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Hermite and Hermite–Fejér interpolation at Pollaczek zeros

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Abstract

In order to approximate functions defined on $(-1, 1)$, which can grow exponentially at ± 1, we introduce an Hermite and an Hermite–Fejér-type interpolation process based at Pollaczek-type zeros. We prove the convergence of these processes in weighted uniform and L^p-norms and provide error estimates which are comparable with the best weighted approximation in suitable function spaces.

Keywords Hermite interpolation · Hermite–Fejér interpolation · Weighted polynomial approximation · Orthogonal polynomials · Pollaczek-type zeros · Exponential weights

Mathematics Subject Classification 41A05 · 41A10

1 Introduction

This paper concerns the weighted polynomial approximation of functions defined on $(-1, 1)$, which can exhibit an exponential growth at the endpoints of the interval, for instance, behaving like $\mathrm{e}^{(1-x)^{-\alpha}(1+x)^{-\beta}}$. This kind of functions do not belong to classical spaces, but rather to spaces with suitable Pollaczek-type weights decaying exponentially at ± 1.

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These weights and the corresponding orthonormal system have been defined and extensively studied in [1] and [2], while the class of functions growing exponentially at ± 1 have been considered in [3] (see also [4, 5] and [6], cf. [7–9] and [10]). Various related approximation processes have been explored in [4, 5, 11–19].

Here, we introduce an Hermite and an Hermite–Fejér-type interpolation process $H^*_{2m+4}(w)$ and $F^*_m(w)$, based on Pollaczek-type zeros. These processes are obtained by applying Hermite and Hermite–Fejér operators to suitable finite sections of the function. A similar approach was introduced in [20] and [21] for Gauss–Laguerre rules and subsequently applied to various polynomial approximation processes (see [4, 10] and the references therein).

We note that there is a vast literature on Hermite and Hermite–Fejér interpolation, much of it due to the Hungarian school, and in this regard, we mention, among others [22–27]. However, these Authors have considered processes based on Jacobi zeros and their generalizations, which are suitable for approximating continuous functions or those with algebraic-type singularities, but are inadequate for functions with exponential behavior at ± 1.

We establish the uniform boundedness and convergence of $H^*_{2m+4}(w)$ and $F^*_m(w)$ in weighted L^p-norms and provide error estimates for functions belonging to the aforementioned spaces. Our results are original and the proofs simple, as they derive the behavior of these interpolation processes from that of the Lagrange interpolation based on the same nodes. Additionally, the error estimates are sharp, since they show that this processes converge with the order of the best polynomial approximation in suitable weighted function spaces.

In brief, this work is focused on the polynomial approximation of functions belonging to proper weighted spaces. Taking into account that the Lagrange operator is defined for continuous functions in L^p and the Hermite operator for functions with a continuous derivative in the first Sobolev space, assuming equal regularity of the functions, we will prove that the order of convergence does not change (with different constants). On the other hand we are going to show that the Hermite-Fejér operator converges uniformly for $f \in C_{\bar{u}}$, in contrast to the Lagrange operator. We would like to highlight that the proofs are new because, instead of evaluating the kernel of the Hermite operator, which is laborious for non-classical orthogonal polynomials, we easily derive the properties of the Hermite operator from those of the Lagrange operator at the same points.

The paper is organized as follows. In Sect. 2, we define the orthonormal polynomials related to Pollaczek-type weights, our Hermite and Hermite–Feér-type interpolating polynomials and the weighted function spaces, and recall some results about weighted polynomial approximation and Lagrange interpolation at Pollaczek-type zeros. In Sect. 3, we state our main results. In Sect. 4, we recall some results useful for the proofs and prove our main results. Finally, in the Appendix, we provide some technical proofs.

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$ or $\mathcal{C}_a$ when $\mathcal{C}$ depends on a. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$. Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integer and positive real numbers, respectively.

2 Basic facts

2.1 Orthonormal system

Let us consider the weight function

$$w(x) = (1-x^2)^\beta e^{-(1-x^2)^{-\alpha}} =: v^\beta(x)\sigma(x), \tag{1}$$

where $v^\beta(x) = (1-x^2)^\beta$, $\sigma(x) = e^{-(1-x^2)^{-\alpha}}$, $\beta \geq 0$, $\alpha > 0$, $x \in (-1, 1)$.

We point out that w is a nonclassical weight function, does not satisfy the doubling condition and, for $\alpha \geq 1/2$, does not belong to the Szegő class (see [28] and [27]). Nevertheless, the weight w belongs to a wide class of exponential weights defined by Levin and Lubinsky in [1] and [2], as it was checked in [3]. In particular, setting $w(x) = e^{-Q(x)}$ with $Q(x) = (1-x^2)^{-\alpha} - \beta\log(1-x^2)$, we can define the Mhaskar–Rakhmanov–Saff number $\bar{a}_\tau = \bar{a}_\tau(w)$, $1 \leq \tau \in \mathbb{R}$, as the positive root of

$$\tau = \frac{2}{\pi}\int_0^1 \bar{a}_\tau t Q'(\bar{a}_\tau t)\frac{dt}{\sqrt{1-t^2}}.$$

The number $\bar{a}_\tau$ is an increasing function of τ, with $\lim_{\tau\to+\infty} \bar{a}_\tau = 1$ and

$$C_1 \tau^{-\frac{1}{\alpha+1/2}} \leq 1 - \bar{a}_\tau \leq C_2 \tau^{-\frac{1}{\alpha+1/2}},$$

where C_1 and C_2 are positive constants independent of τ and α is fixed (see [2, pp. 13, 31]).

Let $\{p_m(w)\}_m$ be the sequence of othonormal polynomials w.r.t. w with positive leading coefficients and x_k be the zeros of $p_m(w)$, located as

$$-a_m < x_1 < x_2 < \ldots < x_m < a_m,$$

where $a_m = a_m(\sqrt{w})$ is the Mhaskar–Rahmanov–Saff number related to $\sqrt{w}$ and

$$1 - a_m \sim \frac{1}{m^{\frac{1}{\alpha+1/2}}}.$$

Remark 1 We note that the coefficients of the three-term recurrence relation for the orthonormal polynomials $\{p_m(w)\}_m$ are not known in explicit form. In order to compute the zeros of $p_m(w)$, we can use the functions "aChebyshevAlgorithm" and "aGaussianNodesWeights" of the MATHEMATICA package "`Orthogonal Polynomials`" [29] and [30] (see also [12] were this procedure has been applied for the weight σ or [31–33] for similar cases).

Moreover, the Mhaskar–Rahmanov–Saff number can be computed by using a procedure similar to the one shown in [12].

We also point out that

$$1 - x_m \sim 1 - a_m \sim \frac{1}{m^{\frac{1}{\alpha+1/2}}},$$

and, if $\alpha \to 0$, we obtain $\frac{1}{m^2}$ at the righ-hand side, as for Jacobi weights.

In order to compute the Mhaskar–Rahmanov–Saff number $a_m = a_m(\sqrt{w})$ we can proceed as done in [12] for the weight σ. Anyway, for m sufficiently large, roughly speaking we can assume that $a_m := 1 - m^{-\frac{1}{\alpha+1/2}}$ and the constants do not affect the rate of convergence.

2.2 Hermite-type polynomial

In order to introduce an Hermite-type interpolation process based at the zeros of

$$q_{m+2}(x) = (a_m^2 - x^2) p_m(w, x),$$

fixed $\theta \in (0, 1)$, we set

$$x_0 := -a_m < x_i < -a_{\theta m} \le x_k \le a_{\theta m} < x_j < a_m =: x_{m+1},$$

and

$$\ell_k(x) = \frac{q_{m+2}(x)}{(x - x_k) q'_{m+2}(x_k)}, \qquad k = 0, 1, \dots, m+1,$$

where $\ell_k \in \mathbb{P}_{m+1}$ are the fundamental Lagrange polynomials.

So, for any continuous function f, with $f' \in C^0(-1, 1)$, we define the Hermite-type polynomial

$$H^*_{2m+4}(w, f, x) = \sum_{|x_k| \le a_{\theta m}} \ell_k^2(x) \left[\nu_k(x) f(x_k) + (x - x_k) f'(x_k) \right]$$

with

$$\nu_k(x) = 1 - 2(x - x_k) \ell'_k(x_k).$$

We note that, denoting by $H_{2m+4}(w, f) \in \mathbb{P}_{2m+3}$ the ordinary Hermite polynomial of f based at the nodes $x_0, x_1, \dots, x_m, x_{m+1}$, namely such that $H_{2m+4}(w, f)(x_i) = f(x_i)$ and $H_{2m+4}(w, f)'(x_i) = f'(x_i)$ $\forall i$, we have

$$H^*_{2m+4}(w, f) = H_{2m+4}(w, \chi_\theta f),$$

where χ_θ is the characteristic function of the interval $[-a_{\theta m}, a_{\theta m}]$.

Naturally, $H^*_{2m+4}(w, f) \in \mathbb{P}_{2m+3}$ and

$$H^*_{2m+4}(w, f)^{(j)}(x_i) = \begin{cases} f^{(j)}(x_i) & |x_i| \le a_{\theta m} \\ \\ 0 & |x_i| > a_{\theta m} \end{cases} \qquad j = 0, 1.$$

So, the nodes $x_i \notin [-a_{\theta m}, a_{\theta m}]$ are double zeros for $H^*_{2m+4}(w, f)$.

We can split $H^*_{2m+4}(w, f)$ as

$$H^*_{2m+4}(w, f, x) = F_m(w, f, x) + G_m(w, f', x),$$

letting

$$F_m(w, f, x) := \sum_{|x_k| \le a_{\theta m}} \ell_k^2(x) v_k(x) f(x_k)$$

be the Hermite–Fejér-type polynomial and

$$G_m(w, f', x) = \sum_{|x_k| \le a_{\theta m}} \ell_k^2(x)(x - x_k) f'(x_k).$$

We also note that in general i $H^*_{2m+4}(w, P) \neq P$

$$H^*_{2m+4}(w, \mathbf{1}, x) \neq 1$$

and

$$F_m(w, f)'(x_i) = 0, \qquad |x_i| \le a_{\theta m}.$$

2.3 Weighted function spaces

We now introduce some function spaces in which we will study our Hermite-type operator $H^*_{2m+4}(w, f)$. These spaces have been introduced and extensively studied in [3] (see also [4, 5] and [6], cfr. [7–9] and [10]). Here we present some basic facts in order to state our main results; other facts will be recalled as needed in the proofs.

Let us consider the weight

$$u(x) = (1 - x^2)^\gamma \mathrm{e}^{-\frac{1}{(1-x^2)^\alpha}} =: v^\gamma(x)\sigma(x) \tag{2}$$

with $\gamma \ge 0$, $\alpha > 0$, $x \in (-1, 1)$.

We can associate to the weight u the following function spaces. For $1 \le p < \infty$, by L^p_u we denote the set of all measurable functions f such that

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_{-1}^1 |fu|^p(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

For $p=\infty$, by a slight abuse of notation, we set

$$L_u^\infty := C_u = \left\{ f \in C^0(-1,1) : \lim_{x\to\pm1} f(x)u(x) = 0 \right\},$$

and we equip this space with the norm

$$\|f\|_{L_u^\infty} := \|fu\|_\infty = \sup_{x\in(-1,1)} |f(x)u(x)|.$$

Note that the limit conditions in the definition of C_u are necessary and sufficient for the validity of the Weierstrass theorem in C_u.

We emphasize that, as mentioned in the Introduction, the functions belonging to the spaces L_u^p can grow exponentially at the endpoints ±1.

For smoother function, we define the Sobolev-type subspaces of L_u^p as

$$W_r^p(u) = \left\{ f \in L_u^p : \ f^{(r-1)} \in AC(-1,1), \ \|f^{(r)}\varphi^r u\|_p < \infty \right\}, \quad 1 \le r \in \mathbb{N},$$

where $1 \le p \le \infty$, $\varphi(x) := \sqrt{1-x^2}$ and $AC(-1,1)$ denotes the set of all functions which are absolutely continuous on every closed subinterval of $(-1,1)$. We equip these spaces with the norm

$$\|f\|_{W_r^p(u)} = \|fu\|_p + \|f^{(r)}\varphi^r u\|_p.$$

In order to introduce some further subspaces of L_u^p, for $1 \le p \le \infty$, $r \ge 1$ and for all sufficiently small $t>0$ (say $t<t_0$), we define the main part of the $r-$th modulus of smoothness as

$$\Omega_\varphi^r(f,t)_{u,p} = \sup_{0<h\le t} \left\| \Delta_{h\varphi}^r(f)\,u \right\|_{L^p(\mathcal{I}_h)},$$

where $\mathcal{I}_h = [-h^*, h^*]$, $h^* = 1 - \mathrm{A}\, h^{1/(\alpha+1/2)}$, $\mathrm{A}>0$ is a fixed constant, and

$$\Delta_{h\varphi}^r f(x) = \sum_{i=0}^r \binom{r}{i} (-1)^i f\left(x + (r-2i)\frac{h\varphi(x)}{2} \right).$$

Then the complete $r-$th modulus of smoothness is given by

$$\omega_\varphi^r(f,t)_{u,p} = \Omega_\varphi^r(f,t)_{u,p} + \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)\,u\|_{L^p[-1,-t^*]} + \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)\,u\|_{L^p[t^*,1]}$$

with $t^* = 1 - \mathrm{A}\, t^{1/(\alpha+1/2)}$ and $\mathrm{A}>0$. We emphasize that the behavior of $\omega_\varphi^r(f,t)_{u,p}$ is independent of the constant A.

Moreover, for any $f \in W_r^p(u)$, with $r \geq 1$ and $1 \leq p \leq \infty$, we have

$$\Omega_\varphi^r(f,t)_{u,p} \sim \sup_{0<h\leq t} \inf_{g\in W_r^p} \left\{ \|(f-g)u\|_{L^p(\mathcal{I}_h)} + h^r \left\| g^{(r)}\varphi^r u \right\|_{L^p(\mathcal{I}_h)} \right\}$$
$$\leq \mathcal{C} \sup_{0<h\leq t} h^r \left\| f^{(r)}\varphi^r u \right\|_{L^p(\mathcal{I}_h)}$$

where $\mathcal{C} \neq \mathcal{C}(f,t)$.

Furthermore, the following equivalence holds

$$\omega_\varphi^r\left(f,\frac{1}{m}\right)_{u,p} \sim \inf_{P_m\in\mathbb{P}_m} \left\{ \|(f-P_m)u\|_p + \frac{1}{m^r}\left\| P_m^{(r)}\varphi^r u \right\|_p \right\}.$$

By means of the $r-$th modulus of smoothness, for $1 \leq p \leq \infty$, we can define the Zygmund-type spaces

$$Z_s^p(u) := Z_{s,r}^p(u) = \left\{ f \in L_u^p \, : \, \sup_{t>0} \frac{\omega_\varphi^r(f,t)_{u,p}}{t^s} < \infty, \; r > s \right\}, \quad 0 < s \in \mathbb{R},$$

equipped with the norm

$$\|f\|_{Z_{s,r}^p(u)} = \|f\|_{L_u^p} + \sup_{t>0} \frac{\omega_\varphi^r(f,t)_{u,p}}{t^s}.$$

2.4 Weighted polynomial approximation

Let us denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m and by

$$E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \|(f-P)\,u\|_p$$

the error of best polynomial approximation in L_u^p, $1 \leq p \leq \infty$. A polynomial realizing the infimum in the previous definition is called polynomial of best approximation for $f \in L_u^p$.

The next theorem collects the Jackson and Stechkin type inequalities and it can be deduced from the results proved in [3, Theorems 3.4, 3.5 and 3.6, p. 175] for the weight $\sigma(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, taking into account that the weight u has a similar behaviour (see also [15, Proposition 2.3, p. 627] and [6, Theorems 4.1 and 4.2, p. 297]).

Theorem 1 *Let $u(x) = (1-x^2)^\gamma \mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}$, with $\alpha > 0$ and $\gamma \geq 0$. For any $f \in L_u^p$, $1 \leq p \leq \infty$, the inequalities*

$$E_m(f)_{u,p} \leq \mathcal{C}\,\omega_\varphi^r\left(f,\frac{1}{m}\right)_{u,p}, \tag{3}$$

$$E_m(f)_{u,p} \leq \mathcal{C}\int_0^{\frac{1}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t}dt \tag{4}$$

and

$$\omega_\varphi^r\left(f,\frac{1}{m}\right)_{u,p} \le \frac{\mathcal{C}}{m^r}\sum_{i=0}^{m}(1+i)^{r-1}E_i(f)_{u,p}\,, \tag{5}$$

hold with $\mathcal{C}$ independent of m and f.

Note that (3) and (5) are similar to the Jackson and Stechkin inequalities proved by Ditzian and Totik for Jacobi weights [7], with a proper different modulus of smoothness, while (4) is a weak form of the Jackson inequality. Moreover, the proof of Stechkin-type inequality (5) is based on the Bernstein inequality (see [6])

$$\|P_m'\varphi u\|_p \le \mathcal{C}m\|P_m u\|_p.$$

Using Theorem 1, we can characterize the weighted function spaces L_u^p, namely by

$$\lim_m E_m(f)_{u,p} = 0 \quad \Leftrightarrow \quad f \in L_u^p.$$

Moreover, we deduce the following more explicit estimates in Sobolev and Zygmund spaces

$$E_m(f)_{u,p} \le \frac{\mathcal{C}}{m^r}\|f\|_{W_r^p(u)}\,, \qquad \forall f \in W_r^p(u)\,,\ r \ge 1\,, \tag{6}$$

and

$$E_m(f)_{u,p} \le \frac{\mathcal{C}}{m^s}\|f\|_{Z_s^p(u)}, \qquad \forall f \in Z_s^p(u),\ s > 0,$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$ and $1 \le p \le \infty$.

Finally, the following embedding theorem have been proved in [6].

Theorem 2 *If $f \in L_u^p$, $1 \le p < \infty$, is such that*

$$\int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty, \qquad r \ge 1,$$

then f is continuous on $(-1, 1)$, while if

$$\int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\nu/p}}\,\mathrm{d}t < \infty, \qquad r \ge 1,$$

where $\nu = (2\alpha+2)/(2\alpha+1)$ then $f \in C_u$.

2.5 Lagrange interpolation

Let us now consider the Lagrange polynomial

$$L_{m+2}^*(w, f, x) = \sum_{|x_k| \le a_{\theta m}} \frac{q_{m+2}(x)}{(x-x_k)q_{m+2}'(x_k)} f(x_k),$$

interpolating a continuous function f at the zeros of $p_m(w)$ belonging to $[-a_{\theta m}, a_{\theta m}]$ and vanishing at the remaining zeros and at $\pm a_m$. The following theorem, proved in [17] will be crucial for our aims.

Theorem 3 *Let $w = v^\beta \sigma$. For any $f \in C(-1,1)$ and for $1 < p < \infty$, there exists a constant $\mathcal{C}_\theta$ depending on $\theta \in (0,1)$ such that*

$$\left\| L^*_{m+2}(w,f) v^\mu \sqrt{\sigma} \right\|_p^p \le \mathcal{C}_\theta \sum_{|x_k| \le a_{\theta m}} \Delta x_k \left| f(x_k) v^\mu(x_k) \sqrt{\sigma(x_k)} \right|^p$$

.$u > -\frac{1}{p}$, if and only if

$$-\frac{1}{p} < \mu - \frac{\beta}{2} + \frac{3}{4} < 1 - \frac{1}{p}.$$

While, for $p = \infty$, we have

$$\left\| L^*_{m+2}(w,f) v^\mu \sqrt{\sigma} \right\|_\infty \le \mathcal{C}_\theta (\log m) \max_{|x_k| \le a_{\theta m}} \left| f(x_k) v^\mu(x_k) \sqrt{\sigma(x_k)}. \right|$$

with $\mu \ge 0$, if and only if

$$0 \le \mu - \frac{\beta}{2} + \frac{3}{4} \le 1.$$

In both cases $\mathcal{C}_\theta$ is independent of f and m.

Finally, we want to give the main idea that justifies the '‘truncation” in the definition of Lagrange and Hermite interpolation based at Pollaczek-type zeros. For any $P_m \in \mathbb{P}_m$ $1 \le p \le \infty$, the restricted range inequalities

$$\|P_m u\|_p \le \mathcal{C} \|P_m u\|_{L^p(A_m)}, \qquad A_m = [-a_m, a_m], \tag{7}$$

and

$$\|P_m u\|_{L^p(A'_{sm})} \le \mathcal{C} e^{-c m^\eta} \|P_m u\|_p, \quad A'_{sm} = [-1,1] \setminus [-a_{sm}, a_{sm}], \; s > 1,$$

hold with $c, \mathcal{C}$ independent of m and P_m, $\eta = 2\alpha/(2\alpha+1)$. From the second inequality, for any $f \in L^p_u$, we deduce

$$\|fu\|_p \le \mathcal{C} \|fu\|_{L^p(A_{\theta m})} + E_M(f)_{u,p}.$$

So, the main part of $\|fu\|_p$ is $\|fu\|_{L^p(A_{\theta m})} = \|\chi f u\|_p$. This suggests to apply this approximation processes only to χf, χ is the characteristic function of $A_{\theta m}$.

3 Main results

Let us first study the bahaviour of our Hermite and Hermite-Fejér type operator in weighted L^p−spaces, showing the boundedness of these operators.

Theorem 4 *Let w and u the weights defined by (1) and (2). For any $f \in L^p_u$, $1 < p < \infty$ such that $\displaystyle\int_0^1 \frac{\Omega_\varphi(f', t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty$, we have*

$$\left\| H^*_{2m+4}(w, f)u \right\|_p \sim \left(\sum_{|x_k|\le a_{\theta m}} \Delta x_k \left[|fu|^p(x_k) + \frac{|f'\varphi u|^p(x_k)}{m^p} \right] \right)^{1/p} \tag{8}$$

and

$$\left\| F^*_m(w, f)u \right\|_p \sim \left(\sum_{|x_k|\le a_{\theta m}} \Delta x_k |fu|^p(x_k) \right)^{1/p} \tag{9}$$

where the constants in $\sim$ are independent of m, f, f', if and only if

$$\frac{uv^2}{w\varphi} \in L^p \quad and \quad \frac{w\varphi}{uv^2} \in L^q, \quad v(x) = 1 - x^2, \ \frac{1}{p} + \frac{1}{q} = 1. \tag{10}$$

Note that the conditions (10) are independent of the exponential part but depend only on the algebraic part of the weights w and u, since they can be written as

$$-\frac{1}{p} < \gamma - \beta + \frac{3}{2} < 1 - \frac{1}{p}.$$

Moreover, this conditions can be obtained from those of Theorem 3 taking the square of $\sqrt{w\varphi}$ and replacing $v^\gamma\sqrt{\sigma}$ with $u = v^\gamma\sigma$. This is in accordance with the results concerning Hermite interpolation at Jacobi zeros [34].

We also remark that, by Theorem 2, the condition $\int_0^1 \frac{\Omega_\varphi(f',t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty$ implies the continuity of f'.

Finally, we emphasize that the constants in $\sim$ in Theorem 4 depend on the truncation parameter θ (see Sect. 4). As for other exponential weights, it is not possible to obtain similar results for the classical Hermite and Hermite–Feér interpolation (see also [12] and [17] for the associated Gaussian rules and Lagrange interpolation, respectively).

Now, let us consider our operators in weighted spaces of continuous functions

Theorem 5 *Let w and u be the weights in (1) and (2). For any $f \in W^\infty_1$, we have*

$$\left\| H^*_{2m+4}(w, f)u \right\|_\infty \le \mathcal{C}(\log m) \left[\|fu\|_\infty + \frac{1}{m} \|f'\varphi u\|_\infty \right] \tag{11}$$

and for any $f \in C_{\bar{u}}$, with $\bar{u}(x) = u(x) \log \frac{\mathrm{e}}{1-x^2}$, we get

$$\left\| F^*_m(w, f)u \right\|_\infty \le \mathcal{C}\|f\bar{u}\|_\infty \tag{12}$$

if and only if the parameters of the weights satisfy

$$0 \le \gamma - \beta + \frac{3}{2} \le 1\,. \tag{13}$$

In both inequalities the constants $\mathcal{C}$ depend on θ but are independent of f and m.

The following theorems show the convergence of our Hermite and Hermite–Fejér operators in weighted L^p-spaces and in weighted uniform metric under the assumptions of Theorems 4 and 5, respectively, providing precise error estimates.

Theorem 6 *Let $1 < p < \infty$. If the weights w and u satisfy the assumptions (10), then, for any $f \in L^p_u$ such that $\int_0^1 \frac{\Omega_\varphi(f', t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty$, we have*

$$\begin{aligned} &\left\| \left[f - H^*_{2m+4}(w, f) \right] u \right\|_p \\ &\le \frac{\mathcal{C}_\theta}{m^{1+1/p}} \int_0^{1/m} \frac{\Omega^r_\varphi(f', t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t + \mathcal{C}_\theta\, \mathrm{e}^{-cm^\eta} \left[\|fu\|_p + \|f'u\|_p \right] \end{aligned} \tag{14}$$

and

$$\left\| \left[f - F^*_m(w, f) \right] u \right\|_p \le \frac{\mathcal{C}_\theta}{m^{1/p}} \int_0^{1/m} \frac{\Omega_\varphi(f, t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t + \mathcal{C}_\theta\, \mathrm{e}^{-cm^\eta} \|fu\|_p\,, \tag{15}$$

where $\eta = \frac{2\alpha}{2\alpha+1}$, $r \ge 1$ is fixed, the constants $\mathcal{C}_\theta$ and c depend on θ, but not on m and f.

Theorem 7 *If the parameters β and γ of the weights w and u satisfy (13), then, for any $f \in W^\infty_1$, we have*

$$\left\| \left[f - H^*_{2m+4}(w, f) \right] u \right\|_\infty \le \mathcal{C} \left[\frac{\log m}{m} E_M(f')_{\varphi u,\infty} + \mathrm{e}^{-cm^\eta} \|f'\varphi u\|_\infty \right] \tag{16}$$

Moreover, with $\bar{u}(x) = u(x) \log \frac{\mathrm{e}}{1-x^2}$, we get

$$\left\| \left[f - F^*_m(w, f) \right] u \right\|_\infty \le \mathcal{C}_\theta \left[\Omega_\varphi \left(f, \frac{\log m}{m} \right)_{\bar{u},\infty} + \mathrm{e}^{-cm^\eta} \|f\bar{u}\|_\infty \right] \tag{17}$$

where $M \sim m$, $\eta = \frac{2\alpha}{2\alpha+1}$, the constants $\mathcal{C}_\theta$ and c depend on θ, but not on m and f.

For instance, if $f \in W^\infty_1(u)$, using also Theorem 1, we deduce

$$\left\| \left[f - H^*_{2m+4}(w, f) \right] u \right\|_\infty \le \mathcal{C} \frac{\log m}{m} \|f\|_{W^\infty_1(u)},$$

for m sufficiently large, namely our Hermite-type polynomial converges to f with the order of the best weighted approximation times $\log m$ (cf. (6)).

To conclude this section we note that, as already mentioned for Theorem 4, in all our main results the constants depend on the truncation parameter θ. However θ is fixed and independent of m, so it only affects the constants and not the order of convergence. Since $\mathcal{C} = O(\frac{1}{\theta})$ and $\mathcal{C} = O(\frac{1}{1-\theta})$, the most appropriate choice seems $\theta = 1/2$ (see Sect. 4 and [17] for more details).

4 Proofs

4.1 Preliminary results

The following estimates involving the orthonormal polynomials $p_m(w)$ and their zeros x_k have been proved by Levin and Lubinsky in [1, 2]

$$|p_m(w,x)|\sqrt{w(x)\sqrt{a_m^2-x^2}} \le \mathcal{C}, \qquad x \in [-a_m, a_m] \tag{18}$$

$$\frac{1}{\left|p_m'(w,x_k)\right|\sqrt{w(x_k)\sqrt{a_m^2-x_k^2}}} \sim \Delta x_k, \quad k=1,\ldots,m, \tag{19}$$

$$|p_m(w,x)|\sqrt{w(x)\sqrt{a_m^2-x^2}} \sim \frac{|x-x_d|}{\Delta x_d} \tag{20}$$

where $x \in (x_1, x_m)$ and x_d in a zero closest to x.

Now, let us set

$$B_k(f,x) := u(x)\ell_k^2(x)\left\{\left(1-2\ell_k'(x_k)(x-x_k)\right)f(x_k)+(x-x_k)f'(x_k)\right\}.$$

Obviously $B_k(f, x_k) = u(x_k)f(x_k)$ and, using (19) and (20), for $|x - x_k| \le \frac{\Delta x_k}{8}$, i.e., $x \in [x_k - \frac{\Delta x_k}{8}, x_k + \frac{\Delta x_k}{8}]$, we have

$$|B_k(f,x)| \le \mathcal{C}\left(|fu|(x_k)+\frac{|f'\varphi u|(x_k)}{m}\right), \qquad |x_k| \le a_{\theta m}, \tag{21}$$

taking also into account that

$$\varphi(x)=\sqrt{1-x^2} \sim \sqrt{a_m^2-x^2} \qquad |x| \le a_{\theta m}.$$

Moreover, for $x \neq x_k$, we can write

$$B_k(f,x)=u(x)\ell_k^2(x)\frac{(x-x_k)}{\Delta x_k}\left[\left(\frac{\Delta x_k}{x-x_k}-2\Delta x_k\ell_k'(x_k)\right)f(x_k)+\Delta x_k f'(x_k)\right].$$

Setting

$$b(x_k) = \frac{\Delta x_k}{x - x_k} - 2\Delta x_k \ell_k'(x_k)$$

with $q_{m+2}(x) = (a_m^2 - x^2)p_m(x)$, we get

$$B_k(f, x) = u(x)q_{m+2}(x)\ell_k(x)\frac{b(x_k)f(x_k) + \Delta x_k f'(x_k)}{q_{m+2}'(x_k)\Delta x_k}$$

and then, with $x \neq x_k$,

$$B_k(f, x) = u(x)q_{m+2}(x)\ell_k(x)F(x_k), \tag{22}$$

where

$$F(x_k) = \frac{b(x_k)f(x_k) + \Delta x_k f'(x_k)}{q_{m+2}'(x_k)\Delta x_k}.$$

Proposition 1 *If* $|x - x_k| > c\Delta x_k$, $c > 0$, *then*

$$|b(x_k)| \leq \mathcal{C} \tag{23}$$

and

$$\left|\frac{1 - 2(x - x_k)\ell_k'(x_k)}{\log \frac{\mathrm{e}}{1 - x_k^2}}\right| \leq \mathcal{C}\left(1 + \frac{1}{\log m}\frac{|x - x_k|}{\Delta x_k}\right) \tag{24}$$

with $|x_k| \leq a_{\theta m}$ *and* $\mathcal{C} = \mathcal{C}(\theta) \neq \mathcal{C}(m, k)$.

The proof of Proposition 1 will be given in the Appendix.

4.2 Proofs of the main results

We are now able to prove Theorems 4, 5, 6 and 7.

Proof of Theorem 4 Let us first prove (8). For $1 < p < \infty$, using the restricted range inequality (7) with $A_m = [-a_{2m+3}, a_{2m+3}]$, we have

$$\begin{aligned}\left\|H_{2m+4}^*(w, f)u\right\|_p &\leq \mathcal{C}\left\|H_{2m+4}^*(w, f)u\right\|_{L^p(A_m)}\\ &= \mathcal{C}\sup_{\|g\|_q=1}\int_{A_m} H_{2m+4}^*(w, f, x)u(x)g(x)\,\mathrm{d}x\\ &=: \mathcal{C}\sup_{\|g\|_q=1}\Gamma(g).\end{aligned}$$

Setting $I_j = \left[x_j - \frac{\delta x_j}{8}, x_j + \frac{\delta x_j}{8}\right]$, $|x_j| \le a_{\theta m}$, and $\mathcal{I} = \cup_{|x_j|\le a_{\theta m}} I_j$, with the notation of the previous Section, we can split the integral $A(g)$ as follows

$$\begin{aligned} A(g) &= \left\{\int_{\mathcal{I}} + \int_{A_m\setminus\mathcal{I}}\right\} \sum_{|x_k|\le a_{\theta m}} B_k(f,x)g(x)\,\mathrm{d}x \\ &= \sum_{|x_j|\le a_{\theta m}} \int_{I_j} \sum_{|x_k|\le a_{\theta m}} B_k(f,x)g(x)\,\mathrm{d}x \\ &\quad + \int_{A_m\setminus\mathcal{I}} \sum_{|x_k|\le a_{\theta m}} B_k(f,x)g(x)\,\mathrm{d}x \\ &= \sum_{|x_j|\le a_{\theta m}} \int_{I_j} B_j(f,x)g(x)\,\mathrm{d}x + \sum_{|x_j|\le a_{\theta m}} \int_{I_j} \sum_{|x_k|\le a_{\theta m},\ k\ne j} B_k(f,x)g(x)\,\mathrm{d}x \\ &\quad + \int_{A_m\setminus\mathcal{I}} \sum_{|x_k|\le a_{\theta m}} B_k(f,x)g(x)\,\mathrm{d}x. \end{aligned}$$

For the first term, since $|x - x_j| \le \frac{\Delta x_j}{8}$, using (21) we have

$$\begin{aligned} &\left|\sum_{|x_j|\le a_{\theta m}} \int_{I_j} B_k(f,x)g(x)\,\mathrm{d}x\right| \\ &\quad \le \mathcal{C} \sum_{|x_j|\le a_{\theta m}} \left[|fu|(x_j) + \frac{|f'\varphi u|(x_j)}{m}\right] \int_{I_j} |g(x)|\,\mathrm{d}x \\ &\quad \le \mathcal{C} \sum_{|x_j|\le a_{\theta m}} \left[|fu|(x_j) + \frac{|f'\varphi u|(x_j)}{m}\right] (\Delta x_j)^{1/p} \left(\int_{I_j} |g(x)|^q\,\mathrm{d}x\right)^{1/q} \\ &\quad \le \mathcal{C} \left(\sum_{|x_j|\le a_{\theta m}} \Delta x_j \left[|fu|^p(x_j) + \frac{|f'\varphi u|^p(x_j)}{m}\right]\right)^{1/p} \|g\|_q\,. \end{aligned}$$

Whereas, using the Hölder inequality and (18) for q_{m+2}, the second and third terms can be estimated as

$$\begin{aligned} &\left|\sum_{|x_j|\le a_{\theta m}} \int_{I_j} \sum_{|x_k|\le a_k,\ k\ne j} B_k(f,x)g(x)\,\mathrm{d}x + \int_{A_m\setminus\mathcal{I}} \sum_{|x_k|\le a_{\theta m}} B_k(f,x)g(x)\,\mathrm{d}x\right| \\ &\quad \le \mathcal{C} \int_{A_m} \sum_{|x_k|\le a_{\theta m}} |B_k(f,x)g(x)|\,\mathrm{d}x \end{aligned}$$

$$= \mathcal{C} \int_{A_m} \sum_{|x_k| \le a_{\theta m}} \left| q_{m+2}(x) L^*_{m+2}(w, F, x) g(x) \right| \, \mathrm{d}x$$

$$\le \mathcal{C} \left\| L^*_{m+2}(w, F) \frac{uv}{\sqrt{w\varphi}} \right\|_{L^p(A_m)} \|g\|_q \, .$$

Now, using Theorem 3, we obtain

$$\left\| L^*_{m+2}(w, F) \frac{uv}{\sqrt{w\varphi}} \right\|^p_{L^p(A_m)} \le \mathcal{C} \sum_{|x_k| \le a_{\theta m}} \Delta x_k \left| F(x_k) \frac{u(x_k) v(x_k)}{\sqrt{w(x_k)\varphi(x_k)}} \right|^p .$$

By (22),

$$\frac{1}{q'_{m+2}(x_k)\Delta x_k} \sim \frac{\sqrt{w(x_k)\varphi(x_k)}}{v(x_k)}$$

and (23), inequality (8) easily follows and the conditions (10) are necessary and sufficient by virtue of Theorem 3.

Formula (9) can be obtained by using similar arguments with $f' = 0$. So, we omit the details. □

Proof of Theorem 5 Let us first prove inequality (11).

If $x \in [-a_{\theta m}, a_{\theta m}]$ and x_d is a zero closest to x, we can write

$$H^*_{2m+4}(w, f, x) u(x) = f(x_d) u(x_d) + \sum_{|x_k| \le a_{\theta m}, k \ne d} B_k(f, x).$$

Using Theorem 3 we have

$$\sum_{|x_k| \le a_{\theta m}, k \ne d} B_k(f, x) \le \mathcal{C} (\log m) \left[\|fu\|_\infty + \frac{1}{m} \|f'\varphi u\|_\infty \right]$$

and inequality (11) follows.

In order to prove inequality (12), let $x \in [-a_{\theta m}, a_{\theta m}]$ and x_d be a zero closest to x. We can write

$$F^*_m(w, f, x) u(x) = f(x_d) u(x_d) + \sum_{|x_k| \le a_{\theta m}, k \ne d} \ell^2_k(x) v_k(x) f(x_k) u(x).$$

By using (24) we get

$$
\begin{aligned}
\left|F_m^*(w,f,x)u(x)\right| &\le |f(x_d)u(x_d)| + \sum_{|x_k|\le a_{\theta m},k\ne d} \left|\ell_k^2(x)\nu_k(x)f(x_k)u(x)\right| \\
&\le |f(x_d)u(x_d)| + \mathcal{C}\sum_{|x_k|\le a_{\theta m},k\ne d} u(x)\frac{\ell_k^2(x)}{u(x_k)}\left(1+\frac{|x-x_k|}{(\log m)\Delta x_k}\right)|f(x_k)|\bar{u}(x_k) \\
&= |f(x_d)u(x_d)| + \mathcal{C}\sum_{|x_k|\le a_{\theta m},k\ne d} u(x)\frac{\ell_k^2(x)}{u(x_k)}|f(x_k)|\bar{u}(x_k) \\
&\quad + \frac{\mathcal{C}}{\log m}\sum_{|x_k|\le a_{\theta m},k\ne d} u(x)\frac{\ell_k^2(x)}{u(x_k)}\frac{|x-x_k|}{\Delta x_k}|f(x_k)|\bar{u}(x_k)\,.
\end{aligned}
$$

By (18) and (19), the first and the second terms at the right hand side can be estimated as

$$
\begin{aligned}
&|f(x_d)u(x_d)| + \mathcal{C}\sum_{|x_k|\le a_{\theta m},k\ne d} u(x)\frac{\ell_k^2(x)}{u(x_k)}|f(x_k)|\bar{u}(x_k) \\
&\quad\le \mathcal{C}\|f\bar{u}\|_\infty\left[1+\sum_{|x_k|\le a_{\theta m},k\ne d}\left(\frac{1-x^2}{1-x_k^2}\right)^{\gamma-\beta+3/2}\left(\frac{\Delta x_k}{x-x_k}\right)^2\right] \\
&\quad\le \mathcal{C}\|f\bar{u}\|_\infty\,,
\end{aligned}
$$

under the assumption (13), i.e. $0\le\gamma-\beta+3/2\le 1$, using the inequality

$$
\sum_{|x_k|\le a_{\theta m},k\ne d}\left(\frac{1-x^2}{1-x_k^2}\right)^{\gamma-\beta+3/2}\left(\frac{\Delta x_k}{x-x_k}\right)^2 \le \mathcal{C}\,, \tag{25}
$$

proved in the Appendix.

Using again (18) and (19), for the third term we have

$$
\begin{aligned}
&\frac{\mathcal{C}}{\log m}\sum_{|x_k|\le a_{\theta m},k\ne d} u(x)\frac{\ell_k^2(x)}{u(x_k)}\frac{|x-x_k|}{\Delta x_k}|f(x_k)|\bar{u}(x_k) \\
&\quad\le \mathcal{C}\frac{\|f\bar{u}\|_\infty}{\log m}\left[1+\sum_{|x_k|\le a_{\theta m},k\ne d}\left(\frac{1-x^2}{1-x_k^2}\right)^{\gamma-\beta+3/2}\frac{\Delta x_k}{|x-x_k|}\right] \\
&\quad\le \mathcal{C}\|f\bar{u}\|_\infty\,,
\end{aligned}
$$

under the assumption (13). So, inequality (12) follows. □

Proof of Theorem 6 Taking into account (8) and [17, Prop. 3.3, p. 73], we get

$$\begin{aligned}\left\| H^*_{2m+4}(w,f)u\right\|_p &\le \mathcal{C}\left(\sum_{|x_k|\le a_{\theta m}} \Delta x_k\left[|fu|^p(x_k)+\frac{|f'\varphi u|^p(x_k)}{m^p}\right]\right)^{1/p}\\ &\le \mathcal{C}\|fu\|_p+\frac{\mathcal{C}}{m}\|f'\varphi u\|_p+\frac{\mathcal{C}_\theta}{m^{1/p}}\int_0^{1/m}\frac{\Omega^r_\varphi(f,t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t\\ &\quad+\frac{\mathcal{C}_\theta}{m^{1+1/p}}\int_0^{1/m}\frac{\Omega^r_\varphi(f',t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t\\ &\le \mathcal{C}\|fu\|_p+\frac{\mathcal{C}}{m}\|f'\varphi u\|_p+\frac{\mathcal{C}_\theta}{m^{1+1/p}}\int_0^{1/m}\frac{\Omega^r_\varphi(f',t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t.\end{aligned}$$

Let $M=\left\lfloor \frac{\theta}{1+\theta}(2m+3)\right\rfloor =: \mathcal{C}_\theta m$. For any polynomial $Q\in\mathbb{P}_M$, we can write

$$Q=H^*_{2m+4}(w,Q)+\Gamma(Q),$$

with

$$\|\Gamma(Q)u\|_p\le \mathcal{C}\mathrm{e}^{-cm^\eta}\|Qu\|_p,$$

where $\mathcal{C}$ and c depend on θ. In particular, if Q is a polynomial of quasi best approximation of $f\in L^p_u$, we have

$$\begin{aligned}&\left\|\left[f-H^*_{2m+4}(w,f)\right]u\right\|_p\\ &\quad=\left\|\left[f-Q-H^*_{2m+4}(w,f-Q)+\Gamma(Q)\right]u\right\|_p\\ &\quad\le \mathcal{C}\|(f-Q)u\|_p+\frac{\mathcal{C}}{m}\|(f'-Q')\varphi u\|_p\\ &\qquad+\frac{\mathcal{C}}{m^{1+1/p}}\int_0^{1/m}\frac{\Omega_\varphi(f'-Q',t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t+\mathcal{C}\mathrm{e}^{-cm^\eta}\|Qu\|_p.\end{aligned}$$

Now, since [3]

$$\mathcal{C}\|(f-Q)u\|_p+\frac{\mathcal{C}}{m}\|(f'-Q')\varphi u\|_p\le \mathcal{C}\omega_\varphi\left(f,\frac{1}{m}\right)_{u,p}+\frac{\mathcal{C}}{m}\|f'\varphi u\|_p\le\frac{\mathcal{C}}{m}\|f'\varphi u\|_p,$$

and estimating $\Omega_\varphi(f'-Q',t)_{\varphi u,p}$, we obtain

$$\begin{aligned}&\left\|\left[f-H^*_{2m+4}(w,f)\right]u\right\|_p\\ &\quad\le\frac{\mathcal{C}}{m}\|f'\varphi u\|_p+\frac{\mathcal{C}}{m^{1+1/p}}\int_0^{1/m}\frac{\Omega_\varphi(f',t)_{\varphi u,p}}{t^{1+1/p}}\,\mathrm{d}t+\mathcal{C}\mathrm{e}^{-cm^\eta}\|Qu\|_p.\end{aligned}\tag{26}$$

Consequently, if $P_{M-1} \in \mathbb{P}_{M-1}$ is a polynomial of quasi best approximation and $P_M \in \mathbb{P}_M$ one of its primitives, using (26) with f replaced by $f - P_M$, we have

$$\begin{aligned}
&\left\|\left[f - H^*_{2m+4}(w, f)\right] u\right\|_p \\
&\leq \|[f - P_M] u\|_p + \left\|H^*_{2m+4}(w, f - P_M) u\right\|_p + \|\Gamma(P_M)u\|_p + \|\Gamma(Q)u\|_p \\
&\leq \frac{\mathcal{C}}{m} E_{M-1}(f')_{\varphi u,p} + \frac{\mathcal{C}}{m^{1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi(f' - P_{M-1}, t)_{\varphi u,p}}{t^{1+1/p}} \,\mathrm{d}t \\
&\quad + \mathcal{C}\, \|\Gamma(Q)u\|_p + \mathcal{C}\, \|\Gamma(P_M)u\|_p \ .
\end{aligned}$$

Using also similar arguments to those in [35, p.280], it follows that

$$\begin{aligned}
&\frac{\mathcal{C}}{m} E_{M-1}(f')_{\varphi u,p} + \frac{\mathcal{C}}{m^{1+1/p}} \int_0^{1/m} \frac{\Omega^r_\varphi(f' - P_{M-1}, t)_{\varphi u,p}}{t^{1+1/p}} \,\mathrm{d}t \\
&\quad \leq \frac{\mathcal{C}}{m^{1+1/p}} \int_0^{1/m} \frac{\Omega^r_\varphi(f', t)_{\varphi u,p}}{t^{1+1/p}} \,\mathrm{d}t
\end{aligned}$$

and

$$\mathcal{C}\, \|\Gamma(Q)u\|_p + \mathcal{C}\, \|\Gamma(P_M)u\|_p \leq \mathcal{C} \mathrm{e}^{-cm^\eta} \left(\|fu\|_p + \|f'\varphi u\|_p\right),$$

so the proof of (14) is complete

We omit the proof of (15) which is similar but simpler than the previous one. □

Proof of Theorem 7 Let us first prove (16). Letting $P_M \in \mathbb{P}_M$, with $M = \lfloor \frac{\theta}{1+\theta}(2m + 3) \rfloor$, be a polynomial of quasi best approximation for $f \in C_u$, we have

$$\begin{aligned}
&\left\|\left[f - H^*_{2m+4}(w, f)\right] u\right\|_\infty \\
&= \left\|\left[(f - P_M) - H^*_{2m+4}(w, f - P_M) + \Gamma(P_M)\right] u\right\|_\infty \\
&\leq \|(f - P_M)u\|_\infty \\
&\quad + \mathcal{C} \log m \left[\|(f - P_M)u\|_\infty + \frac{\left\|(f - P_M)'\varphi u\right\|_\infty}{m} + \|\Gamma(P_M)u\|_\infty\right] \\
&\leq \mathcal{C}\left[\|(f - P_M)u\|_\infty + \frac{\log m}{m} \left\|P'_M \varphi u\right\|_\infty\right] \\
&\quad + \mathcal{C} \frac{\log m}{m} \left\|f'\varphi u\right\|_\infty + \|\Gamma(P_M)u\|_\infty ,
\end{aligned}$$

and then

$$\left\|\left[f - H^*_{2m+4}(w, f)\right] u\right\|_\infty \leq \mathcal{C} \frac{\log m}{m} \left\|f'\varphi u\right\|_\infty + \|\Gamma(P_M)u\|_\infty . \qquad (27)$$

Consequently, if $q_{m-1} \in \mathbb{P}_{M-1}$ is of quasi best approximation for f' in $C_{\varphi u}$ and $Q_M \in \mathbb{P}_M$ is one of its primitives, using (27) with f replaced by $f - Q_M$, we obtain

$$\begin{aligned}
&\left\| \left[f - H^*_{2m+4}(w, f) \right] u \right\|_\infty \\
&\quad \le \|(f - Q_M)u\|_\infty + \left\| H^*_{2m+4}(w, f - Q_M)]u \right\|_\infty + \|\Gamma(Q_M)u\|_\infty \\
&\quad \le \mathcal{C}\frac{\log m}{m} E_{M-1}(f')_{\varphi u,\infty} + (\|\Gamma(P_M)u\|_\infty + \|\Gamma(Q_M)u\|_\infty) \\
&\quad \le \mathcal{C}\frac{\log m}{m} E_{M-1}(f')_{\varphi u,\infty} + \mathcal{C}\mathrm{e}^{-cM^\eta}\|f\|_{W_1^\infty(u)},
\end{aligned}$$

and then (16).

In order to prove (17), we note that for any $P_M \in \mathbb{P}_M$

$$P_M = F^*_m(w, P_M) + G^*_m(w, P_M) + \Gamma(P_M)$$

and

$$\left\| G^*_m(w, P_M)u \right\|_\infty \le \mathcal{C}\frac{\log M}{M}\|P'_M\varphi u\|_\infty.$$

It follows that

$$\begin{aligned}
&\left\| \left[f - F^*_m(w, f) \right] u \right\|_\infty \\
&\quad \le \left\| \left[(f - P_M) - F^*_m(w, f - P_M) \right] u \right\|_\infty + \left\| G^*_m(w, P_M)u \right\|_\infty + \|\Gamma(P_M)u\|_\infty \\
&\quad \le \mathcal{C}\left[\|(f - P_M)\bar{u}\|_\infty + \frac{\log m}{m}\|P'_M\varphi\bar{u}\|_\infty \right] \\
&\quad \le \mathcal{C}\left[\omega_\varphi\left(f, \frac{\log M}{M}\right)_{\bar{u},\infty} + \mathrm{e}^{-cM^\eta}\|f\bar{u}\|_\infty \right],
\end{aligned}$$

which completes the proof. □

Appendix

In order to prove Propoposition 1 we recall that, for any polynomial $P_m \in \mathbb{P}_m$, the following Bernstein inequality [6]

$$\|P'_m\varphi u\|_\infty \le \mathcal{C}m\|P_m u\|_\infty \tag{28}$$

holds with $\mathcal{C}$ independent of m and P_m.

Proof of Propoposition 1 Recalling the definition of ℓ_k, we have

$$\left| \frac{\Delta x_k}{x - x_k} - 2\Delta x_k \ell'_k(x_k) \right| \le \mathcal{C} + \frac{4\Delta x_k}{a_m^2 - x_k^2} + \Delta x_k \left| \frac{p''_m(w, x_k)}{p'_m(w, x_k)} \right|.$$

Taking into accont that $|x_k| \leq a_{\theta m}$ and $a_m^2 - x_k^2 \sim 1 - x_k^2$, for the second addend we get

$$\frac{4\Delta x_k}{a_m^2 - x_k^2} \leq \frac{\mathcal{C}}{m\sqrt{1-x_k^2}} \leq \frac{\mathcal{C}}{m(1-a_{\theta m}^2)} \leq \mathcal{C}.$$

Moreover, using (19), the Bernstein inequality (28) and (18), we obtain

$$\begin{aligned} \Delta x_k \left| \frac{p_m''(w, x_k)}{p_m'(w, x_k)} \right| &\sim \frac{1}{m} \left| \varphi(x_k) \sqrt{w(x_k)\varphi(x_k)} \Delta x_k \, p_m''(w, x_k) \right| \\ &\leq \frac{\mathcal{C}}{m} \left\| \varphi \sqrt{w\varphi} \, p_m'(w) \right\|_\infty \\ &\leq \mathcal{C} \left\| \sqrt{w\varphi} \, p_m(w) \right\|_\infty \leq \mathcal{C}, \end{aligned}$$

and (23) follows.

For (24) we have

$$\left| \frac{1 - 2(x - x_k)\ell_k'(x_k)}{\log \frac{\mathrm{e}}{1-x_k^2}} \right| \leq 1 + \left(\frac{4x_k}{a_m^2 - x_k^2} + \left| \frac{p_m''(w, x_k)}{p_m'(w, x_k)} \right| \right) \frac{|x - x_k|}{\log \frac{\mathrm{e}}{1-x_k^2}}. \tag{29}$$

Now, for the second addend at the right hand side, we get

$$\begin{aligned} \frac{4x_k}{(a_m^2 - x_k^2) \log \frac{\mathrm{e}}{1-x_k^2}} &\leq \frac{\mathcal{C}}{\sqrt{1-x_k^2} \log \frac{\mathrm{e}}{1-x_k^2}} \frac{1}{\sqrt{1-x_k^2}} \\ &\leq \frac{\mathcal{C}}{\log m} \frac{m}{\sqrt{1-x_k^2}}. \end{aligned} \tag{30}$$

In order to estimate the third term, we can rewrite the weight w in (1) as

$$w(x) = \mathrm{e}^{-\left(\frac{1}{(1-x^2)^\alpha} - \log(1-x^2)^\beta\right)} =: \mathrm{e}^{-Q(x)}$$

and, using a result due to Levin and Lubinsky [1, 2]

$$\left| \frac{p_m''(w, x_k)}{p_m'(w, x_k)} \right| \leq \mathcal{C}(1 + Q'(x_k)) \leq \frac{\mathcal{C}}{(1-x_k^2)^{\alpha+1}},$$

we obtain

$$\left|\frac{p_m''(w,x_k)}{p_m'(w,x_k)}\right| \frac{1}{\log\frac{\mathrm{e}}{1-x_k^2}} \le \frac{\mathcal{C}}{(1-x_k^2)^{\alpha+1/2}\log\frac{\mathrm{e}}{1-x_k^2}} \frac{\mathcal{C}}{\sqrt{1-x_k^2}} \\ \le \mathcal{C}\frac{m}{\log m}\frac{1}{\sqrt{1-x_k^2}}. \tag{31}$$

Combining (29), (30) and (31), we get

$$\left|\frac{1-2(x-x_k)\ell_k'(x_k)}{\log\frac{\mathrm{e}}{1-x_k^2}}\right| \le \mathcal{C}\left(1+\frac{|x-x_k|}{(\log m)\Delta x_k}\right),$$

which completes the proof. □

Proof of inequality (25) Taking into account that the term related to x_{d-1} and x_{d+1} can be handled in the same way of that related to x_d, we now prove that the inequality

$$\sum_{|x_k|\le a_{\theta m},k\ne d,d\pm1}\left(\frac{1-x^2}{1-x_k^2}\right)^{\lambda}\left(\frac{\Delta x_k}{x-x_k}\right)^2 \le \mathcal{C}$$

holds if $0\le\lambda\le1$, with x_d a zero closest to $x\in[-a_m,a_m]$.

Without loss of generality, we can assume $-a_m\le x<0$.

Denoting by s this sum, we can split it into three parts

$$s=\sum_{-a_{\theta m}\le x_k\le x_{d-2}}+\sum_{x_{d+2}\le x_k\le x+\frac{1-x}{2}}+\sum_{x+\frac{1-x}{2}<x_k\le a_{\theta m}} =: s_1+s_2+s_3\,. \tag{32}$$

For s_1, since $x_d<x$, $\Delta x_k\le\Delta x_d$ and $\frac{1-x^2}{1-x_k^2}\sim\frac{1+x}{1+t}$, $t\in[x_1,x_{d-1}]$, we have

$$\begin{aligned} s_1 &\le \Delta x_d\int_{x_1}^{x_{d-1}}\left(\frac{1-x^2}{1-t}\right)^{\lambda}\frac{\mathrm{d}t}{(x-t)^2}\\ &\sim \frac{\Delta x_d}{1+x}\int_0^{1-\frac{\Delta x_d}{1+x}}\frac{\mathrm{d}t}{t^{\lambda}(1-t)^2}\\ &\le \frac{\Delta x_d}{1+x}\left[\int_0^{1/2}\frac{\mathrm{d}t}{t^{\lambda}(1-t)^2}+\int_{1/2}^{1-\frac{\Delta x_d}{1+x}}\frac{\mathrm{d}t}{t^{\lambda}(1-t)^2}\right]\\ &\le \frac{\Delta x_d}{1+x}\left[1+\int_{1/2}^{1-\frac{\Delta x_d}{1+x}}\frac{\mathrm{d}t}{(1-t)^2}\right]\\ &\le \mathcal{C}. \end{aligned} \tag{33}$$

For the term s_2 we note that $x_{d+1} \le x_k \le x + \frac{1-x}{2}$ implies

$$1 - x^2 \le \frac{4}{3}(1 - x_k^2).$$

In fact, if $x < x_k \le 0$ then $1 - x^2 < 1 - x_k^2$, whereas, if $0 < x_k < x + \frac{1-x}{2}$ then $1 - x_k^2 \ge 1 - \left(x + \frac{1-x}{2}\right)^2 \ge \frac{3}{4}(1 - x^2)$. It follows that

$$s_2 \le \sum_{x_{d+1} \le x_k \le x + \frac{1-x}{2}} \left(\frac{\Delta x_k}{x_k - x}\right)^2 \le \mathcal{C}\,. \tag{34}$$

Finally, if $x_k > 0$, we have $\frac{1-x^2}{1-x_k^2} \sim \frac{1-x}{1-x_k}$ and $(x_k - x) > 1/2$, whence

$$s_3 \le \frac{\mathcal{C}}{m} \int_{-1}^{1} \left(\frac{1-x}{1-t}\right)^{\lambda} \sqrt{1-t}\,\mathrm{d}t \le \frac{\mathcal{C}}{m}\,. \tag{35}$$

Combining (33), (34) and (35) in (32) our claim follows. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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