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LOWER BOUNDS FOR THE HEIGHT IN GALOIS EXTENSIONS

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Abstract: We prove close to sharp lower bounds for the height of an algebraic number in a Galois extension of $\mathbb{Q}$.

1. Introduction

For an algebraic number α of degree d denote by $h(\alpha) \geq 0$ the absolute logarithmic Weil height, that is

$$h(\alpha) = \frac{1}{d}\left(\log|a| + \sum_i \max\{\log|\alpha_i|, 0\}\right),$$

where a is the leading coefficient of a minimal equation over $\mathbb{Z}$ for α and α_i are its algebraic conjugates. Recall that $h(\alpha) = 0$ if and only if $\alpha = 0$ or α is a root of unity. The well-known Lehmer Problem from 1933 asks whether there is a positive constant c such that

$$h(\alpha) \geq cd^{-1}$$

whenever $\alpha \neq 0$ has degree d and is not a root of unity. This is still unsolved, but the celebrated result of Dobrowolski [7] implies that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that $h(\alpha) \geq c(\varepsilon)d^{-1-\varepsilon}$ (we will not worry about logarithmic refinements in this note).

The inequality in the Lehmer Problem has been established for various classes of α. Thus Breusch [5] proved it for non-reciprocal α, in particular whenever d is odd (see also Smyth [14] for the best possible constant), and David with the first author [1, *Corollaire* 1.7] proved it when $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension. See also their *Corollaire 1.8* for a generalization to extensions that are "almost Galois".

In this note we improve the result in the Galois case, and we even show that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that

$$h(\alpha) \geq c(\varepsilon)d^{-\varepsilon}$$

when $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension. This is related to a problem posed by Smyth during a recent BIRS workshop (see [12, problem 21, p. 17]), who asks for small positive values of $h(\alpha)$ for $\alpha \in \overline{\mathbb{Q}}$ with $\mathbb{Q}(\alpha)/\mathbb{Q}$ Galois.

2. Auxiliary results

We start with a lower bound for the height which is crucial in the proof of the next section.

Theorem 2.1. *Let $K/\mathbb{Q}$ be an abelian extension and let $\alpha_1, \ldots, \alpha_r$ be multiplicatively independent algebraic numbers. Then for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that*

$$\max_i h(\alpha_i) \geq C(\varepsilon)D^{-1/r-\varepsilon}$$

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where $D = [K(\alpha_1, \dots, \alpha_r) : K]$.

This deep result (which we have stated in a simplified form) was proved in several steps. In the special cases $K = \mathbb{Q}$ and $r = 1$, it is the main result of [1] and [3] respectively. The general case (see [6]) was the object of the Ph.D. Thesis of E. Delsinne, under the supervision of the first author.

We now state a result whose proof is implicit in [1, *Corollaire* 6.1].

Lemma 2.2. *Let $F/\mathbb{Q}$ be a Galois extension and $\alpha \in F^\times$. Let ρ be the multiplicative rank of the conjugates $\alpha_1, \dots, \alpha_d$ of α over $\mathbb{Q}$, and suppose $\rho \geq 1$. Then there exists a subfield $L \subseteq F$ which is Galois over $\mathbb{Q}$ of degree $[L : \mathbb{Q}] = n \leq n(\rho)$ and an integer $e \geq 1$ such that $\mathbb{Q}(\zeta_e) \subseteq F$ (for a primitive eth root of unity ζ_e) and $\alpha^e \in L$.*

Proof. Let e be the order of the group of roots of unity in F, so that F contains $\mathbb{Q}(\zeta_e)$. Define $\beta_i = \alpha_i^e$ $(i = 1, \dots, d)$ and $L = \mathbb{Q}(\beta_1, \dots, \beta_d)$. The $\mathbb{Z}$-module

$$\mathcal{M} = \{\beta_1^{a_1} \cdots \beta_d^{a_d} \mid a_1, \dots, a_d \in \mathbb{Z}\}$$

is torsion free (by the choice of e) and so, by the Classification Theorem for abelian groups, is free, of rank ρ. This shows that the action of $\mathrm{Gal}(L/\mathbb{Q})$ over $\mathcal{M}$ defines an injective representation $\mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{GL}_\rho(\mathbb{Z})$. Thus $\mathrm{Gal}(L/\mathbb{Q})$ identifies to a finite subgroup of $\mathrm{GL}_\rho(\mathbb{Z})$. But, by well-known results (see Remark 2.3 below), the cardinalities of the finite subgroups of $\mathrm{GL}_\rho(\mathbb{Z})$ are uniformly bounded by, say, $n = n(\rho)$.

□

Remark 2.3. To quickly see that the order of a finite subgroup of $\mathrm{GL}_\rho(\mathbb{Z})$ is uniformly bounded by some $n(\rho) < \infty$, apply Serre's result [13] which asserts that the reduction mod 3 is injective on the finite subgroups of $\mathrm{GL}_\rho(\mathbb{Z})$. This gives the bound $n(\rho) \leq 3^{\rho^2}$. More precise results are known. Feit [8] (unpublished) shows that the orthogonal group $O_\rho(\mathbb{Z})$ (of order $2^\rho \rho!$) has maximal order for $\rho = 1, 3, 5$ and for $\rho > 10$. For the seven remaining values of ρ, Feit characterizes the corresponding maximal groups. See [9] for more details and for a proof of the weaker statement $n(\rho) \leq 2^\rho \rho!$ for large ρ.

We finally recall a well-known estimate on the Euler's totient function $\phi(\cdot)$ (see for instance [10, Theorem 328, p.267]):

$$\liminf_{n \to \infty} \frac{\phi(n) \log\log n}{n} = e^{-\gamma}. \tag{2.1}$$

3. Main results

We now state two results about α which merely lie in Galois extensions, so are not necessarily generators.

Theorem 3.1. *For any integer $r \geq 1$ and any $\varepsilon > 0$ there is a positive effective constant $c(r, \varepsilon)$ with the following property. Let $F/\mathbb{Q}$ be a Galois extension of degree D and $\alpha \in F^\times$. We assume that there are r conjugates of α over $\mathbb{Q}$ which are multiplicatively independent (so that α is not a root of unity). Then*

$$h(\alpha) \geq c(r, \varepsilon) D^{-1/(r+1)-\varepsilon}.$$

Proof. The new ingredient with respect to *Corollaire 1.7* of [1] is the main result of Delsinne [6], which was not available at that time. We use standard abbreviations like $\ll_\varepsilon, \gg_{r,\varepsilon}$.

Let $\alpha_1, \dots, \alpha_d$ (with $d \le D$) be the conjugates of α over $\mathbb{Q}$ (so that they lie in F). Their multiplicative rank is at least r. If it is strictly bigger, then Theorem 2.1 (with $K = \mathbb{Q}$) applied to $r+1$ independent conjugates gives

$$h(\alpha) \gg_{r,\varepsilon} D^{-1/(r+1)-\varepsilon} .$$

Thus we may assume that the rank is exactly r.

By Lemma 2.2 there exists a number field $L \subseteq F$ of degree $[L:\mathbb{Q}] = n \le n(r)$ and an integer $e \ge 1$ such that $\mathbb{Q}(\zeta_e) \subseteq F$ and $\alpha^e \in L$.

Now let $\varepsilon > 0$. Since $\alpha^e \in L$ and $[L:\mathbb{Q}] \le n$,

$$h(\alpha) = \frac{1}{e} h(\alpha^e) \gg_r \frac{1}{e}. \tag{3.1}$$

On the other hand, the degree of F over the cyclotomic extension $\mathbb{Q}(\zeta_e)$ is $D/\phi(e)$ and $\alpha_1, \dots, \alpha_r \in F$ are multiplicatively independent. By Theorem 2.1 (with $K = \mathbb{Q}(\zeta_e)$) we have

$$h(\alpha) \gg_{r,\varepsilon} (D/\phi(e))^{-1/r-\varepsilon} \gg_{r,\varepsilon} e^{1/r} D^{-1/r-\varepsilon} \tag{3.2}$$

(use (2.1)). Combining (3.1) and (3.2) we get

$$h(\alpha)^{r+1} = h(\alpha)h(\alpha)^r \gg_{r,\varepsilon} D^{-1-r\varepsilon}.$$

□

Taking $r = 1$ we get

Corollary 3.2. *For any $\varepsilon > 0$ there is a positive effective constant $c(\varepsilon)$ with the following property. Let $F/\mathbb{Q}$ be a Galois extension of degree D. Then for any $\alpha \in F^\times$ which is not a root of unity we have*

$$h(\alpha) \ge c(\varepsilon) D^{-1/2-\varepsilon} .$$

For a direct proof of this corollary, which uses [3] instead of the deeper result of [6], see [11, exercise 16.23].

We remark that Corollary 3.2 is optimal: take for F the splitting field of $x^d - 2$, with $D = d\phi(d)$, and $\alpha = 2^{1/d}$. Nevertheless, as mentioned above, this result can be strengthened for a *generator* α of a Galois extension.

Theorem 3.3. *For any $\varepsilon > 0$ there is a positive effective constant $c(\varepsilon)$ with the following property. Let $\alpha \in \overline{\mathbb{Q}}^\times$ be of degree d, not a root of unity, such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois. Then we have*

$$h(\alpha) \ge c(\varepsilon) d^{-\varepsilon} .$$

Proof. Let r be the smallest integer $> 1/\varepsilon$. If $r \ge d$ then $d \le 1 + 1/\varepsilon$ and $h(\alpha) \gg_\varepsilon 1$. So we can assume $r < d$. If r among the conjugates of α are multiplicatively independent, by Theorem 2.1 (with $K = \mathbb{Q}$) we have

$$h(\alpha) \gg_\varepsilon d^{-1/r-\varepsilon} \gg_\varepsilon d^{-2\varepsilon} .$$

Otherwise, the multiplicative rank $\rho \ge 1$ of the conjugates of α is at most $r - 1 \le 1/\varepsilon$. By Lemma 2.2 there exists a number field $L \subseteq \mathbb{Q}(\alpha)$ of degree $[L:\mathbb{Q}] = n \le$

$n(\varepsilon)$ and an integer $e \geq 1$ such that $\mathbb{Q}(\zeta_e) \subseteq \mathbb{Q}(\alpha)$ and $\alpha^e \in L$. As a consequence $L(\alpha)/L$ is of degree $e' \leq e$. The diagram

$$\begin{tikzcd}
 & \mathbb{Q}(\alpha) = L(\alpha) \arrow[d, dash] & \\
 & L(\zeta_e) \arrow[dl, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & & \mathbb{Q}(\zeta_e) \arrow[dl, dash] \\
 & k := L \cap \mathbb{Q}(\zeta_e) \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

shows that the degree of α over $\mathbb{Q}(\zeta_e)$ is

$$[\mathbb{Q}(\alpha) : L(\zeta_e)] \cdot [L(\zeta_e) : \mathbb{Q}(\zeta_e)] = e' \frac{[L(\zeta_e) : \mathbb{Q}(\zeta_e)]}{[L(\zeta_e) : L]}$$

which is

$$e' \frac{[L : k]}{[\mathbb{Q}(\zeta_e) : k]} = e' \frac{[L : \mathbb{Q}]}{[\mathbb{Q}(\zeta_e) : \mathbb{Q}]} = \frac{e'}{\phi(e)} n \leq \frac{e}{\phi(e)} n \ll_\varepsilon d^\varepsilon$$

(use $\phi(e) \leq d$ and (2.1)). By Theorem 2.1 (with $K = \mathbb{Q}(\zeta_e)$ and $r = 1$) we get

$$h(\alpha) \gg_\varepsilon d^{-2\varepsilon} .$$

$\square$

We note that Theorem 3.3 is nearly best possible in the sense that an inequality $h(\alpha) \gg d^\delta$ would be false for any fixed $\delta > 0$. For example for $\alpha = 1 + \zeta_e$ with $d = \phi(e)$ one has $h(\alpha) \leq \log 2$. Or $\alpha = 2^{1/e} + \zeta_e$, whose degree is easily seen to be $e\phi(e)$, with $h(\alpha) \leq 2 \log 2$. But Smyth in [12] quoted above asked whether even $h(\alpha) \gg 1$ is true, a kind of "Galois-Lehmer Problem". We do not know, but it would imply the main result of Amoroso-Dvornicich [2] on abelian extensions, and a slightly weaker result of Amoroso-Zannier [4, Corollary 1.3] on dihedral extensions.

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