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Dynamics of complex bodies with memory and vector-valued microstructure: Existence and uniqueness

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ABSTRACT

We consider bodies with active microstructure described by a vector field with values $\nu \in \mathbb{R}^3$. It complements the macroscopic displacement $u \in \mathbb{R}^3$. We prove existence and uniqueness for the dynamics of such a body under memory effects and a nonlinear microstructural behavior.

1. Introduction

Memory is non-locality in time. Attention to it dates back to L. Boltzmann's and V. Volterra's independent works. An extensive systematic treatment of memory effects for bodies undergoing a large strain regime has been developed in the sixties of the twentieth century (see, e.g., [1–7]). A reprisal of foundational work is evident in the ten years of the twenty-first century, above all for a characterization of free energies (see, e.g., [8–10]). Materials considered are commonly those with response depending on the deformation gradient history. Actions are local in space, distinguished into bulk and contact families. In short, they are what we call *Cauchy's bodies*.

However, physico-chemical phenomena in materials may develop at different time scales when events at different spatial scales are accounted for. Including memory effects in a continuum representation of the mechanical behavior is an indirect way to account for delays in the different scale dynamics.

In particular, here it is aimed to describe micro-to-macro interactions and by the use of order parameter fields to bring at the macro scale information of microscopic events; introducing memory effects connected with these fields is a way to account for the possibility that at micro-scale events occur with different dynamics. Also, depending on the kernel choice, we may describe the influence of critical events in the past, as an abrupt occurrence of damage in a specific time interval.

The analysis of memory effects in the general model-building framework for the mechanics of complex bodies, i.e., those with active microstructure strongly influencing the gross behavior, has been proposed in Ref. [11] with respect to the definition of free energies in linearized setting. In this framework the body morphology is represented not only by the region the body occupies in 3D space. Fields taking values ν into a manifold complement the deformation maps [12]. They bring at gross scale information on the microstructural events characterizing the behavior of a complex body. Specific examples are as follows:

- **Example 1**: In a body made of polymer linear chains embedded into a melt, ν may be taken as the end-to-tail vector of a single molecule attributed to x (or y), when the molecule is schematized as a dumbbell [13–15].
- **Example 2**: In quasicrystals ν describes locally inner degrees of freedom exploited to assure the quasi-periodic arrangement of atoms characterizing quasicrystals [16–18]. Indeed a quasi-periodic arrangement of atoms in (say) 3*D* space (atoms viewed as mass points) can be obtained as the projection of a six-dimensional periodic lattice onto a 3*D* *incommensurate* subspace. The standard displacement in the 6*D* hyperspace has a component along the chosen subspace (the so-called phonon field because it is associated with the propagation of acoustic waves), which is here u, and an orthogonal one (the so-called phason field), which is

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here ν. The phason field is (obviously) insensitive to rigid translations of observers in the chosen subspace (i.e., the physical space) where we project the 6D periodic lattice, because it is orthogonal to that space (such a remark is an example clarifying choices below made).
- **Example 3**: A vector phase field is also appropriate to describe the microstructure of diatomic-type bodies [19]. Locally, every ν represents a vector connecting the two atoms imagined at that point. It plays a role analogous to the end-to-tail vector in Example 1.

Bulk and contact peculiar actions (those with microstructural character) are defined by the power that they perform in the time rate of these microstructure descriptors and satisfy appropriate balance equations derived from invariance principles [20,21]. Here, we refer to that approach and consider the dynamics of a complex body with memory. Its microstructure is described by a vector-valued field ν, with $\nu(t,x) \in \mathbb{R}^3$. It supplements the common displacement field $u = u(t,x) \in \mathbb{R}^3$. With $\mathcal{B} \subset \mathbb{R}^3$ a bounded, simply connected, fit region endowed with Lipschitz boundary $\partial\mathcal{B}$, working in small-strain setting, we consider in $[0,T]\times\mathcal{B}$ the following balance equations of forces and microstructural actions:

$$\begin{aligned} &\rho u_{tt} + \tilde{\gamma} u_t - \mu \Delta u + \lambda_1 \int_0^t \varpi(t-s)\Delta u(s)ds \\ &\quad = \xi \nabla \mathrm{div} u + \kappa \Delta \nu + \bar{\xi}\nabla \mathrm{div}\nu, \end{aligned} \tag{1}$$

$$\begin{aligned} &\varsigma \nu_t - \zeta \Delta \nu + \hat{\eta}|\nu|^{\sigma-1}\nu - \lambda_2 \int_0^t \widetilde{\varpi}(t-s)\nu(s)ds \\ &\quad = \gamma \nabla \mathrm{div}\nu + \kappa \Delta u + \bar{\xi}\nabla \mathrm{div} u - \kappa_0 \nu\ . \end{aligned} \tag{2}$$

The subscript t indicates time derivative. They involve linear constitutive equations and a non-linearity $B(\nu) := \hat{\eta}|\nu|^{\sigma-1}\nu$ in the microstructural self-action structure (for the existence of such an action see [20]). We require $2 \le \sigma \le 3$. Also, we take the constitutive parameters λ_1, λ_2, $\hat{\eta}$, μ, ξ, κ, $\bar{\xi}$, ς, ζ, γ and κ_0 to be positive, and $\mu > \lambda_1$; ρ is the mass density. To characterize memory effects, we take positive kernels $\varpi = \varpi(t)$ and $\widetilde{\varpi} = \widetilde{\varpi}(t)$ (positiveness defined in Section 2, see (5)) and, in particular, we assume $\varpi(t) = \widetilde{\varpi}(t) = \check{\gamma}e^{-\delta t}$, $\delta > 0$ with $\check{\gamma} > 0$. We restrict attention to the case

$$\begin{aligned} &\check{\gamma} < 1, \quad \kappa_0 > 2\lambda_2\check{\gamma}/\delta, \quad \zeta > 2(\kappa + 2\bar{\xi}), \\ &\text{and} \quad \mu > 2(\lambda_1\check{\gamma}/\delta + \kappa + \bar{\xi}). \end{aligned} \tag{3}$$

We adopt initial conditions $u|_{t=0} = u_0$, $u_t|_{t=0} = \dot{u}_0$, $\nu|_{t=0} = \nu_0$, on $\mathcal{B}$ and slip-without-friction-like boundary conditions, given by

$$\begin{aligned} &u \cdot n = 0,\ \ \mathrm{curl}\, u \times n = 0, \ \ \text{on } [0,T]\times\partial\mathcal{B} \\ &\text{and} \quad \nu \cdot n = 0,\ \ \mathrm{curl}\, \nu \times n = 0, \ \ \text{on } [0,T]\times\partial\mathcal{B}\ . \end{aligned} \tag{4}$$

In the balance of momentum (1) the terms $\mu\Delta u$ and $\xi\nabla\mathrm{div}u$ describe the present contribution of the macroscopic stress, while $\kappa\Delta\nu + \bar{\xi}\nabla\mathrm{div}\nu$ the direct influence on the occurrence of stress due to the microstructural effects, a coupling term; the time integral describes the influence of memory on the Cauchy's stress; $\tilde{\gamma}u_t$ is a live load.

In the balance of micro-momentum (2), the terms $\varsigma\nu_t + \hat{\eta}|\nu|^{\sigma-1}\nu$ and $\kappa_0\nu$ report cumulatively the present value of a microstructural self-action (i.e., the one of microstructure on itself), and in particular $\varsigma\nu_t$ is a self-action dissipative component; the time integral accounts for the effects of memory on this type of action, the only ones considered at microstructural level; $\zeta\Delta\nu$ and $\gamma\nabla\mathrm{div}\nu$ account for the microstress due to the microstructure spatial variations while $\kappa\Delta u + \bar{\xi}\nabla\mathrm{div}u$ is a macro-to-micro coupling term. The dynamics of a memory-less version of the present scheme has been discussed in Ref. [22], where the existence of a pertinent weak attractor is shown even in the limit $\tilde{\gamma} = 0$.

2. Deriving Eqs. (1) and (2) from a fully nonlinear setting

2.1. Morphology of bodies with vector-valued microstructure and motions

Consider two isomorphic copies of the three-dimensional real space, namely $\mathbb{R}^3$ and $\tilde{\mathbb{R}}^3$. The isomorphism $\iota : \mathbb{R}^3 \longrightarrow \tilde{\mathbb{R}}^3$ is simply the identification. We select a fit region $\mathcal{B}$ in $\mathbb{R}^3$ as reference (macroscopic) shape of a body (i.e., it is a bounded, simply connected region with surface-like boundary, oriented by the outward unit normal n everywhere to within a finite number of corners and edges), and detect in $\tilde{\mathbb{R}}^3$ all those shapes $\mathcal{B}_a$ that we consider deformed in time with respect to the reference one through time-dependent deformation mappings $(t,x) \longmapsto y := \tilde{y}(t,x) \in \tilde{\mathbb{R}}^3$, $x \in \mathcal{B}$, taken to be differentiable (twice with respect to time) and orientation preserving, so that $\mathcal{B}_a := \tilde{y}(\mathcal{B},t)$. We will write ∇y for $\nabla\tilde{y}(x,t)$. As usual, we define the *displacement field* u as $u(t,x) := \tilde{y}(t,x) - \iota(x)$. It is a Lagrangian field, i.e., one defined over the reference shape.

The condition $|\nabla u| \leqq 1$ defines the *small deformation regime*. When it holds, we avoid distinguishing between reference and current configurations. However, in this section we maintain the distinction and act in the more general setting including large strains because in this way we may have a more clear picture of the scenario in which our analysis is embedded.

The Lagrangian vector field $\dot{y} := \frac{d\tilde{y}(t,x)}{dt}$, which is also such that $\dot{y} = u_t$, indicates the (macroscopic) velocity. At each point x and time t, $\dot{y} = u_t$ belongs to the tangent space to $\mathcal{B}_a$ at $y = \tilde{y}(t,x)$. Consequently, we can consider the velocity field as an Eulerian field, i.e., one defined over $\mathcal{B}_a$, a field that we can write as $(t,y) \longmapsto v := \tilde{v}(t,y)$ and we obviously have $\dot{y} = u_t = v$.

We describe the minute architecture of the material, its microstructure at a certain spatial scale, through a *descriptor field* (a phase field, if one prefers another common nomenclature) taken here to be a 3D differentiable vector field $(t,x) \longmapsto \nu := \tilde{\nu}(t,x) \in \bar{\mathbb{R}}^3$ in Lagrangian representation ($\bar{\mathbb{R}}^3$ is a copy of the 3D real space in principle distinguished, although isomorphic to $\mathbb{R}^3$). We will write $\nabla\nu$ for $\nabla\tilde{\nu}(t,x)$. The map $\tilde{\nu}$ admits an Eulerian representation given by $\bar{\nu} := \tilde{\nu}\circ\tilde{y}^{-1}$. Once again, when we refer to small strain setting, we will not distinguish between the two representation, because (as it common in this setting) we do not distinguish between $\mathcal{B}$ and $\mathcal{B}_a$. However, for the moment, we maintain as above the distinction and will write $\dot{\nu}$ for the time rate of $\tilde{\nu}$, which we will rewrite as ν_t when referring to the small strain setting.

Notice: ν represents some geometric property pertaining to the material element that we attribute to x (or y) in the continuum representation. (Think that we are acting in the sense of field theories.) Examples have been already listed in the Introduction.

Other examples can be listed, as for example the one of elastic microcracked bodies (see, e.g., [23]) but those listed above as sufficiently significant so that we do not dwell further. Of course that ν is a 3D real vector here is a special choice. The general model-building framework for the mechanics of complex bodies foresees that ν should be read in general over a finite-dimensional differentiable manifold [12,20]. When we discuss existence of energy minimizers in elastostatics, we also need that such a manifold (the one of microstructural shapes) be Riemannian and geodesic-complete [21]. Also, when we aim at linearizing, we need to embed the manifold of microstructural shapes into a linear space (the embedding is always possible due to the finite dimensionality but it is not unique) or, when such a manifold is a Lie group, we need to reduce to the pertinent algebra.

2.2. Observers and their changes

An *observer* is a representation (i.e., the assignment of frames of reference) over all the spaces adopted for representing the morphology of a body and its motion [20,21]. Here the spaces involved are the reference one (namely $\mathbb{R}^3$), the physical space (i.e., $\tilde{\mathbb{R}}^3$), and the one in which $\tilde{\nu}$ takes values (namely $\bar{\mathbb{R}}^3$). We consider changes of observers that first leave invariant $\mathbb{R}^3$ and are related by time-dependent orientation-preserving isometries (i.e., rigid-body motions) in the physical space $\tilde{\mathbb{R}}^3$ so that if y is a place evaluated by the first observer, the second one records $y' = a(t) + Q(t)(y - y_0) + y_0$, where $t \longmapsto a(t) \in \mathbb{R}^3$ is a vector-valued smooth map depending only on time, y_0 an arbitrary fixed point, and $t \longmapsto Q(t) \in SO(3)$ a smooth map depending only on time and taking values into the special orthogonal group $SO(3)$. The first observer records a velocity $\dot{y}$ while for the second one it is $\dot{y}' = \dot{a} + \dot{Q}(y - y_0) + Q\dot{y}$, where the superposed dot over a and Q means time derivative as for the other fields. When we pull-back $\dot{y}'$ in the frame of the first observer, we get a velocity $\dot{y}^\diamond$ given by $\dot{y}^\diamond = Q^\mathsf{T}\dot{y}' = Q^\mathsf{T}\dot{a} + Q^\mathsf{T}\dot{Q}(y - y_0) + \dot{y} = \mathfrak{c} + q \times (y - y_0) + \dot{y}$, where the superscript T means standard transposition; $\mathfrak{c} := Q^\mathsf{T}\dot{a}$ is a relative translation velocity between the two observers; q is the axial vector of the skew-symmetric second-rank tensor $Q^\mathsf{T}\dot{Q}$, a relative rotation velocity, indeed; in other words, the skew-symmetric second-rank tensor $q\times = Q^\mathsf{T}\dot{Q}$ is an element of the Lie algebra $\mathfrak{so}(3)$.

Microstructures are in the physical space: their description in terms of the field $\tilde{\nu}$ is only a convenient tool. Changing observers in the physical space may alter the perception of microstructures, depending on the way they are described. Since ν is here a 3*D* real vector, it is influenced by rotations. (Example: Consider the case in which $\nu \in \bar{\mathbb{R}}^3$ is a end-to-tail vector representing a linear polymer chain, the properties of which are attributed to y. A rigid translation of an observer in the physical space leaves invariant the representation of ν because it is transported parallel to itself without any change. At variance, a rotation alters the way the orientation of the molecules is perceived.) Consequently, under rigid-body type changes in observers in the physical space, the value ν recorded by the first observer changes into $\nu' = Q(t)\nu$, and the time rate $\dot{\nu}$ becomes $\dot{\nu}' = \dot{\nu} + \dot{Q}\nu$. By pulling back this last vector in the frame of the first observer, we obtain a new vector $\dot{\nu}^\diamond$ given by $\dot{\nu}^\diamond = \dot{\nu} + Q^\mathsf{T}\dot{Q} = \dot{\nu} + q \times \nu$, which we may write as $\dot{\nu}^\diamond = \dot{\nu} + \mathcal{A}(\nu)q$, with the linear operator $\mathcal{A}(\nu)$ given by $\mathcal{A}(\nu) = -\nu\times$. (A more detailed analysis concerning changes of observers in the mechanics of complex bodies, i.e., those with active microstructure, is in Ref. [21]; it deals with the general case in which ν belongs to a finite-dimensional differentiable manifold generically not embedded into a linear space.)

2.3. External power, invariance, and balance

Actions are those entities involved in changes of body morphology. They are defined by the power that they perform. For them we accept the standard subdivision into bulk and contact families.

By *part* we indicate a subset $\mathfrak{b}$ of $\mathcal{B}$ with non-vanishing volume that is a fit region too as $\mathcal{B}$ is. The *external power* $\mathcal{P}^{ext}_{\mathfrak{b}}$ over $\mathfrak{b}$ along a motion $(\tilde{y}, \tilde{\nu})$ is defined by

$$\mathcal{P}^{ext}_{\mathfrak{b}}(\dot{y}, \dot{\nu}) := \int_{\mathfrak{b}} \left(b^\ddagger \cdot \dot{y} + \beta^\ddagger \cdot \dot{\nu}\right) dx + \int_{\partial\mathfrak{b}} \left(\mathfrak{t}_\partial \cdot \dot{y} + \tau_\partial \cdot \dot{\nu}\right) d\bar{\mathcal{H}}^2(x),$$

where $d\bar{\mathcal{H}}^2(x)$ is the surface measure; the dot indicated duality pairing, identified with the scalar product when the metrics considered are flat as we do here for the sake of simplicity. Subscript ∂ associated with the contact action $\mathfrak{t}_\partial$ indicates that *we presume dependence* of $\mathfrak{t}$ and τ *on the boundary* $\partial\mathfrak{b}$. The covector $b^\ddagger$ indicates bulk actions.

We consider balanced those actions for which the external power is invariant under isometric changes of observers, those defined previously. Formally, we then impose $\mathcal{P}^{ext}_{\mathfrak{b}}(\dot{y}, \dot{\nu}) = \mathcal{P}^{ext}_{\mathfrak{b}}(\dot{y}^\diamond, \dot{\nu}^\diamond)$ and presume that the identity holds *for any choice of* $\mathfrak{b}$, $\mathfrak{c}$, *and* q. The arbitrariness of the relative translational and rotational velocities $\mathfrak{c}$ and q implies the common integral balance of forces and a non-standard balance of couples, namely

$$\int_{\mathfrak{b}} b^\ddagger \;\; dx + \int_{\partial\mathfrak{b}} \mathfrak{t}_\partial \;\; d\bar{\mathcal{H}}^2(x) = 0,$$

$$\int_{\mathfrak{b}} \left((y - y_0) \times b^\ddagger + \mathcal{A}^\mathsf{T}\beta^\ddagger\right) dx + \int_{\partial\mathfrak{b}} \left((y - y_0) \times \mathfrak{t}_\partial + \mathcal{A}^\mathsf{T}\tau_\partial\right) d\bar{\mathcal{H}}^2(x) = 0\ .$$

Cauchy's and Hamel–Noll's theorems directly apply to the first balance. Their techniques can be adapted to the second balance. Pertinent results are summarized in the items below.

- If $\left|b^\ddagger\right|$ is bounded over $\mathcal{B}$ and $\mathfrak{t}_\partial$ depends continuously on x, the action–reaction principle holds first on flat boundaries, and, on its basis, one may further show that $\mathfrak{t}_\partial$ depends on $\partial\mathfrak{b}$ only through the normal n to it in all points where it is well-defined and extends there the action–reaction property, i.e., $\mathfrak{t}_\partial = \mathfrak{t} := \tilde{\mathfrak{t}}(x, t, n) = -\tilde{\mathfrak{t}}(x, t, -n)$. Also, as a function of n, $\tilde{\mathfrak{t}}$ is homogeneous and additive, i.e., there exists a second-rank tensor field $(x, t) \longmapsto P(x, t)$ such that $\tilde{\mathfrak{t}}(x, t, n) = P(x, t)\, n(x)$. This is the standard Cauchy theorem preceded by the Hamel–Noll result; P is the *first Piola–Kirchhoff stress*.
- Since $\mathcal{B}$ is bounded, as above selected, we can choose the arbitrary point y_0 in a way such that the boundedness of $|b^\ddagger|$ implies the one of $|(y - y_0) \times b^\ddagger|$. If in addition $\left|\mathcal{A}^\mathsf{T}\beta^\ddagger\right|$ is bounded over $\mathcal{B}$ and τ_∂ depends continuously on x, the microstructural contact action τ_∂ satisfies a non-standard action–reaction principle and depends on $\partial\mathfrak{b}$ only through the normal n to it in all points where it is well-defined; we have, in fact, $\mathcal{A}^\mathsf{T}\left(\tilde{\tau}(x, t, n) + \tilde{\tau}(x, t, -n)\right) = 0$. Also, as a function of n, $\tilde{\tau}$ is homogeneous and additive, i.e., there exists a second-rank tensor field $(x, t) \longmapsto \mathcal{S}(x, t)$, so called *microstress*, such that $\tilde{\tau}(x, t, n) = \mathcal{S}(x, t)\, n(x)$.
- If both stress fields are in $C^1(\mathcal{B}) \cap C\left(\bar{\mathcal{B}}\right)$ and the bulk actions $x \longmapsto b^\ddagger$, $x \longmapsto \beta^\ddagger$ are continuous over $\mathcal{B}$, the point-wise balance of forces

  $$\mathrm{Div} P + b^\ddagger = 0$$

  holds and there exists a field $x \longmapsto z(x) \in T^*_\nu\mathcal{M}$ such that

  $$\mathrm{Div}\mathcal{S} + \beta^\ddagger - z = 0\ , \qquad \mathrm{skw} P(\nabla y)^\mathsf{T} = \frac{1}{2}\mathsf{e}\left(\mathcal{A}^\mathsf{T} z + \left(\nabla\mathcal{A}^\mathsf{T}\right)\mathcal{S}\right)\ ,$$

  where I is the second-rank identity tensor; moreover,

  $$\mathcal{P}^{ext}_{\mathfrak{b}}(\dot{y}, \dot{\nu}) = \int_{\mathfrak{b}} \left(P \cdot \nabla\dot{y} + z \cdot \dot{\nu} + \mathcal{S} \cdot \nabla\dot{\nu}\right) dx\ ,$$

  with the right-hand side integral called *internal* (or *inner*) *power*.

Details of the pertinent proofs are in Ref. [21]. Since we are distinguishing in this section between $\mathcal{B}$ and $\mathcal{B}_a$, capitalizing the differential operator Div reminds that it includes derivatives with respect to x. Lower case letters, namely div refer to derivatives with respect to y. However, when the analysis is restricted to the small strain regime, as in the subsequent sections, not distinguishing between x and y, as usual in that regime, we will use just div, as in Eqs. (1) and (2).

The bulk actions $b^{\ddagger}$ are assumed to be a sum of inertial (b^{in}) and non-inertial (b) terms, namely $b^{\ddagger} = b^{in} + b$, the inertial ones defined to be such that their power equals a negative of the kinetic energy time derivative. With ρ the mass density, taking as a kinetic energy the standard quadratic form of the velocity, we get $b^{in} = -\rho\ddot{y} = -\rho u_{tt}$. In the present case we do not attribute peculiar relative kinetic energy to the microstructure, an attribution that is not excluded in general (see Refs. [12,20] for discussions on the matter).

The interactions appearing in the previous local balances admit their Eulerian counterparts defined by

$$b_a^{\ddagger} := \det(\nabla y)^{-1} b^{\ddagger} , \quad \boldsymbol{\sigma} := \det(\nabla y)^{-1} P (\nabla y)^{\mathsf{T}} ,$$
$$z_a := \det(\nabla y)^{-1} z ,$$

$$\beta_a^{\ddagger} := \det(\nabla y)^{-1} \beta^{\ddagger} , \quad \mathcal{S}_a := \det(\nabla y)^{-1} \mathcal{S} (\nabla y)^{\mathsf{T}}$$

(remind to distinguish between $\boldsymbol{\sigma}$, a second-rank tensor, the standard Cauchy stress, indeed, and the exponent σ appearing in Eq. (2)). The balance equations in Eulerian representation then follow: we just recall the local balance of forces

$$b_a^{\ddagger} + \operatorname{div} \boldsymbol{\sigma} = 0$$

and the one of microstructural interactions, namely

$$\beta_a^{\ddagger} - z_a + \operatorname{div} \mathcal{S}_a = 0 .$$

The rest follows.

In small deformation setting we have $P \approx \boldsymbol{\sigma}$, $z \approx z_a$, and $\mathcal{S} \approx \mathcal{S}_a$, and, as already mentioned, we 'confuse' $\mathcal{B}$ with $\mathcal{B}_a$, considering all fields depending on x and t.

2.4. Histories and history-dependent free energies

At every $x \in \mathcal{B}$, let us define $\mathfrak{H}^t : \mathbb{R}^+ \longrightarrow M_{3\times 3} \times \mathbb{R}^3 \times M_{3\times 3}$ by

$$\mathfrak{H}^t(s) := (\nabla\tilde{y}(s), \tilde{\nu}(s), \nabla\tilde{\nu}(s)) = (\nabla u(s) + I, \tilde{\nu}(s), \nabla\tilde{\nu}(s))$$

for $0 \le s < t$, where $M_{3\times 3}$ is the space of 3×3 matrices. Thus, $\mathfrak{H}^t$ is the *history* of those state variables listed in the above definition *prior* t. Of course, we admit a prolongation of the history to t and indicate by $\mathfrak{H}$ the present value at t. A *restriction* of an history to the interval $[r, p)$, with $0 < r < p$, is indicated by $\mathfrak{K}_p^r$ and defined by

$$\mathfrak{K}_p^r(s) := (\nabla\tilde{y}(r+s), \tilde{\nu}(r+s), \nabla\tilde{\nu}(r+s)) ,$$

for $0 \le s < p - r$. Given a pair $(\mathfrak{H}^r, \mathfrak{K}_p^r)$, we define a *prolongation* of $\mathfrak{H}^r$ through $\mathfrak{K}_p^r$ as the history given by

$$(\mathfrak{K}_p^r * \mathfrak{H}^r)(s) := \begin{cases} \mathfrak{H}^r(s) & \text{if } 0 \le s < r, \\ \mathfrak{K}_p^r(s) & \text{if } r \le s < p . \end{cases}$$

When both $\mathfrak{H}^r$ and $\mathfrak{K}_p^r$ are differentiable and $\lim_{s\searrow 0} \mathfrak{K}_p^r(s) = \mathfrak{H}^r$, the prolonged history is differentiable too.

Let us assume that P, z, and $\mathcal{S}$ all depend on $\mathfrak{H}$ and $\mathfrak{H}^t$. We say that two histories $\mathfrak{H}^t$ and $\bar{\mathfrak{H}}^t$ are equivalent, and we write in this case $\mathfrak{H}^t \sim \bar{\mathfrak{H}}^t$, when

$$P(\mathfrak{K}_p^r * \mathfrak{H}^r) = P(\mathfrak{H}^r) , \qquad z(\mathfrak{K}_p^r * \mathfrak{H}^r) = z(\mathfrak{H}^r) ,$$
$$\mathcal{S}(\mathfrak{K}_p^r * \mathfrak{H}^r) = \mathcal{S}(\mathfrak{H}^r)$$

for any r and p. In other words, two state histories are equivalent when they are indistinguishable with respect to stress measures (precisely, interaction measures). Then, with this proviso, the state space considered here, i.e., the space of histories, say Σ, can be endowed with a semi-metric (see the explicit definition and the pertinent proof in Ref. [11]). We say that $f : \Sigma \longrightarrow \mathbb{R}$ is a *state function* when $\mathfrak{H}^t \sim \bar{\mathfrak{H}}^t$ implies $f(\mathfrak{H}^t) = f(\bar{\mathfrak{H}}^t)$.

At every $x \in \mathcal{B}$ we define the *work density performed along a history*, and indicate it by $\mathfrak{w}$, as

$$\mathfrak{w}(\mathfrak{H}^t; \mathfrak{H}) := \int_0^t \Big(P(\mathfrak{H}^t; \mathfrak{H})(s) \cdot \nabla\dot{u}(s) + z(\mathfrak{H}^t; \mathfrak{H})(s) \cdot \dot{\nu}(s) + \mathcal{S}(\mathfrak{H}^t; \mathfrak{H})(s) \cdot \nabla\dot{\nu}(s)\Big)\, ds$$

and

$$\mathfrak{w}(\mathfrak{K}_p^r, \mathfrak{H}^r; \mathfrak{H}) = \mathfrak{w}(\mathfrak{K}_p^r * \mathfrak{H}^r; \mathfrak{H}) - \mathfrak{w}(\mathfrak{H}^r; \mathfrak{H}) .$$

The work density performed along a history is evidently additive with respect to prolongations. Also, by adapting straight away a technique in Ref. [11], we can prove that $\mathfrak{w}(\mathfrak{K}_p^r, \cdot; \mathfrak{H})$ is continuous and (above all) is a state function.

We say that $\psi : \Sigma \times M_{3\times 3} \times \mathbb{R}^3 \times M_{3\times 3} \longrightarrow \mathbb{R}$ is a *free energy* in isothermal setting when it is lower semicontinuous over Σ and satisfies the inequality

$$\psi(\mathfrak{K}_p^r * \mathfrak{H}^r; \mathfrak{H}) - \psi(\mathfrak{H}^r; \mathfrak{H}) < \mathfrak{w}(\mathfrak{H}^r; \mathfrak{H}) .$$

Physical admissibility requires also that ψ is a polyconvex function over $M_{3\times 3}$ with respect to ∇y (see Coleman–Noll's result [24]).

2.5. A chain rule and a local form of the Clausius–Duhem inequality

Take $\mathfrak{F} : \Sigma \times M_{3\times 3} \times \mathbb{R}^3 \times M_{3\times 3} \longrightarrow \mathbb{R}$, defined for every $\mathfrak{H} \in M_{3\times 3} \times \mathbb{R}^3 \times M_{3\times 3}$ and every $\mathfrak{H}^t$ such that $\mathfrak{H}^t(s)$ is in the open, connected set $\mathcal{U} \subset M_{3\times 3} \times \mathbb{R}^3 \times M_{3\times 3}$ where $\det \nabla y > 0$ for almost every s. Assume that

(1) $\mathfrak{F}$ is continuously differentiable;
(2) the map $t \longmapsto \mathfrak{H}(t)$ has two continuous derivatives $\dot{\mathfrak{H}}(t)$ and $\ddot{\mathfrak{H}}(t)$; they are values of histories in Σ.

Under these assumptions, the function $h(t) := \mathfrak{F}(\mathfrak{H}^t; \mathfrak{H}(t))$ is continuously differentiable and its derivative is given by

$$\dot{h}(t) = D\mathfrak{F}(\mathfrak{H}^t; \mathfrak{H}(t)) \cdot \dot{\mathfrak{H}}(t) + \delta\mathfrak{F}(\mathfrak{H}^t; \mathfrak{H}(t)|\dot{\mathfrak{H}}^t) ,$$

where $D\mathfrak{F}(\mathfrak{H}^t; \mathfrak{H}(t))$ is a continuous functional taking values in the cotangent space to $\mathcal{U}$ at $\mathfrak{H}$, while $\delta\mathfrak{F}(\mathfrak{H}^t; \mathfrak{H}(t)|\mathfrak{J}^t)$ is a continuous functional depending linearly on $\mathfrak{J}$ and defined on the closed subspace of Σ spanned by the histories $\mathfrak{J}^t$ such that $\mathfrak{H}^t(s) + \mathfrak{J}^t(s) \in \mathcal{U}$.

By taking into account such a chain rule, we take a local isothermal version of the Clausius–Duhem inequality given by

$$\dot{\psi}(\mathfrak{H}^t; \mathfrak{H}) - P(\mathfrak{H}^t; \mathfrak{H}) \cdot \nabla\dot{u} - z(\mathfrak{H}^t; \mathfrak{H}) \cdot \dot{\nu} - \mathcal{S}(\mathfrak{H}^t; \mathfrak{H}) \cdot \nabla\dot{\nu} \le 0 ,$$

presuming its validity for any choice of the rates considered. We also assume that the microstructural self-action z is the sum $z^e(\mathfrak{H}^t; \mathfrak{H}) + z^d(\mathfrak{H})$, where z^d has a current dissipative character, namely it independently satisfies the local dissipation inequality $z^d \cdot \dot{\nu} \ge 0$ for any choice of $\dot{\nu}$. Such an assumption means that we attribute to the microstructure a diffusive character, independently of memory effects (quasicrystals are an example — see [25] and references therein).

As it is customary for the exploitation of Clausius–Duhem's inequality, such arbitrariness implies

$$P(\mathfrak{H}^t; \mathfrak{H}) = \frac{\partial\psi(\mathfrak{H}^t; \mathfrak{H}(t))}{\partial\nabla y} , \qquad z(\mathfrak{H}^t; \mathfrak{H}) = \frac{\partial\psi(\mathfrak{H}^t; \mathfrak{H}(t))}{\partial\nu} ,$$
$$\mathcal{S}(\mathfrak{H}^t; \mathfrak{H}) = \frac{\partial\psi(\mathfrak{H}^t; \mathfrak{H}(t))}{\partial\nabla\nu}$$

and

$$\delta\psi(\mathfrak{H}^t; \mathfrak{H}(t)|\dot{\mathfrak{H}}^t) - z^d(\mathfrak{H}) \cdot \dot{\nu} \le 0 .$$

The last inequality underlines the dissipative character of the memory.

2.6. Special choices leading to Eqs. (1) and (2)

We restrict our attention to the small strain setting, the peculiarities of which we have already recalled.

We choose $\psi(\mathfrak{H}^t;\mathfrak{H}(t))$ with an additive structure $\psi_1(\mathfrak{H})+\psi_2(\mathfrak{H}^t)$ and select

$$\begin{aligned}\psi_1(\mathfrak{H})=\ &\frac{1}{2}\lambda\,(\mathrm{sym}\nabla u\cdot I)^2+\mu\,\mathrm{sym}\nabla u\cdot\mathrm{sym}\nabla u\\&+\frac{1}{2}k_1\,(\nabla\nu\cdot I)^2+k_2\,\mathrm{sym}\nabla\nu\cdot\mathrm{sym}\nabla\nu\\&+k_2'\,\mathrm{skw}\nabla\nu\cdot\mathrm{skw}\nabla\nu\\&+k_3\,(\mathrm{sym}\nabla u\cdot I)\,(\nabla\nu\cdot I)+k_3'\,\mathrm{sym}\nabla\nu\cdot\mathrm{sym}\nabla u\\&+\frac{1}{2}\kappa_0\,|\nu|^2+\hat{\eta}|\nu|^{\sigma}\ .\end{aligned}$$

This is an isotropic energy. The derivation in terms of symmetry group properties of its quadratic part is in Ref. [25]. From previous relations and a choice of $\psi_2(\mathfrak{H}^t)$ involving only the time integrals of $|\nabla u|^2$ and $|\nu|^2$, with λ_1 and λ_2 chosen as to satisfy relations (3) that assure the positive definiteness of the energy, in small strain regime we eventually get

$$\begin{aligned}P\approx\boldsymbol{\sigma}&=\lambda\,(\mathrm{tr}\,(\mathrm{sym}\nabla u))\,I+2\mu\mathrm{sym}\nabla u+k_3\,(\mathrm{tr}\nabla\nu)\,I\\&\quad+k_3'\mathrm{sym}\nabla\nu+\epsilon\nabla\dot{u}-\lambda_1\int_0^t\varpi(t-s)\nabla u(s)ds,\\z\approx z_a&=\kappa_0\nu+\varsigma\dot{\nu}-\lambda_2\int_0^t\tilde{\varpi}(t-s)\nu(s)ds+\hat{\eta}|\nu|^{\sigma-1}\nu,\\\mathcal{S}\approx\mathcal{S}_a&=k_1\,(\mathrm{tr}\nabla\nu)\,I+2k_2\mathrm{sym}\nabla\nu\\&\quad+2k_2'\mathrm{skw}\nabla\nu+k_3\,(\mathrm{tr}(\mathrm{sym}\nabla u))\,I\\&\quad+2k_2'\mathrm{skw}\nabla\nu+k_3\,(\mathrm{tr}(\mathrm{sym}\nabla u))\,I+k_3'\mathrm{sym}\nabla u+\delta\nabla\dot{\nu}\end{aligned}$$

By inserting such structures in the balance equations, we get the system (1)–(2) after setting $\xi=\lambda+\mu$, $\bar{\xi}=k_3+\frac{1}{2}k_3'$, $\zeta=k_2+k_2'$, $\gamma=k_1+k_2-k_2'$, $\kappa=\frac{1}{2}k_3'$. In deriving such constitutive structures we also considered that the inequality $z^d\cdot\dot{\nu}\geq 0$, assumed to be valid for any choice of $\dot{\nu}$, is compatible with $z^d=\varsigma\dot{\nu}$, where ς is a positive constant. We also set $\beta^{\ddagger}=0$ and admit a live load $b=\tilde{\gamma}u_t$. It has a regularizing role. However, we prove our result presuming $\tilde{\gamma}\geq 0$, i.e., we include the (natural) case $\tilde{\gamma}=0$.

As far as we know, the existence result proven in the next sections is the first one concerning the dynamics of complex bodies with memory, although in small strain setting and $\mathbb{R}^3$-valued microstructural descriptors.

3. Functional preliminaries to the analysis of system (1)-(2)

We consider the spaces $H:=\big(L^2(\mathcal{B})\big)^3\cap\{(v\cdot n)|_{\partial\mathcal{B}}=0\}$, and $V:=\big(W^{1,2}(\mathcal{B})\big)^3\cap\{(v\cdot n)|_{\partial\mathcal{B}}=0,\ \text{and}\ (\mathrm{curl}\,v\times n)|_{\partial\mathcal{B}}=0\}$, where the conditions $v\cdot n=0$ on $\partial\mathcal{B}$ and $\mathrm{curl}\,v\times n=0$ on $\partial\mathcal{B}$ are meant, respectively, in a weak sense. V' denotes the dual of V. The space H^k is defined as $H^k:=\big(W^{k,2}(\mathcal{B})\big)^3$, $k\in\mathbb{N}$, with norm $\|\cdot\|_{k,2}$. The norm in L^p spaces will be indicated by $\|\cdot\|_p$. With $(\cdot\,,\,\cdot)$ we denote the L^2-product while with $\langle\cdot\,,\,\cdot\rangle:=\langle\cdot\,,\,\cdot\rangle_{V',V}$ the duality product.

For the sake of conciseness, we define $F(w(t)):=(\varpi*w)(t)=\int_0^t\varpi(t-s)w(s)ds$ and $\|\mathrm{curl}\,w,\mathrm{div}\,w\|^2:=\|\mathrm{curl}\,w\|^2+\|\mathrm{div}\,w\|^2$, for $w\in V$; also $\tilde{\nabla}:=\mathrm{div}\oplus\mathrm{curl}$, so that we write $(\tilde{\nabla}v,\tilde{\nabla}w)=(\mathrm{div}\,v,\mathrm{div}\,w)+(\mathrm{curl}\,v,\mathrm{curl}\,w)$ and $(\tilde{\nabla}w,\tilde{\nabla}w)=\|\tilde{\nabla}w\|^2=\|\mathrm{curl}\,w,\mathrm{div}\,w\|^2$, for $v,w\in V$.

A function $\varpi=\varpi(t)$, $0\leq t\leq T$, is called a positive kernel on $L^2(0,T;H)$ (see, e.g., [26]), if

$$\begin{aligned}\int_0^T\big(F(w(t)),w(t)\big)\,dt&=\int_0^T\int_0^t\varpi(t-s)\big(w(s),w(t)\big)\,ds\,dt\\&=\int_0^T\int_0^t\int_{\mathcal{B}}\varpi(t-s)w(s)\cdot w(t)\,dxdsdt\geq 0\ ,\end{aligned}\tag{5}$$

for all $w\in H$ and every $T>0$. In the previous relation we left understood the dependence on x, as we will do in the rest of the paper, for the sake of conciseness.

More generally ([26, Lemma 2.6]), for $\varpi\in L^1(0,T)$, and $f,g\in L^2(0,T)$ for some $T>0$, we have

$$\begin{aligned}&\int_0^T g^2(t)\left(\int_0^t\varpi(t-s)f(s)\,ds\right)^2dt\\&\leq\left(\int_0^T|\varpi(t)|\,dt\right)^2\int_0^T g^2(t)f^2(t)\,dt\ .\end{aligned}\tag{6}$$

Since we have taken it to be simply connected, there exists $C>0$, only depending on $1<p<+\infty$ and $\mathcal{B}$, such that

$$\begin{aligned}&\|\nabla v\|_p\leq C\big(\|\mathrm{div}\,v\|_p+\|\mathrm{curl}\,v\|_p\big),\quad\text{for all }v\in H^p,\\&\text{with }v\cdot n=0\ \text{ on }\ \partial\mathcal{B},\end{aligned}\tag{7}$$

see [27, Theorem 3.2]. Moreover, the following Poincaré-like inequality holds: $\|v\|_p\leq C\|\nabla v\|_p$, for $v\in H^p$, with $v\cdot n=0$ on $\partial\mathcal{B}$. It will be expedient to reconstruct the whole $W^{1,2}$-norm for the displacement u.

We will omit dx in (most of) the space-integrals while we will keep ds in the time dependent ones.

4. Existence theorem

Let $T>0$. A weak solution (u,ν) of (1)–(2) is meant as defined by smooth compactly supported test functions in $[0,T]$ taking values in V, i.e., elements of $C_0^\infty([0,T];V)$ (see also [28]).

Theorem 4.1. *Take $T>0$, $u_0,\nu_0\in V$, and $\dot{u}_0\in H$. Then system (1)–(2), with initial conditions $u|_{t=0}=u_0$, $u_t|_{t=0}=\dot{u}_0$, $\nu|_{t=0}=\nu_0$ on $\mathcal{B}$, and boundary conditions (4), admits a unique weak solution (u,ν) such that $u,\nu\in L^\infty(0,T;V)$, $u_t\in L^\infty(0,T;H)$, $u_{tt}\in L^2(0,T;V')$, and $\nu_t\in L^2(0,T;H)$.*

The proof follows below.

4.1. Energy estimates

First, proceed formally, with the aim of adopting later a suitable Galerkin's scheme. By testing in $L^2(\mathcal{B})$ Eq. (1) with u_t, integration by parts and conditions (4) lead to

$$\begin{aligned}&\frac{\rho}{2}\frac{d}{dt}\|u_t\|^2+\frac{\mu}{2}\frac{d}{dt}\|\mathrm{curl}\,u\|^2+\frac{\mu+\xi}{2}\frac{d}{dt}\|\mathrm{div}\,u\|^2\\&\quad+\tilde{\gamma}\|u_t\|^2-\lambda_1\big((\varpi*\tilde{\nabla}u)(t),\tilde{\nabla}u_t(t)\big)\\&=-(\kappa+\bar{\xi})\big(\mathrm{div}\,\nu,\mathrm{div}\,u_t\big)-\kappa\big(\mathrm{curl}\,\nu,\mathrm{curl}\,u_t\big).\end{aligned}$$

Time integration for $t\in[0,\tau]$ implies

$$\begin{aligned}&\frac{\rho}{2}\|u_t(\tau)\|^2+\frac{\mu}{2}\|\mathrm{curl}\,u(\tau)\|^2+\frac{\mu+\xi}{2}\|\mathrm{div}\,u(\tau)\|^2\\&\quad+\tilde{\gamma}\int_0^\tau\|u_t(s)\|^2ds-\lambda_1\int_0^\tau\int_0^t\big(\varpi(t-s)\tilde{\nabla}u(s),\tilde{\nabla}u_t(t)\big)dsdt\\&=K_0-\kappa\int_0^\tau\big(\mathrm{div}\,\nu,\mathrm{div}\,u_t\big)ds-\kappa\int_0^\tau\big(\mathrm{curl}\,\nu,\mathrm{curl}\,u_t\big)ds\\&\quad-\bar{\xi}\int_0^\tau\big(\mathrm{div}\,\nu,\mathrm{div}\,u_t\big)ds\triangleq K_0+\sum_{i=1}^3 I_i(\tau),\end{aligned}\tag{8}$$

where $K_0=\frac{\rho}{2}\|\dot{u}_0\|^2+\frac{\mu}{2}\|\mathrm{curl}\,u_0\|^2+\frac{\mu+\xi}{2}\|\mathrm{div}u_0\|^2$. Since $\varpi(t)=\check{\gamma}e^{-\delta t}$, for the last term on the left-hand side of Eq. (8), we compute

$$
\begin{aligned}
-\int_0^\tau\int_0^t &\big(\varpi(t-s)\tilde\nabla u(s),\,\tilde\nabla u_t(t)\big)dsdt\\
&=-\int_0^\tau\int_s^\tau\big(\varpi(t-s)\tilde\nabla u(s),\,\tilde\nabla u_t(t)\big)dtds\\
&=\int_0^\tau\int_s^\tau\big(\partial_t(\varpi(t-s)\tilde\nabla u(s)),\,\tilde\nabla u(t)\big)dtds\\
&\quad-\int_0^\tau\big(\varpi(t-s)\tilde\nabla u(s),\,\tilde\nabla u(t)\big)\Big|_{t=s}^{t=\tau}ds\\
&=-\delta\int_0^\tau\int_s^\tau\big(\varpi(t-s)\tilde\nabla u(s),\,\tilde\nabla u(t)\big)dtds\\
&\quad-\int_0^\tau\big(\varpi(\tau-s)\tilde\nabla u(s),\,\tilde\nabla u(\tau)\big)ds+\int_0^\tau\|\tilde\nabla u(s)\|^2ds\,,
\end{aligned}
\tag{9}
$$

so that, from Eq. (8), we find

$$
\begin{aligned}
&\frac{\rho}{2}\|u_t(\tau)\|^2+\frac{\mu}{2}\|\mathrm{curl}\,u(\tau)\|^2+\frac{\mu+\xi}{2}\|\mathrm{div}\,u(\tau)\|^2\\
&\quad+\tilde\gamma\int_0^\tau\|u_t(s)\|^2ds+\lambda_1\int_0^\tau\|\tilde\nabla u(s)\|^2ds\\
&=K_0+\sum_{i=1}^3 I_i(\tau)+\lambda_1\delta\int_0^\tau\int_s^\tau\big(\varpi(t-s)\tilde\nabla u(s),\,\tilde\nabla u(t)\big)dtds\\
&\quad+\lambda_1\int_0^\tau\big(\varpi(\tau-s)\tilde\nabla u(s),\,\tilde\nabla u(\tau)\big)ds
\end{aligned}
\tag{10}
$$

By setting $\tilde I(u(\tau)):=\int_0^\tau\int_s^\tau\big(\varpi(t-s)\tilde\nabla u(s),\,\tilde\nabla u(t)\big)dtds$, we also have

$$
\begin{aligned}
\tilde I(u(t))&=\int_0^\tau\int_0^t\big(\varpi(t-s)\tilde\nabla u(s),\,\tilde\nabla u(t)\big)dsdt\\
&\le\left(\int_{\mathcal B}\int_0^\tau\left(\int_0^t\varpi(t-s)\tilde\nabla u(s)\,ds\right)^2dt\right)^{\frac12}\left(\int_0^\tau\|\tilde\nabla u(s)\|^2ds\right)^{\frac12}\\
&\le\left(\int_0^\tau\|\varpi*\tilde\nabla u(s)\|^2ds\right)^{\frac12}\left(\int_0^\tau\|\tilde\nabla u(s)\|^2ds\right)^{\frac12}\\
&\le\int_0^\tau\varpi(s)ds\int_0^\tau\|\tilde\nabla u(s)\|^2ds\le\frac{\check\gamma}{\delta}\int_0^\tau\|\tilde\nabla u(s)\|^2ds,
\end{aligned}
\tag{11}
$$

after exploiting the Cauchy–Schwarz and Hölder inequalities, the inequality (6) with $g=1$ and $f=\tilde\nabla u$, and the kernel structure. Similarly, by setting $\hat I(u(\tau)):=\lambda_1\int_0^\tau\big(\varpi(\tau-s)\tilde\nabla u(s),\,\tilde\nabla u(\tau)\big)ds$, we compute

$$
\begin{aligned}
\hat I(u(t))&\le\lambda_1\|\tilde\nabla u(\tau)\|\int_0^\tau\varpi(\tau-s)\|\tilde\nabla u(s)\|ds\\
&\le\lambda_1\sup_{0\le t\le\tau}\|\tilde\nabla u(t)\|^2\int_0^\tau\varpi(s)ds\le\frac{\lambda_1\check\gamma}{\delta}\sup_{0\le t\le\tau}\|\tilde\nabla u(t)\|^2.
\end{aligned}
\tag{12}
$$

By inserting these last two results in the inequality (10), since $\check\gamma<1$, we get

$$
\begin{aligned}
&\frac{\rho}{2}\|u_t(\tau)\|^2+\frac{\mu}{2}\|\mathrm{curl}\,u(\tau)\|^2+\frac{\mu+\xi}{2}\|\mathrm{div}u(\tau)\|^2\\
&\quad+\tilde\gamma\int_0^\tau\|u_t(s)\|^2ds+\lambda_1(1-\check\gamma)\hat C\int_0^\tau\|\nabla u(s)\|^2ds\\
&\le K_0+\sum_{i=1}^3 I_i(\tau)+\frac{\lambda_1\check\gamma}{\delta}\sup_{0\le t\le\tau}\|\tilde\nabla u(t)\|^2,
\end{aligned}
\tag{13}
$$

on the basis of the identity (7), and $\hat C>0$ is a suitable constant. The inequality implies

$$
\begin{aligned}
&\frac{\rho}{2}\sup_{0\le t\le\tau}\|u_t(t)\|^2+\Big(\frac{\mu}{2}-\frac{\lambda_1\check\gamma}{\delta}\Big)\sup_{0\le t\le\tau}\|\tilde\nabla u(t)\|^2\\
&+\tilde\gamma\int_0^\tau\|u_t(s)\|^2ds+\lambda_1(1-\check\gamma)\hat C\int_0^\tau\|\nabla u(s)\|^2ds\le K_0+\sum_{i=1}^3 I_i(\tau),
\end{aligned}
\tag{14}
$$

where $\mu>2\lambda_1\check\gamma/\delta$ thanks to the assumed constitutive restrictions (3).

Also, by testing Eq. (2) with ν_t in $L^2(\mathcal B)$, we get

$$
\begin{aligned}
&\varsigma\|\nu_t\|^2+\frac{\kappa_0}{2}\frac{\mathrm d}{\mathrm{dt}}\|\nu\|^2+\frac{\zeta}{2}\frac{\mathrm d}{\mathrm{dt}}\|\mathrm{curl}\,\nu\|^2+\frac{\zeta+\gamma}{2}\frac{\mathrm d}{\mathrm{dt}}\|\mathrm{div}\,\nu\|^2\\
&+\frac{\hat\eta}{\sigma+1}\frac{\mathrm d}{\mathrm{dt}}\|\nu\|_{\sigma+1}^{\sigma+1}-\lambda_2\int_0^t\big(\varpi(t-s)\nu(s),\,\nu_t(t)\big)ds\\
&=-\kappa\int_{\mathcal B}\mathrm{div}u\cdot\mathrm{div}\,\nu_t-\kappa\int_{\mathcal B}\mathrm{curl}\,u\cdot\mathrm{curl}\,\nu_t-\bar\xi\int_{\mathcal B}\mathrm{div}\,u\cdot\mathrm{div}\,\nu_t.
\end{aligned}
$$

A subsequent integration on $[0,\tau]$ and analogous calculations to those implying relation (9) give

$$
\begin{aligned}
&\varsigma\int_0^\tau\|\nu_t(s)\|^2ds+\frac{\kappa_0}{2}\|\nu(\tau)\|^2+\frac{\zeta}{2}\|\mathrm{curl}\,\nu(\tau)\|^2\\
&\quad+\frac{\zeta+\gamma}{2}\|\mathrm{div}\nu(\tau)\|^2+\frac{\hat\eta}{\sigma+1}\|\nu\|_{\sigma+1}^{\sigma+1}+\lambda_2\int_0^\tau\|\nu(s)\|^2ds\\
&\le\lambda_2\delta\int_0^\tau\int_0^t\big(\varpi(t-s)\nu(s),\,\nu(t)\big)ds\\
&\quad+\lambda_2\int_0^\tau\big(\varpi(\tau-s)\nu(s),\,\nu(\tau)\big)ds+L_0+\sum_{i=1}^3 J_i(\tau),
\end{aligned}
\tag{15}
$$

where $L_0=\frac{\kappa_0}{2}\|\nu_0\|^2+\frac{\zeta}{2}\|\mathrm{curl}\,\nu_0\|^2+\frac{\zeta+\gamma}{2}\|\mathrm{div}\nu_0\|^2+\|\nu_0\|_{\sigma+1}^{\sigma+1}$. For the memory terms on the right-hand side, proceeding as for the inequalities (11)–(12), we have

$$
\begin{aligned}
&\delta\int_0^\tau\int_0^t\big(\varpi(t-s)\nu(s),\,\nu(t)\big)ds\,dt+\int_0^\tau\big(\varpi(\tau-s)\nu(s),\,\nu(\tau)\big)\,ds\\
&\quad\le\check\gamma\int_0^\tau\|\nu(s)\|^2\,ds+\frac{\check\gamma}{\delta}\sup_{0\le t\le\tau}\|\nu(t)\|^2.
\end{aligned}
$$

Direct calculations (see [28, §3]) give

$$
\begin{aligned}
\sum_{i=1}^3\big(I_i(\tau)+J_i(\tau)\big)\ &\le\kappa\|\mathrm{curl}\,u(\tau),\,\mathrm{curl}\,\nu(\tau)\|^2\\
&\quad+(\kappa+\bar\xi)\|\mathrm{div}\,u(\tau),\,\mathrm{div}\,\nu(\tau)\|^2+R_0,
\end{aligned}
\tag{16}
$$

with $R_0=\kappa\big|(\mathrm{curl}\,u_0,\,\mathrm{curl}\,\nu_0)\big|+(\kappa+\bar\xi)\big|(\mathrm{div}\,u_0,\,\mathrm{div}\,\nu_0)\big|$. On the basis of such two last estimates, under the assumptions (3), by summing the inequalities (14), (15), and taking the supremum over $[0,\tau]$, we get

$$
\begin{aligned}
&\frac{\rho}{2}\sup_{0\le t\le\tau}\|u_t(t)\|^2+\left(\frac{\mu}{2}-\frac{\lambda_1\check\gamma}{\delta}-\kappa-\bar\xi\right)\sup_{0\le t\le\tau}\|\tilde\nabla u(t)\|^2\\
&+\left(\frac{\zeta}{2}-\kappa-\bar\xi\right)\sup_{0\le t\le\tau}\|\tilde\nabla\nu(t)\|^2+\left(\frac{\kappa_0}{2}-\frac{\lambda_2\check\gamma}{\delta}\right)\sup_{0\le t\le\tau}\|\nu(t)\|^2\\
&+\frac{\hat\eta}{\sigma+1}\sup_{0\le t\le\tau}\|\nu\|_{\sigma+1}^{\sigma+1}+\tilde\gamma\int_0^\tau\|u_t(s)\|^2ds+\varsigma\int_0^\tau\|\nu_t(s)\|^2ds\\
&+C_1\int_0^\tau\|\nu(s)\|^2\,ds+C_2\int_0^\tau\|\nabla u(s)\|^2ds\le K_0+L_0+R_0,
\end{aligned}
$$

where $C_i=\lambda_i\big(1-\check\gamma\big)\hat C>0$, $i=1,2$ (see (13)), while K_0, L_0, and R_0 are defined by the formulas (8), (15), and (16).

4.1.1. Estimate for u_{tt}

The $L^2(\mathcal{B})$-product of (1) with test $\phi \in V$ and time integration in (s,τ), with $0 < s < \tau < T$, lead to

$$\left|\int_s^\tau \langle u_{tt},\phi\rangle dt\right| \leq \bar{C}\Big((\tau - s) + \int_s^\tau \|u_t(t)\|^2 dt + \int_s^\tau \big(\|\tilde{\nabla}u(t)\|^2 + \|\tilde{\nabla}\nu(t)\|^2\big)dt + \int_s^\tau \|\varpi * \tilde{\nabla}u(t)\|^2 dt\Big)\|\phi\|_{1,2}$$

where $\bar{C} = \bar{C}(\delta,\lambda_1,\mu,\kappa,\xi,\bar{\xi})$ and $\|\varpi * \tilde{\nabla}u(t)\|^2 = \|\varpi * \operatorname{div} u(t), \varpi * \operatorname{curl} u(t)\|^2 = \|\varpi * \operatorname{div} u(t)\|^2 + \|\varpi * \operatorname{curl} u(t)\|^2$, because for the memory term we have $\left|\int_s^\tau \langle F(\tilde{\nabla}u(t)), \tilde{\nabla}\phi\rangle dt\right| \leq c\|\phi\|_{1,2}\int_s^\tau \|\varpi * \tilde{\nabla}u(t)\|dt$, $c > 0$. Thus, by following the same path adopted above, we can conclude that $u_{tt} \in L^2(s,\tau;V')$.

4.1.2. Nonlinearity

Here, $B(\nu) = \hat{\eta}|\nu|^{\sigma-1}\nu$, $2 \leq \sigma \leq 3$, defines a locally Lipschitz operator from V to H. For $w, \tilde{w} \in V$, we have

$$\begin{aligned}
&\|B(w) - B(\tilde{w})\|^2 \\
&\leq C\|(|w|^{\sigma-1} - |\tilde{w}|^{\sigma-1})w\|^2 + C\||\tilde{w}|^{\sigma-1}(w-\tilde{w})\|^2 \\
&\leq C\|w\|_6^2\||w|^{\sigma-1} - |\tilde{w}|^{\sigma-1}\|_3^2 + C\||\tilde{w}|^{\sigma-1}\|_3^2\|w-\tilde{w}\|_6^2 \\
&\leq C\|w\|_{1,2}^2\big(\||w|^{\sigma-2}\|_6^2 + C\||\tilde{w}|^{\sigma-2}\|_6^2\big)\|w-\tilde{w}\|_6^2 \\
&\quad + C\|\tilde{w}\|_{3(\sigma-1)}^{2(\sigma-1)}\|w-\tilde{w}\|_{1,2}^2 \\
&\leq C\|w\|_{1,2}^2\big(\|w\|_{6(\sigma-2)}^{2(\sigma-2)} + \|\tilde{w}\|_{6(\sigma-2)}^{2(\sigma-2)}\big)\|w-\tilde{w}\|_6^2 \\
&\quad + C\|\tilde{w}\|_{3(\sigma-1)}^{2(\sigma-1)}\|w-\tilde{w}\|_{1,2}^2 \\
&\leq C\big[\|w\|_{1,2}^{2(\sigma-1)} + \|w\|_{1,2}^2\|\tilde{w}\|_{1,2}^{2(\sigma-2)} + \|\tilde{w}\|_{1,2}^{2(\sigma-1)}\big]\|w-\tilde{w}\|_{1,2}^2,
\end{aligned} \tag{17}$$

after exploiting Hölder's and Gagliardo–Nirenberg's inequalities (both $3(\sigma-1) \leq 6$ and $6(\alpha-2) \leq 6$ are satisfied) along with $|x^q - y^q| \leq Cq\big(|x|^{q-1} + |y|^{q-1}\big)|x-y|$, for any $x, y \geq 0$, $q \geq 1$, and $C > 0$.

4.2. Galerkin approximation scheme

We need a sequence $\{\omega_k\}_{k\in\mathbb{N}} \subset H^2 \cap \{(v\cdot n)|_{\partial\mathcal{B}} = 0$, and $(\operatorname{curl} v \times n)|_{\partial\mathcal{B}} = 0\}$, which is a complete orthogonal basis of H (for the classical version of the argument adopted here, see, e.g., Ref. [29]). To get $\{\omega_k\}_k$ we make use of the Helmholtz–Leray decomposition. So, after setting $\omega = \omega_k$, we write $\omega = v + \nabla q$, with $\operatorname{div} v = 0$, and we determine v and q as solutions of the following auxiliary eigenvalue problems:

$$\begin{aligned}
&-\mu\Delta v = \lambda v \text{ in } \mathcal{B}, \quad \operatorname{div} v = 0 \text{ in } \mathcal{B}, \\
&\operatorname{curl} v \times n = 0 \text{ on } \partial\mathcal{B}, \quad v\cdot n = 0 \text{ on } \partial\mathcal{B},
\end{aligned} \tag{18}$$

and

$$-\mu\Delta^2 q = \check{\lambda}\Delta q \text{ in } \mathcal{B}, \quad \nabla q\cdot n = 0 \text{ on } \partial\mathcal{B}, \quad q = 0 \text{ on } \partial\mathcal{B}. \tag{19}$$

According to a result in Ref. [30, Theorem 3.3], system (18) admits sequence of eigenfunctions $\{v_k\}_k \subset H^2 \cap \{\operatorname{div} v = 0 \text{ in } \mathcal{B}, (v\cdot n)|_{\partial\mathcal{B}} = 0$, and $(\operatorname{curl} v \times n)|_{\partial\mathcal{B}} = 0\}$ that is a complete orthogonal basis for $\big(L^2(\mathcal{B})\big)^3 \cap \{\operatorname{div} v = 0$ in $\mathcal{B}$ and $(v\cdot n)|_{\partial\mathcal{B}} = 0\}$. Also, from the weak formulation of the second system (19), and the variational spectral theory, we have a sequence of eigenfunctions $\{q_k\}_k \subset W^{2,2} \cap \{p|_{\partial\mathcal{B}} = 0$, and $(\nabla p\cdot n)|_{\partial\mathcal{B}} = 0\}$ that is a complete orthogonal basis for $W^{1,2}(\mathcal{B}) \cap \{p|_{\partial\mathcal{B}} = 0\}$ (see for a similar case [31, Problem (2.8)]). Since $\nabla q_k \cdot n = 0$ on $\partial\mathcal{B}$, the sequence $\{\nabla q_k\}_k$ is a basis for $H^1 \cap \{w = \nabla p, \ (w\cdot n)|_{\partial\mathcal{B}} = 0\}$ and, clearly, $\operatorname{curl}(\nabla q_k) = 0$. As a consequence, $\{(v_k, \nabla q_k)\}_k$ is the desired sequence $\{\omega_k\}_k$, with $\omega_k = v_k + \nabla q_k$, $k \in \mathbb{N}$. We define $X_m := \operatorname{span}\{\omega_1,\ldots,\omega_{d_m}\}$ and indicate by P_m the orthogonal projection from V on X_m. Analogously, but considering the Helmholtz equation, i.e. $-\zeta\Delta\nu + \kappa_0\nu = \breve{\sigma}\nu$, in place of the first equation in (19), we introduce the set $\{\vartheta_r\}_{r\in\mathbb{N}} \subset H^2 \cap \{(v\cdot n)|_{\partial\mathcal{B}} = 0$, and $(\operatorname{curl} v \times n)|_{\partial\mathcal{B}} = 0\}$ that is a complete orthogonal basis of H. Define $Y_m := \operatorname{span}\{\vartheta_1,\ldots,\vartheta_{\delta_m}\}$ and indicate by Π_m the orthogonal projection from V over Y_m. We look for $u^m(t,x) = \sum_{i=1}^{d_m} d_i^m(t)\omega_i(x)$ and $\nu^m(t,x) = \sum_{j=1}^{\delta_m} e_j^m(t)\vartheta_j(x)$, which are, for $(\omega_k,\vartheta_r) \in X_m \times Y_m$, $1 \leq k \leq d_m$, $1 \leq r \leq \delta_m$, and $t \in [0,T]$, solutions of the system of ordinary differential equations

$$\begin{aligned}
&\rho\int_{\mathcal{B}} u_{tt}^m\cdot\omega_k + \tilde{\gamma}\int_{\mathcal{B}} u_t^m\cdot\omega_k + \mu\int_{\mathcal{B}} \operatorname{div} u^m \operatorname{div}\omega_k \\
&\quad + (\mu+\xi)\int_{\mathcal{B}} \operatorname{curl} u^m\cdot\operatorname{curl}\omega_k - \lambda_1\int_{\mathcal{B}} (\varpi * \tilde{\nabla})u^m\cdot\tilde{\nabla}\omega_k \\
&= -\kappa\int_{\mathcal{B}} \operatorname{div}\nu^m \operatorname{div}\omega_k - \kappa\int_{\mathcal{B}} \operatorname{curl}\nu^m\cdot\operatorname{curl}\omega_k \\
&\quad - \bar{\xi}\int_{\mathcal{B}} \operatorname{div}\nu^m \operatorname{div}\omega_k,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\varsigma\int_{\mathcal{B}} \nu_t^m\cdot\vartheta_r + \kappa_0\int_{\mathcal{B}} \nu^m\cdot\vartheta_r + \zeta\int_{\mathcal{B}} \operatorname{curl}\nu^m\cdot\operatorname{curl}\vartheta_r \\
&\quad + (\zeta+\gamma)\int_{\mathcal{B}} \operatorname{div}\nu^m \operatorname{div}\vartheta_r - \lambda_2\int_{\mathcal{B}} (\varpi * \nu^m)\cdot\vartheta_r \\
&\quad + \hat{\eta}\int_{\mathcal{B}} |\nu^m|^{\sigma-1}\nu^m\cdot\vartheta_r \\
&= -\kappa\int_{\mathcal{B}} \operatorname{curl} u^m\cdot\operatorname{curl}\vartheta_r - \kappa\int_{\mathcal{B}} \operatorname{div} u^m \operatorname{div}\vartheta_r - \bar{\xi}\int_{\mathcal{B}} \operatorname{div} u^m \operatorname{div}\vartheta_r.
\end{aligned} \tag{21}$$

Since $B(\nu) = \hat{\eta}|\nu|^{\sigma-1}\nu$ is locally Lipschitz (see (17)), the system (20)–(21) has a unique solution (u^m,ν^m): Existence is due to Carathéodory's theorem and uniqueness follows from Picard's theorem.

4.2.1. Existence and uniqueness of weak solutions to the system (1)–(2)

By using a compactness argument in the style of the Aubin–Lions lemma along with the energy estimates for the Galerkin scheme provided in the previous section, we can extract a subsequence (still labeled by (u^m,ν^m)) such that

$$\begin{aligned}
&u^m \to u \text{ in } L^2(0,T;H),\ L^\infty(0,T;V)_{\text{weak}^\star},\ L^2(0,T;V)_{\text{weak}}; \\
&u_t^m \to u_t \text{ in } L^\infty(0,T;H)_{\text{weak}^\star},\ L^2(0,T;H)_{\text{weak}}, \\
&u_{tt}^m \to u_{tt} \text{ in } L^2(0,T;V')_{\text{weak}}; \quad \nu_t^m \to \nu_t \text{ in } L^2(0,T;H)_{\text{weak}}, \\
&\nu^m \to \nu \text{ in } L^2(0,T;H),\ L^p(0,T;L^p),\ L^\infty(0,T;H)_{\text{weak}^\star}, \\
&\qquad L^{\sigma+1}(0,T;L^{\sigma+1})_{\text{weak}},\ L^2(0,T;V)_{\text{weak}},
\end{aligned}$$

with $1 < p < \sigma+1$. When not specified, it is understood that the convergence is strong (see, e.g., [28,29,32]). Then, it is possible to compute a limit in the Galerkin's formulation (20)–(21).

The limiting pair (u,ν) actually satisfies the weak formulation of system (1)–(2). Indeed, it is sufficient to check the memory terms and the nonlinear term in (2); for the others we exploit a consequence of the above convergence types. Since $u^m \to u$ in $L^2(0,T;V)_{\text{weak}}$, for any $\phi \in L^2(0,T;V)$, fixing $\tau \in [0,T]$, we have

$$\begin{aligned}
&\int_0^\tau \big(\varpi * \tilde{\nabla}u^m(t), \tilde{\nabla}\phi(t)\big)dt = \check{\gamma}\int_0^\tau\int_s^\tau \big(e^{\delta s}\tilde{\nabla}u^m(s), e^{-\delta t}\tilde{\nabla}\phi(t)\big)dtds \\
&= \check{\gamma}\int_0^\tau \Big(\tilde{\nabla}u^m(s), e^{\delta s}\int_s^\tau e^{-\delta t}\tilde{\nabla}\phi(t)dt\Big)ds \\
&\quad \xrightarrow[m\to\infty]{} \check{\gamma}\int_0^\tau \Big(\tilde{\nabla}u(s), e^{\delta s}\int_s^\tau e^{-\delta t}\tilde{\nabla}\phi(t)dt\Big)ds \\
&= \int_0^\tau \big(\varpi * \tilde{\nabla}u(t), \tilde{\nabla}\phi(t)\big)dt,
\end{aligned}$$

because $\int_s^\tau e^{-\delta(t-s)}\tilde{\nabla}\phi(t)dt \in L^2(0,\tau;H)$. Consider $\tau = T$. Jensen's inequality and a slight modification of the estimate (6) (see [26, Proof of Lemma 2.6]), with $g = 1$ and $f = \|\tilde{\nabla}u\|$, imply

$$\int_0^T\int_{\mathcal{B}}\left(\int_s^T e^{-\delta(t-s)}\tilde{\nabla}\phi(t)dt\right)^2 dxds$$
$$\leq \int_0^T\left(\int_0^T e^{-\delta(t-s)}\|\tilde{\nabla}\phi(t)\|dt\right)^2 ds$$
$$\leq \left(\int_0^T e^{\delta t}dt\right)^2 \int_0^T \|\tilde{\nabla}\phi(t)\|^2 dt.$$

Similarly, we have $\int_0^\tau (\varpi * v^m(t), \psi(t))dt \to \int_0^\tau (\varpi * v(t), \psi(t))dt$ as $m \to \infty$, for $\psi \in L^2(0,T;V)$. Since tests in the weak formulation are taken in $C_0^\infty([0,T];V)$, the previous requirements are satisfied.

For $0 \leq \tau \leq T$, and $\psi \in C_0^\infty([0,T];V) \subset L^2(0,T;V) \cap L^{\sigma+1}(0,T;L^{\sigma+1}(\mathcal{B}))$ we also have

$$\int_0^\tau (B(v^m) - B(v), \psi)dt = \int_0^\tau \left(\left(|v^m|^{\sigma-1} - |v|^{\sigma-1}\right)v^m, \psi\right)dt$$
$$+ \int_0^\tau \left(v^m - v, |v|^{\sigma-1}\psi\right)dt = I_1^m + I_2^m \xrightarrow[m\to\infty]{} 0,$$

since $I_2^m \to 0$, as $m \to 0$, due to weak convergence of $v^m \to v$ in $L^{\sigma+1}(0,T;L^{\sigma+1})$ and the inclusion $|v|^{\sigma-1}\psi \in L^{\sigma+1}(0,T;L^{\sigma+1}(\mathcal{B}))$, which holds for $\psi \in L^{\sigma+1}(0,T;L^{\sigma+1}(\mathcal{B}))$. Also, by taking $2 < p < \sigma + 1$, we get

$$\begin{aligned}
I_1^m &\leq C\int_0^\tau\int_{\mathcal{B}} \big||v^m| - |v|\big|\left(|v^m|^{\sigma-2} + |v|^{\sigma-2}\right)|v^m||\psi|\,dxdt \\
&\leq C\int_0^\tau\int_{\mathcal{B}} \big||v^m| - |v|\big|\left(|v^m|^{\sigma-1} + |v|^{\sigma-1}\right)|\psi|\,dxdt \\
&\leq \|v^m - v\|_{L^p(0,T;L^p)}\left(\int_0^\tau\int_{\mathcal{B}}\left(|v^m|^{\frac{(\sigma-1)p}{p-1}}|\psi|^{\frac{p}{p-1}}\right.\right. \\
&\left.\left.\quad +|v|^{\frac{(\sigma-1)p}{p-1}}|\psi|^{\frac{p}{p-1}}\right)dxdt\right)^{\frac{p-1}{p}} \xrightarrow[m\to\infty]{} 0,
\end{aligned} \tag{22}$$

due to the strong convergence of $v^m \to v$ in $L^p(0,T;L^p(\mathcal{B}))$ and because, for $4/3 < p/(p-1) < 2$ and $1 \leq \sigma - 1 \leq 2$, the second integral factor on the right-hand side of inequality (22) is bounded. In fact, we have

$$\int_0^\tau\int_{\mathcal{B}} |v^m|^{\frac{(\sigma-1)p}{p-1}}|\psi|^{\frac{p}{p-1}}dxdt \leq C\int_0^\tau\int_{\mathcal{B}}(1+|v^m|^4)(1+|\psi|^2)dxdt$$
$$\leq C\int_0^\tau\left(\int_{\mathcal{B}}(1+|v^m|^6)dx\right)^{\frac{2}{3}}\left(\int_{\mathcal{B}}(1+|\psi|^6)dx\right)^{\frac{1}{3}}dt,$$

and the conclusion follows by using the embedding $W^{1,2}(\mathcal{B}) \hookrightarrow L^6(\mathcal{B})$, and the regularity of v^m and ψ. For the term involving v the calculations are analogous.

4.3. The key step for uniqueness

Uniqueness of the weak solution follows directly by using the same argument in Ref. [28]. It requires a suitable estimate for the nonlinear term in (2). Let (u, v) and $(\tilde{u}, \tilde{v})$ be two solutions with the same initial data. Take the system (1)–(2) for the differences $(u-\tilde{u}, v-\tilde{v})$ and use as test functions, in the L^2-norm, $u_t - \tilde{u}_t$ and $v_t - \tilde{v}_t$. We obtain inequalities of type (14)–(15) for $(u-\tilde{u}, v-\tilde{v})$ and uniqueness comes from an application of Gronwall's lemma. We must control, however, the nonlinear term. To this aim we can use the following inequality (see also [33]):

$$\begin{aligned}
&\left|\int_{\mathcal{B}}\left(|v|^{\alpha-1}v - |\tilde{v}|^{\alpha-1}\tilde{v}\right)\cdot(v_t - \tilde{v}_t)\right| \\
&\leq C\int_{\mathcal{B}}\left(|v|^{\alpha-1} + |\tilde{v}|^{\alpha-1}\right)|v-\tilde{v}||v_t - \tilde{v}_t| \\
&\leq C\||v|^{\alpha-1} + |\tilde{v}|^{\alpha-1}\|_3\|v-\tilde{v}\|_6\|v_t - \tilde{v}_t\| \\
&\leq C\left(\|v\|_{L^\infty(0,T;V)} + \|\tilde{v}\|_{L^\infty(0,T;V)}\right)\|\nabla(v-\tilde{v})\|\|v_t - \tilde{v}_t\| \\
&\leq C_\varepsilon(T)\|\tilde{\nabla}(v-\tilde{v})\|^2 + \varepsilon\|v_t - \tilde{v}_t\|^2\ ,
\end{aligned}$$

which rests on Hölder's and Young's inequalities, the embedding $W^{1,2}(\mathcal{B}) \hookrightarrow L^6(\mathcal{B})$, the assumption on σ that implies $3(\sigma-1) \leq 6$, the estimate (7), and Poincaré's inequality. Here $\varepsilon > 0$ is small enough in order to control $\varepsilon\|v_t - \tilde{v}_t\|^2$ and reabsorb it on the right-hand side (15) written for $v - \tilde{v}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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