Second-order covariation: it is all about standpoints

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Second-order covariation is a quite recent theoretical construct in the field of Mathematics Education: it differs from the already existing construct of covariation in placing greater emphasis on the role of parameters, with respect to the other variables, as characteristic of a certain family of functions and as relevant in modelling classes of real phenomena. In this contribution, we address the novelty and importance of this type of reasoning based on a teaching experiment concerning modelling of a thermodynamic situation. Starting from the analysis of three episodes, we highlight some features of this construct, and the emerging interpretation as a change in standpoint that results in different graphical representations suitable to interpret globally the specific mathematical situation.

Keywords: Covariation, variables, parameters, graphs.

Introduction to the research problem.

Covariational reasoning is an essential skill necessary to enter deeply into the processes of mathematical modelling (Thompson, 2011): it can help to model "the change and the relationships with appropriate functions and equations, as well as creating, interpreting and translating among symbolic and graphical representations of relationships" (OECD, 2022). Italian mathematics curricula for secondary schools also address the importance of "represent[ing] the same class of phenomena through different approaches" (MIUR, 2010, p. 337). To pursue this goal, using multiple representations (Ainsworth, 2008) turns out to be of great importance for students in order to fully embrace all the aspects of the mathematical situation under investigation. An example of meaningful graphical representation which lends itself to multiple educational purposes consists of the "psychrometric chart" or Carrier diagram. It describes the thermodynamic parameters of moist air at constant air pressure, so it is a graphical representation of an equation of state. This type of representation requires sophisticated forms of covariational reasoning to grasp how magnitudes are varying and to identify their mutual relationships. To better introduce this chart, we comment on a real chart displayed in Figure 1a and/or on a slightly simplified simulation[1] of it created with Wolfram software, shown in Figure 1b. The dry bulb temperature is shown on the horizontal axis while the vertical axis is the absolute humidity. The chart in its complete form contains many parameters: wet bulb temperature, dew point temperature (the temperature at which the air becomes saturated with water vapor), relative humidity, specific enthalpy, and specific volume; indeed, reading a psychrometric chart is very challenging, but if we just focus on temperature (abscissa), absolute humidity (ordinate), and relative humidity (the parameter), the relation between temperature and absolute humidity is given by an exponential-like function (green curve) and a different curve corresponds to each value of the percentage of relative humidity. Indeed, the Carrier diagram shows the mutual relations between the three variables involved crushed onto a two-dimensional

[1] This simulation is available online at this link: https://demonstrations.wolfram.com/ReadingAPsychrometricChart/.

representation: this flattening makes the reading and the interpretation more complicated than a three-dimensional representation in which the three magnitudes assume the same ontological status of variable. Again: why choose relative humidity as the parameter instead of absolute humidity? How would this choice affect the bidimensional representation?

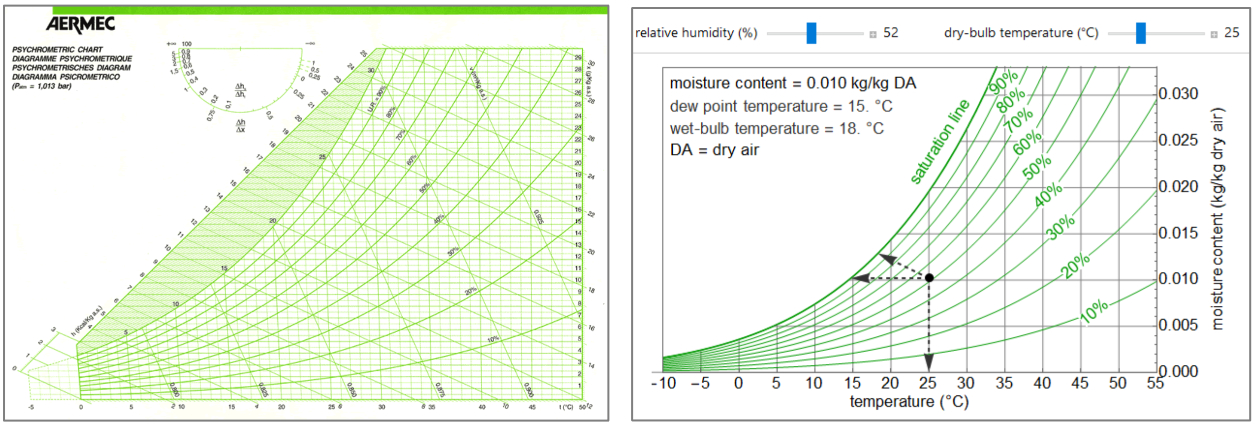

Figure 1: a) An example of psychrometric chart; b) Simulation of a psychrometric chart created with Wolfram software

To answer these questions, the theoretical construct of second-order covariation is introduced. While the construct of covariation focuses on the relation between two or more variables, second-order covariation focuses on the role of characteristic parameters as specific of a certain family of functions. What is defined as being "of second order" depends on the way a specific variable is extrapolated from the assigned scenario and interpreted as a parameter: this determines the standpoint from which the mathematical situation can be represented.

The main purpose of this contribution is to enter the complexity of second-order covariational reasoning required to fully understand the relations between the above magnitudes. Consequently, our research question can be formulated as follows: *how can second-order covariational reasoning be characterized when interpreting representations, like the psychrometric chart, where at least three magnitudes are involved and one of these can be mathematically interpreted as a parameter?*

In this paper, we are going to initially provide a characterization of the construct of second-order covariation referring also to other theoretical contributions in the literature. Then, thanks to the analysis of a few episodes from a teaching experiment involving an Italian secondary school classroom, and based on the interpretation of the psychrometric chart, we are going to explore the covariation emerging in students' reasoning. Finally, we propose a refinement of the construct of second-order covariation which could be interpreted as a change of standpoint suitable to interpret different mathematical representations.

Theoretical framework.

The use of multiple representations strongly supports the learning of mathematical concepts, and among their various functions they provide information that complement each other and/or foster the development of deeper understanding when they support additional information (Ainsworth, 1999). Although multiple representations are beneficial to the learners, they are non-trivial for students to use to relate and identify connections (Ainsworth, 2008). Beyond this, single representations themselves can be challenging: for instance, reading, interpreting, and reasoning about graphs requires grasping the relationship between the values of the magnitudes involved, and so it demands good covariational skills.

Being able to reason covariationally means being able to envision two or more magnitudes that vary simultaneously, as presented in Thompson and Carlson (2017). The last version of this theoretical construct, presented in the same work, and that we are going to refer to as first-order covariation (COV 1), consists of six levels of covariational reasoning ranking from absence of covariational reasoning to smooth continuous covariation. These levels can be interpreted as classes of behaviors or descriptors of a person's ability to engage in covariational reasoning. In this contribution, we consider covariation in a broader epistemological sense, i.e., as the ability to grasp relations of invariance that are more complex than the simple covariation between two magnitudes. Specifically, the construct of second-order covariation (COV 2), recently introduced by Arzarello (2019), is a form of reasoning that consists in suitably envisioning the relationships in which not only variables but also parameters are involved mathematically. In particular, the latter allow to represent classes of real phenomena as families of relations between variables, which are mathematically represented by specific parameters. These determine the peculiarities of the mathematical model. Furthermore, the label “second-order covariation” seems particularly suitable to underline the role played by parameters: indeed, Bloedy-Vinner (2001) already used the expression “second order function” to address those functions whose argument is a parameter and whose output is a function or an equation depending on a specific parameter value.

The necessity of introducing this new order of covariation emerges predominantly from the need of better describing students’ reasoning when dealing with situations of mathematical modelling, such as the law of the inclined plane as presented in Arzarello (2019). However, it also emerges in other studies in literature. For example, Hoofkamp (2011) introduced the term “metavariation” to refer to a variation of the mathematical situation itself and the function as a whole: it is related to the object view of the function and to a qualitative view of the functional dependency and its local and global characteristics. While our approach to second-order covariation mainly focuses on the mathematical and cognitive characterization of this form of covariational reasoning, metavariation emphasizes the instrumentation of second-order covariational processes adopting suitable activities mostly designed with Interactive Geometry Software. This way of reasoning covariationally seems to be more cognitively demanding than the one presented in the already existing taxonomy. Hence, this contribution supports and contributes to the perspective of an enlargement of the theoretical framework that aims at coherently including more complex forms of covariational reasoning and opens up to the possibility that other orders of covariation may exist.

Method.

Participants and task design. The episodes presented and analyzed in this paper are part of a teaching experiment that was held in a 11th grade classroom of 21 students in a scientific-oriented school in Italy. It was conducted at the beginning of the school year 2020/2021 in a mixed modality due to Covid pandemic. Ever since the first year of secondary school the students involved were accustomed to working with different mathematical representations (algebraic, graphical, numerical) and to using Dynamic Geometry Softwares such as GeoGebra. Moreover, in the academic year 2019/2020 the students had already been involved in a teaching experiment concerning the law of the inclined plane, the so-called Galileo experiment (Bagossi, 2021), so they had already experienced covariational reasoning in a context of mathematical modelling. This experimentation had the main purpose of elaborating mathematically on the relationship between temperature and humidity adopting multiple representations and various digital supports. The first task consisted of analyzing the mathematical and physical relationship existing between temperature and humidity through a household experiment during which for an entire day, at regular intervals, students collected the values of temperature and humidity in their rooms with the request to eventually elaborate some hypotheses on a possible relationship between the collected data. In a subsequent lesson, students' hypotheses were discussed, and the teacher commented on a GeoGebra graph displaying some of the data collected. The students, together with their teacher, concluded that the two magnitudes could be related in some way, namely whilst the temperature increases, the humidity decreases, and vice versa. The activity continued with an experiment in class: in a metal pot with some water at room temperature, ice was gradually added until droplets of water appeared on the outside of the pot. At regular intervals of time the students recorded the time spent and the temperature of the water in the pot itself and took note of the temperature when the dew drops appeared (that is the dew point). After a working group session devoted to the understanding of a real psychrometric chart, through a classroom discussion (Discussion 1) led by the teacher, the students investigated the relationships between time, water temperature, room temperature and the reason for the creation of dew drops on the outside of the pot. At the end of the discussion, the teacher provided students with a slightly simplified psychrometric diagram created with GeoGebra (Figure 2a) and a worksheet with a real chart (Figure 1a).

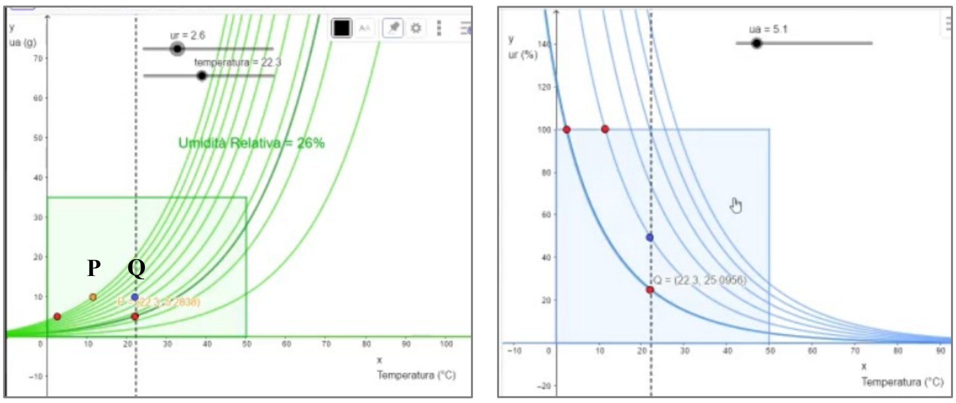

Figure 2: a) GeoGebra applet simulating the psychrometric chart; b) GeoGebra applet describing the relation between relative humidity (y-axis) and temperature (x-axis)

Working in small groups on some guide questions, the students were able to read relative humidity values from the graph representing absolute humidity as a function of temperature and, conversely, to read the values of absolute humidity from a new graph, provided in a second moment, representing relative humidity as a function of temperature (Figure 2b). At the end of the group work, the students discussed with the teacher (Discussion 2) the roles of the magnitudes at stake in the mathematical representations (independent variable, dependent variable, or parameter). We note that the multiple representations involved, such as the real psychrometric chart and the two applets have a complementary function, while the pot experiment with respect to these provides extra information but it will reveal crucial for students to disentangle the meaning of the graphical representations.

Methods of data analysis. The analytic method used in our qualitative study was that of a descriptive coding of the emergent forms of covariational reasoning (Saldana, 2015). A preliminary phase consisted of watching the videos of all the lessons repeatedly in order to identify episodes revealing covariation. During the first cycle of coding, we classified the orders and levels of emerging covariational reasoning describing their features and the quantities involved; then we transcribed the most significant episodes. During the second cycle of coding, we deepened the qualitative description, focusing on the identification of the representations that most influenced the students' reasoning, i.e., the pot experiment or the graphical representations, the use of a qualitative or quantitative narrative, and terms denoting change and movement. Eventually, we revised the coding in light of the qualitative descriptions of the episodes we had elaborated.

Data analysis.

The first episode analyzed here is an excerpt from the 1-hour discussion in presence (Discussion 1) led by the teacher after the working group session on the psychrometric chart. During this episode, students, guided by the teacher, tried to retrace what happened during the experiment with the pot on the psychrometric diagram. The applet reproducing the psychrometric chart was shown on the interactive whiteboard and students had already identified point P representing the point on the saturation line in which the temperature coincides with the temperature of the dew point. Point Q instead has the same ordinate as P and the ambient temperature as abscissa (Figure 2a).

1	Teacher	On the graph, how can I read these passages? We said this is the starting point because we said we do not go from P to Q, but we start from Q. Starting from Q, where did we go?
2	Giorgia	We decreased the temperature hence we moved to the left.
3	Teacher	We decreased the temperature hence we moved to the left. In which way? Did you just decrease the temperature or not? We are during the moment in which you continued to pour and pour [the ice].
4	Emanuele	Only the temperature decreases.
5	Teacher	Only the temperature decreases. And so, on the graph, how do you move?
6	Emanuele	Horizontally.
7	Teacher	Horizontally. We have point Q and we move horizontally to decrease the temperature. Until when do we move horizontally?
8	Emanuele	Until the dew point.
9	Teacher	Until the dew point that is until when we find on which of these green curves?
10	Emanuele	Until that of 100%.

All the episode is centered around a game of displacement between the graphical representation and the experiment facilitated by the mediation of the teacher that constantly asks the students how they would move on the graph, inviting them to relate to the experiment with the GeoGebra applet. This game of displacement results in the interlacing of two different narratives: a qualitative one, used to describe what happened during the experiment that manifests with the use of personal pronouns (e.g., "*we* decreased the temperature" [2]); a quantitative one used to describe how the magnitudes involved are changing (e.g., "Only the temperature decreases" [4]). The students already possess the psychrometric diagram representing the covariation of magnitudes involved and they are moving on this representation: we can recognize the enhancement of a global approach supported by the involved representations.

Discussion 2 was mainly focused on reconstructing the cycle of the pot experiment on the new chart, the one describing the relationship between absolute humidity on the y-axis and temperature on the x-axis. In this second episode, one of the graphs made by the students as homework is shown on the interactive whiteboard (Figure 3). The teacher is commenting on the second step of the experiment the one during which students continued to decrease the temperature in the pot by adding ice cubes and then some drops of water formed on the wall, represented by a horizontal segment, colored in yellow.

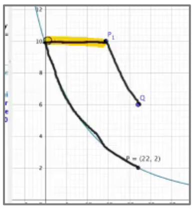

Figure 3: Relative humidity – temperature graph made by one of the students

11	Teacher	What is happening instead on the horizontal trait [of the graph]?
12	Arianna	The relative humidity maintains constant.
13	Teacher	The relative humidity maintains constant.
14	Arianna	And the temperature decreases.
15	Teacher	The temperature decreases. The absolute humidity? Does it decrease or remain constant?
16	Adele	Decreases.
17	Teacher	Decreases. Why?
18	Emanuele	You have the condensation.
19	Teacher	Ok, you have the condensation, and this is what happens practically. But on the graph why? […]
20	Adele	The curve changes.

As in the previous episode, students' linguistic expressions "The relative humidity maintains constant" [12] and "You have the condensation" [18] suggest that the combined and synergic use of the two representations, the chart and the experiment, helped students in reflecting on which magnitudes change and how they change during the different steps of the experiment and the corresponding traits of the chart. Even in this case we can observe the enhancement of a global approach supported by the involved representations.

Finally, this third episode below, coming also from Discussion 2, shows the approach used by the teacher to reflect on the similarities and differences between the two psychrometric charts and it can be intended as a form of conceptualization of the different roles of variables and parameters.

21	Teacher	Are they two different situations/scenarios?
22	Matteo	No.
23	Teacher	No. Why do the two graphs are different if they are not two different situations?
24	Matteo	The value represented on the y-axis is different.
25	Teacher	The value represented on the y-axis is different. If you should make a comparison with something that is not mathematical but concerns real life… We have the same situation/scenario, but the value represented on the y-axis is different… If you should make an analogy…?

26	Giorgia	From the physical point of view, they represent the same thing but from the graphical point of view no… because they are two different values.
27	Teacher	Oh! From the physical point of view, they represent the same thing but from the graphical point of view no because they are two different values. […] Two different situations depending on what?
28	Matteo	A different point of view.

At this point of the discussion, the teacher projects on the interactive whiteboard a new applet showing simultaneously the relationship of absolute humidity versus temperature (Figure 2a) and of relative humidity versus temperature (Figure 2b). The teacher asks to the classroom if the graphs are two different situations or scenarios [21]. Matteo obviously replies no [22], so the teacher asks why the graphs are different if they are not representing different things [23]. When Matteo states that the value represented on the y-axis is different [24], the teacher invites the students to provide a holistic interpretation, an analogy with something non-strictly mathematical [25]. Giorgia suggests that from the physical standpoint the situation is the same: what differs is the graphical representation [26]. Answering a teacher's question [27], Matteo adds that the difference between the two situations depends on "a different point of view" [28]. This interpretation suggests that the different role assumed by variables and parameters does not determine a different situation but a change of standpoint resulting in a different graphical representation.

Discussion.

After a first overall reading of the three episodes presented in the previous section, the question arising spontaneously is: where is covariation? If we think of some typical examples of covariational reasoning such as "A increases, while B decreases", they are surely absent in the excerpts described before. The psychrometric chart at disposal of the students synthetizes in a unique diagram and flattens in two dimensions the relations between three different magnitudes (temperature, absolute humidity, and relative humidity). The students implement forms of reasoning revealing a global approach supported by the adopted representations. In particular, in the physical interpretation of the relations described in the chart, students are deeply supported by the classroom experiment with the pot: all the classroom discussion develops through an interpretation of the diagram with respect to the various steps of the experiment. What the students observe in the last episode is that considering relative humidity as a variable or a parameter does not change the situation, but the perspective with which you look at the situation: the students claim that the relationship is always the same, only expressed in different terms. Second-order covariation is the construct that enables one to read the same mathematical situation from two different standpoints and to recognize a correspondence between the representations: in the real psychrometric chart the parameter is the relative humidity, and this magnitude is the one of second order; in the blue graph (Figure 2b) absolute humidity becomes the parameter and so it is the second order variable determining a different standpoint.

As it is for COV 1, second-order covariation is a cognitive act that cannot be reduced to the reading of a formula choosing to interpret one of the variables involved as a parameter. It is a form of reasoning that consists of envisioning the invariant relationships in a family of functions and this study sheds light on one of its facets that manifests when dealing with graphical representations.

Finally, even if it is not the focus of this paper, we cannot fail to observe the essential role of the teacher in mediating between the multiple representations helping students in relating them.

Moreover, the teacher supports students in better expressing their thoughts and enhances the transition from COV 1 to COV 2, and also from COV 2 to COV 1.

Conclusion.

Second-order covariation requires a complex cognitive engagement: it is a form of covariational reasoning that in activities of mathematical modelling involving multiple representations can be theorized as the ability to read the same mathematical situation from different points of view choosing each time which of the involved variables should be mathematically interpreted as a parameter. The characterization and relevance of second-order covariation still deserve to be deepened through further research.

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