x=\frac{s_{jc}-s_{ij}-s_{ic}}{s_{jc}}\,,\qquad K=k_j+k_c-k_i\,,\qquad \bar{K}_{(1)}=\bar{k}_j+\bar{k}_c=xk_j+k_c\,. \quad \text{(B.15)}$$

Denoting with E_a the energy of parton a in arbitrary frame, and with r the ratio E_i/E_j, in the collinear limit one has

$$\begin{aligned}&s_{ij}\xrightarrow{\mathbf{C}_{ij}}0\,,\qquad s_{ic}\xrightarrow{\mathbf{C}_{ij}}s_{jc}\,r\,,\qquad s_{if}\xrightarrow{\mathbf{C}_{ij}}s_{jf}\,r\,,\\ &x\xrightarrow{\mathbf{C}_{ij}}1-r\,,\qquad K\xrightarrow{\mathbf{C}_{ij}}k_j(1-r)+k_c\,,\qquad \bar{K}_{(1)}\xrightarrow{\mathbf{C}_{ij}}k_j(1-r)+k_c\,,\\ &2\,\bar{k}_j\cdot\bar{k}_f\xrightarrow{\mathbf{C}_{ij}}s_{jf}\,(1-r)\,,\qquad 2\,\bar{k}_c\cdot\bar{k}_f\xrightarrow{\mathbf{C}_{ij}}s_{cf}\,,\qquad 2\,\bar{k}_j\cdot\bar{k}_c\xrightarrow{\mathbf{C}_{ij}}s_{jc}\,(1-r)\,.\end{aligned} \quad \text{(B.16)}$$

- Mapping (icj)

$$
\begin{aligned}
\bar{k}_j &= k_j \,, \\
\bar{k}_c &= x\, k_c \,, \\
\bar{k}_f &= k_f - \frac{2k_f \cdot \big(K + \bar{K}_{(2)}\big)}{\big(K + \bar{K}_{(2)}\big)^2} \big(K + \bar{K}_{(2)}\big) + \frac{2k_f \cdot K}{K^2}\, \bar{K}_{(2)} \,,
\end{aligned}
\tag{B.17}
$$

with

$$
x = \frac{s_{jc} - s_{ij} - s_{ic}}{s_{jc}} \,, \qquad K = k_j + k_c - k_i \,, \qquad \bar{K}_{(2)} = \bar{k}_j + \bar{k}_c = k_j + x k_c \,. \tag{B.18}
$$

Denoting with E_a the energy of parton a in arbitrary frame, and with r the ratio E_i/E_j, in the collinear limit one has

$$
\begin{aligned}
& s_{ij} \xrightarrow{\mathbf{C}_{ij}} 0 \,, \qquad s_{ic} \xrightarrow{\mathbf{C}_{ij}} s_{jc}\, r \,, \qquad s_{if} \xrightarrow{\mathbf{C}_{ij}} s_{jf}\, r \,, \\
& x \xrightarrow{\mathbf{C}_{ij}} 1 - r \,, \qquad K \xrightarrow{\mathbf{C}_{ij}} k_j(1-r) + k_c \,, \qquad \bar{K}_{(2)} \xrightarrow{\mathbf{C}_{ij}} k_j + k_c(1-r) \,, \\
& 2\,\bar{k}_j \cdot \bar{k}_f \xrightarrow{\mathbf{C}_{ij}} s_{jf}\,(1-r) \,, \qquad 2\,\bar{k}_c \cdot \bar{k}_f \xrightarrow{\mathbf{C}_{ij}} s_{cf} \,, \qquad 2\,\bar{k}_j \cdot \bar{k}_c \xrightarrow{\mathbf{C}_{ij}} s_{jc}\,(1-r) \,.
\end{aligned}
\tag{B.19}
$$

Invariants built with the two different momentum mappings coincide in the collinear $\mathbf{C}_{ij}$ limit. The proof of the last relation of Eq. (B.11) is completed by the fact that $\mathbf{C}_{ij}\, x \;=\; \mathbf{C}_{ij}\, x_{[ij]} \;=\; 1 - r$.

B.3 Library of NLO integrals

The analytical results collected in this Section depend on the following functions

$$
\begin{aligned}
A_1(\xi) &\equiv \gamma_E + \Psi^{(0)}(\xi + 1) \,, \\
A_2(\xi) &\equiv \gamma_E - 1 + \Psi^{(0)}(\xi + 2) \;=\; A_1(\xi + 1) - 1 \,, \\
A_3(\xi) &\equiv 1 - \zeta_2 + \Psi^{(1)}(\xi + 2) \,,
\end{aligned}
\tag{B.20}
$$

where $\xi \geqslant 0$, $\gamma_E = 0.5772156649...$ is the Euler-Mascheroni constant, $\Psi^{(n)}(z)$ is the n-th Polygamma function, namely

$$
\Psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \ln[\Gamma(z)] \,, \tag{B.21}
$$

and all functions $A_i(\xi)$ satisfy $A_i(0) = 0$.

B.3.1 Soft counterterms

For $\star\star$ taking value in $\mathrm{FF}, \mathrm{FI}, \mathrm{II}$, we define

$$I^{abc}_{\mathrm{s},\star\star} \equiv \delta_{f_a g}\, I_{\mathrm{s},\star\star}\big(\bar{s}^{(abc)}_{bc}\big)\,, \tag{B.22}$$
$$J^{abc}_{\mathrm{s},\star\star}(x) \equiv \delta_{f_a g}\, J_{\mathrm{s},\star\star}\big(\bar{s}^{(abc)}_{bc}, x\big)\,, \tag{B.23}$$

where the relevant integrals obtained integrating the soft counterterm in Eq. (2.83) are

$$\begin{aligned} I_{\mathrm{s,FF}}(s) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2+\alpha-\epsilon)}{\epsilon^2\,\Gamma(2+\alpha-3\epsilon)} \\ &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\frac{1}{\epsilon^2}+\frac{2}{\epsilon}-\frac{7\pi^2}{12}+6+2\,A_2(\alpha)\left(\frac{1}{\epsilon}+2+A_2(\alpha)\right)-4\,A_3(\alpha)+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.24}$$

$$\begin{aligned} I_{\mathrm{s,FI}}(s) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\,\Gamma(2+\alpha)}{\epsilon^2\,\Gamma(2+\alpha-2\epsilon)} \\ &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\frac{1}{\epsilon^2}+\frac{2}{\epsilon}-\frac{\pi^2}{4}+4+2\,A_2(\alpha)\left(\frac{1}{\epsilon}+2+A_2(\alpha)\right)-2\,A_3(\alpha)+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.25}$$

$$I_{\mathrm{s,II}}(s) = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\,\Gamma(2+\alpha)}{\epsilon^2\,\Gamma(2+\alpha-2\epsilon)} = I_{\mathrm{s,FI}}(s)\,; \tag{B.26}$$

$$\begin{aligned} J_{\mathrm{s,FI}}(s,x) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(2+\alpha-\epsilon)}{(-\epsilon)\,\Gamma(2+\alpha-2\epsilon)}\left(\frac{x^{1+\alpha}}{(1-x)^{1+\epsilon}}\right)_+ \\ &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[-\left(\frac{x^{1+\alpha}}{1-x}\right)_+\left(\frac{1}{\epsilon}+1+A_2(\alpha)\right)+\left(\frac{x^{1+\alpha}\ln(1-x)}{1-x}\right)_+ +\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.27}$$

$$\begin{aligned} J_{\mathrm{s,II}}(s,x) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\epsilon^2\,\Gamma(-2\epsilon)}\left(\frac{x^{1+\alpha}}{(1-x)^{1+2\epsilon}}\right)_+ \\ &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[-\left(\frac{x^{1+\alpha}}{1-x}\right)_+\frac{2}{\epsilon}+4\left(\frac{x^{1+\alpha}\ln(1-x)}{1-x}\right)_+ +\mathcal{O}(\epsilon)\right]. \end{aligned} \tag{B.28}$$

B.3.2 Collinear counterterms

When a collinear splitting occurs in the final state, the relevant integrals are

$$\begin{aligned}
I^{abc}_{\mathrm{hc,F}\star} &\equiv \delta_{\{f_a f_b\}\{q\bar q\}}\, I^{(0\mathrm{g})}_{\mathrm{hc,F}\star}\big(\bar s^{(abc)}_{bc}\big) + \big(\delta_{f_a g}\delta_{f_b\{q,\bar q\}} + \delta_{f_b g}\delta_{f_a\{q,\bar q\}}\big)\, I^{(1\mathrm{g})}_{\mathrm{hc,F}\star}\big(\bar s^{(abc)}_{bc}\big) \\
&\quad + \delta_{f_a g}\delta_{f_b g}\, I^{(2\mathrm{g})}_{\mathrm{hc,F}\star}\big(\bar s^{(abc)}_{bc}\big)\,, \\
J^{abc}_{\mathrm{hc,F}\star}(x) &\equiv \delta_{\{f_a f_b\}\{q\bar q\}}\, J^{(0\mathrm{g})}_{\mathrm{hc,F}\star}\big(\bar s^{(abc)}_{bc}, x\big) + \big(\delta_{f_a g}\delta_{f_b\{q,\bar q\}} + \delta_{f_b g}\delta_{f_a\{q,\bar q\}}\big)\, J^{(1\mathrm{g})}_{\mathrm{hc,F}\star}\big(\bar s^{(abc)}_{bc}, x\big) \\
&\quad + \delta_{f_a g}\delta_{f_b g}\, J^{(2\mathrm{g})}_{\mathrm{hc,F}\star}\big(\bar s^{(abc)}_{bc}, x\big)\,, \\
I^{abc}_{\mathrm{sc,F}\star} &\equiv \delta_{f_a g}\, 2\, C_{f_b}\, I_{\mathrm{sc,F}\star}\big(\bar s^{(abc)}_{bc}\big)\,, \\
J^{abc}_{\mathrm{sc,F}\star}(x) &\equiv \delta_{f_a g}\, 2\, C_{f_b}\, J_{\mathrm{sc,F}\star}\big(\bar s^{(abc)}_{bc}, x\big)\,,
\end{aligned} \tag{B.29}$$

while, if the splitting originates from an initial partonic state, one has

$$\begin{aligned}
J^{abc}_{\mathrm{hc,I}\star}(x) &\equiv \delta_{\{f_a f_{[ab]}\}\{q\bar q\}}\, J^{(0\mathrm{g})}_{\mathrm{hc,I}\star}\big(\bar s^{(abc)}_{bc}, x\big) + \delta_{f_a g}\delta_{f_{[ab]}\{q,\bar q\}}\, J^{(1\mathrm{g}),qg}_{\mathrm{hc,I}\star}\big(\bar s^{(abc)}_{bc}, x\big) \\
&\quad + \delta_{f_{[ab]} g}\delta_{f_a\{q,\bar q\}}\, J^{(1\mathrm{g}),gq}_{\mathrm{hc,I}\star}\big(\bar s^{(abc)}_{bc}, x\big) + \delta_{f_a g}\delta_{f_{[ab]} g}\, J^{(2\mathrm{g})}_{\mathrm{hc,I}\star}\big(\bar s^{(abc)}_{bc}, x\big)\,, \\
I^{abc}_{\mathrm{sc,I}\star} &= \delta_{f_a g}\, 2\, C_{f_b}\, I_{\mathrm{sc,I}\star}\big(\bar s^{(abc)}_{bc}\big)\,, \\
J^{abc}_{\mathrm{sc,I}\star}(x) &= \delta_{f_a g}\, 2\, C_{f_b}\, J_{\mathrm{sc,I}\star}\big(\bar s^{(abc)}_{bc}, x\big)\,,
\end{aligned} \tag{B.30}$$

where $\star = \mathrm{F}, \mathrm{I}$. Explicitly, the integrals resulting from the integration of the collinear counterterms in Eqs. (2.84, 2.85) read as follows.

- Final j, final r:

$$\begin{aligned}
I^{(0\mathrm{g})}_{\mathrm{hc,FF}}(s) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon} 4\, T_R \frac{\Gamma(2-\epsilon)^2\,\Gamma(2+\beta-2\epsilon)}{(-\epsilon)\,\Gamma(4-2\epsilon)\,\Gamma(2+\beta-3\epsilon)} \\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[-\frac{2}{3}\frac{1}{\epsilon} - \frac{16}{9} - \frac{2}{3}\, A_2(\beta) + \mathcal{O}(\epsilon)\right],
\end{aligned} \tag{B.31}$$

$$\begin{aligned}
I^{(1\mathrm{g})}_{\mathrm{hc,FF}}(s) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon} (3-2\epsilon)\, C_F \frac{\Gamma(2-\epsilon)^2\,\Gamma(2+\beta-2\epsilon)}{(-\epsilon)\,\Gamma(4-2\epsilon)\,\Gamma(2+\beta-3\epsilon)} \\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F\left[-\frac{1}{2}\frac{1}{\epsilon} - 1 - \frac{1}{2}\, A_2(\beta) + \mathcal{O}(\epsilon)\right],
\end{aligned} \tag{B.32}$$

$$\begin{aligned}
I^{(2\mathrm{g})}_{\mathrm{hc,FF}}(s) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon} 2\, C_A \frac{\Gamma(2-\epsilon)^2\,\Gamma(2+\beta-2\epsilon)}{(-\epsilon)\,\Gamma(4-2\epsilon)\,\Gamma(2+\beta-3\epsilon)} \\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\frac{1}{3}\frac{1}{\epsilon} - \frac{8}{9} - \frac{1}{3}\, A_2(\beta) + \mathcal{O}(\epsilon)\right],
\end{aligned} \tag{B.33}$$

$$\begin{aligned} I_{\rm sc,FF}(s) &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\,\Gamma(2+\beta-2\epsilon)}{\epsilon^2\,\Gamma(2+\beta-3\epsilon)}\left[\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}-\frac{\Gamma(2+\alpha-2\epsilon)}{\Gamma(2+\alpha-3\epsilon)}\right] \\ &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[-1+\frac{\pi^2}{6}-A_2(\alpha)\left(\frac{1}{\epsilon}+2+\frac{1}{2}A_2(\alpha)+A_2(\beta)\right)\right. \\ &\qquad\qquad\left.+\frac{5}{2}A_3(\alpha)+\mathcal{O}(\epsilon)\right]. \end{aligned} \tag{B.34}$$

- Final j, initial r:

$$\begin{aligned} I^{(0g)}_{\rm hc,FI}(s) &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}4\,T_R\frac{\Gamma(2-\epsilon)^2\,\Gamma(2+\beta)}{(-\epsilon)\,\Gamma(4-2\epsilon)\,\Gamma(2+\beta-\epsilon)} \\ &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}T_R\left[-\frac{2}{3}\frac{1}{\epsilon}-\frac{16}{9}-\frac{2}{3}A_2(\beta)+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.35}$$

$$\begin{aligned} I^{(1g)}_{\rm hc,FI}(s) &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}(3-2\epsilon)\,C_F\frac{\Gamma(2-\epsilon)^2\,\Gamma(2+\beta)}{(-\epsilon)\,\Gamma(4-2\epsilon)\,\Gamma(2+\beta-\epsilon)} \\ &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\left[-\frac{1}{2}\frac{1}{\epsilon}-1-\frac{1}{2}A_2(\beta)+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.36}$$

$$\begin{aligned} I^{(2g)}_{\rm hc,FI}(s) &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}2\,C_A\frac{\Gamma(2-\epsilon)^2\,\Gamma(2+\beta)}{(-\epsilon)\,\Gamma(4-2\epsilon)\,\Gamma(2+\beta-\epsilon)} \\ &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_A\left[-\frac{1}{3}\frac{1}{\epsilon}-\frac{8}{9}-\frac{1}{3}A_2(\beta)+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.37}$$

$$\begin{aligned} J^{(0g)}_{\rm hc,FI}(s,x) &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}4\,T_R\frac{(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(4-2\epsilon)}\left(\frac{x^{1+\beta}}{(1-x)^{1+\epsilon}}\right)_+ \\ &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}T_R\left(\frac{x^{1+\beta}}{1-x}\right)_+\left[\frac{2}{3}+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.38}$$

$$\begin{aligned} J^{(1g)}_{\rm hc,FI}(s,x) &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}(3-2\epsilon)\,C_F\frac{(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(4-2\epsilon)}\left(\frac{x^{1+\beta}}{(1-x)^{1+\epsilon}}\right)_+ \\ &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\left(\frac{x^{1+\beta}}{1-x}\right)_+\left[\frac{1}{2}+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.39}$$

$$\begin{aligned} J^{(2g)}_{\rm hc,FI}(s,x) &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}2\,C_A\frac{(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(4-2\epsilon)}\left(\frac{x^{1+\beta}}{(1-x)^{1+\epsilon}}\right)_+ \\ &= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_A\left(\frac{x^{1+\beta}}{1-x}\right)_+\left[\frac{1}{3}+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.40}$$

$$I_{\rm sc,FI}(s) = \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\,\Gamma(2+\beta)}{\epsilon^2\,\Gamma(2+\beta-\epsilon)}\left[\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}-\frac{\Gamma(2+\alpha-\epsilon)}{\Gamma(2+\alpha-2\epsilon)}\right]$$
$$= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\bigg[-A_2(\alpha)\bigg(\frac{1}{\epsilon}+2+\frac{1}{2}A_2(\alpha)+A_2(\beta)\bigg) + \frac{3}{2}A_3(\alpha)+\mathcal{O}(\epsilon)\bigg], \tag{B.41}$$

$$J_{\rm sc,FI}(s,x) = \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\left(-\frac{1}{\epsilon}\right)\left[\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}-\frac{\Gamma(2+\alpha-\epsilon)}{\Gamma(2+\alpha-2\epsilon)}\right]\left(\frac{x^{1+\beta}}{(1-x)^{1+\epsilon}}\right)_+$$
$$= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left(\frac{x^{1+\beta}}{1-x}\right)_+\Big[A_2(\alpha)+\mathcal{O}(\epsilon)\Big]. \tag{B.42}$$

- Initial j, final r:

$$J^{(0g)}_{\rm hc,IF}(s,x) = \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}T_R\left(1-\frac{2x(1-x)}{1-\epsilon}\right)\frac{(1-x)^{-\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)} \tag{B.43}$$
$$= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}T_R\bigg[\Big(x^2+(1-x)^2\Big)\bigg(-\frac{1}{\epsilon}+\ln(1-x)-A_1(\gamma)\bigg) + 2\,x(1-x)+\mathcal{O}(\epsilon)\bigg],$$

$$J^{(1g),qg}_{\rm hc,IF}(s,x) = \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}C_F\,(1-x)\,(1-\epsilon)\frac{(1-x)^{-\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)} \tag{B.44}$$
$$= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\,(1-x)\left[-\frac{1}{\epsilon}+\ln(1-x)+1-A_1(\gamma)+\mathcal{O}(\epsilon)\right],$$

$$J^{(1g),gq}_{\rm hc,IF}(s,x) = \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}C_F\left(\frac{1+(1-x)^2}{x}-\epsilon x\right)\frac{(1-x)^{-\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)} \tag{B.45}$$
$$= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\left[\frac{1+(1-x)^2}{x}\left(-\frac{1}{\epsilon}+\ln(1-x)-A_1(\gamma)\right)+x+\mathcal{O}(\epsilon)\right],$$

$$J^{(1g)}_{\rm hc,IF}(s,x) \equiv J^{(1g),qg}_{\rm hc,IF}(s,x)+J^{(1g),gq}_{\rm hc,IF}(s,x) \tag{B.46}$$
$$= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}C_F\left(\frac{2}{x}-1-\epsilon\right)\frac{(1-x)^{-\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)}$$
$$= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\left[\left(\frac{2}{x}-1\right)\left(-\frac{1}{\epsilon}+\ln(1-x)-A_1(\gamma)\right)+1+\mathcal{O}(\epsilon)\right],$$

$$J^{(2g)}_{\rm hc,IF}(s,x) = \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}2\,C_A\left(\frac{1-x}{x}+x(1-x)\right)\frac{(1-x)^{-\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)} \tag{B.47}$$
$$= \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}2\,C_A\left(\frac{1-x}{x}+x(1-x)\right)\left[-\frac{1}{\epsilon}+\ln(1-x)-A_1(\gamma)+\mathcal{O}(\epsilon)\right],$$

$$
\begin{aligned}
I_{\mathrm{sc,IF}}(s) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\,\Gamma(1+\gamma-\epsilon)}{\epsilon^2\,\Gamma(1+\gamma-2\epsilon)}\left[\frac{1}{\Gamma(2-\epsilon)}-\frac{\Gamma(2+\alpha)}{\Gamma(2+\alpha-\epsilon)}\right] \qquad \text{(B.48)}\\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[-A_2(\alpha)\left(\frac{1}{\epsilon}+1+\frac{1}{2}A_2(\alpha)+A_1(\gamma)\right)+\frac{1}{2}A_3(\alpha)+\mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$
\begin{aligned}
J_{\mathrm{sc,IF}}(s,x) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)}\left(\frac{x(1-x^\alpha)}{(1-x)^{1+\epsilon}}\right)_+ \qquad \text{(B.49)}\\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\left(\frac{x(x^\alpha-1)}{1-x}\right)_+\left(\frac{1}{\epsilon}+A_1(\gamma)\right)\right.\\
&\qquad\left.+\left(\frac{x(1-x^\alpha)\ln(1-x)}{1-x}\right)_+ +\mathcal{O}(\epsilon)\right].
\end{aligned}
$$

- Initial j, initial r:

$$
\begin{aligned}
J^{(0\mathrm{g})}_{\mathrm{hc,II}}(s,x) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}T_R\left(1-\frac{2x(1-x)}{1-\epsilon}\right)\frac{(1-x)^{-2\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)} \qquad \text{(B.50)}\\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}T_R\left[\left(x^2+(1-x)^2\right)\left(-\frac{1}{\epsilon}+2\ln(1-x)-A_1(\gamma)\right)\right.\\
&\qquad\left.+2\,x(1-x)+\mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$
\begin{aligned}
J^{(1\mathrm{g}),qg}_{\mathrm{hc,II}}(s,x) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}C_F\,(1-x)\,(1-\epsilon)\frac{(1-x)^{-2\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)} \qquad \text{(B.51)}\\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\,(1-x)\left[-\frac{1}{\epsilon}+2\ln(1-x)+1-A_1(\gamma)+\mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$
\begin{aligned}
J^{(1\mathrm{g}),gq}_{\mathrm{hc,II}}(s,x) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}C_F\left(\frac{1+(1-x)^2}{x}-\epsilon x\right)\frac{(1-x)^{-2\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)} \qquad \text{(B.52)}\\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\left[\frac{1+(1-x)^2}{x}\left(-\frac{1}{\epsilon}+2\ln(1-x)-A_1(\gamma)\right)+x+\mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$
\begin{aligned}
J^{(1\mathrm{g})}_{\mathrm{hc,II}}(s,x) &\equiv J^{(1\mathrm{g}),qg}_{\mathrm{hc,II}}(s,x)+J^{(1\mathrm{g}),gq}_{\mathrm{hc,II}}(s,x) \qquad \text{(B.53)}\\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}C_F\left(\frac{2}{x}-1-\epsilon\right)\frac{(1-x)^{-2\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)}\\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\left[\left(\frac{2}{x}-1\right)\left(-\frac{1}{\epsilon}+2\ln(1-x)-A_1(\gamma)\right)+1+\mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$
\begin{aligned}
J^{(2\mathrm{g})}_{\mathrm{hc,II}}(s,x) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}2\,C_A\left(\frac{1-x}{x}+x(1-x)\right)\frac{(1-x)^{-2\epsilon}\,\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)} \qquad \text{(B.54)}\\
&= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}2\,C_A\left(\frac{1-x}{x}+x(1-x)\right)\left[-\frac{1}{\epsilon}+2\ln(1-x)-A_1(\gamma)+\mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$\begin{aligned} I_{\mathrm{sc,II}}(s) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-2\epsilon)\,\Gamma(1+\gamma-\epsilon)}{2\epsilon^2\,\Gamma(1+\gamma-2\epsilon)}\left[\frac{1}{\Gamma(2-2\epsilon)}-\frac{\Gamma(2+\alpha)}{\Gamma(2+\alpha-2\epsilon)}\right] \\ &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[-A_2(\alpha)\left(\frac{1}{\epsilon}+2+A_2(\alpha)+A_1(\gamma)\right)+A_3(\alpha)+\mathcal{O}(\epsilon)\right], \end{aligned} \tag{B.55}$$

$$\begin{aligned} J_{\mathrm{sc,II}}(s,x) &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\gamma-\epsilon)}{(-\epsilon)\,\Gamma(1+\gamma-2\epsilon)}\left(\frac{x(1-x^{\alpha})}{(1-x)^{1+2\epsilon}}\right)_+ \\ &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\left(\frac{x(x^{\alpha}-1)}{1-x}\right)_+\left(\frac{1}{\epsilon}+A_1(\gamma)\right)\right. \\ &\qquad \left.+2\left(\frac{x(1-x^{\alpha})\ln(1-x)}{1-x}\right)_+ +\mathcal{O}(\epsilon)\right]. \end{aligned} \tag{B.56}$$

Appendix C

NNLO Appendices

C.1 Infrared kernels

C.1.1 Soft kernels at tree level

We introduce the kernels associated with the real emission of one or two soft partons, as given in Ref. [200], relevant for both NLO (with the emission of just one parton) and NNLO corrections (with the emission of either one or two partons). We express all kernels in terms of Lorentz-invariant quantities, and using the flavour structures defined in Appendix A. The resulting expressions are

$$\mathcal{I}_{cd}^{(i)} \;=\; f_i^g \,\frac{s_{cd}}{s_{ic}\,s_{id}}\,, \qquad\qquad \mathcal{I}_{cd}^{(ij)} \;=\; f_{ij}^{q\bar{q}}\,2\,T_R\,\mathcal{I}_{cd}^{(q\bar{q})(ij)} - f_{ij}^{gg}\,2\,C_A\,\mathcal{I}_{cd}^{(gg)(ij)}\,, \tag{C.1}$$

where

$$\mathcal{I}_{cd}^{(q\bar{q})(ij)} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2\,s_{[ij]c}\,s_{[ij]d}}\,, \tag{C.2}$$

$$\begin{aligned}\mathcal{I}_{cd}^{(gg)(ij)} = &\ \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2s_{ij}s_{cd}}{s_{ij}^2\,s_{[ij]c}\,s_{[ij]d}} \\ &+ s_{cd}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{s_{[ij]c}\,s_{[ij]d}}\right].\end{aligned}$$

We also define the combinations of eikonal kernels

$$\mathcal{E}_{cd}^{(i)} \equiv \mathcal{I}_{cd}^{(i)} \;=\; f_i^g\,\frac{s_{cd}}{s_{ic}\,s_{id}}\,, \tag{C.3}$$

$$\mathcal{E}_{cd}^{(ij)} \equiv \mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\,\mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\,\mathcal{I}_{dd}^{(ij)} \;=\; f_{ij}^{q\bar{q}}\,2\,T_R\,\mathcal{E}_{cd}^{(q\bar{q})(ij)} - f_{ij}^{gg}\,2\,C_A\,\mathcal{E}_{cd}^{(gg)(ij)}\,, \tag{C.4}$$

with

$$\mathcal{E}_{cd}^{(q\bar{q})(ij)} = \frac{1}{s_{ij}^2}\left[\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{s_{[ij]c}s_{[ij]d}} - \frac{s_{ic}s_{jc}}{s_{[ij]c}^2} - \frac{s_{id}s_{jd}}{s_{[ij]d}^2}\right] - \frac{s_{cd}}{s_{ij}s_{[ij]c}s_{[ij]d}}\,,$$

$$\mathcal{E}_{cd}^{(gg)(ij)} = \frac{1-\epsilon}{s_{ij}^2}\left[\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{s_{[ij]c}s_{[ij]d}} - \frac{s_{ic}s_{jc}}{s_{[ij]c}^2} - \frac{s_{id}s_{jd}}{s_{[ij]d}^2}\right] - 2\,\frac{s_{cd}}{s_{ij}s_{[ij]c}s_{[ij]d}}$$
$$+ s_{cd}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}-s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1-\frac{1}{2}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{s_{[ij]c}s_{[ij]d}}\right]. \qquad \text{(C.5)}$$

C.1.2 Soft kernels at one loop

We introduce kernels associated with the emission of a single-soft gluon at one-loop level, relevant for the soft part of the real-virtual counterterm at NNLO,

$$\tilde{\mathcal{E}}_{cd}^{(i)} \equiv f_i^g\, C_A\,\frac{\Gamma^3(1+\epsilon)\Gamma^4(1-\epsilon)}{\epsilon^2\,\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}\,\frac{s_{cd}}{s_{ic}s_{id}}\left(\frac{e^{\gamma_E}\mu^2 s_{cd}}{s_{ic}s_{id}}\right)^{\epsilon}$$
$$= C_A\,\mathcal{E}_{cd}^{(i)}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{s_{ic}s_{id}}{\mu^2\,s_{cd}}-\frac{5}{2}\zeta_2+\frac{1}{2}\ln^2\frac{s_{ic}s_{id}}{\mu^2\,s_{cd}}+\mathcal{O}(\epsilon)\right],$$
$$\tilde{\mathcal{E}}_{cde}^{(i)} \equiv f_i^g\,\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)}\,\frac{s_{cd}}{s_{ic}s_{id}}\left(\frac{e^{\gamma_E}\mu^2\,s_{de}}{s_{id}s_{ie}}\right)^{\epsilon}$$
$$= \mathcal{E}_{cd}^{(i)}\left[\frac{1}{\epsilon}-\ln\frac{s_{id}s_{ie}}{\mu^2\,s_{de}}+\mathcal{O}(\epsilon)\right], \qquad \text{(C.6)}$$

where ϵ is the dimensional regulator ($d = 4-2\epsilon$).

C.1.3 Collinear and hard-collinear kernels at tree level

In order to define the kernel associated with the tree-level emission of two collinear final-state particles i and j (labelled *single-collinear*), we choose a reference momentum k_r, with $r \neq i, j$, and introduce the following kinematic structures:

$$x_i = \frac{s_{ir}}{s_{[ij]r}}\,, \qquad x_j = \frac{s_{jr}}{s_{[ij]r}}\,, \qquad \tilde{k}_i = x_i\,k_j - x_j\,k_i - (1-2x_j)\frac{s_{ij}}{s_{[ij]r}}\,k_r\,. \qquad \text{(C.7)}$$

Then, the collinear (Altarelli-Parisi) kernels $P_{ij(r)}^{\mu\nu}$ are defined as

$$P_{ij(r)}^{\mu\nu} = -P_{ij(r)}\,g^{\mu\nu} + Q_{ij(r)}^{\mu\nu}\,, \qquad Q_{ij(r)}^{\mu\nu} = Q_{ij(r)}\,d_i^{\mu\nu}\,, \qquad \text{(C.8)}$$

where the azimuthal tensor reads

$$d_i^{\mu\nu} = -g^{\mu\nu} + (d-2)\,\frac{\tilde{k}_i^{\mu}\tilde{k}_i^{\nu}}{\tilde{k}_i^2}\,, \qquad \text{(C.9)}$$

and

$$
\begin{aligned}
P_{ij(r)} &= P^{(0g)}_{ij(r)}\, f^{q\bar q}_{ij} + P^{(1g)}_{ij(r)}\, f^{g}_{i}(f^{q}_{j}+f^{\bar q}_{j}) + P^{(1g)}_{ji(r)}\,(f^{q}_{i}+f^{\bar q}_{i})f^{g}_{j} + P^{(2g)}_{ij(r)}\, f^{gg}_{ij}\,, \qquad \text{(C.10)}\\
Q_{ij(r)} &= T_R\,\frac{2x_ix_j}{1-\epsilon}\, f^{q\bar q}_{ij} - 2\,C_A\, x_ix_j\, f^{gg}_{ij}\,,\\
P^{(0g)}_{ij(r)} &= T_R\left(1-\frac{2x_ix_j}{1-\epsilon}\right),\\
P^{(1g)}_{ij(r)} &= C_F\left[2\,\frac{x_j}{x_i} + (1-\epsilon)x_i\right],\\
P^{(2g)}_{ij(r)} &= 2\,C_A\left(\frac{x_i}{x_j} + \frac{x_j}{x_i} + x_ix_j\right).
\end{aligned}
$$

The hard-collinear kernels $P^{\mathrm{hc},\mu\nu}_{ij(r)}$ are defined as

$$
P^{\mathrm{hc},\mu\nu}_{ij(r)} \equiv P^{\mu\nu}_{ij(r)} + s_{ij}\Big[2\,C_{f_j}\,\mathcal{E}^{(i)}_{jr} + 2\,C_{f_i}\,\mathcal{E}^{(j)}_{ir}\Big]g^{\mu\nu} \equiv -\,P^{\mathrm{hc}}_{ij(r)}\,g^{\mu\nu} + Q^{\mu\nu}_{ij(r)}\,, \qquad \text{(C.11)}
$$

where

$$
P^{\mathrm{hc}}_{ij(r)} = P^{\mathrm{hc},(0g)}_{ij(r)}\, f^{q\bar q}_{ij} + P^{\mathrm{hc},(1g)}_{ij(r)}\, f^{g}_{i}(f^{q}_{j}+f^{\bar q}_{j}) + P^{\mathrm{hc},(1g)}_{ji(r)}\,(f^{q}_{i}+f^{\bar q}_{i})f^{g}_{j} + P^{\mathrm{hc},(2g)}_{ij(r)}\, f^{gg}_{ij}\,, \qquad \text{(C.12)}
$$

$$
P^{\mathrm{hc},(0g)}_{ij(r)} = P^{(0g)}_{ij(r)} = T_R\left(1-\frac{2x_ix_j}{1-\epsilon}\right), \quad P^{\mathrm{hc},(1g)}_{ij(r)} = C_F(1-\epsilon)x_i\,, \quad P^{\mathrm{hc},(2g)}_{ij(r)} = 2\,C_A\,x_ix_j\,.
$$

The kernel describing the emission of three collinear final-state partons i, j and k (labelled *double-collinear*) relies on the choice of a reference momentum k_r, with $r \neq i,j,k$, and on the following kinematic structures,

$$
\begin{aligned}
z_a &= \frac{s_{ar}}{s_{[ijk]r}}\,, \qquad z_{ab} = z_a + z_b\,, \qquad a,b = i,j,k \qquad \text{(C.13)}\\
\tilde k^\mu_a &= k^\mu_a - z_a(k^\mu_i + k^\mu_j + k^\mu_k) - (s_{[ijk]a} - 2\,z_a s_{ijk})\frac{k^\mu_r}{s_{[ijk]r}}\,, \qquad a,b,c = i,j,k\,,\\
\tilde k^2_a &= z_a(z_a s_{ijk} - s_{[ijk]a}) = z_a(s_{bc} - z_{bc}s_{ijk})\,.
\end{aligned}
$$

The double-collinear kernels $P^{\mu\nu}_{ijk(r)}$ are defined as

$$
P^{\mu\nu}_{ijk(r)} \equiv -P_{ijk(r)}\,g^{\mu\nu} + Q^{\mu\nu}_{ijk(r)}\,, \qquad Q^{\mu\nu}_{ijk(r)} = \sum_{a=i,j,k} Q^{a}_{ijk(r)}\,d^{\mu\nu}_a\,. \qquad \text{(C.14)}
$$

The $P_{ijk(r)}$ kernels, organised by flavour structures, are given by

$$\begin{aligned} P_{ijk(r)} &= P^{(0g)}_{ijk(r)}\, f^{q\bar q}_{ij}\,(f^{q'}_k + f^{\bar q'}_k) + P^{(0g)}_{jki(r)}\, f^{q\bar q}_{jk}\,(f^{q'}_i + f^{\bar q'}_i) + P^{(0g)}_{kij(r)}\, f^{q\bar q}_{ik}\,(f^{q'}_j + f^{\bar q'}_j) \\ &+ P^{(0g,id)}_{ijk(r)}(f^q_i f^q_j f^{\bar q}_k + f^{\bar q}_i f^{\bar q}_j f^q_k) + P^{(0g,id)}_{jki(r)}(f^q_j f^q_k f^{\bar q}_i + f^{\bar q}_j f^{\bar q}_k f^q_i) + P^{(0g,id)}_{kij(r)}(f^q_i f^q_k f^{\bar q}_j + f^{\bar q}_i f^{\bar q}_k f^q_j) \\ &+ P^{(1g)}_{ijk(r)}\, f^{q\bar q}_{ij}\, f^g_k + P^{(1g)}_{jki(r)}\, f^{q\bar q}_{jk}\, f^g_i + P^{(1g)}_{kij(r)}\, f^{q\bar q}_{ik}\, f^g_j \\ &+ P^{(2g)}_{ijk(r)}\, f^{gg}_{ij}\,(f^q_k + f^{\bar q}_k) + P^{(2g)}_{jki(r)}\, f^{gg}_{jk}\,(f^q_i + f^{\bar q}_i) + P^{(2g)}_{kij(r)}\, f^{gg}_{ik}\,(f^q_j + f^{\bar q}_j) \\ &+ P^{(3g)}_{ijk(r)}\, f^{ggg}_{ijk}\,, \end{aligned} \tag{C.15}$$

where q' is a quark of flavour equal to or different from that of q; similarly, the azimuthal tensor kernel can be written as

$$Q^a_{ijk(r)} = Q^{(1g),a}_{ijk(r)}\, f^{q\bar q}_{ij}\, f^g_k + Q^{(1g),a}_{jki(r)}\, f^{q\bar q}_{jk}\, f^g_i + Q^{(1g),a}_{kij(r)}\, f^{q\bar q}_{ik}\, f^g_j + Q^{(3g),a}_{ijk(r)}\, f^{ggg}_{ijk}\,. \tag{C.16}$$

The expressions for $P^{(0g)}_{ijk(r)}$, $P^{(0g,id)}_{ijk(r)}$, $P^{(1g)}_{ijk(r)}$, $P^{(2g)}_{ijk(r)}$, and $P^{(3g)}_{ijk(r)}$ read:

$$\begin{aligned} P^{(0g)}_{ijk(r)} &= C_F T_R \Bigg\{ -\frac{s^2_{ijk}}{2s^2_{ij}}\left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}}\right)^2 \\ &\qquad + \frac{s_{ijk}}{s_{ij}}\left[2\,\frac{z_k - z_i z_j}{z_{ij}} + (1-\epsilon) z_{ij}\right] - \frac{1}{2} + \epsilon \Bigg\}\,, \end{aligned} \tag{C.17}$$

$$\begin{aligned} P^{(0g,id)}_{ijk(r)} &= C_F(2C_F - C_A)\Bigg\{ -\frac{s^2_{ijk}\, z_k}{2 s_{jk} s_{ik}}\left[\frac{1+z_k^2}{z_{jk} z_{ik}} - \epsilon\left(\frac{z_{ik}}{z_{jk}} + \frac{z_{jk}}{z_{ik}} + 1 + \epsilon\right)\right] \\ &\qquad + (1-\epsilon)\left[\frac{s_{ij}}{s_{jk}} + \frac{s_{ij}}{s_{ik}} - \epsilon\right] \\ &\qquad + \frac{s_{ijk}}{2s_{jk}}\left[\frac{1 + z_k^2 - \epsilon z_{jk}^2}{z_{ik}} - 2(1-\epsilon)\frac{z_j}{z_{jk}} - \epsilon(1+z_k) - \epsilon^2 z_{jk}\right] \\ &\qquad + \frac{s_{ijk}}{2s_{ik}}\left[\frac{1 + z_k^2 - \epsilon z_{ik}^2}{z_{jk}} - 2(1-\epsilon)\frac{z_i}{z_{ik}} - \epsilon(1+z_k) - \epsilon^2 z_{ik}\right]\Bigg\}\,, \end{aligned} \tag{C.18}$$

$$\begin{aligned} P^{(1g)}_{ijk(r)} &= C_F T_R\left[\frac{2 s_{ijk} s_{ij}}{s_{ik} s_{jk}} + (1-\epsilon)\left(\frac{s_{ik}}{s_{jk}} + \frac{s_{jk}}{s_{ik}} + 2\right) - 2\right] \\ &+ C_A T_R\Bigg[-\frac{s^2_{ijk}}{2s^2_{ij}}\left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}}\right)^2 - \frac{s^2_{ijk}}{s_{ik}s_{jk}} + \frac{s^2_{ijk}}{2s_{ij}}\,\frac{1-2z_k}{z_k z_{ij}}\left(\frac{z_i}{s_{ik}} + \frac{z_j}{s_{jk}}\right) \\ &\qquad + \frac{s_{ijk}}{2 z_k z_{ij}}\left(\frac{z_{ik}}{s_{ik}} + \frac{z_{jk}}{s_{jk}}\right) + \frac{s_{ijk}}{s_{ij}}\,\frac{1 - z_k + 2z_k}{z_k z_{ij}} - \frac{1}{2} + \epsilon\Bigg] \\ &- \sum_{a=i,j,k} Q^a_{ijk(r)}\,, \end{aligned} \tag{C.19}$$

$$
\begin{aligned}
P^{(2g)}_{ijk(r)} &= C_F^2 \left\{ \frac{s_{ijk}^2\, z_k}{2 s_{ik} s_{jk}} \left[\frac{1+z_k^2-\epsilon z_{ij}^2}{z_i z_j} + \epsilon(1-\epsilon) \right] - (1-\epsilon)^2\, \frac{s_{jk}}{s_{ik}} + \epsilon(1-\epsilon) \right. \\
&\qquad \left. + \frac{s_{ijk}}{s_{ik}} \left[\frac{z_k z_{jk} + z_{ik}^3 - \epsilon z_{ik} z_{ij}^2}{z_i z_j} + \epsilon\, z_{ik} + \epsilon^2 \left(1+z_k\right) \right] \right\} \\
&\quad + C_F C_A \left\{ (1-\epsilon) \frac{s_{ijk}^2}{4 s_{ij}^2} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 \right. \\
&\qquad - \frac{s_{ijk}^2\, z_k}{4 s_{ik} s_{jk}} \left[\frac{z_{ij}^2(1-\epsilon) + 2 z_k}{z_i z_j} + \epsilon(1-\epsilon) \right] \\
&\qquad + \frac{s_{ijk}^2}{2 s_{ij} s_{ik}} \left[\frac{z_{ij}^2(1-\epsilon) + 2 z_k}{z_j} + \frac{z_j^2(1-\epsilon) + 2 z_{ik}}{z_{ij}} \right] + \frac{1}{4}(1-\epsilon)(1-2\epsilon) \\
&\qquad + \frac{s_{ijk}}{2 s_{ik}} \left[(1-\epsilon) \frac{z_{ik}^3 + z_k^2 - z_j}{z_j z_{ij}} - 2\epsilon\, \frac{z_{ik}(z_j - z_k)}{z_j z_{ij}} \right. \\
&\qquad\qquad \left. - \frac{z_k z_{jk} + z_{ik}^3}{z_i z_j} + \epsilon\, z_{ik} \frac{z_{ij}^2}{z_i z_j} - \epsilon(1+z_k) - \epsilon^2 z_{ik} \right] \\
&\qquad \left. + \frac{s_{ijk}}{2 s_{ij}} \left[(1-\epsilon) \frac{z_i(2 z_{jk} + z_i^2) - z_j(6 z_{ik} + z_j^2)}{z_j z_{ij}} + 2\epsilon\, \frac{z_k(z_i - 2 z_j) - z_j}{z_j z_{ij}} \right] \right\} \\
&\quad + (i \leftrightarrow j)\,,
\end{aligned}
\tag{C.20}
$$

$$
\begin{aligned}
P^{(3g)}_{ijk(r)} &= C_A^2 \left\{ (1-\epsilon) \frac{s_{ijk}^2}{4 s_{ij}^2} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{3}{4}(1-\epsilon) \right. \\
&\qquad + \frac{s_{ijk}^2}{2 s_{ij} s_{ik}} \left[\frac{2 z_i z_j z_{ik}(1-2z_k)}{z_k z_{ij}} + \frac{1+2z_i+2z_i^2}{z_{ik} z_{ij}} + \frac{1-2 z_i z_{jk}}{z_j z_k} \right. \\
&\qquad\qquad \left. + 2 z_j z_k + z_i(1+2z_i) - 4 \right] \\
&\qquad \left. + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2} \right] \right\} \\
&\quad + (\text{ 5 permutations })\,.
\end{aligned}
\tag{C.21}
$$

The azimuthal kernels $Q^{(1g),a}_{ijk(r)}$ and $Q^{(3g),a}_{ijk(r)}$ are defined according to the following expressions:

$$
\begin{aligned}
Q^{(1g),i}_{ijk(r)} = T_R \frac{\tilde{k}_i^2}{1-\epsilon} \frac{s_{ijk}}{s_{ik} s_{jk}} \Bigg\{ C_A \Bigg[& 1 - \frac{2 z_j}{z_k} \frac{s_{ij} + 2 s_{jk}}{s_{ij}^2} s_{ik} \\
& + \frac{z_i s_{jk} + z_j s_{ik}}{z_{ij} s_{ij}} + \left(\frac{z_i z_j}{z_k z_{ij}} - \frac{1-\epsilon}{2} \right) \frac{s_{ik} - s_{jk}}{s_{ij}} \Bigg] - 2 C_F \Bigg\},
\end{aligned}
$$

$$Q^{(1g),j}_{ijk(r)} = T_R \frac{\tilde{k}_j^2}{1-\epsilon}\frac{s_{ijk}}{s_{ik}s_{jk}}\bigg\{C_A\bigg[1-\frac{2z_i}{z_k}\frac{s_{ij}+2s_{ik}}{s_{ij}^2}s_{jk}+\frac{z_i s_{jk}+z_j s_{ik}}{z_{ij}s_{ij}} + \bigg(\frac{z_i z_j}{z_k z_{ij}}-\frac{1-\epsilon}{2}\bigg)\frac{s_{jk}-s_{ik}}{s_{ij}}\bigg]-2C_F\bigg\},$$

$$Q^{(1g),k}_{ijk(r)} = T_R \frac{\tilde{k}_k^2}{1-\epsilon}\frac{s_{ijk}}{s_{ik}s_{jk}}\bigg\{C_A\bigg[\frac{z_i z_j}{z_k z_{ij}}\frac{4s_{ik}s_{jk}+s_{ij}s_{[ij]k}}{s_{ij}^2}+\frac{z_i-z_j}{2z_{ij}}\frac{s_{ik}-s_{jk}}{s_{ij}}-\epsilon\,\frac{s_{ijk}+s_{ij}}{2s_{ij}}\bigg]+2C_F\,\epsilon\bigg\},$$

$$\sum_{a=i,j,k} Q^{(3g),a}_{ijk(r)} d_a^{\mu\nu} = C_A^2\,\frac{s_{ijk}}{s_{ij}}\bigg\{\ \bigg[\frac{2z_j}{z_k}\frac{1}{s_{ij}}+\bigg(\frac{z_j z_{ik}}{z_k z_{ij}}-\frac{3}{2}\bigg)\frac{1}{s_{ik}}\bigg]\,\tilde{k}_i^2\,d_i^{\mu\nu} + \bigg[\frac{2z_i}{z_k}\frac{1}{s_{ij}}-\bigg(\frac{z_j z_{ik}}{z_k z_{ij}}-\frac{3}{2}-\frac{z_i}{z_k}+\frac{z_i}{z_{ij}}\bigg)\frac{1}{s_{ik}}\bigg]\,\tilde{k}_j^2\,d_j^{\mu\nu} - \bigg[\frac{2z_i z_j}{z_{ij}z_k}\frac{1}{s_{ij}}+\bigg(\frac{z_j z_{ik}}{z_k z_{ij}}-\frac{3}{2}-\frac{z_i}{z_j}+\frac{z_i}{z_{ik}}\bigg)\frac{1}{s_{ik}}\bigg]\,\tilde{k}_k^2\,d_k^{\mu\nu}\bigg\} + (5\text{ permutations})\,. \tag{C.22}$$

The hard-double-collinear kernels $P^{\text{hc},\mu\nu}_{ijk(r)}$ are defined as

$$P^{\text{hc},\mu\nu}_{ijk(r)} \equiv -P^{\text{hc}}_{ijk(r)}\,g^{\mu\nu} + Q^{\mu\nu}_{ijk(r)}\,, \tag{C.23}$$

where $Q^{\mu\nu}_{ijk(r)}$ is given in Eq. (C.14) and

$$P^{\text{hc}}_{ijk(r)} \equiv P_{ijk(r)} - s_{ijk}^2\Big[C_{f_k}\Big(4\,C_{f_k}\mathcal{E}^{(i)}_{kr}\mathcal{E}^{(j)}_{kr} - \mathcal{E}^{(ij)}_{kr}\Big) + (i\leftrightarrow k) + (j\leftrightarrow k)\Big]. \tag{C.24}$$

C.1.4 Collinear and hard-collinear kernels at one loop

The collinear contribution to the real-virtual counterterm at NNLO depends on the one-loop, single-collinear kernel which reads $(r \neq i,j)$:

$$\tilde{P}^{\mu\nu}_{ij(r)} \equiv \frac{\Gamma^2(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}\bigg(\frac{e^{\gamma_E}\mu^2}{s_{ij}}\bigg)^{\epsilon}\bigg\{\frac{C_{f_{[ij]}}}{\epsilon^2}\Big[\rho^{(C)}_{[ij]}+\rho^{(C)}_{ij}F(x_i)+\rho^{(C)}_{ji}F(x_j)\Big]P^{\mu\nu}_{ij(r)} + \hat{P}^{\mu\nu}_{ij(r)}\bigg\}, \tag{C.25}$$

where the function $F(x)$ is defined by

$$F(x) \equiv 1 - {}_2F_1\bigg(1,-\epsilon;1-\epsilon;\frac{x-1}{x}\bigg) = \epsilon\,\ln x + \sum_{n=2}^{+\infty}\epsilon^n\,\text{Li}_n\bigg(\frac{x-1}{x}\bigg)\,, \tag{C.26}$$

and $\hat{P}^{\mu\nu}_{ij(r)}$ reads

$$\hat{P}^{\mu\nu}_{ij(r)} = \left[-g_{\mu\nu} + 4x_i x_j \frac{\tilde{k}_i^\mu \tilde{k}_i^\nu}{\tilde{k}_i^2} \right] \frac{T_R}{1-2\epsilon} \left[\frac{1}{\epsilon}\big(\beta_0 - 3\,C_F\big) + C_A - 2C_F + \frac{C_A + 4\,T_R\,N_f}{3(3-2\epsilon)} \right] f_{ij}^{q\bar{q}}$$
$$- g_{\mu\nu}\, C_F\, \frac{C_A - C_F}{1-2\epsilon} \Big[\big(1 - \epsilon x_i\big) f_i^g (f_j^q + f_j^{\bar{q}}) + \big(1 - \epsilon x_j\big)(f_i^q + f_i^{\bar{q}}) f_j^g \Big]$$
$$+ \frac{\tilde{k}_i^\mu \tilde{k}_i^\nu}{\tilde{k}_i^2}\, 4\,C_A\, \frac{2\,T_R\,N_f - C_A(1-\epsilon)}{(1-2\epsilon)(2-2\epsilon)(3-2\epsilon)} \big(1 - 2\epsilon x_i x_j\big) f_{ij}^{gg}\,. \tag{C.27}$$

The expansion of $\tilde{P}^{\mu\nu}_{ij(r)}$ in the dimensional regulator ϵ gives

$$\tilde{P}^{\mu\nu}_{ij(r)} = P^{\mu\nu}_{ij(r)}\, C_{f_{[ij]}} \Bigg\{ \rho^{(C)}_{[ij]} \Bigg[\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{s_{ij}}{\mu^2} - \frac{1}{2}\bigg(7\,\zeta_2 - \ln^2 \frac{s_{ij}}{\mu^2} \bigg) \Bigg]$$
$$+ \left[\frac{1}{\epsilon} - \ln \frac{s_{ij}}{\mu^2} \right] \Big(\rho^{(C)}_{ij} \ln x_i + \rho^{(C)}_{ji} \ln x_j \Big) + \rho^{(C)}_{ij} \mathrm{Li}_2\bigg(\frac{-x_j}{x_i} \bigg) + \rho^{(C)}_{ji} \mathrm{Li}_2\bigg(\frac{-x_i}{x_j} \bigg) \Bigg\}$$
$$+ \left[-g_{\mu\nu} + 4x_i x_j \frac{\tilde{k}_i^\mu \tilde{k}_i^\nu}{\tilde{k}_i^2} \right] f_{ij}^{q\bar{q}}\, T_R \left[\bigg(\frac{1}{\epsilon} - \ln \frac{s_{ij}}{\mu^2} \bigg) \big(\beta_0 - 3\,C_F\big) + \frac{7}{3} C_A + \frac{5}{3}\beta_0 - 8C_F \right]$$
$$- g_{\mu\nu}\, (f_{ij}^{gq} + f_{ij}^{g\bar{q}})\, C_F\, (C_A - C_F) + \frac{\tilde{k}_i^\mu \tilde{k}_i^\nu}{\tilde{k}_i^2}\, f_{ij}^{gg}\, C_A\, (3C_A - \beta_0) + \mathcal{O}(\epsilon)\,. \tag{C.28}$$

The one-loop collinear kernel $\hat{P}^{\mu\nu}_{ij(r)}$ can be rewritten according to the same structure as in Eq. (C.8),

$$\hat{P}^{\mu\nu}_{ij(r)} = -\hat{P}_{ij(r)}\, g^{\mu\nu} + \hat{Q}^{\mu\nu}_{ij(r)}\,, \qquad \hat{Q}^{\mu\nu}_{ij(r)} = \hat{Q}_{ij(r)}\, d_i^{\mu\nu}\,, \tag{C.29}$$

where we have introduced

$$\hat{P}_{ij(r)} = \frac{T_R}{1-2\epsilon} \left[1 - \frac{2x_i x_j}{1-\epsilon} \right] \left[\frac{1}{\epsilon}\big(\beta_0 - 3\,C_F\big) + C_A - 2C_F + \frac{C_A + 4\,T_R\,N_f}{3(3-2\epsilon)} \right] f_{ij}^{q\bar{q}}$$
$$+ C_F\, \frac{C_A - C_F}{1-2\epsilon} \Big[\big(1 - \epsilon x_i\big) f_i^g (f_j^q + f_j^{\bar{q}}) + \big(1 - \epsilon x_j\big)(f_i^q + f_i^{\bar{q}}) f_j^g \Big]$$
$$+ 4\,C_A\, \frac{C_A(1-\epsilon) - 2\,T_R\,N_f}{(1-2\epsilon)(2-2\epsilon)^2(3-2\epsilon)} \big(1 - 2\epsilon x_i x_j\big) f_{ij}^{gg}\,,$$
$$\hat{Q}_{ij(r)} = 2x_i x_j\, \frac{T_R}{(1-2\epsilon)(1-\epsilon)} \left[\frac{1}{\epsilon}\big(\beta_0 - 3\,C_F\big) + C_A - 2C_F + \frac{C_A + 4\,T_R\,N_f}{3(3-2\epsilon)} \right] f_{ij}^{q\bar{q}}$$
$$+ 4\,C_A\, \frac{2\,T_R\,N_f - C_A(1-\epsilon)}{(1-2\epsilon)(2-2\epsilon)^2(3-2\epsilon)} \big(1 - 2\epsilon x_i x_j\big) f_{ij}^{gg}\,. \tag{C.30}$$

Analogously, the ϵ expansion $\tilde{P}^{\mu\nu}_{ij(r)}$ can be recast in the same form, as

$$\tilde{P}^{\mu\nu}_{ij(r)} = -\tilde{P}_{ij(r)}\, g^{\mu\nu} + \tilde{Q}^{\mu\nu}_{ij(r)}\,, \qquad \tilde{Q}^{\mu\nu}_{ij(r)} = \tilde{Q}_{ij(r)}\, d_i^{\mu\nu}\,, \tag{C.31}$$

where $\tilde{P}_{ij(r)}$ and $\tilde{Q}_{ij(r)}$ are given by ($\mathcal{F} = P, Q$)

$$\tilde{\mathcal{F}}_{ij(r)} = \frac{\Gamma^2(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}\left(\frac{e^{\gamma_E}\mu^2}{s_{ij}}\right)^{\epsilon}\left\{\frac{C_{f_{[ij]}}}{\epsilon^2}\Big[\rho^{(C)}_{[ij]} + \rho^{(C)}_{ij}F(x_i) + \rho^{(C)}_{ji}F(x_j)\Big]\mathcal{F}_{ij(r)} + \hat{\mathcal{F}}_{ij(r)}\right\}. \tag{C.32}$$

The hard-collinear real-virtual kernel, expanded in the regulator ϵ, reads

$$\begin{aligned}
\tilde{P}^{\text{hc},\mu\nu}_{ij(r)} &\equiv \tilde{P}^{\mu\nu}_{ij(r)} - s_{ij}\Big[2\,C_{f_j}\,\tilde{\mathcal{E}}^{(i)}_{jr} + 2\,C_{f_i}\,\tilde{\mathcal{E}}^{(j)}_{ir}\Big]g^{\mu\nu} \qquad \text{(C.33)}\\
&= \tilde{P}^{\text{hc},\mu\nu}_{\text{fin},ij(r)} + C_{f_{[ij]}}\left[\rho^{(C)}_{[ij]}\left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{s_{ij}}{\mu^2}\right) + \frac{1}{\epsilon}\Big(\rho^{(C)}_{ij}\ln x_i + \rho^{(C)}_{ji}\ln x_j\Big)\right]P^{\text{hc},\mu\nu}_{ij(r)}\\
&\quad - \frac{4}{\epsilon}\left[f^g_i C^2_{f_j}\frac{x_j}{x_i}\ln x_j + f^g_j C^2_{f_i}\frac{x_i}{x_j}\ln x_i\right]g^{\mu\nu}\\
&\quad - \frac{T_R}{\epsilon}\big(\beta_0 - 3C_F\big)\,f^{q\bar{q}}_{ij}\left[g_{\mu\nu} - 4x_i x_j\frac{\tilde{k}^\mu_i\tilde{k}^\nu_i}{\tilde{k}^2_i}\right] + \mathcal{O}(\epsilon)\,,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{P}^{\text{hc},\mu\nu}_{\text{fin},ij(r)} &= P^{\text{hc},\mu\nu}_{ij(r)}\,C_{f_{[ij]}}\left\{\rho^{(C)}_{[ij]}\left[\left(\frac{1}{2}\ln^2\frac{s_{ij}}{\mu^2} - \frac{7}{2}\zeta_2\right)\right]\right.\\
&\qquad\left. + \rho^{(C)}_{ij}\left[\text{Li}_2\left(\frac{-x_j}{x_i}\right) - \ln\frac{s_{ij}}{\mu^2}\ln x_i\right] + \rho^{(C)}_{ji}\left[\text{Li}_2\left(\frac{-x_i}{x_j}\right) - \ln\frac{s_{ij}}{\mu^2}\ln x_j\right]\right\}\\
&\quad - g^{\mu\nu}\,2\,f^g_i\,C_{f_j}\frac{x_j}{x_i}\left\{C_A\Big[\ln^2 x_j + 2\,\text{Li}_2(x_i)\Big] + 2C_{f_j}\left[\text{Li}_2\left(\frac{-x_i}{x_j}\right) - \ln\frac{s_{ij}}{\mu^2}\ln x_j\right]\right\}\\
&\quad - g^{\mu\nu}\,2\,f^g_j\,C_{f_i}\frac{x_i}{x_j}\left\{C_A\Big[\ln^2 x_i + 2\,\text{Li}_2(x_j)\Big] + 2C_{f_i}\left[\text{Li}_2\left(\frac{-x_j}{x_i}\right) - \ln\frac{s_{ij}}{\mu^2}\ln x_i\right]\right\}\\
&\quad - \left[g_{\mu\nu} - 4x_i x_j\frac{\tilde{k}^\mu_i\tilde{k}^\nu_i}{\tilde{k}^2_i}\right]f^{q\bar{q}}_{ij}\,T_R\left[\ln\frac{s_{ij}}{\mu^2}\big(3\,C_F - \beta_0\big) + \frac{7}{3}C_A + \frac{5}{3}\beta_0 - 8C_F\right]\\
&\quad - g_{\mu\nu}\,(f^{gq}_{ij} + f^{g\bar{q}}_{ij})\,C_F\,(C_A - C_F) + \frac{\tilde{k}^\mu_i\tilde{k}^\nu_i}{\tilde{k}^2_i}\,f^{gg}_{ij}\,C_A\,(3C_A - \beta_0)\,,
\end{aligned} \tag{C.34}$$

and

$$\begin{aligned}
\text{Li}_2\left(-\frac{x_i}{x_j}\right) &= -\,\text{Li}_2(x_i) - \frac{1}{2}\ln^2 x_j = \text{Li}_2(x_j) + \ln x_i\ln x_j - \frac{1}{2}\ln^2 x_j - \zeta_2\,,\\
\text{Li}_2\left(-\frac{x_j}{x_i}\right) &= -\,\text{Li}_2(x_j) - \frac{1}{2}\ln^2 x_i = \text{Li}_2(x_i) + \ln x_i\ln x_j - \frac{1}{2}\ln^2 x_i - \zeta_2\,.
\end{aligned} \tag{C.35}$$

Equivalently we can write $\tilde{P}^{\text{hc},\mu\nu}_{ij(r)}$ in the form

$$\tilde{P}^{\text{hc},\mu\nu}_{ij(r)} = -\,\tilde{P}^{\text{hc}}_{ij(r)}\,g^{\mu\nu} + \tilde{Q}^{\mu\nu}_{ij(r)}\,, \tag{C.36}$$

with

$$\tilde{P}^{\rm hc}_{ij(r)} \equiv \tilde{P}_{ij(r)} + s_{ij}\Big[2\,C_{f_j}\,\tilde{\mathcal{E}}^{(i)}_{jr} + 2\,C_{f_i}\,\tilde{\mathcal{E}}^{(j)}_{ir}\Big] \tag{C.37}$$
$$= \frac{\Gamma^2(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}\left(\frac{e^{\gamma_E}\mu^2}{s_{ij}}\right)^{\epsilon}\left\{\frac{C_{f_{[ij]}}}{\epsilon^2}\Big[\rho^{(C)}_{[ij]} + \rho^{(C)}_{ij}F(x_i) + \rho^{(C)}_{ji}F(x_j)\Big]P_{ij(r)} + \hat{P}_{ij(r)} \right.$$
$$\left. + 2\,C_A\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\epsilon^2}\left[f^g_i C_{f_j}\left(\frac{x_j}{x_i}\right)^{1+\epsilon} + f^g_j C_{f_i}\left(\frac{x_i}{x_j}\right)^{1+\epsilon}\right]\right\}.$$

C.2 Improved limits

In this Appendix we present three sections that collect the building blocks needed to construct our local counterterms. Specifically, we explicitly define the action of

- improved limits on the double-real matrix element RR (Section C.2.1);
- improved limits on sector functions $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$, $\mathcal{W}_{ijkl}$ (Section C.2.2);
- improved limits on symmetrised sector functions $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$ (Section C.2.3).

The content of each section is organised according to the nature of the singular limits involved, which can be single-unresolved, uniform double-unresolved, and strongly-ordered double-unresolved. The action of improved limits $\overline{\mathbf{L}}$ on matrix elements times sector functions is specified by $\overline{\mathbf{L}}\,RR\,\mathcal{W}_{abcd} \equiv \big(\overline{\mathbf{L}}\,RR\big)\,\big(\overline{\mathbf{L}}\,\mathcal{W}_{abcd}\big)$, and similarly for $\mathcal{Z}$ functions. When acting on sector functions, single-unresolved and strongly-ordered improved limits imply the latter to be evaluated with mapped kinematics. Mapped sector functions are indicated generically as $\overline{\mathcal{W}}$ or $\bar{\mathcal{Z}}$ with no mapping labels in Sections C.2.2, C.2.3, understanding that the actual mapping to be used must be adapted to the one of the matrix elements the sector function is associated to. To be more precise, for each term of an improved limit, the mapping of $\overline{\mathcal{W}}$ or $\bar{\mathcal{Z}}$ is always the same as the first mapping of matrix elements in that term.

To give an explicit example, let us apply this rule to the $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR\,\mathcal{W}_{ijkl}$ contribution to $K^{(\mathbf{12})}_{ijkl}$ counterterm. Starting with the definitions

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR \;\equiv\; \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\left\{\mathcal{E}^{(i)}_{cd}\left[\sum_{e\neq i,k,c,d}\left(\sum_{f\neq i,k,c,d,e}\bar{\mathcal{E}}^{(k)(icd)}_{ef}\,\bar{B}^{(icd,kef)}_{cdef} + 2\,\bar{\mathcal{E}}^{(k)(icd)}_{ed}\,\bar{B}^{(icd,ked)}_{cded}\right)\right.\right.$$
$$\left. + 2\sum_{e\neq i,k,c,d}\bar{\mathcal{E}}^{(k)(idc)}_{ed}\,\bar{B}^{(idc,ked)}_{cded} + 2\,\bar{\mathcal{E}}^{(k)(icd)}_{cd}\Big(\bar{B}^{(icd,kcd)}_{cdcd} + C_A\,\bar{B}^{(icd,kcd)}_{cd}\Big)\right]$$
$$\left. - 2\,C_A\Big[\mathcal{E}^{(i)}_{kc}\,\bar{\mathcal{E}}^{(k)(ick)}_{cd}\,\bar{B}^{(ick,kcd)}_{cd} + \mathcal{E}^{(i)}_{kd}\,\bar{\mathcal{E}}^{(k)(ikd)}_{cd}\,\bar{B}^{(ikd,kcd)}_{cd}\Big]\right\}, \tag{C.38}$$

and

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{\mathrm{s},\,kl}\,\mathcal{W}^{(\alpha)}_{\mathrm{s},\,ij}\,, \tag{C.39}$$

according to the procedure detailed above, the explicit expression for $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR\,\mathcal{W}_{ijkl}$ results in

$$\begin{aligned}
\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR\,\mathcal{W}_{ijkl} \;\equiv&\\
\frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\Bigg\{&\,\mathcal{E}_{cd}^{(i)}\Bigg[\sum_{e\neq i,k,c,d}\Bigg(\sum_{f\neq i,k,c,d,e}\bar{\mathcal{E}}_{ef}^{(k)(icd)}\,\bar{B}_{cdef}^{(icd,kef)}+2\,\bar{\mathcal{E}}_{ed}^{(k)(icd)}\,\bar{B}_{cded}^{(icd,ked)}\Bigg)\overline{\mathcal{W}}_{\mathrm{s},\,kl}^{(icd)} \qquad \text{(C.40)}\\
&+2\sum_{e\neq i,k,c,d}\bar{\mathcal{E}}_{ed}^{(k)(idc)}\,\bar{B}_{cded}^{(idc,ked)}\,\overline{\mathcal{W}}_{\mathrm{s},\,kl}^{(idc)}+2\,\bar{\mathcal{E}}_{cd}^{(k)(icd)}\Big(\bar{B}_{cdcd}^{(icd,kcd)}+C_A\,\bar{B}_{cd}^{(icd,kcd)}\Big)\overline{\mathcal{W}}_{\mathrm{s},\,kl}^{(icd)}\Bigg]\\
&-2\,C_A\Big[\mathcal{E}_{kc}^{(i)}\,\bar{\mathcal{E}}_{cd}^{(k)(ick)}\,\bar{B}_{cd}^{(ick,kcd)}\,\overline{\mathcal{W}}_{\mathrm{s},\,kl}^{(ick)}+\mathcal{E}_{kd}^{(i)}\,\bar{\mathcal{E}}_{cd}^{(k)(ikd)}\,\bar{B}_{cd}^{(ikd,kcd)}\,\overline{\mathcal{W}}_{\mathrm{s},\,kl}^{(ikd)}\Big]\Bigg\}\mathcal{W}_{\mathrm{s},\,ij}^{(\alpha)}\,,
\end{aligned}$$

where it is evident that each $\overline{\mathcal{W}}_{ab}$ contribution is mapped according to the first mapping of the Born matrix element it accompanies.

Finally, we introduce a shorthand notation to simplify the treatment in Section C.2.2: we define single-unresolved improved limits on NLO sector functions as

$$\mathcal{W}_{\mathrm{s},\,ij}^{(\alpha)} \equiv \overline{\mathbf{S}}_i\,\mathcal{W}_{ij}^{(\alpha)} \equiv \frac{\frac{1}{w_{ij}^{\alpha}}}{\sum\limits_{l\neq i}\frac{1}{w_{il}^{\alpha}}}\,, \qquad \mathcal{W}_{\mathrm{s},\,ij} \equiv \mathcal{W}_{\mathrm{s},\,ij}^{(1)}\,, \qquad \text{(C.41)}$$

$$\mathcal{W}_{\mathrm{c},\,ij(r)}^{(\alpha)} \equiv \overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ij}^{(\alpha)} \equiv \frac{e_j^{\alpha}w_{jr}^{\alpha}}{e_i^{\alpha}w_{ir}^{\alpha}+e_j^{\alpha}w_{jr}^{\alpha}}\,, \qquad \mathcal{W}_{\mathrm{c},\,ij(r)} \equiv \mathcal{W}_{\mathrm{c},\,ij(r)}^{(1)}\,, \qquad \text{(C.42)}$$

depending on a reference particle $r \neq i, j$, whose choice will be specified case by case. As for NNLO sector functions, we introduce

$$\hat{\sigma}_{abcd(r)} = \frac{1}{(e_a\,w_{ab}\,w_{ar})^{\alpha}}\frac{1}{(e_c\,w_{cr}+\delta_{bc}\,e_a\,w_{ar})\,w_{cd}}\,, \qquad \text{(C.43)}$$

and

$$\begin{aligned}
\hat{\sigma}_{\{ijk\}(r)} =&\; \hat{\sigma}_{ijjk(r)}+\hat{\sigma}_{ikjk(r)}+\hat{\sigma}_{jiik(r)}+\hat{\sigma}_{jkik(r)}+\hat{\sigma}_{ijkj(r)}+\hat{\sigma}_{ikkj(r)}\\
&+\hat{\sigma}_{kiij(r)}+\hat{\sigma}_{kjij(r)}+\hat{\sigma}_{jiki(r)}+\hat{\sigma}_{jkki(r)}+\hat{\sigma}_{kiji(r)}+\hat{\sigma}_{kjji(r)}\,. \qquad \text{(C.44)}
\end{aligned}$$

C.2.1 Improved limits of RR

Single-unresolved improved limits

For the single-unresolved improved limits we have $(j \neq i)$

$$\overline{\mathbf{S}}_i\, RR \equiv -\mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd}\, \bar{R}^{(icd)}_{cd}\,, \tag{C.45}$$

$$\overline{\mathbf{C}}_{ij}\, RR \equiv \mathcal{N}_1 \frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\, \bar{R}^{(ijr)}_{\mu\nu}\,, \tag{C.46}$$

$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, RR \equiv \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ji}\, RR \;\equiv\; \mathcal{N}_1\, 2\, C_{f_j}\, \mathcal{E}^{(i)}_{jr}\, \bar{R}^{(ijr)}\,; \tag{C.47}$$

$$\overline{\mathbf{HC}}_{ij}\, RR \equiv \overline{\mathbf{C}}_{ij}\left(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\right) RR \;=\; \mathcal{N}_1\, \frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\, \bar{R}^{(ijr)}_{\mu\nu}\,. \tag{C.48}$$

In these equations r must be chosen according to the rule of Eq. (A.13) as $r = r_{ijkl} \neq i, j, k, l$, where i, j, k, l are the indices appearing in the NNLO sector functions multiplying the improved limits $\overline{\mathbf{C}}_{ij}$, $\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}$, $\overline{\mathbf{HC}}_{ij}$. This means that in the topologies $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$ the index $r = r_{ijk}$ is different from the three indices of the sector, while for the topology $\mathcal{W}_{ijkl}$ (i, j, k, l all different) the index $r = r_{ijkl}$ is different from the four indices of the sector. We stress that, having defined $r = r_{ijkl}$, one needs at least five massless partons in Φ_{n+2}, namely three massless final-state partons at Born level. We work under this assumption throughout this thesis.

Uniform double-unresolved improved limits

The double-soft improved limit is given by $(k \neq i)$

$$\begin{aligned}\overline{\mathbf{S}}_{ik}\, RR \equiv \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,k\\ d\neq i,k,c}} \Bigg\{ \mathcal{E}^{(i)}_{cd} \sum_{e\neq i,k,c,d} \Bigg[\sum_{f\neq i,k,c,d,e} \mathcal{E}^{(k)}_{ef}\, \bar{B}^{(icd,kef)}_{cdef} + 4\, \mathcal{E}^{(k)}_{ed}\, \bar{B}^{(icd,ked)}_{cded} \Bigg] \\ + 2\, \mathcal{E}^{(i)}_{cd} \mathcal{E}^{(k)}_{cd}\, \bar{B}^{(icd,kcd)}_{cdcd} + \mathcal{E}^{(ik)}_{cd}\, \bar{B}^{(ikcd)}_{cd} \Bigg\}\,. \end{aligned} \tag{C.49}$$

The soft-collinear improved limit $\overline{\mathbf{SC}}_{ikl}$ and its double-soft version $\overline{\mathbf{S}}_{ik}\, \overline{\mathbf{SC}}_{ikl}$ read ($k \neq i$, $l \neq i, k$, and $r = r_{ikl} \neq i, k, l$ defined with the rule of Eq. (A.13))

$$\begin{aligned}\overline{\mathbf{SC}}_{ikl}\, RR \equiv -\mathcal{N}_1^2\, \frac{P^{\mu\nu}_{kl(r)}}{s_{kl}} \Bigg\{ \sum_{c\neq i,k,l,r} \Bigg[\sum_{d\neq i,k,l,r,c} \mathcal{E}^{(i)}_{cd}\, \bar{B}^{(klr,icd)}_{\mu\nu,cd} + 2\, \mathcal{E}^{(i)}_{cr}\, \bar{B}^{(klr,icr)}_{\mu\nu,cr} \Bigg] \\ + \sum_{c\neq i,k,l} \Bigg[\mathcal{E}^{(i)}_{kc} \left(\rho^{(C)}_{kl}\, \bar{B}^{(lrk,ick)}_{\mu\nu,[kl]c} + \bar{\mathcal{B}}^{(lrk,ick)}_{\mu\nu,[kl]c}\, \tilde{f}^{\,q\bar{q}}_{kl} \right) + (k \leftrightarrow l) \Bigg] \Bigg\}\,, \end{aligned} \tag{C.50}$$

$$\begin{aligned}\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,RR \equiv \overline{\mathbf{S}}_{ki}\,\overline{\mathbf{SC}}_{ikl}\,RR \;\equiv\; \overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ilk}\,RR \\ \equiv -2\,\mathcal{N}_1^2\,\mathcal{E}_{lr}^{(k)}\bigg\{\; C_{f_l}\sum_{c\neq i,k,l,r}\bigg[\sum_{d\neq i,k,l,r,c}\mathcal{E}_{cd}^{(i)}\,\bar{B}_{cd}^{(klr,icd)} + 2\,\mathcal{E}_{cr}^{(i)}\,\bar{B}_{cr}^{(klr,icr)}\bigg] \\ + \sum_{c\neq i,k,l}\Big[C_A\,\mathcal{E}_{kc}^{(i)}\,\bar{B}_{[kl]c}^{(lrk,ick)} + (2C_{f_l}-C_A)\,\mathcal{E}_{lc}^{(i)}\,\bar{B}_{[kl]c}^{(krl,icl)}\Big]\bigg\}\,. \quad \text{(C.51)}\end{aligned}$$

The improved limits $\overline{\mathbf{SC}}_{ijk}$, $\overline{\mathbf{SC}}_{kij}$, $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk}$, $\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ijk}$, $\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{kij}$ can be obtained from these limits with a renaming of indices. For the uniform double-unresolved limits involving $\overline{\mathbf{C}}_{ijk}$, we have ($j\neq i$, $k\neq i,j$ and $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{C}}_{ijk}\,RR \equiv \frac{\mathcal{N}_1^2}{s_{ijk}^2}\,P_{ijk(r)}^{\mu\nu}\,\bar{B}_{\mu\nu}^{(ijkr)}\,; \quad \text{(C.52)}$$

$$\begin{aligned}\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR \equiv \overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ikj}\,RR \;\equiv\; \overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{kij}\,RR \\ \equiv \mathcal{N}_1^2\,C_{f_k}\bigg[4\,C_{f_k}\,\mathcal{E}_{kr}^{(i)}\,\mathcal{E}_{kr}^{(j)} - \mathcal{E}_{kr}^{(ij)}\bigg]\bar{B}^{(ijkr)}\,, \quad \text{(C.53)}\end{aligned}$$

$$\overline{\mathbf{HC}}_{ijk}\,RR \equiv \overline{\mathbf{C}}_{ijk}\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\big)\,RR \;=\; \frac{\mathcal{N}_1^2}{s_{ijk}^2}\,P_{ijk(r)}^{\mathrm{hc},\mu\nu}\,\bar{B}_{\mu\nu}^{(ijkr)}\,; \quad \text{(C.54)}$$

$$\begin{aligned}\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,RR \equiv \overline{\mathbf{C}}_{jki}\,\overline{\mathbf{SC}}_{ijk}\,RR \\ \equiv \mathcal{N}_1^2\,C_{f_{[jk]}}\,\frac{P_{jk(r)}^{\mu\nu}}{s_{jk}}\bigg[\rho_{jk}^{(C)}\mathcal{E}_{jr}^{(i)}\bar{B}_{\mu\nu}^{(krj,irj)} + \rho_{kj}^{(C)}\mathcal{E}_{kr}^{(i)}\bar{B}_{\mu\nu}^{(jrk,irk)}\bigg]\,, \quad \text{(C.55)}\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,RR \equiv \overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ikj}\,\overline{\mathbf{SC}}_{ikj}\,RR = \overline{\mathbf{S}}_{ji}\,\overline{\mathbf{C}}_{jki}\,\overline{\mathbf{SC}}_{ijk}\,RR \\ \equiv 2\mathcal{N}_1^2\,C_{f_k}\,\mathcal{E}_{kr}^{(j)}\Big[C_A\,\mathcal{E}_{jr}^{(i)}\,\bar{B}^{(krj,irj)} + (2C_{f_k}-C_A)\,\mathcal{E}_{kr}^{(i)}\,\bar{B}^{(jrk,irk)}\Big]\,, \quad \text{(C.56)}\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SHC}}_{ijk}\,RR \equiv \overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\big)\,RR \\ \equiv \mathcal{N}_1^2\,C_{f_{[jk]}}\,\frac{P_{jk(r)}^{\mathrm{hc},\mu\nu}}{s_{jk}}\bigg[\rho_{jk}^{(C)}\mathcal{E}_{jr}^{(i)}\bar{B}_{\mu\nu}^{(krj,irj)} + \rho_{kj}^{(C)}\mathcal{E}_{kr}^{(i)}\bar{B}_{\mu\nu}^{(jrk,irk)}\bigg]\,, \quad \text{(C.57)}\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{SHC}}_{ijk}\big(1-\overline{\mathbf{C}}_{ijk}\big)\;RR \equiv \overline{\mathbf{SC}}_{ijk}\big(1-\overline{\mathbf{C}}_{ijk}\big)\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\big)\,RR \quad \text{(C.58)} \\ \equiv -\mathcal{N}_1^2\,\frac{P_{jk(r)}^{\mathrm{hc},\mu\nu}}{s_{jk}}\bigg\{\sum_{c\neq i,j,k,r}\bigg[\sum_{d\neq i,j,k,r,c}\mathcal{E}_{cd}^{(i)}\,\bar{B}_{\mu\nu,cd}^{(jkr,icd)} + 2\,\mathcal{E}_{cr}^{(i)}\,\bar{B}_{\mu\nu,cr}^{(jkr,icr)}\bigg] \\ + \sum_{c\neq i,j,k}\bigg[\mathcal{E}_{jc}^{(i)}\Big(\rho_{jk}^{(C)}\,\bar{B}_{\mu\nu,[jk]c}^{(krj,icj)} + \bar{\mathcal{B}}_{\mu\nu,[jk]c}^{(krj,icj)}\,\tilde{f}_{jk}^{\,q\bar{q}}\Big) + (j\leftrightarrow k)\bigg] \\ + C_{f_{[jk]}}\bigg[\rho_{jk}^{(C)}\mathcal{E}_{jr}^{(i)}\,\bar{B}_{\mu\nu}^{(krj,irj)} + \rho_{kj}^{(C)}\mathcal{E}_{kr}^{(i)}\,\bar{B}_{\mu\nu}^{(jrk,irk)}\bigg]\bigg\}\,.\end{aligned}$$

Finally, the limits involving $\overline{\mathbf{C}}_{ijkl}$ are given by ($j \neq i$, $k \neq i, j$, $l \neq i, j, k$ and $r = r_{ijkl} \neq i, j, k, l$)

$$\overline{\mathbf{C}}_{ijkl}\, RR \equiv \mathcal{N}_1^2\, \frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\, \frac{P^{\rho\sigma}_{kl(r)}}{s_{kl}}\, \bar{B}^{(ijr,klr)}_{\mu\nu\rho\sigma}\,, \tag{C.59}$$

$$\begin{aligned}\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijkl}\, RR &\equiv \overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{jikl}\, RR \;\equiv\; \overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijlk}\, RR \;\equiv\; \overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{jilk}\, RR \\ &\equiv 4\,\mathcal{N}_1^2\, C_{f_j}\, C_{f_l}\, \mathcal{E}^{(i)}_{jr}\, \mathcal{E}^{(k)}_{lr}\, \bar{B}^{(ijr,klr)}\,;\end{aligned} \tag{C.60}$$

$$\begin{aligned}\overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{ijkl}\, RR &\equiv \overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{klij}\, RR \;\equiv\; \overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{jikl}\, RR \;\equiv\; \overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{klji}\, RR \\ &\equiv 2\,\mathcal{N}_1^2\, C_{f_j}\, \mathcal{E}^{(i)}_{jr}\, \frac{P^{\mu\nu}_{kl(r)}}{s_{kl}}\, \bar{B}^{(ijr,klr)}_{\mu\nu}\,;\end{aligned} \tag{C.61}$$

$$\begin{aligned}\overline{\mathbf{HC}}_{ijkl}\, RR &\equiv \overline{\mathbf{C}}_{ijkl}\,\big(1 + \overline{\mathbf{S}}_{ik} + \overline{\mathbf{S}}_{jk} + \overline{\mathbf{S}}_{il} + \overline{\mathbf{S}}_{jl} - \overline{\mathbf{SC}}_{ikl} - \overline{\mathbf{SC}}_{jkl} - \overline{\mathbf{SC}}_{kij} - \overline{\mathbf{SC}}_{lij}\big)\, RR \\ &= \mathcal{N}_1^2\, \frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\, \frac{P^{\mathrm{hc},\rho\sigma}_{kl(r)}}{s_{kl}}\, \bar{B}^{(ijr,klr)}_{\mu\nu\rho\sigma}\,.\end{aligned} \tag{C.62}$$

Strongly-ordered double-unresolved improved limits

The improved limit $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}$ is given by ($k \neq i$)

$$\begin{aligned}\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\, RR \equiv \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,k\\ d\neq i,k,c}} \Bigg\{ \mathcal{E}^{(i)}_{cd} \Bigg[\sum_{e\neq i,k,c,d} \Bigg(\sum_{f\neq i,k,c,d,e} \bar{\mathcal{E}}^{(k)(icd)}_{ef}\, \bar{B}^{(icd,kef)}_{cdef} + 2\,\bar{\mathcal{E}}^{(k)(icd)}_{ed}\, \bar{B}^{(icd,ked)}_{cded} \Bigg) \\ + 2 \sum_{e\neq i,k,c,d} \bar{\mathcal{E}}^{(k)(idc)}_{ed}\, \bar{B}^{(idc,ked)}_{cded} + 2\,\bar{\mathcal{E}}^{(k)(icd)}_{cd} \Big(\bar{B}^{(icd,kcd)}_{cdcd} + C_A\, \bar{B}^{(icd,kcd)}_{cd} \Big) \Bigg] \\ - 2\, C_A \Big[\mathcal{E}^{(i)}_{kc}\, \bar{\mathcal{E}}^{(k)(ick)}_{cd}\, \bar{B}^{(ick,kcd)}_{cd} + \mathcal{E}^{(i)}_{kd}\, \bar{\mathcal{E}}^{(k)(ikd)}_{cd}\, \bar{B}^{(ikd,kcd)}_{cd} \Big] \Bigg\}\,.\end{aligned} \tag{C.63}$$

For $\overline{\mathbf{S}}_i\,\overline{\mathbf{SC}}_{ikl}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}$ we have ($k \neq i$, $l \neq i, k$, and $r = r_{ikl} \neq i, k, l$)

$$\begin{aligned}\overline{\mathbf{S}}_i\,\overline{\mathbf{SC}}_{ikl}\, RR \equiv -\mathcal{N}_1^2 \sum_{c\neq i,k,l} \Bigg\{ \sum_{d\neq i,k,l,c} \mathcal{E}^{(i)}_{cd}\, \frac{\bar{P}^{(icd)\mu\nu}_{kl(r)}}{\bar{s}^{(icd)}_{kl}}\, \bar{B}^{(icd,klr)}_{\mu\nu,cd} \\ + \Bigg[\mathcal{E}^{(i)}_{kc}\, \frac{\bar{P}^{(ikc)\mu\nu}_{kl(r)}}{2\,\bar{s}^{(ikc)}_{kl}} \Big(\rho^{(C)}_{kl}\, \bar{B}^{(ikc,lrk)}_{\mu\nu,[kl]c} + \bar{\mathcal{B}}^{(ikc,lrk)}_{\mu\nu,[kl]c}\, \tilde{f}^{\,q\bar{q}}_{kl} \Big) + (k \leftrightarrow l) \Bigg] \\ + \Bigg[\mathcal{E}^{(i)}_{kc}\, \frac{\bar{P}^{(ick)\mu\nu}_{kl(r)}}{2\,\bar{s}^{(ick)}_{kl}} \Big(\rho^{(C)}_{kl}\, \bar{B}^{(ick,lrk)}_{\mu\nu,[kl]c} + \bar{\mathcal{B}}^{(ick,lrk)}_{\mu\nu,[kl]c}\, \tilde{f}^{\,q\bar{q}}_{kl} \Big) + (k \leftrightarrow l) \Bigg] \Bigg\}\,,\end{aligned} \tag{C.64}$$

$$
\begin{aligned}
\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,RR \equiv\, & \overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ilk}\,RR \\
\equiv\, & -\mathcal{N}_1^2 \sum_{c\neq i,k,l} \bigg[\, 2\,C_{f_l} \sum_{d\neq i,k,l,c} \mathcal{E}_{cd}^{(i)}\,\bar{\mathcal{E}}_{lr}^{(k)(icd)}\,\bar{B}_{cd}^{(icd,klr)} \\
& + C_A\,\mathcal{E}_{kc}^{(i)} \Big(\bar{\mathcal{E}}_{lr}^{(k)(ikc)}\,\bar{B}_{lc}^{(ikc,lrk)} + \bar{\mathcal{E}}_{lr}^{(k)(ick)}\,\bar{B}_{lc}^{(ick,lrk)} \Big) \\
& + (2C_{f_l} - C_A)\,\mathcal{E}_{lc}^{(i)} \Big(\bar{\mathcal{E}}_{lr}^{(k)(ilc)}\,\bar{B}_{lc}^{(ilc,krl)} + \bar{\mathcal{E}}_{lr}^{(k)(icl)}\,\bar{B}_{lc}^{(icl,krl)} \Big) \bigg] .
\end{aligned} \tag{C.65}
$$

Combining the previous definitions we have ($j \neq i$, $k \neq i, j$, and $r = r_{ijk} \neq i, j, k$)

$$
\begin{aligned}
\overline{\mathbf{S}}_i\,\overline{\mathbf{SHC}}_{ijk}\,RR \equiv\, & \overline{\mathbf{S}}_i\,\overline{\mathbf{SC}}_{ijk}\left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik}\right) RR \qquad (C.66) \\
\equiv\, & -\mathcal{N}_1^2 \sum_{c\neq i,j,k} \Bigg\{ \sum_{d\neq i,j,k,c} \mathcal{E}_{cd}^{(i)}\, \frac{\bar{P}_{jk(r)}^{(icd)\mathrm{hc},\mu\nu}}{\bar{s}_{jk}^{(icd)}}\, \bar{B}_{\mu\nu,cd}^{(icd,jkr)} \\
& + \left[\mathcal{E}_{jc}^{(i)}\, \frac{\bar{P}_{jk(r)}^{(ijc)\mathrm{hc},\mu\nu}}{2\,\bar{s}_{jk}^{(ijc)}} \Big(\rho_{jk}^{(C)}\,\bar{B}_{\mu\nu,[jk]c}^{(ijc,krj)} + \bar{\mathcal{B}}_{\mu\nu,[jk]c}^{(ijc,krj)}\,\tilde{f}_{jk}^{\,q\bar{q}} \Big) + (j \leftrightarrow k) \right] \\
& + \left[\mathcal{E}_{jc}^{(i)}\, \frac{\bar{P}_{jk(r)}^{(icj)\mathrm{hc},\mu\nu}}{2\,\bar{s}_{jk}^{(icj)}} \Big(\rho_{jk}^{(C)}\,\bar{B}_{\mu\nu,[jk]c}^{(icj,krj)} + \bar{\mathcal{B}}_{\mu\nu,[jk]c}^{(icj,krj)}\,\tilde{f}_{jk}^{\,q\bar{q}} \Big) + (j \leftrightarrow k) \right] \Bigg\} .
\end{aligned}
$$

For the strongly-ordered double-unresolved limits involving $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}$, we have ($j \neq i$, $k \neq i, j$, $r = r_{ijk} \neq i, j, k$)

$$
\begin{aligned}
& \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})RR \;\equiv \\
& \mathcal{N}_1^2 \frac{C_{f_{[jk]}}}{2} \Bigg\{ \rho_{jk}^{(C)}\,\mathcal{E}_{jr}^{(i)} \left[\frac{\bar{P}_{jk(r)}^{(ijr)\mu\nu}}{\bar{s}_{jk}^{(ijr)}} \Big(\bar{B}_{\mu\nu}^{(ijr,jkr)} - \bar{B}_{\mu\nu}^{(ijr,krj)} \Big) + \frac{\bar{P}_{jk(r)}^{(irj)\mu\nu}}{\bar{s}_{jk}^{(irj)}} \Big(\bar{B}_{\mu\nu}^{(irj,jkr)} - \bar{B}_{\mu\nu}^{(irj,krj)} \Big) \right] \\
& \quad + \rho_{kj}^{(C)}\,\mathcal{E}_{kr}^{(i)} \left[\frac{\bar{P}_{jk(r)}^{(ikr)\mu\nu}}{\bar{s}_{jk}^{(ikr)}} \Big(\bar{B}_{\mu\nu}^{(ikr,jkr)} - \bar{B}_{\mu\nu}^{(ikr,jrk)} \Big) + \frac{\bar{P}_{jk(r)}^{(irk)\mu\nu}}{\bar{s}_{jk}^{(irk)}} \Big(\bar{B}_{\mu\nu}^{(irk,jkr)} - \bar{B}_{\mu\nu}^{(irk,jrk)} \Big) \right] \\
& \quad - \rho_{[jk]}^{(C)}\,\mathcal{E}_{jk}^{(i)} \left[\frac{\bar{P}_{jk(r)}^{(ijk)\mu\nu}}{\bar{s}_{jk}^{(ijk)}} \bar{B}_{\mu\nu}^{(ijk,jkr)} + \frac{\bar{P}_{jk(r)}^{(ikj)\mu\nu}}{\bar{s}_{jk}^{(ikj)}} \bar{B}_{\mu\nu}^{(ikj,jkr)} \right] \Bigg\} ,
\end{aligned} \tag{C.67}
$$

$$
\begin{aligned}
& \overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})RR \;\equiv\; \overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ikj}(1-\overline{\mathbf{SC}}_{ikj})RR \\
& \equiv \mathcal{N}_1^2\, C_{f_k} \Bigg\{ C_A\,\mathcal{E}_{jr}^{(i)} \Big[\bar{\mathcal{E}}_{kr}^{(j)(ijr)} \big(\bar{B}^{(ijr,jkr)} - \bar{B}^{(ijr,krj)} \big) + \bar{\mathcal{E}}_{kr}^{(j)(irj)} \big(\bar{B}^{(irj,jkr)} - \bar{B}^{(irj,krj)} \big) \Big] \\
& \quad + (2C_{f_k} - C_A)\,\mathcal{E}_{kr}^{(i)} \Big[\bar{\mathcal{E}}_{kr}^{(j)(ikr)} \big(\bar{B}^{(ikr,jkr)} - \bar{B}^{(ikr,jrk)} \big) + \bar{\mathcal{E}}_{kr}^{(j)(irk)} \big(\bar{B}^{(irk,jkr)} - \bar{B}^{(irk,jrk)} \big) \Big] \\
& \quad + C_A\,\mathcal{E}_{jk}^{(i)} \Big[\bar{\mathcal{E}}_{kr}^{(j)(ijk)}\,\bar{B}^{(ijk,jkr)} + \bar{\mathcal{E}}_{kr}^{(j)(ikj)}\,\bar{B}^{(ikj,jkr)} \Big] \Bigg\} ,
\end{aligned} \tag{C.68}
$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{HC}}^{(\mathbf{s})}_{ijk}\,RR \;\equiv\; \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\right)\left(1-\overline{\mathbf{SC}}_{ijk}\right)RR \tag{C.69}$$
$$= \mathcal{N}_1^{\,2}\frac{C_{f_{[jk]}}}{2}\Bigg\{\rho^{(C)}_{jk}\,\mathcal{E}^{(i)}_{jr}\left[\frac{\bar{P}^{(ijr)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\left(\bar{B}^{(ijr,jkr)}_{\mu\nu}-\bar{B}^{(ijr,krj)}_{\mu\nu}\right)+\frac{\bar{P}^{(irj)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(irj)}_{jk}}\left(\bar{B}^{(irj,jkr)}_{\mu\nu}-\bar{B}^{(irj,krj)}_{\mu\nu}\right)\right]$$
$$+\,\rho^{(C)}_{kj}\,\mathcal{E}^{(i)}_{kr}\left[\frac{\bar{P}^{(ikr)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ikr)}_{jk}}\left(\bar{B}^{(ikr,jkr)}_{\mu\nu}-\bar{B}^{(ikr,jrk)}_{\mu\nu}\right)+\frac{\bar{P}^{(irk)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(irk)}_{jk}}\left(\bar{B}^{(irk,jkr)}_{\mu\nu}-\bar{B}^{(irk,jrk)}_{\mu\nu}\right)\right]$$
$$-\,\rho^{(C)}_{[jk]}\,\mathcal{E}^{(i)}_{jk}\left[\frac{\bar{P}^{(ijk)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijk)}_{jk}}\bar{B}^{(ijk,jkr)}_{\mu\nu}+\frac{\bar{P}^{(ikj)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ikj)}_{jk}}\bar{B}^{(ikj,jkr)}_{\mu\nu}\right]\Bigg\}.$$

For $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}$ we have ($j\neq i$, $k\neq i,j$, $r=r_{ijkl}\neq i,j,k,l$, $r'=r_{ijk}\neq i,j,k$ in sector $\mathcal{W}_{ijkl}$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv -\mathcal{N}_1^{\,2}\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\Bigg\{\sum_{c\neq i,j,k,r'}\Bigg[\sum_{d\neq i,j,k,r',c}\bar{\mathcal{E}}^{(k)(ijr)}_{cd}\,\bar{B}^{(ijr,kcd)}_{\mu\nu,cd}+2\,\bar{\mathcal{E}}^{(k)(ijr)}_{cr'}\,\bar{B}^{(ijr,kcr')}_{\mu\nu,cr'}\Bigg]$$
$$+\,2\sum_{c\neq i,j,k}\bar{\mathcal{E}}^{(k)(ijr)}_{jc}\,\bar{B}^{(ijr,kcj)}_{\mu\nu,jc}\Bigg\}, \tag{C.70}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{SC}}_{kji}\,RR$$
$$\equiv -\,2\,\mathcal{N}_1^{\,2}\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\Bigg\{\sum_{c\neq i,j,k,r'}\Bigg[\sum_{d\neq i,j,k,r',c}\bar{\mathcal{E}}^{(k)(ijr)}_{cd}\,\bar{B}^{(ijr,kcd)}_{cd}+2\,\bar{\mathcal{E}}^{(k)(ijr)}_{cr'}\,\bar{B}^{(ijr,kcr')}_{cr'}\Bigg]$$
$$+\,2\sum_{c\neq i,j,k}\bar{\mathcal{E}}^{(k)(ijr)}_{jc}\,\bar{B}^{(ijr,kcj)}_{jc}\Bigg], \tag{C.71}$$

$$\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv \overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\right)\overline{\mathbf{SC}}_{kij}\,RR$$
$$= -\mathcal{N}_1^{\,2}\,\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\Bigg\{\sum_{c\neq i,j,k,r'}\Bigg[\sum_{d\neq i,j,k,r',c}\bar{\mathcal{E}}^{(k)(ijr)}_{cd}\,\bar{B}^{(ijr,kcd)}_{\mu\nu,cd}+2\,\bar{\mathcal{E}}^{(k)(ijr)}_{cr'}\,\bar{B}^{(ijr,kcr')}_{\mu\nu,cr'}\Bigg]$$
$$+\,2\sum_{c\neq i,j,k}\bar{\mathcal{E}}^{(k)(ijr)}_{jc}\bar{B}^{(ijr,kcj)}_{\mu\nu,jc}\Bigg\}. \tag{C.72}$$

The improved limits $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR$, $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR$ and their combination $\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR$ appear in the sector topology $\mathcal{W}_{ijjk}$ only, and are given by ($j\neq i$ and $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}RR \equiv -\mathcal{N}_1^{2}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\frac{P_{ij(r)}}{s_{ij}}\,\bar{\mathcal{E}}^{(j)(ijr)}_{cd}+\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\left[\frac{\bar{k}^{(ijr)}_{c,\mu}}{\bar{s}^{(ijr)}_{jc}}-\frac{\bar{k}^{(ijr)}_{d,\mu}}{\bar{s}^{(ijr)}_{jd}}\right]\left[\frac{\bar{k}^{(ijr)}_{c,\nu}}{\bar{s}^{(ijr)}_{jc}}-\frac{\bar{k}^{(ijr)}_{d,\nu}}{\bar{s}^{(ijr)}_{jd}}\right]\Bigg\}\bar{B}^{(ijr,jcd)}_{cd}, \tag{C.73}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{S}}_{ji}\,RR \;\equiv\; -\,2\,\mathcal{N}_1^{\,2}\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\bar{\mathcal{E}}^{(j)(ijr)}_{cd}\,\bar{B}^{(ijr,jcd)}_{cd}\,, \tag{C.74}$$

$$\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR \equiv \overline{\mathbf{C}}_{ij}\big(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\big)\,\overline{\mathbf{S}}_{ij}RR = \tag{C.75}$$
$$-\,\mathcal{N}_1^2\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\left[\frac{P^{\rm hc}_{ij(r)}}{s_{ij}}\,\bar{\mathcal{E}}^{(j)(ijr)}_{cd}+\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\left(\frac{\bar{k}^{(ijr)}_{c,\mu}}{\bar{s}^{(ijr)}_{jc}}-\frac{\bar{k}^{(ijr)}_{d,\mu}}{\bar{s}^{(ijr)}_{jd}}\right)\left(\frac{\bar{k}^{(ijr)}_{c,\nu}}{\bar{s}^{(ijr)}_{jc}}-\frac{\bar{k}^{(ijr)}_{d,\nu}}{\bar{s}^{(ijr)}_{jd}}\right)\right]\bar{B}^{(ijr,jcd)}_{cd}.$$

For the strongly-ordered double-unresolved limits involving $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}$, we have ($j\neq i$, $k\neq i,j$, $r=r_{ijk}\neq i,j,k$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR \equiv \mathcal{N}_1^2\Bigg\{\frac{P_{ij(r)}}{s_{ij}}\frac{\bar{P}^{(ijr)\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\bar{B}^{(ijr,jkr)}_{\mu\nu}+2\,C_A\,\bar{\mathcal{E}}^{(k)(ijr)}_{jr}\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{B}^{(ijr,jkr)}_{\mu\nu} \tag{C.76}$$
$$-\,2\,C_{f_k}\,\bar{\mathcal{E}}^{(j)(ijr)}_{kr}\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{\tilde{\bar{k}}^{(ijr)}_{\mu}\tilde{\bar{k}}^{(ijr)}_{\nu}}{\big(\tilde{\bar{k}}^{(ijr)}\big)^2}\,\bar{B}^{(ijr,jkr)}\Bigg\},$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{C}}_{jik}\,RR \;\equiv\; 2\,\mathcal{N}_1^2\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\frac{\bar{P}^{(ijr)\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\,\bar{B}^{(ijr,jkr)}_{\mu\nu}\,, \tag{C.77}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}RR \equiv 2\,\mathcal{N}_1^2\,C_{f_k}\,\bar{\mathcal{E}}^{(j)(ijr)}_{kr}\left\{\frac{P_{ij(r)}}{s_{ij}}-\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{\tilde{\bar{k}}^{(ijr)}_{\mu}\tilde{\bar{k}}^{(ijr)}_{\nu}}{\big(\tilde{\bar{k}}^{(ijr)}\big)^2}\right\}\bar{B}^{(ijr,jkr)}\,, \tag{C.78}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{S}}_{ji}\,\overline{\mathbf{C}}_{jik}\,RR \;\equiv\; 4\,\mathcal{N}_1^2\,C_{f_j}\,C_{f_k}\,\mathcal{E}^{(i)}_{jr}\,\bar{\mathcal{E}}^{(j)(ijr)}_{kr}\,\bar{B}^{(ijr,jkr)}\,, \tag{C.79}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv 2\,\mathcal{N}_1^2\,C_{f_{[ij]}}\,\bar{\mathcal{E}}^{(k)(ijr)}_{jr}\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{B}^{(ijr,krj)}_{\mu\nu}\,, \tag{C.80}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{C}}_{jik}\,\overline{\mathbf{SC}}_{kji}\,RR \;\equiv\; 4\,\mathcal{N}_1^2\,C^2_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\bar{\mathcal{E}}^{(k)(ijr)}_{jr}\,\bar{B}^{(ijr,krj)}\,, \tag{C.81}$$

$$\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{HC}}^{\,(\mathbf{c})}_{ijk}\,RR\,\mathcal{Z}_{ijk} \equiv \overline{\mathbf{C}}_{ij}\big(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\big)\,\overline{\mathbf{C}}_{ijk}\,\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{SC}}_{kij}\big)\,RR \tag{C.82}$$
$$=\mathcal{N}_1^2\,\frac{P^{\rm hc}_{ij(r)}}{s_{ij}}\,\frac{\bar{P}^{(ijr){\rm hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\,\bar{B}^{(ijr,jkr)}_{\mu\nu}$$
$$-\,2\,\mathcal{N}_1^2\,C_{f_{[ij]}}\,\bar{\mathcal{E}}^{(k)(ijr)}_{jr}\,\frac{P^{{\rm hc},\mu\nu}_{ij(r)}}{s_{ij}}\,\Big(\bar{B}^{(ijr,krj)}_{\mu\nu}-\bar{B}^{(ijr,kjr)}_{\mu\nu}\Big)\,.$$

Finally the limits involving $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}$ are given by ($j \neq i$, $k \neq i,j$, $l \neq i,j,k$ and $r = r_{ijkl} \neq i,j,k,l$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,RR \equiv \mathcal{N}_1^2\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{\bar{P}^{(ijr)\rho\sigma}_{kl(r)}}{\bar{s}^{(ijr)}_{kl}}\,\bar{B}^{(ijr,klr)}_{\mu\nu\rho\sigma}\,, \tag{C.83}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,RR \equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{C}}_{jikl}\,RR \;\equiv\; 2\,\mathcal{N}_1^2\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}\,\frac{\bar{P}^{(ijr)\rho\sigma}_{kl(r)}}{\bar{s}^{(ijr)}_{kl}}\bar{B}^{(ijr,klr)}_{\rho\sigma}\,, \tag{C.84}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,RR \equiv \overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijlk}\,RR \;\equiv\; 2\,\mathcal{N}_1^2\,C_{f_l}\,\frac{P^{\mu\nu}_{ij(r)}}{s_{ij}}\,\bar{\mathcal{E}}^{(k)(ijr)}_{lr}\,\bar{B}^{(ijr,klr)}_{\mu\nu}\,, \tag{C.85}$$

$$\begin{aligned}\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,RR &\equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{SC}}_{kji}\,\overline{\mathbf{C}}_{jikl}\,RR \;\equiv\; \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijlk}\,RR \\ &\equiv \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ji}\,\overline{\mathbf{SC}}_{kji}\,\overline{\mathbf{C}}_{jilk}\,RR \\ &\equiv 4\,\mathcal{N}_1^2\,C_{f_j}C_{f_l}\,\mathcal{E}^{(i)}_{jr}\,\mathcal{E}^{(k)}_{lr}\,\bar{B}^{(ijr,klr)}\,,\end{aligned} \tag{C.86}$$

$$\begin{aligned}\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{HC}}^{(\mathbf{c})}_{ijkl}\,RR &\equiv \overline{\mathbf{C}}_{ij}\,\big(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\big)\,\overline{\mathbf{C}}_{ijkl}\,\big(1-\overline{\mathbf{SC}}_{kij}-\overline{\mathbf{SC}}_{lij}\big)\,RR \\ &= \mathcal{N}_1^2\,\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\,\frac{\bar{P}^{(ijr)\mathrm{hc},\rho\sigma}_{kl(r)}}{\bar{s}^{(ijr)}_{kl}}\,\bar{B}^{(ijr,klr)}_{\mu\nu\rho\sigma}\,.\end{aligned} \tag{C.87}$$

C.2.2 Improved limits of $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$, $\mathcal{W}_{ijkl}$

Single-unresolved improved limits

For the single-unresolved improved limits we have ($j \neq i$, $k \neq i$, $l \neq i,k$ and $r = r_{ijkl} \neq i,j,k,l$)

$$\overline{\mathbf{S}}_i\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{kl}\,\mathcal{W}^{(\alpha)}_{\mathrm{s},\,ij}\,, \tag{C.88}$$

$$\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{kl}\,\mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,, \tag{C.89}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{kl}\,. \tag{C.90}$$

Uniform double-unresolved improved limits

The double-soft improved limit is given by ($j \neq i$, $k \neq i$, $l \neq i,k$)

$$\overline{\mathbf{S}}_{ik}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ijkl}}{\sum_{b\neq i}\sum_{d\neq i,k}\sigma_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\sigma_{kbid}}\,. \tag{C.91}$$

The soft-collinear improved limits $\overline{\mathbf{SC}}_{ikl}$ and $\overline{\mathbf{SC}}_{kij}$ as well as their double-soft versions $\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}$ and $\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{kij}$ read ($j \neq i$, $k \neq i$, $l \neq i, k$)

$$\overline{\mathbf{SC}}_{ikl}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\frac{\sigma_{kl}}{w_{kr}}}{\sum_{b\neq i}\sigma_{ib}^{(\alpha)}\left(\frac{\sigma_{kl}}{w_{kr}}+\frac{\sigma_{lk}}{w_{lr}}\right)+\frac{\sigma_{kl}^{(\alpha)}}{w_{kr}}\sum_{d\neq i,k}\sigma_{id}+\frac{\sigma_{lk}^{(\alpha)}}{w_{lr}}\sum_{d\neq i,l}\sigma_{id}}\,,\quad r=r_{ikl}\,, \tag{C.92}$$

$$\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkl} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}}\sigma_{kl}}{\sum_{b\neq k}\sigma_{kb}^{(\alpha)}\left(\frac{\sigma_{ij}}{w_{ir}}+\frac{\sigma_{ji}}{w_{jr}}\right)+\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}}\sum_{d\neq i,k}\sigma_{kd}+\frac{\sigma_{ji}^{(\alpha)}}{w_{jr}}\sum_{d\neq k,j}\sigma_{kd}}\,,\quad r=r_{ijk}, \tag{C.93}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\sigma_{kl}}{\sum_{b\neq i}\sigma_{ib}^{(\alpha)}\sigma_{kl}+\sigma_{kl}^{(\alpha)}\sum_{d\neq i,k}\sigma_{id}}\,,\quad r=r_{ikl}\,, \tag{C.94}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\sigma_{kl}}{\sum_{b\neq k}\sigma_{kb}^{(\alpha)}\sigma_{ij}+\sigma_{ij}^{(\alpha)}\sum_{d\neq i,k}\sigma_{kd}}\,,\quad r=r_{ijk}\,. \tag{C.95}$$

For the uniform double-unresolved limits involving $\overline{\mathbf{C}}_{ijk}$, we have ($j \neq i$, $k \neq i, j$ and $r = r_{ijk} \neq i, j, k$)

$$\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \frac{\hat{\sigma}_{ijjk(r)}}{\hat{\sigma}_{\{ijk\}(r)}}\,,\qquad \overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijkj} \equiv \frac{\hat{\sigma}_{ijkj(r)}}{\hat{\sigma}_{\{ijk\}(r)}}\,; \tag{C.96}$$

$$\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \frac{\hat{\sigma}_{ijjk(r)}}{\hat{\sigma}_{ijjk(r)}+\hat{\sigma}_{ikjk(r)}+\hat{\sigma}_{jiik(r)}+\hat{\sigma}_{jkik(r)}}\,, \tag{C.97}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijkj} \equiv \frac{\hat{\sigma}_{ijkj(r)}}{\hat{\sigma}_{ijkj(r)}+\hat{\sigma}_{ikkj(r)}+\hat{\sigma}_{kiij(r)}+\hat{\sigma}_{kjij(r)}}\,; \tag{C.98}$$

$$\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{jk}}{w_{jr}}}{\frac{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}{w_{ir}^{\alpha}}\left(\frac{\sigma_{jk}}{w_{jr}}+\frac{\sigma_{kj}}{w_{kr}}\right)+\frac{\sigma_{jk}^{(\alpha)}}{w_{jr}^{\alpha}}\frac{\sigma_{ik}}{w_{ir}}+\frac{\sigma_{kj}^{(\alpha)}}{w_{kr}^{\alpha}}\frac{\sigma_{ij}}{w_{ir}}}\,, \tag{C.99}$$

$$\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{W}_{ijkj} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}{\frac{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}{w_{ir}^{\alpha}}\left(\frac{\sigma_{jk}}{w_{jr}}+\frac{\sigma_{kj}}{w_{kr}}\right)+\frac{\sigma_{jk}^{(\alpha)}}{w_{jr}^{\alpha}}\frac{\sigma_{ik}}{w_{ir}}+\frac{\sigma_{kj}^{(\alpha)}}{w_{kr}^{\alpha}}\frac{\sigma_{ij}}{w_{ir}}}\,, \tag{C.100}$$

$$\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkj} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}{\frac{\sigma_{kj}^{(\alpha)}+\sigma_{ki}^{(\alpha)}}{w_{kr}^{\alpha}}\left(\frac{\sigma_{ji}}{w_{jr}}+\frac{\sigma_{ij}}{w_{ir}}\right)+\frac{\sigma_{ji}^{(\alpha)}}{w_{jr}^{\alpha}}\frac{\sigma_{ki}}{w_{kr}}+\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}\,; \tag{C.101}$$

$$\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{jk}}{w_{jr}}}{\frac{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{jk}}{w_{jr}}+\frac{\sigma_{jk}^{(\alpha)}}{w_{jr}^{\alpha}}\frac{\sigma_{ik}}{w_{ir}}}\,, \tag{C.102}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{W}_{ijkj} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}{\frac{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}+\frac{\sigma_{kj}^{(\alpha)}}{w_{kr}^{\alpha}}\frac{\sigma_{ij}}{w_{ir}}}\,, \tag{C.103}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkj} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}{\frac{\sigma_{kj}^{(\alpha)}+\sigma_{ki}^{(\alpha)}}{w_{kr}^{\alpha}}\frac{\sigma_{ij}}{w_{ir}}+\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}^{\alpha}}\frac{\sigma_{kj}}{w_{kr}}}\,. \tag{C.104}$$

Finally the limits involving $\overline{\mathbf{C}}_{ijkl}$ are given by ($j \neq i$, $k \neq i, j$, $l \neq i, j, k$ and $r = r_{ijkl} \neq i, j, k, l$)

$$\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \frac{\frac{\sigma_{ijkl}}{w_{ir}w_{kr}}}{\frac{\sigma_{ijkl}+\sigma_{klij}}{w_{ir}w_{kr}}+\frac{\sigma_{ijlk}+\sigma_{lkij}}{w_{ir}w_{lr}}+\frac{\sigma_{jikl}+\sigma_{klji}}{w_{jr}w_{kr}}+\frac{\sigma_{jilk}+\sigma_{lkji}}{w_{jr}w_{lr}}}\,, \tag{C.105}$$

$$\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\sigma_{kl}}{\sigma_{ij}^{(\alpha)}\sigma_{kl}+\sigma_{kl}^{(\alpha)}\sigma_{ij}}\,, \tag{C.106}$$

$$\overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \frac{\sigma_{ij}^{(\alpha)}\frac{\sigma_{kl}}{w_{kr}}}{\sigma_{ij}^{(\alpha)}\left(\frac{\sigma_{kl}}{w_{kr}}+\frac{\sigma_{lk}}{w_{lr}}\right)+\left(\frac{\sigma_{kl}^{(\alpha)}}{w_{kr}}+\frac{\sigma_{lk}^{(\alpha)}}{w_{lr}}\right)\sigma_{ij}}\,, \tag{C.107}$$

$$\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \frac{\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}}\sigma_{kl}}{\sigma_{kl}^{(\alpha)}\left(\frac{\sigma_{ij}}{w_{ir}}+\frac{\sigma_{ji}}{w_{jr}}\right)+\left(\frac{\sigma_{ij}^{(\alpha)}}{w_{ir}}+\frac{\sigma_{ji}^{(\alpha)}}{w_{jr}}\right)\sigma_{kl}}\,. \tag{C.108}$$

Strongly-ordered double-unresolved improved limits

The improved limit $\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}$ is given by ($j \neq i$, $k \neq i$, $l \neq i, k$)

$$\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{\mathrm{s},\,kl}\,\mathcal{W}_{\mathrm{s},\,ij}^{(\alpha)}\,. \tag{C.109}$$

For $\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ikl}$ and $\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}$ we have ($j \neq i$, $k \neq i$, $l \neq i, k$, and $r = r_{ikl} \neq i, k, l$)

$$\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ikl}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{\mathrm{c},\,kl(r)}\,\mathcal{W}_{\mathrm{s},\,ij}^{(\alpha)}\,, \tag{C.110}$$

$$\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,\mathcal{W}_{ijkl} \equiv \mathcal{W}_{\mathrm{s},\,ij}^{(\alpha)}\,. \tag{C.111}$$

For the strongly-ordered double-unresolved limits involving $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}$, we have ($j \neq i$, $k \neq i, j$, $r = r_{ijk} \neq i, j, k$ and $\tau = jk, kj$)

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\mathcal{W}_{ij\tau} \equiv \overline{\mathcal{W}}_{\mathrm{c},\tau(r)}\,\frac{\sigma_{ij}^{(\alpha)}}{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}\,, \tag{C.112}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\mathcal{W}_{ijjk} \equiv \frac{\sigma_{ij}^{(\alpha)}}{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}\,, \tag{C.113}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\mathcal{W}_{ijkj} \equiv \frac{\sigma_{ij}^{(\alpha)}}{\sigma_{ij}^{(\alpha)}+\sigma_{ik}^{(\alpha)}}\,. \tag{C.114}$$

For $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}$ we have ($j \neq i$, $k \neq i$, $l \neq i, k$, and $r = r_{ijkl} \neq i, j, k, l$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkl} \equiv \mathcal{W}_{\mathrm{c},\,ij(r)}^{(\alpha)}\,\overline{\mathcal{W}}_{\mathrm{s},\,kl}\,; \tag{C.115}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{\mathrm{s},\,kl}\,. \tag{C.116}$$

The improved limits $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR\,\mathcal{W}_{ijjk}$ and $\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,RR\,\mathcal{W}_{ijjk}$ read ($j \neq i$, $k \neq i, j$ and $r = r_{ijk} \neq i, j, k$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\mathcal{W}_{ijjk} \equiv \mathcal{W}_{\mathrm{c},\,ij(r)}^{(\alpha)}\,\overline{\mathcal{W}}_{\mathrm{s},\,jk}\,; \tag{C.117}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\mathcal{W}_{ijjk} \equiv \overline{\mathcal{W}}_{\mathrm{s},\,jk}\,. \tag{C.118}$$

For the strongly-ordered double-unresolved limits involving $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}$, we have ($j \neq i$, $k \neq i, j$, $r = r_{ijk} \neq i, j, k$, and $\tau = jk, kj$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ij\tau} \equiv \mathcal{W}_{\mathrm{c},\,ij(r)}^{(\alpha)}\,\overline{\mathcal{W}}_{\mathrm{c},\,\tau(r)}\,; \tag{C.119}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ij\tau} \equiv \overline{\mathcal{W}}_{\mathrm{c},\,\tau(r)}\,; \tag{C.120}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijjk} \equiv \mathcal{W}_{\mathrm{c},\,ij(r)}^{(\alpha)}\,; \tag{C.121}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\mathcal{W}_{ijjk} \equiv 1\,; \tag{C.122}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkj} \equiv \mathcal{W}_{\mathrm{c},\,ij(r)}^{(\alpha)}\,; \tag{C.123}$$

$$\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,\mathcal{W}_{ijkj} \equiv 1\,. \tag{C.124}$$

Finally the limits involving $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}$ are given by ($j \neq i$, $k \neq i,j$, $l \neq i,j,k$ and $r = r_{ijkl} \neq i,j,k,l$)

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,\overline{\mathcal{W}}_{\mathrm{c},\,kl(r)}\,; \tag{C.125}$$

$$\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \overline{\mathcal{W}}_{\mathrm{c},\,kl(r)}\,; \tag{C.126}$$

$$\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv \mathcal{W}^{(\alpha)}_{\mathrm{c},\,ij(r)}\,; \tag{C.127}$$

$$\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,\mathcal{W}_{ijkl} \equiv 1\,. \tag{C.128}$$

C.2.3 Improved limits of $\mathcal{Z}_{ijk}$, $\mathcal{Z}_{ijkl}$

Single-unresolved improved limits

For the single-unresolved improved limits in $K^{(\mathbf{1})}_{\{ijk\}}$ we have ($j \neq i$, $k \neq i,j$)

$$\overline{\mathbf{S}}_{i}\,\mathcal{Z}_{ijk} = \bar{\mathcal{Z}}_{jk}\Big(\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij} + \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ik}\Big)\,, \qquad \overline{\mathbf{HC}}_{ij}\,\mathcal{Z}_{ijk} = \bar{\mathcal{Z}}_{jk}\,; \tag{C.129}$$

while for $K^{(\mathbf{1})}_{\{ijkl\}}$ we have ($j \neq i$, $k \neq i,j$)

$$\overline{\mathbf{S}}_{i}\,\mathcal{Z}_{ijkl} = \bar{\mathcal{Z}}_{kl}\,\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij}\,, \qquad \overline{\mathbf{HC}}_{ij}\,\mathcal{Z}_{ijkl} = \bar{\mathcal{Z}}_{kl}\,. \tag{C.130}$$

Uniform double-unresolved improved limits

For $K^{(\mathbf{2})}_{\{ijk\}}$ we have ($j \neq i$, $k \neq i,j$, and $r = r_{ijk} \neq i,j,k$)

$$\begin{aligned}
\overline{\mathbf{S}}_{ik}\,\mathcal{Z}_{ijk} &= \frac{\sigma_{ikkj} + \sigma_{ijkj} + \sigma_{kiij} + \sigma_{kjij}}{\sum_{b\neq i}\sum_{d\neq i,k}\sigma_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i}\sigma_{kbid}}\,, \\
\overline{\mathbf{SC}}_{ijk}\,\mathcal{Z}_{ijk} &= \frac{\left(\sigma^{(\alpha)}_{ij} + \sigma^{(\alpha)}_{ik}\right)\left(\frac{\sigma_{jk}}{w_{jr}} + \frac{\sigma_{kj}}{w_{kr}}\right) + \frac{\sigma^{(\alpha)}_{jk}}{w_{jr}}\sigma_{ik} + \frac{\sigma^{(\alpha)}_{kj}}{w_{kr}}\sigma_{ij}}{\sum_{b\neq i}\sigma^{(\alpha)}_{ib}\left(\frac{\sigma_{jk}}{w_{jr}} + \frac{\sigma_{kj}}{w_{kr}}\right) + \frac{\sigma^{(\alpha)}_{jk}}{w_{jr}}\sum_{d\neq i,j}\sigma_{id} + \frac{\sigma^{(\alpha)}_{kj}}{w_{kr}}\sum_{d\neq i,k}\sigma_{id}}\,, \\
\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,\mathcal{Z}_{ijk} &= \frac{\left(\sigma^{(\alpha)}_{ij} + \sigma^{(\alpha)}_{ik}\right)\sigma_{jk} + \sigma^{(\alpha)}_{jk}\sigma_{ik}}{\sum_{b\neq i}\sigma^{(\alpha)}_{ib}\sigma_{jk} + \sigma^{(\alpha)}_{jk}\sum_{d\neq i,j}\sigma_{id}}\,, \\
\overline{\mathbf{HC}}_{ijk}\,\mathcal{Z}_{ijk} &= 1\,, \\
\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SHC}}_{ijk}\,\mathcal{Z}_{ijk} &= 1\,.
\end{aligned} \tag{C.131}$$

For $K^{(\mathbf{2})}_{\{ijkl\}}$ one has ($j \neq i$, $k \neq i, j$, $l \neq i, j, k$, and $r = r_{ikl} \neq i, k, l$)

$$\begin{aligned}
\overline{\mathbf{S}}_{ik}\, \mathcal{Z}_{ijkl} \,&=\, \frac{\sigma_{ijkl} + \sigma_{klij}}{\sum_{b\neq i}\sum_{d\neq i,k}\sigma_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i}\sigma_{kbid}}\,, \\
\overline{\mathbf{SC}}_{ikl}\, \mathcal{Z}_{ijkl} \,&=\, \frac{\sigma^{(\alpha)}_{ij}\left(\frac{\sigma_{kl}}{w_{kr}} + \frac{\sigma_{lk}}{w_{lr}}\right) + \left(\frac{\sigma^{(\alpha)}_{kl}}{w_{kr}} + \frac{\sigma^{(\alpha)}_{lk}}{w_{lr}}\right)\sigma_{ij}}{\sum_{b\neq i}\sigma^{(\alpha)}_{ib}\left(\frac{\sigma_{kl}}{w_{kr}} + \frac{\sigma_{lk}}{w_{lr}}\right) + \frac{\sigma^{(\alpha)}_{kl}}{w_{kr}}\sum_{d\neq i,k}\sigma_{id} + \frac{\sigma^{(\alpha)}_{lk}}{w_{lr}}\sum_{d\neq i,l}\sigma_{id}}\,, \\
\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\, \mathcal{Z}_{ijkl} \,&=\, \frac{\sigma^{(\alpha)}_{ij}\sigma_{kl} + \sigma^{(\alpha)}_{kl}\sigma_{ij}}{\sum_{b\neq i}\sigma^{(\alpha)}_{ib}\sigma_{kl} + \sigma^{(\alpha)}_{kl}\sum_{d\neq i,k}\sigma_{id}}\,, \\
\overline{\mathbf{HC}}_{ijkl}\, \mathcal{Z}_{ijkl} \,&=\, 1\,.
\end{aligned} \tag{C.132}$$

Strongly-ordered double-unresolved improved limits

For $K^{(\mathbf{12})}_{\{ijk\}}$ one has ($j \neq i$, $k \neq i, j$)

$$\begin{aligned}
\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\, \mathcal{Z}_{ijk} \,&=\, \bar{\mathcal{Z}}_{\mathrm{s},\,jk}\left(\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij} + \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ik}\right), \\
\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SHC}}_{ijk}\, \mathcal{Z}_{ijk} \,&=\, \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij} + \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ik}\,, \\
\overline{\mathbf{S}}_{i}\,\overline{\mathbf{HC}}^{(\mathbf{s})}_{ijk}\, \mathcal{Z}_{ijk} \,&=\, 1\,, \\
\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{S}}_{ij}\, \mathcal{Z}_{ijk} \,&=\, \bar{\mathcal{Z}}_{\mathrm{s},\,jk}\,, \\
\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{SC}}_{kij}\, \mathcal{Z}_{ijk} \,&=\, \bar{\mathcal{Z}}_{\mathrm{s},\,kj}\,, \\
\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{HC}}^{(\mathbf{c})}_{ijk}\, \mathcal{Z}_{ijk} \,&=\, 1\,.
\end{aligned} \tag{C.133}$$

For $K^{(\mathbf{12})}_{\{ijkl\}}$ one has ($j \neq i$, $k \neq i, j$, $l \neq i, j, k$)

$$\begin{aligned}
\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\, \mathcal{Z}_{ijkl} \,&=\, \bar{\mathcal{Z}}_{\mathrm{s},\,kl}\,\mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij}\,, \\
\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SHC}}_{ikl}\, \mathcal{Z}_{ijkl} \,&=\, \mathcal{Z}^{(\alpha)}_{\mathrm{s},\,ij}\,, \\
\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{SC}}_{kij}\, \mathcal{Z}_{ijkl} \,&=\, \bar{\mathcal{Z}}_{\mathrm{s},\,kl}\,, \\
\overline{\mathbf{HC}}_{ij}\,\overline{\mathbf{HC}}^{(\mathbf{c})}_{ijkl}\, \mathcal{Z}_{ijkl} \,&=\, 1\,.
\end{aligned} \tag{C.134}$$

C.3 The consistency of the double-real contribution $RR_{\rm sub}$

The finiteness of the subtracted double-real contribution $RR_{\rm sub}$ in Eq. (3.33) is achieved once the integrability of

$$\begin{aligned}
RR\,\mathcal{W}_{ijjk} - K^{(\mathbf{1})}_{ijjk} - \left(K^{(\mathbf{2})}_{ijjk} - K^{(\mathbf{12})}_{ijjk}\right) &\to \text{integrable}\,,\\
RR\,\mathcal{W}_{ijkj} - K^{(\mathbf{1})}_{ijkj} - \left(K^{(\mathbf{2})}_{ijkj} - K^{(\mathbf{12})}_{ijkj}\right) &\to \text{integrable}\,,\\
RR\,\mathcal{W}_{ijkl} - K^{(\mathbf{1})}_{ijkl} - \left(K^{(\mathbf{2})}_{ijkl} - K^{(\mathbf{12})}_{ijkl}\right) &\to \text{integrable}\,,
\end{aligned} \tag{C.135}$$

has been proven. In the following sections, topology by topology, we provide a detailed list of all the relevant consistency relations that establish the locality of our singularity-cancellation procedure. When an entry of the following lists involves a difference between two improved limits, it implies that both contributions exhibit the same leading singular behaviour in that particular limit. In cases where more than two improved limits are involved in a given consistency condition, then it means that all of these limits display the same leading singular behaviour.

C.3.1 Topology $\mathcal{W}_{ijjk}$

Let us prove that

$$RR\,\mathcal{W}_{ijjk} - K^{(\mathbf{1})}_{ijjk} - \left(K^{(\mathbf{2})}_{ijjk} - K^{(\mathbf{12})}_{ijjk}\right) \to \text{integrable}\,, \tag{C.136}$$

in the proper singular limits of this topology (namely the first line of Eq. (3.37)):

$$\mathbf{S}_i\,, \qquad \mathbf{C}_{ij}\,, \qquad \mathbf{S}_{ij}\,, \qquad \mathbf{C}_{ijk}\,, \qquad \mathbf{SC}_{ijk}\,. \tag{C.137}$$

The counterterms of this topology are

$$\begin{aligned}
K^{(\mathbf{1})}_{ijjk} &= \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i\right)\right] RR\,\mathcal{W}_{ijjk}\,,\\
K^{(\mathbf{2})}_{ijjk} &= \left[\overline{\mathbf{S}}_{ij} + \overline{\mathbf{SC}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}\right) + \overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}\right)\left(1-\overline{\mathbf{SC}}_{ijk}\right)\right] RR\,\mathcal{W}_{ijjk}\,,\\
K^{(\mathbf{12})}_{ijjk} &= \Big[\overline{\mathbf{S}}_i\left(\overline{\mathbf{S}}_{ij} + \overline{\mathbf{SC}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}\right) + \overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}\right)\left(1-\overline{\mathbf{SC}}_{ijk}\right)\right)\\
&\qquad + \overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i\right)\left(\overline{\mathbf{S}}_{ij} + \overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}\right)\right)\Big] RR\,\mathcal{W}_{ijjk}\,.
\end{aligned} \tag{C.138}$$

The consistency of this topology is achieved through the consistency relations:

- Limit $\mathbf{S}_i$

$$
\begin{aligned}
\mathbf{S}_i \,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_i \,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,; \qquad (C.139)
\end{aligned}
$$

- Limit $\mathbf{C}_{ij}$

$$
\begin{aligned}
\mathbf{C}_{ij}\,(1-\overline{\mathbf{C}}_{ij})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{C}}_{ij})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{C}}_{ij})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{C}}_{ij})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{C}}_{ij})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{C}}_{ij})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{C}}_{ij})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{C}}_{ij})\, RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,; \qquad (C.140)
\end{aligned}
$$

- Limit $\mathbf{S}_{ij}$

$$
\begin{aligned}
\mathbf{S}_{ij}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,;
\end{aligned}
\tag{C.141}
$$

- Limit $\mathbf{C}_{ijk}$

$$
\begin{aligned}
\mathbf{C}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,;
\end{aligned}
\tag{C.142}
$$

- Limit $\mathbf{SC}_{ijk}$

$$
\begin{aligned}
\mathbf{SC}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,.
\end{aligned}
\tag{C.143}
$$

In addition to the proper singular limits of $RR\,\mathcal{W}_{ijjk}$, the improved limits of RR have spurious limits, which must be consistently compensated by the sector functions and/or by other improved limits[1] (see first line of Eq. (3.38)):

- Limit $\mathbf{C}_{ir}$

$$
\begin{aligned}
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,;
\end{aligned}
\tag{C.144}
$$

[1] In the list we omit the other singular limits which are trivially compensated by the sector functions such as $\mathbf{C}_{kr}$, $\mathbf{C}_{ikr}$, and $\mathbf{C}_{jkr}$.

- Limit $\mathbf{C}_{jr}$

$$\begin{aligned}
\mathbf{C}_{jr}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{jr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{jr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{jr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,;
\end{aligned} \tag{C.145}$$

- Limit $\mathbf{C}_{ijr}$

$$\begin{aligned}
\mathbf{C}_{ijr}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijr}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijr}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{ijr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,.
\end{aligned} \tag{C.146}$$

To complete our analysis, we must also examine the consistency relations for secondary limits that are not suppressed by sector functions when taken in specific limits. As specified in Eq. (3.39), we verify

- Limit $\mathbf{S}_{j}$

$$\begin{aligned}
\mathbf{S}_{j}\,\overline{\mathbf{S}}_{i}(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{j}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{S}}_{ij})(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{S}_{j}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}(1-\overline{\mathbf{S}}_{ij})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,;
\end{aligned} \tag{C.147}$$

- Limit $\mathbf{C}_{jk}$

$$\begin{aligned}
\mathbf{C}_{jk}\,\overline{\mathbf{S}}_{i}(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{jk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{jk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,,\\
\mathbf{C}_{jk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijjk} &\to \text{integrable}\,.
\end{aligned} \tag{C.148}$$

C.3.2 Topology $\mathcal{W}_{ijkj}$

Let us prove that

$$RR\,\mathcal{W}_{ijkj} - K^{(\mathbf{1})}_{ijkj} - \left(K^{(\mathbf{2})}_{ijkj} - K^{(\mathbf{12})}_{ijkj}\right) \to \text{integrable}\,, \tag{C.149}$$

in the proper singular limits of this topology (namely the second line of Eq. (3.37)):

$$\mathbf{S}_i\,, \qquad \mathbf{C}_{ij}\,, \qquad \mathbf{S}_{ik}\,, \qquad \mathbf{C}_{ijk}\,, \qquad \mathbf{SC}_{ijk}\,, \qquad \mathbf{SC}_{kij}\,. \tag{C.150}$$

The counterterms of this topology are

$$\begin{aligned}
K^{(\mathbf{1})}_{ijkj} &= \Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i\right)\Big]\, RR\,\mathcal{W}_{ijkj}\,,\\
K^{(\mathbf{2})}_{ijkj} &= \Big[\overline{\mathbf{S}}_{ik} + \left(\overline{\mathbf{SC}}_{ijk} + \overline{\mathbf{SC}}_{kij}\right)\left(1-\overline{\mathbf{S}}_{ik}\right) + \overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ik}\right)\left(1-\overline{\mathbf{SC}}_{ijk}-\overline{\mathbf{SC}}_{kij}\right)\Big] RR\,\mathcal{W}_{ijkj}\,,\\
K^{(\mathbf{12})}_{ijkj} &= \Big[\overline{\mathbf{S}}_i\left(\overline{\mathbf{S}}_{ik} + \overline{\mathbf{SC}}_{ijk}\left(1-\overline{\mathbf{S}}_{ik}\right) + \overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ik}\right)\left(1-\overline{\mathbf{SC}}_{ijk}\right)\right)\\
&\qquad + \overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i\right)\left(\overline{\mathbf{SC}}_{kij} + \overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{SC}}_{kij}\right)\right)\Big]\, RR\,\mathcal{W}_{ijkj}\,.
\end{aligned} \tag{C.151}$$

The consistency of this topology is achieved through the consistency:

- Limit $\mathbf{S}_i$

$$\begin{aligned}
\mathbf{S}_i\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\, RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,;
\end{aligned} \tag{C.152}$$

- Limit $\mathbf{C}_{ij}$

$$
\begin{aligned}
\mathbf{C}_{ij}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,;
\end{aligned}
\tag{C.153}
$$

- Limit $\mathbf{S}_{ik}$

$$
\begin{aligned}
\mathbf{S}_{ik}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,;
\end{aligned}
\tag{C.154}
$$

- Limit $\mathbf{C}_{ijk}$

$$
\begin{aligned}
\mathbf{C}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,; \qquad \text{(C.155)}
\end{aligned}
$$

- Limit $\mathbf{SC}_{ijk}$

$$
\begin{aligned}
\mathbf{SC}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{ijk}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{i})\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,; \qquad \text{(C.156)}
\end{aligned}
$$

- Limit $\mathbf{SC}_{kij}$

$$
\begin{aligned}
\mathbf{SC}_{kij}\,(1-\overline{\mathbf{SC}}_{kij})\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,(\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij})\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,(1-\overline{\mathbf{S}}_{ik})\,(1-\overline{\mathbf{C}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,. \qquad \text{(C.157)}
\end{aligned}
$$

In addition to the proper singular limits of $RR\,\mathcal{W}_{ijkj}$, the improved limits of RR have spurious limits, which must be consistently compensated by the sector functions and/or by other improved limits[2] (see second line of Eq. (3.38)):

- Limit $\mathbf{C}_{ir}$

$$
\begin{aligned}
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,; \qquad \text{(C.158)}
\end{aligned}
$$

[2] In the list we omit the other singular limits which are trivially compensated by the sector functions such as $\mathbf{C}_{jr}$, $\mathbf{C}_{ijr}$, and $\mathbf{C}_{jkr}$.

- Limit $\mathbf{C}_{kr}$

$$\begin{aligned}
\mathbf{C}_{kr}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,;
\end{aligned} \tag{C.159}$$

- Limit $\mathbf{C}_{ikr}$

$$\begin{aligned}
\mathbf{C}_{ikr}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ikr}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ikr}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ikr}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{ikr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,.
\end{aligned} \tag{C.160}$$

To complete our analysis, we must also examine the consistency relations for secondary limits that are not suppressed by sector functions when taken in specific limits. As specified in Eq. (3.39), we verify

- Limit $\mathbf{S}_{k}$

$$\begin{aligned}
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{S}}_{ik})(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ijk}(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,;
\end{aligned} \tag{C.161}$$

- Limit $\mathbf{C}_{jk}$

$$\begin{aligned}
\mathbf{C}_{jk}\,\overline{\mathbf{S}}_{i}(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{jk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{jk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,,\\
\mathbf{C}_{jk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathcal{W}_{ijkj} &\to \text{integrable}\,.
\end{aligned} \tag{C.162}$$

C.3.3 Topology $\mathcal{W}_{ijkl}$

Let us prove that

$$RR\,\mathcal{W}_{ijkl} - K^{(\mathbf{1})}_{ijkl} - \Big(K^{(\mathbf{2})}_{ijkl} - K^{(\mathbf{12})}_{ijkl}\Big) \to \text{integrable}\,, \tag{C.163}$$

in the proper singular limits of this topology (namely the third line of Eq. (3.37)):

$$\mathbf{S}_i\,, \qquad \mathbf{C}_{ij}\,, \qquad \mathbf{S}_{ik}\,, \qquad \mathbf{SC}_{ikl}\,, \qquad \mathbf{SC}_{kij}\,, \qquad \mathbf{C}_{ijkl}\,. \tag{C.164}$$

The counterterms of this topology are

$$\begin{aligned}
K^{(\mathbf{1})}_{ijkl} &= \Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}\,\big(1-\overline{\mathbf{S}}_i\big)\Big]\,RR\,\mathcal{W}_{ijkl}\,,\\
K^{(\mathbf{2})}_{ijkl} &= \Big[\overline{\mathbf{S}}_{ik} + \big(\overline{\mathbf{SC}}_{ikl} + \overline{\mathbf{SC}}_{kij}\big)\,\big(1-\overline{\mathbf{S}}_{ik}\big) + \overline{\mathbf{C}}_{ijkl}\,\big(1+\overline{\mathbf{S}}_{ik} - \overline{\mathbf{SC}}_{ikl} - \overline{\mathbf{SC}}_{kij}\big)\Big]RR\,\mathcal{W}_{ijkl}\,,\\
K^{(\mathbf{12})}_{ijkl} &= \Big[\overline{\mathbf{S}}_i\,\Big(\overline{\mathbf{S}}_{ik} + \overline{\mathbf{SC}}_{ikl}\,\big(1-\overline{\mathbf{S}}_{ik}\big)\Big)\\
&\qquad + \overline{\mathbf{C}}_{ij}\,\big(1-\overline{\mathbf{S}}_i\big)\,\Big(\overline{\mathbf{SC}}_{kij} + \overline{\mathbf{C}}_{ijkl}\,\big(1-\overline{\mathbf{SC}}_{kij}\big)\Big)\Big]RR\,\mathcal{W}_{ijkl}\,.
\end{aligned} \tag{C.165}$$

The consistency of this topology is achieved through the consistency:

- Limit $\mathbf{S}_i$

$$\begin{aligned}
\mathbf{S}_i\,(1-\overline{\mathbf{S}}_i)\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_i)\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{S}}_i)\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_i)\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_i)\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{SC}}_{ikl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ijkl}\,(\overline{\mathbf{S}}_{ik}-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_i)\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{S}}_i)\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_i\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,\overline{\mathbf{SC}}_{ijk}\,(1-\overline{\mathbf{S}}_i)\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,;
\end{aligned} \tag{C.166}$$

- Limit $\mathbf{C}_{ij}$

$$
\begin{aligned}
\mathbf{C}_{ij}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,(\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,(\overline{\mathbf{SC}}_{ikl}-\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,;
\end{aligned}
\tag{C.167}
$$

- Limit $\mathbf{S}_{ik}$

$$
\begin{aligned}
\mathbf{S}_{ik}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{C}}_{ijkl}\Big[(1-\overline{\mathbf{SC}}_{kij})\Big(1-\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\Big)+\overline{\mathbf{S}}_{ik}-\overline{\mathbf{SC}}_{ikl}\Big]\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{ik}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,;
\end{aligned}
\tag{C.168}
$$

- Limit $\mathbf{SC}_{kij}$

$$
\begin{aligned}
\mathbf{SC}_{kij}\,(1-\overline{\mathbf{SC}}_{kij})\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,(\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij})\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{SC}}_{kij})\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{C}}_{ijkl}\Big[\overline{\mathbf{S}}_{ik}-\overline{\mathbf{SC}}_{ikl}+\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{SC}}_{kij})\Big]RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{kij}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,;
\end{aligned}
\tag{C.169}
$$

- Limit $\mathbf{SC}_{ikl}$

$$
\begin{aligned}
\mathbf{SC}_{ikl}\,(1-\overline{\mathbf{SC}}_{ikl})\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{ikl}\Big[\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,(1-\overline{\mathbf{C}}_{ijkl})+\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{SC}}_{ikl})\Big]\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{ikl}\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{SC}}_{ikl})\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{ikl}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{SC}_{ikl}\Big[\overline{\mathbf{C}}_{ijkl}\,(\overline{\mathbf{S}}_{ik}-\overline{\mathbf{SC}}_{kij})-\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{i})\,(1-\overline{\mathbf{C}}_{ijkl})\Big]RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,;
\end{aligned}
\tag{C.170}
$$

- Limit $\mathbf{C}_{ijkl}$

$$
\begin{aligned}
\mathbf{C}_{ijkl}\,(1-\overline{\mathbf{C}}_{ijkl})\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ijkl}\,\overline{\mathbf{S}}_{i}\,(1-\overline{\mathbf{SC}}_{ikl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ijkl}\,(\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}-\overline{\mathbf{SC}}_{ikl})\,(1-\overline{\mathbf{C}}_{ijkl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ijkl}\Big[\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{SC}}_{ikl}-\overline{\mathbf{SC}}_{kij}+\overline{\mathbf{C}}_{ijkl})&\\
+\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ijkl})\Big]RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ijkl}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{C}}_{ijkl})\,(1-\overline{\mathbf{C}}_{ij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ijkl}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}\,(1-\overline{\mathbf{SC}}_{ikl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,.
\end{aligned}
\tag{C.171}
$$

In addition to the proper singular limits of $RR\,\mathcal{W}_{ijkl}$, the improved limits of RR have spurious limits, which must be consistently compensated by the sector functions and/or by other improved limits[3] (see third line of Eq. (3.38)):

- Limit $\mathbf{C}_{ir}$

$$
\begin{aligned}
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{SC}}_{ikl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ijkl}\,(\overline{\mathbf{SC}}_{kij}-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{ir}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{S}}_{i})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,;
\end{aligned}
\tag{C.172}
$$

[3]In the list we omit the other singular limits which are trivially compensated by the sector functions such as $\mathbf{C}_{jr}$ and $\mathbf{C}_{lr}$.

- Limit $\mathbf{C}_{kr}$

$$\begin{aligned}
\mathbf{C}_{kr}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{C}}_{ijkl}\,(\overline{\mathbf{SC}}_{ikl}-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ikl}\,(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kr}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}\,(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,.
\end{aligned}\tag{C.173}$$

To complete our analysis, we must also examine the consistency relations for secondary limits that are not suppressed by sector functions when taken in specific limits. As specified in Eq. (3.39), we verify

- Limit $\mathbf{S}_{k}$

$$\begin{aligned}
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{SC}}_{ikl}(1-\overline{\mathbf{S}}_{ik})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{S}_{k}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijkl}(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,;
\end{aligned}\tag{C.174}$$

- Limit $\mathbf{C}_{kl}$

$$\begin{aligned}
\mathbf{C}_{kl}\,\overline{\mathbf{S}}_{i}(1-\overline{\mathbf{SC}}_{ikl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kl}\,\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{C}}_{ijkl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kl}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{C}}_{ijkl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kl}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ik}(1-\overline{\mathbf{SC}}_{ikl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kl}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}(1-\overline{\mathbf{C}}_{ijkl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,,\\
\mathbf{C}_{kl}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{SC}}_{kij}(1-\overline{\mathbf{C}}_{ijkl})\,RR\,\mathcal{W}_{ijkl} &\to \text{integrable}\,.
\end{aligned}\tag{C.175}$$

C.4 Integration of azimuthal contributions

The azimuthal parts of the collinear kernels $Q^{\mu\nu}_{ij(r)}$, $\tilde{Q}^{\mu\nu}_{ij(r)}$ and $Q^{\mu\nu}_{ijk(r)}$, defined in Appendix C.1, contain $\tilde{k}^{\mu}_{a}\tilde{k}^{\nu}_{a}$, where $a=i$ for $P^{\mu\nu}_{ij(r)}$, $\tilde{P}^{\mu\nu}_{ij(r)}$ and $a=i,j,k$ for $P^{\mu\nu}_{ijk(r)}$. In all counterterms, $Q^{\mu\nu}_{ij(r)}$ has to be integrated in the single-radiative phase space $d\Phi^{(ijr)}_{\rm rad}$, $d\Phi^{(irj)}_{\rm rad}$ or $d\Phi^{(jri)}_{\rm rad}$, while $\tilde{Q}^{\mu\nu}_{ij(r)}$ and $Q^{\mu\nu}_{ijk(r)}$ are always integrated in $d\Phi^{(ijr)}_{\rm rad}$ and $d\Phi^{(ijkr)}_{\rm rad,2}$, respectively. In all cases, when integrating $Q^{\mu\nu}_{ij(r)}$ and $\tilde{Q}^{\mu\nu}_{ij(r)}$ in their single-radiative phase space, or $Q^{\mu\nu}_{ijk(r)}$ in its double-radiative phase space, the integral of the tensor structure $\tilde{k}^\mu_a \tilde{k}^\nu_a$ must be a symmetric rank-2 tensor constructed combining $g^{\mu\nu}$ and mapped momenta, see [18]. Thus

$$\int d\Phi^{(\tau)}_{\rm rad}\, f(\{k\})\, \tilde{k}^\mu_a \tilde{k}^\nu_a = A\, g^{\mu\nu} + B\, \bar{k}^{(\tau)\mu} \bar{k}^{(\tau)\nu} + C\Big(\bar{k}^{(\tau)\mu} \bar{k}^{(\tau)\nu}_q + \bar{k}^{(\tau)\mu}_q \bar{k}^{(\tau)\nu}\Big) + D\, \bar{k}^{(\tau)\mu}_q \bar{k}^{(\tau)\nu}_q\,, \tag{C.176}$$

where $\tau = ijr, irj, jri, ijkr$, $q = r$ if $\tau = ijr, irj, jri$, $q = r$ if $\tau = ijkr$, and

$$\bar{k}^{(ijr)} = \bar{k}^{(ijr)}_j\,, \qquad \bar{k}^{(irj)} = \bar{k}^{(irj)}_j\,, \qquad \bar{k}^{(jri)} = \bar{k}^{(jri)}_i\,, \qquad \bar{k}^{(ijkr)} = \bar{k}^{(ijkr)}_k\,. \tag{C.177}$$

Since $\tilde{k}_a$ is orthogonal to $\bar{k}^{(\tau)\mu}$ and $\bar{k}^{(\tau)\mu}_q$, so must be also its integrals. This leads to the conditions $D = 0$ and $A + C\, \bar{k}^{(\tau)} \cdot \bar{k}^{(\tau)}_q = 0$. We have

$$\int d\Phi^{(\tau)}_{\rm rad}\, f(\{k\})\, \tilde{k}^\mu_a \tilde{k}^\nu_a = A\left[g^{\mu\nu} - \frac{\bar{k}^{(\tau)\mu} \bar{k}^{(\tau)\nu}_q + \bar{k}^{(\tau)\mu}_q \bar{k}^{(\tau)\nu}}{\bar{k}^{(\tau)} \cdot \bar{k}^{(\tau)}_q}\right] + B\, \bar{k}^{(\tau)\mu} \bar{k}^{(\tau)\nu}\,. \tag{C.178}$$

In all counterterms this tensor is contracted with either

$$\bar{R}^{(\tau)}_{\mu\nu}\,, \qquad \bar{B}^{(\tau)}_{\mu\nu}\,, \qquad \bar{B}^{(\tau,\dots)}_{\mu\nu}\,, \qquad \text{or} \qquad \left[\frac{\bar{k}^{(\tau)}_{c,\mu}}{\bar{s}^{(\tau)}_{jc}} - \frac{\bar{k}^{(\tau)}_{d,\mu}}{\bar{s}^{(\tau)}_{jd}}\right]\left[\frac{\bar{k}^{(\tau)}_{c,\nu}}{\bar{s}^{(\tau)}_{jc}} - \frac{\bar{k}^{(\tau)}_{d,\nu}}{\bar{s}^{(\tau)}_{jd}}\right]. \tag{C.179}$$

As a consequence, the terms proportional to $\bar{k}^{(\tau)\mu}$ or to $\bar{k}^{(\tau)\nu}$ vanish, and just $A\, g^{\mu\nu}$ contributes. On the other hand, since $\bar{k}^{(\tau)}$ is on shell, A can be obtained as follows:

$$g_{\mu\nu} \int d\Phi^{(\tau)}_{\rm rad}\, f(\{k\})\, \tilde{k}^\mu_a \tilde{k}^\nu_a \;=\; A\,(d-2) \quad\Longrightarrow\quad A \;=\; \frac{1}{d-2} \int d\Phi^{(\tau)}_{\rm rad}\, f(\{k\})\, \tilde{k}^2_a\,. \tag{C.180}$$

Thus in all counterterms we can subtitute

$$\int d\Phi^{(\tau)}_{\rm rad}\, f(\{k\})\, \tilde{k}^\mu_a \tilde{k}^\nu_a \;\to\; A\, g^{\mu\nu} \;=\; \int d\Phi^{(\tau)}_{\rm rad}\, f(\{k\})\, \frac{g^{\mu\nu}}{d-2}\, \tilde{k}^2_a\,, \tag{C.181}$$

and the integrals of $Q^{\mu\nu}_{ij(r)}$, $\tilde{Q}^{\mu\nu}_{ij(r)}$ and $Q^{\mu\nu}_{ijk(r)}$ vanish in all counterterms:

$$\begin{aligned}
\int d\Phi^{(\tau)}_{\rm rad}\, \frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}} &= \int d\Phi^{(\tau)}_{\rm rad}\, \frac{Q_{ij(r)}}{s_{ij}}\left[-\, g^{\mu\nu} + (d-2)\, \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i}\right] \;\to\; 0\,, \qquad \tau = ijr, irj, jri\,;\\
\int d\Phi^{(\tau)}_{\rm rad}\, \frac{\tilde{Q}^{\mu\nu}_{ij(r)}}{s_{ij}} &= \int d\Phi^{(\tau)}_{\rm rad}\, \frac{\tilde{Q}_{ij(r)}}{s_{ij}}\left[-\, g^{\mu\nu} + (d-2)\, \frac{\tilde{k}^\mu_i \tilde{k}^\nu_i}{\tilde{k}^2_i}\right] \;\to\; 0\,, \qquad \tau = ijr\,;\\
\int d\Phi^{(\tau)}_{\rm rad,2}\, \frac{Q^{\mu\nu}_{ijk(r)}}{s^2_{ijk}} &= \sum_{a=i,j,k} \int d\Phi^{(\tau)}_{\rm rad,2}\, \frac{Q^{(a)}_{ijk(r)}}{s^2_{ijk}}\left[-\, g^{\mu\nu} + (d-2)\, \frac{\tilde{k}^\mu_a \tilde{k}^\nu_a}{\tilde{k}^2_a}\right] \;\to\; 0\,, \qquad \tau = ijkr\,.
\end{aligned} \tag{C.182}$$

C.5 Constituent integrals

In the following we report the constituent integrals relevant for the analytic integration of all counterterms at NNLO. Such integrals are schematically denoted as J_t^ℓ, where t indicates the type of integral, while ℓ is a set of labels whose different indices denote distinguished particles.
The soft integrated kernel is

$$J_{\rm s}^{ilm} \;\equiv\; \mathcal{N}_1 \int d\Phi_{\rm rad}^{(ilm)}\, \mathcal{E}_{lm}^{(i)} \;\equiv\; \delta_{f_i g}\, J_s(\bar{s}_{lm}^{(ilm)})\,, \tag{C.183}$$

with

$$\begin{aligned} J_{\rm s}(s) \,=\, & \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \\ =\, & \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\bigg[\;\frac{1}{\epsilon^2}+\frac{2}{\epsilon}+6-\frac{7}{12}\pi^2+\left(18-\frac{7}{6}\pi^2-\frac{25}{3}\zeta_3\right)\epsilon \\ & +\left(54-\frac{7}{2}\pi^2-\frac{50}{3}\zeta_3-\frac{71}{1440}\pi^4\right)\epsilon^2+\mathcal{O}(\epsilon^3)\bigg]. \end{aligned} \tag{C.184}$$

The double-soft integrated kernels read

$$\begin{aligned} J_{\rm s\otimes s}^{ijcdef} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(icd,jef)}\, \mathcal{E}_{cd}^{(i)}\, \mathcal{E}_{ef}^{(j)} \;\equiv\; J_{\rm s\otimes s}^{(4)}\left(\bar{s}_{cd}^{(icd,jef)}, \bar{s}_{ef}^{(icd,jef)}\right) f_{ij}^{gg}\,, \\ J_{\rm s\otimes s}^{ijcde} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(icd,jed)}\, \mathcal{E}_{cd}^{(i)}\, \mathcal{E}_{ed}^{(j)} \;\equiv\; J_{\rm s\otimes s}^{(3)}\left(\bar{s}_{cd}^{(icd,jed)}, \bar{s}_{ed}^{(icd,jed)}\right) f_{ij}^{gg}\,, \\ J_{\rm s\otimes s}^{ijcd} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(ijcd)}\, \mathcal{E}_{cd}^{(i)}\, \mathcal{E}_{cd}^{(j)} \;\equiv\; J_{\rm s\otimes s}^{(2)}\left(\bar{s}_{cd}^{(ijcd)}\right) f_{ij}^{gg}\,, \\ J_{\rm ss}^{ijcd} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(ijcd)}\, \mathcal{E}_{cd}^{(ij)} \;\equiv\; 2\,T_R\, J_{\rm ss}^{(\rm q\bar{q})}\left(\bar{s}_{cd}^{(ijcd)}\right) f_{ij}^{q\bar{q}} - 2\,C_A\, J_{\rm ss}^{(\rm gg)}\left(\bar{s}_{cd}^{(ijcd)}\right) f_{ij}^{gg}\,, \end{aligned} \tag{C.185}$$

with

$$J^{(4)}_{s\otimes s}(s,s') = \left(\frac{\alpha_S}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} \left[\frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(16 - \frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2} + \left(60 - \frac{14}{3}\pi^2 - \frac{50}{3}\zeta_3\right)\frac{1}{\epsilon} + 216 - \frac{56}{3}\pi^2 - \frac{200}{3}\zeta_3 + \frac{29}{120}\pi^4 + \mathcal{O}(\epsilon)\right],$$

$$J^{(3)}_{s\otimes s}(s,s') = \left(\frac{\alpha_S}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} \left[\frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(17 - \frac{4}{3}\pi^2\right)\frac{1}{\epsilon^2} + \left(70 - \frac{16}{3}\pi^2 - \frac{68}{3}\zeta_3\right)\frac{1}{\epsilon} + 284 - \frac{68}{3}\pi^2 - \frac{272}{3}\zeta_3 + \frac{13}{90}\pi^4 + \mathcal{O}(\epsilon)\right],$$

$$J^{(2)}_{s\otimes s}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} \left[\frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(18 - \frac{3}{2}\pi^2\right)\frac{1}{\epsilon^2} + \left(76 - 6\pi^2 - \frac{74}{3}\zeta_3\right)\frac{1}{\epsilon} + 312 - 27\pi^2 - \frac{308}{3}\zeta_3 + \frac{49}{120}\pi^4 + \mathcal{O}(\epsilon)\right],$$

$$J^{(q\bar{q})}_{ss}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} \left[\frac{1}{6}\frac{1}{\epsilon^3} + \frac{17}{18}\frac{1}{\epsilon^2} + \left(\frac{116}{27} - \frac{7}{36}\pi^2\right)\frac{1}{\epsilon} + \frac{1474}{81} - \frac{131}{108}\pi^2 - \frac{19}{9}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{(gg)}_{ss}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} \left[\frac{1}{2}\frac{1}{\epsilon^4} + \frac{35}{12}\frac{1}{\epsilon^3} + \left(\frac{487}{36} - \frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2} + \left(\frac{1562}{27} - \frac{269}{72}\pi^2 - \frac{77}{6}\zeta_3\right)\frac{1}{\epsilon} + \frac{19351}{81} - \frac{3829}{216}\pi^2 - \frac{1025}{18}\zeta_3 - \frac{23}{240}\pi^4 + \mathcal{O}(\epsilon)\right]. \quad \text{(C.186)}$$

The soft real-virtual integrated kernels are

$$\tilde{J}^{icd}_s \equiv \mathcal{N}_1 \int d\Phi^{(icd)}_{\rm rad}\, \tilde{\mathcal{E}}^{(i)}_{cd} \equiv \delta_{f_i g}\, C_A\, \tilde{J}_s\left(\bar{s}^{(icd)}_{cd}\right),$$

$$J^{icd(e)}_{\Delta s} \equiv \mathcal{N}_1 \frac{2}{\epsilon^2} \int d\Phi^{(icd)}_{\rm rad}\, \mathcal{E}^{(i)}_{cd} \left[\left(\frac{s_{ed}}{\bar{s}^{(icd)}_{ed}}\right)^{-\epsilon} - 1\right] \equiv f^g_i\, J^{(3)}_{\Delta s}\left(\bar{s}^{(icd)}_{cd}\right),$$

$$J^{icd}_{\Delta s} \equiv \mathcal{N}_1 \frac{1}{\epsilon^2} \int d\Phi^{(icd)}_{\rm rad}\, \mathcal{E}^{(i)}_{cd} \left[\left(\frac{s_{cd}}{\bar{s}^{(icd)}_{cd}}\right)^{-\epsilon} - 1\right] \equiv f^g_i\, J^{(2)}_{\Delta s}\left(\bar{s}^{(icd)}_{cd}\right),$$

$$\tilde{J}^{icde}_s \equiv \mathcal{N}_1 \int d\Phi^{(icd)}_{\rm rad}\, \tilde{\mathcal{E}}^{(i)}_{cde}, \quad \text{(C.187)}$$

with

$$\tilde{J}_{\mathrm{s}}(s) = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-2\epsilon}\frac{\Gamma^3(1+\epsilon)\Gamma^3(1-\epsilon)}{4\,\epsilon^4\,\Gamma(1+2\epsilon)\Gamma(2-4\epsilon)} \tag{C.188}$$
$$= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\left[\frac{1}{4\epsilon^4}+\frac{1}{\epsilon^3}+\left(4-\frac{7}{24}\pi^2\right)\frac{1}{\epsilon^2}+\left(16-\frac{7}{6}\pi^2-\frac{14}{3}\zeta_3\right)\frac{1}{\epsilon} + 64-\frac{14}{3}\pi^2-\frac{56}{3}\zeta_3-\frac{7}{480}\pi^4+\mathcal{O}(\epsilon)\right],$$

$$J^{(3)}_{\Delta\mathrm{s}}(s) = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\left(2-\frac{\pi^2}{3}\right)\frac{1}{\epsilon^2}+\left(16-\frac{2}{3}\pi^2-12\,\zeta_3\right)\frac{1}{\epsilon} + 92-\frac{7}{2}\pi^2-24\,\zeta_3-\frac{7}{18}\pi^4+\mathcal{O}(\epsilon)\right],$$

$$J^{(2)}_{\Delta\mathrm{s}}(s) = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\left(2-\frac{\pi^2}{3}\right)\frac{1}{\epsilon^2}+\left(14-\frac{2}{3}\pi^2-10\,\zeta_3\right)\frac{1}{\epsilon} + 74-\frac{23}{6}\pi^2-20\,\zeta_3-\frac{7}{36}\pi^4+\mathcal{O}(\epsilon)\right],$$

$$\sum_{\substack{c\neq i,d\neq i,c\\ e\neq i,c,d}}\tilde{J}_{\mathrm{s}}^{icde}\,B_{cde} = -\,f_i^g\,\frac{\alpha_{\mathrm{S}}}{2\pi}\sum_{\substack{c\neq i,d\neq i,c\\ e\neq i,c,d}}B_{cde}\left[\frac{1}{2}\ln\frac{\bar{s}_{ce}}{\bar{s}_{de}}\ln^2\frac{\bar{s}_{cd}}{\mu^2}+\frac{1}{6}\ln^3\frac{\bar{s}_{ce}}{\bar{s}_{de}}+\mathrm{Li}_3\left(-\frac{\bar{s}_{ce}}{\bar{s}_{de}}\right)+\mathcal{O}(\epsilon)\right].$$

The hard-collinear integrated kernels are given by

$$J_{\mathrm{hc}}^{ijr} \equiv \mathcal{N}_1\int d\Phi_{\mathrm{rad}}^{(ijr)}\,\frac{P^{\mathrm{hc}}_{ij(r)}}{s_{ij}}$$
$$\equiv J^{(0\mathrm{g})}_{\mathrm{hc}}\left(\bar{s}^{(ijr)}_{jr}\right)\,f^{q\bar{q}}_{ij} + J^{(1\mathrm{g})}_{\mathrm{hc}}\left(\bar{s}^{(ijr)}_{jr}\right)\,(f^{gq}_{ij}+f^{g\bar{q}}_{ij}) + J^{(2\mathrm{g})}_{\mathrm{hc}}\left(\bar{s}^{(ijr)}_{jr}\right)\,f^{gg}_{ij}\,, \tag{C.189}$$

where

$$J^{(0\mathrm{g})}_{\mathrm{hc}}(s) = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)}\,T_R\,\frac{-2}{3-2\epsilon}$$
$$= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}T_R\left[-\frac{2}{3}\frac{1}{\epsilon}-\frac{16}{9}-\left(\frac{140}{27}-\frac{7}{18}\pi^2\right)\epsilon - \left(\frac{1252}{81}-\frac{28}{27}\pi^2-\frac{50}{9}\zeta_3\right)\epsilon^2+\mathcal{O}(\epsilon^3)\right],$$

$$J^{(1\mathrm{g})}_{\mathrm{hc}}(s) = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)}\,C_F\left(-\frac{1}{2}\right)$$
$$= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}C_F\left[-\frac{1}{2}\frac{1}{\epsilon}-1-\left(3-\frac{7}{24}\pi^2\right)\epsilon-\left(9-\frac{7}{12}\pi^2-\frac{25}{6}\zeta_3\right)\epsilon^2+\mathcal{O}(\epsilon^3)\right],$$

$$J_{\rm hc}^{(2g)}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{e^{\gamma_E}\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)}\,C_A\left(-\frac{1}{3-2\epsilon}\right) \tag{C.190}$$
$$= \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\frac{1}{3}\frac{1}{\epsilon}-\frac{8}{9}-\left(\frac{70}{27}-\frac{7}{36}\pi^2\right)\epsilon-\left(\frac{626}{81}-\frac{14}{27}\pi^2-\frac{25}{9}\zeta_3\right)\epsilon^2+\mathcal{O}(\epsilon^3)\right].$$

A useful combination of these constituent integrals is

$$J_{\rm hc}^{k}(s) = (f_k^q+f_k^{\bar q})\,J_{\rm hc}^{(1g)}(s) + f_k^g\left[N_f\,J_{\rm hc}^{(0g)}(s)+\frac{1}{2}\,J_{\rm hc}^{(2g)}(s)\right]$$
$$= \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\frac{\gamma_k^{\rm hc}}{\epsilon}+\phi_k^{\rm hc}+\mathcal{O}(\epsilon)\right]. \tag{C.191}$$

The hard double-collinear integrated kernels are given by

$$J_{\rm hcc}^{ijkr} \equiv \mathcal{N}_1^2\int d\Phi_{\rm rad,2}^{(ijkr)}\,\frac{P_{ijk(r)}^{\rm hc}}{s_{ijk}^2}$$
$$\equiv J_{\rm hcc}^{(0g)}\left(\bar s_{kr}^{(ijkr)}\right)(f_{ijk}^{q\bar q q'}+f_{ijk}^{q\bar q\bar q'}) + J_{\rm hcc}^{(0g,id)}\left(\bar s_{kr}^{(ijkr)}\right)(f_{ijk}^{q\bar q q}+f_{ijk}^{q\bar q\bar q}) \tag{C.192}$$
$$+ J_{\rm hcc}^{(1g)}\left(\bar s_{kr}^{(ijkr)}\right)f_{ijk}^{q\bar q g} + J_{\rm hcc}^{(2g)}\left(\bar s_{kr}^{(ijkr)}\right)(f_{ijk}^{ggq}+f_{ijk}^{gg\bar q}) + J_{\rm hcc}^{(3g)}\left(\bar s_{kr}^{(ijkr)}\right)f_{ijk}^{ggg}\,,$$

with

$$J_{\rm hcc}^{(0g)}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F T_R\left[\frac{1}{6}\frac{1}{\epsilon^2}+\left(\frac{13}{36}+\frac{1}{9}\pi^2\right)\frac{1}{\epsilon}-\frac{119}{216}+\frac{17}{108}\pi^2+\frac{14}{3}\zeta_3+\mathcal{O}(\epsilon)\right],$$

$$J_{\rm hcc}^{(0g,id)}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F\left(2C_F-C_A\right)$$
$$\times\left[-\left(\frac{13}{8}-\frac{1}{4}\pi^2+\zeta_3\right)\frac{1}{\epsilon}-\frac{227}{16}+\pi^2+\frac{17}{2}\zeta_3-\frac{11}{120}\pi^4+\mathcal{O}(\epsilon)\right],$$

$$J_{\rm hcc}^{(1g)}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}$$
$$\times\left\{C_F T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{31}{9}\frac{1}{\epsilon^2}-\left(\frac{889}{54}-\pi^2\right)\frac{1}{\epsilon}-\frac{23833}{324}+\frac{31}{6}\pi^2+\frac{160}{9}\zeta_3+\mathcal{O}(\epsilon)\right]\right.$$
$$\left.+\,C_A T_R\left[-\frac{1}{\epsilon^3}-\frac{89}{18}\frac{1}{\epsilon^2}-\left(\frac{1211}{54}-\frac{3}{2}\pi^2\right)\frac{1}{\epsilon}-\frac{2620}{27}+\frac{89}{12}\pi^2+\frac{80}{3}\zeta_3+\mathcal{O}(\epsilon)\right]\right\},$$

$$J_{\rm hcc}^{(2g)}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg\{C_F^2\Bigg[-\frac{2}{\epsilon^3} - \frac{37}{4}\frac{1}{\epsilon^2} - \left(\frac{307}{8} - 3\,\pi^2 + 4\,\zeta_3\right)\frac{1}{\epsilon} - \frac{2361}{16} + \frac{111}{8}\,\pi^2 + \frac{136}{3}\,\zeta_3 - \frac{\pi^4}{3} + \mathcal{O}(\epsilon)\Bigg] + C_F C_A\Bigg[-\frac{1}{2}\frac{1}{\epsilon^3} - \frac{23}{12}\frac{1}{\epsilon^2} - \left(\frac{241}{36} - \frac{1}{18}\,\pi^2 - 4\,\zeta_3\right)\frac{1}{\epsilon} - \frac{4609}{216} + \frac{53}{216}\,\pi^2 - \frac{47}{6}\,\zeta_3 + \frac{7}{20}\,\pi^4 + \mathcal{O}(\epsilon)\Bigg]\Bigg\},$$

$$J_{\rm hcc}^{(3g)}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A^2\Bigg[-\frac{5}{2}\frac{1}{\epsilon^3} - \frac{77}{6}\frac{1}{\epsilon^2} - \left(48 - \frac{11}{4}\,\pi^2 + 3\,\zeta_3\right)\frac{1}{\epsilon} - \frac{16943}{108} + \frac{61}{4}\,\pi^2 + \frac{56}{3}\,\zeta_3 - \frac{9}{40}\,\pi^4 + \mathcal{O}(\epsilon)\Bigg]. \quad \text{(C.193)}$$

For the hard-collinear times hard-collinear integrated kernels we have

$$\begin{aligned}
J_{\rm hc\otimes hc}^{ijklr} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(ijr,klr)}\,\frac{P_{ij(r)}^{\rm hc}(s_{ir},s_{jr})}{s_{ij}}\,\frac{P_{kl(r)}^{\rm hc}(s_{kr},s_{lr})}{s_{kl}} \\
&\equiv J_{\rm hc\otimes hc}^{\rm qqqq}\left(\bar{s}_{jr}^{(ijr,klr)}\bar{s}_{lr}^{(ijr,klr)}\right) f_{ij}^{q\bar{q}} f_{kl}^{q'\bar{q}'} \\
&+ J_{\rm hc\otimes hc}^{\rm qqqg}\left(\bar{s}_{jr}^{(ijr,klr)}\bar{s}_{lr}^{(ijr,klr)}\right)\left[f_{ij}^{q\bar{q}}(f_{kl}^{gq'}+f_{kl}^{g\bar{q}'})+(f_{ij}^{gq'}+f_{ij}^{g\bar{q}'})f_{kl}^{q\bar{q}}\right] \\
&+ J_{\rm hc\otimes hc}^{\rm qqgg}\left(\bar{s}_{jr}^{(ijr,klr)}\bar{s}_{lr}^{(ijr,klr)}\right)(f_{ij}^{q\bar{q}} f_{kl}^{gg} + f_{ij}^{gg} f_{kl}^{q\bar{q}}) \\
&+ J_{\rm hc\otimes hc}^{\rm qgqg}\left(\bar{s}_{jr}^{(ijr,klr)}\bar{s}_{lr}^{(ijr,klr)}\right)(f_{ij}^{gq} + f_{ij}^{g\bar{q}})(f_{kl}^{gq'} + f_{kl}^{g\bar{q}'}) \\
&+ J_{\rm hc\otimes hc}^{\rm qggg}\left(\bar{s}_{jr}^{(ijr,klr)}\bar{s}_{lr}^{(ijr,klr)}\right)\left[(f_{ij}^{gq}+f_{ij}^{g\bar{q}})f_{kl}^{gg}+f_{ij}^{gg}(f_{kl}^{gq}+f_{kl}^{g\bar{q}})\right] \\
&+ J_{\rm hc\otimes hc}^{\rm gggg}\left(\bar{s}_{jr}^{(ijr,klr)}\bar{s}_{lr}^{(ijr,klr)}\right) f_{ij}^{gg} f_{kl}^{gg}\,, \qquad \text{(C.194)}
\end{aligned}$$

with

$$
\begin{aligned}
J^{\mathrm{qqqq}}_{\mathrm{hc}\otimes\mathrm{hc}}(ss') &= \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R^2 \left[\frac{4}{9}\,\frac{1}{\epsilon^2} + \frac{64}{27}\,\frac{1}{\epsilon} + \frac{284}{27} - \frac{16}{27}\,\pi^2 + \mathcal{O}(\epsilon)\right] , \\
J^{\mathrm{qqqg}}_{\mathrm{hc}\otimes\mathrm{hc}}(ss') &= \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R\, C_F \left[\frac{1}{3}\,\frac{1}{\epsilon^2} + \frac{14}{9}\,\frac{1}{\epsilon} + \frac{181}{27} - \frac{4}{9}\,\pi^2 + \mathcal{O}(\epsilon)\right] , \\
J^{\mathrm{qqgg}}_{\mathrm{hc}\otimes\mathrm{hc}}(ss') &= \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R\, C_A \left[\frac{2}{9}\,\frac{1}{\epsilon^2} + \frac{32}{27}\,\frac{1}{\epsilon} + \frac{142}{27} - \frac{8}{27}\,\pi^2 + \mathcal{O}(\epsilon)\right] , \\
J^{\mathrm{qgqg}}_{\mathrm{hc}\otimes\mathrm{hc}}(ss') &= \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_F^2 \left[\frac{1}{4}\,\frac{1}{\epsilon^2} + \frac{1}{\epsilon} + \frac{17}{4} - \frac{1}{3}\,\pi^2 + \mathcal{O}(\epsilon)\right] , \\
J^{\mathrm{qggg}}_{\mathrm{hc}\otimes\mathrm{hc}}(ss') &= \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_A\, C_F \left[\frac{1}{6}\,\frac{1}{\epsilon^2} + \frac{7}{9}\,\frac{1}{\epsilon} + \frac{181}{54} - \frac{2}{9}\,\pi^2 + \mathcal{O}(\epsilon)\right] , \\
J^{\mathrm{gggg}}_{\mathrm{hc}\otimes\mathrm{hc}}(ss') &= \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_A^2 \left[\frac{1}{9}\,\frac{1}{\epsilon^2} + \frac{16}{27}\,\frac{1}{\epsilon} + \frac{71}{27} - \frac{4}{27}\,\pi^2 + \mathcal{O}(\epsilon)\right] . \qquad \text{(C.195)}
\end{aligned}
$$

The soft-times-hard-collinear integrated kernels read

$$
\begin{aligned}
J_{\rm s\otimes hc}^{jkricd} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(jkr,icd)} \, \frac{P_{jk(r)}^{\rm hc}}{s_{jk}} \, \mathcal{E}_{cd}^{(i)} \\
&\equiv f_i^g \bigg[J_{\rm s\otimes hc}^{4(1g)}\Big(\bar s_{kr}^{(\mu)}, \bar s_{cd}^{(\mu)}\Big) f_{jk}^{q\bar q} \\
&\quad + J_{\rm s\otimes hc}^{4(2g)}\Big(\bar s_{kr}^{(\mu)}, \bar s_{cd}^{(\mu)}\Big)(f_{jk}^{gq}+f_{jk}^{g\bar q}) + J_{\rm s\otimes hc}^{4(3g)}\Big(\bar s_{kr}^{(\mu)}, \bar s_{cd}^{(\mu)}\Big) f_{jk}^{gg} \bigg]_{\mu=jkr,icd} , \\
J_{\rm s\otimes hc}^{jkricr} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(jkr,icr)} \, \frac{P_{jk(r)}^{\rm hc}}{s_{jk}} \, \mathcal{E}_{cr}^{(i)} \\
&\equiv f_i^g \bigg[J_{\rm s\otimes hc}^{3(1g)}\Big(\bar s_{kr}^{(\mu)}, \bar s_{cr}^{(\mu)}\Big) f_{jk}^{q\bar q} \\
&\quad + J_{\rm s\otimes hc}^{3(2g)}\Big(\bar s_{kr}^{(\mu)}, \bar s_{cr}^{(\mu)}\Big)(f_{jk}^{gq}+f_{jk}^{g\bar q}) + J_{\rm s\otimes hc}^{3(3g)}\Big(\bar s_{kr}^{(\mu)}, \bar s_{cr}^{(\mu)}\Big) f_{jk}^{gg} \bigg]_{\mu=jkr,icr} , \\
J_{\rm s\otimes hc}^{krjic} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(krj,icj)} \, \frac{P_{jk(r)}^{\rm hc}}{s_{jk}} \, \mathcal{E}_{jc}^{(i)} \\
&\equiv f_i^g \bigg[J_{\rm s\otimes hc}^{3(1g)}\Big(\bar s_{jr}^{(\mu)}, \bar s_{jc}^{(\mu)}\Big) f_{jk}^{q\bar q} \\
&\quad + J_{\rm s\otimes hc}^{3(2g)}\Big(\bar s_{jr}^{(\mu)}, \bar s_{jc}^{(\mu)}\Big)(f_{jk}^{gq}+f_{jk}^{g\bar q}) + J_{\rm s\otimes hc}^{3(3g)}\Big(\bar s_{jr}^{(\mu)}, \bar s_{jc}^{(\mu)}\Big) f_{jk}^{gg} \bigg]_{\mu=krj,icj} , \\
J_{\rm s\otimes hc}^{krjir} &\equiv \mathcal{N}_1^2 \int d\Phi_{\rm rad,2}^{(\mu)} \, \frac{P_{jk(r)}^{\rm hc}}{s_{jk}} \, \mathcal{E}_{jr}^{(i)} \\
&\equiv f_i^g \bigg[\; J_{\rm s\otimes hc}^{gqq}\Big(\bar s_{jr}^{(\mu)}\Big) f_{jk}^{q\bar q} + J_{\rm s\otimes hc}^{ggq}\Big(\bar s_{jr}^{(\mu)}\Big) f_j^g (f_k^q+f_k^{\bar q}) \\
&\quad + J_{\rm s\otimes hc}^{gqg}\Big(\bar s_{jr}^{(\mu)}\Big)(f_j^q+f_j^{\bar q}) f_k^g + J_{\rm s\otimes hc}^{ggg}\Big(\bar s_{jr}^{(\mu)}\Big) f_{jk}^{gg} \bigg]_{\mu=\{krj,irj;\; krj,ijr\}} , \qquad \text{(C.196)}
\end{aligned}
$$

with

$$
\begin{aligned}
J_{\rm s\otimes hc}^{4(1g)}(s,s') &= \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R \bigg[-\frac{2}{3}\frac{1}{\epsilon^3} - \frac{28}{9}\frac{1}{\epsilon^2} - \left(\frac{344}{27} - \frac{7}{9}\pi^2\right)\frac{1}{\epsilon} \\
&\qquad - \frac{3928}{81} + \frac{98}{27}\pi^2 + \frac{100}{9}\zeta(3) + \mathcal{O}(\epsilon) \bigg], \\
J_{\rm s\otimes hc}^{4(2g)}(s,s') &= \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_F \bigg[-\frac{1}{2}\frac{1}{\epsilon^3} - \frac{2}{\epsilon^2} - \left(8 - \frac{7}{12}\pi^2\right)\frac{1}{\epsilon} - 30 + \frac{7}{3}\pi^2 + \frac{25}{3}\zeta(3) + \mathcal{O}(\epsilon) \bigg], \\
J_{\rm s\otimes hc}^{4(3g)}(s,s') &= \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_A \bigg[-\frac{1}{3}\frac{1}{\epsilon^3} - \frac{14}{9}\frac{1}{\epsilon^2} - \left(\frac{172}{27} - \frac{7}{18}\pi^2\right)\frac{1}{\epsilon} \\
&\qquad - \frac{1964}{81} + \frac{49}{27}\pi^2 + \frac{50}{9}\zeta(3) + \mathcal{O}(\epsilon) \bigg],
\end{aligned}
$$

$$J^{3(1g)}_{s\otimes hc}(s,s') = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{28}{9}\frac{1}{\epsilon^2}-\left(\frac{362}{27}-\frac{8}{9}\pi^2\right)\frac{1}{\epsilon} -\frac{4504}{81}+\frac{112}{27}\pi^2+\frac{136}{9}\zeta(3)+\mathcal{O}(\epsilon)\right],$$

$$J^{3(2g)}_{s\otimes hc}(s,s') = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_F\left[-\frac{1}{2}\frac{1}{\epsilon^3}-\frac{2}{\epsilon^2}-\left(\frac{17}{2}-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon} -35+\frac{8}{3}\pi^2+\frac{34}{3}\zeta(3)+\mathcal{O}(\epsilon)\right],$$

$$J^{3(3g)}_{s\otimes hc}(s,s') = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon} C_A\left[-\frac{1}{3}\frac{1}{\epsilon^3}-\frac{14}{9}\frac{1}{\epsilon^2}-\left(\frac{181}{27}-\frac{4}{9}\pi^2\right)\frac{1}{\epsilon} -\frac{2252}{81}+\frac{56}{27}\pi^2+\frac{68}{9}\zeta(3)+\mathcal{O}(\epsilon)\right],$$

$$J^{gqq}_{s\otimes hc}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{28}{9}\frac{1}{\epsilon^2}-\left(\frac{344}{27}-\frac{17}{18}\pi^2\right)\frac{1}{\epsilon} -\frac{4225}{81}+\frac{128}{27}\pi^2+\frac{139}{9}\zeta(3)+\mathcal{O}(\epsilon)\right],$$

$$J^{gqg}_{s\otimes hc}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F\left[-\frac{1}{2}\frac{1}{\epsilon^3}-\frac{2}{\epsilon^2}-\left(9-\frac{5}{6}\pi^2\right)\frac{1}{\epsilon} -38+\frac{19}{6}\pi^2+\frac{101}{6}\zeta(3)+\mathcal{O}(\epsilon)\right],$$

$$J^{ggq}_{s\otimes hc}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F\left[-\frac{1}{2}\frac{1}{\epsilon^3}-\frac{2}{\epsilon^2}-\left(8-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon} -32+\frac{17}{6}\pi^2+\frac{59}{6}\zeta(3)+\mathcal{O}(\epsilon)\right], \tag{C.197}$$

$$J^{ggg}_{s\otimes hc}(s) = \left(\frac{\alpha_S}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A\left[-\frac{1}{3}\frac{1}{\epsilon^3}-\frac{14}{9}\frac{1}{\epsilon^2}-\left(\frac{199}{27}-\frac{5}{9}\pi^2\right)\frac{1}{\epsilon} -\frac{2477}{81}+\frac{119}{54}\pi^2+\frac{101}{9}\zeta(3)+\mathcal{O}(\epsilon)\right].$$

Finally the hard-collinear real-virtual integrated kernels read

$$\tilde{J}^{ijr}_{\rm hc} \equiv \mathcal{N}_1 \int d\Phi^{(ijr)}_{\rm rad} \, \frac{\tilde{P}^{\rm hc}_{ij(r)}}{s_{ij}} \; \equiv \; \tilde{J}^{(0{\rm g})}_{\rm hc}\Big(\bar{s}^{(ijr)}_{jr}\Big) f^{q\bar{q}}_{ij} + \tilde{J}^{(1{\rm g})}_{\rm hc}\Big(\bar{s}^{(ijr)}_{jr}\Big)(f^{gq}_{ij}+f^{g\bar{q}}_{ij}) + \tilde{J}^{(2{\rm g})}_{\rm hc}\Big(\bar{s}^{(ijr)}_{jr}\Big) f^{gg}_{ij} \,,$$

$$\begin{aligned}
J^{ijr}_{\Delta{\rm hc}} &\equiv \mathcal{N}_1 \, \frac{2}{\epsilon^2} \int d\Phi^{(ijr)}_{\rm rad} \, \frac{P^{\rm hc}_{ij(r)}}{s_{ij}} \left[\left(\frac{s_{cr}}{\bar{s}^{(ijr)}_{cr}} \right)^{-\epsilon} - 1 \right] \\
&\equiv J^{(0{\rm g})}_{\Delta{\rm hc}}\Big(\bar{s}^{(ijr)}_{jr}\Big) \, f^{q\bar{q}}_{ij} + J^{(1{\rm g})}_{\Delta{\rm hc}}\Big(\bar{s}^{(ijr)}_{jr}\Big) \, (f^{gq}_{ij}+f^{g\bar{q}}_{ij}) + J^{(2{\rm g})}_{\Delta{\rm hc}}\Big(\bar{s}^{(ijr)}_{jr}\Big) \, f^{gg}_{ij} \,,
\end{aligned}$$

$$\begin{aligned}
J^{ijrc}_{\Delta{\rm hc}} &\equiv \mathcal{N}_1 \, \frac{2}{\epsilon^2} \int d\Phi^{(ijr)}_{\rm rad} \left\{ \frac{P^{\rm hc}_{ij(r)}}{s_{ij}} \left[1 - \left(\frac{\bar{s}^{(ijr)}_{jc}}{s_{[ij]r}} \right)^{-\epsilon} \right] \right. \\
&\qquad\qquad \left. + \, 2C_{f_j} \mathcal{E}^{(i)}_{jr} \left[1 - \left(\frac{s_{jr}}{s_{[ij]r}} \right)^{-\epsilon} \right] + 2C_{f_i} \mathcal{E}^{(j)}_{ir} \left[1 - \left(\frac{s_{ir}}{s_{[ij]r}} \right)^{-\epsilon} \right] \right\} \\
&\equiv \left[J^{(0{\rm g})}_{\Delta{\rm hc,A}}\Big(\bar{s}^{(ijr)}_{jr}\Big) + J^{(0{\rm g})}_{\Delta{\rm hc,B}}\Big(\bar{s}^{(ijr)}_{jc}\Big) \right] f^{q\bar{q}}_{ij} + \left[J^{(1{\rm g})}_{\Delta{\rm hc,A}}\Big(\bar{s}^{(ijr)}_{jr}\Big) + J^{(1{\rm g})}_{\Delta{\rm hc,B}}\Big(\bar{s}^{(ijr)}_{jc}\Big) \right] (f^{gq}_{ij}+f^{g\bar{q}}_{ij}) \\
&\quad + \left[J^{(2{\rm g})}_{\Delta{\rm hc,A}}\Big(\bar{s}^{(ijr)}_{jr}\Big) + J^{(2{\rm g})}_{\Delta{\rm hc,B}}\Big(\bar{s}^{(ijr)}_{jc}\Big) \right] f^{gg}_{ij} \,,
\end{aligned}$$

$$\begin{aligned}
J^{jri,c}_{\Delta{\rm hc}} &\equiv \mathcal{N}_1 \, \frac{\rho^{(C)}_{ij}}{\epsilon^2} \int d\Phi^{(jri)}_{\rm rad} \, \frac{P^{\rm hc}_{ij(r)}}{s_{ij}} \left[\left(\frac{\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir}} \right)^{-\epsilon} - \left(\frac{s_{ir}\bar{s}^{(jri)}_{ic}}{\bar{s}^{(jri)}_{ir} s_{ic}} \right)^{-\epsilon} \right] \\
&\equiv \left[J^{\rm qq}_{\Delta{\rm hc,A}}\Big(\bar{s}^{(jri)}_{ir}\Big) + J^{\rm qq}_{\Delta{\rm hc,B}}\Big(\bar{s}^{(jri)}_{ic}\Big) \right] f^{q\bar{q}}_{ij} + \left[J^{\rm qg}_{\Delta{\rm hc,A}}\Big(\bar{s}^{(jri)}_{ir}\Big) + J^{\rm qg}_{\Delta{\rm hc,B}}\Big(\bar{s}^{(jri)}_{ic}\Big) \right] (f^{q}_{i}+f^{\bar{q}}_{i}) f^{g}_{j} \\
&\quad + \left[J^{\rm gq}_{\Delta{\rm hc,A}}\Big(\bar{s}^{(jri)}_{ir}\Big) + J^{\rm gq}_{\Delta{\rm hc,B}}\Big(\bar{s}^{(jri)}_{ic}\Big) \right] f^{g}_{i} (f^{q}_{j}+f^{\bar{q}}_{j}) + \left[J^{\rm gg}_{\Delta{\rm hc,A}}\Big(\bar{s}^{(jri)}_{ir}\Big) + J^{\rm gg}_{\Delta{\rm hc,B}}\Big(\bar{s}^{(jri)}_{ic}\Big) \right] f^{gg}_{ij} \,,
\end{aligned}$$

$$\tilde{J}^{jri,c}_{\Delta{\rm hc}} \equiv \frac{\mathcal{N}_1}{\epsilon^2} \int d\Phi^{(jri)}_{\rm rad} \, \frac{T_R}{s_{ij}} \left(1 - \frac{2}{1-\epsilon} \, \frac{s_{ir}s_{jr}}{s^2_{[ij]r}} \right) \left[\left(\frac{\bar{s}^{(jri)}_{ic}}{\mu^2} \right)^{-\epsilon} - \left(\frac{\bar{s}^{(jri)}_{ic}}{s_{ic}} \right)^{-\epsilon} \right] \equiv \tilde{J}^{c}_{\Delta{\rm hc}}\Big(\bar{s}^{(jri)}_{ir}, \bar{s}^{(jri)}_{ic}\Big) \,, \tag{C.198}$$

where

$$\begin{aligned}
\tilde{J}^{(0g)}_{\rm hc}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon} T_R \Bigg\{ & N_f T_R\left[\frac{4}{9}\frac{1}{\epsilon^2}+\frac{64}{27}\frac{1}{\epsilon}+\frac{284}{27}-\frac{2}{3}\pi^2+\mathcal{O}(\epsilon)\right] \\
&+C_F\left[\frac{2}{3}\frac{1}{\epsilon^3}+\frac{31}{9}\frac{1}{\epsilon^2}+\left(\frac{431}{27}-\pi^2\right)\frac{1}{\epsilon}+\frac{5506}{81}-\frac{31}{6}\pi^2-\frac{124}{9}\zeta_3+\mathcal{O}(\epsilon)\right] \\
&+C_A\left[-\frac{1}{3\epsilon^3}-\frac{31}{18}\frac{1}{\epsilon^2}-\left(\frac{211}{27}-\frac{1}{2}\pi^2\right)\frac{1}{\epsilon}-\frac{5281}{162}+\frac{31}{12}\pi^2+\frac{62}{9}\zeta_3+\mathcal{O}(\epsilon)\right]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\tilde{J}^{(1g)}_{\rm hc}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_F \Bigg\{ & C_F\Bigg[-\left(\frac{5}{4}-\frac{\pi^2}{3}\right)\frac{1}{\epsilon^2}-\left(\frac{15}{2}-\frac{2}{3}\pi^2-10\zeta_3\right)\frac{1}{\epsilon} \\
&\qquad -\frac{141}{4}+\frac{109}{24}\pi^2+20\zeta_3-\frac{7}{45}\pi^4+\mathcal{O}(\epsilon)\Bigg] \\
&+C_A\left[\frac{1}{4\epsilon^3}+\frac{1}{2\epsilon^2}+\left(1-\frac{\pi^2}{24}-4\zeta_3\right)\frac{1}{\epsilon}+\frac{9}{4}+\frac{7}{12}\pi^2-\frac{67}{6}\zeta_3-\frac{11}{90}\pi^4+\mathcal{O}(\epsilon)\right]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\tilde{J}^{(2g)}_{\rm hc}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A \Bigg\{ & N_f T_R\left[\frac{1}{3}\frac{1}{\epsilon}+\frac{25}{9}+\mathcal{O}(\epsilon)\right] \\
&+C_A\Bigg[\frac{1}{6}\frac{1}{\epsilon^3}-\left(\frac{28}{9}-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2}-\left(\frac{61}{3}-\frac{7}{4}\pi^2-12\,\zeta_3\right)\frac{1}{\epsilon} \\
&\qquad -\frac{15805}{162}+\frac{38}{3}\pi^2+\frac{221}{9}\zeta_3-\frac{5}{9}\pi^4+\mathcal{O}(\epsilon)\Bigg]\Bigg\}; \qquad \text{(C.199)}
\end{aligned}$$

$$\begin{aligned}
J^{(0g)}_{\Delta\rm hc}(s) &= \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[-\left(\frac{4}{3}-\frac{2}{9}\pi^2\right)\frac{1}{\epsilon}-\frac{104}{9}+\frac{16}{27}\pi^2+8\zeta_3+\mathcal{O}(\epsilon)\right], \\
J^{(1g)}_{\Delta\rm hc}(s) &= \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F\left[-\left(1-\frac{\pi^2}{6}\right)\frac{1}{\epsilon}-8+\frac{\pi^2}{3}+6\zeta_3+\mathcal{O}(\epsilon)\right], \\
J^{(2g)}_{\Delta\rm hc}(s) &= \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\left(\frac{2}{3}-\frac{\pi^2}{9}\right)\frac{1}{\epsilon}-\frac{52}{9}+\frac{8}{27}\pi^2+4\zeta_3+\mathcal{O}(\epsilon)\right]; \qquad \text{(C.200)}
\end{aligned}$$

$$J^{(0g)}_{\Delta hc,A}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[-\frac{4}{3}\frac{1}{\epsilon^3} - \frac{32}{9}\frac{1}{\epsilon^2} - \left(\frac{280}{27} - \frac{7}{9}\pi^2\right)\frac{1}{\epsilon} - \frac{2504}{81} + \frac{56}{27}\pi^2 + \frac{100}{9}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{(0g)}_{\Delta hc,B}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[\frac{4}{3}\frac{1}{\epsilon^3} + \frac{32}{9}\frac{1}{\epsilon^2} + \left(\frac{244}{27} - \frac{5}{9}\pi^2\right)\frac{1}{\epsilon} + \frac{1784}{81} - \frac{40}{27}\pi^2 - \frac{52}{9}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{(1g)}_{\Delta hc,A}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F\left[-\frac{1}{\epsilon^3} - \left(6 - \frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2} - \left(30 - \frac{23}{12}\pi^2 - 16\zeta_3\right)\frac{1}{\epsilon} - 126 + \frac{49}{6}\pi^2 + \frac{121}{3}\zeta_3 + \frac{\pi^4}{9} + \mathcal{O}(\epsilon)\right], \quad \text{(C.201)}$$

$$J^{(1g)}_{\Delta hc,B}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_F\left[\frac{1}{\epsilon^3} + \frac{2}{\epsilon^2} + \left(5 - \frac{5}{12}\pi^2\right)\frac{1}{\epsilon} + 12 - \frac{5}{6}\pi^2 - \frac{13}{3}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{(2g)}_{\Delta hc,A}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\frac{2}{3}\frac{1}{\epsilon^3} - \left(\frac{88}{9} - \frac{4}{3}\pi^2\right)\frac{1}{\epsilon^2} - \left(\frac{1436}{27} - \frac{55}{18}\pi^2 - 32\zeta_3\right)\frac{1}{\epsilon} - \frac{18748}{81} + \frac{406}{27}\pi^2 + \frac{626}{9}\zeta_3 + \frac{2}{9}\pi^4 + \mathcal{O}(\epsilon)\right],$$

$$J^{(2g)}_{\Delta hc,B}(s) = \frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[\frac{2}{3}\frac{1}{\epsilon^3} + \frac{16}{9}\frac{1}{\epsilon^2} + \left(\frac{122}{27} - \frac{5}{18}\pi^2\right)\frac{1}{\epsilon} + \frac{892}{81} - \frac{20}{27}\pi^2 - \frac{26}{9}\zeta_3 + \mathcal{O}(\epsilon)\right];$$

$$J^{qq}_{\Delta hc,A}(s)=\frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[\frac{2}{3}\frac{1}{\epsilon^3} + \frac{16}{9}\frac{1}{\epsilon^2} + \left(\frac{122}{27} - \frac{4}{9}\pi^2\right)\frac{1}{\epsilon} + \frac{1108}{81} - \frac{44}{27}\pi^2 - \frac{47}{9}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{qq}_{\Delta hc,B}(s)=\frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} T_R\left[-\frac{2}{3}\frac{1}{\epsilon^3} - \frac{16}{9}\frac{1}{\epsilon^2} - \left(\frac{140}{27} - \frac{7}{18}\pi^2\right)\frac{1}{\epsilon} - \frac{1252}{81} + \frac{28}{27}\pi^2 + \frac{50}{9}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{qg}_{\Delta hc,A}(s)=\frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}(2C_F - C_A)\left[\frac{1}{2}\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2} + \left(\frac{7}{2} - \frac{11}{24}\pi^2\right)\frac{1}{\epsilon} + \frac{23}{2} - \frac{2}{3}\pi^2 - \frac{32}{3}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{qg}_{\Delta hc,B}(s)=\frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}(2C_F - C_A)\left[-\frac{1}{2\epsilon^3} - \frac{1}{\epsilon^2} - \left(3 - \frac{7}{24}\pi^2\right)\frac{1}{\epsilon} - 9 + \frac{7}{12}\pi^2 + \frac{25}{6}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{gq}_{\Delta hc,A}(s)=\frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[\frac{1}{2\epsilon^3} + \frac{1}{\epsilon^2} + \left(\frac{5}{2} - \frac{7}{24}\pi^2\right)\frac{1}{\epsilon} + \frac{13}{2} - \frac{5}{6}\pi^2 - \frac{5}{3}\zeta_3 + \mathcal{O}(\epsilon)\right], \quad \text{(C.202)}$$

$$J^{gq}_{\Delta hc,B}(s)=\frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\frac{1}{2\epsilon^3} - \frac{1}{\epsilon^2} - \left(3 - \frac{7}{24}\pi^2\right)\frac{1}{\epsilon} - 9 + \frac{7}{12}\pi^2 + \frac{25}{6}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{gg}_{\Delta hc,A}(s)=\frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[\frac{1}{3\epsilon^3} + \frac{8}{9}\frac{1}{\epsilon^2} + \left(\frac{88}{27} - \frac{11}{36}\pi^2\right)\frac{1}{\epsilon} + \frac{716}{81} - \frac{17}{54}\pi^2 - \frac{64}{9}\zeta_3 + \mathcal{O}(\epsilon)\right],$$

$$J^{gg}_{\Delta hc,B}(s)=\frac{\alpha_S}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon} C_A\left[-\frac{1}{3\epsilon^3} - \frac{8}{9}\frac{1}{\epsilon^2} - \left(\frac{70}{27} - \frac{7}{36}\pi^2\right)\frac{1}{\epsilon} - \frac{626}{81} + \frac{14}{27}\pi^2 + \frac{25}{9}\zeta_3 + \mathcal{O}(\epsilon)\right].$$

Bibliography

[1] G. Bertolotti, P. Torrielli, S. Uccirati and M. Zaro, *Local analytic sector subtraction for initial- and final-state radiation at NLO in massless QCD*, *JHEP* **12** (2022) 042 [2209.09123]. (Cited on pages 27 and 118).

[2] G. Bertolotti, L. Magnea, G. Pelliccioli, A. Ratti, C. Signorile-Signorile, P. Torrielli et al., *NNLO subtraction for any massless final state: a complete analytic expression*, *JHEP* **07** (2023) 140 [2212.11190]. (Cited on page 27).

[3] A.M. Litke and R. Wilson, *Electron-positron collisions*, *Scientific American* **229** (1973) 104. (Cited on page 1).

[4] www.jstor.org/stable/24923224. (Cited on page 1).

[5] Particle Data Group collaboration, *Review of Particle Physics*, *PTEP* **2022** (2022) 083C01. (Cited on page 1).

[6] ATLAS collaboration, *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett. B* **716** (2012) 1 [1207.7214]. (Cited on page 1).

[7] CMS collaboration, *Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC*, *Phys. Lett. B* **716** (2012) 30 [1207.7235]. (Cited on page 1).

[8] FCC collaboration, *FCC-ee: The Lepton Collider: Future Circular Collider Conceptual Design Report Volume 2*, *Eur. Phys. J. ST* **228** (2019) 261. (Cited on page 2).

[9] CEPC Study Group collaboration, *CEPC Conceptual Design Report: Volume 2 - Physics & Detector*, 1811.10545. (Cited on page 2).

[10] CLIC collaboration, *The CLIC Potential for New Physics*, 1812.02093. (Cited on page 2).

[11] P. Bambade et al., *The International Linear Collider: A Global Project*, 1903.01629. (Cited on page 2).

[12] R. Abdul Khalek et al., *Science Requirements and Detector Concepts for the Electron-Ion Collider: EIC Yellow Report*, *Nucl. Phys. A* **1026** (2022) 122447 [2103.05419]. (Cited on page 2).

[13] C. Aime et al., *Muon Collider Physics Summary*, 2203.07256. (Cited on page 2).

[14] Muon Collider collaboration, *A Muon Collider Facility for Physics Discovery*, 2203.08033. (Cited on page 2).

[15] FCC collaboration, *FCC-hh: The Hadron Collider: Future Circular Collider Conceptual Design Report Volume 3*, *Eur. Phys. J. ST* **228** (2019) 755. (Cited on page 2).

[16] S. Frixione, Z. Kunszt and A. Signer, *Three jet cross-sections to next-to-leading order*, *Nucl. Phys. B* **467** (1996) 399 [hep-ph/9512328]. (Cited on pages 3, 23, 27, 33, 47, and 64).

[17] S. Frixione, *A General approach to jet cross-sections in QCD*, *Nucl. Phys. B* **507** (1997) 295 [hep-ph/9706545]. (Cited on pages 3, 23, and 33).

[18] S. Catani and M.H. Seymour, *A General algorithm for calculating jet cross-sections in NLO QCD*, *Nucl. Phys.* **B485** (1997) 291 [hep-ph/9605323]. (Cited on pages 3, 8, 23, 28, 35, 39, and 187).

[19] S. Catani, S. Dittmaier, M.H. Seymour and Z. Trocsanyi, *The Dipole formalism for next-to-leading order QCD calculations with massive partons*, *Nucl. Phys. B* **627** (2002) 189 [hep-ph/0201036]. (Cited on pages 3 and 23).

[20] Z. Nagy and D.E. Soper, *General subtraction method for numerical calculation of one loop QCD matrix elements*, *JHEP* **09** (2003) 055 [hep-ph/0308127]. (Cited on pages 3 and 24).

[21] G. Bevilacqua, M. Czakon, M. Kubocz and M. Worek, *Complete Nagy-Soper subtraction for next-to-leading order calculations in QCD*, *JHEP* **10** (2013) 204 [1308.5605]. (Cited on pages 3 and 24).

[22] S. Frixione and M. Grazzini, *Subtraction at NNLO*, *JHEP* **06** (2005) 010 [hep-ph/0411399]. (Cited on pages 3 and 26).

[23] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, *Antenna subtraction at NNLO*, *JHEP* **09** (2005) 056 [hep-ph/0505111]. (Cited on pages 3 and 24).

[24] A. Daleo, A. Gehrmann-De Ridder, T. Gehrmann and G. Luisoni, *Antenna subtraction at NNLO with hadronic initial states: initial-final configurations*, *JHEP* **01** (2010) 118 [0912.0374]. (Cited on pages 3 and 24).

[25] R. Boughezal, A. Gehrmann-De Ridder and M. Ritzmann, *Antenna subtraction at NNLO with hadronic initial states: double real radiation for initial-initial configurations with two quark flavours*, *JHEP* **02** (2011) 098 [1011.6631]. (Cited on pages 3 and 24).

[26] T. Gehrmann and P.F. Monni, *Antenna subtraction at NNLO with hadronic initial states: real-virtual initial-initial configurations*, *JHEP* **12** (2011) 049 [1107.4037]. (Cited on pages 3 and 24).

[27] A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, *Antenna subtraction at NNLO with hadronic initial states: double real initial-initial configurations*, *JHEP* **10** (2012) 047 [1207.5779]. (Cited on pages 3 and 24).

[28] T. Gehrmann, E.W.N. Glover and M. Marcoli, *The colourful antenna subtraction method*, 2310.19757. (Cited on pages 3 and 25).

[29] G. Somogyi, Z. Trocsanyi and V. Del Duca, *Matching of singly- and doubly-unresolved limits of tree-level QCD squared matrix elements*, *JHEP* **06** (2005) 024 [hep-ph/0502226]. (Cited on pages 3 and 25).

[30] G. Somogyi, Z. Trocsanyi and V. Del Duca, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Regularization of doubly-real emissions*, *JHEP* **01** (2007) 070 [hep-ph/0609042]. (Cited on pages 3 and 25).

[31] G. Somogyi and Z. Trocsanyi, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Regularization of real-virtual emission*, *JHEP* **01** (2007) 052 [hep-ph/0609043]. (Cited on pages 3 and 25).

[32] G. Somogyi and Z. Trocsanyi, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Integrating the subtraction terms. I.*, *JHEP* **08** (2008) 042 [0807.0509]. (Cited on pages 3 and 25).

[33] U. Aglietti, V. Del Duca, C. Duhr, G. Somogyi and Z. Trocsanyi, *Analytic integration of real-virtual counterterms in NNLO jet cross sections. I.*, *JHEP* **09** (2008) 107 [0807.0514]. (Cited on pages 3 and 25).

[34] P. Bolzoni, S.-O. Moch, G. Somogyi and Z. Trocsanyi, *Analytic integration of real-virtual counterterms in NNLO jet cross sections. II.*, *JHEP* **08** (2009) 079 [0905.4390]. (Cited on pages 3 and 25).

[35] P. Bolzoni, G. Somogyi and Z. Trocsanyi, *A subtraction scheme for computing QCD jet cross sections at NNLO: integrating the iterated singly-unresolved subtraction terms*, *JHEP* **01** (2011) 059 [1011.1909]. (Cited on pages 3 and 25).

[36] V. Del Duca, G. Somogyi and Z. Trocsanyi, *Integration of collinear-type doubly unresolved counterterms in NNLO jet cross sections*, *JHEP* **06** (2013) 079 [1301.3504]. (Cited on pages 3 and 25).

[37] G. Somogyi, *A subtraction scheme for computing QCD jet cross sections at NNLO: integrating the doubly unresolved subtraction terms*, *JHEP* **04** (2013) 010 [1301.3919]. (Cited on pages 3 and 25).

[38] G. Somogyi and F. Tramontano, *Fully exclusive heavy quark-antiquark pair production from a colourless initial state at NNLO in QCD*, *JHEP* **11** (2020) 142 [2007.15015]. (Cited on pages 3 and 25).

[39] M. Czakon, *A novel subtraction scheme for double-real radiation at NNLO*, *Phys. Lett. B* **693** (2010) 259 [1005.0274]. (Cited on pages 3 and 25).

[40] M. Czakon, *Double-real radiation in hadronic top quark pair production as a proof of a certain concept*, *Nucl. Phys. B* **849** (2011) 250 [1101.0642]. (Cited on pages 3 and 25).

[41] R. Boughezal, K. Melnikov and F. Petriello, *A subtraction scheme for NNLO computations*, *Phys. Rev. D* **85** (2012) 034025 [1111.7041]. (Cited on pages 3 and 25).

[42] M. Czakon and D. Heymes, *Four-dimensional formulation of the sector-improved residue subtraction scheme*, *Nucl. Phys. B* **890** (2014) 152 [1408.2500]. (Cited on pages 3 and 25).

[43] F. Caola, K. Melnikov and R. Röntsch, *Nested soft-collinear subtractions in NNLO QCD computations*, *Eur. Phys. J. C* **77** (2017) 248 [1702.01352]. (Cited on pages 3 and 25).

[44] F. Devoto, K. Melnikov, R. Röntsch, C. Signorile-Signorile and D.M. Tagliabue, *A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation*, 2310.17598. (Cited on pages 3 and 26).

[45] S. Catani and M. Grazzini, *An NNLO subtraction formalism in hadron collisions and its application to Higgs boson production at the LHC*, *Phys. Rev. Lett.* **98** (2007) 222002 [hep-ph/0703012]. (Cited on pages 3 and 26).

[46] R. Bonciani, S. Catani, M. Grazzini, H. Sargsyan and A. Torre, *The q_T subtraction method for top quark production at hadron colliders*, *Eur. Phys. J.* **C75** (2015) 581 [1508.03585]. (Cited on pages 3 and 26).

[47] R. Boughezal, C. Focke, X. Liu and F. Petriello, *W-boson production in association with a jet at next-to-next-to-leading order in perturbative QCD*, *Phys. Rev. Lett.* **115** (2015) 062002 [1504.02131]. (Cited on pages 3 and 26).

[48] R. Boughezal, C. Focke, W. Giele, X. Liu and F. Petriello, *Higgs boson production in association with a jet at NNLO using jettiness subtraction*, *Phys. Lett. B* **748** (2015) 5 [1505.03893]. (Cited on pages 3 and 26).

[49] J. Gaunt, M. Stahlhofen, F.J. Tackmann and J.R. Walsh, *N-jettiness Subtractions for NNLO QCD Calculations*, *JHEP* **09** (2015) 058 [1505.04794]. (Cited on pages 3 and 26).

[50] M. Cacciari, F.A. Dreyer, A. Karlberg, G.P. Salam and G. Zanderighi, *Fully Differential Vector-Boson-Fusion Higgs Production at Next-to-Next-to-Leading Order*, *Phys. Rev. Lett.* **115** (2015) 082002 [1506.02660]. (Cited on pages 3 and 26).

[51] F. Herzog, *Geometric IR subtraction for final state real radiation*, *JHEP* **08** (2018) 006 [1804.07949]. (Cited on pages 3 and 26).

[52] G.F.R. Sborlini, F. Driencourt-Mangin, R. Hernandez-Pinto and G. Rodrigo, *Four-dimensional unsubtraction from the loop-tree duality*, *JHEP* **08** (2016) 160 [1604.06699]. (Cited on pages 3 and 23).

[53] G.F.R. Sborlini, F. Driencourt-Mangin and G. Rodrigo, *Four-dimensional unsubtraction with massive particles*, *JHEP* **10** (2016) 162 [1608.01584]. (Cited on pages 3 and 23).

[54] Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni and B. Ruijl, *Numerical Loop-Tree Duality: contour deformation and subtraction*, *JHEP* **04** (2020) 096 [1912.09291]. (Cited on pages 3 and 23).

[55] Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni and B. Ruijl, *Manifestly Causal Loop-Tree Duality*, 2009.05509. (Cited on pages 3 and 23).

[56] Z. Capatti, V. Hirschi, A. Pelloni and B. Ruijl, *Local Unitarity: a representation of differential cross-sections that is locally free of infrared singularities at any order*, *JHEP* **04** (2021) 104 [2010.01068]. (Cited on pages 3 and 23).

[57] Z. Capatti, V. Hirschi and B. Ruijl, *Local unitarity: cutting raised propagators and localising renormalisation*, *JHEP* **10** (2022) 120 [2203.11038]. (Cited on pages 3 and 23).

[58] H.A. Chawdhry, M.L. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to three-photon production at the LHC*, *JHEP* **02** (2020) 057 [1911.00479]. (Cited on pages 3 and 27).

[59] S. Kallweit, V. Sotnikov and M. Wiesemann, *Triphoton production at hadron colliders in NNLO QCD*, *Phys. Lett. B* **812** (2021) 136013 [2010.04681]. (Cited on pages 3 and 27).

[60] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to diphoton production with an additional jet at the LHC*, *JHEP* **09** (2021) 093 [2105.06940]. (Cited on pages 3 and 27).

[61] M. Czakon, A. Mitov and R. Poncelet, *Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC*, *Phys. Rev. Lett.* **127** (2021) 152001 [2106.05331]. (Cited on pages 3 and 27).

[62] H.B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, *Next-to-next-to-leading order QCD corrections to $Wb\bar{b}$ production at the LHC*, *Phys. Rev. D* **106** (2022) 074016 [2205.01687]. (Cited on pages 3 and 27).

[63] S. Catani, S. Devoto, M. Grazzini, S. Kallweit, J. Mazzitelli and C. Savoini, *Higgs Boson Production in Association with a Top-Antitop Quark Pair in Next-to-Next-to-Leading Order QCD*, *Phys. Rev. Lett.* **130** (2023) 111902 [2210.07846]. (Cited on pages 3 and 27).

[64] L. Buonocore, S. Devoto, S. Kallweit, J. Mazzitelli, L. Rottoli and C. Savoini, *Associated production of a W boson and massive bottom quarks at next-to-next-to-leading order in QCD*, *Phys. Rev. D* **107** (2023) 074032 [2212.04954]. (Cited on pages 3 and 27).

[65] M. Alvarez, J. Cantero, M. Czakon, J. Llorente, A. Mitov and R. Poncelet, *NNLO QCD corrections to event shapes at the LHC*, *JHEP* **03** (2023) 129 [2301.01086]. (Cited on pages 3 and 27).

[66] S. Badger, M. Czakon, H.B. Hartanto, R. Moodie, T. Peraro, R. Poncelet et al., *Isolated photon production in association with a jet pair through next-to-next-to-leading order in QCD*, 2304.06682. (Cited on pages 3 and 27).

[67] L. Buonocore, S. Devoto, M. Grazzini, S. Kallweit, J. Mazzitelli, L. Rottoli et al., *Associated production of a W boson with a top-antitop quark pair: next-to-next-to-leading order QCD predictions for the LHC*, 2306.16311. (Cited on pages 3 and 27).

[68] A. Huss, J. Huston, S. Jones and M. Pellen, *Les Houches 2021—physics at TeV colliders: report on the standard model precision wishlist*, *J. Phys. G* **50** (2023) 043001 [2207.02122]. (Cited on pages 3 and 27).

[69] F. Caola, W. Chen, C. Duhr, X. Liu, B. Mistlberger, F. Petriello et al., *The Path forward to N^3LO*, in *Snowmass 2021*, 3, 2022 [2203.06730]. (Cited on pages 3 and 27).

[70] G.F. Sterman, *An Introduction to quantum field theory*, Cambridge University Press (8, 1993). (Cited on page 5).

[71] M.E. Peskin and D.V. Schroeder, *An Introduction to quantum field theory*, Addison-Wesley, Reading, USA (1995). (Cited on pages 5 and 8).

[72] R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and collider physics*, vol. 8, Cambridge University Press (2, 2011), 10.1017/CBO9780511628788. (Cited on pages 5 and 6).

[73] P. Nason, *Introduction to perturbative QCD*, in *11th Jorge Andre Swieca Summer School on Particle and Fields*, pp. 409–486, 1, 2001. (Cited on pages 5 and 16).

[74] G.P. Salam, *Elements of QCD for hadron colliders*, in *2009 European School of High-Energy Physics*, 11, 2010 [1011.5131]. (Cited on page 5).

[75] C.G. Bollini and J.J. Giambiagi, *Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter*, *Nuovo Cim. B* **12** (1972) 20. (Cited on page 8).

[76] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Five-Loop Running of the QCD coupling constant*, *Phys. Rev. Lett.* **118** (2017) 082002 [1606.08659]. (Cited on page 10).

[77] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *The five-loop beta function of Yang-Mills theory with fermions*, *JHEP* **02** (2017) 090 [1701.01404]. (Cited on page 10).

[78] K.G. Chetyrkin, G. Falcioni, F. Herzog and J.A.M. Vermaseren, *Five-loop renormalisation of QCD in covariant gauges*, *JHEP* **10** (2017) 179 [1709.08541]. (Cited on page 10).

[79] D.J. Gross and F. Wilczek, *Ultraviolet behavior of non-abelian gauge theories*, *Phys. Rev. Lett.* **30** (1973) 1343. (Cited on page 10).

[80] H.D. Politzer, *Reliable perturbative results for strong interactions?*, *Phys. Rev. Lett.* **30** (1973) 1346. (Cited on page 10).

[81] R.P. Feynman, *Very high-energy collisions of hadrons*, *Phys. Rev. Lett.* **23** (1969) 1415. (Cited on page 11).

[82] J.D. Bjorken and E.A. Paschos, *Inelastic electron-proton and γ-proton scattering and the structure of the nucleon*, *Phys. Rev.* **185** (1969) 1975. (Cited on page 11).

[83] R. Angeles-Martinez et al., *Transverse Momentum Dependent (TMD) parton distribution functions: status and prospects*, *Acta Phys. Polon. B* **46** (2015) 2501 [1507.05267]. (Cited on page 11).

[84] V. Bertone, M. Cacciari, S. Frixione and G. Stagnitto, *The partonic structure of the electron at the next-to-leading logarithmic accuracy in QED*, *JHEP* **03** (2020) 135 [1911.12040]. (Cited on page 11).

[85] F. Garosi, D. Marzocca and S. Trifinopoulos, *LePDF: Standard Model PDFs for high-energy lepton colliders*, *JHEP* **09** (2023) 107 [2303.16964]. (Cited on page 11).

[86] S. Frixione and G. Stagnitto, *The muon parton distribution functions*, 2309.07516. (Cited on page 11).

[87] S. Ferrario Ravasio, G. Limatola and P. Nason, *Infrared renormalons in kinematic distributions for hadron collider processes*, *JHEP* **06** (2021) 018 [2011.14114]. (Cited on page 12).

[88] F. Caola, S. Ferrario Ravasio, G. Limatola, K. Melnikov and P. Nason, *On linear power corrections in certain collider observables*, *JHEP* **01** (2022) 093 [2108.08897]. (Cited on page 12).

[89] F. Caola, S. Ferrario Ravasio, G. Limatola, K. Melnikov, P. Nason and M.A. Ozcelik, *Linear power corrections to e^+e^- shape variables in the three-jet region*, *JHEP* **12** (2022) 062 [2204.02247]. (Cited on page 12).

[90] S. Makarov, K. Melnikov, P. Nason and M.A. Ozcelik, *Linear power corrections to single top production processes at the LHC*, *JHEP* **05** (2023) 153 [2302.02729]. (Cited on page 12).

[91] S. Makarov, K. Melnikov, P. Nason and M.A. Ozcelik, *Linear power corrections to top quark pair production in hadron collisions*, 2308.05526. (Cited on page 12).

[92] F. Bloch and A. Nordsieck, *Note on the Radiation Field of the electron*, *Phys. Rev.* **52** (1937) 54. (Cited on pages 14 and 21).

[93] T. Kinoshita, *Mass singularities of Feynman amplitudes*, *J. Math. Phys.* **3** (1962) 650. (Cited on pages 14, 21, and 31).

[94] T.D. Lee and M. Nauenberg, *Degenerate Systems and Mass Singularities*, *Phys. Rev.* **133** (1964) B1549. (Cited on pages 14, 21, and 31).

[95] J. Grammer, G. and D. Yennie, *Improved treatment for the infrared divergence problem in quantum electrodynamics*, *Phys.Rev.* **D8** (1973) 4332. (Cited on page 21).

[96] J.C. Collins, D.E. Soper and G.F. Sterman, *Factorization of Hard Processes in QCD*, *Adv. Ser. Direct. High Energy Phys.* **5** (1989) 1 [hep-ph/0409313]. (Cited on pages 19 and 21).

[97] Y.L. Dokshitzer, *Calculation of the Structure Functions for Deep Inelastic Scattering and e+ e- Annihilation by Perturbation Theory in Quantum Chromodynamics.*, *Sov. Phys. JETP* **46** (1977) 641. (Cited on pages 20, 36, and 139).

[98] V.N. Gribov and L.N. Lipatov, *Deep inelastic e p scattering in perturbation theory*, *Sov. J. Nucl. Phys.* **15** (1972) 438. (Cited on pages 20, 36, and 139).

[99] G. Altarelli and G. Parisi, *Asymptotic Freedom in Parton Language*, *Nucl. Phys. B* **126** (1977) 298. (Cited on pages 20, 36, and 139).

[100] G. Curci, W. Furmanski and R. Petronzio, *Evolution of Parton Densities Beyond Leading Order: The Nonsinglet Case*, *Nucl. Phys. B* **175** (1980) 27. (Cited on page 20).

[101] W. Furmanski and R. Petronzio, *Singlet Parton Densities Beyond Leading Order*, *Phys. Lett. B* **97** (1980) 437. (Cited on page 20).

[102] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three loop splitting functions in QCD: The Nonsinglet case*, *Nucl. Phys. B* **688** (2004) 101 [hep-ph/0403192]. (Cited on page 20).

[103] A. Vogt, S. Moch and J.A.M. Vermaseren, *The Three-loop splitting functions in QCD: The Singlet case*, *Nucl. Phys. B* **691** (2004) 129 [hep-ph/0404111]. (Cited on page 20).

[104] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*, *JHEP* **10** (2017) 041 [1707.08315]. (Cited on page 21).

[105] G. Falcioni, F. Herzog, S. Moch and A. Vogt, *Four-loop splitting functions in QCD – The quark-quark case*, *Phys. Lett. B* **842** (2023) 137944 [2302.07593]. (Cited on page 21).

[106] G. Falcioni, F. Herzog, S. Moch and A. Vogt, *Four-loop splitting functions in QCD – The gluon-to-quark case*, 2307.04158. (Cited on page 21).

[107] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T.-Z. Yang, *Complete N_f^2 contributions to four-loop pure-singlet splitting functions*, 2308.07958. (Cited on page 21).

[108] G. Falcioni, F. Herzog, S. Moch, J. Vermaseren and A. Vogt, *The double fermionic contribution to the four-loop quark-to-gluon splitting function*, 2310.01245. (Cited on page 21).

[109] S. Moch, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt, *Additional moments and x-space approximations of four-loop splitting functions in QCD*, 2310.05744. (Cited on page 21).

[110] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T.-Z. Yang, *The $N_f C_F^3$ contribution to the non-singlet splitting function at four-loop order*, 2310.12240. (Cited on page 21).

[111] N. Agarwal, L. Magnea, C. Signorile-Signorile and A. Tripathi, *The infrared structure of perturbative gauge theories*, *Phys. Rept.* **994** (2023) 1 [2112.07099]. (Cited on page 21).

[112] A. Sen, *Asymptotic Behavior of the Wide Angle On-Shell Quark Scattering Amplitudes in Nonabelian Gauge Theories*, *Phys. Rev.* **D28** (1983) 860. (Cited on page 21).

[113] J.C. Collins, *Sudakov form-factors*, *Adv. Ser. Direct. High Energy Phys.* **5** (1989) 573 [hep-ph/0312336]. (Cited on page 21).

[114] L. Magnea and G.F. Sterman, *Analytic continuation of the Sudakov form-factor in QCD*, *Phys. Rev. D* **42** (1990) 4222. (Cited on pages 21 and 109).

[115] G.F. Sterman, *Partons, factorization and resummation, TASI 95*, in *QCD and beyond. Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI-95, Boulder, USA, June 4-30, 1995*, pp. 327–408, 1995 [hep-ph/9606312]. (Cited on page 21).

[116] S. Catani, *The Singular behavior of QCD amplitudes at two loop order*, *Phys. Lett.* **B427** (1998) 161 [hep-ph/9802439]. (Cited on pages 21, 56, 95, 108, and 110).

[117] G.F. Sterman and M.E. Tejeda-Yeomans, *Multiloop amplitudes and resummation*, *Phys. Lett.* **B552** (2003) 48 [hep-ph/0210130]. (Cited on pages 21 and 108).

[118] L.J. Dixon, L. Magnea and G.F. Sterman, *Universal structure of subleading infrared poles in gauge theory amplitudes*, *JHEP* **0808** (2008) 022 [0805.3515]. (Cited on page 21).

[119] T. Becher and M. Neubert, *Infrared singularities of scattering amplitudes in perturbative QCD*, *Phys.Rev.Lett.* **102** (2009) 162001 [0901.0722]. (Cited on page 21).

[120] E. Gardi and L. Magnea, *Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes*, *JHEP* **0903** (2009) 079 [0901.1091]. (Cited on pages 21, 108, and 109).

[121] T. Becher and M. Neubert, *On the Structure of Infrared Singularities of Gauge-Theory Amplitudes*, *JHEP* **06** (2009) 081 [0903.1126]. (Cited on pages 21, 108, 109, and 110).

[122] E. Gardi and L. Magnea, *Infrared singularities in QCD amplitudes*, *Nuovo Cim.* **C32N5-6** (2009) 137 [0908.3273]. (Cited on page 21).

[123] I. Feige and M.D. Schwartz, *Hard-Soft-Collinear Factorization to All Orders*, *Phys. Rev.* **D90** (2014) 105020 [1403.6472]. (Cited on pages 21 and 108).

[124] Ø. Almelid, C. Duhr and E. Gardi, *Three-loop corrections to the soft anomalous dimension in multileg scattering*, *Phys. Rev. Lett.* **117** (2016) 172002 [1507.00047]. (Cited on page 21).

[125] Ø. Almelid, C. Duhr, E. Gardi, A. McLeod and C.D. White, *Bootstrapping the QCD soft anomalous dimension*, *JHEP* **09** (2017) 073 [1706.10162]. (Cited on page 21).

[126] J.M. Campbell and E.W.N. Glover, *Double unresolved approximations to multiparton scattering amplitudes*, *Nucl. Phys.* **B527** (1998) 264 [hep-ph/9710255]. (Cited on page 21).

[127] S. Catani and M. Grazzini, *Collinear factorization and splitting functions for next-to-next-to-leading order QCD calculations*, *Phys. Lett.* **B446** (1999) 143 [hep-ph/9810389]. (Cited on page 21).

[128] Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, *The infrared behavior of one loop QCD amplitudes at next-to-next-to leading order*, *Phys. Rev.* **D60** (1999) 116001 [hep-ph/9903516]. (Cited on pages 21 and 97).

[129] S. Catani and M. Grazzini, *Infrared factorization of tree level QCD amplitudes at the next-to-next-to-leading order and beyond*, *Nucl. Phys.* **B570** (2000) 287 [hep-ph/9908523]. (Cited on pages 21 and 63).

[130] D.A. Kosower, *All order collinear behavior in gauge theories*, *Nucl. Phys.* **B552** (1999) 319 [hep-ph/9901201]. (Cited on page 21).

[131] S. Catani and M. Grazzini, *The soft gluon current at one loop order*, *Nucl. Phys.* **B591** (2000) 435 [hep-ph/0007142]. (Cited on pages 21 and 97).

[132] V. Del Duca, A. Frizzo and F. Maltoni, *Factorization of tree QCD amplitudes in the high-energy limit and in the collinear limit*, *Nucl. Phys.* **B568** (2000) 211 [hep-ph/9909464]. (Cited on page 21).

[133] S.D. Badger and E.W.N. Glover, *Two loop splitting functions in QCD*, *JHEP* **07** (2004) 040 [hep-ph/0405236]. (Cited on page 21).

[134] C. Duhr and T. Gehrmann, *The two-loop soft current in dimensional regularization*, *Phys. Lett.* **B727** (2013) 452 [1309.4393]. (Cited on page 21).

[135] Y. Li and H.X. Zhu, *Single soft gluon emission at two loops*, *JHEP* **11** (2013) 080 [1309.4391]. (Cited on page 21).

[136] P. Banerjee, P.K. Dhani and V. Ravindran, *Gluon jet function at three loops in QCD*, *Phys. Rev. D* **98** (2018) 094016 [1805.02637]. (Cited on page 21).

[137] R. Brüser, Z.L. Liu and M. Stahlhofen, *Three-Loop Quark Jet Function*, *Phys. Rev. Lett.* **121** (2018) 072003 [1804.09722]. (Cited on page 21).

[138] L.J. Dixon, E. Herrmann, K. Yan and H.X. Zhu, *Soft gluon emission at two loops in full color*, *JHEP* **05** (2020) 135 [1912.09370]. (Cited on page 21).

[139] S. Catani, D. Colferai and A. Torrini, *Triple (and quadruple) soft-gluon radiation in QCD hard scattering*, *JHEP* **01** (2020) 118 [1908.01616]. (Cited on page 21).

[140] V. Del Duca, C. Duhr, R. Haindl, A. Lazopoulos and M. Michel, *Tree-level splitting amplitudes for a gluon into four collinear partons*, *JHEP* **10** (2020) 093 [2007.05345]. (Cited on page 21).

[141] S. Catani and L. Cieri, *Multiple soft radiation at one-loop order and the emission of a soft quark–antiquark pair*, *Eur. Phys. J. C* **82** (2022) 97 [2108.13309]. (Cited on page 21).

[142] V. Del Duca, C. Duhr, R. Haindl and Z. Liu, *Tree-level soft emission of a quark pair in association with a gluon*, *JHEP* **01** (2023) 040 [2206.01584]. (Cited on page 21).

[143] M. Czakon, F. Eschment and T. Schellenberger, *Revisiting the double-soft asymptotics of one-loop amplitudes in massless QCD*, *JHEP* **04** (2023) 065 [2211.06465]. (Cited on page 21).

[144] M. Czakon and S. Sapeta, *Complete collection of one-loop triple-collinear splitting operators for dimensionally-regulated QCD*, *JHEP* **07** (2022) 052 [2204.11801]. (Cited on page 21).

[145] S. Catani, L. Cieri, D. Colferai and F. Coradeschi, *Soft gluon–quark–antiquark emission in QCD hard scattering*, *Eur. Phys. J. C* **83** (2023) 38 [2210.09397]. (Cited on page 21).

[146] W.T. Giele, E.W.N. Glover and D.A. Kosower, *Higher order corrections to jet cross-sections in hadron colliders*, *Nucl. Phys. B* **403** (1993) 633 [hep-ph/9302225]. (Cited on pages 22 and 95).

[147] W.T. Giele, E.W.N. Glover and D.A. Kosower, *The inclusive two jet triply differential cross-section*, *Phys. Rev.* **D52** (1995) 1486 [hep-ph/9412338]. (Cited on page 22).

[148] R.M. Prisco and F. Tramontano, *Dual subtractions*, *JHEP* **06** (2021) 089 [2012.05012]. (Cited on page 23).

[149] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J.-C. Winter, *From loops to trees by-passing Feynman's theorem*, *JHEP* **09** (2008) 065 [0804.3170]. (Cited on page 23).

[150] G. Rodrigo, S. Catani, T. Gleisberg, F. Krauss and J.-C. Winter, *From multileg loops to trees (by-passing Feynman's Tree Theorem)*, *Nucl. Phys. B Proc. Suppl.* **183** (2008) 262 [0807.0531]. (Cited on page 23).

[151] I. Bierenbaum, S. Catani, P. Draggiotis and G. Rodrigo, *A Tree-Loop Duality Relation at Two Loops and Beyond*, *JHEP* **10** (2010) 073 [1007.0194]. (Cited on page 23).

[152] I. Bierenbaum, S. Buchta, P. Draggiotis, I. Malamos and G. Rodrigo, *Tree-Loop Duality Relation beyond simple poles*, *JHEP* **03** (2013) 025 [1211.5048]. (Cited on page 23).

[153] J. de Jesús Aguilera-Verdugo et al., *A Stroll through the Loop-Tree Duality*, *Symmetry* **13** (2021) 1029 [2104.14621]. (Cited on page 23).

[154] Z. Nagy and D.E. Soper, *Parton showers with quantum interference*, *JHEP* **09** (2007) 114 [0706.0017]. (Cited on page 24).

[155] Z. Nagy and D.E. Soper, *Parton showers with quantum interference: Leading color, spin averaged*, *JHEP* **03** (2008) 030 [0801.1917]. (Cited on page 24).

[156] Z. Nagy and D.E. Soper, *Parton showers with quantum interference: Leading color, with spin*, *JHEP* **07** (2008) 025 [0805.0216]. (Cited on page 24).

[157] S. Frixione, P. Nason and C. Oleari, *Matching NLO QCD computations with Parton Shower simulations: the POWHEG method*, *JHEP* **11** (2007) 070 [0709.2092]. (Cited on page 24).

[158] T. Gleisberg and F. Krauss, *Automating dipole subtraction for QCD NLO calculations*, *Eur. Phys. J.* **C53** (2008) 501 [0709.2881]. (Cited on page 24).

[159] R. Frederix, T. Gehrmann and N. Greiner, *Automation of the Dipole Subtraction Method in MadGraph/MadEvent*, *JHEP* **09** (2008) 122 [0808.2128]. (Cited on page 24).

[160] T. Gleisberg, S. Hoeche, F. Krauss, M. Schonherr, S. Schumann, F. Siegert et al., *Event generation with SHERPA 1.1*, *JHEP* **02** (2009) 007 [0811.4622]. (Cited on page 24).

[161] M. Czakon, C.G. Papadopoulos and M. Worek, *Polarizing the Dipoles*, *JHEP* **08** (2009) 085 [0905.0883]. (Cited on page 24).

[162] R. Frederix, S. Frixione, F. Maltoni and T. Stelzer, *Automation of next-to-leading order computations in QCD: The FKS subtraction*, *JHEP* **10** (2009) 003 [0908.4272]. (Cited on page 24).

[163] K. Hasegawa, S. Moch and P. Uwer, *AutoDipole: Automated generation of dipole subtraction terms*, *Comput. Phys. Commun.* **181** (2010) 1802 [0911.4371]. (Cited on page 24).

[164] S. Alioli, P. Nason, C. Oleari and E. Re, *A general framework for implementing NLO calculations in shower Monte Carlo programs: the POWHEG BOX*, *JHEP* **06** (2010) 043 [1002.2581]. (Cited on page 24).

[165] S. Platzer and S. Gieseke, *Dipole Showers and Automated NLO Matching in Herwig++*, *Eur. Phys. J.* **C72** (2012) 2187 [1109.6256]. (Cited on page 24).

[166] J. Alwall, R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer et al., *The automated computation of tree-level and next-to-leading order differential cross sections, and their matching to parton shower simulations*, *JHEP* **07** (2014) 079 [1405.0301]. (Cited on pages 24, 117, and 118).

[167] T. Sjöstrand, S. Ask, J.R. Christiansen, R. Corke, N. Desai, P. Ilten et al., *An introduction to PYTHIA 8.2*, *Comput. Phys. Commun.* **191** (2015) 159 [1410.3012]. (Cited on page 24).

[168] J. Reuter, B. Chokoufe, A. Hoang, W. Kilian, M. Stahlhofen, T. Teubner et al., *Automation of NLO processes and decays and POWHEG matching in WHIZARD*, *J. Phys. Conf. Ser.* **762** (2016) 012059 [1602.06270]. (Cited on page 24).

[169] Sherpa collaboration, *Event Generation with Sherpa 2.2*, *SciPost Phys.* **7** (2019) 034 [1905.09127]. (Cited on page 24).

[170] D.A. Kosower, *Antenna factorization of gauge theory amplitudes*, *Phys. Rev. D* **57** (1998) 5410 [hep-ph/9710213]. (Cited on page 24).

[171] D.A. Kosower, *Multiple singular emission in gauge theories*, *Phys. Rev. D* **67** (2003) 116003 [hep-ph/0212097]. (Cited on page 24).

[172] F.V. Tkachov, *A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions*, *Phys. Lett. B* **100** (1981) 65. (Cited on page 25).

[173] K. Chetyrkin and F. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops*, *Nucl. Phys. B* **192** (1981) 159. (Cited on page 25).

[174] J. Currie, E.W.N. Glover and J. Pires, *Next-to-Next-to Leading Order QCD Predictions for Single Jet Inclusive Production at the LHC*, *Phys. Rev. Lett.* **118** (2017) 072002 [1611.01460]. (Cited on page 25).

[175] J. Currie, E.W.N. Glover and S. Wells, *Infrared Structure at NNLO Using Antenna Subtraction*, *JHEP* **04** (2013) 066 [1301.4693]. (Cited on page 25).

[176] P. Jakubčík, M. Marcoli and G. Stagnitto, *The parton-level structure of e^+e^- to 2 jets at N3LO*, *JHEP* **01** (2023) 168 [2211.08446]. (Cited on page 25).

[177] O. Braun-White, N. Glover and C.T. Preuss, *A general algorithm to build real-radiation antenna functions for higher-order calculations*, *JHEP* **06** (2023) 065 [2302.12787]. (Cited on page 25).

[178] X. Chen, P. Jakubčík, M. Marcoli and G. Stagnitto, *The parton-level structure of Higgs decays to hadrons at N3LO*, *JHEP* **06** (2023) 185 [2304.11180]. (Cited on page 25).

[179] O. Braun-White, N. Glover and C.T. Preuss, *A general algorithm to build mixed real and virtual antenna functions for higher-order calculations*, 2307.14999. (Cited on page 25).

[180] T. Binoth and G. Heinrich, *Numerical evaluation of phase space integrals by sector decomposition*, *Nucl. Phys. B* **693** (2004) 134 [hep-ph/0402265]. (Cited on page 25).

[181] C. Signorile-Signorile and D.M. Tagliabue, *Advances in the nested soft-collinear subtraction scheme*, *PoS* **RADCOR2023** (2023) 075 [2308.11982]. (Cited on page 26).

[182] L. Buonocore, S. Kallweit, L. Rottoli and M. Wiesemann, *Linear power corrections for two-body kinematics in the qT subtraction formalism*, *Phys. Lett. B* **829** (2022) 137118 [2111.13661]. (Cited on page 26).

[183] S. Camarda, L. Cieri and G. Ferrera, *Fiducial perturbative power corrections within the $\mathbf{q}_T$ subtraction formalism*, *Eur. Phys. J. C* **82** (2022) 575 [2111.14509]. (Cited on page 26).

[184] S. Catani, S. Devoto, M. Grazzini, S. Kallweit and J. Mazzitelli, *Top-quark pair production at the LHC: Fully differential QCD predictions at NNLO*, *JHEP* **07** (2019) 100 [1906.06535]. (Cited on page 26).

[185] S. Catani, S. Devoto, M. Grazzini, S. Kallweit and J. Mazzitelli, *Bottom-quark production at hadron colliders: fully differential predictions in NNLO QCD*, *JHEP* **03** (2021) 029 [2010.11906]. (Cited on page 26).

[186] L. Buonocore, M. Grazzini, J. Haag, L. Rottoli and C. Savoini, *Exploring slicing variables for jet processes*, 2307.11570. (Cited on page 26).

[187] M. Grazzini, S. Kallweit and M. Wiesemann, *Fully differential NNLO computations with MATRIX*, *Eur. Phys. J. C* **78** (2018) 537 [1711.06631]. (Cited on pages 26 and 27).

[188] C.W. Bauer, S. Fleming and M.E. Luke, *Summing Sudakov logarithms in $B \to X_s\gamma$in effective field theory.*, *Phys. Rev. D* **63** (2000) 014006 [hep-ph/0005275]. (Cited on page 26).

[189] C.W. Bauer, S. Fleming, D. Pirjol and I.W. Stewart, *An Effective field theory for collinear and soft gluons: Heavy to light decays*, *Phys. Rev. D* **63** (2001) 114020 [hep-ph/0011336]. (Cited on page 26).

[190] C.W. Bauer and I.W. Stewart, *Invariant operators in collinear effective theory*, *Phys. Lett. B* **516** (2001) 134 [hep-ph/0107001]. (Cited on page 26).

[191] C.W. Bauer, D. Pirjol and I.W. Stewart, *Soft collinear factorization in effective field theory*, *Phys. Rev. D* **65** (2002) 054022 [hep-ph/0109045]. (Cited on page 26).

[192] C.W. Bauer, S. Fleming, D. Pirjol, I.Z. Rothstein and I.W. Stewart, *Hard scattering factorization from effective field theory*, *Phys. Rev. D* **66** (2002) 014017 [hep-ph/0202088]. (Cited on page 26).

[193] J.M. Campbell and R.K. Ellis, *An Update on vector boson pair production at hadron colliders*, *Phys. Rev.* **D60** (1999) 113006 [hep-ph/9905386]. (Cited on page 26).

[194] J.M. Campbell, R.K. Ellis and C. Williams, *Vector boson pair production at the LHC*, *JHEP* **07** (2011) 018 [1105.0020]. (Cited on page 26).

[195] R. Boughezal, J.M. Campbell, R.K. Ellis, C. Focke, W. Giele, X. Liu et al., *Color singlet production at NNLO in MCFM*, *Eur. Phys. J. C* **77** (2017) 7 [1605.08011]. (Cited on pages 26 and 27).

[196] J. Campbell and T. Neumann, *Precision Phenomenology with MCFM*, *JHEP* **12** (2019) 034 [1909.09117]. (Cited on page 26).

[197] W.J. Torres Bobadilla et al., *May the four be with you: Novel IR-subtraction methods to tackle NNLO calculations*, *Eur. Phys. J. C* **81** (2021) 250 [2012.02567]. (Cited on pages 26 and 27).

[198] M. Czakon, Z. Kassabov, A. Mitov, R. Poncelet and A. Popescu, *HighTEA: High energy Theory Event Analyser*, 2304.05993. (Cited on page 27).

[199] L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, *Local analytic sector subtraction at NNLO*, *JHEP* **12** (2018) 107 [1806.09570]. (Cited on pages 27, 33, 63, 65, and 66).

[200] L. Magnea, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, *Analytic integration of soft and collinear radiation in factorised QCD cross sections at NNLO*, *JHEP* **02** (2021) 037 [2010.14493]. (Cited on pages 28, 60, 71, 78, 81, 97, 104, 105, and 151).

[201] F. Febres Cordero, A. von Manteuffel and T. Neumann, *Computational Challenges for Multi-loop Collider Phenomenology: A Snowmass 2021 White Paper*, *Comput. Softw. Big Sci.* **6** (2022) 14 [2204.04200]. (Cited on page 28).

[202] L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, *Factorisation and Subtraction beyond NLO*, *JHEP* **12** (2018) 062 [1809.05444]. (Cited on page 28).

[203] L. Magnea, C. Milloy, C. Signorile-Signorile, P. Torrielli and S. Uccirati, *Strongly-ordered infrared limits for subtraction counterterms from factorisation*, in *16th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory 2022*, 9, 2022 [2209.06102]. (Cited on page 28).

[204] Z. Nagy, *Next-to-leading order calculation of three jet observables in hadron hadron collision*, *Phys. Rev. D* **68** (2003) 094002 [hep-ph/0307268]. (Cited on page 47).

[205] V. Del Duca, N. Deutschmann and S. Lionetti, *Momentum mappings for subtractions at higher orders in QCD*, *JHEP* **12** (2019) 129 [1910.01024]. (Cited on page 68).

[206] V. Del Duca, C. Duhr, A. Kardos, G. Somogyi and Z. Trócsányi, *Three-Jet Production in Electron-Positron Collisions at Next-to-Next-to-Leading Order Accuracy*, *Phys. Rev. Lett.* **117** (2016) 152004 [1603.08927]. (Cited on page 78).

[207] V. Del Duca, C. Duhr, A. Kardos, G. Somogyi, Z. Szőr, Z. Trócsányi et al., *Jet production in the CoLoRFulNNLO method: event shapes in electron-positron collisions*, *Phys. Rev. D* **94** (2016) 074019 [1606.03453]. (Cited on page 78).

[208] A. Grozin, J. Henn and M. Stahlhofen, *On the Casimir scaling violation in the cusp anomalous dimension at small angle*, *JHEP* **10** (2017) 052 [1708.01221]. (Cited on page 109).

[209] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*, *Phys. Lett. B* **782** (2018) 627 [1805.09638]. (Cited on page 109).

[210] S. Weinzierl, *Does one need the $O(\epsilon)-$ and $O(\epsilon^2)$-terms of one-loop amplitudes in an NNLO calculation ?*, *Phys. Rev. D* **84** (2011) 074007 [1107.5131]. (Cited on page 113).

[211] R. Frederix, S. Frixione, V. Hirschi, D. Pagani, H.S. Shao and M. Zaro, *The automation of next-to-leading order electroweak calculations*, *JHEP* **07** (2018) 185 [1804.10017]. (Cited on pages 117 and 118).

[212] S. Lionetti, *Subtraction of Infrared Singularities at Higher Orders in QCD*, Ph.D. thesis, ETH, Zurich (main), 2018. 10.3929/ethz-b-000332748. (Cited on page 118).

[213] V. Hirschi, S. Lionetti and A. Schweitzer, *One-loop weak corrections to Higgs production*, *JHEP* **05** (2019) 002 [1902.10167]. (Cited on page 118).

[214] M. Becchetti, R. Bonciani, V. Del Duca, V. Hirschi, F. Moriello and A. Schweitzer, *Next-to-leading order corrections to light-quark mixed QCD-EW contributions to Higgs boson production*, *Phys. Rev. D* **103** (2021) 054037 [2010.09451]. (Cited on page 118).

[215] R. Bonciani, V. Del Duca, H. Frellesvig, M. Hidding, V. Hirschi, F. Moriello et al., *Next-to-leading-order QCD corrections to Higgs production in association with a jet*, *Phys. Lett. B* **843** (2023) 137995 [2206.10490]. (Cited on page 118).

[216] V. Hirschi, R. Frederix, S. Frixione, M.V. Garzelli, F. Maltoni and R. Pittau, *Automation of one-loop QCD corrections*, *JHEP* **05** (2011) 044 [1103.0621]. (Cited on page 118).

[217] G.P. Lepage, *VEGAS: AN ADAPTIVE MULTIDIMENSIONAL INTEGRATION PROGRAM*, . (Cited on page 121).

[218] G.P. Lepage, *Adaptive multidimensional integration: VEGAS enhanced*, *J. Comput. Phys.* **439** (2021) 110386 [2009.05112]. (Cited on page 121).

[219] J. Butterworth et al., *PDF4LHC recommendations for LHC Run II*, *J. Phys. G* **43** (2016) 023001 [1510.03865]. (Cited on page 121).

[220] A. Buckley, J. Ferrando, S. Lloyd, K. Nordström, B. Page, M. Rüfenacht et al., *LHAPDF6: parton density access in the LHC precision era*, *Eur. Phys. J. C* **75** (2015) 132 [1412.7420]. (Cited on page 122).

[221] M. Cacciari, G.P. Salam and G. Soyez, *The anti-k_t jet clustering algorithm*, *JHEP* **04** (2008) 063 [0802.1189]. (Cited on page 122).

[222] M. Cacciari, G.P. Salam and G. Soyez, *FastJet User Manual*, *Eur. Phys. J. C* **72** (2012) 1896 [1111.6097]. (Cited on page 122).

[223] S. Catani, Y.L. Dokshitzer, M.H. Seymour and B.R. Webber, *Longitudinally invariant K_t clustering algorithms for hadron hadron collisions*, *Nucl. Phys. B* **406** (1993) 187. (Cited on page 123).

[224] S.D. Ellis and D.E. Soper, *Successive combination jet algorithm for hadron collisions*, *Phys. Rev. D* **48** (1993) 3160 [hep-ph/9305266]. (Cited on page 123).