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A NOTE ON THE DENSITY OF k-FREE POLYNOMIAL SETS, HAAR MEASURE AND GLOBAL FIELDS

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Abstract. In this work we investigate the general relation between the density of a subset of the ring of integers D of a general global field and the Haar measure of its closure in the profinite completion $\widehat{D}$. We then study a specific family of sets, the preimages of k-free elements (for any given $k \in \mathbb{N} \setminus \{0,1\}$) via one variable polynomial maps, showing that under some hypotheses their asymptotic density always exists and it is precisely the Haar measure of the closure in $\widehat{D}$ of their set.

1. Introduction

This work is intended to be part of a more general treatment that we develop in [4]. Our main purpose is to present a reasonably general notion of density on any *global ring* (see Definition 1.1) which shall include the most commonly used ones and to investigate the relation between it and the Haar measure on the profinite completion. Several researches have been already undertaken in this respect for the subsets of $\mathbb{N}$, see for example [3], while in [9] a reasonably general first axiomatization of the main properties of a density in the framework of a general notion has been proposed. The very nice work [12] of G. Micheli also offers a notion of asymptotic density on a positive characteristic global ring and we have been inspired by such an approach (see section 2).

We will discuss a concrete example of a general class of subsets of a global ring which are largely studied in number theory, the k-free elements, showing that the asymptotic density of their one-variable polynomial preimages exists and it is equal to the Haar measure of the closure of their set in the profinite completion of the ring. The k-free elements (see [13] for a complete survey) are the natural generalization of square-free numbers of $\mathbb{N}$, which have been analyzed in many of their most relevant aspects by the distinguished work of many researchers, among whom B. Poonen (see [14]) and M. Bhargava (see [2]). In particular, in [14] it is computed the asymptotic density of the *square-free* preimage of any *multivariate* polynomial in $\mathbb{Z}^n$ under the assumption of abc conjecture, while in [2] it is developed a general method to determine the asymptotic density of square-free *preimages* of multivariate integer polynomials that are invariants for the action of an algebraic group on a vector space. We will analyze such sets in the simpler case of one variable polynomial preimages, but for any $k \geq 2$ and on a quite special class of Dedekind domains D, which we call *global rings*, as a concrete application of the proposed techniques.

Definition 1.1. We call **global ring** a Dedekind domain D which is either the ring of integers of a number field or the ring of rational functions of a smooth projective and irreducible curve over a finite field[1] $\mathbb{F}_q$, regular outside a chosen nonempty and finite set $\mathcal{S}$ of points of the curve.

Let $k \in \mathbb{N} \setminus \{0,1\}$. The k-free elements of D are all the elements $x \in D$ such that $v_{\mathfrak{p}}(x) \in \{0,1,...,k-1\}$ for every finite place $v_{\mathfrak{p}}$ (associated to a non-zero prime ideal $\mathfrak{p}$ of D). Our main result will consist in showing that the polynomial (separable and in one variable) preimage of the k-free elements of a general global ring D has asymptotic density, which is precisely the Haar measure of its closure in $\widehat{D}$.

A natural development of this work would be to extend it to the more general case of the ring of $\mathcal{S}$-integers in any given number field. In other words, given $\mathcal{S}$ a finite set of places including (but not limited to) the places at infinity, one may consider the same problem on the ring consisting of v-integers for every $v \notin \mathcal{S}$ (see Definition 2.1 for the function field analog). No conceptual obstacles

[1] See for example [19], section 3.2, for a quite complete overview.

seem to stand in the way but the technicalities which are involved in providing a precise analog of the crucial estimate (5) appear sufficiently long to be avoided in the present treatment.

In section 2 we will provide a generalization of the notion of density (which will be discussed extensively in [4]) for a global ring and we will discuss more in detail how to treat global rings in order to attach to them an asymptotic density which extends the typical one on $\mathbb{N}$, showing that it satisfies the general requirements mentioned before. We will then recall the main properties of the Haar measure on a compact topological group (like $\widehat{D}$), showing that the general notion of density previously introduced easily implies the identification between density and measure of the closure of the ideals of D in $\widehat{D}$, which form a fundamental basis of open neighborhoods of 0. In particular, all their cosets form a basis of closed and open neighborhoods of the points of $\widehat{D}$. This will be one of the keys in our proof of the main Theorem (see Theorem 3.2). The main Theorem we prove in the present work will be treated then in section 3. It represents a class of examples (for every global ring) of sets for which we can prove the existence of asymptotic density by directly showing that it is precisely the Haar measure of the closure of such sets. It extends in particular to the number field case, and by the use of different techniques, the analogous result proved by K. Ramsay for the global function fields (see [16]).

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2. Preliminaries

As anticipated, we will be interested in a quite large class of Dedekind domains, called global rings. As introduced in the previous section, a global ring is either the ring of integers of a number field or a ring of regular functions in the following sense. Let F be a global function field, of positive characteristic. As known to the experts in function field arithmetic, F has a powerful geometric meaning represented by a smooth projective irreducible curve Σ_F over a finite field $\mathbb{F}_q$ which is attached to it, and of which it is the field of rational functions (see for example [19], Appendix B for a general introduction).

Definition 2.1. Given a chosen nonempty finite set $\mathcal{S}$ of places of F, we define:

$$D_{\mathcal{S}} := \{x \in F,\ v(x) \geq 0,\ \forall v \notin \mathcal{S}\}.$$

This is called a global ring of positive characteristic and we will briefly indicate it as $D := D_{\mathcal{S}}$.

All global rings D satisfy the two following conditions, which are crucial to introduce a general notion of density, including all the most commonly used ones: asymptotic, logarithmic, uniform and analytic density (see [9] for subsets of $\mathbb{N}$ and [4], section 4 for a more general case of a global ring).

(A1) D is countable;
(A2) all non-zero ideals of D have finite index.

Global rings only form a proper subclass of the more general class of Dedekind domains of the form described above. For example one can localize any global ring at any maximal ideal, obtaining a Dedekind domain, still satisfying (A1) and (A2), but which clearly cannot be a global ring anymore.

To introduce the notion of density in a general fashion we proceed as follows.

We call respectively **upper** and **lower** density on D the following two maps:

$$d^+, d^- : 2^D \longrightarrow [0,1]$$

such that the following conditions hold for every $X, Y \subseteq D$:

(Dn1) $d^-(D) = 1$;
(Dn2) $d^-(X) \leqslant d^+(X)$;
(Dn3) $X \subseteq Y$ implies $d^-(X) \leqslant d^-(Y)$ and $d^+(X) \leqslant d^+(Y)$;
(Dn4) if D is the disjoint union of X and Y, then $d^+(X) + d^-(Y) = 1$;
(Dn5) if X and Y are disjoint, then

$$d^+(X \cup Y) \leqslant d^+(X) + d^+(Y) \tag{1}$$

and

$$d^{-}(X \cup Y) \geqslant d^{-}(X) + d^{-}(Y)\,; \tag{2}$$

(Dn6) for every $a \in D$, one has

$$d^{-}(\{x + a \mid x \in X\}) = d^{-}(X)$$

and

$$d^{+}(\{x + a \mid x \in X\}) = d^{+}(X)$$

(Dn7) for every ideal $\mathfrak{a}$ of D, we have that

$$d^{-}(\mathfrak{a}) = d^{+}(\mathfrak{a}) = \frac{1}{||\mathfrak{a}||}$$

where we denote $||\mathfrak{a}||$ the index of $\mathfrak{a}$ in D.

In the whole document we will use the convention that if κ is an infinite cardinal, then $1/\kappa = 0$. This shows in particular that the 0 ideal has density 0 as its index is infinite.

Definition 2.2. We say that the density of a set $S \subseteq D$ **exists** if $d^{+}(S) = d^{-}(S)$ and we call it $d(S)$.

Lemma 2.3. *Assume d^{+} and d^{-} satisfy conditions* (Dn4) *and* (Dn5)*. If $Y \subseteq D$ has a density, then the equality*

$$d^{+}(X) = d(Y) - d^{-}(Y - X) \tag{3}$$

holds for every $X \subseteq Y$.

Proof. Let X be a subset of Y and Z the complement of Y in D. Then $Z \cup (Y - X)$ is the complement of X and it follows

$$d^{+}(X) \overset{\text{by (Dn4)}}{=} 1 - d^{-}(Z \cup (Y - X)) \overset{\text{by (2)}}{\leqslant} 1 - d^{-}(Z) - d^{-}(Y - X) = d^{+}(Y) - d^{-}(Y - X)\,,$$

that is, $d^{+}(X) + d^{-}(Y - X) \leqslant d^{+}(Y)$.

On the other hand, the complement of $Y - X$ is $Z \cup X$ and thus

$$d^{-}(Y - X) \overset{\text{by (Dn4)}}{=} 1 - d^{+}(Z \cup X) \overset{\text{by (1)}}{\geqslant} 1 - d^{+}(Z) - d^{+}(X) = d^{-}(Y) - d^{+}(X)\,,$$

that is, $d^{+}(X) + d^{-}(Y - X) \geqslant d^{-}(Y)$.

Therefore, if the density of Y exists then one has $d(Y) = d^{+}(X) + d^{-}(Y - X)$. □

We now extend the quite common notion of asymptotic density to a general global ring. It is necessary to provide two different specific constructions depending on the characteristic zero or finite characteristic setting. The main difficulty is represented by introducing a meaningful metric on D in a canonical way, which is normally not possible. Indeed, if D is the ring of integers of a number field it does not necessarily embed discretely in $\mathbb{C}$ and must be seen instead as a lattice of $\mathbb{R}^r \times \mathbb{C}^s$, where r is the number of distinct real embeddings of ∞ into D and s is the number of distinct couples of complex conjugate extensions of the Archimedean place to D.

Definition 2.4. Let $k \in \mathbb{N} \setminus \{0, 1\}$. The k-free elements of D are all the elements $x \in D$ such that:

$$v_{\mathfrak{p}}(x) \in \{0, 1, ..., k - 1\}$$

for every finite place $v_{\mathfrak{p}}$ (corresponding to a non-zero prime ideal $\mathfrak{p}$ of D).

2.0.1. *The number field case.* Let D be the ring of integers of a number field F. As anticipated, it can be embedded into $\mathbb{R}^r \times \mathbb{C}^s$ in the following way. Note that $r + 2s = [F : \mathbb{Q}]$. Let us call:

$$N := [F : \mathbb{Q}].$$

Let $F = \mathbb{Q}(\alpha)$. Let $\alpha_i := \sigma_i(\alpha)$ be the general conjugate of α by any F-isomorphism σ_i. The map:

$$\alpha \mapsto (\alpha_i)$$

embeds then F into $\mathbb{R}^r \times \mathbb{C}^s$ and the $\mathbb{Z}$-basis of D into a $\mathbb{R}$-basis of $\mathbb{R}^r \times \mathbb{C}^s$, making D a rank N lattice in $\mathbb{R}^r \times \mathbb{C}^s$. We define the following metric on $\mathbb{R}^r \times \mathbb{C}^s$:

$$|x| := \max_i\{|x_i|\}$$

where $x = (x_1, ..., x_{r+2s})$ is the generic element of $\mathbb{R}^r \times \mathbb{C}^s$. Note that up to the canonical $\mathbb{R}$-vector space isomorphism $\mathbb{C} \simeq \mathbb{R}^2$ which assigns to $\mathbb{C}$ the basis $\{1, i\}$, we can identify $\mathbb{R}^r \times \mathbb{C}^s$ with $\mathbb{R}^N$ and for brevity we will identify the two vector spaces from now on.

We now define as follows the open balls induced by the given metric:

$$B(x, r) := \{y \in \mathbb{R}^N,\ |x - y| < r\}$$

where $x \in \mathbb{R}^N$ and $r > 0$. Note that the definition we have provided ensures that a ball only contains finitely many elements of D (see for example [8], chapter 5). Indeed, choose a $\mathbb{Z}$-basis $x_1, ..., x_N$ of D. The matrix

$$(\sigma_i(x_j))$$

is nonsingular because the field extension $F/\mathbb{Q}$ is nontrivial and separable (see for example [11], page 96 item 39 a.), hence if we call $B_D(x, r)$ the set of all the elements of $B(x, r)$ which belong to D via the identification of D with a lattice of $\mathbb{R}^N$, the system of inequalities which describe all points of $B_D(0, r)$

$$\left(\left|\sum_{j=1}^{N} n_j \sigma_i(x_j)\right| < r\right)_{0\leq i\leq N-1, 1\leq j\leq N}$$

can only have finitely many solutions $(n_1, ..., n_N) \in \mathbb{Z}^N$. Analogous argument repeats for the balls centered in arbitrary points other than 0.

Definition 2.5. Let X be a subset of D. We define as follows the upper and lower **asymptotic density** of X:

$$d_{as}{}^{+}(X) := \limsup_{r\to\infty} \frac{|X \cap B(0, r)|}{|D \cap B(0, r)|} \quad \text{and} \quad d_{as}{}^{-}(X) := \liminf_{r\to\infty} \frac{|X \cap B(0, r)|}{|D \cap B(0, r)|}.$$

If $d_{as}{}^{+}(X) = d_{as}{}^{-}(X)$ we say that X has asymptotic density, which we call $d_{as}(X)$.

2.0.2. *The positive characteristic case.* Let F be a global field in positive characteristic. Then, let us consider a finite set $\mathcal{S}$ of places of the corresponding projective curve Σ_F over $\mathbb{F}_q$. To each $v \in \mathcal{S}$ we can attach a non-Archimedean metric on F as follows. For every place v, let $\mathcal{O}_v$ be the v-valuation ring and let $\mathfrak{m}_v$ be its maximal ideal. We call $\deg_v$ the degree of $\mathcal{O}_v/\mathfrak{m}_v$ as $\mathbb{F}_q$-vector space. Then, we set the following absolute value:

$$|x|_v = q^{-v(x)\deg_v} \textit{ for every } v \in \mathcal{S}$$

for every $x \in F$. Let us call F_v the completion of F with respect to such a metric. We define the following Dedekind domain:

$$D =: D_{\mathcal{S}} := \bigcap_{v\notin\mathcal{S}} \mathcal{O}_v.$$

We can therefore consider on it the metric induced by $\prod_{v\in\mathcal{S}} F_v$, given as usual as:

$$|x| := \max_{v\in\mathcal{S}}\{|x_v|_v\}$$

where $x = (x_v)$ is the general tuple in $\prod_{v\in\mathcal{S}} F_v$ and each factor F_v is endowed by its corresponding non-Archimedean metric $|\cdot|_v$ defined by v as above. We then define all open balls in such a space as follows:

$$B(x, r) := \{y \in \prod_{v\in\mathcal{S}} F_v,\ |x - y| < r\}$$

for each $x \in \prod_{v\in\mathcal{S}} F_v$ and $r > 0$. We call again $B_D(x, r)$ the intersection of $B(x, r)$ with the image of D by the embedding $F \hookrightarrow \prod_{v\in\mathcal{S}} F_v$ analogous to the one described above. In particular, the diagonal embedding of F into $\prod_{v\in\mathcal{S}} F_v$ is such that each F_v gives to the elements of D the corresponding metric attached to v. Our aim in the discussion we are about to make below is to show that $B_D(x, r)$ has at most finitely many elements, being a Riemann-Roch space and therefore having finite dimension on the finite field $\mathbb{F}_q$ (see [19], chapter 1).

We point out that the treatment we are about to develop was already known essentially by several important works made in the past: we mention J. V. Armitage (see [1]), M. Eichler ([5]) and K.

Mahler ([10]).

We start by providing the necessary definitions.

Definition 2.6. Let Σ be an algebraic curve over $\mathbb{F}_q$. The group of **divisors** on Σ is the free abelian group generated by the places of Σ, i.e. formed by all formal sums

$$\sum_{v \text{ places of } \Sigma} n_v[v]$$

with $n_v \in \mathbb{Z}$ for every place v and $n_v = 0$ for all but finitely many places. The **degree** of a divisor $\mathcal{D} = \sum n_v[v]$ is defined as:

$$\deg(\mathcal{D}) := \sum_{v \text{ places of } \Sigma} n_v.$$

See also [19], Definition 1.4.1, page 16. The following partial order relation is established between divisors:

$$\mathcal{D}_1 \geq \mathcal{D}_2 \Longleftrightarrow n_{v,1} \geq n_{v,2} \ \forall\, v \text{ place of } \Sigma$$

where

$$\mathcal{D}_i = \sum_{v \text{ places of } \Sigma} n_{v,i}[v]$$

for $i = 1, 2$.

Let F be a global function field and let Σ_F be as before the smooth projective curve associated to F as previously mentioned. Let $v(x) \in \mathbb{Z}$ be the order of an element $x \in F^*$ at v for any place v of F. The **principal divisors** of F are divisors on Σ_F defined as follows:

$$div(x) := \sum_{v \text{ places of } F} v(x)[v] \quad \text{where } x \in F^*.$$

It is well known (see [19], Theorem 1.4.11) that

$$\deg(div(x)) = 0$$

for every $x \in F^*$.

Definition 2.7. Let $\mathcal{D}$ be a divisor on Σ_F. The **Riemann-Roch space** associated to $\mathcal{D}$ is defined as follows:

$$\mathcal{L}(\mathcal{D}) := \{x \in F,\ div(x) \geq -\mathcal{D}\} \cup \{0\}.$$

The dimension of a Riemann-Roch space over $\mathbb{F}_q$ is **finite** and strictly related to the **genus** of Σ_F (see [19], section 1.5).

Lemma 2.8. *The balls induced by the metric described above on the image of D embedded into $\prod_{v\in\mathcal{S}} F_v$ are Riemann-Roch spaces of positive divisors with support contained in $\mathcal{S}$ related to Σ_F, while all such spaces are a finer but equivalent basis of neighborhoods of* 0.

Proof. All balls centered at 0 *are Riemann-Roch spaces*: In the following, we always assume $r \in q^{\mathbb{Z}}$. To each such r, it corresponds a positive divisor $\mathcal{D}_r$ of Σ_F, with support in $\mathcal{S}$, such that:

$$B_D(0, r) = \mathcal{L}(\mathcal{D}_r).$$

Indeed, let $x \in B_D(0, r)$ for a given $r > 0$. Let $v(x) \in \mathbb{Z}$ be the order of x at v for any place v of F. By the definition we gave to $D = D_{\mathcal{S}}$ it follows that we can have that $v(x) < 0$ only if $v \in \mathcal{S}$. For every $x \in B_D(0, r) \setminus \{0\}$ we have that:

$$div(x) = \sum_{v \text{ places of } F} v(x)[v].$$

Note that if $r < q$ then $B_D(0, r) = \{0\}$ because $v(x)$ can be negative only if $v \in \mathcal{S}$ and $\deg(div(x)) = 0$ for every $x \in F^*$. Remembering that we are assuming without loss of generality (since each place of F corresponds to a discrete valuation) that $r \in q^{\mathbb{Z}}$:

$$B_D(0, r) = \{x \in D,\ v(x)\deg_v \geq -\log_q(r),\ \forall v \text{ places of } F\}.$$

Indeed, we set the following divisor:

$$\mathcal{D}_r := \sum_{v\in\mathcal{S}} \left\lfloor \frac{\log_q(r)}{\deg_v} \right\rfloor \cdot [v].$$

It then follows that $div(x) = \sum_{v \text{ places of } F} v(x)[v] \geq -\mathcal{D}_r$ for every $x \in B_D(0,r)$ (note that $v(x) \geq 0$ for every $v \notin \mathcal{S}$ and that $-\left\lfloor \frac{\log_q(r)}{\deg_v} \right\rfloor = \left\lceil -\frac{\log_q(r)}{\deg_v} \right\rceil$, while $v(x) \in \mathbb{Z}$). Also, if $x \in F$ is such that $div(x) \geq -\mathcal{D}_r$, this implies that $v(x) \geq -\left\lfloor \frac{\log_q(r)}{\deg_v} \right\rfloor \geq -\frac{\log_q(r)}{\deg_v}$ for each $v \in \mathcal{S}$, which means that $x \in B_D(0,r)$.

Riemann-Roch spaces are equivalent to balls centered at 0: Let $\mathcal{D} = \sum_{v \text{ places of } F} M_v[v]$ be a positive divisor with support in $\mathcal{S}$. If $x \in \mathcal{L}(\mathcal{D}) \setminus \{0\}$ it follows that $div(x) = \sum_{v \text{ places of } F} v(x)[v]$ is such that $v(x) \geq -M_v$ for every v places of F. By assumption, $M_v = 0$ for every $v \notin \mathcal{S}$ and $M_v \geq 0$ for every $v \in \mathcal{S}$. By definition:

$$\mathcal{L}(\mathcal{D}) = \{x \in D,\ v(x) \geq -M_v,\ \forall v \text{ places of } F\} \cup \{0\}$$

then if we call $r := \max_{v\in\mathcal{S}}\{q^{M_v \deg_v}\}$ and $s := \min_{v\in\mathcal{S}}\{q^{M_v \deg_v}\}$ we easily see that:

$$B_D(0,s) \subseteq \mathcal{L}(\mathcal{D}) \subseteq B_D(0,r).$$

□

Since the sum is a continuous operation, we conclude that all translations of the Riemann-Roch spaces by any point $x \in D$ are a basis of neighborhoods of all points of D, equivalent to the open balls we introduced before. This immediately implies now that $B_D(x,r)$ has at most finitely many elements, because the Riemann-Roch spaces all have **finite** dimension on $\mathbb{F}_q$.

In particular, the definition of asymptotic density we gave for number fields (see Definition 2.5) also extends to the global function fields case.

Lemma 2.9. *Let $\mathfrak{a}$ be an ideal of $D_\mathcal{S}$. Let $B_\mathfrak{a}(0,t)$ be the set of all points of $\mathfrak{a}$ embedded into $B_D(0,t)$ as described before. We have that:*

$$\frac{|B_\mathfrak{a}(0,t)|}{|B_D(0,t)|} \sim_{t\to+\infty} \frac{1}{||\mathfrak{a}||}.$$

Proof. Let $\mathcal{D}_t$ be the divisor associated to t as explained before. We know that:

$$B_D(0,t) = \mathcal{L}(\mathcal{D}_t).$$

By Remark 2.8 and the Riemann-Roch Theorem (see [19], Theorem 1.5.15), we have that if t is sufficiently large it follows that:

$$|B(0,t)| = q^{l_t}$$

where:

$$l_t = \deg(\mathcal{D}_t) - g + 1$$

and g is the genus of Σ_F. Let:

$$\mathfrak{a} = \prod_{i=1}^{n} \mathfrak{m}_i^{m_i}$$

be the decomposition of $\mathfrak{a}$ in prime factors, each one of them corresponds to a place not belonging to $\mathcal{S}$. More specifically, if P_i is the point of Σ_F associated to $\mathfrak{m}_i$ for each $i = 1, ..., n$, $\mathfrak{a}$ is the ideal of all the rational functions on the curve which not only are regular outside of the points associated to the places of $\mathcal{S}$, but vanish with order at least m_i at P_i for each $i = 1, ..., r$. Therefore:

$$\mathcal{D}_{\mathfrak{a},t} := \mathcal{D}_t - \sum_{i=1}^{r} m_i \cdot [v_{P_i}]$$

is such that:

$$B_\mathfrak{a}(0,t) = \mathcal{L}(\mathcal{D}_{\mathfrak{a},t}).$$

By the same arguments as before we have that:

$$|B_{\mathfrak{a}}(0,t)| \sim_{t\to+\infty} q^{\deg(\mathcal{D}_{\mathfrak{a},t})+1-g}. \tag{4}$$

It follows that:

$$\frac{|B_{\mathfrak{a}}(0,t)|}{|B_D(0,t)|} \sim_{t\to+\infty} q^{-\sum_{i=1}^{n} m_i \deg_{\mathfrak{m}_i}}.$$

By the definition of degree of a place we have given before, it is now easy to see that:

$$q^{\sum_{i=1}^{n} m_i \deg_{\mathfrak{m}_i}} = ||\mathfrak{m}_1||^{m_1}...||\mathfrak{m}_n||^{m_n} = ||\mathfrak{a}||.$$

□

Remark 2.10. It is worthwhile to remark that the description we gave of $|B_{\mathfrak{a}}(0,t)|$ (the number of elements of $\mathfrak{a}$ embedded into the ball of radius r) in the proof above provides a function field analogue of [8, VI,§2, Theorem 2]. More specifically, it suggests that while the volume formula for the fundamental parallelogram of a number field F is $\frac{\sqrt{|disc(F)|}}{2^s}$, if F is the global function field associated to a curve Σ_F of genus g over $\mathbb{F}_q$ then the "fundamental parallelogram" $(\prod_{v\in\mathcal{S}} F_v)/D$ (remember that D embeds diagonally in $\prod_{v\in\mathcal{S}} F_v$) in this setting will be expected having volume q^{g-1}. This agrees with the computation made by A. Weil in [20], pag. 13 item $\mathfrak{b}$, in which the measure of the adelic quotient $\mathbb{A}_F/F \simeq (\prod_{v\in\mathcal{S}} F_v)/D \times \widehat{F}$ (corresponding to the volume of the fundamental parallelogram of D embedded into F_∞) is precisely q^{g-1}.

2.1. The asymptotic density on a global ring.

Proposition 2.11. *Let D be a global ring. The pair $({d_{as}}^+, {d_{as}}^-)$ satisfies all conditions (Dn1) - (Dn7).*

Proof. It is a straightforward remark that ${d_{as}}^+$ and ${d_{as}}^-$ satisfy (Dn1) - (Dn5), mainly because of the set-theoretic nature of such conditions. To show (Dn6) and (Dn7) is instead more delicate and it will be done case by case. We start by proving (Dn6) and (Dn7) for a number field first.
Let $a \in D$ and $X \subseteq D$, where D is the ring of integers of a number field F. It is clear that $|(a+X)\cap B(0,r)| = |X \cap B(-a,r)|$ for every $r > 0$. The metric we have chosen also implies:

$$\mathrm{Vol}(B(0,r)) = \mathrm{Vol}(B(-a,r)) = r^N$$

where the volume $\mathrm{Vol}(B(-a,r))$ of the ball of center $-a$ and radius $r > 0$ (which is actually a cube of side length r in $\mathbb{R}^N$) will be r^N. In particular, as $B(0,r)\cup B(-a,r) \subseteq B(0,|a|+r)$, it follows that:

$$\mathrm{Vol}(B(0,|a|+r)\setminus B(0,r)) = \mathrm{Vol}(B(0,|a|+r)\setminus B(-a,r)) = O_{r\to+\infty}(r^{N-1}).$$

We remind that a must be thought as a point of $\mathbb{R}^N$ and $|a| = \max_{i=1,...,N}\{|a|_i\}$. We also know that:

$$\frac{|X\cap B(0,|a|+r)| - |X\cap B(0,r)|}{|D\cap B(0,r)|} \leq \frac{\mathrm{Vol}(B(0,|a|+r)\setminus B(0,r))}{|D\cap B(0,r)|}.$$

By [8, VI,§2, Theorem 2] (see also below) we know that $|D\cap B(0,r)| = O_{r\to+\infty}(r^N)$. Hence the limit for r tending to infinity of the latter value will be 0. Therefore, by repeating the same argument with $B(0,|a|+r)$ and $B(-a,r)$ we obtain:

$$\frac{|X\cap B(0,r)|}{|D\cap B(0,r)|} \sim_{r\to+\infty} \frac{|X\cap B(0,|a|+r)|}{|D\cap B(0,r)|} \sim_{r\to+\infty} \frac{|(a+X)\cap B(0,r)|}{|D\cap B(0,r)|}.$$

It is now easy to see that (Dn6) follows. Now let $\mathfrak{a}$ be an ideal of D. Assume D be a ring of integers of a number field. By [8, VI,§2, Theorem 2], if we see $\mathfrak{a}$ as a sub-lattice of D embedded into $\mathbb{R}^N$ by an $\mathbb{R}$-vector space isomorphism, we have that:

$$|B_{\mathfrak{a}}(0,r)| = \frac{\mathrm{Vol}(B(0,1))}{\mathrm{Vol}(\Delta_{\mathfrak{a}})} r^N + O_{r\to+\infty}(r^{N-1}) \tag{5}$$

for r sufficiently large, where $\Delta_{\mathfrak{a}}$ is the fundamental domain of $\mathfrak{a}$. We know that $\mathrm{Vol}(B(0,1)) = 1$. Repeating the same remark by replacing $\mathfrak{a}$ by D we have:

$$\frac{|B_{\mathfrak{a}}(0,r)|}{|B_D(0,r)|} = \frac{\mathrm{Vol}(\Delta_{\mathfrak{a}})^{-1} r^N + O_{r\to+\infty}(r^{N-1})}{\mathrm{Vol}(\Delta_D)^{-1} r^N + O_{r\to+\infty}(r^{N-1})}$$

which yields:

$$\frac{|B_{\mathfrak{a}}(0,r)|}{|B_D(0,r)|} \sim_{r\to+\infty} \frac{1}{||\mathfrak{a}||}.$$

This because it is well known (see for example [11], page 135) that $\mathrm{Vol}(\Delta_{\mathfrak{a}}) = \mathrm{Vol}(\Delta_D)||\mathfrak{a}||$. This shows (Dn7). Now let D be a global ring of positive characteristic. By the non-Archimedean metric on $\prod_{v\in\mathcal{S}} F_v$ it is also easy to see that $B(0,r) = B(-a,r)$ for any $r > |a|$. This implies (Dn6) immediately again as $|(a+X)\cap B(0,r)| = |X\cap B(-a,r)|$. By Lemma 2.9 we now have that:

$$d(\mathfrak{a}) = \frac{1}{||\mathfrak{a}||}.$$

□

Although the discussion above has shown that the asymptotic density defined on any global ring satisfies the conditions (Dn1) - (Dn7) we have previously set, we would like to warn the reader that such a density may still show undesired features. For example, one can remark that in the case of the ring of integers of a number field of degree $N > 1$ the asymptotic density of a given subset X of such a ring is not necessarily preserved by a unimodular linear automorphism of $\mathbb{R}^N$ or by the choice of a metric equivalent to the one given on page 4. We illustrate the meaning of this in the following example.

Example 2.12. Consider D the ring of Gaussian integers $\mathbb{Z}[i]$ and let us define the following subset:

$$X := \{x+iy,\ x,y\in\mathbb{N}\}.$$

It is not hard to see that X has asymptotic density 1/4. Now, if we identify $\mathbb{C}$ with $\mathbb{R}^2$ as an $\mathbb{R}$-vector space by assuming on it the canonical basis $\{1,i\}$, we also see X as the subset $\mathbb{N}^2$ of $\mathbb{R}^2$. If we take on $\mathbb{R}^2$ the topology generated by the "slanted" squared balls obtained by modifying the "straight" ones after the linear transformation of $\mathbb{R}^2$ represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

the set X will now have density 1/8 with respect to the new topology, although the two topologies are clearly equivalent to each other. A more detailed discussion of this kind phenomena can be found in [4], Remarks 4.19.

2.2. **Haar measure and profinite completion.** We introduce now the key tools developed in [4]. The main idea consists to study specific subsets of D (like the one we specifically analyze in this work) by taking their closure in $\widehat{D}$. The main advantage by such an operation relies in being allowed to deal with the Haar measure on the compact ring:

$$\widehat{D} := \varprojlim_{\mathfrak{a}\subset D} D/\mathfrak{a}$$

where $\mathfrak{a}$ ranges over all nonzero ideals of D. This tool is expected to allow interpreting algebraically the general notion of density by a different point of view and what we propose in this work is precisely a class of concrete examples where the two notions are strictly and very easily related. In the following, given any subset X of D we will denote as $\widehat{X}$ the closure of X in $\widehat{D}$.

Lemma 2.13. *An ideal of $\widehat{D}$ is closed if and only if it is principal.*

Proof. One implication is trivial: since $\widehat{D}$ is compact and multiplication is continuous, $a\widehat{D}$ must be closed for every $a\in\widehat{D}$.

As for the converse, we start with the observation that by the Chinese Remainder Theorem one has that

$$\widehat{D} = \prod_{\mathfrak{p}\in\mathrm{Max}(D)} D_{\mathfrak{p}}$$

where $D_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic completion of D. Hence we can express $x\in\widehat{D}$ as $x = (x_{\mathfrak{p}})_{\mathfrak{p}}$, with $x_{\mathfrak{p}}\in\prod_{\mathfrak{p}\in\mathrm{Max}(D)} D_{\mathfrak{p}}$. For any set S of non-zero prime ideals of D, let $e_S = (e_{S,\mathfrak{p}})_{\mathfrak{p}}$ be defined by

$$e_{S,\mathfrak{p}} := \begin{cases} 1 & \text{if } \mathfrak{p}\in S \\ 0 & \text{if } \mathfrak{p}\notin S \end{cases}.$$

Then $e_S\widehat{D}$ is a subring of $\widehat{D}$, isomorphic to $\prod_{\mathfrak{p}\in S} D_{\mathfrak{p}}$.

Also, for every non-zero prime $\mathfrak{p}$ choose $\tilde{u}_{\mathfrak{p}} \in D$ satisfying $v_{\mathfrak{p}}(\tilde{u}_{\mathfrak{p}}) = 1$ and put $u_{\mathfrak{p}} := e_{\{\mathfrak{p}\}}\tilde{u}_{\mathfrak{p}}$. The subring $e_{\{\mathfrak{p}\}}\widehat{D}$ is a principal ideal domain having $u_{\mathfrak{p}}$ as a uniformizer.

Let I be any ideal of $\widehat{D}$. By the above, $e_{\{\mathfrak{p}\}}I$ is a principal ideal and we have $e_{\{\mathfrak{p}\}}I = u_{\mathfrak{p}}^{v_{\mathfrak{p}}(I)}\widehat{D}$ for some $v_{\mathfrak{p}}(I) \in \mathbb{N}\cup\{\infty\}$. If S is any finite set of non-zero primes, then the equality

$$e_S = \sum_{\mathfrak{p}\in S} e_{\{\mathfrak{p}\}}$$

implies

$$e_S I = \sum_{\mathfrak{p}\in S} e_{\{\mathfrak{p}\}} I = a_{S,I}\widehat{D}\,,$$

with $a_{S,I} = \sum u_{\mathfrak{p}}^{v_{\mathfrak{p}}(I)}$. Moreover, $a_{S,I} \in I$, since $e_S \cdot I \subseteq I$.

Let $a_I \in \widehat{D}$ be the point corresponding to $(u_{\mathfrak{p}}^{v_{\mathfrak{p}}(I)})_{\mathfrak{p}}$ by the Chinese Remainder Theorem. The inequality

$$v_{\mathfrak{p}}(x) \geqslant v_{\mathfrak{p}}(I) = v_{\mathfrak{p}}(a_I)$$

holds for every $x \in I$ and every $\mathfrak{p}$, proving the inclusion $I \subseteq a_I\widehat{D}$.

Moreover a_I is an accumulation point of the set $\{a_{S,I}\}$ (where S varies among all finite subsets of non-zero primes). Indeed, let U be any open neighborhood of a_I. Without loss of generality, we can assume $U = \prod_{\mathfrak{p}} U_{\mathfrak{p}}$, where each $U_{\mathfrak{p}}$ is open in $D_{\mathfrak{p}}$ and $U_{\mathfrak{p}} = D_{\mathfrak{p}}$ for every $\mathfrak{p}$ outside of a finite set T; but then $a_{S,I} \in U$ if $T \subseteq S$. If I is closed this yields $a_I \in I$ and hence $a_I\widehat{D} \subseteq I$. In the general case, one gets the equality $\hat{I} = a_I\widehat{D}$. □

Remark 2.14. We would like to remark that Lemma 2.13 remains valid under the only assumption that D is a Dedekind domain. The “residually finite” hypothesis was used to prove the first implication (all principal ideals of $\widehat{D}$ are closed) and it can be removed up to a slightly harder argument, which we leave to the reader. The argument remains valid and identical in proving the other implication. Lemma 2.13 easily implies also that an ideal of $\widehat{D}$ is principal if and only if it is finitely generated, which is also equivalent to be of the form

$$\prod_{\mathfrak{p}\in \mathrm{Max}(D)} \mathfrak{a}_{\mathfrak{p}}$$

with $\mathfrak{a}_{\mathfrak{p}}$ any ideal of $D_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Max}(D)$. Indeed, as sum and multiplication are both continuous it follows that a finitely generated ideal of $\widehat{D}$ is closed, while an ideal in the shape as above is the limit of a converging sequence of principal ideals in the sense explained in the proof of Lemma 2.13.

As $\widehat{D}$ is a topological compact Hausdorff ring we can define the Haar measure on it. Let us briefly recall the general definition of an Haar measure.

Definition 2.15. Let G be a topological compact Hausdorff group. The family of all its closed subsets generates the Borel σ-algebra $\mathcal{B}$, on which it is defined a measure

$$\mu : \mathcal{B} \longrightarrow [0, 1]$$

called Haar measure, satisfying the following requirements.

(1) Let H be a closed subgroup of G. Then

$$\mu(H) = \frac{1}{(G : H)}.$$

(2) μ is invariant with respect to the group operation (In other words, all cosets of any closed subgroup of G have all the same measure).
(3) For every open (non-empty) subset U of G we have that $\mu(U) > 0$.

Note that the group G being compact, it is also well known to be unimodular, which gives no ambiguity to condition (2).

Theorem 2.16. *All measures on a given group G as above, which satisfy condition (2) and (3), are the same up to positive constant multiple and are countably additive.*

Proof. See for example [7], Theorem C, page 263. □

Note that as G is clearly a closed subgroup of itself it follows that $\mu(G) = 1$, hence by the statement above the Haar measure on G is unique.

Also, in the case of our interest, when G is a profinite group, the proof of Theorem 2.16 (and indeed of the existence of the Haar measure on G) is considerably easier: see for example [6], section 18.1 and 18.2.

We now give some useful consequences of the given properties of the Haar measure on $\widehat{D}$.

Lemma 2.17. *Let $\mathfrak{a}$ be an ideal of D. We have that:*

$$d(\mathfrak{a}) = \mu(\widehat{\mathfrak{a}}).$$

Proof. Immediate from (Dn7) and by the ring isomorphism:

$$\widehat{D}/\widehat{\mathfrak{a}} \simeq D/\mathfrak{a}$$

for any ideal $\mathfrak{a}$ of D. This is constructed by composing the quotient map by $\widehat{\mathfrak{a}}$ with the diagonal embedding:

$$D \hookrightarrow \widehat{D}.$$

□

We note that (Dn6) agrees with the invariance by sum of μ, extending the equality above to all cosets of $\mathfrak{a}$.

Lemma 2.18. *A subset X of $\widehat{D}$ is at the same time closed and open if and only if there exists a nonzero ideal $\mathfrak{a}$ of D such that X is a finite union of cosets of $\widehat{\mathfrak{a}}$.*

Proof. We know that all the closures of ideals of D in $\widehat{D}$ form a basis of open neighborhoods of 0 in the Tychonoff topology of $\widehat{D}$. Because of (A2) and the quotient identification above we can conclude that they are all at the same time closed and open, and the same holds for all their cosets because $\widehat{D}$ is a topological ring. We therefore conclude that the cosets of all closures of the ideals of D in $\widehat{D}$ form a basis for the closed and opens. Because $\widehat{D}$ is compact, this makes all open covers finite without loss of generality. Now, given $\mathfrak{a}_1, ..., \mathfrak{a}_n$ ideals of D and $\hat{x}_1, ..., \hat{x}_n \in \widehat{D}$ such that

$$X = \bigcup_{i=1}^{n} (\hat{x}_i + \widehat{\mathfrak{a}}_i)$$

we define

$$\mathfrak{a} := \bigcap_{i=1}^{n} \mathfrak{a}_i.$$

As

$$\widehat{\mathfrak{a}} \subseteq \bigcap_{i=1}^{n} \widehat{\mathfrak{a}}_i$$

we call

$$m_i := (\widehat{\mathfrak{a}}_i : \widehat{\mathfrak{a}})$$

for $i = 1, ..., n$. For every $i = 1, ..., n$ there exist $\hat{y}_{1,i}, ..., \hat{y}_{m_i,i} \in \widehat{D}$ such that

$$\widehat{\mathfrak{a}}_i = \bigcup_{j=1}^{m_i} (\hat{y}_{j,i} + \widehat{\mathfrak{a}})$$

which proves the existence of some finite subset $\mathfrak{A}$ of $\widehat{D}$ such that

$$X = \bigcup_{\hat{y} \in \mathfrak{A}} (\hat{y} + \widehat{\mathfrak{a}})$$

showing then the first implication. The opposite one is immediate. □

2.3. **Supernatural elements.** We introduce here a slightly abstract notion, extending the notion of *supernatural integers*[2] of $\mathbb{Z}$ to D. The supernatural integers of $\mathbb{Z}$ are defined to be the set $\mathcal{S}(\mathbb{Z})$ of all formal products

$$\prod_{p \text{ primes of } \mathbb{Z}} p^{n_p}$$

where $n_p \in \mathbb{N} \cup \{\infty\}$. (If $n_p = \infty$ for all p, the product is 0.) It is not hard to see that the supernatural integers are in bijection with the principal ideals of $\widehat{\mathbb{Z}}$. Indeed, by Remark 2.14 we know that the principal ideals of $\widehat{\mathbb{Z}}$ are exactly those of the following shape:

$$\prod_{p \text{ primes of } \mathbb{Z}} p^{n_p}\mathbb{Z}_p$$

where $n_p \in \mathbb{N} \cup \{\infty\}$, and this induces an identification with the infinite strings (p^{n_p}) with n_p as before, which are obviously in bijection with the supernatural integers. Let us call $\widehat{D}^*$ the unit group[3] of $\widehat{D}$. We call:

$$\mathcal{S}(D) := \widehat{D}/\widehat{D}^*$$

the set of all *supernatural elements* of D.

Remark 2.19. $\mathcal{S}(D)$ corresponds to all principal ideals of $\widehat{D}$.

Proof. Given $x, y \in \widehat{D}$, we have that $x\widehat{D} = y\widehat{D}$ if and only if $v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}(y)$ for every $\mathfrak{p} \in \mathrm{Max}(D)$. The reason is that

$$x\widehat{D} = \{z \in \widehat{D},\ v_{\mathfrak{p}}(z) \geq v_{\mathfrak{p}}(x),\ \text{for all } \mathfrak{p} \in \mathrm{Max}(D)\}$$

for every $x \in \widehat{D}$. Therefore, $x\widehat{D} = y\widehat{D}$ if and only if for every $\mathfrak{p} \in \mathrm{Max}(D)$ there exists $u_{\mathfrak{p}} \in (D_{\mathfrak{p}})^*$ such that $x_{\mathfrak{p}} = u_{\mathfrak{p}} y_{\mathfrak{p}}$. The strings of units in $\prod_{\mathfrak{p} \in \mathrm{Max}(D)} D_{\mathfrak{p}}$ being correspondent to the units of $\widehat{D}$ via the obvious identification through the Chinese Remainder Theorem, we hence conclude that the condition above is equivalent to x and y being the same up to a unit multiple. $\square$

Remark 2.19 hence generalizes the notion of supernatural integers $\widehat{\mathbb{Z}}/\widehat{\mathbb{Z}}^*$ to D, a notion based morally on the clear identification between the natural numbers and the (principal) ideals of $\mathbb{Z}$.
Clearly, there is no canonical way to extend a density (upper or lower) from D to $\widehat{D}$, but if $\mathfrak{a}$ is an ideal of D we know that $\widehat{\mathfrak{a}}$ belongs to the basis of open neighborhoods of 0. In particular, if X is a subset of $\widehat{D}$ with non-empty intersection with D we define:

$$\widetilde{X}_{\mathfrak{a}} := \widehat{\pi}_{\mathfrak{a}}^{-1}(\widehat{\pi}_{\mathfrak{a}}(X))$$

where:

$$\widehat{\pi}_{\mathfrak{a}} : \widehat{D} \twoheadrightarrow \widehat{D}/\widehat{\mathfrak{a}}$$

is the usual projection map. If:

$$\pi_{\mathfrak{a}} : D \twoheadrightarrow D/\mathfrak{a}$$

is the usual projection map on D it follows that:

$$\widetilde{X}_{\mathfrak{a}} \cap D = \pi_{\mathfrak{a}}^{-1}(\pi_{\mathfrak{a}}(X \cap D))$$

by the canonical ring isomorphism $\widehat{D}/\widehat{\mathfrak{a}} \simeq D/\mathfrak{a}$ discussed before, which induces a one-to-one correspondence between the cosets of $\widehat{a}$ in $\widehat{D}$ and the cosets of $\mathfrak{a}$ in D. Therefore $\widetilde{X}_{\mathfrak{a}} \cap D$ is the disjoint union of the cosets of $\mathfrak{a}$ represented by each point of $X \cap D$. Therefore its density clearly exists. Hence by (Dn7) and Lemma 2.17:

$$d(\widetilde{X}_{\mathfrak{a}}) := d(\widetilde{X}_{\mathfrak{a}} \cap D) = \mu(\widetilde{X}_{\mathfrak{a}}).$$

This gives full meaning to this apparent abuse of notation and agrees with the conditions (Dn1) - (Dn7) which every density must satisfy.

Note that the topology of $\widehat{D}$ induces the corresponding quotient topology on $\mathcal{S}(D)$. Such a construction allows us to prove the following results.

[2]See for example [18] (a primary reference) or [6], section 22.8., for a more modern treatment.

[3]It is important not to confuse the group of units of $\widehat{D}$ with the closure of D^* in $\widehat{D}$, which is completely different and with totally different size, as we show in [4].

Let $\sigma \in \mathcal{S}(D)$ be a principal ideal of $\widehat{D}$. We define:

$$\widetilde{X}_\sigma := \widehat{\pi}_\sigma^{-1}(\overline{\widehat{\pi}_\sigma(X)})$$

for every $X \subseteq \widehat{D}$, where $\widehat{\pi}_\sigma : \widehat{D} \twoheadrightarrow \widehat{D}/\sigma$ is the usual projection map and $\overline{\widehat{\pi}_\sigma(X)}$ is the closure of $\widehat{\pi}_\sigma(X)$ in $\widehat{D}/\sigma$. Note that such closure is taken with respect to the quotient topology on $\widehat{D}/\sigma$, which is not necessarily the discrete one because not all principal ideals of $\widehat{D}$ are closures of ideals of D. We now take the convention, given a subset X of $\widehat{D}$, to call $\widehat{X}$ its closure.

Lemma 2.20. *Let $\mathcal{T}$ be a subset of $\mathcal{S}(D)$. Then for every subset X of $\widehat{D}$, we have*

$$\widehat{X} = \bigcap_{\sigma \in \mathcal{T}} \widetilde{X}_\sigma \tag{6}$$

if 0 is an accumulation point of $\mathcal{T}$.

Proof. By definition each $\widetilde{X}_\sigma$ is a closed set containing X. Hence $\widehat{X}$ is contained in the intersection on the right-hand side of (6).

Vice versa, let $z \in \widehat{D}$ be in the complement of $\widehat{X}$. By definition of the topology on $\widehat{D}$, there is an ideal $\mathfrak{a}$ of D such that $(z + \mathfrak{a}\widehat{D}) \cap \widehat{X} = \emptyset$ - that is, $\hat{\pi}_\mathfrak{a}(z) \notin \hat{\pi}_\mathfrak{a}(X)$. The assumption on $\mathcal{T}$ implies that there is some $\sigma \in \mathcal{T}$ such that $\sigma \subseteq \widehat{\mathfrak{a}}$. Hence $\hat{\pi}_\sigma(z) \notin \hat{\pi}_\sigma(X)$, so $z \notin \widetilde{X}_\sigma$. This shows that z is not in the right-hand side of (6). □

Remark 2.21. We now remark that there exists a natural embedding of the ideals of D into $\mathcal{S}(D)$. By Lemma 2.13 we know that $\mathcal{S}(D)$ represents all closed ideals of $\widehat{D}$. Therefore, if $\mathfrak{a}$ is an ideal of D, its closure $\widehat{\mathfrak{a}}$ in $\widehat{D}$ will correspond to an element of $\mathcal{S}(D)$. The association is obviously injective.

Remarks 2.22.

1. In Lemma 2.20 one can take $\mathcal{T}$ the subset of all those $\sigma \in \mathcal{S}(D)$ corresponding to ideals $\mathfrak{a} \subset D$ to obtain $\widehat{X} = \cap \widetilde{X}_\mathfrak{a}$.
2. From (A1) and (A2) we have that $\mathcal{S}(D)$ is second-countable. Indeed, the closures of the ideals of D form a base of open neighborhoods of 0 and hence, shifting by x, of any $x \in \widehat{D}$; and $\widehat{D}$ contains a countable dense subset, namely D. Thus the closures of cosets of ideals in D form a countably infinite base for the topology of $\widehat{D}$. In particular, 0 has a countable basis of open neighborhoods: it follows that for any $\mathcal{T}$ having 0 as an accumulation point there is a countable set $\mathcal{T}' \subseteq \mathcal{T}$ having the same property.

We now conclude this section by proving the two following technical lemmas which will be needed crucially in the proof of the main result of this paper. They will be used in the proof of Theorem 3.2 to provide an upper bound for the upper asymptotic density of the set of k-free integers (the final object of our study).

Lemma 2.23. *For every $X \subseteq \widehat{D}$ and $\mathcal{T} \subseteq \mathcal{S}(D)$ having 0 as a limit point,*

$$\mu(\widehat{X}) = \lim_{\sigma \to 0} \mu(\widetilde{X}_\sigma) \,, \tag{7}$$

where the limit is taken letting σ vary in $\mathcal{T}$.

Proof. One has $\mu(\widehat{X}) \leqslant \mu(\widetilde{X}_\sigma)$ for every σ, because $\widehat{X} \subseteq \widetilde{X}_\sigma$ holds by definition. The equality (7) then follows from (6) and Remark 2.22.2. □

Lemma 2.24. *Let X be a subset of $\widehat{D}$. Then one has the inequality*

$$d^+(X \cap D) \leqslant \mu(\widehat{X}). \tag{8}$$

Proof. By Remark 2.22.1 we can take a subset $\mathcal{T}$ of $\mathcal{S}(D)$ having 0 as an accumulation point and such that all elements of $\mathcal{T}$ are closures in $\widehat{D}$ of ideals of D (which by Lemma 2.13 are all elements of $\mathcal{S}(D)$). By (Dn7) we then have that for each $\sigma \in \mathcal{T}$ the density of $\widetilde{X}_\sigma \cap D$ exists and it is precisely $\mu(\widetilde{X}_\sigma)$. The inclusion $X \subset \widetilde{X}_\sigma$ implies $d^+(X \cap D) \leqslant d(\widetilde{X}_\sigma \cap D)$ for every $\sigma \in \mathcal{T}$ and hence

$$d^+(X \cap D) \leqslant \limsup_{\sigma \to 0\,,\, \sigma \in \mathcal{T}} d(\widetilde{X}_\sigma) = \limsup_{\sigma \to 0\,,\, \sigma \in \mathcal{T}} \mu(\widetilde{X}_\sigma) = \mu(\widehat{X})$$

where the last equality follows from Lemma 2.23. □

3. The main Theorem

We now present the main result of this work, after first proving the following Lemma.

Lemma 3.1. *Let $k \in \mathbb{N} \setminus \{0,1\}$. Let $f(x) \in D[x]$ be a separable polynomial of degree d and $V = \text{Spec } D[x]/(f)$ be the scheme corresponding to its zeroes (viewed as a (Spec D)-scheme). Then*

$$|V(D/\mathfrak{p}^k)| \leqslant d$$

for almost every prime ideal $\mathfrak{p}$ in D.

We recall that

$$V(D/\mathfrak{p}^k) = \{x \in D/\mathfrak{p}^k,\ \textit{such that}\ f(x) = 0\}$$

by definition of V.

Proof. We show that the natural map

$$\pi^{\mathfrak{p}}_{\mathfrak{p}^k} : V(D/\mathfrak{p}^k) \to V(D/\mathfrak{p})$$

is injective for all but finitely many $\mathfrak{p}$.

Assume $\mathfrak{p}^k | (f(a))$ for some $a \in D$. Then

$$f(x) = (x-a)g(x) + f(a)$$

for some $g(x) \in D[x]$. Clearly, this is implied by the fact that $x - a$ is monic. Hence for every $s \in \mathfrak{p}$

$$f(a+s) \equiv sg(a+s) \mod \mathfrak{p}^k$$

shows that if $a+s$ yields a zero of $f(x)$ modulo $\mathfrak{p}^k$ and $s \in \mathfrak{p} \setminus \mathfrak{p}^k$, then

$$g(a+s) \in \mathfrak{p}\,. \tag{9}$$

The latter condition holds only if $g(a) \in \mathfrak{p}$, which implies that $\pi_{\mathfrak{p}}(a)$ is a double root of $f(x)$ in $D/\mathfrak{p}$. Since $f(x)$ has no multiple roots, its discriminant $disc(f)$ is not zero. Since $f(x)$ has a double root in $D/\mathfrak{p}$ only if $\mathfrak{p}$ contains $disc(f)$, condition (9) holds only for finitely many $\mathfrak{p}$.

□

Theorem 3.2. *Let D be a global ring. Let $k \in \mathbb{N} \setminus \{0,1\}$. Let $f \in D[x]$ be a separable polynomial. If D is the ring of integers of a number field, suppose that f has degree at most $\frac{k}{N}$. The set*

$$X := \{a \in D,\ f(a) \text{ is } k-\text{free}\}$$

has asymptotic density, which is precisely $\mu(\widehat{X})$, with μ the usual Haar measure on $\widehat{D}$.

Proof. Let V be the scheme corresponding to the zeroes of $f(x)$. Let $\mathfrak{p}$ be a prime ideal of D. Let $C_{\mathfrak{p}}$ denote the complement of $V(D/\mathfrak{p}^k)$ in $D/\mathfrak{p}^k$. Consider the maps $\widehat{\pi}_{\mathfrak{p}^k} : \widehat{D} \twoheadrightarrow D/\mathfrak{p}^k$ and $\pi_{\mathfrak{p}^k} : D \twoheadrightarrow D/\mathfrak{p}^k$. Define

$$\widetilde{X}_{\mathfrak{p}} := \widehat{\pi}^{-1}_{\mathfrak{p}^k}(C_{\mathfrak{p}})$$

and

$$X_{\mathfrak{p}} := \pi^{-1}_{\mathfrak{p}^k}(C_{\mathfrak{p}}) = \widetilde{X}_{\mathfrak{p}} \cap D\,.$$

Note that:

$$\widehat{X_{\mathfrak{p}}} \subseteq \widetilde{X_{\mathfrak{p}}}.$$

Indeed, the projection maps are made continuous by the choice of the quotient topology on $\widehat{D}/\widehat{\mathfrak{p}} \simeq D/\mathfrak{p}$, which turns out easily to be the discrete topology. Hence $\widetilde{X}_{\mathfrak{p}}$ is closed, which implies that it contains all its limit points and in particular the limit points of $X_{\mathfrak{p}}$. Then one clearly has

$$X = \bigcap_{\mathfrak{p}} X_{\mathfrak{p}}\,. \tag{10}$$

Moreover, the closure $\widehat{X}$ of X is contained in $\cap_{\mathfrak{p}} \widehat{X}_{\mathfrak{p}} \subseteq \cap_{\mathfrak{p}} \widetilde{X}_{\mathfrak{p}}$.

Let $x \in \mathbb{N}$. Put

$$\widetilde{Y}_x := \bigcap_{||\mathfrak{p}|| \leqslant x} \widetilde{X}_{\mathfrak{p}}$$

and

$$Y_x := \widetilde{Y}_x \cap D = \bigcap_{||\mathfrak{p}|| \leqslant x} X_{\mathfrak{p}} \,. \tag{11}$$

We indicate

$$\widetilde{Y}_\infty := \bigcap_{\mathfrak{p} \in \mathrm{Max}(D)} X_{\mathfrak{p}}.$$

Every $\widetilde{X}_{\mathfrak{p}}$ is closed and open. Now, by [17, Proposition 13] only finitely many prime ideals in D have a fixed norm. Hence the sets $\widetilde{Y}_x$ form a decreasing family of closed and open subsets of $\widehat{D}$, each containing $\widehat{X}$. To ease the notation we will always mean from now on, when referring to a density, the *asymptotic density*, which we will simply denote as d^+, d^- or d where it exists instead of d_{as}. As $\widetilde{Y}_x$ is closed and open, we have by Proposition 2.11, Lemma 2.17 and Lemma 2.18 that Y_x has asymptotic density, with

$$d(Y_x) = \mu(\widetilde{Y}_x) \,.$$

Therefore we obtain by (8)

$$d^+(X) \leqslant \mu(\widehat{X}) \leqslant \mu(\widetilde{Y}_\infty) = \lim_{x \to \infty} \mu(\widetilde{Y}_x) = \lim_{x \to \infty} d(Y_x) \tag{12}$$

and the theorem follows if we can prove

$$d^-(X) \geqslant \mu(\widetilde{Y}_\infty) \,.$$

Since Y_x has a density, we have by Lemma 2.3

$$d^-(X) = d(Y_x) - d^+(Y_x - X) \,. \tag{13}$$

By (10) and (11), we find

$$Y_x - X = Y_x - \bigcap_{||\mathfrak{p}|| > x} X_{\mathfrak{p}} \subseteq D - \bigcap_{||\mathfrak{p}|| > x} X_{\mathfrak{p}} = \bigcup_{||\mathfrak{p}|| > x} \pi_{\mathfrak{p}^k}^{-1}(V(D/\mathfrak{p}^k)) =: A_{x,\infty} \,.$$

In particular $d^+(Y_x - X) \leqslant d^+(A_{x,\infty})$. Thus it is enough to show

$$\lim_{x \to \infty} d^+(A_{x,\infty}) = 0 \,, \tag{14}$$

since then (13) yields

$$d^-(X) = \lim_{x \to \infty} \big(d(Y_x) - d^+(Y_x - X)\big) = \lim_{x \to \infty} d(Y_x) = \mu(\widetilde{Y}_\infty) \,.$$

Thus we need to estimate

$$d^+(A_{x,\infty}) := \limsup_{y \to \infty} \frac{|A_{x,\infty} \cap B(0,y)|}{|B_D(0,y)|} \,.$$

We now proceed separately distinguishing the number field case by the positive characteristic case.

Number field case: Let

$$c := \max_{v | \infty} \{|\gamma|_v\}$$

where γ is the leading coefficient of f in D. Define

$$A_{x,z} := \bigcup_{x < ||\mathfrak{p}|| \leqslant z} \pi_{\mathfrak{p}^k}^{-1}(V(D/\mathfrak{p}^k)) \,,$$

so that we can write $A_{x,\infty} = A_{x,z} \cup A_{z,\infty}$. Note that for $y \gg 0$ one has

$$||(f(\alpha))|| < 2c^N y^{Nd} \tag{15}$$

(where d is the degree of $f(x)$) for every $\alpha \in D$ such that $|\alpha|_v \in (0, y)$ for every $v|\infty$. Indeed, it is clear that for every place v above ∞, if $|\alpha|_v = |\sigma_v(\alpha)|$ (where σ_v indicates the general embedding of F into $\mathbb{C}$ or $\mathbb{R}$, and we call $|\cdot|_v$ the absolute value induced by a given choice of v), then $|f(\sigma_v(\alpha))| < 2cy^d$ if y is sufficiently large. As it is well known (see for example [11], Theorem 22c) that $||(f(\alpha))|| = |N_{F/\mathbb{Q}}(f(\alpha))|$, this completes the proof of (15). It follows that $||\mathfrak{p}||^k > 2c^N y^{Nd}$ implies

$$f(\alpha) \in \mathfrak{p}^k \iff f(\alpha) = 0 \,.$$

Thus

$$|A_{2^{1/k}c^{N/k}y^{Nd/k},\infty} \cap B(0,y)| \le d, \tag{16}$$

and we have by (5)

$$d^+(A_{x,\infty}) = \limsup_{y\to\infty} \frac{|A_{x,2^{1/k}c^{N/k}y^{Nd/k}} \cap B(0,y)|}{\mathrm{Vol}(\Delta_D)^{-1}y^N}.$$

For every prime $\mathfrak{p}$, denote $c_\mathfrak{p} := |V(D/\mathfrak{p}^k)|$. Then for every $z > 0$ one has

$$|\pi_{\mathfrak{p}^k}^{-1}(V(D/\mathfrak{p}^k))\cap B(0,z)| \leqslant c_\mathfrak{p}\left(\frac{z^N}{\mathrm{Vol}(\Delta_D)||\mathfrak{p}^k||} + O_{z\to+\infty}(z^{N-1})\right) \leqslant d\left(\frac{z^N}{\mathrm{Vol}(\Delta_D)||\mathfrak{p}^k||} + O_{z\to+\infty}(z^{N-1})\right)$$

for all but finitely many prime ideals $\mathfrak{p}$ of D (where the first inequality follows from (5) and the last inequality comes from Lemma 3.1). Hence we have that:

$$|A_{x,2^{1/k}c^{N/k}y^{Nd/k}} \cap B(0,y)| \leqslant \sum_{x<||\mathfrak{p}||\leqslant 2^{1/k}c^{N/k}y^{Nd/k}} |\pi_{\mathfrak{p}^k}^{-1}(V(D/\mathfrak{p}^k)) \cap B(0,y)| \leqslant$$

$$\leqslant y^N d \sum_{x<||\mathfrak{p}||\leqslant 2^{1/k}c^{N/k}y^{Nd/k}} \frac{1}{\mathrm{Vol}(\Delta_D)||\mathfrak{p}^k||} + \pi(x, 2^{1/k}c^{N/k}y^{Nd/k})O_{y\to+\infty}(y^{N-1}),$$

where $\pi(x,z)$ denotes the number of prime ideals of D with norm between x and z. Therefore

$$\frac{|A_{x,\infty}\cap B(0,y)|}{|B_D(0,y)|} \lesssim_{y\to+\infty} \frac{|A_{x,\infty}\cap B(0,y)|}{y^N} \lesssim_{y\to+\infty} d\sum_{x<||\mathfrak{p}||} \frac{1}{\mathrm{Vol}(\Delta_D)||\mathfrak{p}^k||} + \frac{\pi(x,2^{1/k}c^{N/k}y^{Nd/k})y^{N-1}+d}{y^N}$$

where the occurrence of d in the last numerator follows from (16). Dedekind zeta functions converge absolutely in the half-plane $s > 1$ (see e.g. [8, VIII,§2]): hence $\sum ||\mathfrak{p}||^{-k}$ converges and therefore its tail converges to 0. Thus we have (14) if we can bound

$$\limsup_{y\to\infty} \frac{\pi(x, 2^{1/k}c^{N/k}y^{Nd/k})}{y}.$$

By the generalized Landau's prime numbers Theorem (see [8, XV,§5, Theorem 4])

$$\limsup_{y\to\infty} \frac{\pi(x,2^{1/k}c^{N/k}y^{Nd/k})}{y} \le \limsup_{y\to\infty} \frac{\pi(0,2^{1/k}c^{N/k}y^{Nd/k})}{y} = \limsup_{y\to\infty} \frac{2^{1/k}c^{N/k}y^{Nd/k}}{y(1/k\log 2 + N/k\log c + Nd/k\log y)} = 0$$

since $d \le k/N$.

Positive characteristic case: Define as before

$$c := \max_{v\in\mathcal{S}}\{|\gamma|_v\}$$

and

$$A_{x,z} := \bigcup_{x<||\mathfrak{p}||\leqslant z} \pi_{\mathfrak{p}^k}^{-1}(V(D/\mathfrak{p}^k)),$$

so that we can write $A_{x,\infty} = A_{x,z} \cup A_{z,\infty}$. Note that for $y \gg 0$ we have that $|f(\alpha)|_v < cy^d$ for every $\alpha \in D$ such that $|\alpha|_v \in (0,y)$ for every $v \in \mathcal{S}$ as the absolute value we use is non-Archimedean. Now one has:

$$||(f(\alpha))|| = \prod_{v\in\mathcal{S}} q^{-\deg_v v(f(\alpha))} = \prod_{v\in\mathcal{S}} |f(\alpha)|_v < c^{|\mathcal{S}|}y^{|\mathcal{S}|d}.$$

It follows that $||\mathfrak{p}||^k > c^{|\mathcal{S}|}y^{|\mathcal{S}|d}$ (where d is the degree of $f(x)$) implies

$$f(\alpha) \in \mathfrak{p}^k \iff f(\alpha) = 0.$$

Again we remark that $|A_{c^{|\mathcal{S}|/k}y^{|\mathcal{S}|d/k},\infty} \cap B(0,y)| \le d$, and we have by (4)

$$d^+(A_{x,\infty}) = \limsup_{y\to\infty} \frac{|A_{x,c^{|\mathcal{S}|/k}y^{|\mathcal{S}|d/k}} \cap B(0,y)|}{q^{1-g}y^{|\mathcal{S}|}}.$$

For every prime $\mathfrak{p}$, denote $c_\mathfrak{p} := |V(D/\mathfrak{p}^k)|$. Then for every $z > 0$ one has by (4)

$$|\pi_{\mathfrak{p}^k}^{-1}(V(D/\mathfrak{p}^k)) \cap B(0,z)| \leqslant c_\mathfrak{p}\left(\frac{z^{|\mathcal{S}|}q^{1-g}}{||\mathfrak{p}||^k}\right) \leqslant d\left(\frac{z^{|\mathcal{S}|}q^{1-g}}{||\mathfrak{p}^k||}\right)$$

for all but finitely many prime ideals $\mathfrak{p}$ of D (where the first inequality follows from (4) and the last inequality comes from Lemma 3.1). Hence

$$|A_{x,c^{|\mathcal{S}|/k}y^{|\mathcal{S}|d/k}}\cap B(0,y)| \leqslant \sum_{x<||\mathfrak{p}||\leqslant c^{|\mathcal{S}|/k}y^{|\mathcal{S}|d/k}} |\pi_{\mathfrak{p}^k}^{-1}(V(D/\mathfrak{p}^k))\cap B(0,y)| \leqslant q^{1-g}y^{|\mathcal{S}|}d \sum_{x<||\mathfrak{p}||\leqslant c^{|\mathcal{S}|/k}y^{|\mathcal{S}|d/k}} \frac{1}{||\mathfrak{p}^k||}\,.$$

Therefore

$$\frac{|A_{x,\infty}\cap B(0,y)|}{|B_D(0,y)|} < d\sum_{x<||\mathfrak{p}||}\frac{1}{||\mathfrak{p}^k||}$$

for y sufficiently large. Again, since $\sum ||\mathfrak{p}||^{-k}$ converges, we have (14). □

Remark 3.3. The key idea of using the limit (14) was suggested by [15].

Corollary 3.4. *Let X be defined as in Theorem 3.2. We have that*

$$d(X) = \prod_{\mathfrak{p}\in\mathrm{Max}(D)}\left(1-\frac{c_{\mathfrak{p}}}{||\mathfrak{p}||^k}\right)$$

where

$$c_{\mathfrak{p}} := |V(D/\mathfrak{p}^k)|$$

with $V(D/\mathfrak{p}^k)$ defined as in Lemma 3.1.

Proof. Let $\widetilde{Y}_\infty$ and $\widetilde{X}_{\mathfrak{p}}$ be defined as in the proof of Theorem 3.2. As Theorem 3.2 proves in particular (see (12)) that $d(X)=\mu(\widetilde{Y}_\infty)$, let us define $Z_{\mathfrak{p}}$ as the projection of $\widetilde{X}_{\mathfrak{p}}$ in $D_{\mathfrak{p}}$ (the $\mathfrak{p}$-adic completion of D) via the isomorphism given by the Chinese Remainder Theorem, for every $\mathfrak{p}\in\mathrm{Max}(D)$. We clearly have that

$$\widetilde{Y}_\infty = \bigcap_{\mathfrak{p}\in\mathrm{Max}(D)}\widetilde{X}_{\mathfrak{p}} = \prod_{\mathfrak{p}\in\mathrm{Max}(D)} Z_{\mathfrak{p}}.$$

Therefore

$$d(X) = \mu\left(\prod_{\mathfrak{p}\in\mathrm{Max}(D)} Z_{\mathfrak{p}}\right) \overset{(*)}{=} \prod_{\mathfrak{p}\in\mathrm{Max}(D)}\mu_{\mathfrak{p}}(Z_{\mathfrak{p}})\,,$$

where $\mu_{\mathfrak{p}}$ is the Haar measure on $D_{\mathfrak{p}}$ and the second equality follows from Lemma 2.23. Indeed, for any finite subset S of Max(D), let $\sigma_S := \prod_{\mathfrak{p}\in S}\mathfrak{p}^\infty$. Then one has

$$\widetilde{X}_{\sigma_S} := \prod_{\mathfrak{p}\in S} Z_{\mathfrak{p}}\times\prod_{\mathfrak{p}\in\mathrm{Max}(D)\setminus S} D_{\mathfrak{p}}$$

and the definition of product measure yields

$$\mu(\widetilde{X}_{\sigma_S}) = \prod_{\mathfrak{p}\in S}\mu_{\mathfrak{p}}(Z_{\mathfrak{p}})\,.$$

Now one just has to apply Lemma 2.23 with $\mathcal{T}=\{\sigma_S\}$ (S varying among all finite subsets of Max(D)) to obtain the equality $(*)$. By the definition we gave of $Z_{\mathfrak{p}}$ it is now easy to see that $Z_{\mathfrak{p}}$ is the closure of $X_{\mathfrak{p}}$ in $D_{\mathfrak{p}}$. In particular we have that

$$\mu_{\mathfrak{p}}(Z_{\mathfrak{p}}) = \frac{|C_{\mathfrak{p}}|}{||\mathfrak{p}||^k}$$

for every $\mathfrak{p}\in\mathrm{Max}(D)$. Therefore, as $|C_{\mathfrak{p}}| = ||\mathfrak{p}||^k - c_{\mathfrak{p}}$ we have that

$$d(X) = \mu(\widehat{X}) = \prod_{\mathfrak{p}\in\mathrm{Max}(D)}\left(1-\frac{c_{\mathfrak{p}}}{||\mathfrak{p}||^k}\right).$$

□

It is now immediate to check that in the specific example of the trivial polynomial $f(x) = x$ on $\mathbb{Z}$ the formula above agrees with the well known asymptotic density of the k-free elements to be $1/\zeta(k)$. An analogous argument can be repeated to compute such a density in the case of a linear polynomial $f(x) = ax + b$. One easily computes

$$c_{\mathfrak{p}} = \begin{cases} \|\mathfrak{p}\|^{\min\{v_{\mathfrak{p}}(a),k\}} & \text{if } v_{\mathfrak{p}}(a) \leqslant v_{\mathfrak{p}}(b) \text{ or } v_{\mathfrak{p}}(a) > v_{\mathfrak{p}}(b) \geqslant k \\ 0 & \text{if } v_{\mathfrak{p}}(a) > v_{\mathfrak{p}}(b) \text{ and } v_{\mathfrak{p}}(b) < k \end{cases}$$

and hence, if there is no $\mathfrak{p}$ such that $\min\{v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(b)\} \geqslant k$, then

$$d(X) = \prod_{\mathfrak{p} \text{ s.t. } 0 < v_{\mathfrak{p}}(a) \leqslant v_{\mathfrak{p}}(b)} \left(1 - \frac{\|\mathfrak{p}\|^{\min\{v_{\mathfrak{p}}(a),k\}}}{\|\mathfrak{p}\|^k}\right) \prod_{\mathfrak{p} \text{ s.t. } 0 = v_{\mathfrak{p}}(a)} \left(1 - \frac{1}{\|\mathfrak{p}\|^k}\right)$$

is a rational multiple of $1/\zeta_D(k)$, where we call ζ_D the zeta function of D. Obviously, if we have $\min\{v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(b)\} \geqslant k$ for some $\mathfrak{p}$, then $d(X) = 0$.

References

[1] J. V. Armitage: Algebraic functions and an analogue of the geometry of numbers: The Riemann-Roch theorem. *Arch. Math.* **18** (1967), 383-393.

[2] M. Bhargava: The geometric sieve and the density of squarefree values of invariant polynomials. arXiv:1402.0031.

[3] R. C. Buck: The measure theoretic approach to density. *Amer. J. Math.* **68** (1946), 560-580.

[4] L. Demangos - I. Longhi: Densities on Dedekind domains, completions and Haar measure. arXiv:2009.04229

[5] M. Eichler: *Einführung in die Theorie der algebraischen Zahlen und Funktionen.* Birkhäuser 1963

[6] M. D. Fried - M. Jarden: *Field Arithmetic.* Springer, 2005

[7] P. Halmos: *Measure Theory.* D. van Nostrand and Co., 1950.

[8] S. Lang: *Algebraic number theory.* Second edition. Graduate Texts in Mathematics, **110**. Springer-Verlag, 1994.

[9] P. Leonetti - S. Tringali: On the notions of upper and lower density. *Proc. Edinb. Math. Soc. (2)* **63** (2020), no. 1, 139-167

[10] K. Mahler: An analogue to Minkowski's geometry of numbers in a field of series. *Ann. of Math. (2)* **42** (1941), 488-522.

[11] D. Marcus: *Number fields.* Springer, 1977

[12] G. Micheli: A local to global principle for densities over function fields. arXiv:1701.01178v1

[13] F. Pappalardi: A survey on k-freeness. *Number theory*, Ramanujan Math. Soc. Lect. Notes Ser. **1**, 71 - 88. Ramanujan Math. Soc., 2005.

[14] B. Poonen: Squarefree values of multivariable polynomials. *Duke Math. J.* **118** (2003), no. 2, 353-373

[15] B. Poonen - M. Stoll: A local-global principle for densities. *Topics in number theory* (University Park, PA, 1997), 241-244. Math. Appl., **467**, Kluwer Acad. Publ., 1999.

[16] K. Ramsay: Square-free values of polynomials in one variable over function fields. *Int. Math. Res. Notes* 1992, no. 4, 97-102

[17] P. Samuel: About Euclidean rings. *J. Algebra* **19** (1971), 282-301

[18] E. Steinitz: Algebraische Theorie der Körper. *J. Reine Angew. Math.* **137** (1910), 167-309.

[19] H. Stichtenoth: *Algebraic Function Fields and Codes.* Springer, 2008

[20] A Weil: *Adeles and Algebraic Groups.* Progress in Mathematics **23**, Birkhauser (1982)

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