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STRONG UNIQUE CONTINUATION FROM THE BOUNDARY FOR THE SPECTRAL FRACTIONAL LAPLACIAN

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Abstract. We investigate unique continuation properties and asymptotic behaviour at boundary points for solutions to a class of elliptic equations involving the spectral fractional Laplacian. An extension procedure leads us to study a degenerate or singular equation on a cylinder, with a homogeneous Dirichlet boundary condition on the lateral surface and a non-homogeneous Neumann condition on the basis. For the extended problem, by an Almgren-type monotonicity formula and a blow-up analysis, we classify the local asymptotic profiles at the edge where the transition between boundary conditions occurs. Passing to traces, an analogous blow-up result and its consequent strong unique continuation property is deduced for the nonlocal fractional equation.

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1. Introduction and statement of the main results

In this paper, we prove the strong unique continuation property and derive local asymptotics from a point $x_0 \in \partial\Omega$ for the solutions to the following equation

$$(-\Delta)^s u = hu \quad \text{on } \Omega, \tag{1.1}$$

where $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^N$ is a bounded Lipschitz domain whose boundary is $C^{1,1}$ in a neighbourhood of x_0, h is a measurable function on Ω satisfying suitable summability properties, which will be more specifically clarified below (see (1.7)), $N > 2s$ and $(-\Delta)^s$ is the so-called *spectral* fractional Laplacian.

Several results are available in the literature about the spectral fractional Laplacian and its interpretations. See [1], [21], and references therein for a detailed overview. We mention that regularity properties for stationary

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equations are discussed in [16], while existence and uniqueness results for evolution equations governed by the spectral fractional Laplacian are established in [4]. More closely related to the present paper are the results in [29], where a strong unique continuation principle at nodal points is proved for fractional powers of some divergence-type elliptic operators, including the case of the spectral fractional Laplacian. The techniques used in [29] are inspired by those introduced in [12], which are based on a combination of a monotonicity formula for an Almgren-type frequency function and a blow-up analysis. This local approach is made possible by the extension results by [26], Theorem 1.1 and [7], Theorem 2.5.

The development of a monotonicity formula for the extended problem presents new difficulties when dealing with boundary points. Indeed, since the point x_0 from which the unique continuation is sought after lies on $\partial\Omega$, the geometry of $\partial\Omega$ can interfere with the monotonicity argument. This issue arises in the study of boundary unique continuation also in the local case, which has been treated in [2, 3, 13, 19, 28] by monotonicity methods. In the present paper, we face this difficulty by straightening the boundary with a local diffeomorphism; this transfers the information about the geometry of $\partial\Omega$ into a coefficient matrix in the operator, which turns out to be a perturbation of the identity if the boundary is regular enough, see Section 3. Secondly, a Pohozaev type identity is needed to differentiate the frequency function and to develop the monotonicity argument. To this aim, we rely on a more general result contained in [14], Proposition 2.3, which is based on a Sobolev-type regularity theory for a class of degenerate and singular problems. Furthermore, a blow-up analysis provides a detailed description of the asymptotic behaviour of solutions to (1.1) at x_0, giving a complete classification of the order of homogeneity of asymptotic profiles, see Theorem 1.2 below. For this purpose, an important role is played by an eigenvalue problem on a half-sphere under a symmetry condition, see (1.19).

The extension problem corresponding to (1.1) consists of a degenerate or singular equation on the cylinder $\Omega \times (0, +\infty)$; a homogeneous Dirichlet boundary condition is imposed on the lateral surface $\partial\Omega \times (0, +\infty)$ and a weighted Neumann-type derivative on the basis $\Omega \times \{0\}$ is equal to the right hand side of (1.1), see (1.17). Therefore, the formulation of the problem in terms of the extension leads us to study what happens near a point of the edge, at which a transition between boundary conditions of a different type takes place. We observe that this situation is quite different from the one that occurs in [10], where unique continuation from boundary points is studied for the *restricted fractional Laplacian*; indeed, the extension problem corresponding to the case treated in [10] is a degenerate or singular problem with mixed conditions that vary on a flat basis rather than on an edge. In fact, the analysis carried out in the present paper highlights different asymptotic behaviors at the boundary for the two operators, unlike what happens at internal points, where the locally equivalent form of the extended problems induces the same blow-up profiles.

In order to introduce a suitable functional setting and give a weak formulation of (1.1), we recall the definition of the spectral fractional Laplacian, which can be given in terms of the Dirichlet eigenvalues of the Laplacian, see *e.g.* [8], [21] and [1]. From classical spectral theory, the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta\varphi = \mu\varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases}$$

admits an increasing and diverging sequence $\{\mu_k\}_{k\in\mathbb{N}\setminus\{0\}}$ of positive eigenvalues (repeated according to their multiplicity). Furthermore, there exists an orthonormal basis of $L^2(\Omega)$ made of the corresponding eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}\setminus\{0\}}$. Every $v \in L^2(\Omega)$ can be expanded with respect to the basis $\{\varphi_k\}_{k\in\mathbb{N}\setminus\{0\}}$ as

$$v = \sum_{k=1}^{\infty} (v, \varphi_k)_{L^2(\Omega)}\varphi_k \quad \text{in } L^2(\Omega),$$

where $(v, \varphi_k)_{L^2(\Omega)}$ is the L^2-scalar product, *i.e.* $(v_1, v_2)_{L^2(\Omega)} = \int_\Omega v_1 v_2 \, \mathrm{d}x$.

We introduce the functional space

$$\mathbb{H}^s(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{k=1}^{\infty} \mu_k^s (v, \varphi_k)^2_{L^2(\Omega)} < +\infty \right\}$$

which is a Hilbert space with respect to the scalar product

$$(v_1, v_2)_{\mathbb{H}^s(\Omega)} := \sum_{k=0}^{\infty} \mu_k^s (v_1, \varphi_k)_{L^2(\Omega)} (v_2, \varphi_k)_{L^2(\Omega)}, \quad v_1, v_2 \in \mathbb{H}^s(\Omega). \tag{1.2}$$

A more explicit characterization of the space $\mathbb{H}^s(\Omega)$ is provided by the interpolation theory, see [4], Section 3.1.3 and [20]:

$$\mathbb{H}^s(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1-s} = \begin{cases} H_0^s(\Omega), & \text{if } s \in (0,1) \setminus \{\frac{1}{2}\}, \\ H_{00}^{1/2}(\Omega), & \text{if } s = \frac{1}{2}. \end{cases}$$

Here, denoting as $H^s(\Omega)$ the usual fractional Sobolev space $W^{s,2}(\Omega)$, $H_0^s(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$, and

$$H_{00}^{1/2}(\Omega) := \left\{ u \in H_0^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{u^2(x)}{d(x, \partial\Omega)} \, \mathrm{d}x < +\infty \right\},$$

where $d(x, \partial\Omega) := \inf\{|x-y| : y \in \partial\Omega\}$. We recall that $H^s(\Omega) = H_0^s(\Omega)$ if $s \in (0, \frac{1}{2}]$, see [20]. Moreover, if $s \neq \frac{1}{2}$, the trivial extension by 0 outside Ω defines a linear and continuous operator from $H_0^s(\Omega)$ into $H^s(\mathbb{R}^N)$, see [6], Remark 2.5 and Proposition B.1. On the other hand, the trivial extension defines a linear and continuous operator from $H_{00}^{1/2}(\Omega)$ into $H^{1/2}(\mathbb{R}^N)$, as one can easily deduce from estimate (B.2) in [6]. Then

$$\begin{aligned} \iota : \mathbb{H}^s(\Omega) &\to H^s(\mathbb{R}^N), \\ v \mapsto \tilde{v} &= \begin{cases} v, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \end{aligned} \tag{1.3}$$

is a linear and continuous operator.

It is easy to verify that, if $v \in \mathbb{H}^s(\Omega)$, then the series $\sum_{k=1}^\infty \mu_k^s (v, \varphi_k)_{L^2(\Omega)} \varphi_k$ converges in the dual space $(\mathbb{H}^s(\Omega))^*$ to some $F \in (\mathbb{H}^s(\Omega))^*$ such that ${}_{(\mathbb{H}^s(\Omega))^*}\langle F, \varphi_k \rangle_{\mathbb{H}^s(\Omega)} = \mu_k^s (v, \varphi_k)_{L^2(\Omega)}$. Hence, for every $v \in \mathbb{H}^s(\Omega)$, we can define its spectral fractional Laplacian as

$$(-\Delta)^s v = \sum_{k=1}^{\infty} \mu_k^s (v, \varphi_k)_{L^2(\Omega)} \varphi_k \in (\mathbb{H}^s(\Omega))^*. \tag{1.4}$$

Actually, the spectral fractional Laplacian is the Riesz isomorphism between $\mathbb{H}^s(\Omega)$ endowed with the scalar product (1.2) and its dual $(\mathbb{H}^s(\Omega))^*$, *i.e.*

$${}_{(\mathbb{H}^s(\Omega))^*}\langle (-\Delta)^s v_1, v_2 \rangle_{\mathbb{H}^s(\Omega)} = (v_1, v_2)_{\mathbb{H}^s(\Omega)} \quad \text{for all } v_1, v_2 \in \mathbb{H}^s(\Omega). \tag{1.5}$$

The spectral fractional Laplacian defined in (1.4) is a different operator from the usual fractional Laplacian defined by the Fourier transform as

$$\mathcal{F}((-\Delta)^s v)(\xi) := |\xi|^{2s} \widehat{v}(\xi) \tag{1.6}$$

for any $v \in \mathcal{S}(\mathbb{R}^N)$. Indeed, the spectral fractional Laplacian depends on the domain Ω and it is a global operator in Ω, while the fractional Laplacian is a global operator on the whole $\mathbb{R}^N$. Moreover, the eigenfunctions of the spectral fractional Laplacian coincide with the eigenfunctions of the Dirichlet Laplacian, hence they are smooth up to the boundary if Ω is sufficiently regular; on the other hand, the eigenfunctions of the restricted fractional Laplacian, defined by restricting the operator in (1.6) to act only on functions vanishing outside Ω, are only Hölder continuous, see [24].

Within the functional setting introduced above, we can give the notion of weak solution to (1.1). To this purpose, we assume that

$$h \in W^{1,\frac{N}{2s}+\varepsilon}(\Omega) \tag{1.7}$$

for some $\varepsilon \in (0,1)$. We note that it is not restrictive to assume ε small. In view of (1.5), we say that a function $u \in \mathbb{H}^s(\Omega)$ is a weak solution to (1.1) if

$$(u,\phi)_{\mathbb{H}^s(\Omega)} = \int_\Omega h(x)u(x)\phi(x)\,\mathrm{d}x \qquad \text{for any } \phi \in C_c^\infty(\Omega). \tag{1.8}$$

The right hand side in (1.8) is finite in view of (1.7), the Hölder's inequality, and the following fractional Sobolev inequality

$$\|v\|_{L^{2_s^*}(\Omega)} \le \mathcal{K}_{N,s}\,\|v\|_{H^s(\Omega)} \qquad \text{for any } v \in H_0^s(\Omega),$$

where

$$2_s^* := \frac{2N}{N-2s}, \tag{1.9}$$

and $\mathcal{K}_{N,s} > 0$ is a positive constant depending only on N and s, see *e.g.* [11], Theorem 6.5 and [6], Remark 2.5 and Proposition B.1.

In order to establish a unique continuation property at a fixed point $x_0 \in \partial\Omega$, we need to assume some regularity on the boundary of Ω near x_0; more precisely, we assume that there exist a radius $R > 0$ and a function g such that

$$g \in C^{1,1}(\mathbb{R}^{N-1},\mathbb{R}) \tag{1.10}$$

and, up to rigid motions, letting $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$,

$$\partial\Omega \cap B'_R(x_0) = \{(x', x_N) \in B'_R(x_0) : x_N = g(x')\}, \tag{1.11}$$

$$\Omega \cap B'_R(x_0) = \{(x', x_N) \in B'_R(x_0) : x_N < g(x')\}, \tag{1.12}$$

where, for any $r > 0$ and $x \in \mathbb{R}^N$,

$$B'_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}. \tag{1.13}$$

The spectral fractional Laplacian defined in (1.4) turns out to be a nonlocal operator on Ω. As we intend to use an approach based on local doubling inequalities, which are deduced from an Almgren-type monotonicity formula in the spirit of [15], it is quite natural to deal with the local realization of the spectral fractional Laplacian. This is obtained by the extension procedure described in [7] (see also [26] and [8]) which transforms (1.1) into a singular or degenerate problem on a cylinder contained in a $N+1$-dimensional space.

More precisely, we consider the half-space $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0,\infty)$, whose total variable is denoted as $z = (x,t) \in \mathbb{R}^N \times [0,\infty)$. For any open set $E \subseteq \mathbb{R}^N \times (0,\infty)$, let $H^1(E, t^{1-2s})$ be the completion of $C_c^\infty(\overline{E})$

with respect to the norm

$$\|\phi\|_{H^1(E,t^{1-2s})} := \left(\int_E t^{1-2s}(\phi^2 + |\nabla\phi|^2)\,\mathrm{d}z\right)^{\frac{1}{2}}.$$

By [18], Theorems 11.11, 11.2, Remarks 11.12-(iii) and the extension theorems for weighted Sobolev spaces with weights in the Muckenhoupt's A_2 class proved in [9], for any open Lipschitz set $E \subseteq \mathbb{R}^N \times (0,\infty)$, the space $H^1(E, t^{1-2s})$ can be characterized as

$$H^1(E,t^{1-2s}) = \left\{v \in W^{1,1}_{\mathrm{loc}}(E) : \int_E t^{1-2s}(v^2 + |\nabla v|^2)\,\mathrm{d}z < +\infty\right\}.$$

We define

$$\mathcal{C}_\Omega := \Omega \times (0,+\infty), \quad \partial_L \mathcal{C}_\Omega := \partial\Omega \times [0,+\infty), \tag{1.14}$$

and

$$H^1_{0,L}(\mathcal{C}_\Omega, t^{1-2s}) := \overline{C^\infty_c(\mathcal{C}_\Omega \cup \Omega)}^{\|\cdot\|_{H^1(\mathcal{C}_\Omega,t^{1-2s})}},$$

i.e $H^1_{0,L}(\mathcal{C}_\Omega, t^{1-2s})$ is the closure in $H^1(\mathcal{C}_\Omega, t^{1-2s})$ of $C^\infty_c(\mathcal{C}_\Omega \cup \Omega)$. Furthermore there exists a linear and continuous trace operator

$$\mathrm{Tr}_\Omega : H^1_{0,L}(C_\Omega, t^{1-2s}) \to \mathbb{H}^s(\Omega) \tag{1.15}$$

which is also onto (see [8], Prop. 2.1). Moreover, in [8] it is observed that, for every $v \in \mathbb{H}^s(\Omega)$, the minimization problem

$$\min_{\substack{w\in H^1_{0,L}(\mathcal{C}_\Omega,t^{1-2s})\\ \mathrm{Tr}_\Omega(w)=v}} \left\{\int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla w(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right\}$$

has a unique minimizer $\mathcal{H}(v) = V \in H^1_{0,L}(\mathcal{C}_\Omega, t^{1-2s})$ which solves

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla V) = 0, & \text{in } \mathcal{C}_\Omega,\\ \mathrm{Tr}_\Omega(V) = v, & \text{on } \Omega\times\{0\},\\ V = 0, & \text{on } \partial\Omega\times[0,+\infty),\\ -\lim_{t\to0^+} t^{1-2s}\frac{\partial V}{\partial t} = \kappa_{s,N}(-\Delta)^s v, & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.16}$$

where $\kappa_{s,N} > 0$ is a positive constant depending only on N and s. Equation (1.16) has to be interpreted in a weak sense, that is

$$\int_{\mathcal{C}_\Omega} t^{1-2s}\nabla V\cdot\nabla\phi\,\mathrm{d}z = \kappa_{s,N}(v, \mathrm{Tr}_\Omega(\phi))_{\mathbb{H}^s(\Omega)} \quad \text{for all } \phi \in H^1_{0,L}(\mathcal{C}_\Omega, t^{1-2s}),$$

in view of (1.5). Hence, if $u \in \mathbb{H}^s(\Omega)$ solves (1.1), then its extension $\mathcal{H}(u) = U \in H^1_{0,L}(\mathcal{C}_\Omega, t^{1-2s})$ weakly solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0, & \text{in } \mathcal{C}_\Omega, \\ \operatorname{Tr}_\Omega(U) = u, & \text{on } \Omega \times \{0\}, \\ U = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ -\lim_{t\to 0^+} t^{1-2s}\frac{\partial U}{\partial t} = \kappa_{s,N} h u, & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.17}$$

according to (1.16), namely

$$\int_{\mathcal{C}_\Omega} t^{1-2s}\nabla U \cdot \nabla \phi \, \mathrm{d}z = \kappa_{s,N} \int_\Omega h u \operatorname{Tr}_\Omega(\phi) \, \mathrm{d}x \quad \text{for all } \phi \in H^1_{0,L}(\mathcal{C}_\Omega, t^{1-2s}). \tag{1.18}$$

The asymptotic behavior at $x_0 \in \partial\Omega$ of any solution U of (1.17), and consequently of any solution u of (1.1), turns out to be related to the eigenvalues of the following problem

$$\begin{cases} -\operatorname{div}_{\mathbb{S}}(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}} Y) = \mu\, \theta_{N+1}^{1-2s}\, Y, & \text{on } \mathbb{S}^+ \\ \lim_{\theta_{N+1}\to 0^+} \theta_{N+1}^{1-2s}\, \nabla_{\mathbb{S}} Y \cdot \nu = 0, & \text{on } \mathbb{S}', \\ Y \in H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta_{N+1}^{1-2s}), \end{cases} \tag{1.19}$$

where

$$\begin{aligned} &\mathbb{S} := \{\theta = (\theta', \theta_N, \theta_{N+1}) \in \mathbb{R}^{N+1} : |\theta'|^2 + \theta_N^2 + \theta_{N+1}^2 = 1\}, \\ &\mathbb{S}^+ := \{\theta = (\theta', \theta_N, \theta_{N+1}) \in \mathbb{S} : \theta_{N+1} > 0\}, \\ &\mathbb{S}' := \partial\mathbb{S}^+ = \{\theta = (\theta', \theta_N, \theta_{N+1}) \in \mathbb{S} : \theta_{N+1} = 0\}, \end{aligned}$$

and ν is the outer normal vector to $\mathbb{S}^+$ on $\mathbb{S}'$, that is $\nu = -(0, \dots, 0, 1)$. We consider the weighted space

$$L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s}) := \left\{ \Psi : \mathbb{S}^+ \to \mathbb{R} \text{ measurable} : \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \Psi^2 \, \mathrm{d}S < +\infty \right\},$$

where $\mathrm{d}S$ denotes the volume element on N-dimensional spheres. In order to introduce the space $H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ where problem (1.19) is formulated, we first denote by $H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ the completion of $C^\infty(\overline{\mathbb{S}^+})$ with respect to the norm

$$\|\phi\|_{H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})} := \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} (\phi^2 + |\nabla_{\mathbb{S}}\phi|^2) \, \mathrm{d}S \right)^{1/2}.$$

Then we define

$$H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta_{N+1}^{1-2s}) := \{\Psi \in H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s}) : \Psi(\theta', \theta_N, \theta_{N+1}) = -\Psi(\theta', -\theta_N, \theta_{N+1})\}. \tag{1.20}$$

It is easy to verify that $H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ is a closed subspace of $H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})$.

A function $Y \in H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ is an eigenfunction of (1.19) if $Y \not\equiv 0$ and

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}\, \nabla_{\mathbb{S}} Y \cdot \nabla_{\mathbb{S}} \Psi \, \mathrm{d}S = \mu \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} Y \Psi \, \mathrm{d}S \tag{1.21}$$

for all $\Psi \in H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta_{N+1}^{1-2s})$.

By classical spectral theory, the set of the eigenvalues of problem (1.19) is an increasing and diverging sequence of positive real numbers $\{\mu_m\}_{m\in\mathbb{N}\setminus\{0\}}$. In Appendix A we explicitly determine the sequence $\{\mu_m\}_{m\in\mathbb{N}\setminus\{0\}}$, obtaining that, for all $m \in \mathbb{N}\setminus\{0\}$,

$$\mu_m = \begin{cases} m^2 + m(N-2s), & \text{if } N > 1, \\ (2m-1)^2 + (2m-1)(N-2s), & \text{if } N = 1. \end{cases} \tag{1.22}$$

Let, for future reference,

$$V_m \text{ be the eigenspace of problem (1.19) associated to the eigenvalue } \mu_m, \tag{1.23}$$

$$M_m \text{ be the dimension of } V_m, \tag{1.24}$$

$$\{Y_{m,k} : m \in \mathbb{N}\setminus\{0\} \text{ and } k \in \{1,\dots,M_m\}\} \text{ be an orthonormal basis of } L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \tag{1.25}$$
$$\text{such that } \{Y_{m,k} : k = 1,\dots,M_m\} \text{ is a basis of } V_m.$$

Remark 1.1. Let Y be an eigenfunction of (1.19) associated to the eigenvalue $m^2 + m(N-2s)$. Then Y cannot vanish identically on $\mathbb{S}'$.

Indeed, if $Y \equiv 0$ on $\mathbb{S}'$, the function $V(r\theta) := r^m Y(\theta)$ would solve $\operatorname{div}(t^{1-2s}\nabla V) = 0$ on $\mathbb{R}^{N+1}_+$, satisfying both Neumann and Dirichlet boundary condition on $\mathbb{R}^N \times \{0\}$. This would contradict the unique continuation principle for elliptic equations with weights in the Muckenhoupt A_2 class, see [15], [27], and [23], Proposition 2.2.

The main result of the present paper is a complete classification of asymptotic blow-up profiles at a point $x_0 \in \partial\Omega$ for solutions of (1.16) and, in turn, for the corresponding solutions of (1.1).

Theorem 1.2. *Let $N > 2s$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $x_0 \in \partial\Omega$ and assume that there exist $R > 0$ and a function g satisfying (1.10), (1.11), and (1.12). Let u be a non-trivial solution of (1.1) in the sense of (1.8), with h satisfying (1.7). Then there exists $m_0 \in \mathbb{N}\setminus\{0\}$ (which is odd in the case $N = 1$) and an eigenfunction Y of (1.19) associated to the eigenvalue $m_0^2 + m_0(N-2s)$, such that*

$$\lambda^{-m_0} u(\lambda x + x_0) \to |x|^{m_0} \widehat{Y}\left(\frac{x}{|x|}, 0\right) \quad \text{as } \lambda \to 0^+ \quad \text{in } H^s(B_1'),$$

where $B_1' := B_1'(0)$ has been defined in (1.13), u is trivially extended to zero outside Ω as in (1.3), and

$$\widehat{Y}(\theta', \theta_N, \theta_{N+1}) = \begin{cases} Y(\theta', \theta_N, \theta_{N+1}), & \text{if } \theta_N < 0, \\ 0, & \text{if } \theta_N \geq 0. \end{cases} \tag{1.26}$$

Unlike the analogous result for the restricted fractional Laplacian established in [10], the order of homogeneity of limit profiles does not depend on s and it is always an integer. This is a consequence of the regularity of the eigenfunctions of (1.19), see Appendix A for further details. In particular, the eigenfunctions of (1.19), after an even reflection through the equator $\theta_{N+1} = 0$, turn out to be smooth thanks to [25], Theorem 1.1; therefore, they are much more regular than the solutions of the corresponding problem on the half-sphere appearing in [10] and presenting mixed boundary conditions, which are responsible for a lower regularity.

Theorem 1.2 is proved by passing to the trace in the following blow-up result for solutions of the extended problem (1.17).

Theorem 1.3. *Let $N > 2s$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $x_0 \in \partial\Omega$ and assume that there exist $R > 0$ and a function g satisfying (1.10), (1.11), and (1.12). Let U be a non-trivial solution to (1.17) in the sense of (1.18), with h satisfying (1.7). Then there exist $m_0 \in \mathbb{N}\setminus\{0\}$ (which is odd in the case $N = 1$) and* *eigenfunction Y of (1.19), associated to the eigenvalue $m_0^2+m_0(N-2s)$, such that, letting $z_0=(x_0,0)$,*

$$\lambda^{-m_0}U(\lambda z+z_0)\to |z|^{m_0}\widehat{Y}\left(\frac{z}{|z|}\right) \quad \text{as } \lambda\to 0^+ \quad \text{in } H^1(B_1^+,t^{1-2s}), \tag{1.27}$$

where $B_1^+=\{z=(x,t)\in\mathbb{R}^N\times(0,+\infty):|z|<1\}$ and U is trivially extended to zero outside $\mathcal{C}_\Omega$.

In Theorem 6.1 a more precise characterization of the function $\widehat{Y}$ appearing in (1.26) and (1.27) is given, by writing it as a linear combination of the eigenfunctions $Y_{m_0,k}$ with coefficients computed in (5.45).

From Remark 1.1, Theorem 1.2 and Theorem 1.3 we deduce the following unique continuation principles.

Corollary 1.4. *Let $N>2s$ and $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitz domain. Let $x_0\in\partial\Omega$ and assume that there exist $R>0$ and a function g satisfying (1.10), (1.11), and (1.12). Let u be a solution to (1.1) in the sense of (1.8) and U be a solution to (1.17) in the sense of (1.18), with h satisfying (1.7).*

(i) If $u(x)=O\left(|x-x_0|^k\right)$ as $x\to x_0$ for any $k\in\mathbb{N}$, then $u\equiv 0$ in Ω.
(ii) If $U(z)=O\left(|z-(x_0,0)|^k\right)$ as $z\to(x_0,0)$ for any $k\in\mathbb{N}$, then $U\equiv 0$ on $\mathcal{C}_\Omega$.

The paper is organized as follows. In Section 2 we fix some notation used throughout the paper and recall some preliminary results concerning functional inequalities and trace operators. In Section 3 we apply the local diffeomorphism introduced in [2], see also [10], Section 2, to write an equivalent formulation of problem (1.17) on a domain with a straightened lateral boundary in a neighbourhood of x_0, see (3.6). In Section 4 we study the Almgren-type frequency function associated to the auxiliary problem (3.6) and prove its boundedness, which is used in Section 5 to develop a blow-up analysis. Finally in Section 6 we prove our main results and in Appendix A we compute the eigenvalues of problem (1.19).

2. Notations and preliminaries

In this section we present some notation used throughout the paper and prove some preliminary results concerning functional inequalities and trace operators.

For every $r>0$, let

$$B_r^+:=\{z\in\mathbb{R}^{N+1}_+:|z|<r\}, \qquad S_r^+:=\{z\in\mathbb{R}^{N+1}_+:|z|=r\},$$
$$B_r':=\{x\in\mathbb{R}^N:|x|<r\}, \qquad S_r':=\{x\in\mathbb{R}^N:|x|=r\}.$$

For every $r>0$ we define the space

$$H^1_{0,S_r^+}(B_r^+,t^{1-2s}):=\overline{C_c^\infty(B_r^+\cup B_r')}^{\|\cdot\|_{H^1(B_r^+,t^{1-2s})}},$$

as the closure in $H^1(B_r^+,t^{1-2s})$ of $C_c^\infty(B_r^+\cup B_r')$.

Remark 2.1. Since $B_r^+\subset B_r'\times(0,+\infty)$, the trivial extension to 0 is a linear and continuous operator from $H^1_{0,S_r^+}(B_r^+,t^{1-2s})$ to $H^1_{0,L}(\mathcal{C}_{B_r'},t^{1-2s})$.

Proposition 2.2. *For every $r>0$ there exists a linear and continuous trace operator*

$$\mathrm{Tr}:H^1(B_r^+,t^{1-2s})\to H^s(B_r')$$

such that the restriction of Tr to $H^1_{0,S_r^+}(B_r^+,t^{1-2s})$ coincides with the restriction of $\mathrm{Tr}_{B_r'}$ to $H^1_{0,S_r^+}(B_r^+,t^{1-2s})$. In particular, for every $r>0$,

$$\mathrm{Tr}(H^1_{0,S_r^+}(B_r^+,t^{1-2s}))\subseteq\mathbb{H}^s(B_r').$$

Proof. Thanks to Remark 2.1, the operator $\mathrm{Tr}_{B'_r}$ defined in (1.15) is well defined on $H^1_{0,S^+_r}(B^+_r, t^{1-2s})$ and $\mathrm{Tr}_{B'_r}(H^1_{0,S^+_r}(B^+_r, t^{1-2s})) \subseteq \mathbb{H}^s(B'_r)$. Furthermore, as observed in [17], Proposition 2.1 and [5, 20], there exists a linear, continuous trace operator $\mathrm{Tr} : H^1(B^+_r, t^{1-2s}) \to H^s(B'_r)$. For every $u \in C^\infty_c(B^+_r \cup B'_r)$, we have $\mathrm{Tr}(u) = u_{|_{B'_r\times\{0\}}} = \mathrm{Tr}_{B'_r}(u)$. By density we conclude that Tr and $\mathrm{Tr}_{B'_r}$ are equal on $H^1_{0,S^+_r}(B^+_r, t^{1-2s})$. □

We observe that $H^1(B^+_r, t^{1-2s}) \subset W^{1,1}(B^+_r)$, hence, denoting as Tr_1 the classical trace operator from $W^{1,1}(B^+_r)$ to $L^1(S^+_r)$, we can consider its restriction to $H^1(B^+_r, t^{1-2s})$, still denoted as Tr_1; from [22], Theorem 19.7 and the Divergence Theorem one can easily deduce that, for any $r > 0$, such a restriction is a linear, continuous trace operator

$$\mathrm{Tr}_1 : H^1(B^+_r, t^{1-2s}) \to L^2(S^+_r, t^{1-2s}) \tag{2.1}$$

which is also compact. With a slight abuse of notation, from now on we will simply write v instead of $\mathrm{Tr}_1(v)$ on S^+_r.

We recall from [12], Lemma 2.6 the following Sobolev-type inequality with boundary terms.

Proposition 2.3. *There exists a constant $\mathcal{S}_{N,s} > 0$ such that, for all $r > 0$ and $v \in H^1(B^+_r, t^{1-2s})$,*

$$\left(\int_{B'_r} |\operatorname{Tr}(v)|^{2^*_s}\,\mathrm{d}x\right)^{\frac{2}{2^*_s}} \le \mathcal{S}_{N,s}\left(\int_{B^+_r} t^{1-2s}|\nabla v|^2\,\mathrm{d}z + \frac{N-2s}{2r}\int_{S^+_r} t^{1-2s}v^2\,\mathrm{d}S\right), \tag{2.2}$$

where 2^*_s is defined as in (1.9).

The following inequality will be used to obtain estimates on the Almgren frequency function.

Proposition 2.4. *Let ω_N be the N-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$. For any $r > 0$, $v \in H^1(B^+_r, t^{1-2s})$ and $f \in L^{\frac{N}{2s}+\varepsilon}(B'_r)$ with $\varepsilon > 0$, we have*

$$\int_{B'_r} f|\operatorname{Tr}(v)|^2\,\mathrm{d}x \le \eta_f(r)\left(\int_{B^+_r} t^{1-2s}|\nabla v|^2\,\mathrm{d}z + \frac{N-2s}{2r}\int_{S^+_r} t^{1-2s}v^2\,\mathrm{d}S\right), \tag{2.3}$$

where

$$\eta_f(r) := \mathcal{S}_{N,s}\,\omega_N^{\frac{4s^2\varepsilon}{N(N+2s\varepsilon)}}\,\|f\|_{L^{\frac{N}{2s}+\varepsilon}(B'_r)}\,r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}. \tag{2.4}$$

Proof. By the Hölder inequality

$$\int_{B'_r} f|\operatorname{Tr}(v)|^2\,\mathrm{d}x \le \|\operatorname{Tr}(v)\|^2_{L^{2^*_s}(B'_r)}\,\|f\|_{L^{\frac{N}{2s}+\varepsilon}(B'_r)}\,\omega_N^{\frac{4s^2\varepsilon}{N(N+2s\varepsilon)}}\,r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}.$$

Then (2.3) follows from (2.2). □

We also recall the following Hardy-type inequality with boundary terms from [12], Lemma 2.4.

Proposition 2.5. *For any $r > 0$ and any $v \in H^1(B^+_r, t^{1-2s})$*

$$\left(\frac{N-2s}{2}\right)^2\int_{B^+_r} t^{1-2s}\frac{|v(z)|^2}{|z|^2}\,\mathrm{d}z \le \int_{B^+_r} t^{1-2s}\left(\nabla v\cdot\frac{z}{|z|}\right)^2\mathrm{d}z + \left(\frac{N-2s}{2r}\right)\int_{S^+_r} t^{1-2s}v^2\,\mathrm{d}S. \tag{2.5}$$

The following Poincaré-type inequality directly follows from (2.5): for all $r > 0$ and $v \in H^1(B_r^+, t^{1-2s})$

$$\int_{B_r^+} t^{1-2s} v^2 \,\mathrm{d}z \le \frac{4r}{(N-2s)^2}\left(r\int_{B_r^+} t^{1-2s}|\nabla v|^2 \,\mathrm{d}z + \frac{N-2s}{2}\int_{S_r^+} t^{1-2s} v^2 \,\mathrm{d}S\right). \tag{2.6}$$

Remark 2.6. As a consequence of (2.6) and by continuity of the trace operator (2.1), for every $r > 0$

$$\left(\int_{S_r^+} t^{1-2s} v^2 \,\mathrm{d}S + \int_{B_r^+} t^{1-2s}|\nabla v|^2 \,\mathrm{d}z\right)^{1/2}$$

is an equivalent norm on $H^1(B_r^+, t^{1-2s})$.

3. Straightening the boundary

Let $x_0 \in \partial\Omega$, $R > 0$ and g satisfy (1.10), (1.11), and (1.12). Up to a suitable choice of the coordinate system, it is not restrictive to assume that

$$x_0 = 0, \quad g(0) = 0, \quad \nabla g(0) = 0.$$

We use the local diffeomorphism F constructed in [10], Section 2 (see also [2]) to straighten the boundary of $\mathcal{C}_\Omega$ in a neighbourhood of 0; for the sake of clarity and completeness we summarize its properties in Propositions 3.1 and 3.2 below, referring to [10], Section 2 for their proofs. We consider the variable $z = (y, t) \in \mathbb{R}^N \times [0, \infty)$ with $y = (y', y_N) = (y_1, \cdots, y_N)$. For future reference we define

$$M_N := \left(\begin{array}{c|c|c} \mathrm{Id}_{N-1} & 0 & 0 \\ \hline 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \qquad M_N' := \left(\begin{array}{c|c} \mathrm{Id}_{N-1} & 0 \\ \hline 0 & -1 \end{array}\right), \tag{3.1}$$

where Id_{N-1} is the identity $(N-1)\times(N-1)$ matrix.

Proposition 3.1 ([10], Sect. 2)**.** *There exist $F = (F_1, \dots, F_{N+1}) \in C^{1,1}(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ and $r_0 > 0$ such that $F\big|_{B_{r_0}} : B_{r_0} \to F(B_{r_0})$ is a diffeomorphism of class $C^{1,1}$,*

$$\begin{aligned} &F(y', 0, 0) = (y', g(y'), 0) \quad \text{for all } y' \in \mathbb{R}^{N-1}, \\ &F_N(y', y_N, t) = y_N + g(y') \quad \text{for all } (y', y_N, t) \in \mathbb{R}^{N-1}\times\mathbb{R}\times\mathbb{R}, \\ &F_{N+1}(y, t) = t, \quad \text{for all } (y, t) \in \mathbb{R}^N \times \mathbb{R}, \\ &\alpha(y, t) := \det J_F(y, t) > 0 \quad \text{in } B_{r_0}, \end{aligned}$$

and

$$F(\{(y', y_N, t) \in B_{r_0}^+ : y_N = 0\}) = \partial_L \mathcal{C}_\Omega \cap F(B_{r_0}^+), \tag{3.2}$$

$$F(\{(y', y_N, t) \in B_{r_0}^+ : y_N < 0\}) = \mathcal{C}_\Omega \cap F(B_{r_0}^+), \tag{3.3}$$

where $\partial_L \mathcal{C}_\Omega$ is defined in (1.14) and $J_F(y, t)$ is the Jacobian matrix of F. Furthermore the following properties hold:

i) J_F depends only on the variable y and

$$J_F(y', y_N) = J_F(y) = \mathrm{Id}_{N+1} + O(|y|) \quad \text{as } |y| \to 0^+,$$

where Id_{N+1} *denotes the identity* $(N+1)\times(N+1)$ *matrix and* $O(|y|)$ *denotes a matrix with all entries being* $O(|y|)$ *as* $|y| \to 0^+$*;*

ii) $\alpha(y) = \det J_F(y) = 1 + O(|y'|^2) + O(y_N)$ *as* $|y'| \to 0^+$ *and* $y_N \to 0$*;*

iii) $\frac{\partial F_i}{\partial t} = \frac{\partial F_{N+1}}{\partial y_i} = 0$ *for any* $i = 1, \dots, N$ *and* $\frac{\partial F_{N+1}}{\partial t} = 1$.

For every $r > 0$, let

$$\mathcal{Q}_r := \{(y', y_N, t) \in B_r^+ : y_N < 0\}, \tag{3.4}$$

so that $F(\mathcal{Q}_{r_0}) = \mathcal{C}_\Omega \cap F(B_{r_0}^+)$ in view of (3.3). If $U \in H^1_{0,L}(\mathcal{C}_\Omega, t^{1-2s})$ solves (1.17), then the function

$$W = U \circ F \in H^1(\mathcal{Q}_{r_0}, t^{1-2s}) \tag{3.5}$$

is a weak solution to

$$\begin{cases} \operatorname{div}(t^{1-2s} A \nabla W) = 0, & \text{in } \mathcal{Q}_{r_0}, \\ -\lim_{t\to 0^+} t^{1-2s} \alpha \frac{\partial W}{\partial t} = \kappa_{s,N} \bar{h} W, & \text{on } \mathcal{Q}'_{r_0}, \end{cases} \tag{3.6}$$

where $\mathcal{Q}'_r := \{(y', y_N) \in B'_r : y_N < 0\}$ for all $r > 0$, $A = A(y)$ is the $(N+1)\times(N+1)$ matrix-valued function given by

$$A(y) := (J_F(y))^{-1} (J_F(y)^{-1})^T |\det J_F(y)|,$$

and

$$\bar{h}(y) = \alpha(y) h(F(y, 0)). \tag{3.7}$$

As observed in [10], Section 2, A has $C^{0,1}$ entries $\left(a_{ij}\right)_{i,j=1}^{N+1}$ and can be written as

$$A(y) = A(y', y_N) = \left(\begin{array}{c|c} D(y', y_N) & 0 \\ \hline 0 & \alpha(y', y_N) \end{array}\right), \tag{3.8}$$

with

$$D(y', y_N) = \left(\begin{array}{c|c} \mathrm{Id}_{N-1} + O(|y'|^2) + O(y_N) & O(y_N) \\ \hline O(y_N) & 1 + O(|y'|^2) + O(y_N) \end{array}\right), \tag{3.9}$$

where Id_{N-1} is the identity $(N-1)\times(N-1)$ matrix, $O(y_N)$ and $O(|y'|^2)$ denote blocks of matrices with all elements being $O(y_N)$ as $y_N \to 0$ and $O(|y'|^2)$ as $|y'| \to 0$ respectively. In particular, in view of (3.8)-(3.9) we have

$$a_{Nj}(y', 0) = a_{jN}(y', 0) = 0 \quad \text{for all } j = 1, \dots, N-1. \tag{3.10}$$

Having in mind to reflect our problem through the hyperplane $y_N = 0$, we define

$$\widetilde{A}(y', y_N) := \begin{cases} A(y', y_N), & \text{if } y_N \le 0, \\ M_N A(y', -y_N) M_N, & \text{if } y_N > 0, \end{cases} \tag{3.11}$$

$$\widetilde{D}(y', y_N) := \begin{cases} D(y', y_N), & \text{if } y_N \leq 0, \\ M'_N D(y', -y_N) M'_N, & \text{if } y_N > 0, \end{cases} \tag{3.12}$$

with M_N, M'_N as in (3.1), and

$$\widetilde{\alpha}(y', y_N) := \begin{cases} \alpha(y', y_N), & \text{if } y_N \leq 0, \\ \alpha(y', -y_N), & \text{if } y_N > 0, \end{cases} \tag{3.13}$$

where $\alpha(y) = \det J_F(y)$. We observe that the Lipschitz continuity of A and (3.10) imply that the entries of $\widetilde{A}$ are of class $C^{0,1}$. Furthermore, $\widetilde{A}$ is symmetric and, possibly choosing r_0 smaller from the beginning,

$$\|\widetilde{A}(y)\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})} \leq 2 \quad \text{and} \quad \frac{1}{2}|z|^2 \leq \widetilde{A}(y)z \cdot z \leq 2|z|^2 \quad \text{for all } z \in \mathbb{R}^{N+1},\ y \in \overline{B'_{r_0}}, \tag{3.14}$$

where $\|\cdot\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})}$ denotes the operator norm on the space of bounded linear operators from $\mathbb{R}^{N+1}$ into itself. We also observe that (3.8)-(3.9) imply the expansion

$$\widetilde{A}(y) = \mathrm{Id}_{N+1} + O(|y|) \quad \text{as } |y| \to 0^+. \tag{3.15}$$

Letting $\widetilde{A}$ and $\widetilde{D}$ be as in (3.11)–(3.12), we define

$$\mu(z) := \frac{\widetilde{A}(y)z \cdot z}{|z|^2} \quad \text{and} \quad \beta(z) := \frac{\widetilde{A}(y)z}{\mu(z)} \quad \text{for every } z = (y,t) \in \overline{B^+_{r_0}} \setminus \{0\}, \tag{3.16}$$

and

$$\beta'(y) := \frac{\widetilde{D}(y)y}{\mu(y,0)} \quad \text{for every } y \in \overline{B'_{r_0}}. \tag{3.17}$$

For every $z = (z_1, \dots, z_{N+1}) \in \mathbb{R}^{N+1}$ and $y \in \overline{B'_{r_0}}$, $\mathrm{d}\widetilde{A}(y)zz$ is defined as the vector of $\mathbb{R}^{N+1}$ with i-th component given by

$$(\mathrm{d}\widetilde{A}(y)zz)_i = \sum_{h,k=1}^{N+1} \frac{\partial \widetilde{a}_{kh}}{\partial z_i}(y) z_h z_k, \quad i = 1, \dots, N+1, \tag{3.18}$$

where $(\widetilde{a}_{k,h})_{k,h=1}^{N+1}$ are the entries of the matrix $\widetilde{A}$ in (3.11).

Proposition 3.2. *Let μ, β, and β' be as in (3.16)–(3.17). Then, possibly choosing r_0 smaller from the beginning, we have*

$$\frac{1}{2} \leq \mu(z) \leq 2 \quad \textit{for any } z \in \overline{B^+_{r_0}} \setminus \{0\}, \tag{3.19}$$

$$\mu(z) = 1 + O(|z|), \quad \nabla\mu(z) = O(1) \quad \textit{as } |z| \to 0^+. \tag{3.20}$$

Moreover β and β' are well-defined and

$$\beta(z) = z + O(|z|^2) = O(|z|) \quad \textit{as } |z| \to 0^+, \tag{3.21}$$

$$J_\beta(z) = \widetilde{A}(y) + O(|z|) = \mathrm{Id}_{N+1} + O(|z|), \quad \mathrm{div}(\beta)(z) = N + 1 + O(|z|) \quad \textit{as } |z| \to 0^+, \tag{3.22}$$

$$\beta'(y) = y + O(|y|^2) = O(|y|), \quad \mathrm{div}(\beta')(y) = N + O(|y|) \quad \text{as } |y| \to 0^+. \tag{3.23}$$

Proof. (3.19) easily follows from (3.14). We refer to [10], Lemma 2.1 for the proof of (3.20). As a direct consequence, β and β' are well-defined. From (3.21) and (3.22), whose proof is contained in [10], Lemma 2.2, we derive (3.23), after noting that β' coincides with the first N-components of the vector β. □

Remark 3.3. From the Lipschitz continuity of $\widetilde{A}$ observed above and Proposition 3.2 we have

$$\begin{gathered}\widetilde{A} \in C^{0,1}(B_{r_0}^+, \mathbb{R}^{(N+1)^2}),\ \mu \in C^{0,1}(B_{r_0}^+),\ \frac{1}{\mu} \in C^{0,1}(B_{r_0}^+),\ \beta \in C^{0,1}(B_{r_0}^+, \mathbb{R}^{N+1}) \\ J_\beta \in L^\infty(B_{r_0}^+, \mathbb{R}^{(N+1)^2}),\ \mathrm{div}(\beta) \in L^\infty(B_{r_0}^+),\ \beta' \in L^\infty(B'_{r_0}, \mathbb{R}^N),\ \mathrm{div}(\beta') \in L^\infty(B'_{r_0}).\end{gathered} \tag{3.24}$$

Remark 3.4. If $v \in H^1_{0,L}(\mathcal{C}_\Omega, t^{1-2s})$, then $(v \circ F)|_{\mathcal{Q}_{r_0}} \in H^1(\mathcal{Q}_{r_0}, t^{1-2s})$ by Proposition 3.1, and

$$(v \circ F)(z) = 0 \quad \text{for any } z \in \{(y', y_N, t) \in B_{r_0}^+ : y_N = 0\} \tag{3.25}$$

in view of (3.2). Equality (3.25) is meant in the sense of the classical theory of traces for Sobolev spaces; this is possible thanks to the fact that $H^1(E, t^{1-2s}) \subset W^{1,1}(E)$ for any bounded open set $E \subseteq \mathbb{R}^N \times (0, \infty)$.

If W is a solution to (3.6), let $\widetilde{W}$ be defined as follows

$$\widetilde{W}(y', y_N, t) := \begin{cases} W(y', y_N, t), & \text{if } (y', y_N, t) \in \mathcal{Q}_{r_0}, \\ -W(y', -y_N, t), & \text{if } (y', y_N, t) \in B_{r_0}^+ \text{ and } y_N > 0. \end{cases} \tag{3.26}$$

For the sake of convenience we will still denote $\widetilde{W}$ with W. Letting $\bar{h}$ be defined in (3.7), we also consider the following function

$$\widetilde{h}(y', y_N) := \begin{cases} \bar{h}(y', y_N), & \text{if } (y', y_N) \in \mathcal{Q}'_{r_0}, \\ \bar{h}(y', -y_N), & \text{if } (y', y_N) \in B'_{r_0}, \text{ and } y_N > 0. \end{cases} \tag{3.27}$$

It is easy to verify that $W \in H^1(B_{r_0}^+, t^{1-2s})$ thanks to Remark 3.4 and

$$\widetilde{h} \in W^{1, \frac{N}{2s} + \varepsilon}(B'_{r_0}) \tag{3.28}$$

thanks to (1.7), (3.7) and Proposition 3.1. Furthermore W weakly solves

$$\begin{cases} \mathrm{div}(t^{1-2s} \widetilde{A} \nabla W) = 0, & \text{on } B_{r_0}^+, \\ -\lim_{t \to 0^+} t^{1-2s} \widetilde{\alpha} \frac{\partial W}{\partial t} = \kappa_{s,N} \widetilde{h}\, \mathrm{Tr}(W), & \text{on } B'_{r_0}, \end{cases} \tag{3.29}$$

with $\widetilde{\alpha}$ defined in (3.13), $\widetilde{h}$ in (3.27) and $\widetilde{A}$ in (3.11), namely

$$\int_{B_{r_0}^+} t^{1-2s} \widetilde{A} \nabla W \cdot \nabla \phi \, \mathrm{d}z = \kappa_{s,N} \int_{B'_{r_0}} \widetilde{h}\, \mathrm{Tr}(W)\, \mathrm{Tr}(\phi)\, \mathrm{d}y \quad \text{for all } \phi \in H^1_{0, S_{r_0}^+}(B_{r_1}^+, t^{1-2s}). \tag{3.30}$$

Thanks to Proposition 2.2, (3.28) and the Hölder inequality, the second member of (3.30) is well-defined.

Remark 3.5. In [14], Theorem 2.1, it is proved that, if $W \in H^1(B_{r_0}^+, t^{1-2s})$ is a weak solution to (3.30) with $\widetilde{A}$ and $\widetilde{h}$ satisfying (3.8), (3.11), (3.24), (3.19), (3.28), then

$$\nabla_x W \in H^1(B_r^+, t^{1-2s}) \quad \text{and} \quad t^{1-2s}\frac{\partial W}{\partial t} \in H^1(B_r^+, t^{2s-1}) \tag{3.31}$$

for all $r \in (0, r_0)$. Furthermore

$$\|\nabla_x W\|_{H^1(B_r^+, t^{1-2s})} + \left\| t^{1-2s}\frac{\partial W}{\partial t} \right\|_{H^1(B_r^+, t^{2s-1})} \le C\,\|W\|_{H^1(B_{r_0}^+, t^{1-2s})}$$

for a positive constant $C > 0$ depending only on N, s, r, r_0, $\|\widetilde{h}\|_{W^{1,\frac{N}{2s}}(B'_{r_0})}$, $\|\widetilde{A}\|_{W^{1,\infty}(B_{r_0}^+, \mathbb{R}^{(N+1)^2})}$ (but independent of W).

Remark 3.6. If $W \in H^1(B_{r_0}^+, t^{1-2s})$ is a weak solution to (3.30), the regularity result (3.31) and (2.1) ensure that, for all $\phi \in H^1(B_{r_0}^+, t^{1-2s})$ and $r \in (0, r_0)$, $t^{1-2s}\,\mathrm{Tr}_1(\widetilde{D}\nabla_x W \cdot x)\,\mathrm{Tr}_1\,\phi \in L^1(S_r^+)$; moreover the function

$$r \mapsto \int_{S_r^+} t^{1-2s}(\widetilde{D}\nabla_x W \cdot x)\phi\,\mathrm{d}S$$

is continuous in $(0, r_0)$. Furthermore, since $t^{1-2s}\frac{\partial W}{\partial t} \in H^1(B_r^+, t^{2s-1})$ for all $r \in (0, r_0)$ by (3.31), for all $\phi \in H^1(B_{r_0}^+, t^{1-2s})$ and $r \in (0, r_0)$ we also have $t^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial t}t\phi \in W^{1,1}(B_r^+)$, so that $\mathrm{Tr}_1(t^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial t}t\phi) \in L^1(S_r^+)$; moreover the function

$$r \mapsto \int_{S_r^+} t^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial t}t\phi\,\mathrm{d}S$$

is continuous in $(0, r_0)$. We conclude that, for all $\phi \in H^1(B_{r_0}^+, t^{1-2s})$, the function

$$t^{1-2s}(\widetilde{A}\nabla W \cdot z)\phi = t^{1-2s}(\widetilde{D}\nabla_x W \cdot x)\phi + t^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial t}t\phi$$

has a trace on S_r^+ for all $r \in (0, r_0)$ and the function

$$r \mapsto \int_{S_r^+} t^{1-2s}(\widetilde{A}\nabla W \cdot z)\phi\,\mathrm{d}S$$

is continuous in $(0, r_0)$.

The following result provides an integration by parts formula which will be useful in Section 5.

Proposition 3.7. *Let W be a weak solution to (3.29). For all $r \in (0, r_0)$ and $\phi \in H^1(B_{r_0}^+, t^{1-2s})$*

$$\int_{B_r^+} t^{1-2s}\widetilde{A}\nabla W \cdot \nabla\phi\,\mathrm{d}z = \frac{1}{r}\int_{S_r^+} t^{1-2s}(\widetilde{A}\nabla W \cdot z)\phi\,\mathrm{d}S + \kappa_{s,N}\int_{B'_r} \widetilde{h}\,\mathrm{Tr}(W)\,\mathrm{Tr}(\phi)\,\mathrm{d}x. \tag{3.32}$$

Proof. By density it is enough to prove (3.32) for $\phi \in C^\infty(\overline{B^+_{r_0}})$. Let $r \in (0, r_0)$. For every $n \in \mathbb{N}$, let

$$\eta_n(z) := \begin{cases} 1, & \text{if} \quad 0 \le |z| \le r - \frac{1}{n}, \\ n(r - |z|), & \text{if} \quad r - \frac{1}{n} \le |z| \le r, \\ 0, & \text{if} \quad |z| \ge r. \end{cases}$$

Testing (3.30) with $\phi\eta_n$ and passing to the limit as $n \to \infty$, we obtain (3.32) thanks to the integral mean value theorem and Remark 3.6. □

Remark 3.8. For all $r \in (0, r_0]$ and any $v \in H^1(B^+_r, t^{1-2s})$, thanks to (2.3), (3.14) and (3.19),

$$\int_{B^+_r} t^{1-2s}|\nabla v|^2 \,\mathrm{d}z \le 2\int_{B^+_r} t^{1-2s}\widetilde{A}\nabla v\cdot\nabla v \,\mathrm{d}z - 2\kappa_{N,s}\int_{B'_r} \widetilde{h}|\operatorname{Tr}(v)|^2 \,\mathrm{d}x \\ + 2\kappa_{N,s}\eta_{\tilde{h}}(r)\left(\int_{B^+_r} t^{1-2s}|\nabla v|^2 \,\mathrm{d}z + \frac{N-2s}{r}\int_{S^+_r} t^{1-2s}\mu v^2 \,\mathrm{d}S\right).$$

Therefore, if $\eta_{\tilde{h}}(r) < \frac{1}{2\kappa_{N,s}}$,

$$\int_{B^+_r} t^{1-2s}|\nabla v|^2 \,\mathrm{d}z \le \frac{2}{1-2\kappa_{N,s}\eta_{\tilde{h}}(r)}\left(\int_{B^+_r} t^{1-2s}\widetilde{A}\nabla v\cdot\nabla v \,\mathrm{d}z - \kappa_{N,s}\int_{B'_r} \widetilde{h}|\operatorname{Tr}(v)|^2 \,\mathrm{d}x\right) \\ + \frac{2(N-2s)\kappa_{N,s}\eta_{\tilde{h}}(r)}{(1-2\kappa_{N,s}\eta_{\tilde{h}}(r))r}\int_{S^+_r} t^{1-2s}\mu v^2 \,\mathrm{d}S. \tag{3.33}$$

4. The monotonicity formula

Let W be a non-trivial weak solution of (3.29). For any $r \in (0, r_0]$ we define the height function and the energy function as

$$H(r) := \frac{1}{r^{N+1-2s}}\int_{S^+_r} t^{1-2s}\mu W^2 \,\mathrm{d}S, \tag{4.1}$$

$$D(r) := \frac{1}{r^{N-2s}}\left(\int_{B^+_r} t^{1-2s}\widetilde{A}\nabla W\cdot\nabla W \,\mathrm{d}z - \kappa_{N,s}\int_{B'_r} \widetilde{h}|\operatorname{Tr} W|^2 \,\mathrm{d}x\right), \tag{4.2}$$

respectively. Eventually choosing r_0 smaller from the beginning, we may assume that

$$\eta_{\tilde{h}}(r) < \frac{1}{4\kappa_{N,s}} \quad \text{for all } r \in (0, r_0], \tag{4.3}$$

so that (3.33) holds for every $r \in (0, r_0]$.

Proposition 4.1. *Let H and D be as in (4.1) and (4.2). Then $H \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and*

$$H'(r) = \frac{2}{r^{N+1-2s}}\int_{S^+_r} t^{1-2s}\mu W\frac{\partial W}{\partial\nu} \,\mathrm{d}S + H(r)O(1) \quad \text{as } r \to 0^+ \tag{4.4}$$

in the sense of distributions and almost everywhere, where ν is the outer normal vector to B_r^+ on S_r^+, i.e $\nu(z) := \frac{z}{|z|}$. Moreover, almost everywhere we have

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s}(\widetilde{A}\nabla W \cdot \nu) W \,\mathrm{d}S + H(r)O(1) \qquad \text{as } r \to 0^+ \tag{4.5}$$

and

$$H'(r) = \frac{2}{r} D(r) + H(r)O(1) \qquad \text{as } r \to 0^+. \tag{4.6}$$

Proof. The proof is similar to that of [10], Lemma 3.1 thus we omit it. □

Proposition 4.2. *We have $H(r) > 0$ for every $r \in (0, r_0]$.*

Proof. Let us assume by contradiction that there exists $r \in (0, r_0]$ such that $H(r) = 0$. Then, from (4.1) and (3.19) we deduce that $W \equiv 0$ on S_r^+. Thus we can test (3.30) with W, obtaining that

$$\begin{aligned} 0 &= \int_{B_r^+} t^{1-2s}\widetilde{A}\nabla W \cdot \nabla W \,\mathrm{d}z - \kappa_{N,s}\int_{B'_r} \widetilde{h}|\operatorname{Tr}(W)|^2 \,\mathrm{d}x \\ &\geq \left(\frac{1}{2} - \kappa_{N,s}\eta_{\widetilde{h}}(r)\right) \|\nabla W\|^2_{L^2(B_r^+, t^{1-2s})}, \end{aligned}$$

thanks to (3.33). Then, by (4.3) we can conclude that $W \equiv 0$ on B_r^+; this implies that $W \equiv 0$ on $B_{r_0}^+$ by classical unique continuation principles for second order elliptic operators with Lipschitz coefficients (see *e.g.* [15]), giving rise to a contradiction. □

The following proposition contains a Pohozaev-type identity for problem (3.29). For its proof we refer to [14], Proposition 2.3, where a more general version is established exploiting some Sobolev-type regularity results.

Proposition 4.3 ([14], Prop. 2.3)**.** *Let W be a weak solution to equation (3.29). Then, for a.e. $r \in (0, r_0)$,*

$$\begin{aligned} &\int_{S_r^+} t^{1-2s}\widetilde{A}\nabla W \cdot \nabla W \,\mathrm{d}S - \kappa_{N,s}\int_{S'_r} \widetilde{h}|\operatorname{Tr}(W)|^2 \,\mathrm{d}S' \\ &= 2\int_{S_r^+} t^{1-2s}\frac{|\widetilde{A}\nabla W \cdot \nu|^2}{\mu} \,\mathrm{d}S - \frac{\kappa_{N,s}}{r}\int_{B'_r} (\operatorname{div_y}(\beta')\widetilde{h} + \beta' \cdot \nabla\widetilde{h})|\operatorname{Tr}(W)|^2 \,\mathrm{d}y \\ &\quad + \frac{1}{r}\int_{B_r^+} t^{1-2s}\widetilde{A}\nabla W \cdot \nabla W \operatorname{div}(\beta) \,\mathrm{d}z - \frac{2}{r}\int_{B_r^+} t^{1-2s} J_\beta(\widetilde{A}\nabla W) \cdot \nabla W \,\mathrm{d}z \\ &\quad + \frac{1}{r}\int_{B_r^+} t^{1-2s}(\mathrm{d}\widetilde{A}\,\nabla W\,\nabla W) \cdot \beta \,\mathrm{d}z + \frac{1-2s}{r}\int_{B_r^+} t^{1-2s}\frac{\widetilde{\alpha}}{\mu}\widetilde{A}\nabla W \cdot \nabla W \,\mathrm{d}z, \end{aligned} \tag{4.7}$$

where μ and β are defined in (3.16), $\widetilde{\alpha}$ in (3.13), β' in (3.17), ν is the outer normal vector to B_r^+ on S_r^+, i.e $\nu(z) = \frac{z}{|z|}$, and $\mathrm{d}S'$ denotes the volume element on $(N-1)$-dimensional spheres.

Remark 4.4. As in Remark 3.6, by the Coarea Formula we have

$$\int_{B'_{r_0}} |\widetilde{h}||\operatorname{Tr}(W)|^2 \,\mathrm{d}x = \int_0^{r_0} \left(\int_{S'_\rho} |\widetilde{h}||\operatorname{Tr}(W)|^2 \,\mathrm{d}S'\right) \mathrm{d}\rho,$$

hence $\rho \to \int_{S'_\rho} \widetilde{h}|\operatorname{Tr}(W)|^2 \, dS'$ is a well-defined $L^1(0,r_0)$-function, as a consequence of (3.28), (2.2) and the Hölder inequality.

Proposition 4.5. *Let D be as in (4.2). Then $D \in W^{1,1}_{\mathrm{loc}}((0,r_0])$ and*

$$D'(r) = 2r^{2s-N}\int_{S_r^+} t^{1-2s}\frac{|\widetilde{A}\nabla W\cdot\nu|^2}{\mu}\,dS + O\left(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)\left[D(r)+\frac{N-2s}{2}H(r)\right] \tag{4.8}$$

as $r \to 0^+$, in the sense of distributions and almost everywhere.

Proof. By the Coarea Formula $D \in W^{1,1}_{\mathrm{loc}}((0,r_0])$ and

$$\begin{aligned} D'(r) = (2s-N)r^{2s-N-1}&\left(\int_{B_r^+} t^{1-2s}\widetilde{A}\nabla W\cdot\nabla W\,dz - \kappa_{N,s}\int_{B'_r}\widetilde{h}|\operatorname{Tr}(W)|^2\,dx\right)\\ &+ r^{2s-N}\left(\int_{S_r^+} t^{1-2s}\widetilde{A}\nabla W\cdot\nabla W\,dS - \kappa_{N,s}\int_{S'_r}\widetilde{h}|\operatorname{Tr}(W)|^2\,dS'\right) \end{aligned}\tag{4.9}$$

a.e. and in the sense of distributions in $(0,r_0)$. Using (4.7) to estimate the second term on the right hand side of (4.9), for a.e. $r\in(0,r_0)$ we have

$$\begin{aligned} D'(r) =&(2s-N)r^{2s-N-1}\left(\int_{B_r^+} t^{1-2s}\widetilde{A}\nabla W\cdot\nabla W\,dz - \kappa_{N,s}\int_{B'_r}\widetilde{h}|\operatorname{Tr}(W)|^2\,dx\right)\\ &+ r^{2s-N}\left(2\int_{S_r^+} t^{1-2s}\frac{|\widetilde{A}\nabla W\cdot\nu|^2}{\mu}\,dS - \frac{\kappa_{N,s}}{r}\int_{B'_r}(\operatorname{div}_{\mathrm{y}}(\beta')\widetilde{h}+\beta'\cdot\nabla\widetilde{h})|\operatorname{Tr}(W)|^2\,dy\right)\\ &+ r^{2s-N}\left(\frac{1}{r}\int_{B_r^+} t^{1-2s}\widetilde{A}\nabla W\cdot\nabla W\operatorname{div}(\beta)\,dz - \frac{2}{r}\int_{B_r^+} t^{1-2s}J_\beta(\widetilde{A}\nabla W)\cdot\nabla W\,dz\right)\\ &+ r^{2s-N}\left(\frac{1}{r}\int_{B_r^+} t^{1-2s}(d\widetilde{A}\nabla W\nabla W)\cdot\beta\,dz + \frac{1-2s}{r}\int_{B_r^+} t^{1-2s}\frac{\widetilde{\alpha}}{\mu}\widetilde{A}\nabla W\cdot\nabla W\,dz\right). \end{aligned}\tag{4.10}$$

Furthermore, thanks to point ii) of Proposition 3.1, (3.13), (3.14), (3.19), (3.20), (3.21), (3.22), and (3.33), we deduce that

$$\begin{aligned} &r^{2s-N-1}\int_{B_r^+} t^{1-2s}\left[\left(2s-N+\operatorname{div}(\beta)+(1-2s)\tfrac{\widetilde{\alpha}}{\mu}\right)\widetilde{A}\nabla W\cdot\nabla W - 2J_\beta(\widetilde{A}\nabla W)\cdot\nabla W\right]dz\\ &+ r^{2s-N-1}\int_{B_r^+} t^{1-2s}(d\widetilde{A}\,\nabla W\,\nabla W)\cdot\beta\,dz = O(r)\,r^{2s-N-1}\int_{B_r^+} t^{1-2s}|\nabla W|^2\,dz\\ &= O(1)\left[D(r)+\frac{N-2s}{2}H(r)\right] \text{ as } r\to 0^+, \end{aligned}\tag{4.11}$$

where we used also the fact that $d\widetilde{A}\,\nabla W\,\nabla W = O(1)|\nabla W|^2$ as $r \to 0^+$ by (3.18) and (3.24).

In addition, recalling that $\tilde{h} \in W^{1,\frac{N}{2s}+\varepsilon}(B'_{r_1})$, from (2.3), (2.4), (3.24) and (3.33) it follows that

$$r^{2s-N-1} \int_{B'_r} [(2s - N + \operatorname{div}_y(\beta'))\widetilde{h} + \beta' \cdot \nabla\widetilde{h}]|\operatorname{Tr}(W)|^2 \, \mathrm{d}x = O\left(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)\left[D(r) + \frac{N-2s}{2}H(r)\right] \tag{4.12}$$

as $r \to 0^+$. Combining (4.10), (4.11) and (4.12), we obtain (4.8). □

For every $r \in (0, r_0]$ we define the *frequency function*

$$\mathcal{N}(r) := \frac{D(r)}{H(r)}. \tag{4.13}$$

Definition (4.13) is well-posed thanks to Proposition 4.2.

Proposition 4.6. *We have $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and*

$$\mathcal{N}(r) > -\frac{N-2s}{2} \quad \text{for every } r \in (0, r_0]. \tag{4.14}$$

Furthermore, if $\nu(z) := \frac{z}{|z|}$ is the outer normal vector to B_r^+ on S_r^+ and

$$\mathcal{V}(r) := 2r\frac{\left(\int_{S_r^+} t^{1-2s}\mu W^2 \, \mathrm{d}S\right)\left(\int_{S_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu} \, \mathrm{d}S\right) - \left(\int_{S_r^+} t^{1-2s} W A\nabla W \cdot \nu \, \mathrm{d}S\right)^2}{\left(\int_{S_r^+} t^{1-2s}\mu W^2 \, \mathrm{d}S\right)^2},$$

then

$$\mathcal{V}(r) \geq 0 \quad \text{for a.e. } r \in (0, r_0) \tag{4.15}$$

and, for a.e. $r \in (0, r_0)$,

$$\mathcal{N}'(r) - \mathcal{V}(r) = O\left(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)\left[\mathcal{N}(r) + \frac{N-2s}{2}\right] \quad \text{as } r \to 0^+. \tag{4.16}$$

Proof. Since $D \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and $\frac{1}{H} \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ by Proposition 4.1 and Proposition 4.2, then $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}((0, r_0])$. Furthermore we recall that (3.33) holds for every $r \in (0, r_1]$, thus

$$\mathcal{N}(r) \geq -\kappa_{N,s}(N-2s)\eta_{\tilde{h}}(r), \tag{4.17}$$

for every $r \in (0, r_0]$ and, in virtue of this, (4.14) directly follows from (4.3). Moreover (4.15) is a consequence of the Cauchy-Schwarz inequality in $L^2(S_r^+, t^{1-2s})$. From (4.5), (4.6) and (4.8) we deduce that

$$\begin{aligned}
\mathcal{N}'(r) &= \frac{D'(r)H(r) - D(r)H'(r)}{(H(r))^2} = \frac{D'(r)H(r) - \frac{r}{2}(H'(r))^2 + O(r)H(r)H'(r)}{(H(r))^2} \\
&= \mathcal{V}(r) + O(r) + O(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}})\left[\mathcal{N}(r) + \frac{N-2s}{2}\right] \\
&\quad + \frac{O(r^{-N+2s})}{H(r)}\int_{S_r^+} t^{1-2s}(A\nabla W\cdot\nu)W \, \mathrm{d}S
\end{aligned} \tag{4.18}$$

as $r \to 0^+$. In order to deal with the last term in (4.18), we observe that, for a.e. $r \in (0, r_0)$,

$$\int_{S_r^+} t^{1-2s}(A\nabla W \cdot \nu)W \,\mathrm{d}S = r^{N-2s}D(r) + H(r)O(r^{N+1-2s}) \quad \text{as } r \to 0^+,$$

in virtue of (4.5) and (4.6). Thus, substituting into (4.18), we conclude that

$$\mathcal{N}'(r) = \mathcal{V}(r) + O(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}) \left[\mathcal{N}(r) + \frac{N-2s}{2}\right] \quad \text{as } r \to 0^+,$$

where we have used that $\frac{4s^2\varepsilon}{N+2s\varepsilon} < 1$ since $\varepsilon \in (0,1)$ and $N > 2s$. Estimate (4.16) is thereby proved. $\square$

Proposition 4.7. *There exists a constant $C > 0$ such that, for every $r \in (0, r_0]$,*

$$\mathcal{N}(r) \le C. \tag{4.19}$$

Proof. From (4.15) and (4.16) we deduce that there exists a constant $c > 0$ such that

$$\left(\mathcal{N}(r) + \frac{N-2s}{2}\right)' \ge -c\, r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}} \left(\mathcal{N}(r) + \frac{N-2s}{2}\right) \quad \text{for a.e. } r \in (0, r_1), \tag{4.20}$$

for some $r_1 \in (0, r_0)$ sufficiently small. Hence, thanks to (4.14), we are allowed to divide each member of (4.20) by $\mathcal{N}(r) + \frac{N-2s}{2}$, obtaining that

$$\left(\log\left(\mathcal{N}(r) + \frac{N-2s}{2}\right)\right)' \ge -c\, r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}} \quad \text{for a.e. } r \in (0, r_1).$$

Then, integrating over (r, r_1) with $r < r_1$, we have

$$\mathcal{N}(r) \le -\frac{N-2s}{2} + \exp\left(c\,\frac{N+2s\varepsilon}{4s^2\varepsilon} r_1^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)\left(\mathcal{N}(r_1) + \frac{N-2s}{2}\right) \quad \text{for every } r \in (0, r_1),$$

which proves (4.19), taking into account the continuity of $\mathcal{N}$ in $(0, r_0]$. $\square$

Proposition 4.8. *There exists the limit*

$$\gamma := \lim_{r \to 0^+} \mathcal{N}(r). \tag{4.21}$$

Moreover γ is finite and $\gamma \ge 0$.

Proof. Combining (4.19) and (4.20), we infer that

$$\left(\mathcal{N}(r) + \frac{N-2s}{2}\right)' \ge -c\, r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}} \left(C + \frac{N-2s}{2}\right) \tag{4.22}$$

for a.e. $r \in (0, r_1)$, hence

$$\left(\frac{N-2s}{2} + \mathcal{N}(r) + c\left(\frac{N-2s}{2} + C\right)\frac{N+2s\varepsilon}{4s^2\varepsilon} r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)' \ge 0 \quad \text{for a.e. } r \in (0, r_1).$$

From this, it follows in particular that the limit γ in (4.21) exists. Moreover, by (4.14) and (4.19) γ is finite, whereas (4.17) implies that $\gamma \geq 0$. ☐

Proposition 4.9. *There exist $c_0, \bar{c} > 0$ and $\bar{r} \in (0, r_0)$ such that*

$$H(r) \leq c_0\, r^{2\gamma} \quad \text{for all } r \in (0, r_0] \tag{4.23}$$

and

$$H(Rr) \leq R^{\bar{c}}\, H(r) \quad \text{for all } R \geq 1 \text{ and } r \in \left(0, \tfrac{\bar{r}}{R}\right]. \tag{4.24}$$

Furthermore, for any $\sigma > 0$ there exists a constant $c_\sigma > 0$ such that

$$H(r) \geq c_\sigma r^{2\gamma+\sigma} \quad \text{for all } r \in (0, r_0]. \tag{4.25}$$

Proof. By (4.21) we have $\mathcal{N}(r) = \gamma + \int_0^r \mathcal{N}'(t)\,\mathrm{d}t$; hence from (4.6) it follows that

$$\frac{H'(r)}{H(r)} = \frac{2}{r}\mathcal{N}(r) + O(1) = \frac{2}{r}\int_0^r \mathcal{N}'(t)\,\mathrm{d}t + \frac{2\gamma}{r} + O(1). \tag{4.26}$$

From (4.22) and up to choosing r_1 smaller, it follows that, for a.e. $r \in (0, r_1)$,

$$\frac{H'(r)}{H(r)} \geq -\kappa r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}} + \frac{2\gamma}{r}$$

for some positive constant $\kappa > 0$. Then an integration over (r, r_1) yields

$$\log\left(\frac{H(r_1)}{H(r)}\right) \geq -\kappa\frac{N+2s\varepsilon}{4s^2\varepsilon}\left(r_1^{\frac{4s^2\varepsilon}{N+2s\varepsilon}} - r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right) + \log\left(\frac{r_1}{r}\right)^{2\gamma}$$

and thus

$$H(r) \leq \frac{H(r_1)}{r_1^{2\gamma}}\exp\left(\kappa\frac{N+2s\varepsilon}{4s^2\varepsilon}r_1^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)r^{2\gamma}$$

for all $r \in (0, r_1]$, thus implying (4.23) thanks to the continuity of H in $(0, r_0]$.

To prove (4.24), we observe that (4.26) and (4.19) imply that, for some $\bar{r} \in (0, r_0)$ and $\bar{c} > 0$,

$$\frac{H'(r)}{H(r)} \leq \frac{\bar{c}}{r} \quad \text{for all } r \in (0, \bar{r}),$$

whose integration over (r, rR) directly gives (4.24).

In view of Proposition 4.8, for any $\sigma > 0$ there exists $r_\sigma \in (0, r_0]$ such that

$$\frac{H'(r)}{H(r)} = \frac{2}{r}\mathcal{N}(r) + O(1) \leq \frac{2\gamma+\sigma}{r} \quad \text{for all } r \in (0, r_\sigma].$$

Integrating over (r, r_σ) and recalling that H is continuous in $(0, r_0]$, we deduce (4.25). ☐

Proposition 4.10. *There exists the limit $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ and it is finite.*

Proof. By (4.23) it is sufficient to show that the limit does exist. In view of (4.6) we have

$$\begin{aligned}\left(\frac{H(r)}{r^{2\gamma}}\right)' &= \frac{r^{2\gamma}H'(r)-2\gamma r^{2\gamma-1}H(r)}{r^{4\gamma}} = 2r^{-2\gamma-1}(D(r)-\gamma H(r)) + r^{-2\gamma}O(1)H(r)\\ &= 2r^{-2\gamma-1}H(r)\left(\mathcal{N}(r)-\gamma+rO(1)\right)\\ &= 2r^{-2\gamma-1}H(r)\left(\int_0^r \left[\mathcal{N}'(t)-\mathcal{V}(t)\right]\mathrm{d}t + \int_0^r \mathcal{V}(t)\,\mathrm{d}t + rO(1)\right)\end{aligned}$$

as $r\to 0^+$. Integrating over $(r,\tilde r)$ with $\tilde r\in(0,r_0)$ small, we obtain that

$$\begin{aligned}\frac{H(\tilde r)}{\tilde r^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} &= \int_r^{\tilde r} 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \mathcal{V}(t)\,\mathrm{d}t\right)\mathrm{d}\rho\\ &\quad+\int_r^{\tilde r}\left[2\rho^{-2\gamma}H(\rho)O(1)+2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho\left[\mathcal{N}'(t)-\mathcal{V}(t)\right]\mathrm{d}t\right)\right]\mathrm{d}\rho.\end{aligned} \tag{4.27}$$

Letting

$$f(\rho) := 2\rho^{-2\gamma}H(\rho)O(1)+2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho\left[\mathcal{N}'(t)-\mathcal{V}(t)\right]\mathrm{d}t\right),$$

from (4.16), (4.19) and (4.23) it follows that $f\in L^1(0,\tilde r)$ and hence there exists the limit

$$\lim_{r\to 0^+}\int_r^{\tilde r} f(\rho)\,\mathrm{d}\rho = \int_0^{\tilde r} f(\rho)\,\mathrm{d}\rho < +\infty.$$

On the other hand, in view of (4.15), there exists the limit

$$\lim_{r\to 0^+}\int_r^{\tilde r} 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \mathcal{V}(t)\,\mathrm{d}t\right)\mathrm{d}\rho.$$

Therefore we can conclude thanks to (4.27). □

5. The blow-up analysis

In the present section, we aim to classify the possible vanishing orders of solutions to (3.29). To this purpose, let W be a non-trivial weak solution to (3.29) and H be defined in (4.1). For any $\lambda\in(0,r_0]$, we consider the function

$$V^\lambda(z) := \frac{W(\lambda z)}{\sqrt{H(\lambda)}}. \tag{5.1}$$

It is easy to verify that V^λ weakly solves

$$\begin{cases}\operatorname{div}(t^{1-2s}\widetilde{A}(\lambda\cdot)\nabla V^\lambda)=0, & \text{on } B^+_{r_0\lambda^{-1}},\\ -\lim_{t\to 0^+} t^{1-2s}\widetilde{\alpha}(\lambda\cdot)\frac{\partial V^\lambda}{\partial t} = \kappa_{s,N}\lambda^{2s}\widetilde{h}(\lambda\cdot)\operatorname{Tr}(V^\lambda), & \text{on } B'_{r_0\lambda^{-1}},\end{cases}$$

where we have defined $\widetilde{\alpha}$ in (3.13). It follows that, for any $\lambda \in (0, r_0]$,

$$\int_{B_1^+} t^{1-2s} \widetilde{A}(\lambda \cdot) \nabla V^\lambda \cdot \nabla \phi \, \mathrm{d}z - \kappa_{s,N} \lambda^{2s} \int_{B_1'} \widetilde{h}(\lambda \cdot) \operatorname{Tr}(V^\lambda) \operatorname{Tr}(\phi) \, \mathrm{d}y = 0 \tag{5.2}$$

for every $\phi \in H^1_{0,S_1^+}(B_1^+, t^{1-2s})$. Furthermore by (4.1) and (5.1)

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \mu(\lambda\theta) |V^\lambda(\theta)|^2 \, \mathrm{d}S = 1 \quad \text{for any } \lambda \in (0, r_0]. \tag{5.3}$$

Proposition 5.1. *For every $R \geq 1$, the family of functions $\{V^\lambda : \lambda \in (0, \frac{\bar{r}}{R}]\}$ is bounded in $H^1(B_R^+, t^{1-2s})$.*

Proof. By (3.33) and (4.24), for all $\lambda \in (0, \frac{\bar{r}}{R}]$ with $\bar{r}$ as in Lemma 4.9, we have

$$\begin{aligned}\int_{B_R^+} t^{1-2s} |\nabla V^\lambda|^2 \, \mathrm{d}z &= \frac{\lambda^{2s-N}}{H(\lambda)} \int_{B_{\lambda R}^+} t^{1-2s} |\nabla W|^2 \, \mathrm{d}z \leq \frac{\lambda^{2s-N} R^{\bar{c}}}{H(\lambda R)} \int_{B_{\lambda R}^+} t^{1-2s} |\nabla W|^2 \, \mathrm{d}z \\ &\leq \frac{2R^{\bar{c}+N-2s}}{1 - 2\kappa_{N,s} \eta_{\tilde{h}}(\lambda R)} \mathcal{N}(\lambda R) + \frac{2(N-2s) R^{\bar{c}+N-2s} \kappa_{N,s} \eta_{\tilde{h}}(\lambda R)}{1 - 2\kappa_{N,s} \eta_{\tilde{h}}(\lambda R)},\end{aligned}$$

which, together with (4.3) and (4.19), allows us to deduce that $\{\nabla V^\lambda : \lambda \in (0, \frac{\bar{r}}{R}]\}$ is uniformly bounded in $L^2(B_R^+, t^{1-2s})$. On the other hand, (3.19), a scaling argument, and (4.24) imply that

$$\int_{S_R^+} t^{1-2s} |V^\lambda|^2 \mathrm{d}S = \frac{\lambda^{-N-1+2s}}{H(\lambda)} \int_{S_{R\lambda}^+} t^{1-2s} W^2 \mathrm{d}S \leq 2R^{N+1-2s} \frac{H(R\lambda)}{H(\lambda)} \leq 2R^{N+1-2s+\bar{c}},$$

so that the claim follows from (2.6). □

Proposition 5.2. *Let W be a non-trivial weak solution to (3.29). Let γ be as in Proposition 4.8. There exists $m_0 \in \mathbb{N} \setminus \{0\}$ (which is odd in the case $N = 1$) such that*

$$\gamma = m_0. \tag{5.4}$$

Furthermore, for any sequence $\{\lambda_n\}$ such that $\lambda_n \to 0^+$ as $n \to \infty$, there exist a subsequence $\{\lambda_{n_k}\}$ and an eigenfunction Ψ of problem (1.19) associated with the eigenvalue $\mu_{m_0} = m_0^2 + m_0(N - 2s)$ such that $\|\Psi\|_{L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})} = 1$ and

$$\frac{W(\lambda_{n_k} z)}{\sqrt{H(\lambda_{n_k})}} \to |z|^\gamma \Psi\left(\frac{z}{|z|}\right) \text{ as } k \to +\infty \quad \text{strongly in } H^1(B_1^+, t^{1-2s}). \tag{5.5}$$

Proof. Let W be a non-trivial weak solution to (3.29) and $\{\lambda_n\}$ be a sequence such that $\lambda_n \to 0^+$ as $n \to +\infty$. Thanks to Proposition 5.1, there exist a subsequence $\{\lambda_{n_k}\}$ and $V \in H^1(B_1^+, t^{1-2s})$ such that

$$V^{\lambda_{n_k}} \rightharpoonup V \quad \text{weakly in } H^1(B_1^+, t^{1-2s}) \text{ as } k \to +\infty. \tag{5.6}$$

For sufficiently large k we have $\lambda_{n_k} \in (0, r_0)$ and thus $B_1^+ \subset B_{r_0/\lambda_{n_k}}^+$, hence from (5.2) we deduce that

$$\int_{B_1^+} t^{1-2s} \widetilde{A}(\lambda_{n_k} \cdot) \nabla V^{\lambda_{n_k}} \cdot \nabla \phi \, \mathrm{d}z = \kappa_{s,N} \lambda_{n_k}^{2s} \int_{B_1'} \widetilde{h}(\lambda_{n_k} \cdot) \operatorname{Tr}(V^{\lambda_{n_k}}) \operatorname{Tr}(\phi) \, \mathrm{d}y \tag{5.7}$$

for every $\phi \in H^1_{0,S_1^+}(B_1^+, t^{1-2s})$. In order to study what happens as $k \to +\infty$, we notice that the term on the left hand side of (5.7) can be rewritten as follows

$$\begin{aligned}&\int_{B_1^+} t^{1-2s} \widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}} \cdot \nabla\phi \, \mathrm{d}z \\ &= \int_{B_1^+} t^{1-2s}(\widetilde{A}(\lambda_{n_k}\cdot) - \mathrm{Id}_{N+1})\nabla V^{\lambda_{n_k}} \cdot \nabla\phi \, \mathrm{d}z + \int_{B_1^+} t^{1-2s}\nabla V^{\lambda_{n_k}} \cdot \nabla\phi \, \mathrm{d}z.\end{aligned} \tag{5.8}$$

Therefore, in view of (3.15), Proposition 5.1 and (5.6), we conclude that

$$\lim_{k\to+\infty} \int_{B_1^+} t^{1-2s} \widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}} \cdot \nabla\phi \, \mathrm{d}z = \int_{B_1^+} t^{1-2s}\nabla V \cdot \nabla\phi \, \mathrm{d}z. \tag{5.9}$$

As for the right hand side in (5.7), we have

$$\begin{aligned}&\left|\lambda_{n_k}^{2s} \int_{B_1'} \widetilde{h}(\lambda_{n_k}\cdot) \operatorname{Tr}(V^{\lambda_{n_k}}) \operatorname{Tr}(\phi) \, \mathrm{d}y\right| \\ &\leq \lambda_{n_k}^{2s} \eta_{\widetilde{h}(\lambda_{n_k}\cdot)}(1) \left(\int_{B_1^+} t^{1-2s}|\nabla\phi|^2 \, \mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{B_1^+} t^{1-2s}|\nabla V^{\lambda_{n_k}}|^2 \, \mathrm{d}z + \frac{N-2s}{2}\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}|V^{\lambda_{n_k}}|^2 \, \mathrm{d}S\right)^{\frac{1}{2}}\end{aligned} \tag{5.10}$$

thanks to Hölder's inequality and (2.3). By (2.4) and the change of variable $x \mapsto \lambda_{n_k}x$, we obtain that

$$\begin{aligned}\lambda_{n_k}^{2s} \eta_{\widetilde{h}(\lambda_{n_k}\cdot)}(1) &= \mathcal{S}_{N,s}\omega_N^{\frac{4s^2\varepsilon}{N(N+2s\varepsilon)}} \lambda_{n_k}^{2s} \|\widetilde{h}(\lambda_{n_k}\cdot)\|_{L^{\frac{N}{2s}+\varepsilon}(B_1')} \\ &= \mathcal{S}_{N,s}\omega_N^{\frac{4s^2\varepsilon}{N(N+2s\varepsilon)}} \|\widetilde{h}\|_{L^{\frac{N}{2s}+\varepsilon}(B'_{\lambda_{n_k}})} \lambda_{n_k}^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}.\end{aligned} \tag{5.11}$$

Putting together (5.10) and (5.11), thanks to Proposition 5.1, (5.3), and (3.19) we infer that

$$\lim_{k\to+\infty} \lambda_{n_k}^{2s} \int_{B_1'} \widetilde{h}(\lambda_{n_k}\cdot) \operatorname{Tr}(V^{\lambda_{n_k}}) \operatorname{Tr}(\phi) \, \mathrm{d}y = 0. \tag{5.12}$$

Passing to the limit as $k \to +\infty$ in (5.7) we conclude that V weakly solves the following problem:

$$\begin{cases}\operatorname{div}(t^{1-2s}\nabla V) = 0, & \text{in } B_1^+, \\ \lim_{t\to 0^+} t^{1-2s}\frac{\partial V}{\partial t} = 0, & \text{on } B_1'.\end{cases} \tag{5.13}$$

In particular V is smooth on B_1^+ and $V \not\equiv 0$ since, by (3.20), (5.6) and the compactness of the trace operator in (2.1), (5.3) leads to

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^2 \, \mathrm{d}S = 1. \tag{5.14}$$

Now we aim to show that, along a further subsequence,

$$V^{\lambda_{n_k}} \to V \quad \text{strongly in } H^1(B_1^+, t^{1-2s}) \text{ as } k \to +\infty. \tag{5.15}$$

To this purpose, we first notice that a change of variables in (3.32) yields

$$\int_{B_1^+} t^{1-2s}\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot\nabla\phi\,\mathrm{d}z-\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot z\,\phi\,\mathrm{d}S$$
$$=\kappa_{s,N}\lambda_{n_k}^{2s}\int_{B_1'}\widetilde{h}(\lambda_{n_k}\cdot)\operatorname{Tr}(V^{\lambda_{n_k}})\operatorname{Tr}(\phi)\,\mathrm{d}y \tag{5.16}$$

for any $\phi\in H^1(B_1^+,t^{1-2s})$ and k sufficiently large.

From Proposition 5.1 and the regularity result contained in [14], Theorem 2.1 and recalled in Remark 3.5, it follows that $\{\nabla_x V^{\lambda_{n_k}}\}$ and $\left\{\operatorname{Tr}_1\left(t^{1-2s}\frac{\partial V^{\lambda_{n_k}}}{\partial t}\right)\right\}$ are uniformly bounded in k in the spaces $H^1(B_1^+,t^{1-2s})$ and $H^1(B_1^+,t^{2s-1})$ respectively. Then, by the continuity of the trace operator Tr_1 from $H^1(B_1^+,t^{1-2s})$ to $L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})$ and from $H^1(B_1^+,t^{2s-1})$ to $L^2(\mathbb{S}^+,\theta_{N+1}^{2s-1})$, we have that $\{\operatorname{Tr}_1(\nabla_x V^{\lambda_{n_k}})\}$ is bounded in $\left(L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})\right)^N$ and $\left\{\operatorname{Tr}_1\left(t^{1-2s}\frac{\partial V^{\lambda_{n_k}}}{\partial t}\right)\right\}$ is bounded in $L^2(\mathbb{S}^+,\theta_{N+1}^{2s-1})$. Therefore

$$\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}|\nabla V^{\lambda_{n_k}}|^2\,\mathrm{d}S=\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}|\nabla_x V^{\lambda_{n_k}}|^2\,\mathrm{d}S+\int_{\mathbb{S}^+}\theta_{N+1}^{2s-1}\left|\theta_{N+1}^{1-2s}\frac{\partial V^{\lambda_{n_k}}}{\partial t}\right|^2\mathrm{d}S$$

is bounded uniformly with respect to k. Taking into account (3.15), it follows that there exists $f\in L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})$ such that, up to a further subsequence,

$$\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot z\rightharpoonup f\quad\text{ weakly in } L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})\text{ as } k\to+\infty. \tag{5.17}$$

Thus by (5.9) and after proving (5.12) when $\phi\in H^1(B_1^+,t^{1-2s})$ with the same argument (*i.e.* combining (2.3) with (5.11)), passing to the limit as $k\to+\infty$ in (5.16) we obtain that

$$\int_{B_1^+}t^{1-2s}\nabla V\cdot\nabla\phi\,\mathrm{d}z=\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}f\phi\,\mathrm{d}S \tag{5.18}$$

for any $\phi\in H^1(B_1^+,t^{1-2s})$. Furthermore, by (5.17), combined with (5.6) and compactness of the trace operator in (2.1), we have

$$\lim_{k\to+\infty}\int_{\mathbb{S}^+}t^{1-2s}\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot z\;V^{\lambda_{n_k}}\,\mathrm{d}S=\int_{\mathbb{S}^+}t^{1-2s}fV\,\mathrm{d}S. \tag{5.19}$$

Hence, testing (5.16) with $V^{\lambda_{n_k}}$ itself, taking into account (5.19), using (5.12) with $\phi=V^{\lambda_{n_k}}$, and passing to the limit as $k\to+\infty$, we deduce that

$$\lim_{k\to+\infty}\int_{B_1^+}t^{1-2s}\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot\nabla V^{\lambda_{n_k}}\,\mathrm{d}z=\int_{\mathbb{S}^+}t^{1-2s}fV\,\mathrm{d}S,$$

which, by (5.18) tested with V, implies that

$$\lim_{k\to+\infty}\int_{B_1^+}t^{1-2s}A(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot\nabla V^{\lambda_{n_k}}\,\mathrm{d}z=\int_{B_1^+}t^{1-2s}|\nabla V|^2\mathrm{d}z. \tag{5.20}$$

Writing the left hand side in (5.20) as in (5.8), by (3.15) and Proposition 5.1 we infer that

$$\lim_{k\to+\infty}\int_{B_1^+} t^{1-2s}|\nabla V^{\lambda_{n_k}}|^2\,\mathrm{d}z = \int_{B_1^+} t^{1-2s}|\nabla V|^2\mathrm{d}z.$$

This convergence, together with (5.6), allows us to conclude that $\nabla V^{\lambda_{n_k}} \to \nabla V$ in $L^2(B_1^+, t^{1-2s})$. In conclusion, combining this with the compactness of the trace operator given in (2.1), (5.15) easily follows from Remark 2.6.

For any $r \in (0,1]$ and $k \in \mathbb{N}$ we define

$$H_k(r) := \frac{1}{r^{N+1-2s}}\int_{S_r^+} t^{1-2s}\mu(\lambda_{n_k}\cdot)|V^{\lambda_{n_k}}|^2\,\mathrm{d}S,$$
$$D_k(r) := \frac{1}{r^{N-2s}}\left(\int_{B_r^+} t^{1-2s}\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot\nabla V^{\lambda_{n_k}}\,\mathrm{d}z - k_{s,N}\lambda_{n_k}^{2s}\int_{B_r'}\widetilde{h}(\lambda_{n_k}\cdot)|\operatorname{Tr}(V^{\lambda_{n_k}})|^2\,\mathrm{d}y\right),$$

and

$$H_V(r) := \frac{1}{r^{N+1-2s}}\int_{S_r^+} t^{1-2s}V^2\,\mathrm{d}S, \quad D_V(r) := \frac{1}{r^{N-2s}}\int_{B_r^+} t^{1-2s}|\nabla V|^2\,\mathrm{d}z.$$

By Proposition 4.2 in the case $\widetilde{h}=0$, $\widetilde{A}=\mathrm{Id}_{N+1}$ and $\mu=1$, it is clear that $H_V(r)>0$ for any $r\in(0,1]$. Thus the frequency function

$$\mathcal{N}_V(r) := \frac{D_V(r)}{H_V(r)} \quad r\in(0,1]$$

is well defined. Furthermore by (4.21), (5.15), a change of variables, and a combination of (2.3) and (5.11), we have

$$\gamma = \lim_{k\to+\infty}\mathcal{N}(\lambda_{n_k}r) = \lim_{k\to+\infty}\frac{D_k(r)}{H_k(r)} = \mathcal{N}_V(r) \quad \text{for any } r\in(0,1] \tag{5.21}$$

and hence $\mathcal{N}_V'(r)=0$ for a.e. $r\in(0,1]$. Arguing as in Proposition 4.6 in the case $\widetilde{h}=0$, $\widetilde{A}=\mathrm{Id}_{N+1}$ and $\mu=1$, we can prove that

$$\mathcal{N}_V'(r) = 2r\frac{\left(\int_{S_r^+} t^{1-2s}V^2\,\mathrm{d}S\right)\left(\int_{S_r^+} t^{1-2s}|\nabla V\cdot\nu|^2\,\mathrm{d}S\right) - \left(\int_{S_r^+} t^{1-2s}V(\nabla V\cdot\nu)\,\mathrm{d}S\right)^2}{\left(\int_{S_r^+} t^{1-2s}V^2\,\mathrm{d}S\right)^2}.$$

Therefore we conclude that

$$\left(\int_{S_r^+} t^{1-2s}V^2\,\mathrm{d}S\right)\left(\int_{S_r^+} t^{1-2s}|\nabla V\cdot\nu|^2\,\mathrm{d}S\right) = \left(\int_{S_r^+} t^{1-2s}V\,(\nabla V\cdot\nu)\,\mathrm{d}S\right)^2 \quad \text{a.e. } r\in(0,1)$$

where $\nu=\frac{z}{|z|}$, *i.e.* equality holds in the Cauchy-Schwartz inequality for the vectors V and $\nabla V\cdot\nu$ in $L^2(S_r^+, t^{1-2s})$ for a.e. $r\in(0,1)$. It follows that in polar coordinates

$$\frac{\partial V}{\partial r}(r\theta) = \rho(r)V(r\theta) \quad \text{for a.e. } r\in(0,1) \text{ and for any } \theta\in\mathbb{S}^+, \tag{5.22}$$

for some function $r \mapsto \rho(r)$. By (5.22) we have

$$\int_{S_r^+} t^{1-2s} V(\nabla V \cdot \nu)\, \mathrm{d}S = \rho(r) \int_{S_r^+} t^{1-2s} V^2 \, \mathrm{d}S. \tag{5.23}$$

In the case $\widetilde{h} = 0$, $A = \mathrm{Id}_{N+1}$ and $\mu = 1$, (4.4) boils down to $H_V' = \frac{2}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} V \frac{\partial V}{\partial \nu}\, \mathrm{d}S$, since the perturbative term involves $\nabla \mu$, which now trivially equals 0. From this and (5.23) we deduce that $\rho(r) = \frac{H_V'(r)}{2H_V(r)}$. At this point, we exploit (4.6) which, in the case $\widetilde{h} = 0$, $A = \mathrm{Id}_{N+1}$ and $\mu = 1$, becomes $H_V'(r) = \frac{2}{r} D_V(r)$ and thus implies

$$\rho(r) = \frac{1}{r} \mathcal{N}_V(r) = \frac{\gamma}{r},$$

where we used also (5.21). Then an integration over $(r, 1)$ of (5.22) for any fixed $\theta \in \mathbb{S}^+$ yields

$$V(r\theta) = r^\gamma V(\theta) = r^\gamma \Psi(\theta) \quad \text{ for any } (r,\theta) \in (0,1] \times \mathbb{S}^+, \tag{5.24}$$

where $\Psi := V|_{\mathbb{S}^+}$. In view of [12], Lemma 2.1, (5.13) becomes

$$\gamma(N - 2s + \gamma) r^{-1-2s+\gamma} \theta_{N+1}^{1-2s} \Psi(\theta) + r^{-1-2s+\gamma} \operatorname{div}_{\mathbb{S}^+}(\theta_{N+1}^{1-2s} \nabla_{\mathbb{S}^+} \Psi(\theta)) = 0$$

for any $(r,\theta) \in (0,1] \times \mathbb{S}^+$, together with the boundary condition $\lim_{\theta_{N+1} \to 0^+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}} \Psi \cdot \nu = 0$ on $\mathbb{S}'$. Since V^λ is odd with respect to y_N for any $\lambda \in (0, r_0]$ by (5.1) and (3.26), then also V is odd with respect to y_N, so that $\Psi \in H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta_{N+1}^{1-2s})$. By (5.24) and (5.14) we have $\|\Psi\|_{L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})} = 1$, so that $\Psi \not\equiv 0$ is an eigenfunction of problem (1.19) associated to the eigenvalue $\gamma(\gamma + N - 2s)$. From (1.22) it follows that there exists $m_0 \in \mathbb{N} \setminus \{0\}$ (which is odd in the case $N = 1$) such that $\gamma(\gamma + N - 2s) = m_0(m_0 + N - 2s)$. Therefore, since $\gamma \geq 0$ by Proposition 4.8, we conclude that $\gamma = m_0$ thus proving (5.4). Moreover (5.5) follows from (5.15) and (5.24). □

In Proposition 4.10, we have shown that there exists the limit $\lim_{\lambda \to 0^+} \lambda^{-2\gamma} H(\lambda)$ and it is non-negative. Now we prove that $\lim_{\lambda \to 0^+} \lambda^{-2\gamma} H(\lambda) > 0$.

To this end we define, for every $\lambda \in (0, r_0]$, $m \in \mathbb{N} \setminus \{0\}$, $k \in \{1, \dots, M_m\}$,

$$\varphi_{m,k}(\lambda) := \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} W(\lambda\theta) Y_{m,k}(\theta)\, \mathrm{d}S, \tag{5.25}$$

i.e $\{\varphi_{m,k}(\lambda)\}_{m,k}$ are the Fourier coefficients of $W(\lambda \cdot)$ with respect to the orthonormal basis $\{Y_{m,k}\}_{m,k}$ introduced in (1.25). For every $\lambda \in (0, r_0]$, $m \in \mathbb{N} \setminus \{0\}$, $k \in \{1, \dots, M_m\}$, we also define

$$\begin{aligned} \Upsilon_{m,k}(\lambda) := &- \int_{B_\lambda^+} t^{1-2s} (\widetilde{A} - \mathrm{Id}_{N+1}) \nabla W \cdot \frac{1}{|z|} \nabla_{\mathbb{S}} Y_{m,k}\big(\tfrac{z}{|z|}\big)\, \mathrm{d}z \\ &+ \int_{S_\lambda^+} t^{1-2s} (\widetilde{A} - \mathrm{Id}_{N+1}) \nabla W \cdot \frac{z}{|z|} Y_{m,k}\big(\tfrac{z}{|z|}\big)\, \mathrm{d}S \\ &+ \kappa_{N,s} \int_{B_\lambda'} \widetilde{h}(y) \operatorname{Tr}(W) \operatorname{Tr}\Big(Y_{m,k}\big(\tfrac{y}{|y|}\big)\Big)\, \mathrm{d}y, \end{aligned} \tag{5.26}$$

where Id_{N+1} is the identity $(N+1) \times (N+1)$ matrix.

Proposition 5.3. *Let γ be as in (4.21) and let $m_0 \in \mathbb{N} \setminus \{0\}$ be such that $\gamma = m_0$ according to Proposition 5.2. For every $k \in \{1, \dots, M_{m_0}\}$ and $r \in (0, r_0]$*

$$\varphi_{m_0,k}(\lambda) = \lambda^{m_0} \left(\frac{\varphi_{m_0,k}(r)}{r^{m_0}} + \frac{m_0 r^{-2m_0-N+2s}}{2m_0+N-2s} \int_0^r \rho^{m_0-1} \Upsilon_{m_0,k}(\rho)\, \mathrm{d}\rho \right) + \lambda^{m_0} \frac{m_0+N-2s}{2m_0+N-2s} \int_\lambda^r \rho^{-m_0-N-1+2s} \Upsilon_{m_0,k}(\rho)\, \mathrm{d}\rho + O\left(\lambda^{m_0+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right) \tag{5.27}$$

as $\lambda \to 0^+$.

Proof. Let $k \in \{1, \dots, M_{m_0}\}$ and $\phi \in \mathcal{D}(0, r_0)$. Testing (3.30) with $|z|^{-N-1+2s}\phi(|z|)Y_{m_0,k}\big(\frac{z}{|z|}\big)$, since $Y_{m_0,k}$ solves (1.21), we obtain that $\varphi_{m_0,k}$ satisfies

$$-\varphi''_{m_0,k} - \frac{N+1-2s}{\lambda}\varphi'_{m_0,k} + \frac{\mu_{m_0}}{\lambda^2}\varphi_{m_0,k} = \zeta_{m_0,k} \tag{5.28}$$

in the sense of distributions in $(0, r_0)$, where

$$\begin{aligned} {}_{\mathcal{D}'(0,r_0)}\langle \zeta_{m_0,k}, \phi \rangle_{\mathcal{D}(0,r_0)} := \kappa_{N,s} &\int_0^{r_0} \frac{\phi(\lambda)}{\lambda^{2-2s}} \left(\int_{\mathbb{S}'} \widetilde{h}(\lambda\theta') \operatorname{Tr}(W(\lambda \cdot))(\theta') Y_{m_0,k}(\theta', 0)\, \mathrm{d}S' \right) \mathrm{d}\lambda \\ &- \int_0^{r_0} \left(\int_{S_\lambda^+} t^{1-2s}(A - \mathrm{Id}_{N+1})\nabla W \cdot \nabla\big(|z|^{-N-1+2s}\phi(|z|)Y_{m_0,k}\big(\tfrac{z}{|z|}\big)\big)\, \mathrm{d}S \right) \mathrm{d}\lambda. \end{aligned}$$

Furthermore, it is easy to verify that $\Upsilon_{m_0,k} \in L^1(0, r_0)$ and

$$\Upsilon'_{m_0,k}(\lambda) = \lambda^{N+1-2s}\zeta_{m_0,k}(\lambda)$$

in the sense of distributions in $(0, r_0)$. Then equation (5.28) can be rewritten as follows

$$-\left(\lambda^{2m_0+N+1-2s}(\lambda^{-m_0}\varphi_{m_0,k}(\lambda))'\right)' = \lambda^{m_0}\Upsilon'_{m_0,k}(\lambda) \tag{5.29}$$

in the sense of distributions in $(0, r_0)$. Integrating (5.29) over (λ, r) for any $r \in (0, r_0]$, we obtain that there exists a constant $c_{m_0,k}(r) \in \mathbb{R}$ which depends only on m_0, k, r, such that

$$(\lambda^{-m_0}\varphi_{m_0,k}(\lambda))' = -\lambda^{-m_0-N-1+2s}\Upsilon_{m_0,k}(\lambda) - m_0\lambda^{-2m_0-N-1+2s}\left(c_{m_0,k}(r) + \int_\lambda^r \rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\, \mathrm{d}\rho \right)$$

in the sense of distributions in $(0, r_0)$. In particular we deduce that $\varphi_{m_0,k} \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and a further integration over (λ, r) gives

$$\begin{aligned} \varphi_{m_0,k}(\lambda) = &\lambda^{m_0}\left(\frac{\varphi_{m_0,k}(r)}{r^{m_0}} - \frac{m_0 c_{m_0,k}(r)}{(2m_0+N-2s)r^{2m_0+N-2s}} \right) \\ &+ \lambda^{m_0}\frac{m_0+N-2s}{2m_0+N-2s}\int_\lambda^r \rho^{-m_0-N-1+2s}\Upsilon_{m_0,k}(\rho)\, \mathrm{d}\rho \\ &+ \frac{m_0\lambda^{-m_0-N+2s}}{2m_0+N-2s}\left(c_{m_0,k}(r) + \int_\lambda^r \rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\, \mathrm{d}\rho \right) \end{aligned} \tag{5.30}$$

for every $\lambda, r \in (0, r_0]$. Now we claim that

$$\int_0^{r_0} \rho^{-m_0-N-1+2s} |\Upsilon_{m_0,k}(\rho)| \,\mathrm{d}\rho < +\infty. \tag{5.31}$$

By the Hölder inequality, a change of variables, (3.15), (5.1), Proposition 5.1, and (4.23) we have

$$\begin{aligned}
&\lambda^{-m_0-N-1+2s} \left| \int_{B_\lambda^+} t^{1-2s} (\widetilde{A} - \mathrm{Id}_{N+1}) \nabla W \cdot \frac{1}{|z|} \nabla_{\mathbb{S}} Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \,\mathrm{d}z \right| \\
&\leq \lambda^{-m_0-N-1+2s} \left(\int_{B_\lambda^+} t^{1-2s} |(\widetilde{A} - \mathrm{Id}_{N+1}) \nabla W|^2 \,\mathrm{d}z \right)^{\frac{1}{2}} \left(\int_{B_\lambda^+} \frac{t^{1-2s}}{|z|^2} \left| \nabla_{\mathbb{S}} Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \right|^2 \,\mathrm{d}z \right)^{\frac{1}{2}} \\
&\leq \lambda^{-m_0-1} O(\lambda) \sqrt{H(\lambda)} \left(\int_{B_1^+} t^{1-2s} |\nabla V^\lambda|^2 \,\mathrm{d}z \right)^{\frac{1}{2}} \left(\int_{B_1^+} \frac{t^{1-2s}}{|z|^2} \left| \nabla_{\mathbb{S}} Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \right|^2 \,\mathrm{d}z \right)^{\frac{1}{2}} \\
&\leq \mathrm{const}\, \lambda^{-m_0} \sqrt{H(\lambda)} \leq \mathrm{const},
\end{aligned} \tag{5.32}$$

where we used the fact that

$$\begin{aligned}
\int_{B_1^+} \frac{t^{1-2s}}{|z|^2} \left| \nabla_{\mathbb{S}} Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \right|^2 \,\mathrm{d}z &= \int_0^1 \rho^{N-1-2s} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} Y_{m_0,k}(\theta)|^2 \,\mathrm{d}S \right) \mathrm{d}\rho \\
&= \frac{m_0^2 + m_0(N-2s)}{N-2s}.
\end{aligned}$$

Dealing with the second term of (5.26), from an integration by parts, the Hölder inequality, (3.15) (5.1), Proposition 5.1, and (4.23) it follows that, for every $r \in (0, r_0]$,

$$\begin{aligned}
&\int_0^r \lambda^{-m_0-N-1+2s} \left| \int_{S_\lambda^+} t^{1-2s} (\widetilde{A} - \mathrm{Id}_{N+1}) \nabla W \cdot \frac{z}{|z|} Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \,\mathrm{d}S \right| \mathrm{d}\lambda \\
&\leq \mathrm{const} \int_0^r \lambda^{-m_0-N+2s} \left(\int_{S_\lambda^+} t^{1-2s} |\nabla W| \left| Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \right| \,\mathrm{d}S \right) \mathrm{d}\lambda \\
&= \mathrm{const} \Bigg(r^{-m_0-N+2s} \int_{B_r^+} t^{1-2s} |\nabla W| \left| Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \right| \,\mathrm{d}z \\
&\qquad + (m_0 + N - 2s) \int_0^r \lambda^{-m_0-N-1+2s} \bigg(\int_{B_\lambda^+} t^{1-2s} |\nabla W| \left| Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \right| \,\mathrm{d}z \bigg) \mathrm{d}\lambda \Bigg) \\
&\leq \mathrm{const} \left(r^{-m_0+1} \sqrt{H(r)} + \int_0^r \lambda^{-m_0} \sqrt{H}(\lambda) \,\mathrm{d}\lambda \right) \leq \mathrm{const}\, r,
\end{aligned} \tag{5.33}$$

taking into account that

$$\int_{B_\lambda^+} t^{1-2s} \left| Y_{m_0,k}\big(\tfrac{z}{|z|}\big) \right|^2 \,\mathrm{d}z = \frac{\lambda^{N+2-2s}}{N+2-2s}.$$

By the Hölder inequality the third term in (5.26) can be estimated as

$$
\begin{aligned}
&\lambda^{-m_0-N-1+2s}\left|\int_{B'_\lambda}\widetilde{h}(y)\,\mathrm{Tr}(W)\,\mathrm{Tr}\left(Y_{m_0,k}\left(\tfrac{y}{|y|}\right)\right)\,\mathrm{d}y\right| \qquad (5.34)\\
&\leq \lambda^{-m_0-N-1+2s}\left(\int_{B'_\lambda}|\widetilde{h}(y)||\,\mathrm{Tr}(W)|^2\,\mathrm{d}y\right)^{\frac12}\left(\int_{B'_\lambda}|\tilde{h}(y)|\left|\mathrm{Tr}\left(Y_{m_0,k}\left(\tfrac{y}{|y|}\right)\right)\right|^2\mathrm{d}y\right)^{\frac12}\\
&\leq \lambda^{-m_0-N-1+2s}\eta_{|\tilde{h}|}(\lambda)\left(\int_{B^+_\lambda}t^{1-2s}|\nabla W|^2\,\mathrm{d}z+\frac{N-2s}{2\lambda}\int_{S^+_\lambda}t^{1-2s}W^2\,\mathrm{d}S\right)^{\frac12}\\
&\qquad\times\left(\int_{B^+_\lambda}t^{1-2s}\left|\nabla Y_{m_0,k}\left(\tfrac{z}{|z|}\right)\right|^2\mathrm{d}z+\frac{N-2s}{2\lambda}\int_{S^+_\lambda}t^{1-2s}\left|Y_{m_0,k}\left(\tfrac{z}{|z|}\right)\right|^2\mathrm{d}S\right)^{\frac12}\\
&\leq \lambda^{-m_0-1}\eta_{|\tilde{h}|}(\lambda)\sqrt{H(\lambda)}\left(\int_{B^+_1}t^{1-2s}|\nabla V^\lambda|^2\mathrm{d}z+(N-2s)\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}\mu(\lambda\theta)|V^\lambda|^2\,\mathrm{d}S\right)^{\frac12}\\
&\qquad\times\left(\lambda^2\int_{B^+_1}t^{1-2s}\left|\nabla Y_{m_0,k}\left(\tfrac{z}{|z|}\right)\right|^2\mathrm{d}z+\frac{N-2s}{2}\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}|Y_{m_0,k}(\theta)|^2\,\mathrm{d}S\right)^{\frac12}\\
&\leq \mathrm{const}\,\lambda^{-m_0-1}\eta_{|\tilde{h}|}(\lambda)\sqrt{H(\lambda)}\leq \mathrm{const}\,\lambda^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}},
\end{aligned}
$$

in view of (2.3), (2.4), (3.19), (4.23), (5.1), (5.3) and Proposition 5.1. Collecting estimates (5.32), (5.33) and (5.34) we deduce that, for every $r\in(0,r_0]$,

$$\int_0^r\rho^{-m_0-N-1+2s}|\Upsilon_{m_0,k}(\rho)|\,\mathrm{d}\rho\leq\mathrm{const}\left(r+\int_0^r\rho^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\,\mathrm{d}\rho\right)\leq\mathrm{const}\,r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}, \tag{5.35}$$

thus proving (5.31). Moreover we have

$$\int_0^{r_0}\rho^{m_0-1}|\Upsilon_{m_0,k}(\rho)|\,\mathrm{d}\rho<+\infty, \tag{5.36}$$

as a consequence of (5.31), since in a neighbourhood of 0, $\rho^{m_0-1}\leq\rho^{-m_0-N-1+2s}$.

Now we claim that, for every $r\in(0,r_0]$,

$$c_{m_0,k}(r)+\int_0^r\rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\,\mathrm{d}\rho=0 \tag{5.37}$$

To prove (5.37) we argue by contradiction. If there exists $r\in(0,r_0]$ such that (5.37) does not hold true, then by (5.30), (5.31) and (5.36)

$$\varphi_{m_0,k}(\lambda)\sim\frac{m_0\lambda^{-m_0-N+2s}}{2m_0+N-2s}\left(c_{m_0,k}(r)+\int_0^r\rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\,\mathrm{d}\rho\right)\quad\text{as }\lambda\to0^+.$$

From this, it follows that

$$\int_0^{r_0}\lambda^{N-1-2s}|\varphi_{m_0,k}(\lambda)|^2d\lambda=+\infty, \tag{5.38}$$

since $N-2s+2m_0>0$. On the other hand, from (5.25), the Parseval identity and (2.5) we deduce the following estimate

$$\int_0^{r_0} \lambda^{N-1-2s}|\varphi_{m_0,k}(\lambda)|^2\,\mathrm{d}\lambda \le \int_0^{r_0} \lambda^{N-1-2s}\left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}|W(\lambda\theta)|^2\,\mathrm{d}S\right)\mathrm{d}\lambda$$
$$= \int_0^{r_0} \lambda^{-2}\left(\int_{S_\lambda^+} t^{1-2s}|W|^2\,\mathrm{d}S\right)\mathrm{d}\lambda = \int_{B_{r_0}^+} t^{1-2s}\frac{|W(z)|^2}{|z|^2}\,\mathrm{d}z < +\infty,$$

which contradicts (5.38). Hence (5.37) is proved. From (5.37) and (5.35) it follows that, for every $r\in(0,r_0]$,

$$\lambda^{-m_0-N+2s}\left|c_{m_0,k}(r)+\int_\lambda^r \rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\,\mathrm{d}\rho\right| = \lambda^{-m_0-N+2s}\left|\int_0^\lambda \rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\,\mathrm{d}\rho\right|$$
$$\le \lambda^{-m_0-N+2s}\left(\lambda^{2m_0+N-2s}\int_0^\lambda \rho^{-m_0-N-1+2s}|\Upsilon_{m_0,k}(\rho)|\,\mathrm{d}\rho\right) \le \mathrm{const}\,\lambda^{m_0+\frac{4s^2\varepsilon}{N+2s\varepsilon}}. \tag{5.39}$$

We finally deduce (5.27) combining (5.30), (5.37) and (5.39). $\square$

Proposition 5.4. *Let γ be as in (4.21). Then*

$$\lim_{\lambda\to 0^+}\lambda^{-2\gamma}H(\lambda)>0. \tag{5.40}$$

Proof. By (3.20), the Parseval identity and (5.25) we have

$$H(\lambda)=\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}\mu(\lambda\theta)|W(\lambda\theta)|^2\,\mathrm{d}S=(1+O(\lambda))\sum_{m=1}^{\infty}\sum_{k=1}^{M_m}|\varphi_{m,k}(\lambda)|^2. \tag{5.41}$$

Let $m_0\in\mathbb{N}\setminus\{0\}$ be such that $\gamma=m_0$ according to Proposition 5.2. We argue by contradiction and assume that $0=\lim_{\lambda\to0^+}\lambda^{-2\gamma}H(\lambda)=\lim_{\lambda\to0^+}\lambda^{-2m_0}H(\lambda)$. In view of (5.41) this would imply that

$$\lim_{\lambda\to0^+}\lambda^{-m_0}\varphi_{m_0,k}(\lambda)=0 \quad \text{for every } k\in\{1,\dots,M_{m_0}\}.$$

Therefore, from (5.27) it follows that, for all $k\in\{1,\dots,M_{m_0}\}$ and $r\in(0,r_0]$,

$$\frac{\varphi_{m_0,k}(r)}{r^{m_0}} + \frac{m_0 r^{-2m_0-N+2s}}{2m_0+N-2s}\int_0^r \rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\,\mathrm{d}\rho + \frac{m_0+N-2s}{2m_0+N-2s}\int_0^r \rho^{-m_0-N-1+2s}\Upsilon_{m_0,k}(\rho)\,\mathrm{d}\rho = 0,$$

so that, substituting into (5.27), we obtain that

$$\varphi_{m_0,k}(\lambda)=-\frac{m_0+N-2s}{2m_0+N-2s}\lambda^{m_0}\int_0^\lambda \rho^{-m_0-N-1+2s}\Upsilon_{m_0,k}(\rho)\,\mathrm{d}\rho+O\left(\lambda^{m_0+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)$$

as $\lambda\to0^+$. Hence, from (5.35) we infer that

$$\varphi_{m_0,k}(\lambda)=O\left(\lambda^{m_0+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right) \quad \text{as } \lambda\to0^+ \quad \text{for all } k\in\{1,\dots,M_{m_0}\}. \tag{5.42}$$

Moreover, estimate (4.25) with $\sigma = \frac{2s^2\varepsilon}{N+2s\varepsilon}$ implies that

$$\frac{1}{\sqrt{H(\lambda)}} = O\left(\lambda^{-m_0-\frac{2s^2\varepsilon}{N+2s\varepsilon}}\right) \quad \text{as } \lambda \to 0^+. \tag{5.43}$$

Since

$$\varphi_{m_0,k}(\lambda) = \sqrt{H(\lambda)} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^\lambda(\theta) Y_{m_0,k}(\theta)\, \mathrm{d}S \quad \text{for all } k \in \{1,\dots,M_{m_0}\}$$

by (5.25) and (5.1), from (5.42) and (5.43) we deduce that

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^\lambda(\theta)\Psi(\theta)\, \mathrm{d}S = O\left(\lambda^{\frac{2s^2\varepsilon}{N+2s\varepsilon}}\right) \quad \text{as } \lambda \to 0^+, \tag{5.44}$$

for every $\Psi \in \mathrm{Span}\{Y_{m_0,k} : k \in \{1,\dots M_{m_0}\}\}$. By (1.24), (1.25), (2.1) and Proposition 5.2, for any sequence $\lambda_n \to 0^+$, there exist a subsequence $\lambda_{n_h} \to 0^+$ and $\Psi \in \mathrm{Span}\{Y_{m_0,k} : k \in \{1,\dots M_{m_0}\}\}$ such that $\|\Psi\|_{L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})} = 1$ and

$$\lim_{h\to+\infty} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^{\lambda_{n_h}}(\theta)\Psi(\theta)\, \mathrm{d}S = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\Psi|^2\, \mathrm{d}S = 1,$$

thus contradicting (5.44). □

Theorem 5.5. *Let W be a non-trivial weak solution to (3.29). Let γ be as in (4.21) and $m_0 \in \mathbb{N}\setminus\{0\}$ be such that $\gamma = m_0$, according to Proposition 5.2. Let $\{Y_{m_0,k}\}_{k\in\{1,\dots,M_{m_0}\}}$ be as in (1.25), with V_{m_0} and M_{m_0} defined as in (1.23) and (1.24) respectively. Then*

$$\lambda^{-m_0} W(\lambda z) \to |z|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k Y_{m_0,k}\left(\frac{z}{|z|}\right) \quad \textit{as } \lambda \to 0^+ \quad \textit{strongly in } H^1(B_1^+, t^{1-2s}),$$

where $(\beta_1,\dots,\beta_{M_{m_0}}) \neq (0,\dots,0)$ and, for every $k \in \{1,\dots,M_{m_0}\}$,

$$\beta_k = \frac{\varphi_{m_0,k}(r)}{r^{m_0}} + \frac{m_0 r^{-2m_0-N+2s}}{(2m_0+N-2s)} \int_0^r \rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\, \mathrm{d}\rho + \frac{m_0+N-2s}{2m_0+N-2s} \int_0^r \rho^{-m_0-N-1+2s}\Upsilon_{m_0,k}(\rho)\, \mathrm{d}\rho, \tag{5.45}$$

for all $r \in (0, r_0]$, where $\varphi_{m_0,k}$ is defined in (5.25) and $\Upsilon_{m_0,k}$ in (5.26).

Proof. From Proposition 5.2, (1.25), and (5.40) it follows that, for any sequence $\{\lambda_n\}$ such that $\lambda_n \to 0^+$ as $n \to \infty$, there exist a subsequence $\{\lambda_{n_h}\}$ and real numbers $\beta_1,\dots,\beta_{M_{m_0}}$ such that $(\beta_1,\dots,\beta_{M_{m_0}}) \neq (0,\dots,0)$ and

$$\lambda_{n_h}^{-m_0} W(\lambda_{n_h} z) \to |z|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k Y_{m_0,k}\left(\frac{z}{|z|}\right) \quad \text{as } h \to +\infty \quad \text{strongly in } H^1(B_1^+, t^{1-2s}). \tag{5.46}$$

We claim that the numbers $\beta_1, \dots \beta_{M_{m_0}}$ depend neither on the sequence $\{\lambda_n\}$ nor on its subsequence $\{\lambda_{n_h}\}$. Letting $\varphi_{m_0,k}$ be as (5.25), for every $k \in \{1, \dots, M_{m_0}\}$

$$\lim_{h\to+\infty} \lambda_{n_h}^{-m_0} \varphi_{m_0,k}(\lambda_{n_h}) = \lim_{h\to+\infty} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \lambda_{n_h}^{-m_0} W(\lambda_{n_h}\theta) Y_{m_0,k}(\theta)\, \mathrm{d}S = \beta_k, \tag{5.47}$$

thanks to (5.46) and the compactness of the trace operator in (2.1). Combining (5.47) and (5.27) we obtain that, for every $r \in (0, r_0]$, $\beta_k = \lim_{h\to+\infty} \lambda_{n_h}^{-m_0} \varphi_{m_0,k}(\lambda_{n_h})$ is equal to the right hand side in (5.45), thus proving the claim. By Urysohn's subsequence principle we conclude that the convergence in (5.46) holds as $\lambda \to 0^+$, hence the proof is complete. □

6. Proofs of the main results

The proof of Theorem 1.3 is obtained as a consequence of the following result.

Theorem 6.1. *Let $N > 2s$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain such that $0 \in \partial\Omega$ and (1.10)–(1.12) are satisfied with $x_0 = 0$ for some function g and $R > 0$. Let U be a non-trivial solution to (1.17) in the sense of (1.18), with h satisfying (1.7), and let*

$$\widehat{U}(z) = \begin{cases} U(z), & \text{if } z \in \mathcal{C}_\Omega \cap F(B_{r_0}^+), \\ 0, & \text{if } z \in F(B_{r_0}^+) \setminus \mathcal{C}_\Omega, \end{cases} \tag{6.1}$$

with F and r_0 being as in Proposition 3.1. Then there exist $m_0 \in \mathbb{N} \setminus \{0\}$ (which is odd in the case $N = 1$) such that

$$\lambda^{-m_0} \widehat{U}(\lambda z) \to |z|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k \widehat{Y}_{m_0,k}\left(\frac{z}{|z|}\right) \quad \text{as } \lambda \to 0^+ \quad \text{strongly in } H^1(B_1^+, t^{1-2s}), \tag{6.2}$$

where M_{m_0} is as in (1.24),

$$\widehat{Y}_{m_0,k}(\theta', \theta_N, \theta_{N+1}) = \begin{cases} Y_{m_0,k}(\theta', \theta_N, \theta_{N+1}), & \text{if } \theta_N < 0, \\ 0, & \text{if } \theta_N \geq 0, \end{cases} \tag{6.3}$$

with $\{Y_{m_0,k}\}_{k \in \{1,\dots,M_{m_0}\}}$ being as in (1.25), and the coefficients β_k satisfy (5.45).

Proof. If U is a non-trivial solution of (1.17), then the function W defined in (3.5) and (3.26) belongs to $H^1(B_{r_0}^+, t^{1-2s})$ and is a non-trivial weak solution to (3.29). Letting

$$\widehat{W}(z) = \begin{cases} W(z), & \text{if } z \in \mathcal{Q}_{r_0}, \\ 0, & \text{if } z \in B_{r_0}^+ \setminus \mathcal{Q}_{r_0}, \end{cases}$$

where $\mathcal{Q}_{r_0}$ is defined in (3.4), by Remark 3.4 we have $\widehat{W} \in H^1(B_{r_0}^+, t^{1-2s})$. Moreover Theorem 5.5 implies that

$$\lambda^{-m_0} \widehat{W}(\lambda z) \to \widehat{\Phi}(z) \quad \text{strongly in } H^1(B_1^+, t^{1-2s}) \quad \text{as } \lambda \to 0^+,$$

where

$$\widehat{\Phi}(z) = |z|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k \widehat{Y}_{m_0,k}\left(\frac{z}{|z|}\right)$$

with β_k as in (5.45). Hence, by homogeneity,

$$\lambda^{-m_0}\widehat{W}(\lambda z) \to \widehat{\Phi}(z) \quad \text{strongly in } H^1(B_r^+, t^{1-2s}) \quad \text{as } \lambda \to 0^+ \quad \text{for all } r > 1. \tag{6.4}$$

We note that

$$\lambda^{-m_0}\widehat{U}(\lambda z) = \lambda^{-m_0}\widehat{W}(\lambda G_\lambda(z)) \quad \text{and} \quad \nabla\left(\frac{\widehat{U}(\lambda\cdot)}{\lambda^{m_0}}\right) = \nabla\left(\frac{\widehat{W}(\lambda\cdot)}{\lambda^{m_0}}\right)(G_\lambda(z))J_{G_\lambda}(z) \tag{6.5}$$

where

$$G_\lambda(z) := \frac{1}{\lambda}F^{-1}(\lambda z) \quad \text{for any } \lambda \in (0,1] \text{ and } z \in \frac{1}{\lambda}F(B_{r_0^+}).$$

From Proposition 3.1 we deduce that

$$G_\lambda(z) = z + O(\lambda) \quad \text{and} \quad J_{G_\lambda}(z) = \mathrm{Id}_{N+1} + O(\lambda) \quad \text{as } \lambda \to 0^+$$

uniformly respect to $z \in B_1^+$. It follows that, if $f_\lambda \to f$ in $L^2(B_r^+, t^{1-2s})$ as $\lambda \to 0^+$ for some $r > 1$, then $f_\lambda \circ G_\lambda \to f$ in $L^2(B_1^+, t^{1-2s})$ as $\lambda \to 0^+$. Then we conclude in view of (6.4) and (6.5). □

Proof of Theorem 1.3. It follows directly from Theorem 6.1 up to a translation. □

Passing to traces in (6.2) we obtain the following blow-up result for solutions to (1.1).

Theorem 6.2. *Let $N > 2s$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain such that $0 \in \partial\Omega$ and (1.10)–(1.12) are satisfied with $x_0 = 0$ for some function g and $R > 0$. Let $u \in \mathbb{H}^s(\Omega)$ be a non-trivial solution of (1.1) in the sense of (1.8), with h satisfying (1.7), and let $\widehat{u}(x) = \iota(u)$ with ι defined in (1.3). Then there exists $m_0 \in \mathbb{N} \setminus \{0\}$ (which is odd in the case $N = 1$) such that*

$$\lambda^{-m_0}\widehat{u}(\lambda x) \to |x|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k \widehat{Y}_{m_0,k}\left(\frac{x}{|x|}, 0\right) \quad \text{as } \lambda \to 0^+ \quad \text{strongly in } H^s(B_1'),$$

where M_{m_0} is as in (1.24), $\{\widehat{Y}_{m_0,k}\}_{k\in\{1,\dots,M_{m_0}\}}$ are defined in (6.3) and the coefficients β_k satisfy (5.45).

Proof. As observed in [8] and recalled at page 6, if $u \in \mathbb{H}^s(\Omega)$ is a non-trivial solution of (1.1), then its extension $\mathcal{H}(u) = U$ is non-trivial solution to (1.17). Hence the corresponding function $\widehat{U}$ defined in (6.1) satisfies (6.2) by Theorem 6.1. Since $\widehat{u} = \mathrm{Tr}(\widehat{U})$, the conclusion follows from Proposition 2.2. □

Proof of Theorem 1.2. It follows directly from Theorem 6.2 up to a translation. □

Appendix A. Neumann eigenvalues on the half-sphere under a symmetry condition

In order to determine the eigenvalues of (1.19), we first need the following preliminary lemma.

Lemma A.1. *Let $m, N \in \mathbb{N} \setminus \{0\}$ and let $u \in C^m(\mathbb{R}^N) \setminus \{0\}$ be a positively homogeneous function of degree m, i.e*

$$u(\lambda x) = \lambda^m u(x) \quad \text{for every } \lambda > 0 \text{ and } x \in \mathbb{R}^N. \tag{A.1}$$

Then u is a homogeneous polynomial of degree m.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ be a multindex, $|\alpha| := \sum_{i=1}^N \alpha_i$, and $x^\alpha = x_1^{\alpha_1} \dots x_N^{\alpha_N}$ for any vector $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. By Taylor's Theorem with Lagrange remainder centered at 0, for any $x \in \mathbb{R}^N$ there exists $t \in [0,1]$ such that

$$u(x) = \sum_{|\alpha|<m} c_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(0) x^\alpha + \sum_{|\alpha|=m} c_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(tx) x^\alpha,$$

where $c_\alpha > 0$ are positive constants depending on α and $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$ stands for $\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$. By (A.1), one can easily prove that $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$ is a positively homogeneous function of degree $m - |\alpha|$ for all α with $|\alpha| \le m$. Thus, combining this fact with the continuity of $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$, it is clear that $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}(0) = 0$ for every $\alpha \in \mathbb{N}^N$ with $|\alpha| < m$. On the other hand, for every $\alpha \in \mathbb{N}^N$ with $|\alpha| = m$, $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$ is constant and exactly equal to $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}(0)$, being a homogeneous function of degree 0. It follows that

$$u(x) = \sum_{|\alpha|=m} c_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(0) x^\alpha \quad \text{for every } x \in \mathbb{R}^N,$$

hence proving the claim. □

Proposition A.2. *All the eigenvalues of problem (1.19) are characterized by formula (1.22).*

Proof. We start by proving that if μ is an eigenvalue of (1.19), then $\mu = m^2 + m(N - 2s)$ for some $m \in \mathbb{N} \setminus \{0\}$. If μ is an eigenvalue, then there exists a non-trivial solution Y of (1.19). A direct computation shows that Y is a weak solution to (1.19) if and only if the function

$$U(z) := |z|^\gamma Y\left(\frac{z}{|z|}\right), \quad z \in \mathbb{R}^{N+1}_+,$$

with

$$\gamma := -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu}, \tag{A.2}$$

belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s})$, is odd with respect to y_N and weakly solves

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \lim_{t \to 0^+} t^{1-2s} \frac{\partial U}{\partial \nu} = 0, & \text{on } \mathbb{R}^N. \end{cases} \tag{A.3}$$

Hence, if μ is an eigenvalue of (1.19), there exists a solution U of (A.3) which is odd with respect to y_N and positively homogeneous of degree γ. The regularity result in [25], Theorem 1.1 ensures that $U \in C^\infty(\overline{B_1^+})$. Then there exists $m \in \mathbb{N} \setminus \{0\}$ such that $\gamma = m$ and so $\mu = m^2 + m(N-2s)$ thanks to (A.2). We notice that the case $m = 0$ is excluded since in that case $\mu = 0$ and 0 is not an eigenvalue. Indeed, if by contradiction 0 is an eigenvalue, letting Y be an eigenfunction of (1.19) with associated eigenvalue 0 and choosing in (1.21) $\Psi = Y$, we would have Y constant and $Y \not\equiv 0$, hence $Y \notin H^1_{\rm odd}(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ which is a contradiction (see (1.20)).

Viceversa, in order to prove that the numbers given in (1.22) are eigenvalues of (1.19), we need to show that, for any fixed $m \in \mathbb{N} \setminus \{0\}$, there actually exist an eigenfunction associated to $m^2 + m(N-2s)$ if $N > 1$ and an eigenfunction associated to $(2m-1)^2 + (2m-1)(N-2s)$ if $N = 1$. Equivalently, for any fixed $m \in \mathbb{N} \setminus \{0\}$ we have to find a non-trivial solution to (A.3) which is odd with respect to y_N and positively homogeneous with degree m if $N > 1$ and $2m-1$ if $N = 1$. To this end, we observe that equation $\operatorname{div}(t^{1-2s}\nabla U) = 0$ can be rewritten as

$$\Delta U + \frac{1-2s}{t} U_t = 0. \tag{A.4}$$

We first consider the case $N = 1$. If $n = 2m-1$ with $m \in \mathbb{N} \setminus \{0\}$, we consider the following homogeneous polynomial of degree $2m-1$, odd with respect to y_1,

$$U_{1,m}(y_1, t) := \sum_{k=0}^{m-1} a_k y_1^{2k+1} t^{2m-2k-2}, \tag{A.5}$$

with $a_0, \dots, a_{m-1} \in \mathbb{R}$. A direct computation shows that $U_{1,m}$ is a solution of (A.3), and equivalently of (A.4), if and only if

$$a_k = \frac{-2[(m-k)^2 - s(m-k)]}{k(2k+1)} a_{k-1} \quad \text{for all } k \in \{1, \dots, m-1\}.$$

Thus, for example choosing $a_0 := 1$, we have constructed a non-trivial solution to (A.3) which is odd with respect to y_1 and positively homogeneous of degree $2m-1$.

To complete the proof of (1.22) in the case $N = 1$, it remains to show that, if $n = 2m$ with $m \in \mathbb{N} \setminus \{0\}$, then $n^2 + n(N-2s)$ is not an eigenvalue of (1.19). To this aim, we argue by contradiction and assume that $(2m)^2 + 2m(N-2s)$ is an eigenvalue of (1.19) associated to an eigenfunction Ψ. Then the function defined as

$$U(z) = |z|^\gamma \Psi\left(\frac{z}{|z|}\right), \quad z = (y_1, t) \in \mathbb{R}^2_+,$$

with

$$\gamma = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + (2m)^2 + 2m(N-2s)} = 2m$$

is a non-trivial solution to (A.3), odd with respect to y_1. Hence, if we consider the even reflection of U with respect to t, namely the function $\widetilde{U}(y_1, t) := U(y_1, |t|)$, $\widetilde{U}$ is a solution of $\operatorname{div}(|t|^{1-2s}\nabla \widetilde{U}) = 0$ in $\mathbb{R}^2$. Then, by [25], Theorem 1.1 we deduce that $\widetilde{U} \in C^\infty(\mathbb{R}^2)$. Moreover, $\widetilde{U}$ is positively homogeneous of degree $\gamma = 2m$, therefore from Lemma A.1 it follows that $\widetilde{U}$ is a homogeneous polynomial of degree $2m$, namely

$$\widetilde{U}(y_1, t) = \sum_{k=0}^{2m} a_k y_1^{2m-k} t^k$$

where $a_k = 0$ if k is odd since $\widetilde{U}$ is even with respect to t. In this way $\tilde{U}$ turns out to be even also with respect to y_1 and this contradicts the fact that U is non-trivial and odd with respect to y_1.

If $N = 2$ and $m \in \mathbb{N} \setminus \{0\}$ is odd, then we consider $U_2(y_1, y_2, t) := U_{1,n}(y_2, t)$, where $U_{1,n}$ is defined in (A.5) and $n \in \mathbb{N} \setminus \{0\}$ is such that $m = 2n - 1$. Such U_2 is a positively homogeneous solution of (A.3) of degree m, odd with respect to y_2. If $m \in \mathbb{N} \setminus \{0\}$ is even, *i.e* $m = 2n$ with $n \in \mathbb{N} \setminus \{0\}$, then we define

$$U_3(y_1, y_2, t) := \sum_{k=0}^{n-1} a_k y_1^{2k+1} y_2^{2n-2k-1},$$

with $a_0, \dots, a_{n-1} \in \mathbb{R}$. A direct computation shows that U_3 is a solution of (A.3), and equivalently of (A.4), if and only if

$$a_{k+1} = \frac{-[2(n-k)^2 - 3n + 3k + 1]}{(2k^2 + 5k + 3)} a_k \quad \text{for all } k \in \{0, \dots, n-2\}.$$

Then, choosing for example again $a_0 = 1$, we obtain that U_3 is a solution of (A.3) which is positively homogeneous of degree m and odd with respect to y_2, as desired.

If $N > 2$, for any $m \in \mathbb{N} \setminus \{0\}$ there exists a harmonic homogeneous polynomial $P \not\equiv 0$ in the variables $y_1, \dots, y_{N-1}$, of degree $m - 1$. Then $U_4(y_1, \dots, y_{N-1}, y_N, t) := P(y_1, \dots, y_{N-1})\, y_N$ is a non trivial solution to (A.3) which is odd with respect to y_N and positively homogeneous of degree m.

□

Author contributions

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