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Local C^2-smooth spline quasi-interpolation methods

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ABSTRACT

In this paper we construct new univariate local C^2 quasi-interpolating splines having specific polynomial reproduction properties. The splines are directly determined by setting their Bernstein-Bézier coefficients to appropriate combinations of the given data values. In certain cases we obtain a family of quasi-interpolating operators satisfying the required conditions, so we fix some extra properties (interpolation of the vertices, extra locality, extra polynomial reproduction) in order to compute unique approximants. We also provide numerical results confirming the theoretical ones.

1. Introduction

In many mathematical problems and scientific applications it is important to approximate functions and data accurately and in this context quasi-interpolation is a useful tool for its peculiar properties (see e.g. the recent book [1] for a general overview on quasi-interpolation): for example a nice property, if compared to interpolation, is that quasi-interpolation does not require the solution of any system of equations and moreover, since quasi-interpolation does not require that the approximant exactly matches the data at certain points, this could be useful if we are dealing with noisy data. Usually, a quasi-interpolating spline for a function f is expressed as linear combination of basis functions for the considered spline space and functions values (see e.g. [2]). In this paper we propose an alternative approach where the spline is directly determined by setting its Bernstein-Bézier coefficients to appropriate combinations of the given data values. This technique, initially proposed in the bivariate case in [3] has been extended in [4–6], considered also in the 1D case in [7] and applied to digital elevation models in [8]. In particular, in this work we propose the construction of new 1D C^2 local quasi-interpolating splines of degree 3, 4 and 5 having specific polynomial reproduction properties. The approximants here constructed have the same approximation properties of those proposed in [7], although they are more local, in the sense that for constructing the spline in a specific interval they use data values that are closer to such an interval with respect to the approximants in [7].

Here is an outline of the paper. In Section 2 we define the spline spaces and the quasi-interpolating operators. In particular we obtain, in certain cases, a family of quasi-interpolating operators satisfying the required conditions, so we fix some extra properties (interpolation of the vertices, extra locality, extra polynomial reproduction) in order to compute unique approximants. In Section 3 we present numerical results confirming the theoretical ones and in Section 4 we provide some conclusions.

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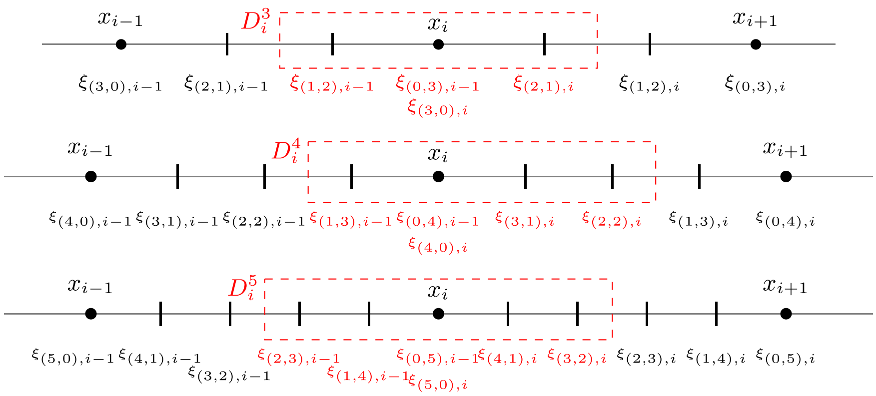

Fig. 1. The sets D_i^d, for $d = 3, 4, 5$.

2. Spline spaces and quasi-interpolating operators

In this paper, we consider a uniform partition $\mathcal{P} := a + h\mathbb{Z}$ of $\mathbb{R}$ generated by the knots $x_i := a + i\, h$, $i \in \mathbb{Z}$, with $a \in \mathbb{R}$, $h > 0$, and define the subintervals $I_i := [x_i, x_{i+1}]$. We are interested in the space

$$S_d(\mathcal{P}) := \left\{ s \in C^2(\mathbb{R}) : s_i := s_{|I_i} \in \mathbb{P}_d,\ i \in \mathbb{Z} \right\},$$

of C^2-continuous polynomial splines of degree $d = 3, 4, 5$ on $\mathbb{R}$, where $\mathbb{P}_d$ is the space of polynomials of degree less than or equal to d. We want to construct quasi-interpolation operators (QIOs) $Q_{d,k} : C(\mathbb{R}) \longrightarrow S_d(\mathcal{P})$ exact on $\mathbb{P}_k$, $k \le d$, expressed in Bernstein-Bézier (BB-) form. Indeed, since $x \in I_i$ can be written from its barycentric coordinates $(1-t, t)$, $t \in [0, 1]$, with respect to I_i as $x = (1-t)\, x_i + t x_{i+1}$, the quasi-interpolating (QI) spline $Q_{d,k} f$ in each subinterval I_i can be expressed in terms of the Bernstein polynomials relative to I_i

$$Q_{d,k} f_{|I_i} = \sum_{|\alpha|=d} b_{\alpha,i}^{d,k}(f)\, \mathfrak{B}_{\alpha,i}, \tag{2.1}$$

where $\alpha := (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, $|\alpha| := \alpha_1 + \alpha_2$, $\mathfrak{B}_{\alpha,i}$ are the Bernstein polynomials on I_i, i.e.

$$\mathfrak{B}_{\alpha,i}(x) := \frac{d!}{\alpha_1! \alpha_2!} (1-t)^{\alpha_1}\, t^{\alpha_2},\ x \in I_i,$$

with $t := \frac{x - x_i}{h}$. The coefficients $b_{\alpha,i}^{d,k}(f)$ are called the BB-coefficients of $Q_{d,k} f$. They are naturally associated with the domain points $\xi_{\alpha,i} := \frac{\alpha_1}{d} x_i + \frac{\alpha_2}{d} x_{i+1}$ in I_i determined by the barycentric coordinates $\left(\frac{\alpha_1}{d}, \frac{\alpha_2}{d}\right)$, $|\alpha| = d$. Since the partition is uniform, it is sufficient to define the BB-coefficients associated with the domain points in a set D_i^d such that $\Xi_d = \cup_{i \in \mathbb{Z}} D_i^d$, where Ξ_d is the set of all domain points. In particular (see Fig. 1):

$$\begin{aligned} D_i^3 &:= \{\xi_{(1,2),i-1}, \xi_{(0,3),i-1} \equiv \xi_{(3,0),i}, \xi_{(2,1),i}\} \\ D_i^4 &:= \{\xi_{(1,3),i-1}, \xi_{(0,4),i-1} \equiv \xi_{(4,0),i}, \xi_{(3,1),i}, \xi_{(2,2),i}\} \\ D_i^5 &:= \{\xi_{(2,3),i-1}, \xi_{(1,4),i-1}, \xi_{(0,5),i-1} \equiv \xi_{(5,0),i}, \xi_{(4,1),i}, \xi_{(3,2),i}\} \end{aligned}$$

Moreover, we want that the QIO is local, so we define the BB-coefficients in (2.1) from the values of f at specific points lying in a neighbourhood of I_i and depending on the degree k of polynomial reproduction. In particular, each $b_{\alpha,i}^{d,k}(f)$ is defined by using the so-called mask M_α^k, whose elements are denoted by $M_{\alpha,j}^k$, i.e. $M_\alpha^k := (M_{\alpha,j}^k)_{j=1}^{2k+1} \in \mathbb{R}^{2k+1}$, based on $2k+1$ points contained in $[x_{i-1}, x_{i+1}]$ (see Fig. 2):

$$b_{\alpha,i}^{d,k}(f) = \sum_{j=1}^{2k+1} M_{\alpha,j}^k f\left(x_i + \frac{h}{k}(j - k - 1)\right). \tag{2.2}$$

For each domain point $\xi_{\alpha,i}$ in the set D_i^d we have to determine the values $M_{\alpha,j}^k$, $j = 1, \dots, 2k+1$ of the corresponding mask M_α^k ensuring C^2 continuity and exactness on $\mathbb{P}_k$, for $k \le d$.

In the cubic case the structure of BB-coefficients given by (2.2) only allows the exactness of the quasi-interpolant on $\mathbb{P}_1$.

Proposition 1. *The masks* $M_{(3,0)}^1 := \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$, $M_{(2,1)}^1 := \left(0, \frac{2}{3}, \frac{1}{3}\right)$, *and* $M_{(1,2)}^1 := \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ *give rise to the unique C^2 cubic QI defined by (2.1) and (2.2) which is exact on $\mathbb{P}_1$.*

Proof. In general, the QI $Q_{d,k} f$ given by (2.1) and (2.2) is C^1-continuous at x_i if, and only if,

$$b_{(d,0),i}^{d,k}(f) = \frac{1}{2}\left(b_{(1,d-1),i-1}^{d,k}(f) + b_{(d-1,1),i}^{d,k}(f)\right). \tag{2.3}$$

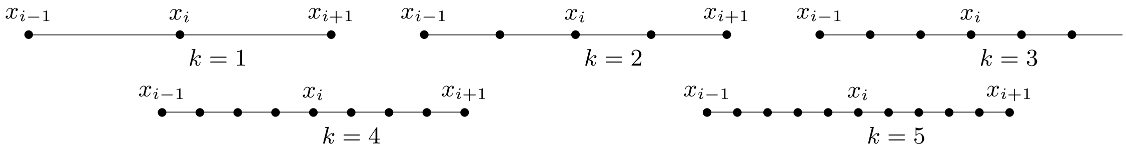

Fig. 2. The points involved in the construction of $b_{\alpha,i}^{d,k}(f)$, for $k = 1, 2, 3, 4, 5$.

Moreover, C^2-continuity at x_i is equivalent to the condition

$$-b_{(2,d-2),i-1}^{d,k}(f) + 2b_{(1,d-1),i-1}^{d,k}(f) - 2b_{(d-1,1),i}^{d,k}(f) + b_{(d-2,2),i}^{d,k}(f) = 0. \tag{2.4}$$

On the other hand, the exactness of $Q_{d,k}$ on $\mathbb{P}_k$ is achieved by imposing that the BB-coefficients of the monomials $m_k(x) := x^k$, $k \in \mathbb{N} \cup \{0\}$, equal those of $Q_{d,k} m_k$ for the considered values of k. These conditions give rise to constraints that together with (2.3) and (2.4) produce a system of linear equations whose solutions can be determined, in general, using a Computer Algebra System. When $d = 3$ and $k = 1$, that system has a unique solution providing the masks appearing in the statement. □

Many other possibilities exist for C^2 quartic and quintic QIs. The masks depend on free parameters, which allows the construction of quasi-interpolants with additional properties useful in practice.

Proposition 2. *The masks* $M_{(4,0)}^1 := \left(-\frac{1}{8} + \alpha, \frac{5}{4} - 2\alpha, -\frac{1}{8} + \alpha\right)$, $M_{(3,1)}^1 := \left(-\frac{1}{4} + \alpha, \frac{5}{4} - 2\alpha, \alpha\right)$, $M_{(2,2)}^1 := \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $M_{(1,3)}^1 := \left(\alpha, \frac{5}{4} - 2\alpha, -\frac{1}{4} + \alpha\right)$, $\alpha \in \mathbb{R}$, *provide* C^2 *quartic QIs exact on* $\mathbb{P}_1$.

Analogously, the masks $M_{(5,0)}^1 := (\alpha, -2\alpha + 1, \alpha)$, $M_{(4,1)}^1 := \left(-\frac{1}{5} + \beta, \frac{6}{5} - 2\beta, \beta\right)$, $M_{(3,2)}^1 := \left(-\frac{2}{5} + \gamma, \frac{7}{5} - 2\gamma, \gamma\right)$, $M_{(1,4)}^1 := \left(\frac{1}{5} + 2\alpha - \beta, \frac{4}{5} - 4\alpha + 2\beta, 2\alpha - \beta\right)$, $M_{(2,3)}^1 := \left(\frac{2}{5} + 4\alpha - 4\beta + \gamma, \frac{3}{5} - 8\alpha + 8\beta - 2\gamma, 4\alpha - 4\beta + \gamma\right)$, $\alpha, \beta, \gamma \in \mathbb{R}$, *give rise to* C^2 *quintic QIs also exact on* $\mathbb{P}_1$.

Proof. Similarly to Proposition 1, we impose the exactness of $Q_{d,1}$ on $\mathbb{P}_1$, by requiring that the BB-coefficients of the monomial $m_1(x) := x$ equal those of $Q_{d,1} m_1$, for $d = 4, 5$ and we consider (2.3) and (2.4) with $k = 1$ and $d = 4$ and 5, respectively. The obtained linear systems are solved by using a Computer Algebra System, getting the masks appearing in the statement, that depend on some free parameters. □

In contrast to the cubic case, it is possible to obtain quartic and quintic QIOs exact on $\mathbb{P}_2$.

Proposition 3. *QIOs exact on* $\mathbb{P}_2$ *are produced respectively in the quartic and quintic cases by the masks*

- $M_{(4,0)}^2 := \left(\frac{5}{24} - \alpha - 3\beta, -\frac{2}{3} + 3\alpha + 8\beta, \frac{7}{4} - 3\alpha - 6\beta, -\frac{1}{3} + \alpha, \frac{1}{24} + \beta\right)$,
  $M_{(3,1)}^2 := \left(\frac{1}{4} - \alpha - 3\beta, -1 + 3\alpha + 8\beta, \frac{7}{4} - 3\alpha - 6\beta, \alpha, \beta\right)$, $M_{(2,2)}^2 := \left(0, 0, -\frac{1}{6}, \frac{4}{3}, -\frac{1}{6}\right)$,
  $M_{(1,3)}^2 := \left(\frac{1}{6} - \alpha - 3\beta, -\frac{1}{3} + 3\alpha + 8\beta, \frac{7}{4} - 3\alpha - 6\beta, -\frac{2}{3} + \alpha, \frac{1}{12} + \beta\right)$, $\alpha, \beta \in \mathbb{R}$,
- $M_{(5,0)}^2 := \left(-\frac{3}{20} - \frac{3}{16}\alpha - \frac{1}{2}\beta - \frac{3}{2}\gamma + \frac{1}{16}\delta, \frac{2}{5} + \frac{1}{2}\alpha + \frac{3}{2}\beta + 4\gamma, \frac{7}{10} - \frac{3}{8}\alpha - 3\gamma - \frac{3}{2}\beta - \frac{3}{8}\delta,\right.$
  $\left.\frac{1}{2}\beta + \frac{1}{2}\delta, \frac{1}{20} + \frac{1}{16}\alpha + \frac{1}{2}\gamma - \frac{3}{16}\delta\right)$, $M_{(2,3)}^2 := \left(-\frac{1}{5} - \varepsilon - 3\zeta, \frac{6}{5} + 3\varepsilon + 8\zeta, -3\varepsilon - 6\zeta, \varepsilon, \zeta\right)$,
  $M_{(4,1)}^2 := \left(-\frac{1}{10} - \frac{3}{8}\alpha + \frac{1}{8}\delta, \alpha, 1 - \frac{3}{4}\alpha - \frac{3}{4}\delta, \delta, \frac{1}{10} + \frac{1}{8}\alpha - \frac{3}{8}\delta\right)$,
  $M_{(3,2)}^2 := \left(-\frac{3}{4}\alpha + 2\beta + 6\gamma + \frac{1}{4}\delta - \varepsilon - 3\zeta, -\frac{2}{5} + 2\alpha - 6\beta - 16\gamma + 3\varepsilon + 8\zeta,\right.$
  $\left.\frac{6}{5} - \frac{3}{2}\alpha + 6\beta + 12\gamma - \frac{3}{2}\delta - 3\varepsilon - 6\zeta, -2\beta + 2\delta + \varepsilon, \frac{1}{5} + \frac{1}{4}\alpha - 2\gamma - \frac{3}{4}\delta + \zeta\right)$,
  $M_{(1,4)}^2 := \left(-\frac{1}{5} - \beta - 3\gamma, \frac{4}{5} + 3\beta + 8\gamma, \frac{2}{5} - 3\beta - 6\gamma, \beta, \gamma\right)$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$.

The following result shows the masks yielding C^2-quartic and quintic QIOs exact on $\mathbb{P}_3$. Three and nine degrees of freedom appear, respectively.

Proposition 4. *QIOs exact on* $\mathbb{P}_3$ *are produced in the quartic by the masks*

$$M^3_{(4,0)} := \Big(\tfrac{3}{16} + \tfrac{2}{9}\alpha - \tfrac{1}{3}\beta - \tfrac{4}{9}\gamma, -\tfrac{39}{80} - \tfrac{4}{5}\alpha + \tfrac{4}{5}\beta + \gamma, \tfrac{3}{16} + \alpha, \tfrac{11}{8} - \tfrac{4}{9}\alpha - \tfrac{4}{3}\beta - \tfrac{10}{9}\gamma, -\tfrac{3}{16} + \beta, -\tfrac{3}{16} + \gamma, \tfrac{9}{80} + \tfrac{1}{45}\alpha - \tfrac{2}{15}\beta - \tfrac{4}{9}\gamma\Big),\ M^3_{(2,2)} := \Big(0,0,0,-\tfrac{1}{4},\tfrac{3}{4},\tfrac{3}{4},-\tfrac{1}{4}\Big),$$

$$M^3_{(3,1)} := \Big(\tfrac{1}{4} + \tfrac{2}{9}\alpha - \tfrac{1}{3}\beta - \tfrac{4}{9}\gamma, -\tfrac{27}{40} - \tfrac{4}{5}\alpha + \tfrac{4}{5}\beta + \gamma, \alpha, \tfrac{11}{8} - \tfrac{4}{9}\alpha - \tfrac{4}{3}\beta - \tfrac{10}{9}\gamma, \beta, \gamma, \tfrac{1}{20} + \tfrac{1}{45}\alpha - \tfrac{2}{15}\beta - \tfrac{4}{9}\gamma\Big),$$

$$M^3_{(1,3)} := \Big(\tfrac{1}{8} + \tfrac{2}{9}\alpha - \tfrac{1}{3}\beta - \tfrac{4}{9}\gamma, -\tfrac{3}{10} - \tfrac{4}{5}\alpha + \tfrac{4}{5}\beta + \gamma, \tfrac{3}{8} + \alpha, \tfrac{11}{8} - \tfrac{4}{9}\alpha - \tfrac{4}{3}\beta - \tfrac{10}{9}\gamma, -\tfrac{3}{8} + \beta, -\tfrac{3}{8} + \gamma, \tfrac{7}{40} + \tfrac{1}{45}\alpha - \tfrac{2}{15}\beta - \tfrac{4}{9}\gamma\Big), \quad \alpha,\beta,\gamma \in \mathbb{R}.$$

And, in the quintic case, by the following masks, with $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta,\theta,\kappa \in \mathbb{R}$, they are

$$M^3_{(5,0)} := \Big(\tfrac{2}{25} - \tfrac{2}{9}\alpha + \tfrac{1}{2}\beta + 2\gamma + 5\delta - \tfrac{1}{15}\varepsilon + \tfrac{1}{90}\zeta, -\tfrac{9}{20} + \tfrac{1}{2}\alpha - 2\beta - \tfrac{15}{2}\gamma - 18\delta, \tfrac{9}{10} + 3\beta + 10\gamma + \tfrac{45}{2}\delta + \tfrac{1}{2}\varepsilon, \tfrac{3}{10} - \tfrac{5}{9}\alpha - 2\beta - 5\gamma - 10\delta - \tfrac{2}{3}\varepsilon - \tfrac{2}{9}\zeta, \tfrac{1}{2}\beta + \tfrac{1}{2}\zeta, \tfrac{27}{100} + \tfrac{1}{2}\alpha + \tfrac{1}{2}\gamma + \tfrac{2}{5}\varepsilon - \tfrac{2}{5}\zeta, -\tfrac{1}{10} - \tfrac{2}{9}\alpha + \tfrac{1}{2}\delta - \tfrac{1}{6}\varepsilon + \tfrac{1}{9}\zeta\Big),$$

$$M^3_{(4,1)} := \Big(-\tfrac{1}{25} - \tfrac{4}{9}\alpha - \tfrac{2}{15}\varepsilon + \tfrac{1}{45}\zeta, \alpha, \varepsilon, \tfrac{7}{10} - \tfrac{10}{9}\alpha - \tfrac{4}{3}\varepsilon - \tfrac{4}{9}\zeta, \zeta, \tfrac{27}{50} + \alpha + \tfrac{4}{5}\varepsilon - \tfrac{4}{5}\zeta, -\tfrac{1}{5} - \tfrac{4}{9}\alpha - \tfrac{1}{3}\varepsilon + \tfrac{2}{9}\zeta\Big),$$

$$M^3_{(3,2)} := \Big(-\tfrac{97}{20} - \tfrac{80}{9}\alpha - 20\delta - \tfrac{20}{3}\varepsilon + \tfrac{40}{9}\zeta + \eta + 4\theta + 10\kappa, 18 + 32\alpha + 72\delta + 24\varepsilon - 16\zeta - 4\eta - 15\theta - 36\kappa, -\tfrac{477}{20} - 40\alpha - 90\delta - 30\varepsilon + 20\zeta + 6\eta + 20\theta + 45\kappa, \tfrac{121}{10} + \tfrac{160}{9}\alpha + 40\delta + \tfrac{40}{3}\varepsilon - \tfrac{80}{9}\zeta - 4\eta - 10\theta - 20\kappa, \eta, \theta, -\tfrac{2}{5} - \tfrac{8}{9}\alpha - 2\delta - \tfrac{2}{3}\varepsilon + \tfrac{4}{9}\zeta + \kappa\Big),$$

$$M^3_{(1,4)} := \Big(\tfrac{1}{5} + \beta + 4\gamma + 10\delta, -\tfrac{9}{10} - 4\beta - 15\gamma - 36\delta, \tfrac{9}{5} + 6\beta + 20\gamma + 45\delta, -\tfrac{1}{10} - 4\beta - 10\gamma - 20\delta, \beta, \gamma, \delta\Big),$$

$$M^3_{(2,3)} := \Big(-\tfrac{437}{100} - 8\alpha + 2\beta + 8\gamma - \tfrac{32}{5}\varepsilon + \tfrac{22}{5}\zeta + \eta + 4\theta + 10\kappa, \tfrac{81}{5} + 30\alpha - 8\beta - 30\gamma + 24\varepsilon - 16\zeta - 4\eta - 15\theta - 36\kappa, -\tfrac{81}{4} - 40\alpha + 12\beta + 40\gamma - 32\varepsilon + 20\zeta + 6\eta + 20\theta + 45\kappa, \tfrac{21}{2} + 20\alpha - 8\beta - 20\gamma + 16\varepsilon - 8\zeta - 4\eta - 10\theta - 20\kappa, 2\beta + \eta - 2\zeta, -\tfrac{27}{25} - 2\alpha + 2\gamma - \tfrac{8}{5}\varepsilon + \tfrac{8}{5}\zeta + \theta, \kappa\Big).$$

Only the quartic and quintic cases exact on $\mathbb{P}_4$ remain to be considered, since it is not possible to construct with the proposed procedure quintic QIs exact on $\mathbb{P}_5$.

Proposition 5. *The following masks define QIOs exact on $\mathbb{P}_4$ and C^2 quartic and quintic QIs:*

- $$M^4_{(4,0)} := \Big(\tfrac{43}{126} - \tfrac{5}{56}\alpha - \tfrac{3}{28}\beta + \tfrac{5}{28}\gamma + \tfrac{15}{56}\delta, -\tfrac{80}{63} + \tfrac{4}{21}\alpha + \tfrac{5}{21}\beta - \tfrac{10}{21}\gamma - \tfrac{20}{21}\delta, \tfrac{5}{3} + \delta, -\tfrac{8}{9} + \gamma, 1 - \tfrac{5}{12}\alpha - \tfrac{5}{6}\beta - \tfrac{5}{6}\gamma - \tfrac{5}{12}\delta, \tfrac{8}{9} + \beta, -\tfrac{5}{3} + \alpha, \tfrac{80}{63} - \tfrac{20}{21}\alpha - \tfrac{10}{21}\beta + \tfrac{5}{21}\gamma + \tfrac{4}{21}\delta, -\tfrac{43}{126} + \tfrac{15}{56}\alpha + \tfrac{5}{28}\beta - \tfrac{3}{28}\gamma - \tfrac{5}{56}\delta\Big),\ M^4_{(2,2)} := \Big(0,0,0,0,\tfrac{13}{18},-\tfrac{32}{9},\tfrac{20}{3},-\tfrac{32}{9},\tfrac{13}{18}\Big),$$

  $$M^4_{(3,1)} := \Big(\tfrac{9}{56} - \tfrac{5}{56}\alpha - \tfrac{3}{28}\beta + \tfrac{5}{28}\gamma + \tfrac{15}{56}\delta, -\tfrac{8}{21} + \tfrac{4}{21}\alpha + \tfrac{5}{21}\beta - \tfrac{10}{21}\gamma - \tfrac{20}{21}\delta, \delta, \gamma, 1 - \tfrac{5}{12}\alpha - \tfrac{5}{6}\beta - \tfrac{5}{6}\gamma - \tfrac{5}{12}\delta, \beta, \alpha, \tfrac{8}{21} - \tfrac{20}{21}\alpha - \tfrac{10}{21}\beta + \tfrac{5}{21}\gamma + \tfrac{4}{21}\delta, -\tfrac{9}{56} + \tfrac{15}{56}\alpha + \tfrac{5}{28}\beta - \tfrac{3}{28}\gamma - \tfrac{5}{56}\delta\Big),$$

  $$M^4_{(1,3)} := \Big(\tfrac{263}{504} - \tfrac{5}{56}\alpha - \tfrac{3}{28}\beta + \tfrac{5}{28}\gamma + \tfrac{15}{56}\delta, -\tfrac{136}{63} + \tfrac{4}{21}\alpha + \tfrac{5}{21}\beta - \tfrac{10}{21}\gamma - \tfrac{20}{21}\delta, \tfrac{10}{3} + \delta, -\tfrac{16}{9} + \gamma, 1 - \tfrac{5}{12}\alpha - \tfrac{5}{6}\beta - \tfrac{5}{6}\gamma - \tfrac{5}{12}\delta, \tfrac{16}{9} + \beta, -\tfrac{10}{3} + \alpha, \tfrac{136}{63} - \tfrac{20}{21}\alpha - \tfrac{10}{21}\beta + \tfrac{5}{21}\gamma + \tfrac{4}{21}\delta, -\tfrac{263}{504} + \tfrac{15}{56}\alpha + \tfrac{5}{28}\beta - \tfrac{3}{28}\gamma - \tfrac{5}{56}\delta\Big), \quad \alpha,\beta,\gamma,\delta \in \mathbb{R},$$

- $$M^4_{(5,0)} := \Big(-\tfrac{17}{210} - \tfrac{5}{112}\alpha - \tfrac{3}{56}\beta + \tfrac{5}{56}\gamma + \tfrac{15}{112}\delta + \tfrac{3}{64}\varepsilon + \tfrac{1}{8}\zeta + \tfrac{5}{32}\eta - \tfrac{7}{64}\theta, \tfrac{16}{105} + \tfrac{2}{21}\alpha + \tfrac{5}{42}\beta - \tfrac{5}{21}\gamma - \tfrac{10}{21}\delta + \tfrac{1}{2}\theta, \tfrac{4}{15} - \tfrac{5}{8}\varepsilon - \tfrac{3}{2}\zeta - \tfrac{7}{4}\eta - \tfrac{7}{8}\theta + \tfrac{1}{2}\delta, -\tfrac{64}{75} + \tfrac{3}{2}\varepsilon + \tfrac{16}{5}\zeta + \tfrac{7}{2}\eta + \tfrac{7}{10}\theta + \tfrac{1}{2}\gamma, \tfrac{8}{5} - \tfrac{5}{24}\delta - \tfrac{5}{12}\gamma - \tfrac{45}{32}\varepsilon - \tfrac{5}{12}\beta - \tfrac{9}{4}\zeta - \tfrac{5}{24}\alpha - \tfrac{35}{16}\eta - \tfrac{7}{32}\theta, \tfrac{1}{2}\beta + \tfrac{1}{2}\varepsilon, \tfrac{1}{2}\alpha + \tfrac{1}{2}\zeta, \tfrac{1}{2}\eta - \tfrac{16}{105} - \tfrac{10}{21}\alpha - \tfrac{5}{21}\beta + \tfrac{5}{42}\gamma + \tfrac{2}{21}\delta, \tfrac{71}{1050} + \tfrac{15}{112}\alpha + \tfrac{5}{56}\beta - \tfrac{3}{56}\gamma - \tfrac{5}{112}\delta - \tfrac{1}{64}\varepsilon - \tfrac{3}{40}\zeta - \tfrac{7}{32}\eta + \tfrac{1}{320}\theta\Big),$$

  $$M^4_{(4,1)} := \Big(-\tfrac{1}{30} + \tfrac{3}{32}\varepsilon + \tfrac{1}{4}\zeta + \tfrac{5}{16}\eta - \tfrac{7}{32}\theta, \theta, \tfrac{8}{15} - \tfrac{5}{4}\varepsilon - 3\zeta - \tfrac{7}{2}\eta - \tfrac{7}{4}\theta, -\tfrac{128}{75} + 3\varepsilon + \tfrac{32}{5}\zeta + 7\eta + \tfrac{7}{5}\theta, \tfrac{11}{5} - \tfrac{45}{16}\varepsilon - \tfrac{9}{2}\zeta - \tfrac{35}{8}\eta - \tfrac{7}{16}\delta, \varepsilon, \zeta, \eta, \tfrac{1}{150} - \tfrac{1}{32}\varepsilon - \tfrac{3}{20}\zeta - \tfrac{7}{16}\eta + \tfrac{1}{160}\theta\Big),$$

  $$M^4_{(3,2)} := \Big(-\tfrac{19}{20} + \tfrac{45}{32}\kappa + \tfrac{5}{4}\lambda + \tfrac{9}{16}\mu - \tfrac{5}{32}\nu, \tfrac{128}{35} - 5\kappa - \tfrac{32}{7}\lambda - \tfrac{15}{7}\mu + \tfrac{5}{7}\nu, -\tfrac{64}{15} + \tfrac{21}{4}\kappa + 5\lambda + \tfrac{5}{2}\mu - \tfrac{5}{4}\nu, \nu, \tfrac{5}{2} - \tfrac{35}{16}\kappa - \tfrac{5}{2}\lambda - \tfrac{15}{8}\mu - \tfrac{5}{16}\nu, \mu, \lambda, \kappa, \tfrac{5}{84} - \tfrac{15}{32}\kappa - \tfrac{5}{28}\lambda - \tfrac{5}{112}\mu + \tfrac{1}{244}\nu\Big),$$

  $$M^4_{(1,4)} := \Big(-\tfrac{9}{70} - \tfrac{5}{56}\alpha - \tfrac{3}{28}\beta + \tfrac{5}{28}\gamma + \tfrac{15}{56}\delta, \tfrac{32}{105} + \tfrac{4}{21}\alpha + \tfrac{5}{21}\beta - \tfrac{10}{21}\gamma - \tfrac{20}{21}\delta, \delta, \gamma, 1 - \tfrac{5}{12}\alpha - \tfrac{5}{6}\beta - \tfrac{5}{6}\gamma - \tfrac{5}{12}\delta, \beta, \alpha, -\tfrac{32}{105} - \tfrac{20}{21}\alpha - \tfrac{10}{21}\beta + \tfrac{5}{21}\gamma + \tfrac{4}{21}\delta, \tfrac{9}{70} + \tfrac{15}{56}\alpha + \tfrac{5}{28}\beta - \tfrac{3}{28}\gamma - \tfrac{5}{56}\delta\Big),$$

$$M^4_{(2,3)} := \Big(-\tfrac{479}{420} - \tfrac{5}{28}\alpha - \tfrac{3}{14}\beta + \tfrac{5}{14}\gamma + \tfrac{15}{28}\delta - \tfrac{3}{16}\varepsilon - \tfrac{1}{2}\zeta - \tfrac{5}{8}\eta + \tfrac{7}{16}\theta + \tfrac{45}{32}\kappa + \tfrac{5}{4}\lambda + \tfrac{9}{16}\mu - \tfrac{5}{32}\nu,$$
$$\tfrac{64}{15} + \tfrac{8}{21}\alpha + \tfrac{10}{21}\beta - \tfrac{20}{21}\gamma - \tfrac{40}{21}\delta - 2\theta - 5\kappa - \tfrac{32}{7}\lambda - \tfrac{15}{7}\mu + \tfrac{5}{7}\nu, -\tfrac{16}{3} + 2\delta + \tfrac{5}{2}\varepsilon + 6\zeta + 7\eta + \tfrac{7}{2}\theta$$
$$+\tfrac{21}{4}\kappa + 5\lambda + \tfrac{5}{2}\mu - \tfrac{5}{4}\nu, \tfrac{256}{75} + 2\gamma - 6\varepsilon - \tfrac{64}{5}\zeta - 14\eta - \tfrac{14}{5}\theta + \nu,$$
$$\tfrac{1}{10} - \tfrac{5}{6}\alpha - \tfrac{5}{3}\beta - \tfrac{5}{3}\gamma - \tfrac{5}{6}\delta + \tfrac{45}{8}\varepsilon + 9\zeta + \tfrac{35}{4}\eta + \tfrac{7}{8}\theta - \tfrac{35}{16}\kappa - \tfrac{5}{2}\lambda - \tfrac{15}{8}\mu - \tfrac{5}{16}\nu, 2\beta - 2\varepsilon + \mu,$$
$$2\alpha - 2\zeta + \lambda, -\tfrac{64}{105} - \tfrac{40}{21}\alpha - \tfrac{20}{21}\beta + \tfrac{10}{21}\gamma + \tfrac{8}{21}\delta - 2\eta + \kappa, \tfrac{91}{300} + \tfrac{15}{28}\alpha + \tfrac{5}{14}\beta - \tfrac{3}{14}\gamma - \tfrac{5}{28}\delta + \tfrac{1}{16}\varepsilon + \tfrac{3}{10}\zeta$$
$$+\tfrac{7}{8}\eta - \tfrac{1}{80}\theta - \tfrac{15}{32}\kappa - \tfrac{5}{28}\lambda - \tfrac{5}{112}\mu + \tfrac{1}{224}\nu\Big).$$

To prove Propositions 3–5, we can follow the proof of Proposition 2, by imposing the exactness on $\mathbb{P}_2$, $\mathbb{P}_3$ and $\mathbb{P}_4$, respectively.

Almost all masks depend on free parameters and they can be chosen in order to obtain operators having interesting properties: for example here we consider interpolation at the knots, i.e. we require that $Q_{d,k}f(x_i) = f(x_i)$. Imposing interpolation, for quartic splines we obtain:

- $k=1$: $M^1_{(4,0)} := (0,1,0)$, $M^1_{(3,1)} := \left(-\frac{1}{8}, 1, \frac{1}{8}\right)$, $M^1_{(2,2)} := \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $M^1_{(1,3)} := \left(\frac{1}{8}, 1, -\frac{1}{8}\right)$;
- $k=2$: $M^2_{(4,0)} := (0,0,1,0,0)$, $M^2_{(3,1)} := \left(\frac{1}{24}, -\frac{1}{3}, 1, \frac{1}{3}, -\frac{1}{24}\right)$, $M^2_{(2,2)} := \left(0,0,-\frac{1}{6}, \frac{4}{3}, -\frac{1}{6}\right)$, $M^2_{(1,3)} := \left(-\frac{1}{24}, \frac{1}{3}, 1, -\frac{1}{3}, \frac{1}{24}\right)$; it is also possible to verify that with these masks we obtain exactness on $\mathbb{P}_3$
- $k=3$: $M^3_{(4,0)} := (0,0,0,1,0,0,0)$, $M^3_{(3,1)} := \left(\frac{1}{16}, -\frac{3}{16}, -\frac{3}{16}, 1, \frac{3}{16}, \frac{3}{16}, -\frac{1}{16}\right)$, $M^3_{(2,2)} := \left(0,0,0,-\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, -\frac{1}{4}\right)$, $M^3_{(1,3)} := \left(-\frac{1}{16}, \frac{3}{16}, \frac{3}{16}, 1, -\frac{3}{16}, -\frac{3}{16}, \frac{1}{16}\right)$,
- $k=4$: $M^4_{(4,0)} := (0,0,0,0,1,0,0,0,0)$, $M^4_{(3,1)} := \left(-\frac{13}{72}, \frac{8}{9}, -\frac{5}{3}, \frac{8}{9}, 1, -\frac{8}{9}, \frac{5}{3}, -\frac{8}{9}, \frac{13}{72}\right)$, $M^4_{(2,2)} := \left(0,0,0,0,\frac{13}{18}, -\frac{32}{9}, \frac{20}{3}, -\frac{32}{9}, \frac{13}{18}\right)$, $M^4_{(1,3)} := \left(\frac{13}{72}, -\frac{8}{9}, \frac{5}{3}, -\frac{8}{9}, 1, \frac{8}{9}, -\frac{5}{3}, \frac{8}{9}, -\frac{13}{72}\right)$.

Imposing interpolation, for quintic splines some parameters remains free, so, in order to have masks as local as possible we can fix them to zero, obtaining masks based on points on only the interval $[x_{i-1}, x_i]$, otherwise we can ask for extra polynomial reproduction. We propose some possibilities:

- $k=1$: imposing locality we obtain $M^1_{(5,0)} := (0,1,0)$, $M^1_{(4,1)} := \left(-\frac{1}{5}, \frac{6}{5}, 0\right)$, $M^1_{(3,2)} := \left(-\frac{2}{5}, \frac{7}{5}, 0\right)$, $M^1_{(2,3)} := \left(\frac{2}{5}, \frac{3}{5}, 0\right)$, $M^1_{(1,4)} := \left(\frac{1}{5}, \frac{4}{5}, 0\right)$; imposing exactness on $\mathbb{P}_2$ we obtain $M^1_{(5,0)} := (0,1,0)$, $M^1_{(4,1)} := \left(-\frac{1}{10}, 1, \frac{1}{10}\right)$, $M^1_{(3,2)} := \left(-\frac{3}{20}, \frac{9}{10}, \frac{1}{4}\right)$, $M^1_{(2,3)} := \left(\frac{1}{4}, \frac{9}{10}, -\frac{3}{20}\right)$, $M^1_{(1,4)} := \left(\frac{1}{10}, 1, -\frac{1}{10}\right)$;
- $k=2$: imposing locality we obtain $M^2_{(5,0)} := (0,0,1,0,0)$, $M^2_{(4,1)} := \left(\frac{1}{5}, -\frac{4}{5}, \frac{8}{5}, 0, 0\right)$, $M^2_{(3,2)} := \left(\frac{3}{5}, -2, \frac{12}{5}, 0, 0\right)$, $M^2_{(2,3)} := \left(-\frac{1}{5}, \frac{6}{5}, 0, 0, 0\right)$, $M^2_{(1,4)} := \left(-\frac{1}{5}, \frac{4}{5}, \frac{2}{5}, 0, 0\right)$; imposing exactness on $\mathbb{P}_4$ we obtain the masks in [7, Proposition 11];
- $k=3$: imposing locality we obtain $M^3_{(5,0)} := (0,0,0,1,0,0,0)$, $M^3_{(4,1)} := \left(-\frac{1}{5}, \frac{9}{10}, -\frac{9}{5}, \frac{21}{10}, 0,0,0\right)$, $M^3_{(3,2)} := \left(-\frac{17}{20}, \frac{18}{5}, -\frac{117}{20}, \frac{41}{10}, 0,0,0\right)$, $M^3_{(2,3)} := \left(-\frac{1}{20}, 0, \frac{27}{20}, -\frac{3}{10}, 0,0,0\right)$, $M^3_{(1,4)} := \left(\frac{1}{5}, -\frac{9}{10}, \frac{9}{5}, -\frac{1}{10}, 0,0,0\right)$; imposing exactness on $\mathbb{P}_4$ and fixing to zero the remaining free parameters, we obtain $M^3_{(5,0)} := (0,0,0,1,0,0,0)$, $M^3_{(4,1)} := \left(-\frac{1}{20}, \frac{3}{10}, -\frac{9}{10}, \frac{3}{2}, \frac{3}{20}, 0, 0\right)$, $M^3_{(3,2)} := \left(-\frac{11}{80}, \frac{3}{4}, -\frac{63}{40}, \frac{5}{4}, \frac{57}{80}, 0, 0\right)$, $M^3_{(2,3)} := \left(\frac{1}{16}, -\frac{9}{20}, \frac{81}{40}, -\frac{3}{4}, \frac{9}{80}, 0, 0\right)$, $M^3_{(1,4)} := \left(\frac{1}{20}, -\frac{3}{10}, \frac{9}{10}, \frac{1}{2}, -\frac{3}{20}, 0, 0\right)$;
- $k=4$: $M^4_{(5,0)} := (0,0,0,0,1,0,0,0,0)$, $M^4_{(4,1)} := \left(\frac{1}{5}, -\frac{16}{15}, \frac{12}{5}, -\frac{16}{5}, \frac{8}{3}, 0,0,0,0\right)$, $M^4_{(3,2)} := \left(\frac{17}{15}, -\frac{88}{15}, \frac{62}{5}, -\frac{40}{3}, \frac{20}{3}, 0,0,0,0\right)$, $M^4_{(2,3)} := \left(\frac{1}{3}, -\frac{8}{5}, \frac{14}{5}, -\frac{8}{15}, 0,0,0,0,0\right)$, $M^4_{(1,4)} := \left(-\frac{1}{5}, \frac{16}{15}, -\frac{12}{5}, \frac{16}{5}, -\frac{2}{3}, 0,0,0,0\right)$.

We remark that other conditions can be imposed in order to obtain operators with interesting features (see [7]).

Regarding the approximation properties of the constructed QIOs, thanks to standard results in approximation theory (see e.g. [9]), we can state the following theorem.

Theorem 6. *For a function f enough regular, there exists a constant C independent of f and h such that*

$$\left\|Q_{d,k}f - f\right\|_{\infty,I_i} \le C h^{k+1} \left\|f^{(k+1)}\right\|_{\infty,I_i}, \quad \text{for all } i \in \mathbb{Z}.$$

3. Numerical results

In this section we test the performance of the proposed QIs (considering those obtained by imposing extra locality) for the function

$$f(x) = \frac{3}{4}e^{-2(9x-2)^2} - \frac{1}{5}e^{-(9x-7)^2-(9x-4)^2} + \frac{1}{2}e^{-(9x-7)^2-\frac{1}{4}(9x-3)^2} + \frac{3}{4}e^{\frac{1}{10}(-9x-1)-\frac{1}{49}(9x+1)^2}, \ x \in [0,1]$$

Table 1
Maximum absolute errors and numerical convergence order.

n	$E_{3,1}$	$N_{3,1}$	$E_{4,1}$	$N_{4,1}$	$E_{4,2}$	$N_{4,2}$	$E_{4,3}$	$N_{4,3}$	$E_{4,4}$	$N_{4,4}$
4	3.16(−01)	–	4.01(−01)	–	2.52(−01)	–	2.42(−01)	–	3.41(−01)	–
8	2.91(−01)	1.2	1.87(−01)	1.1	4.94(−02)	2.4	5.45(−02)	2.1	2.06(−02)	4.1
16	1.28(−01)	1.2	8.03(−02)	1.2	5.77(−03)	3.1	6.50(−03)	3.1	8.97(−04)	4.5
32	3.67(−02)	1.8	1.56(−02)	2.4	2.86(−04)	4.3	3.21(−04)	4.3	4.48(−05)	4.3
64	9.70(−03)	1.9	3.80(−03)	2.0	1.86(−05)	3.9	2.07(−05)	4.0	1.52(−06)	4.9
128	2.46(−03)	2.0	9.32(−04)	2.0	1.15(−06)	4.0	1.28(−06)	4.0	4.83(−08)	5.0
256	6.16(−04)	2.0	2.31(−04)	2.0	7.14(−08)	4.0	7.93(−08)	4.0	1.52(−09)	5.0
512	1.54(−04)	2.0	5.77(−05)	2.0	4.46(−09)	4.0	4.96(−09)	4.0	4.75(−11)	5.0
1024	3.86(−05)	2.0	1.44(−05)	2.0	2.79(−10)	4.0	3.10(−10)	4.0	1.48(−12)	5.0
n	$E_{5,1}$	$N_{5,1}$	$E_{5,2}$	$N_{5,2}$	$E_{5,3}$	$N_{5,3}$	$E_{5,4}$	$N_{5,4}$		
4	6.06(−01)	–	6.84(−01)	–	2.42(−01)	–	6.14(−01)	–		
8	3.40(−01)	0.8	1.10(−01)	2.6	1.03(−01)	1.2	5.38(−02)	3.5		
16	9.71(−02)	1.8	2.87(−02)	1.9	1.10(−02)	3.2	1.67(−03)	5.0		
32	2.07(−02)	2.2	4.12(−03)	2.8	6.68(−04)	4.0	9.02(−05)	4.2		
64	5.42(−03)	1.9	5.22(−04)	3.0	4.95(−05)	3.8	3.23(−06)	4.8		
128	1.36(−03)	2.0	6.52(−05)	3.0	3.25(−06)	3.9	1.04(−07)	5.0		
256	3.39(−04)	2.0	8.15(−06)	3.0	2.04(−07)	4.0	3.27(−09)	5.0		
512	8.47(−05)	2.0	1.02(−06)	3.0	1.28(−08)	4.0	1.02(−10)	5.0		
1024	2.12(−05)	2.0	1.27(−07)	3.0	7.98(−10)	4.0	3.20(−12)	5.0		

that is the 1D-version of the well-known Franke's function. We compute the maximum absolute error $E_{d,k}f := \max_{u\in G} |f(u) - Q_{d,k}f(u)|$, for a sequence of uniform mesh with knots $x_i = ih$, $i = 0,\ldots,n$, for increasing values of n, with $h = \frac{1}{n}$, using a uniform mesh G of evaluation points in the domain made of 25 points in each interval $[x_i, x_{i+1})$. We also compute the corresponding numerical convergence orders $N_{d,k}$. The results reported in Table 1 confirm the theoretical ones in Theorem 6.

4. Conclusions

In this paper we have proposed the construction of univariate local C^2 quasi-interpolating splines having specific polynomial reproduction properties. In certain cases we have obtained a family of quasi-interpolating operators satisfying the required conditions, so we have fixed some extra properties (interpolation of the vertices, locality, extra polynomial reproduction) in order to compute unique approximants. We have also provided numerical tests confirming the theoretical ones.

We remark that in the numerical tests we have evaluated the function f outside the interval $[0,1]$ in order to be able to construct masks for BB-coefficients associated with domain points near the boundary. Another approach is to construct ad hoc masks for those cases.

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Data availability

Data will be made available on request.

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