



# Bott–Chern harmonic forms and primitive decompositions on compact almost Kähler manifolds

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## Abstract

Let  $(X, J, \omega)$  be a compact  $2n$ -dimensional almost Kähler manifold. We prove primitive decompositions for Bott–Chern and Aeppli harmonic forms in special bidegrees and show that such bidegrees are optimal. We also show how the spaces of primitive Bott–Chern, Aeppli, Dolbeault and  $\partial$ -harmonic forms on  $(X, J, \omega)$  are related.

**Keywords** Almost complex manifold · Almost Kähler manifold · Harmonic forms · Aeppli

**Mathematics Subject Classification** 32Q60 · 53C15 · 58A14

## 1 Introduction

Let  $(X, J, \omega)$  be an almost Hermitian manifold of real dimension  $2n$ . Denote by

$$L : \Lambda^k X \rightarrow \Lambda^{k+2} X, \quad \alpha \mapsto \omega \wedge \alpha$$

the Lefschetz operator, and by

$$\Lambda : \Lambda^k X \rightarrow \Lambda^{k-2} X, \quad \Lambda := *^{-1} L *$$

its dual, where  $*$  :  $\Lambda^k X \rightarrow \Lambda^{2n-k}$  is the Hodge  $*$  operator. A  $k$ -form  $\alpha \in \Lambda^k X$ , for  $k \leq n$ , is said to be *primitive* if  $\Lambda \alpha = 0$ , or equivalently if  $L^{n-k+1} \alpha = 0$ . Given that, the following vector bundle decomposition holds

$$\Lambda^k X = \bigoplus_{r \geq \max(k-n, 0)} L^r (P^{k-2r} X), \quad (1)$$

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where we denoted by

$$P^s X := \ker (\Lambda : \Lambda^s X \rightarrow \Lambda^{s-2} X)$$

the bundle of primitive  $s$ -forms. The operators  $L$  and  $\Lambda$  extend to smooth sections, in particular to smooth  $k$ -forms  $A^k := \Gamma(X, \Lambda^k X)$  and to smooth  $(p, q)$ -forms  $A^{p,q} := \Gamma(X, \Lambda^{p,q} X)$ . We also set  $P^s := \Gamma(X, P^s X)$  and  $P^{p,q} := \Gamma(X, P^{p,q} X)$ , where  $P^{p,q} X := P^{p+q} X \cap \Lambda^{p,q} X$  is the bundle of primitive  $(p, q)$ -forms.

If  $(X, J, \omega)$  is a compact Kähler manifold, then the Lefschetz decomposition theorem says that the primitive decomposition of forms (1) descends to de Rham cohomology, i.e.,

$$H_{dR}^k X = \bigoplus_{r \geq \max(k-n, 0)} L^r (\ker (\Lambda : H_{dR}^{k-2r} X \rightarrow H_{dR}^{k-2r-2} X)).$$

Cirici and Wilson recently proved a generalized Lefschetz decomposition theorem for compact almost Kähler manifolds. Denote by  $\Delta_d := dd^* + d^*d$  the Hodge Laplacian, where  $d$  is the exterior differential and  $d^* := - * d *$  is its formal adjoint. The space of harmonic  $(p, q)$ -forms  $\ker \Delta_d \cap A^{p,q}$  will be indicated by  $\mathcal{H}_d^{p,q}$ . They showed, see [3, Corollary 5.4], that if  $(X, J, \omega)$  is a compact almost Kähler manifold, then

$$\mathcal{H}_d^{p,q} = \bigoplus_{r \geq \max(p+q-n, 0)} L^r (\mathcal{H}_d^{p-r, q-r} \cap P^{p-r, q-r}). \tag{2}$$

Let  $(X, J, \omega)$  be an almost Hermitian manifold, then other natural spaces of harmonic forms can be introduced. The exterior differential decomposes into  $d = \mu + \partial + \bar{\partial} + \bar{\mu}$ , and we set  $\partial^* = - * \bar{\partial} *$ ,  $\bar{\partial}^* = - * \partial *$  as the formal adjoints of  $\partial$ ,  $\bar{\partial}$ , where  $*$  is the  $\mathbb{C}$ -linear extension of the real Hodge  $*$  operator. Recall that

$$\Delta_\partial = \partial \bar{\partial}^* + \bar{\partial}^* \partial, \quad \Delta_{\bar{\partial}} = \bar{\partial} \partial^* + \partial^* \bar{\partial},$$

are respectively the  $\partial$  and  $\bar{\partial}$ , or Dolbeault, Laplacians, and

$$\Delta_{BC} = \partial \bar{\partial} \bar{\partial}^* \partial^* + \bar{\partial}^* \partial^* \partial \bar{\partial} + \partial^* \bar{\partial} \bar{\partial}^* \partial + \bar{\partial}^* \partial \partial^* \bar{\partial} + \partial^* \partial + \bar{\partial}^* \bar{\partial},$$

and

$$\Delta_A = \partial \bar{\partial} \bar{\partial}^* \partial^* + \bar{\partial}^* \partial^* \partial \bar{\partial} + \partial \bar{\partial}^* \bar{\partial} \partial^* + \bar{\partial} \partial^* \partial \bar{\partial}^* + \partial \bar{\partial}^* + \bar{\partial} \bar{\partial}^*,$$

are respectively the Bott–Chern and the Aepli Laplacians. Denote by

$$\mathcal{H}_\partial^{p,q}, \quad \mathcal{H}_{\bar{\partial}}^{p,q}, \quad \mathcal{H}_{BC}^{p,q}, \quad \mathcal{H}_A^{p,q},$$

the kernels of these Laplacians intersected with the space of  $(p, q)$ -forms. If  $X$  is compact these spaces are finite-dimensional but they do not have a cohomological counterpart. In fact, the almost complex Dolbeault, Bott–Chern and Aepli cohomology groups might be infinite dimensional (see [2], [4]).

In the integrable case, i.e., when  $d = \partial + \bar{\partial}$  and  $(X, J)$  is a complex manifold, if we further assume that  $X$  is compact and we endow  $(X, J)$  with any Hermitian metric  $\omega$ , all these spaces of harmonic  $(p, q)$ -forms have a cohomological meaning and, if  $\omega$  is Kähler, they coincide, i.e.,

$$\mathcal{H}_d^{p,q} = \mathcal{H}_\partial^{p,q} = \mathcal{H}_{\bar{\partial}}^{p,q} = \mathcal{H}_{BC}^{p,q} = \mathcal{H}_A^{p,q}.$$

Considered that, we are interested in understanding whether the primitive decomposition of harmonic forms (2) holds also for these spaces of harmonic forms, and how these spaces are related on a given compact almost Kähler manifold of real dimension  $2n$ .

Another motivation for this problem is the following. In [7] and [8] Holt and Zhang studied Dolbeault harmonic forms on the Kodaira-Thurston manifold to answer a famous question of Kodaira and Spencer which appeared as Problem 20 in Hirzebruch’s 1954 problem list [5]. Introducing an effective method to solve the PDE system associated to Dolbeault harmonic forms on the Kodaira-Thurston manifold, they proved that the dimension of the space of Dolbeault harmonic forms depends on the choice of the almost Hermitian metric. In [12], Tomassini and the second author answered again to the same question, with a different approach, analyzing locally conformally almost Kähler metrics on almost complex 4-manifolds. In [9], Tomassini and the first author introduced Bott–Chern and Aeppli harmonic forms on almost Hermitian manifolds and studied their relation with Dolbeault harmonic forms. See also [6] and [10] for recent results concerning the dimension of the spaces of Dolbeault and Bott–Chern harmonic  $(1, 1)$ -forms on compact almost Hermitian 4-manifolds. In particular, in [7, Proposition 6.1], in [12, Theorem 3.6] and in [9, Corollary 4.4], the primitive decomposition of  $(1, 1)$ -forms is used to deduce, in fact, primitive decompositions of Dolbeault and Bott–Chern harmonic  $(1, 1)$ -forms on a compact almost Hermitian 4-manifold. Given that, the study of primitive decompositions of Dolbeault, Bott–Chern, Aeppli harmonic forms can be seen as a generalisation of the just mentioned results in higher dimension  $2n \geq 4$  and for every bidegree  $(p, q)$ .

The case of primitive decompositions of Dolbeault harmonic forms is studied in [1]. In this paper, we are interested in studying primitive decompositions of Bott–Chern and Aeppli harmonic forms on a given  $2n$ -dimensional compact almost Kähler manifold  $(X, J, \omega)$ . Indeed, we prove

**Theorem 1.1** (Theorems 3.2, 3.3, 3.4) *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension  $2n$ . Then,*

$$\begin{aligned} \mathcal{H}_{BC}^{1,1} &= \mathbb{C}\omega \oplus \left( \mathcal{H}_{BC}^{1,1} \cap P^{1,1} \right), \\ \mathcal{H}_A^{1,1} &= \mathbb{C}\omega \oplus \left( \mathcal{H}_A^{1,1} \cap P^{1,1} \right), \\ \mathcal{H}_{BC}^{n-1,n-1} &= \mathbb{C}\omega^{n-1} \oplus L^{n-2} \left( \mathcal{H}_A^{1,1} \cap P^{1,1} \right), \\ \mathcal{H}_A^{n-1,n-1} &= \mathbb{C}\omega^{n-1} \oplus L^{n-2} \left( \mathcal{H}_{BC}^{1,1} \cap P^{1,1} \right). \end{aligned}$$

This, in particular, generalizes the decomposition of  $\mathcal{H}_{BC}^{1,1}$  in real dimension 4 of [9, Corollary 4.4] to higher dimensions. As a corollary of Theorem 1.1, [7, Proposition 6.1] and [9, Corollary 4.4], we derive

**Corollary 1.2** (Corollaries 3.5, 3.6) *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension 4. Then,*

$$\mathcal{H}_d^{1,1} = \mathcal{H}_\partial^{1,1} = \mathcal{H}_{\bar{\partial}}^{1,1} = \mathcal{H}_{BC}^{1,1} = \mathcal{H}_A^{1,1}.$$

Considered that the spaces of primitive Bott–Chern and Aeppli harmonic  $(1, 1)$ -forms turned out to be useful for the above decompositions, and that the same holds for  $\partial$  and Dolbeault harmonic  $(1, 1)$ -forms by the results of [1], we study the relations among all these spaces of primitive harmonic forms on compact almost Kähler manifolds. In particular, we analyze inclusions and non inclusions between these spaces. We prove

**Proposition 1.3** (Propositions 4.1 and 4.2) *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension  $2n$ . Then, for  $p + q \leq n$ ,*

$$\begin{aligned} \mathcal{H}_{BC}^{p,q} \cap P^{p,q} &= \mathcal{H}_{\bar{\partial}}^{p,q} \cap \mathcal{H}_{\partial}^{p,q} \cap P^{p,q}, \\ \mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q} &\subseteq \mathcal{H}_A^{p,q} \cap P^{p,q}. \end{aligned}$$

Moreover, for  $p + q = n$ ,

$$\mathcal{H}_{BC}^{p,q} \cap P^{p,q} = \mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q} = \mathcal{H}_{\partial}^{p,q} \cap P^{p,q} = \mathcal{H}_A^{p,q} \cap P^{p,q}.$$

In Proposition 4.4 we show that such inclusions are in general strict, and provide other non inclusions.

Finally, we show that the primitive decompositions of Bott–Chern and Aeppli harmonic forms obtained for the bidegrees  $(1, 1)$  and  $(n - 1, n - 1)$  are exclusive for these bidegrees. In fact, working on an explicit almost Kähler structure on the Iwasawa manifold, we show that the natural primitive decomposition of Bott–Chern harmonic forms one would expect on  $(2, 1)$ -forms in real dimension 6 does not hold.

The paper is organised in the following way. In section 2 we introduce some preliminaries of almost Hermitian geometry, including some observations on the other possible definitions of the Bott–Chern and Aeppli Laplacians. In section 3, we write down some trivial decompositions of Bott–Chern and Aeppli harmonic forms for the special bidegrees  $(p, 0)$ ,  $(0, q)$ ,  $(n, n - p)$  and  $(n - q, n)$ , and then we prove the non trivial decompositions for the bidegrees  $(1, 1)$  and  $(n - 1, n - 1)$  stated in Theorem 1.1. In section 4 we study the possible inclusions and non inclusions among the spaces of primitive  $\partial, \bar{\partial}$ , Bott–Chern and Aeppli harmonic forms. Finally, in section 5, we analyze primitive decompositions of Bott–Chern and Aeppli harmonic forms in dimension 6.

## 2 Preliminaries of almost Hermitian geometry

Throughout this paper, we will only consider connected manifolds without boundary. Let  $(X, J)$  be an almost complex manifold of dimension  $2n$ , i.e., a  $2n$ -differentiable manifold together with an almost complex structure  $J$ , that is  $J \in \text{End}(TX)$  and  $J^2 = -\text{id}$ . The complexified tangent bundle  $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$  decomposes into the two eigenspaces of  $J$  associated to the eigenvalues  $i, -i$ , which we denote respectively by  $T^{1,0}X$  and  $T^{0,1}X$ , giving

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X.$$

Denoting by  $\Lambda^{1,0}X$  and  $\Lambda^{0,1}X$  the dual vector bundles of  $T^{1,0}X$  and  $T^{0,1}X$ , respectively, we set

$$\Lambda^{p,q}X = \bigwedge^p \Lambda^{1,0}X \wedge \bigwedge^q \Lambda^{0,1}X$$

to be the vector bundle of  $(p, q)$ -forms, and let  $A^{p,q} = \Gamma(X, \Lambda^{p,q}X)$  be the space of smooth sections of  $\Lambda^{p,q}X$ . We denote by  $A^k = \Gamma(X, \Lambda^k X)$  the space of  $k$ -forms. Note that  $\Lambda^k X \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}X$ .

Let  $f \in C^\infty(X, \mathbb{C})$  be a smooth function on  $X$  with complex values. Its differential  $df$  is contained in  $A^1 \otimes \mathbb{C} = A^{1,0} \oplus A^{0,1}$ . On complex 1-forms, the exterior differential acts as

$$d : A^1 \otimes \mathbb{C} \rightarrow A^2 \otimes \mathbb{C} = A^{2,0} \oplus A^{1,1} \oplus A^{0,2}.$$

Therefore, it turns out that the differential operates on  $(p, q)$ -forms as

$$d : A^{p,q} \rightarrow A^{p+2,q-1} \oplus A^{p+1,q} \oplus A^{p,q+1} \oplus A^{p-1,q+2},$$

where we denote the four components of  $d$  by

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}.$$

From the relation  $d^2 = 0$ , we derive

$$\begin{cases} \mu^2 = 0, \\ \mu\partial + \partial\mu = 0, \\ \partial^2 + \mu\bar{\partial} + \bar{\partial}\mu = 0, \\ \partial\bar{\partial} + \bar{\partial}\partial + \mu\bar{\mu} + \bar{\mu}\mu = 0, \\ \bar{\partial}^2 + \bar{\mu}\partial + \partial\bar{\mu} = 0, \\ \bar{\mu}\bar{\partial} + \bar{\partial}\bar{\mu} = 0, \\ \bar{\mu}^2 = 0. \end{cases}$$

Let  $(X, J)$  be an almost complex manifold. If the almost complex structure  $J$  is induced from a complex manifold structure on  $X$ , then  $J$  is called integrable. It is equivalent to the decomposition of the exterior differential as  $d = \partial + \bar{\partial}$ .

A Riemannian metric on  $X$  for which  $J$  is an isometry is called almost Hermitian. Let  $g$  be an almost Hermitian metric, the 2-form  $\omega$  such that

$$\omega(u, v) = g(Ju, v) \quad \forall u, v \in \Gamma(TX)$$

is called the fundamental form of  $g$ . We will call  $(X, J, \omega)$  an almost Hermitian manifold. We denote by  $h$  the Hermitian extension of  $g$  on the complexified tangent bundle  $T_{\mathbb{C}}X$ , and by the same symbol  $g$  the  $\mathbb{C}$ -bilinear symmetric extension of  $g$  on  $T_{\mathbb{C}}X$ . Also denote by the same symbol  $\omega$  the  $\mathbb{C}$ -bilinear extension of the fundamental form  $\omega$  of  $g$  on  $T_{\mathbb{C}}X$ . Thanks to the elementary properties of the two extensions  $h$  and  $g$ , we may want to consider  $h$  as a Hermitian operator  $T^{1,0}X \times T^{1,0}X \rightarrow \mathbb{C}$  and  $g$  as a  $\mathbb{C}$ -bilinear operator  $T^{1,0}X \times T^{0,1}X \rightarrow \mathbb{C}$ . Recall that  $h(u, v) = g(u, \bar{v})$  for all  $u, v \in \Gamma(T^{1,0}X)$ .

Let  $(X, J, \omega)$  be an almost Hermitian manifold of real dimension  $2n$ . Extend  $h$  on  $(p, q)$ -forms and denote the Hermitian inner product by  $\langle \cdot, \cdot \rangle$ . Let  $*$  :  $A^{p,q} \rightarrow A^{n-q,n-p}$  the  $\mathbb{C}$ -linear extension of the standard Hodge  $*$  operator on Riemannian manifolds with respect to the volume form  $\text{Vol} = \frac{\omega^n}{n!}$ , i.e.,  $*$  is defined by the relation

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle \text{Vol} \quad \forall \alpha, \beta \in A^{p,q}.$$

Then the operators

$$d^* = - * d *, \quad \mu^* = - * \bar{\mu} *, \quad \partial^* = - * \bar{\partial} *, \quad \bar{\partial}^* = - * \partial *, \quad \bar{\mu}^* = - * \mu *,$$

are the formal adjoint operators respectively of  $d, \mu, \partial, \bar{\partial}, \bar{\mu}$ . Recall that  $\Delta_d = dd^* + d^*d$  is the Hodge Laplacian, and, as in the integrable case, set

$$\Delta_{\partial} = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},$$

respectively as the  $\partial$  and  $\bar{\partial}$  Laplacians. Again, as in the integrable case, set

$$\Delta_{BC} = \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\partial + \bar{\partial}^*\bar{\partial},$$

and

$$\Delta_A = \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \partial\bar{\partial}^*\bar{\partial}\partial^* + \bar{\partial}\partial^*\partial\bar{\partial}^* + \partial\partial^* + \bar{\partial}\bar{\partial}^*,$$

respectively as the Bott–Chern and the Aepli Laplacians. Note that

$$*\Delta_{BC} = \Delta_A * \quad \Delta_{BC}* = *\Delta_A. \tag{3}$$

If  $X$  is compact, then we easily deduce the following relations, for any  $k$ -form  $\beta$  and any  $(p, q)$ -form  $\alpha$ ,

$$\begin{cases} \Delta_d\beta = 0 & \iff d\beta = 0, d*\beta = 0, \\ \Delta_{\partial}\alpha = 0 & \iff \partial\alpha = 0, \bar{\partial}*\alpha = 0, \\ \Delta_{\bar{\partial}}\alpha = 0 & \iff \bar{\partial}\alpha = 0, \partial*\alpha = 0, \\ \Delta_{BC}\alpha = 0 & \iff \partial\alpha = 0, \bar{\partial}\alpha = 0, \partial\bar{\partial}*\alpha = 0, \\ \Delta_A\alpha = 0 & \iff \partial*\alpha = 0, \bar{\partial}*\alpha = 0, \partial\bar{\partial}\alpha = 0, \end{cases}$$

which characterize the spaces of harmonic forms

$$\mathcal{H}_d^k, \mathcal{H}_{\partial}^{p,q}, \mathcal{H}_{\bar{\partial}}^{p,q}, \mathcal{H}_{BC}^{p,q}, \mathcal{H}_A^{p,q},$$

defined as the spaces of forms which are in the kernel of the associated Laplacians. All these Laplacians are elliptic operators on the almost Hermitian manifold  $(X, J, \omega)$  (cf. [5], [9]), implying that all the spaces of harmonic forms are finite dimensional when the manifold is compact. Denote by

$$b^k, h_{\partial}^{p,q}, h_{\bar{\partial}}^{p,q}, h_{BC}^{p,q}, h_A^{p,q}$$

respectively the real dimension of  $\mathcal{H}_d^k$  and the complex dimensions of  $\mathcal{H}_{\partial}^{p,q}, \mathcal{H}_{\bar{\partial}}^{p,q}, \mathcal{H}_{BC}^{p,q}, \mathcal{H}_A^{p,q}$ .

**Remark 2.1** By (3), note that

$$*\mathcal{H}_{BC}^{p,q} = \mathcal{H}_A^{n-q,n-p}, \quad *\mathcal{H}_A^{p,q} = \mathcal{H}_{BC}^{n-q,n-p}. \tag{4}$$

In the following, it will be often useful to study the spaces  $\mathcal{H}_{BC}^{p,q}$  and  $\mathcal{H}_A^{p,q}$  for  $p + q \leq n$  in order to obtain information respectively on the spaces  $\mathcal{H}_A^{n-q,n-p}$  and  $\mathcal{H}_{BC}^{n-q,n-p}$ .

**Remark 2.2** We observe that, since the operators  $\partial$  and  $\bar{\partial}$  do not anticommute in the non integrable setting, we could have made a different choice for the Bott–Chern and Aepli Laplacians, namely we could have taken

$$\begin{aligned} \Delta_{BC,2} &= \bar{\partial}\partial\partial^*\bar{\partial}^* + \partial^*\bar{\partial}^*\bar{\partial}\partial + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} + \partial^*\partial \\ \Delta_{A,2} &= \bar{\partial}\partial\partial^*\bar{\partial}^* + \partial^*\bar{\partial}^*\bar{\partial}\partial + \bar{\partial}\partial^*\partial\bar{\partial}^* + \partial\bar{\partial}^*\bar{\partial}\partial^* + \bar{\partial}\bar{\partial}^* + \partial\partial^* \end{aligned}$$

However, we notice that they differ by conjugation, namely

$$\bar{\Delta}_{BC} = \Delta_{BC,2}, \quad \bar{\Delta}_A = \Delta_{A,2}.$$

Hence,

$$\alpha \in \text{Ker } \Delta_{BC} \iff \bar{\alpha} \in \text{Ker } \Delta_{BC,2},$$

therefore it is not restrictive to study only the space  $\mathcal{H}_{BC}^{\bullet,\bullet}(X)$ . A similar argument shows that it is not restrictive to study only the space  $\mathcal{H}_A^{\bullet,\bullet}(X)$ .

### 3 Primitive decompositions of Bott–Chern harmonic forms

In the following we are going to show that, in special bidegrees, we have natural primitive decompositions for Bott–Chern harmonic forms on compact almost Kähler manifolds. We first need to introduce some notations and recall some well known facts about primitive forms. Let  $(X, J, \omega)$  be a  $2n$ -dimensional almost Hermitian manifold. We denote with

$$L : \Lambda^k X \rightarrow \Lambda^{k+2} X, \quad \alpha \mapsto \omega \wedge \alpha$$

the Lefschetz operator and with

$$\Lambda : \Lambda^k X \rightarrow \Lambda^{k-2} X, \quad \Lambda = *^{-1} L *$$

its dual. A differential  $k$ -form  $\alpha_k$  on  $X$ , for  $k \leq n$ , is said to be *primitive* if  $\Lambda \alpha_k = 0$ , or equivalently  $L^{n-k+1} \alpha_k = 0$ . Then, the following vector bundle decomposition holds (see e.g., [14, p. 26, Théorème 3])

$$\Lambda^k X = \bigoplus_{r \geq \max(k-n, 0)} L^r (P^{k-2r} X), \tag{5}$$

where we denoted

$$P^s X := \ker (\Lambda : \Lambda^s X \rightarrow \Lambda^{s-2} X)$$

the bundle of primitive  $s$ -forms. Accordingly to such decomposition, given any  $k$ -form  $\alpha_k \in \Lambda^k X$ , we can write

$$\alpha_k = \sum_{r \geq \max(k-n, 0)} \frac{1}{r!} L^r \beta_{k-2r}, \tag{6}$$

where  $\beta_{k-2r} \in P^{k-2r} X$ , namely

$$\Lambda \beta_{k-2r} = 0,$$

or equivalently

$$L^{n-k+2r+1} \beta_{k-2r} = 0.$$

Furthermore, the decomposition above is compatible with the bidegree decomposition on the bundle of complex  $k$ -forms  $\Lambda_{\mathbb{C}}^k X$  induced by  $J$ , that is

$$P_{\mathbb{C}}^k X = \bigoplus_{p+q=k} P^{p,q} X,$$

where

$$P^{p,q} X = P_{\mathbb{C}}^k X \cap \Lambda^{p,q} X.$$

For any given  $\beta_k \in P^k X$ , we have the following formula (cf. [14, p. 23, Théorème 2])

$$* L^r \beta_k = (-1)^{\frac{k(k+1)}{2}} \frac{r!}{(n-k-r)!} L^{n-k-r} J \beta_k. \tag{7}$$

Let us set  $P^s := \Gamma(X, P^s X)$  and  $P^{p,q} := \Gamma(X, P^{p,q} X)$ . We recall that the map  $L^h : \Lambda^k X \rightarrow \Lambda^{k+2h} X$  is injective for  $h+k \leq n$  and is surjective for  $h+k \geq n$ .

**Remark 3.1** In [3, Corollary 5.4] it is proven that on a  $2n$ -dimensional compact almost Kähler manifold such primitive decompositions pass to  $d$ -harmonic forms, namely

$$\mathcal{H}_d^{p,q} = \bigoplus_{r \geq \max(p+q-n, 0)} L^r (\mathcal{H}_d^{p-r, q-r} \cap P^{p-r, q-r}).$$

In fact, this holds true also for the spaces of harmonic forms introduced in [11]. More precisely, setting

$$\bar{\delta} := \bar{\partial} + \mu, \quad \delta := \partial + \bar{\mu}$$

one has, on compact almost Kähler manifolds, for every  $p, q$  ([11, Proposition 6.2, Theorem 6.7])

$$\mathcal{H}_d^{p,q} = \mathcal{H}_{\bar{\delta}}^{p,q} = \mathcal{H}_{\delta}^{p,q}.$$

Here we are interested in investigating when the decomposition (5) descends to Bott–Chern and Aeppli harmonic forms. Note that, since  $(p, 0)$ -forms and  $(0, q)$ -forms are trivially primitive, we immediately derive for  $p, q \leq n$

$$\begin{aligned} \mathcal{H}_{BC}^{p,0} &= \mathcal{H}_{BC}^{p,0} \cap P^{p,0}, & \mathcal{H}_{BC}^{0,q} &= \mathcal{H}_{BC}^{0,q} \cap P^{0,q}, \\ \mathcal{H}_A^{p,0} &= \mathcal{H}_A^{p,0} \cap P^{p,0}, & \mathcal{H}_A^{0,q} &= \mathcal{H}_A^{0,q} \cap P^{0,q}. \end{aligned}$$

Applying the Hodge  $*$  operator to the previous trivial primitive decompositions of the spaces of Bott–Chern and Aeppli harmonic forms, from (4) and (7) we easily obtain respectively

$$\begin{aligned} \mathcal{H}_A^{n,n-p} &= L^{n-p} \left( \mathcal{H}_{BC}^{p,0} \cap P^{p,0} \right), & \mathcal{H}_A^{n-q,n} &= L^{n-q} \left( \mathcal{H}_{BC}^{0,q} \cap P^{0,q} \right), \\ \mathcal{H}_{BC}^{n,n-p} &= L^{n-p} \left( \mathcal{H}_A^{p,0} \cap P^{p,0} \right), & \mathcal{H}_{BC}^{n-q,n} &= L^{n-q} \left( \mathcal{H}_A^{0,q} \cap P^{0,q} \right). \end{aligned}$$

In particular, taking  $p = q = n$  we obtain

$$\mathcal{H}_{BC}^{n,0} = \mathcal{H}_A^{n,0} \quad \text{and} \quad \mathcal{H}_{BC}^{0,n} = \mathcal{H}_A^{0,n}.$$

In fact, this can be obtained directly using formula (7) as done in Proposition 4.2 recalling that  $(n, 0)$ -forms and  $(0, n)$ -forms are trivially primitive. We find more interesting primitive decompositions when we look at the space of Bott–Chern and Aeppli harmonic forms of bidegree  $(1, 1)$ .

**Theorem 3.2** *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension  $2n$ . Then,*

$$\mathcal{H}_{BC}^{1,1} = \mathbb{C}\omega \oplus \left( \mathcal{H}_{BC}^{1,1} \cap P^{1,1} \right).$$

**Proof** Let  $\psi \in \mathcal{H}_{BC}^{1,1}$ , i.e.,  $\psi \in A^{1,1}$  and

$$\partial\psi = 0, \quad \bar{\partial}\psi = 0, \quad \partial\bar{\partial} * \psi = 0. \tag{8}$$

By (6), we derive

$$\psi = f\omega + \gamma,$$

where  $f$  is a smooth function with complex values on  $X$ , and  $\gamma$  is a primitive  $(1, 1)$ -form, i.e.,  $\Delta\gamma = 0$ . Since both  $f$  and  $\gamma$  are primitive forms, we apply (7) to compute  $*\psi$ . We obtain

$$*\psi = \frac{\omega^{n-1}}{(n-1)!} f - \frac{\omega^{n-2}}{(n-2)!} \wedge \gamma.$$



Now, from (8) and from the assumption that the metric is almost Kähler, it follows that

$$\begin{aligned} 0 &= \partial\psi = \partial f \wedge \omega + \partial\gamma, \\ 0 &= \bar{\partial}\psi = \bar{\partial} f \wedge \omega + \bar{\partial}\gamma, \\ 0 &= \frac{\omega^{n-1}}{(n-1)!} \wedge \partial\bar{\partial} f - \frac{\omega^{n-2}}{(n-2)!} \wedge \partial\bar{\partial}\gamma \\ &= \frac{\omega^{n-1}}{(n-1)!} \wedge \partial\bar{\partial} f - \frac{\omega^{n-2}}{(n-2)!} \wedge \partial(-\bar{\partial} f \wedge \omega) \\ &= \frac{\omega^{n-1}}{(n-1)!} \wedge \partial\bar{\partial} f + \frac{\omega^{n-1}}{(n-2)!} \wedge \partial\bar{\partial} f \\ &= \left( \frac{1}{(n-1)!} + \frac{1}{(n-2)!} \right) \omega^{n-1} \wedge \partial\bar{\partial} f. \end{aligned}$$

Arguing like in [9, Theorem 4.3] or in [12, Proposition 3.4], one can show that the differential operator  $L : C^\infty(X, \mathbb{C}) \rightarrow C^\infty(X, \mathbb{C})$  defined by

$$L : f \mapsto -i * (\partial\bar{\partial} f \wedge \omega^{n-1})$$

is strongly elliptic and, being  $f \in \text{Ker } L$ , it follows that  $f$  is a complex constant by the maximum principle. Since  $f \in \mathbb{C}$ , the equations in (8) are equivalent to

$$\partial\gamma = 0, \quad \bar{\partial}\gamma = 0.$$

Note that  $\partial\gamma = \bar{\partial}\gamma = 0$  and  $\Lambda\gamma = 0$  implies  $\partial\bar{\partial} * \gamma = 0$ . Summing up, we showed that if  $\psi = f\omega + \gamma \in \mathcal{H}_{BC}^{1,1}$ , where  $f \in C^\infty(X, \mathbb{C})$  and  $\gamma \in P^{1,1}$ , then  $f \in \mathbb{C}$  and  $\gamma \in \mathcal{H}_{BC}^{1,1}$  proving the inclusion  $\subseteq$  of the statement. The converse inclusion  $\supseteq$  is trivial, therefore the theorem is proved.  $\square$

For Aeppli harmonic  $(1, 1)$ -forms, we find the following similar decomposition.

**Theorem 3.3** *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension  $2n$ . Then,*

$$\mathcal{H}_A^{1,1} = \mathbb{C}\omega \oplus \left( \mathcal{H}_A^{1,1} \cap P^{1,1} \right).$$

**Proof** Let  $\psi \in \mathcal{H}_A^{1,1}$ , i.e.,  $\psi \in A^{1,1}$  and

$$\partial * \psi = 0, \quad \bar{\partial} * \psi = 0, \quad \partial\bar{\partial}\psi = 0. \tag{9}$$

By (6), we derive

$$\psi = f\omega + \gamma,$$

where  $f$  is a smooth function with complex values on  $X$ , and  $\gamma$  is a primitive  $(1, 1)$ -form, i.e.,  $\Lambda\gamma = 0$ . Since both  $f$  and  $\gamma$  are primitive forms, we apply (7) to compute  $*\psi$ . We obtain

$$*\psi = \frac{\omega^{n-1}}{(n-1)!} f - \frac{\omega^{n-2}}{(n-2)!} \wedge \gamma.$$

Now, from (8) and from the assumption that the metric is almost Kähler, it follows that

$$\begin{aligned}
 0 &= \partial * \psi = \frac{\omega^{n-1}}{(n-1)!} \wedge \partial f - \frac{\omega^{n-2}}{(n-2)!} \wedge \partial \gamma, \\
 0 &= \bar{\partial} * \psi = \frac{\omega^{n-1}}{(n-1)!} \wedge \bar{\partial} f - \frac{\omega^{n-2}}{(n-2)!} \wedge \bar{\partial} \gamma,
 \end{aligned}
 \tag{10}$$

$$0 = \partial \bar{\partial} f \wedge \omega + \partial \bar{\partial} \gamma.
 \tag{11}$$

We apply  $L^{n-2}$  to (11), obtaining

$$0 = \omega^{n-1} \wedge \partial \bar{\partial} f + \omega^{n-2} \wedge \partial \bar{\partial} \gamma.$$

We apply  $\partial$  to (10), deriving

$$0 = \frac{\omega^{n-1}}{(n-1)!} \wedge \partial \bar{\partial} f - \frac{\omega^{n-2}}{(n-2)!} \wedge \partial \bar{\partial} \gamma.$$

Combining the last two equations, we find

$$\begin{aligned}
 0 &= \frac{\omega^{n-1}}{(n-1)!} \wedge \partial \bar{\partial} f + \frac{\omega^{n-1}}{(n-2)!} \wedge \partial \bar{\partial} f \\
 &= \left( \frac{1}{(n-1)!} + \frac{1}{(n-2)!} \right) \omega^{n-1} \wedge \partial \bar{\partial} f.
 \end{aligned}$$

Arguing like in Theorem 3.2, it follows that  $f$  is a complex constant. Since  $f \in \mathbb{C}$ , the equations in (9) are equivalent to

$$\partial * \gamma = 0, \quad \bar{\partial} * \gamma = 0, \quad \partial \bar{\partial} \gamma = 0.$$

Summing up, we showed that if  $\psi = f\omega + \gamma \in \mathcal{H}_A^{1,1}$ , where  $f \in C^\infty(X, \mathbb{C})$  and  $\gamma \in P^{1,1}$ , then  $f \in \mathbb{C}$  and  $\gamma \in \mathcal{H}_A^{1,1}$  proving the inclusion  $\subseteq$  of the statement. The converse inclusion  $\supseteq$  is trivial, therefore the theorem is proved.  $\square$

As a corollary of the previous results we obtain also the following decompositions of the spaces of  $(n-1, n-1)$  Bott–Chern and Aeppli harmonic forms.

**Theorem 3.4** *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension  $2n$ . Then, the following decompositions hold*

$$\mathcal{H}_{BC}^{n-1, n-1} = \mathbb{C} \omega^{n-1} \oplus L^{n-2} \left( \mathcal{H}_A^{1,1} \cap P^{1,1} \right),
 \tag{12}$$

$$\mathcal{H}_A^{n-1, n-1} = \mathbb{C} \omega^{n-1} \oplus L^{n-2} \left( \mathcal{H}_{BC}^{1,1} \cap P^{1,1} \right).
 \tag{13}$$

**Proof** Decompositions (12) and (13) follow respectively from the decompositions of Theorems 3.3 and 3.2 applying the Hodge star operator and using formulae (7) and (4).  $\square$

From Theorems 3.2, 3.3 and 3.4, it immediately follows

**Corollary 3.5** *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension 4. Then,*

$$\mathcal{H}_{BC}^{1,1} = \mathcal{H}_A^{1,1} = \mathbb{C} \omega \oplus \left( \mathcal{H}_{BC}^{1,1} \cap P^{1,1} \right).$$

Combining Corollary 3.5 with [7, Proposition 6.1] and [9, Corollary 4.4], we derive

**Corollary 3.6** *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension 4. Then,*

$$\mathcal{H}_d^{1,1} = \mathcal{H}_\partial^{1,1} = \mathcal{H}_{\bar{\partial}}^{1,1} = \mathcal{H}_{BC}^{1,1} = \mathcal{H}_A^{1,1}.$$

These last two results will be generalized in Proposition 4.2.

**Remark 3.7** Notice that, in fact, on 4-dimensional almost Hermitian manifolds the primitive decomposition of  $\mathcal{H}_{BC}^{1,1}$  was proved in [9, Corollary 4.4], where it is shown that

$$\mathcal{H}_{BC}^{1,1} \cap P^{1,1} = \{\alpha \in A^{1,1} : \Delta_d \alpha = 0, * \alpha = -\alpha\}.$$

### 4 Relations among the spaces of primitive harmonic forms

We saw that the spaces of primitive Bott–Chern and Aeppli harmonic forms are important in Theorems 3.2, 3.3 and 3.4. Moreover, the spaces of primitive  $\partial$ - and Dolbeault harmonic forms play a similar role as shown in [1]. Let us then study the possible inclusions and non inclusions between these spaces.

**Proposition 4.1** *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension  $2n$ . Then, for  $p + q \leq n$ ,*

$$\mathcal{H}_{BC}^{p,q} \cap P^{p,q} = \mathcal{H}_{\bar{\partial}}^{p,q} \cap \mathcal{H}_\partial^{p,q} \cap P^{p,q}, \tag{14}$$

$$\mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q} \subseteq \mathcal{H}_A^{p,q} \cap P^{p,q}. \tag{15}$$

In particular,

$$\mathcal{H}_{BC}^{p,q} \cap P^{p,q} \subseteq \mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q},$$

$$\mathcal{H}_{BC}^{p,q} \cap P^{p,q} \subseteq \mathcal{H}_\partial^{p,q} \cap P^{p,q},$$

$$\mathcal{H}_{BC}^{p,q} \cap P^{p,q} \subseteq \mathcal{H}_A^{p,q} \cap P^{p,q}.$$

**Proof** We start by showing (14). Let  $\alpha \in \mathcal{H}_{\bar{\partial}}^{p,q} \cap \mathcal{H}_\partial^{p,q} \cap P^{p,q}$ , i.e.,

$$\partial \alpha = \bar{\partial} \alpha = \partial * \alpha = \bar{\partial} * \alpha = 0.$$

Hence,  $\partial \alpha = \bar{\partial} \alpha = \partial \bar{\partial} * \alpha = 0$  and so  $\alpha \in \mathcal{H}_{BC}^{p,q}$ . Notice that in fact we did not use that  $\alpha$  is a primitive form. Viceversa, let  $\alpha \in \mathcal{H}_{BC}^{p,q} \cap P^{p,q}$ , i.e.,

$$\partial \alpha = \bar{\partial} \alpha = \partial \bar{\partial} * \alpha = 0.$$

Since  $\alpha$  is primitive,  $* \alpha = c \omega^{n-p-q} \wedge \alpha$ , with  $c = \frac{(-1)^{\frac{(p+q)(p+q+1)}{2}}}{(n-p-q)!} i^{p-q}$  and since  $\omega$  is  $d$ -closed we have

$$\partial * \alpha = c \partial(\omega^{n-p-q} \wedge \alpha) = c \omega^{n-p-q} \wedge \partial \alpha = 0$$

and similarly

$$\bar{\partial} * \alpha = c \bar{\partial}(\omega^{n-p-q} \wedge \alpha) = c \omega^{n-p-q} \wedge \bar{\partial} \alpha = 0$$

so  $\alpha \in \mathcal{H}_{\bar{\partial}}^{p,q} \cap \mathcal{H}_\partial^{p,q} \cap P^{p,q}$ .

Now we show (15). Let  $\alpha \in \mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q}$ , i.e.,

$$\bar{\partial} \alpha = \partial * \alpha = 0.$$

In particular  $\partial\bar{\partial}\alpha = \partial * \alpha = 0$ . To prove  $\bar{\partial} * \alpha = 0$ , which implies  $\alpha \in \mathcal{H}_A^{p,q} \cap P^{p,q}$ , recall that  $*\alpha = c \omega^{n-p-q} \wedge \alpha$ , and again since  $\omega$  is  $d$ -closed we have

$$\bar{\partial} * \alpha = c \bar{\partial}(\omega^{n-p-q} \wedge \alpha) = c \omega^{n-p-q} \wedge \bar{\partial}\alpha = 0. \quad \square$$

In fact, without any assumption on the almost Hermitian metric, when the total degree of the forms is half the dimension, all the spaces of primitive harmonic forms coincide.

**Proposition 4.2** *Let  $(X, J, \omega)$  be a compact almost Hermitian manifold of dimension  $2n$ . Then, for  $p + q = n$ ,*

$$\mathcal{H}_{BC}^{p,q} \cap P^{p,q} = \mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q} = \mathcal{H}_{\partial}^{p,q} \cap P^{p,q} = \mathcal{H}_A^{p,q} \cap P^{p,q}.$$

**Proof** Let  $\alpha \in P^{p,q}$  for  $p + q = n$ , then by Formula (7)  $*\alpha = c_{p,q}\alpha$ , with  $c_{p,q} = (-1)^{\frac{n(n+1)}{2}} i^{p-q}$ . Therefore,

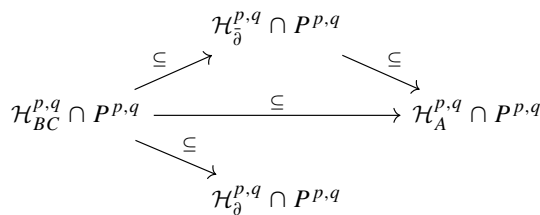
$$\partial\alpha = 0 \iff \partial * \alpha = 0 \iff \bar{\partial} * \alpha = 0$$

and

$$\bar{\partial}\alpha = 0 \iff \bar{\partial} * \alpha = 0 \iff \partial * \alpha = 0.$$

The equalities follow then directly from the definitions. For instance, to prove  $\mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q} \subseteq \mathcal{H}_{BC}^{p,q} \cap P^{p,q}$ , let  $\alpha \in \mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q}$ , then  $\bar{\partial}\alpha = 0$  and  $\partial * \alpha = 0$ . Hence, by the previous observation one has also that  $\bar{\partial} * \alpha = 0$  and  $\partial\alpha = 0$ , giving in particular that  $\partial\bar{\partial} * \alpha = 0$ ,  $\bar{\partial}\alpha = 0$  and  $\partial\alpha = 0$  which means that  $\alpha \in \mathcal{H}_{BC}^{p,q} \cap P^{p,q}$ . Viceversa, to prove  $\mathcal{H}_{BC}^{p,q} \cap P^{p,q} \subseteq \mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q}$ , let  $\alpha \in \mathcal{H}_{BC}^{p,q} \cap P^{p,q}$ , then  $\bar{\partial}\alpha = 0$ ,  $\partial\alpha = 0$  and  $\partial\bar{\partial} * \alpha = 0$ . Hence, by the previous observation one has also that  $\partial * \alpha = 0$  which means that  $\alpha \in \mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q}$ . The other inclusions can be proved similarly.  $\square$

For  $p + q < n$ , the inclusions of Proposition 4.1 can be summed up in the following diagram.



Notice that even though we have the inclusion  $\mathcal{H}_{\bar{\partial}}^{p,q} \cap P^{p,q} \subseteq \mathcal{H}_A^{p,q} \cap P^{p,q}$ , we cannot expect a similar inclusion for  $\mathcal{H}_{\partial}^{p,q} \cap P^{p,q}$  in  $\mathcal{H}_A^{p,q} \cap P^{p,q}$ . Indeed, since  $\partial$  and  $\bar{\partial}$  do not anticommute, the condition  $\partial\alpha = 0$  does not imply  $\partial\bar{\partial}\alpha = 0$ . See Proposition 4.4 for more details.

Combining Proposition 4.1 for  $(p, q) = (1, 1)$  with Theorems 3.2, 3.3 and 3.4 we obtain the following

**Corollary 4.3** *Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension  $2n$ . Then,*

$$\mathcal{H}_{BC}^{1,1}(X) \subseteq \mathcal{H}_A^{1,1}(X),$$

and

$$\mathcal{H}_A^{n-1,n-1}(X) \subseteq \mathcal{H}_{BC}^{n-1,n-1}(X).$$

Now we study whether the inclusions in Proposition 4.1 are strict, and in general if there are other inclusions between the spaces of primitive harmonic forms.

**Proposition 4.4** *There exists a compact almost Kähler 6-manifold  $(X, J, \omega)$  such that*

$$\mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1} \not\subseteq \mathcal{H}_{BC}^{1,1} \cap P^{1,1}, \tag{16}$$

$$\mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1} \not\subseteq \mathcal{H}_{\partial}^{1,1} \cap P^{1,1}, \tag{17}$$

$$\mathcal{H}_A^{1,1} \cap P^{1,1} \not\subseteq \mathcal{H}_{BC}^{1,1} \cap P^{1,1}, \tag{18}$$

$$\mathcal{H}_A^{1,1} \cap P^{1,1} \not\subseteq \mathcal{H}_{\partial}^{1,1} \cap P^{1,1}, \tag{19}$$

$$\mathcal{H}_{\partial}^{1,1} \cap P^{1,1} \not\subseteq \mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1}, \tag{20}$$

$$\mathcal{H}_{\partial}^{1,1} \cap P^{1,1} \not\subseteq \mathcal{H}_{BC}^{1,1} \cap P^{1,1}, \tag{21}$$

$$\mathcal{H}_{\partial}^{1,1} \cap P^{1,1} \not\subseteq \mathcal{H}_A^{1,1} \cap P^{1,1}. \tag{22}$$

**Proof** We refer to Example 4.5 for the proof of this Proposition. □

**Example 4.5** We recall the following construction of [1]. Let  $X = \mathbb{T}^6 = \mathbb{Z}^6 \backslash \mathbb{R}^6$  be the 6-dimensional torus with  $(x^1, x^2, x^3, y^1, y^2, y^3)$  coordinates on  $\mathbb{R}^6$ . Let  $g = g(x^3, y^3)$  be a non-constant function on  $\mathbb{T}^6$ . We define an almost complex structure  $J$  setting as global co-frame of  $(1, 0)$ -forms

$$\varphi^1 := e^g dx^1 + i e^{-g} dy^1, \quad \varphi^2 := dx^2 + i dy^2, \quad \varphi^3 := dx^3 + i dy^3.$$

The structure equations are

$$d\varphi^1 = V_3(g)\varphi^{3\bar{1}} - \bar{V}_3(g)\varphi^{\bar{1}3}, \quad d\varphi^2 = d\varphi^3 = 0,$$

where  $\{V_1, V_2, V_3\}$  denotes the global frame of vector fields dual to  $\{\varphi^1, \varphi^2, \varphi^3\}$ . Notice that in particular  $J$  is not integrable. Then, the  $(1, 1)$ -form

$$\omega := \frac{i}{2}\varphi^{1\bar{1}} + \frac{i}{2}\varphi^{2\bar{2}} + \frac{i}{2}\varphi^{3\bar{3}}$$

is a compatible symplectic structure, namely  $(J, \omega)$  is an almost Kähler structure on  $\mathbb{T}^6$ .

We will show that the form  $\varphi^{2\bar{1}}$  verifies claims (16), (17), (18), (19), while the form  $\varphi^{1\bar{2}}$  verifies (20), (21), (22).

First, note that

$$\bar{\partial}\varphi^{2\bar{1}} = 0$$

and

$$\partial * \varphi^{2\bar{1}} = -\omega \wedge \partial\varphi^{2\bar{1}} = \overline{V_3(g)}\omega \wedge \varphi^{12\bar{3}} = 0,$$

thus

$$\varphi^{2\bar{1}} \in \mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1} \subseteq \mathcal{H}_A^{1,1} \cap P^{1,1}.$$

On the other hand

$$\partial\varphi^{2\bar{1}} = -\overline{V_3(g)}\varphi^{12\bar{3}} \neq 0$$

and so

$$\varphi^{2\bar{1}} \notin \mathcal{H}_{\partial}^{1,1} \cap P^{1,1} \supseteq \mathcal{H}_{BC}^{1,1} \cap P^{1,1}.$$

This proves (16), (17), (18), (19).

Now, note that

$$\partial\varphi^{1\bar{2}} = 0$$

and

$$\bar{\partial} * \varphi^{1\bar{2}} = -\omega \wedge \bar{\partial}\varphi^{1\bar{2}} = -V_3(g)\omega \wedge \varphi^{3\bar{1}\bar{2}} = 0,$$

thus

$$\varphi^{1\bar{2}} \in \mathcal{H}_\partial^{1,1} \cap P^{1,1}.$$

On the other hand

$$\partial\bar{\partial}\varphi^{1\bar{2}} = \partial(V_3(g)\varphi^{3\bar{1}\bar{2}}) = -V_3(g)\overline{V_3(g)}\varphi^{3\bar{3}1\bar{2}} \neq 0$$

and so

$$\varphi^{1\bar{2}} \notin \mathcal{H}_A^{1,1} \cap P^{1,1} \supseteq \mathcal{H}_\partial^{1,1} \cap P^{1,1} \supseteq \mathcal{H}_{BC}^{1,1} \cap P^{1,1}.$$

This proves (20), (21), (22). □

Combining Propositions 4.1 and 4.4 one finds the following diagram of strict inclusions.

$$\begin{array}{ccc}
 & \mathcal{H}_\partial^{p,q} \cap P^{p,q} & \\
 \swarrow \subsetneq & & \searrow \subsetneq \\
 \mathcal{H}_{BC}^{p,q} \cap P^{p,q} & \xrightarrow{\subsetneq} & \mathcal{H}_A^{p,q} \cap P^{p,q} \\
 \searrow \subsetneq & & \\
 & \mathcal{H}_\partial^{p,q} \cap P^{p,q} & 
 \end{array} \tag{23}$$

The remaining non-inclusions of Propositions 4.4 (which are not already included in diagram (23)) can be summed up in the following diagram.

$$\begin{array}{ccc}
 & \mathcal{H}_\partial^{p,q} \cap P^{p,q} & \\
 & \updownarrow \not\subset & \\
 \mathcal{H}_{BC}^{p,q} \cap P^{p,q} & & \mathcal{H}_A^{p,q} \cap P^{p,q} \\
 & \downarrow \not\subset & \swarrow \not\subset \\
 & \mathcal{H}_\partial^{p,q} \cap P^{p,q} & \nwarrow \not\subset
 \end{array} \tag{24}$$

It remains open to understand if  $\mathcal{H}_A^{p,q} \cap P^{p,q}$  is either contained or not in  $\mathcal{H}_\partial^{p,q} \cap P^{p,q}$  in general.

**Remark 4.6** We notice that in [11] the spaces of Bott–Chern and Aeppli harmonic forms were introduced using the operators  $\bar{\delta}$  and  $\delta$ . In fact, it was shown that with respect to such operators, on compact almost Kähler manifolds one has the usual equalities (that are true for Kähler manifolds), namely (cf. [11, Proposition 6.2, Theorem 6.7, Proposition 6.10, Corollary 6.12])

$$\mathcal{H}_{BC(\delta, \bar{\delta})}^{p,q}(X) = \mathcal{H}_{A(\delta, \bar{\delta})}^{p,q}(X) = \mathcal{H}_{\bar{\delta}}^{p,q}(X) = \mathcal{H}_\delta^{p,q}(X) = \mathcal{H}_d^{p,q}(X).$$

### 5 Primitive decompositions of harmonic forms in dimension 6

Let  $(X, J, \omega)$  be a compact almost Kähler manifold of dimension  $2n$ . In Sect. 3 we saw that the primitive decompositions of  $(p, q)$ -forms descend to Bott–Chern and Aeppli harmonic forms for the special bidegrees  $(1, 1)$ ,  $(p, 0)$  and  $(0, q)$ . By Bott–Chern and Aeppli duality, we saw that we can also deduce primitive decompositions for the bidegrees  $(n - 1, n - 1)$ ,  $(n, n - p)$  and  $(n - q, n)$ . But do these decompositions hold only for these special bidegrees? Are there other bidegrees with nice primitive decompositions of the spaces of Bott–Chern and Aeppli harmonic forms? For  $2n = 2, 4$ , since the previous bidegrees are all the possible bidegrees, the situation is well understood. Therefore, to answer our question, we should investigate what happens in the dimension  $2n = 6$ .

If  $2n = 6$ , then the only bidegrees for which we do not still have primitive decompositions of the spaces of Bott–Chern and Aeppli harmonic forms are  $(2, 1)$ ,  $(1, 2)$ . Let us focus on the bidegree  $(2, 1)$ . The primitive decomposition of forms reads as

$$A^{2,1} = P^{2,1} \oplus L(A^{1,0}).$$

Passing to Bott–Chern harmonic forms, it is immediate to see that

$$\mathcal{H}_{BC}^{2,1} \supseteq \left(\mathcal{H}_{BC}^{2,1} \cap P^{2,1}\right) \oplus L\left(\mathcal{H}_{BC}^{1,0}\right). \tag{25}$$

However, for Aeppli harmonic forms, a similar inclusion does not hold, because in general

$$L\left(\mathcal{H}_A^{1,0}\right) \not\subseteq \mathcal{H}_A^{2,1}.$$

Indeed, let  $\alpha \in \mathcal{H}_A^{1,0}$ . For bidegree reasons,  $\partial\bar{\partial}\alpha = 0$  and  $\bar{\partial} * \alpha = 0$  or, equivalently, since  $\alpha$  is primitive,  $\partial\bar{\partial}\alpha = 0$  and  $\omega^2 \wedge \bar{\partial}\alpha = 0$ . Note that this does not imply, in general, that  $L\alpha = \omega \wedge \alpha \in \mathcal{H}_A^{2,1}$ , indeed we cannot conclude that  $\partial * (\omega \wedge \alpha) = -i\omega \wedge \partial\alpha$  is equal to zero. Therefore, we will focus only on Bott–Chern harmonic forms. At this point, we could hope that the inclusion of (25) is indeed an identity. In fact, it does not happen, as it is shown by the following

**Proposition 5.1** *There exists a compact almost Kähler 6-dimensional manifold  $(X, J, \omega)$  such that*

$$\mathcal{H}_{BC}^{2,1} \neq \left(\mathcal{H}_{BC}^{2,1} \cap P^{2,1}\right) \oplus L\left(\mathcal{H}_{BC}^{1,0}\right)$$

**Proof** We refer to Example 5.2 for the proof of this Proposition. □

**Example 5.2** Let  $X := \mathbb{Z}[i]^3 \setminus (\mathbb{C}^3, \cdot)$  be the Iwasawa manifold, where the group structure on  $\mathbb{C}^3$  is defined by

$$(w^1, w^2, w^3) \cdot (z^1, z^2, z^3) = (w^1 + z^1, w^2 + z^2, w^3 + w^1 z^2 + z^3).$$

The standard complex structure of  $\mathbb{C}^3$  induces, on  $X$ , the complex structure given by

$$\psi^1 = dz^1, \quad \psi^2 = dz^2, \quad \psi^3 = -z^1 dz^2 + dz^3$$

being a global coframe of  $(1, 0)$ -forms. The complex structure equations are

$$d\psi^1 = 0, \quad d\psi^2 = 0, \quad d\psi^3 = -\psi^{12}.$$

If we set

$$\psi^1 = e^1 + ie^2, \quad \psi^2 = e^3 + ie^4, \quad \psi^3 = e^5 + ie^6,$$

then the real structure equations are

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = -e^{13} + e^{24}, \quad de^6 = -e^{14} - e^{23}.$$

Let us consider the non integrable left-invariant almost complex structure  $J$  given by

$$\varphi^1 = e^1 + ie^6, \quad \varphi^2 = e^2 + ie^5, \quad \varphi^3 = e^3 + ie^4$$

being a global coframe of  $(1, 0)$ -forms. By a direct computation the structure equations become (cf. also [13])

$$\begin{aligned} 4d\varphi^1 &= -\varphi^{13} - i\varphi^{23} + \varphi^{1\bar{3}} + \varphi^{3\bar{1}} - i\varphi^{2\bar{3}} + i\varphi^{3\bar{2}} + \varphi^{\bar{1}3} - i\varphi^{\bar{2}3}, \\ 4d\varphi^2 &= -i\varphi^{13} + \varphi^{23} - i\varphi^{1\bar{3}} + i\varphi^{3\bar{1}} - \varphi^{2\bar{3}} - \varphi^{3\bar{2}} - i\varphi^{\bar{1}3} - \varphi^{\bar{2}3}, \\ d\varphi^3 &= 0. \end{aligned}$$

Endow  $(X, J)$  with the left-invariant almost Kähler structure given by

$$\omega = 2(e^{16} + e^{25} + e^{34}) = i(\varphi^{1\bar{1}} + \varphi^{2\bar{2}} + \varphi^{3\bar{3}}).$$

First, we do the following observation that will allow us to work with only left-invariant forms (cf. [1, Lemma 5.2]). Take  $\eta \in A^{2,1}$  and assume it is left-invariant. By (5), it follows that

$$\eta = \alpha + L\beta,$$

with  $\alpha \in A^{2,1}$  primitive, i.e.,  $L\alpha = 0$  and  $\beta \in A^{1,0}$  ( $\beta$  is in fact primitive for bidegree reasons). We apply  $L$  and find  $L\eta = L^2\beta$ . Note that  $L\eta$ , and so  $L^2\beta$ , are left-invariant, and that  $L^2 : \Lambda^1 X \rightarrow \Lambda^5 X$  is an isomorphism at the level of the exterior algebra. Therefore, also  $\beta$  is left-invariant. Now, since  $L\beta$  and  $\eta$  are left-invariant, it follows that also  $\alpha$  is left-invariant. Summing up, if  $\eta \in A^{2,1}$  is left-invariant,  $\eta = \alpha + L\beta$  and  $L\alpha = 0$ , then  $\alpha$  and  $\beta$  are left-invariant, too.

We want to find an element  $\eta \in A^{2,1}$  which is contained in  $\mathcal{H}_{BC}^{2,1}$  but is not contained in

$$\left(\mathcal{H}_{BC}^{2,1} \cap P^{2,1}\right) \oplus L\left(\mathcal{H}_{BC}^{1,0}\right).$$

Thanks to the previous argument, if  $\eta \in \mathcal{H}_{BC}^{2,1}$  is left-invariant and  $\eta = \alpha + L\beta$ , with  $\alpha \in \mathcal{H}_{BC}^{2,1} \cap P^{2,1}$  and  $\beta \in \mathcal{H}_{BC}^{1,0}$ , then  $\alpha$  and  $\beta$  are left-invariant.

A long, but direct and straightforward computation, shows that the space of left-invariant Bott–Chern harmonic  $(2, 1)$ -forms is

$$\mathbb{C} \langle \varphi^{13\bar{1}} + \varphi^{23\bar{2}}, \varphi^{13\bar{2}} + \varphi^{23\bar{1}} - 2i\varphi^{23\bar{2}} \rangle,$$

while it is easy to verify that the space of left-invariant forms which are contained in  $L\left(\mathcal{H}_{BC}^{1,0}\right)$  is

$$\mathbb{C} \langle \varphi^{13\bar{1}} + \varphi^{23\bar{2}} \rangle.$$

Since  $L(\varphi^{13\bar{2}} + \varphi^{23\bar{1}} - 2i\varphi^{23\bar{2}}) = -2iL(\varphi^{23\bar{2}}) \neq 0$ , it means that  $\varphi^{13\bar{2}} + \varphi^{23\bar{1}} - 2i\varphi^{23\bar{2}}$  is not primitive. Therefore  $\varphi^{13\bar{2}} + \varphi^{23\bar{1}} - 2i\varphi^{23\bar{2}}$  is a left-invariant, Bott–Chern harmonic  $(2, 1)$ -form, but it is not contained in

$$\left(\mathcal{H}_{BC}^{2,1} \cap P^{2,1}\right) \oplus L\left(\mathcal{H}_{BC}^{1,0}\right).$$



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## References

1. Cattaneo, A., Tardini, N., Tomassini, A.: Primitive decompositions of Dolbeault harmonic forms on compact almost-Kähler manifolds. *Rev. Union Mat. Argent.* (2022). [arXiv:2201.09273](https://arxiv.org/abs/2201.09273)
2. Cirici, J., Wilson, S.O.: Dolbeault cohomology for almost complex manifolds. *Adv. Math.* **391** (2021)
3. Cirici, J., Wilson, S.O.: Topological and geometric aspects of almost Kähler manifolds via harmonic theory. *Sel. Math. New Ser.* **26**(35) (2020)
4. Coelho, R., Placini, G., Stelzig, J.: Maximally non-integrable almost complex structures: an  $h$ -principle and cohomological properties. *Sel. Math. New Ser.* **28**(5), 83, 25 (2022)
5. Hirzebruch, F.: Some problems on differentiable and complex manifolds. *Ann. Math.* **60**(2), 213–236 (1954)
6. Holt, T.: Bott-Chern and  $\bar{\partial}$  Harmonic Forms on Almost Hermitian 4-manifolds. *Math. Z.* **302**(1), 47–72 (2022)
7. Holt, T., Zhang, W.: Harmonic forms on the Kodaira-Thurston manifold. *Adv. Math.* **400**(108277), 30 (2022)
8. Holt, T., Zhang, W.: Kodaira-Spencer, Almost Kähler., problem. *Math. Res. Lett.* (2021). [arXiv:2010.12545](https://arxiv.org/abs/2010.12545)
9. Piovani, R., Tomassini, A.: Bott-Chern laplacian on almost Hermitian manifolds. *Math. Z.* **301**(3), 2685–2707 (2022)
10. Piovani, R., Tomassini, A.: On the dimension of Dolbeault harmonic  $(1, 1)$ -forms on almost Hermitian 4-manifolds. *Pure Appl. Math. Q.* **18**(3), 1187–1201 (2022)
11. Tardini, N., Tomassini, A.: Differential operators on almost-Hermitian manifolds and harmonic forms. *Complex Manifolds* **7**(1), 106–128 (2020)
12. Tardini, N., Tomassini, A.:  $\bar{\partial}$ -harmonic forms on 4-dimensional almost-hermitian manifolds. *Math. Res. Lett* (2021). [arXiv:2104.10594](https://arxiv.org/abs/2104.10594)
13. Tardini, N., Tomassini, A.: Almost-complex invariants of families of six-dimensional solvmanifolds. *Complex Manifolds* **9**(1), 238–260 (2022)
14. Weil, A.: Introduction à l'Étude des Variétés kähéliennes, p. 175. Publications de l'Institut de Mathématique de l'Université de Nancago VI, Hermann, Paris (1958)

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