UNIVERSITÀ DEGLI STUDI DI TORINO

DIPARTIMENTO DI MATEMATICA “GIUSEPPE PEANO”

Dottorato di ricerca in Matematica pura e applicata

XXXIII ciclo

Uniform Martin’s conjecture, locally

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Anni accademici: 2017-2018, 2018-2019, 2019-2020
Settore scientifico-disciplinare: MAT/01 – Logica matematica

Abstract

We start with the observation that, while the reason for a countable Borel equivalence relation not being Borel reducible to another one has always to do with the Borel structure of them in their globality, this might not be the case when dealing with *uniform* Borel reducibility. We show in Chapter 1 that the impossibility of finding a uniform Borel reductions can be ruled out by global as well as local phenomena.

As Slaman and Steel's result that Martin's conjecture holds for uniformly degree invariant functions gives limitations to the possible behaviors of uniform homomorphisms of Turing equivalence to itself, we investigate in Chapter 2 whether the reason for these limitations can be retrieved locally. We discover that this is the case for the first part of the result. We then present some corollaries of this fact, such as the possibility of embedding the structure of real numbers pre-ordered by Turing reducibility inside the structure of equivalence relations on $\mathbb{N}$ pre-ordered by computable reducibility. Encouraged by these results, Chapter 3 continues with an investigation about the internal structure of single Turing degrees provided by Turing reductions, addressing questions such as: which degrees can we embed in which other degrees? And: what is the theory for a cone of them?

In Chapter 4, we come back to the question whether the second half of uniform Martin's conjecture also arises locally, and here we notice that the answer is harder to give. Still, we provide some partial results, such as the one joint with Patrick Lutz that a theorem of Lachlan — which was strengthened by Steel's proof of part two of uniform Martin's conjecture — does arise locally.

Finally, in Chapter 5, we study hypotheses that, despite being apparently weaker than uniformity, still enable one to carry out the uniformity arguments to prove Martin's conjecture. We consider several such hypotheses: although they all seem weaker than ordinary uniform degree invariance, we prove that most of them actually coincide with it from a global point of view, while, for most of them, it remains open whether they differ from a local perspective.

Contents

Chapter 1

Introduction

The main topic of this dissertation is uniform Martin's conjecture and how much of it follows from local arguments. In this chapter, we give a short history of how the ideas around Martin's conjecture were born, as well as our idea of retrieving uniform Martin's conjecture locally.

1.1 Introduction to Martin's conjecture

Martin's conjecture is one of the most important open problems of computability theory. Its aim is to shed a light on one of the most fundamental phenomena exhibited by Turing degree since the very early days.

Indeed, till the mid '50s, all known Turing degrees were comparable, and in fact well-ordered by Turing reducibility, with successors given by the Turing jump. Most famously, Post proposed, in his 1944 seminal paper [Pos44], the celebrated problem of finding a computably enumerable set of natural numbers whose Turing degree is neither $[0]_T$ nor $[0]'_T$. The solution provided independently by Friedberg and Muchnik one decade later is just as famous. If $(\varphi_i^x)_{i\in\omega}$ is a standard enumeration of computable-in-x partial functions, where $x \in 2^\omega$, $W_i^x = \mathrm{dom}(\varphi_i^x)$ is the i-th c.e.-in-x set and $R_i^x = W_i^x \oplus x$, then the relativized version of Friedberg and Muchnik result states the following:

Theorem 1.1.1 (Friedberg-Muchnik)**.** *There are $a, b \in \omega$ such that, for every $x \in 2^\omega$:*

$$R_a^x \not\leq_T R_b^x \text{ and } R_b^x \not\leq_T R_a^x.$$

This solved Post's problem because R_b^x trivially satisfies $x \leq_T R_b^x \leq_T x'$, so the Turing degree of R_a^x cannot be $[x]_T$ (for in that case, we would have $R_a^x \leq_T R_b^x$), nor can be $[x]'_T$ (for in that case we would have $R_b^x \leq_T R_a^x$).

Corollary 1.1.2. *There exists $a \in \omega$ such that, for every $x \in 2^\omega$:*

$$x <_T R_a^x <_T x'.$$

This solution apparently gives strong evidence for totally rejecting the idea that $[0]_T$ and $[0]'_T$ are the only c.e. Turing degrees: it shows that there are plenty of non-comparable degrees and also there are intermediate degrees between every degree and its jump, also supplying c.e. operators witnessing both situations at the same time. To be precise, $x \mapsto R_a^x$ and $x \mapsto R_b^x$ in Theorem 1.1.1 exhibit, given any $x \in 2^\omega$, two incomparable c.e.-in-x degrees strictly in-between the degrees of x and x'. However, if we just give a Turing degree $[x]_T$, we need to choose a representative in $[x]_T$ in order to have these c.e. operators give us two incomparable c.e.-in-$[x]_T$ degrees strictly in-between $[x]_T$ and $[x]'_T$, and these may vary depending on the choice of the representative. This observation led Sacks to ask the following question. Say that a function $f : A \to 2^\omega$, where $A \subseteq 2^\omega$, is **degree invariant** or **Turing invariant** (abbreviated DI or TI) if, for all $x, y \in A$ one has

$$x \equiv_T y \implies f(x) \equiv_T f(y).$$

Question 1.1.3 (Sacks, [Sac67]). *Does there exist a degree invariant c.e. operator $x \mapsto W_e^x$ such that*

$$x <_T W_e^x <_T x'$$

for all $x \in 2^\omega$?

This question is open to this day and, in fact, we are not only unable of finding a degree invariant c.e. operator always providing an intermediate degree, but we are also not able of exhibiting *any* degree invariant function $f : 2^\omega \to 2^\omega$ such that $x <_T f(x) <_T x'$ for all $x \in 2^\omega$ without using the Axiom of Choice. Thus, even though the problem of finding an intermediate degree between any degree and its jump has been solved for a long time, in a sense we can say that we still do not know any *canonical* way of picking one.

This relates to the empirical phenomenon that Turing degrees that correspond to problems occurring in common mathematical practice are organized in a very simple substructure of the tangled structure of Turing degrees: such "natural" Turing degrees appear to be well-ordered by $\leq_T$, and there seems to be no "natural" Turing degree strictly between a "natural" Turing degree and its Turing jump.

Martin's conjecture is a long-standing open problem whose aim was to provide a precise mathematical formalization of these observations. The leading idea is to formalize the notion of "natural" Turing degree as a suitable equivalence class of "definable" functions over Turing degrees. The intuition is that, as Steel explains in the introduction of [Ste82], "natural" Turing degrees are supposed to have a definition that can be relativized, and the process of relativizing their definition is supposed to induce "definable" TI functions. Precisely, 'definable' is formalized setting Martin's conjecture under the Axiom of Determinacy (AD).

Before we state the conjecture, we recall the basic definitions that occur in it. Upward Turing cones, i.e. sets of the form

$$\{ x \in 2^\omega \mid x \geq_T z \},$$

are usually referred to just as **cones**, in this context. A set $A \subseteq 2^\omega$ is said to be **Turing-invariant** if it is closed under $\equiv_T$. Turing Determinacy (TD) denotes the statement that every Turing-invariant subset of 2^ω either contains a cone or is disjoint from a cone. Martin's celebrated cone theorem [Mar68] states that TD follows from AD. The importance of TD lies in the fact that it enables to define a natural notion of *largeness* for Turing-invariant sets: the map

$$\mu(A) = \begin{cases} 1 & \text{if } A \text{ contains a cone} \\ 0 & \text{otherwise} \end{cases}$$

defines, under TD, a measure on the σ-algebra of Turing-invariant subsets of 2^ω.

Recall the definition of the Turing jump of $x \in 2^\omega$:

$$x'(n) = \begin{cases} 1 & \text{if } n \in \mathrm{dom}(\varphi_n^x), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, given two TI functions $f, g : 2^\omega \to 2^\omega$, one defines

$$f \leq_M g \iff f(x) \leq_T g(x) \text{ on a cone.}$$

Conjecture 1.1.4 (Martin)**.** *In* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$*, the following are conjectured:*

I. *if $f : 2^\omega \to 2^\omega$ is Turing invariant, then either $f(x) \geq_T x$ on a cone or there exists $y \in 2^\omega$ such that $f(x) \equiv_T y$ on a cone;*

II. *the set of TI functions f such that $f \geq_M \mathrm{id}_{2^\omega}$ is pre-well-ordered by $\leq_M$; moreover, if such an f has $\leq_M$-rank α, then f' (defined by $f'(x) := f(x)'$) has $\leq_M$-rank $\alpha + 1$.*

Thus, the first of the conjecture seems to be the less relevant, as it just says the apparent technicality that definable Turing invariant functions that are not constant on a cone (and hence trivial) must be above the identity on a cone. On the other hand, the second part seems the most fundamental, as it states the core of the idea behind Martin's conjecture. So, for example, part II, and not part I, implies a strong negative answer to Sacks' Question 1.1.3. However, part I, and not part II, re-sparkled the interest on Martin's conjecture after the discovery of its profound consequences in the theory of countable Borel equivalence relations (see [Tho09]).

Although Martin's conjecture is still open in its generality, some restrictions of it to particular classes of functions have been proved. The most celebrated example is arguably that of uniformly degree invariant functions, which we are going to introduce in the next section.

As a convention, *uniform* Martin's conjecture refers to Martin's conjecture restricted to *uniformly* degree invariant functions, *projective* Martin's conjecture refers to Martin's conjecture restricted to *projective* degree invariant functions only, and so on.

1.2 Uniform Martin's conjecture

Let $(\varphi_i^x)_{i\in\mathbb{N}}$ be the standard numbering of partial unary computable-in-x functions, where the oracle x is a function from $\mathbb{N}$ to $\mathbb{N}$. Given $x, y \in 2^\omega$ and $i, j \in \mathbb{N}$, we say that $x \leq_T y$ via i if $x = \varphi_i^y$, and we say that $x \equiv_T y$ via (i, j) if $x \geq_T y$ via i and $x \leq_T y$ via j.

A function $f : A \to 2^\omega$, with $A \subseteq 2^\omega$, is said to be **order-preserving** if, for all $x, y \in A$, one has

$$x \leq_T y \implies f(x) \leq_T f(y),$$

whereas it is said to be **uniformly order-preserving** (abbreviated **UOP**) if every time we have $x, y \in A$ such that $x \leq_T y$ via i, we can choose *uniformly in x and y* an index j such that $f(x) \leq_T f(y)$ via j. In other words, f is UOP if there is a function (which we shall call **uniformity function** for f) $u : \mathbb{N} \to \mathbb{N}$ such that

$$x \leq_T y \text{ via } i \implies f(x) \leq_T f(y) \text{ via } u(i)$$

for all $x, y \in A$. Similarly, f is said to be **uniformly Turing invariant** or **uniformly degree invariant** (abbreviated **UTI** or UDI) if there is a function (again, called uniformity function) $u : \mathbb{N}^2 \to \mathbb{N}^2$ such that, for all $x, y \in A$,

$$x \equiv_T y \text{ via } (i, j) \implies f(x) \equiv_T f(y) \text{ via } u(i, j).$$

Uniformly degree invariant functions were introduced by Lachlan in a paper in which he proved that, if we also ask the c.e. operator in Sacks' question (i.e. Question 1.1.3) to be *uniformly* degree invariant, then the answer is negative.

Theorem 1.2.1 (Lachlan, [Lac75])**.** *There is no uniformly degree invariant solution to Post's problem; that is, there is no $e \in \omega$ such that the e-th c.e. operator $x \mapsto W_e^x$ is uniformly degree invariant and satisfies, for all $x \in 2^\omega$,*

$$x <_T W_e^x <_T x'.$$

A few years later, Steel showed that this follows from the fact that part II of Martin's conjecture holds for all definable UTI functions.

Theorem 1.2.2 (Steel, [Ste82])**.** *Part II of Martin's conjecture holds for the class of uniformly degree invariant functions.*

Slaman and Steel then proved that also the other half of uniform Martin's conjecture holds.

Theorem 1.2.3 (Slaman and Steel, [SS88])**.** *Part I of Martin's conjecture holds for the class of uniformly degree invariant functions.*

UTI functions are especially crucial, not only because they constitute a class of TI functions for which we can prove Martin's conjecture, but also because they are essentially the only definable TI functions that we know. Indeed, in [Ste82], Steel formulated a conjecture (that strengthens Martin's, in the light of the previous results) that postulates that all TI functions are UTI up to equivalence on a cone, under Determinacy.

Conjecture 1.2.4 (Steel)**.** *Under* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$*, every Turing invariant function* $f : 2^\omega \to 2^\omega$ *is* $\equiv_M$*-equivalent to a uniformly Turing invariant one.*

So it might be that, even though Martin's conjecture hinges on specific properties of Turing equivalence that do not generalize easily to others equivalence relations, it is actually the consequence of a principle of uniformity under Determinacy that is much more likely to be featured by a broad class of equivalence relations. This is the point of view presented in [MSS16], where it is shown that the analog of Martin's conjecture for arithmetic equivalence $\equiv_A$ fails, but the analog of Steel's conjecture for $\equiv_A$ remains open.

1.3 Uniform Borel reducibility

What renewed the interest on Martin's conjecture was the discovery of its connections with the theory of countable Borel equivalence relations. Recall that, if E and F are equivalence relations on X and Y respectively, a **homomorphism** from E to F is a function $f : X \to Y$ such that, for all $x, y \in X$,

$$x \; E \; y \implies f(x) \; F \; f(y).$$

Furthermore, f is called a **reduction** from E to F if, for all $x, y \in X$,

$$x \; E \; y \iff f(x) \; F \; f(y).$$

For the rest of this chapter, X and Y will denote standard Borel spaces, that is, spaces with a Borel structure that can be induced by a complete, separable metric on the space. If E and F are equivalence relations on X and Y respectively, and there is a reduction f from E to F which is Borel, then E is said to be **Borel reducible** to F and we write $E \leq_B F$. The study of the structure of $\leq_B$ on certain families of equivalence relations, such as the family of analytic equivalence relations, has been a trend topic in descriptive set theory sometimes referred to as invariant descriptive set theory (see, for instance, [Gao08]). The name 'invariant' comes from the fact that establishing an easy reduction from an equivalence relation E to equality means providing easy invariants for E, whereas establishing an easy reduction from an equivalence relation E to another equivalence relation F means providing easy invariants *up to* F for E. The latter circumstance can be interpreted as E being no more complicated than F. For this reason, Borel reducibility is used as a tool to compare the complexity of classification problems that can be identified with equivalence relations on Polish spaces.

A particular family on which Borel reducibility $\leq_B$ has been intensively studied (see, for instance, [JKL02]) is the family of **countable Borel equivalence relations**: equivalence relations which are Borel (as subsets of $X \times X$, where X is the standard Borel space providing their domain) and whose equivalence classes are countable.

Definition 1.3.1. A countable Borel equivalence relation E is said to be universal if, for every countable Borel equivalence relation F, one has $F \leq_B E$.

Of course, Turing equivalence is such an equivalence relation, and an open problem in this area is whether $\equiv_T$ is a *universal* countable Borel equivalence relation, meaning the following:

Conjecture 1.3.2 (Kechris)**.** *Turing equivalence $\equiv_T$ is a countable Borel equivalence relation.*

The first observation of a link between Martin's conjecture and the theory of countable Borel equivalence relations was the discovery that Martin's conjecture settles this problem on the negative side.

Theorem 1.3.3 ([DS00])**.** *Part I of Martin's conjecture contradicts Kechris' conjecture 1.3.2.*

Deeper connections were discovered by Thomas, who showed in [Tho09] that part I of Martin's conjecture would have many major consequences on the structure of weak Borel reducibility[1] between countable Borel equivalence relations.

Like in the setting of Martin's conjecture, where uniformly Turing invariant functions play an important role, the notion of uniformity for Borel reductions also gained attention. In an unpublished work, Montalbán, Reimann and Slaman studied the notion of uniformity with respect to group actions in order to disprove a suitable uniform version of Kechris' conjecture. Their starting point was the observation that all known proofs of universality for countable Borel equivalence relations generated by a group action always yield to *uniform* universality (we shall explain what this means in a moment). Thus, they defined a group action whose orbit equivalence relation is Turing equivalence and proved that the latter is not uniformly universal with respect to the former. This result, plus their initial empirical observation, might suggest that Kechris' conjecture does not hold. Let us explain the notion of uniform universality.

By a theorem of Feldman and Moore, countable Borel equivalence relations are exactly the orbit equivalence relations of the Borel actions of countable groups.

Theorem 1.3.4 (Feldman-Moore)**.** *If E is a countable Borel equivalence relation on a standard Borel space X, then there is a countable group G and a Borel action $\cdot$*

$$\cdot : G \times X \to X$$

such that, for all $x, y \in X$,

$$x \mathrel{E} y \iff \exists g \in G : g \cdot x = y.$$

Vice versa, it is obvious that every orbit equivalence relation of a Borel action of a countable group is a countable Borel equivalence relation. For orbit equivalence relations, the notion of uniform homomorphism / reduction is the following:

[1] A weak Borel reduction is defined as a countable-to-one Borel homomorphism.

Definition 1.3.5. Suppose that, for $i \in \{1,2\}$, E_i is the orbit equivalence relation of the Borel action $\cdot_i$ of the countable group G_i on the standard Borel space X_i, and let f be a homomorphism / reduction from E_1 to E_2. We say f is a **uniform** homomorphism / reduction (with respect to $\cdot_1$ and $\cdot_2$) if there is a function $u : G_1 \to G_2$ such that

$$g \cdot_1 x = y \implies u(g) \cdot_2 f(x) = f(y)$$

for all $g \in G_1$ and $x, y \in X_1$.

When there is a uniform Borel reduction from $(E_1, \cdot_1)$ to $(E_2, \cdot_2)$, we say that $(E_1, \cdot_1)$ is uniformly Borel reducible to $(E_2, \cdot_2)$. Thus, a countable Borel equivalence relation E is said to be **uniformly universal** with respect to a countable Borel action[2] $\cdot_E$ generating it, if for all countable Borel equivalence relation F and countable Borel action $\cdot_F$ generating it, $(F, \cdot_F)$ is uniformly Borel reducible to $(E, \cdot_E)$.

Notice that the definition of uniformly Turing invariant function is not too far from Definition 1.3.5. Let us say that $x \mathrel{E_1} y$ via g with respect to $\cdot_1$ when $g \cdot_1 x = y$. Then, $f : X_1 \to X_2$ is a uniform homomorphism when there is $u : G_1 \to G_2$ such that

$$x \mathrel{E_1} y \text{ via } g \text{ with respect to } \cdot_1 \implies f(x) \mathrel{E_2} f(y) \text{ via } u(g) \text{ with respect to } \cdot_2.$$

The problem in reducing the definition of uniformly Turing invariant function to Definition 1.3.5 is that there is no group action $\cdot$ such that

$$x \equiv_T y \text{ via } (i,j) \iff (i,j) \cdot x = y$$

for all $(i,j) \in \omega^2$ and $x, y \in 2^\omega$. There are indeed two problems, namely the presence of the inverse and most importantly the seriality of the action: there exists $(i,j) \in \omega^2$ such that

- for some $x \in 2^\omega$ there is some other $y \in 2^\omega$ for which $x \equiv_T y$ via (i,j);
- for some $z \in 2^\omega$ there is no $w \in 2^\omega$ for which $z \equiv_T w$ via (i,j).

The concept of generating family of partial Borel functions introduced in [MSS16] has exactly the aim of overcoming these issues and unifying these two definitions of uniformity into a single one.

Definition 1.3.6. Let X be a standard Borel space. A **generating family of partial Borel functions** on X is a countable family $\{\psi_i\}_{i \in \omega}$ (the indexing is often omitted for brevity) of partial Borel[3] functions on X that contains the identity function on X and is closed under composition. The quasi-order generated by such a family $\{\psi_i\}$ is denoted by $\leq_{\{\psi_i\}}$ and is defined by

$$x \geq_{\{\psi_i\}} y \iff \exists k \in \omega : \psi_k(x) = y.$$

[2] By this, we mean a Borel action of a countable group.

[3] Precisely, their their graph is Borel.

The equivalence relation generated by $\{\psi_i\}$ is the symmetrization of $\leq_{\{\psi_i\}}$, i.e.

$$x \, E_{\{\psi_i\}} \, y \iff \exists (i,j) \in \omega^2 : \big(\psi_i(x) = y \text{ and } \psi_j(y) = x\big).$$

If $\psi_k(x) = y$, we say $x \geq_{\{\psi_i\}} y$ via k, whereas if $\psi_i(x) = y$ and $\psi_j(y) = x$, we say that $x \, E_{\{\psi_i\}} \, y$ via (i,j). Then, a homomorphism / reduction f from $E_{\{\psi_i\}}$ to $E_{\{\theta_i\}}$ is **uniform** (with respect to $\{\psi_i\}$ and $\{\theta_i\}$) if there is a function $u : \omega^2 \to \omega^2$ such that

$$x \, E_{\{\psi_i\}} \, y \text{ via } (i,j) \implies f(x) \, E_{\{\theta_i\}} \, f(y) \text{ via } u(i,j).$$

The function u is again called **uniformity function**. We shall call a homomorphism which is uniform with respect to $\{\psi_i\}$ and $\{\theta_i\}$ a "$\{\psi_i\}$ to $\{\theta_i\}$ uniformly invariant function" (or just "$\{\psi_i\}$ **to** $\{\theta_i\}$ **UI** function").

A Borel action $\cdot$ of a countable group $G = \{g_i\}_{i\in\omega}$ on a standard Borel space X naturally induces the generating family of partial Borel functions $\{\psi_{g_i}\}$, where ψ_{g_i} is the total function on X defined by $\psi_{g_i}(x) = g_i \cdot x$. Observe that, in the hypotheses of Definition 1.3.5, f is a uniform homomorphism / reduction with respect to to $\cdot_1$ and $\cdot_2$ if and only if it is uniform with respect to the generating families of Borel functions that $\cdot_1$ and $\cdot_2$ induce.

Moreover, [MSS16] suggests using the generating family of partial Borel functions $\{\Phi_i\}$ on 2^ω defined as

$$\Phi_i(x) = \begin{cases} \varphi_i^x & \text{if } \varphi_i^x \in 2^\omega, \text{ i.e. if } \varphi_i^x \text{ is total and is } \{0,1\}\text{-valued} \\ \text{undefined} & \text{otherwise} \end{cases}$$

so that $E_{\{\Phi_i\}} = \equiv_T$ and homomorphisms from $\equiv_T$ to $\equiv_T$ which are uniform with respect to $\{\Phi_i\}$ coincide with uniformly Turing invariant functions. To see that this family is in fact closed under composition, let $c(i,j)$ be the program that, on input n and with oracle x tries to computes $\varphi_j^x(n)$ and, if that computation converges, tries to compute and output $\varphi_i^{\varphi_j^x}(n)$. Notice that, for all $x \in 2^\omega$,

$$\varphi_{c(i,j)}^x \in 2^\omega \iff \varphi_i^x \in 2^\omega \text{ and } \varphi_j^{\varphi_i^x} \in 2^\omega$$

and, in that case, $\varphi_{c(i,j)}$ and $\varphi_i^{\varphi_j^x}$ are equal. Thus,

$$\Phi_{c(i,j)} = \begin{cases} \varphi_i^{\varphi_j^x} & \varphi_j^x \in 2^\omega \text{ and } \varphi_i^{\varphi_j^x} \in 2^\omega \\ \text{undefined} & \text{otherwise} \end{cases}$$

which means that $\Phi_{c(i,j)}$ coincides with the composition $\Phi_i \circ \Phi_j$. Notice that c is a computable function; in fact, sometimes (e.g. in [Mar17]) it is required that there is a computable operation on indices providing the index for the composition.

1.4 Concrete examples on equality mod finite

In this section, we present different families of Borel partial functions generating the simplest non-trivial countable Borel equivalence relation, namely equality mod finite, which are not uniformly Borel bi-reducible to each other. The main point of these toy examples is that, in some cases, the reason for the non-existence of a uniform Borel reduction can be retrieved locally, i.e. from the structure induced by the generating family of Borel functions on single equivalence classes, whereas other times there is a global obstruction to the existence of such a reduction even though the structure induced by the two different families of Borel functions is indistinguishable.

Recall that equality mod finite, denoted E_0, is the relation on 2^ω defined by

$$xE_0y \iff \exists n \in \omega, \forall m \geq n : x(m) = y(m).$$

It is easy to see that E_0 is Borel and its every equivalence class is countable: the equivalence class of $x \in 2^\omega$ can be obtained considering the overwriting of each of the countably many $s \in 2^{<\omega}$ onto x. Overwriting can indeed be viewed as a way of generating E_0: for $s \in 2^{<\omega}$, let ow_s be the function overwriting s onto its argument, that is

$$\mathrm{ow}_s(x)(n) = \begin{cases} s(n) & n < |s| \\ x(n) & \text{otherwise.} \end{cases}$$

It is clear that each ow_s is Borel and $\mathrm{ow}_s \circ \mathrm{ow}_t$ is ow_r where r is gotten overwriting s onto t.[4] Furthermore, if ε denotes the empty string, then ow_ε is the identity function. Hence, $\{\,\mathrm{ow}_s\,\}_{s\in 2^{<\omega}}$ is a generating family of partial Borel functions and $E_{\{\,\mathrm{ow}_s\,\}} = E_0$.

Another natural way of generating E_0 is the following: think of $x \in 2^\omega$ as a formal series

$$\sum_{n=0}^{\infty} x(n)2^n.$$

Then, for instance, $011011011\ldots$ would represent the formal series

$$2^1 + 2^2 + 2^4 + 2^5 + 2^7 + 2^8 \ldots \tag{1.4.1}$$

Think of the additive action of $\mathbb{Z}$ on the space of these formal series: so for example 5 acts on (1.4.1) giving

$$\begin{aligned} &(2^0 + 2^2) + (2^1 + 2^2 + 2^4 + 2^5 + 2^7 + 2^8 \ldots) = \\ &= 2^0 + 2^1 + 2 \cdot 2^2 + 2^4 + 2^5 + 2^7 + 2^8 \ldots \\ &= 2^0 + 2^1 + 2^3 + 2^4 + 2^5 + 2^7 + 2^8 \ldots \end{aligned}$$

[4] Of course, r is just s if s is loner than t.

Analogously, the action of -5 on (1.4.1) is

$$(-2^0 - 2^2) + (2^1 + 2^2 + 2^4 + 2^5 + 2^7 + 2^8 \dots) = \\ = 2^0 + 2^4 + 2^5 + 2^7 + 2^8 \dots$$

Translating this in 2^ω, we get that 5 and -5 act on $011011011\dots$ giving $110111011\dots$ and $100011011\dots$ respectively. It is easy to verify that this is indeed a group action and that, for all x and y in the same E_0-class, there is some $k \in \mathbb{Z}$ acting on x giving y as a result. This action is dubbed *odometer*. Its orbit equivalence relation coincides with E_0 with the exception that the constant sequences $000\dots$ and $111\dots$ are in the same odometer orbit whereas they are of course in different E_0-classes. Odometer orbit equivalence and E_0 thus coincide on $X_2 := 2^\omega \setminus ([000\dots]_{E_0} \cup [111\dots]_{E_0})$. To avoid this problem, let us stipulate from now on that E_0 has X_2 as its domain.[5]

Like every group action, the odometer induces a generating family of Borel functions, so that we can talk about uniformity with respect to the odometer. Let od_2 be the map adding 1 in the sense of the odometer. Notice that this function has an inverse ${\mathrm{od}_2}^{-1}$ which is the map subtracting 1 in the sense of the odometer. Let us denote, as usual,

$${\mathrm{od}_2}^k = \begin{cases} \underbrace{\mathrm{od}_2 \circ \cdots \circ \mathrm{od}_2}_{k} & k > 0 \\ \underbrace{{\mathrm{od}_2}^{-1} \circ \cdots \circ {\mathrm{od}_2}^{-1}}_{-k} & k < 0 \\ \mathrm{id}_{X_2} & k = 0. \end{cases}$$

Then, $\{ {\mathrm{od}_2}^k \}_{k \in \mathbb{Z}}$ is a generating family of Borel functions that generates E_0 on X_2. For brevity, if ${\mathrm{od}_2}^k(x) = y$, we just say that $x \; E_0 \; y$ via k instead of $x \; E_0 \; y$ via $(k, -k)$.

Overwriting and odometer are two essentially different ways of generating E_0, as witnessed by the following:

Proposition 1.4.1 (folklore)**.** *There is no uniform Borel reduction from E_0 equipped with the odometer to E_0 equipped with overwriting.*

The most interesting thing about Proposition 1.4.1 is that it has a *local* reason, that is, it arises from the differences that single E_0-classes feature when they are equipped with odometer rather than overwriting.

Lemma 1.4.2. *Every uniform homomorphism from a single E_0-class equipped with the odometer to a single E_0-class equipped with overwriting is constant.*

From this lemma, it follows that every uniform reduction from E_0 equipped with the odometer to E_0 equipped with overwriting must be a reduction from E_0 to $=_{2^\omega}$, so Proposition 1.4.1 follows from the following basic fact in Descriptive Set Theory (see [HKL90]).

[5] This is harmless since E_0 on 2^ω (i.e. the usual E_0) and E_0 on X_2 are Borel bi-reducible, as witnessed by the map $x \mapsto 01^\frown x$.

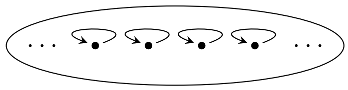

Figure 1.1: $\mathrm{od}_2{}^0$ on a E_0-class: an arrow goes from x to y if and only if $x\ E_0\ y$ via 0 with respect to the odometer.

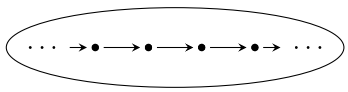

Figure 1.2: od_2 on a E_0-class: an arrow goes from x to y if and only if $x\ E_0\ y$ via 1 with respect to the odometer.

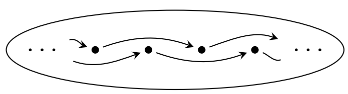

Figure 1.3: $\mathrm{od}_2{}^2$ on a E_0-class: an arrow goes from x to y if and only if $x\ E_0\ y$ via 2 with respect to the odometer.

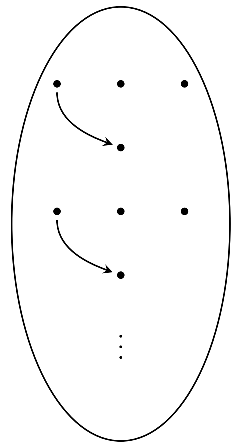

Figure 1.4: E_0-class via a fixed $(s,t) \in 2^{<\omega} \times 2^{<\omega}$, $(s,t) \neq (\varepsilon, \varepsilon)$. There is an arrow from x to y iff $x\ E_0\ y$ via (s,t) with respect to overwriting. If $(s,t) = (\varepsilon,\varepsilon)$, then the picture would look the same as figure 1.1.

Fact 1.4.3 (Non-smoothness of E_0)**.** *There is no Borel reduction from E_0 to $=_{2^\omega}$.*

Then, let us prove Lemma 1.4.2.

Proof of Lemma 1.4.2. Let $f : [x]_{E_0} \to 2^\omega$ and $u : \mathbb{Z} \to 2^{<\omega} \times 2^{<\omega}$ such that, for all $y, z \in [x]_{E_0}$

$$y\ E_0\ z \text{ via } k \text{ with respect to the odometer } \implies f(y)\ E_0\ f(z) \text{ via } u(k) \text{ with respect to overwriting.}$$

In other words, f is a uniform homomorphism and u is a uniformity function for it; notice that $\mathrm{ran}(f)$ is necessarily contained in a single E_0-class.

Recall that we denote by ε the empty string. The key point is that u must map every $k \in \mathbb{Z}$ to $(\varepsilon, \varepsilon)$: x admits y and z in the same E_0-class such that $z\ E_0\ y$ via k and $y\ E_0\ x$ via k, so this implies $f(z)\ E_0\ f(y)$ via $u(k)$ and $f(y)\ E_0\ f(x)$ via $u(k)$. But this means $u(k) = (\varepsilon, \varepsilon)$, because E_0 via any other $(s,t) \in 2^\omega \times 2^\omega$ does not feature pre-predecessors (see figures 1.1–1.4). Thus, for all y in $[x]_{E_0}$, $f(x)$ and $f(y)$ are E_0-equivalent via $(\varepsilon, \varepsilon)$, that is, they are equal. $\square$

Proposition 1.4.4. *E_0 equipped with overwriting is uniformly Borel reducible to E_0 equipped with odometer, as witnessed by the identity function.*

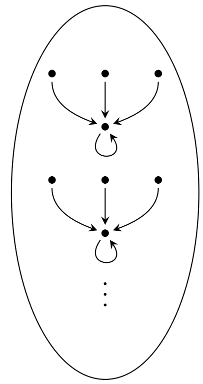

Figure 1.5: Overwriting the string s on a single E_0-class: there is an arrow x to y if and only if $\mathrm{ow}_s(x) = y$. The presence of fixed points is the key for Remark 1.4.5.

Proof. First notice that, if $x\ E_0\ y$ via (s,t), we can suppose that s and t have same length. For if, say, s is shorter than t, then $x(n) = y(n) = t(n)$ for $|s| \leq n < |t|$. Thus, if we call $s \star t$ the result of overwriting s onto t, then $s \star t$ is as long as t and $x\ E_0\ y$ via $(s \star t, t)$. Then, if we view $s \star t$ and s as two natural numbers written in base 2, it is obvious that there is an integer k which, added or subtracted to $s \star t$, gives t. Hence, $x\ E_0\ y$ via k with respect to the odometer. □

Remark 1.4.5. Sometimes, it happens that a uniform homomorphism or reduction f admits a uniformity function $u : \omega^2 \to \omega^2$ of the form

$$u : (i,j) \mapsto (v(i), v(j)),$$

with $v : \omega \to \omega$. This cannot be the case for a Borel uniform reduction from E_0 with overwriting to E_0 with the odometer, and the reason for this is again local. Indeed, for all $s \in 2^{<\omega}$ and all x, $\mathrm{ow}_s(x)$ is in $[x]_{E_0}$ and is fixed by ow_s, i.e. $\mathrm{ow}_s(\mathrm{ow}_s(x)) = \mathrm{ow}_s(x)$. Then, $v(s)$ must be 0, because it must fix $f(\mathrm{ow}_s(x))$, and no other $k \in \mathbb{Z}$ does. Then, v must be identically zero, and this means, like in the argument for proving Proposition 1.4.1, that f would be a Borel reduction from E_0 to $=_{2^\omega}$, which is impossible by Fact 1.4.3.

There are also cases in which the reason for the non-existence of a uniform Borel reduction *cannot* be detected locally. This is the case of the odometer on 3^ω and on 2^ω respectively.[6]

[6] Of course, odometer on 3^ω works as addition with carry in base 3 instead of base 2. When we consider the odometer on 3^ω we actually have to remove the E_0-classes of constant sequences. E_0 on this new set is Borel bi-embeddable with E_0 on 3^ω and, hence, with E_0 on 2^ω.

Theorem 1.4.6 (Marks, private communication)**.** *There is no uniform Borel reduction from E_0 on 3^ω equipped with the odometer to E_0 on 2^ω equipped with the odometer.*

Proof. Suppose by contradiction that there is such a uniform Borel reduction f and u is a uniformity function for it. We stress that odometer on 2^ω is actually defined on

$$X_2 := 2^\omega \setminus \big([000\dots]_{E_0} \cup [111\dots]_{E_0}\big)$$

and similarly the odometer on 3^ω is defined on the set X_3 of elements of 3^ω that are not eventually constant. So f is actually a function from X_3 to X_2. Let od_3 denote the map over X_3 that adds 1 in the sense of the odometer in base 3. Of course, this map is continuous and has an inverse (the map that subtracts 1), which is also continuous. This means, od_3 is a homeomorphism of X_3 onto itself. Define a graph G on X_3 by saying that $x, y \in X_3$ are adjacent iff $\mathrm{od}_3(x) = y$ or $\mathrm{od}_3(y) = x$. We can exploit the uniformity of f to give a 2-coloring of G, that is, a map $c : X_3 \to 2$ that sends adjacent vertices to different numbers (colors). Indeed, if $x, y \in X_3$ are adjacent in the graph G, i.e. $x\ E_0\ y$ via 1 or $y\ E_0\ x$ via 1, then $f(x)\ E_0\ f(y)$ via $u(1)$ or $f(y)\ E_0\ f(x)$ via $u(1)$. Notice that $u(1)$ cannot be 0, otherwise f would be a Borel reduction from E_0 to $=_{2^\omega}$, so if k is the first non-null digit in the binary representation of $u(1)$, $f(x)$ and $f(y)$ differ in the k-th digit. This means that

$$c : X_3 \to 2; x \mapsto f(x)(k)$$

is a 2-coloring of the vertices of G. This 2-coloring is also Borel, because f is. By the Baire Category Theorem, either $c^{-1}(0)$ or $c^{-1}(1)$ has to be non-meager. Since this non-meager set is Borel, it has the Baire property and is therefore comeager on some basic open set

$$N_s = \{\, x \in X_3 \mid x \text{ extends } s \,\}.$$

If ℓ is the length of s, notice that ${\mathrm{od}_3}^{3^\ell}[N_s] \subseteq N_s$ and ${\mathrm{od}_3}^{-3^\ell}[N_s] \subseteq N_s$, so that ${\mathrm{od}_3}^{3^\ell} \restriction N_s$ is a homeomorphism of N_s onto itself. Since 3^ℓ is odd, ${\mathrm{od}_3}^{3^\ell}$ sends vertices of one color to vertices of the other color: ${\mathrm{od}_3}^{3^\ell}[c^{-1}(0) \cap N_s] = c^{-1}(1) \cap N_s$ and ${\mathrm{od}_3}^{3^\ell}[c^{-1}(1) \cap N_s] = c^{-1}(0) \cap N_s$, so we have a homeomorphism that maps a comeager set onto a meager set, contradiction. $\square$

Even if not uniformly, E_0 on 3^ω is Borel reducible to E_0 on 2^ω, as they both are hyperfinite non-smooth Borel equivalence relations ([DJK94]).

1.5 Local vs global obstructions

Our point in discussing these examples on E_0 is that the obstruction to the existence of uniform Borel reductions can be local or global. In the case of Proposition 1.4.1, it *is* local, because it can be detected from the structure induced by the generating families of Borel functions on single equivalence classes:

the obstruction to the existence of a "global" Borel uniform reduction can be derived — using a general fact (namely, Fact 1.4.3) — from obstructions to the existence of non-trivial uniform homomorphisms defined on single equivalence classes. On the other hand, the obstruction given by Theorem 1.4.6 is purely global, as the structures induced by $\{\, {\mathrm{od}_3}^k \,\}_k$ and $\{\, {\mathrm{od}_2}^k \,\}_k$ on single equivalence classes are isomorphic: for all E_0-classes $C \subseteq X_3$ and $D \subseteq X_2$, there is a bijective $f : C \to D$ such that

$$ {\mathrm{od}_3}^k(x) = y \iff {\mathrm{od}_2}^k(f(x)) = f(y), $$

for all $x, y \in C$ and $k \in \mathbb{Z}$. The reason why one cannot extend such local isomorphisms to a global uniform reduction is that one can build isomorphisms between (E_0 on 3^ω)-classes and (E_0 on 2^ω)-classes *once representatives are chosen among those classes*, and Fact 1.4.3 tells us we cannot choose a representative in each E_0-class in a Borel way.

We point out that Theorems 1.2.1, 1.2.2 and 1.2.3 are also establishing obstructions to the existence of certain kinds of uniform homomorphisms: this time, the equivalence relation is Turing equivalence and the generating family of partial Borel functions is given by Turing reductions. So, the main question we are going to investigate in the next chapters is the following:

Question 1.5.1. *Are the obstructions established by Theorems 1.2.1, 1.2.2 and 1.2.3 local?*

We are going to prove that the answer is positive for Theorems 1.2.1 and 1.2.3, respectively in Chapters 4 and 2. Moreover, in Chapter 4 we are going to investigate what happens for 1.2.2.

Chapter 2

Part I of uniform Martin's conjecture, locally

In this chapter, we answer Question 1.5.1 for what concerns part I of uniform Martin's conjecture, showing that the obstruction proven by Theorem 1.2.3 arises locally similarly to how the obstruction to Proposition 1.4.1 turned out to be local in Chapter 1.

2.1 The result

The proof given by Slaman and Steel in [SS88] of part I of uniform Martin's conjecture makes use of the Axiom of Determinacy, as they use a variation of a Wadge game[1] such that the existence of a winning strategy for player I or II implies $f \geq_M \mathrm{id}_{2^\omega}$ or $f \leq_M \mathrm{id}_{2^\omega}$ respectively, and then they present two different arguments that prove that $f <_M \mathrm{id}_{2^\omega}$ implies that f is constant (up to $\equiv_T$) on a cone.

Here, we shall present a proof of the following improvement of their result which does not use games and that clearly displays a local cause.

Theorem 2.1.1. *Assume* TD *and let* $f : 2^\omega \to 2^\omega$ *be UTI on a cone. Then, either* $f(x) \geq_T x$ *on a cone, or there exists* $y \in 2^\omega$ *such that* $f(x) = y$ *on a cone.*

Let us stress the differences between the results: Slaman and Steel showed that, under AD, UTI functions are either increasing or constant *up to Turing equivalence* on a cone. By contrast, Theorem 2.1.1 tells us that, under the sole assumption of TD, UTI functions are either increasing or *literally* constant on a cone.

In Chapter 1, we showed that Proposition 1.4.1 is a global obstruction which is caused by the local obstruction Lemma 1.4.2 via a general property of

[1] Steel had already used variations of Wadge games to prove part II of uniform Martin's conjecture in [Ste82].

reductions which are Borel, namely Fact 1.4.3. In a very similar way, our proof of Theorem 2.1.1 will show that the global dichotomy in part I of uniform Martin's conjecture arises from an analogous local dichotomy via a general property of TI functions under TD.

Theorem 2.1.2. *Let $x \in 2^\omega$ and $f : [x]_T \to 2^\omega$ be UTI. Then, either $f(x) \geq_T x$ or f is constant.*

We shall refer to this Theorem as the local uniform Martin's conjecture part I. Before we prove it, let us show how easily Theorem 2.1.1 descends from it.

Proof of Theorem 2.1.1 *from Theorem* 2.1.2. Suppose f is UTI on the cone above z. Consider

$$A = \{ x \in 2^\omega \mid f \restriction [x]_T \text{ is constant} \}.$$

Of course, A is Turing-invariant, so by TD either $2^\omega \setminus A$ or A contains a cone. In the former case — say $2^\omega \setminus A$ contains the cone above w — given any $x \geq_T z \oplus w$, we can apply Theorem 2.1.2 and deduce $f(x) \geq_T x$. Otherwise, if A contains a cone, next folklore Fact applies. □

Fact 2.1.3. ($\mathsf{ZF} + \mathsf{CC}_\mathbb{R} + \mathsf{TD}$) *Suppose $f : 2^\omega \to 2^\omega$ is such that the following holds for all x, y in a cone:*

$$x \equiv_T y \implies f(x) = f(y).$$

Then, f is literally constant on a cone.

Recall that $\mathsf{CC}_\mathbb{R}$ stands for the Axiom of Countable Choice for Reals, which assert that every function F from natural numbers to non-empty sets of reals admits a choice function, that is a function c from natural numbers to reals such that, for all $n \in \omega$, $c(n) \in F(n)$.

The easy yet classic argument for Fact 2.1.3 is probably found for the first time in [SS88], in the form of a remark that AD implies there is no choice function on the Turing degrees. We present it here for the reader's convenience.

Proof of Fact 2.1.3. Suppose that the hypothesis holds in the cone based in z. Then, the sets of the form

$$\{ x \geq_T z \mid f(x)(i) = j \}$$

are Turing-invariant, and so TD implies that, for all i, there's a cone of x's on which the i-th digit of $f(x)$ is constant. Use $\mathsf{CC}_\mathbb{R}$ to choose, for all $i \in \omega$, a $z_i \in 2^\omega$ such that $x \mapsto f(x)(i)$ is constant on the cone above z_i. Hence, f is constant on the cone above $\bigoplus_i z_i$.[2] □

[2]For the definition of $\bigoplus z_i$, see page 18.

Another advantage of Theorem 2.1.2 is that it enables to calibrate the strength of the statement of part I of uniform Martin's conjecture over ZF. We now have two different statements for uniform Martin's conjecture part I, namely the original Theorem 1.2.3 and our slight improvement Theorem 2.1.1. However, it is an easy corollary of Theorem 2.1.2 that they are both equivalent to TD over $\mathsf{ZF} + \mathsf{CC}_{\mathbb{R}}$. If we exploit a recent result of Peng and Yu, we can prove that the equivalence holds over ZF alone.

Theorem 2.1.4 (Peng and Yu, [PY20])**.** $\mathsf{ZF} + \mathsf{TD}$ *implies* $\mathsf{CC}_{\mathbb{R}}$.

Corollary 2.1.5. *The following statements are equivalent over* ZF*:*

(a) TD*;*

(b) *the statement of Fact* 2.1.3*;*

(c) *for all* $f : 2^\omega \to 2^\omega$ *which is UTI on a cone, either* $f(x) \geq_T x$ *on a cone, or* f *is literally constant on a cone;*

(d) *for all* $f : 2^\omega \to 2^\omega$ *which is UTI on a cone, either* $f(x) \geq_T x$ *on a cone, or* f *is constant up to* $\equiv_T$ *on a cone.*

Proof. (a) $\implies$ (b) because Fact 2.1.3 can be proved in $\mathsf{ZF} + \mathsf{TD} + \mathsf{CC}_{\mathbb{R}}$, and hence in $\mathsf{ZF} + \mathsf{TD}$ by Theorem 2.1.4. Notice that also (b) implies (a) because, to prove that a Turing-invariant $A \subseteq 2^\omega$ either contains or is disjoint from a cone, it is enough to prove that its "characteristic function"

$$f_A(x) = \begin{cases} 111\ldots & x \in A \\ 000\ldots & x \notin A \end{cases}$$

is constant on a cone.

To see (b) $\implies$ (c), it is enough to know that Theorem 2.1.2 is provable in ZF and the proof of Theorem 2.1.1 from Theorem 2.1.2 only used (b) and TD, which follows from (b) as we just showed.

(c) $\implies$ (d) is trivial. Let us prove (a) from (d). Fix $A \subseteq 2^\omega$ which is Turing invariant (i.e. closed under $\equiv_T$), and define

$$f(x) = \begin{cases} \underline{0} = 000\ldots & x \in A, \\ \underline{0}' & x \notin A. \end{cases}$$

Of course, f is UTI: it is easy to witness $f(x) \equiv_T f(y)$, as this only happens when $f(x)$ and $f(y)$ are equal. Clearly, $f \not\geq_M \mathrm{id}_{2^\omega}$, so by ($d$) we get that f is constant on a cone up to $\equiv_T$. Then, either $f(x) \equiv_T \underline{0}$ on a cone, or $f(x) \equiv_T \underline{0}'$ on a cone. In the former case, A contains a cone, in the latter one, the complement of A does. □

In [CWY10], the authors calibrated the strength of part II of uniform Martin's conjecture for projective functions.

Theorem 2.1.6 (Chong, Wang and Yu, [CWY10]). *Over* ZFC*, part II of projective uniform Martin's conjecture is equivalent to Projective Determinacy (which, by unpublished work by Woodin, is equivalent over* ZFC *to Projective Turing Determinacy).*

Using the same proof of Corollary 2.1.5, one can easily get that, under ZFC, projective Turing Determinacy is equivalent to (either statement of) part I of uniform Martin's conjecture. Putting all together, we get the following:

Theorem 2.1.7. *The following are equivalent over* ZFC*:*

- *Projective Determinacy;*
- *Projective Turing Determinacy;*
- *part I of projective uniform Martin's conjecture;*
- *part II of projective uniform Martin's conjecture.*

2.2 The proof

We now address the proof of Theorem 2.1.2. The argument itself is very short and easy, but we need a few preliminaries and notation first. Recall that the join (or merge) of $x, y \in 2^\omega$, is the element of 2^ω denoted by $x \oplus y$ and defined by

$$(x \oplus y)(n) = \begin{cases} x\left(\frac{n}{2}\right) & \text{if } n \text{ is even,} \\ y\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, the join (or merge) of a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of 2^ω is the element of 2^ω denoted by $\bigoplus_n x_n$ defined by

$$\left(\bigoplus_n x_n\right)(\langle i, j\rangle) = x_j(i),$$

where $\langle \cdot, \cdot \rangle$ is a computable bijection between $\mathbb{N}^2$ and $\mathbb{N}$ chosen once for all. We shall say that x_j is the j-th column of $\bigoplus_n x_n$, while $n \mapsto x_n(i)$ is its i-th row. The following fact easily descends from the existence of a universal oracle Turing machine.

Fact 2.2.1. *Fix $y \in 2^\omega$ and a computable $t : \mathbb{N} \to \mathbb{N}$. If*

$$\bigoplus_n \varphi^y_{t(n)}$$

is in 2^ω, then it is Turing reducible to y.

Lemma 2.2.2 (Computable uniformity function lemma). *Let $A \subseteq 2^\omega$ be such that, for all $x \in A$, the concatenations $0^\frown x$ and $1^\frown x$ belong to A as well. Let $f : A \to 2^\omega$ be either UOP or UTI. In either case, there is a computable uniformity function for f.*

Proof. First, suppose f is UOP and u is a uniformity function for it. Let $a, b, c \in \mathbb{N}$ be such that $\varphi_c^x = 1^\frown x$, $\varphi_b^x = 0^\frown x$ and

$$\varphi_a^{0^e 1^\frown x} = \varphi_e^x$$

for all $x \in 2^\omega$ ($0^e 1$ is shorthand for $\underbrace{0 \dots 0}_{e} 1$). The argument will essentially be that the behavior of u is ruled out in a computable way by $u(a)$, $u(b)$ and $u(c)$. To see this, let ij denote the composition of computer programs i and j, so that

$$\varphi_i^{\varphi_j^x} = \varphi_{ij}^x.$$

There's no ambiguity in writing ijk as this operation is associative. Now, fix $x \in A$ and $e \in \mathbb{N}$ such that φ_e^x is in A and notice that we have $\varphi_e^x = \varphi_{ab^e c}^x$. Therefore,

$$f(\varphi_e^x) = f\big(\varphi_{ab^e c}^x\big) \leq_T f\big(\varphi_{b^e c}^x\big) \text{ via } u(a),$$

for $\varphi_{ab^e c}^x \leq_T \varphi_{b^e c}^x$ via a and $\varphi_{b^e c}^x = 0^e 1^\frown x$ is in A by the hypothesis on A, because x is A.[3] By iterating this, we finally reach

$$f(\varphi_e^x) \leq_T f(x) \text{ via } u(a)u(b)^e u(c).$$

Thus, setting $v(e) \coloneqq u(a)u(b)^e u(c)$, we get that v is a uniformity function for f, and since $u(a)$, $u(b)$, $u(c)$ are three fixed natural numbers and the composition of computer programs is computable, v is computable, too.

When f is UTI, the argument is analogous. This time, define

$$(i,j)(k,l) \coloneqq (ik, lj)$$

and let $d \in \mathbb{N}$ be such that, for all $x \in 2^\omega$,

$$\varphi_d^{0^i 1 0^j 1^\frown x} = 0^j 1 0^i 1^\frown \varphi_i^x.$$

Also pick $m \in \mathbb{N}$ such that $\varphi_m^x(n) = x(n+1)$ for all n, so that we have $x \equiv_T 1^\frown x$ via (c, m) and $1^\frown x \equiv_T x$ via (m, c).

Now observe that, for $x, y \in 2^\omega$:

$$\begin{aligned} x \equiv_T y \text{ via } (i,j) &\iff (0^i 1 0^j 1^\frown x) \equiv_T (0^j 1 0^i 1^\frown y) \text{ via } (d,d) \\ &\iff x \equiv_T (0^j 1 0^i 1^\frown y) \text{ via } (d,d)(b,m)^i(b,c)(b,m)^j(c,m) \\ &\iff x \equiv_T y \text{ via } (m,c)(m,b)^i(m,c)(m,b)^j(d,d)(b,m)^i(b,c)(b,m)^j(c,m). \end{aligned}$$

Thus, if u is a uniformity function for f, we can set

$$v(i,j) \coloneqq u(m,c)u(m,b)^i u(m,c)u(m,b)^j u(d,d)u(b,m)^i u(b,c)u(b,m)^j u(c,m)$$

and the same argument as before gives us that v is a computable uniformity function for f. □

[3] The author wishes to thank Kirill Gura for pointing out the necessity of the hypothesis that A is closed under initial appending of 0's and 1's in order to carry on this argument.

Proof of Theorem 2.1.2. Suppose f is not constant, so that there is $z \equiv_T x$ such that $f(x) \neq f(z)$. Obviously, there is a computable function r such that

$$\varphi^x_{r(n)} = \begin{cases} x & \text{if } x(n) = 1, \\ z & \text{if } x(n) = 0. \end{cases}$$

Also obviously, there is $e \in \mathbb{N}$ such that

$$\varphi^x_e = \varphi^z_e = x.$$

If we call $\varphi^x_{r(n)} = x_n$, we thus have $x \equiv_T x_n$ via $(r(n), e)$. So, if u is a computable uniformity function for f (which exists by Lemma 2.2.2), we get

$$f(x) \equiv_T f(x_n) \text{ via } u(r(n), e).$$

Notice that

$$f(x_n) = \begin{cases} f(x) & \text{if } x(n) = 1, \\ f(z) & \text{if } x(n) = 0. \end{cases}$$

So, supposing that $f(x)$ and $f(z)$ differ on the k-th digit, the k-th row of $\bigoplus_n f(x_n)$ is either x or $i \mapsto 1 - x(i)$, and hence

$$\bigoplus_n f(x_n) \geq_T x.$$

But also $f(x) \geq_T \bigoplus_n f(x_n)$ by Fact 2.2.1, because $f(x) \geq_T f(x_n)$ via the first coordinate of $u(r(n), e)$, which is a computable function of n. □

2.3 The arithmetic case

Since there exist many reducibility notions in computability theory, hence families of partial Borel functions generating quasi-orders (hence equivalence relations), we could ask how much of the local phenomenon we discovered about Turing degrees also holds for these similar notions of degrees. We shall notice that Theorem 2.1.2 is actually quite peculiar of the Turing case.

An example of such a similar reducibility is arithmetic reducibility. Recall that a real x is said to be arithmetically reducible to a real y — and this is written $x \leq_A y$ if there is some $n \in \omega$ such that $x \leq_T y^{(n)}$. By Post's theorem, this is equivalent to x being definable in first-order arithmetic with a parameter for y. Of course, x and y are called arithmetically equivalent when they are arithmetically reducible to each other, and this is denoted by $x \equiv_A y$. As for Turing reducibility and equivalence, we can formulate a "via" version for arithmetic reducibility and equivalence: we can say that $x \leq_T y$ via (i, n) if $x \leq_T y^{(n)}$ via i. This enables to talk about uniformly arithmetically invariant (**UAI**) functions and formulate an arithmetic analog of Martin's conjecture in the uniform version as well. However, Slaman and Steel proved in the early '90s (even though the proof was only published in [MSS16]) how to construct a counter-example for part I of the arithmetic uniform Martin's conjecture, that is, a uniformly arithmetically invariant $g : 2^\omega \to 2^\omega$ which is neither constant on a cone up to $\equiv_A$, nor such that $g(x) \geq_A x$ on a cone.[4] Of course, since part I of the arithmetic uniform Martin's conjecture fails globally, it also has to fail locally: the best we can do to generalize Theorem 2.1.2 to the arithmetic case is the following:

Proposition 2.3.1. *If $f : [x]_A \to 2^\omega$ is uniformly arithmetically invariant, then either f is constant or $x \leq_T f(x)^{(\omega)}$.*

Proof. If f is uniformly arithmetically invariant, then $f^{(\omega)}$ is arithmetic- to many-one- uniformly invariant, hence uniformly Turing invariant. Then, by Theorem 2.1.2, either $f(x)^{(\omega)} \geq_T x$ or $f^{(\omega)}$ is constant on $[x]_A$, and in the latter case f itself must be constant. □

In fact, the analog of Fact 2.2.1 that would need to generalize the proof of Theorem 2.1.2 to the arithmetic case would be that, for total computable s and t,

$$\bigoplus_n \varphi_{t(n)}^{y^{(s(n))}}$$

is Turing reducible to $y^{(\omega)}$. We cannot do any better than that (essentially, because there is a universal Turing machine but no universal arithmetic reduction).

Notice we proved Proposition 2.3.1 *from* and not *like* Theorem 2.1.2, using the above analog of Fact 2.2.1: indeed, there is one more point of the proof of Theorem 2.1.2 that does not (at least, not immediately) adapt to the arithmetic case, namely Lemma 2.2.2. There is no problem to adapt it to uniformly arithmetically order-preserving functions, though.

Lemma 2.3.2. *Let $A \subseteq 2^\omega$ be closed under arithmetic equivalence, and suppose $f : A \to 2^\omega$ is uniformly arithmetically order-preserving. Then, there is a computable uniformity function for f.*

Proof. The same as the proof of Lemma 2.2.2 for uniformly (Turing) order-preserving functions, but instead of just considering Turing (hence, arithmetic) reductions a, b and c defined there, also use an arithmetical reduction δ that takes every $x \in 2^\omega$ to the Turing jump of x. Then, clearly, $x \leq_T y^{(n)}$ via i if and only if $x \leq_A y$ via $ab^i c\delta^n$; furthermore, if x and y are in A, then A also contains all the intermediate reals that are gotten by applying reductions a, b, c, δ in $ab^i c\delta^n$ one after the other. Thus, the proof continues as that of Lemma 2.2.2 without a hitch. □

[4] Here, 'cone' might mean either Turing upward cone or arithmetical upward cone, since g is arithmetically invariant and the arithmetical cone above z coincides with the closure under $\equiv_A$ of the Turing cone above z.

A hitch, however, occurs when trying to the same result for UAI functions. Of course, there is no problem in getting invertible arithmetic reductions that play the role of a, b, c and δ, but the matter is: how to combine them effectively when we are given i, j and m, n such that $x^{(m)} \geq_T y$ via i and $y^{(n)} \geq_T x$ via i? The trick of appending a code for i, j, m, n at the beginning of the reals does not seem to help because — again — there is no universal arithmetic reduction, which means there is no arithmetic reduction that performs the n-th iteration of the jump, according to the n that finds coded at the beginning of the oracle. A solution could be found if we could compute from input i, j, m, n an output i', j', m', n' such that $x^{(m')} \equiv_T y^{(n')}$ via (i', j'). However, this is impossible in general, as shown by the following result.

Theorem 2.3.3 (Sacks, [Sac67])**.** *There exists $x \in 2^\omega$ such that, for all $n \in \omega$,*

$$0^{(n)} <_T x^{(n)} <_T 0^{(n+1)}$$

The x in the previous theorem is arithmetically equivalent to 0, but no finite jump of x is Turing equivalent to no finite jump of 0. Hence, we ask the following:

Question 2.3.4. *Is there a uniformly arithmetically invariant function that admits no computable uniformity function?*

Going back to Martin's conjecture, we said the argument in [MSS16] that disproves the arithmetic analog of Martin's conjecture does not disprove part II alone. In other words, it does not disprove the statement that the set of arithmetically invariant functions f such that $f(x) \geq_A x$ on a cone is pre-well-ordered by the relation $\leq_A^\triangledown$:

$$f \leq_A^\triangledown g \iff \exists y,\ \forall x \geq_A y : f(x) \leq_A g(x).$$

As the authors note, the arithmetic analog of Steel's Conjecture 1.2.4 — that is, the conjecture that every arithmetically invariant function is arithmetically equivalent on a cone to a uniformly arithmetically invariant one — remains open. So Steel's conjecture might be the right conjecture to generalize to other notions of reducibility, whereas Martin's seems to be peculiar of Turing reducibility.

2.4 Variance functions

If we examine the proof of Theorem 2.1.2, we can observe that it holds not only when $f : [x]_T \to 2^\omega$ is uniformly Turing invariant, but it suffices that f admits a *computable* function v such that, for all y,

$$x \equiv_T y \text{ via } (i, j) \implies f(x) \equiv_T f(y) \text{ via } v(i, j). \tag{2.4.1}$$

In other words, v need not be a uniformity function for f, as the previous formula need not hold for all elements of $[x]_T$, but just for x. That is, f only need admit a computable variance function in x, in the sense of the following definition.

Definition 2.4.1. If $x \in A \subseteq 2^\omega$ and $f : A \to 2^\omega$, we say $v : \omega^2 \to \omega^2$ is a **variance function** of f in x (with respect to Turing equivalence via) if, for all $y \in A$, (2.4.1) holds.

Observe we do not require f to be Turing invariant, but if it has a variance function in x, then it must map the degree of x into a single degree.

Actually, in the proof of Theorem 2.1.2, we do not even need all of v to be computable, but just its first component to be so. Let us call the projection on the first coordinate of a variance function a downward variance function.

Definition 2.4.2. If $x \in A \subseteq 2^\omega$ and $f : A \to 2^\omega$ is Turing invariant, we say $v : \omega^2 \to \omega$ is a **downward variance function** of f in x if, for all $y \in A$,

$$x \equiv_T y \text{ via } (i, j) \implies f(x) \geq_T f(y) \text{ via } v(i, j).$$

On the other hand, we say v is a **upward variance function** of f in x if, for all $y \in A$,

$$x \equiv_T y \text{ via } (i, j) \implies f(x) \leq_T f(y) \text{ via } v(i, j).$$

Then, notice that the argument we used to prove Theorem 2.1.2 actually leads to:

Theorem 2.4.3. *If v is a downward variance function of f in x, then either f is constant or $f(x) \oplus v \geq_T x$.*

We shall say more about variance and downward variance functions in Chapter 5, but now let us generalize this notion with respect to arbitrary generating families of reductions, which will be useful later on.

Definition 2.4.4. Let $f : X \to Y$ be any function, $\{ \Psi_i \}$ and $\{ \Omega_i \}$ be generating family of reductions on X and Y respectively, and fix $x \in X$ and $y \in Y$. We say $v : \omega \to \omega$ is a variance function of f in (x, y) with respect to $\{ \Psi_i \}$ and $\{ \Omega_i \}$ if, for all $z \in X$ and $e \in \omega$:

$$x \geq_{\{ \Psi_i \}} z \text{ via } e \implies y \geq_{\{ \Omega_i \}} f(z) \text{ via } v(e).$$

We say $v : \omega \to \omega$ is a variance function in x if it is a variance function in $(x, f(x))$.

Definition 2.4.2 is essentially Definition 2.4.4 with respect to the family of Turing bi-reductions. This is our default choice when we say 'variance function' unless other generating family of reductions are specified explicitly or implicitly from the context. Furthermore, note that a downward variance function is just a variance function with respect to Turing bi-reductions on the domain and Turing reductions on the codomain.

2.5 Left inverses of UTI functions

We present here another application of the local version of part I of uniform Martin's conjecture. Let us start observing that uniformly Turing invariant functions defined on single degrees are either constant or injective.

Lemma 2.5.1 (Kirill Gura's injectivity lemma, private communication). *If $f : [x]_T \to 2^\omega$ is uniformly Turing invariant and non-constant, then f is injective.*

Proof. Suppose, towards a contradiction, that there are $y, z \in [x]_T$, $y \neq z$ such that $f(y) = f(z)$. By the hypothesis that f is non-constant, there must be an element in $[x]_T$ whose image is not $f(y)$, for instance $f(x) \neq f(y)$. Notice that x differs from y and z as it has a different image via f. Let n be such that $x \upharpoonright n \neq y \upharpoonright n$, and let $k, l \in \omega$ be such that $\varphi_k^x = y$ and $\varphi_l^y = z$. Consider i such that, for all $w \in 2^\omega$,

$$\varphi_i^w = \begin{cases} \varphi_k^w & \text{if } w \upharpoonright n = x \upharpoonright n, \\ \varphi_l^w & \text{otherwise.} \end{cases}$$

Thus, we have $\varphi_i^x = y$ and $\varphi_i^y = z$. Analogously, we have j such that $\varphi_j^y = x$ and $\varphi_j^z = y$, so that $x \equiv_T y$ via (i, j) and $y \equiv_T z$ via (i, j). By uniformity of f, there is $(i', j') \in \omega^2$ such that $f(x) \equiv_T f(y)$ and $f(y) \equiv_T f(z)$ both via (i', j'). Hence, $\varphi_{j'}^{f(z)} = f(y)$ and $\varphi_{j'}^{f(y)} = f(x)$, but $f(x) \neq f(y) = f(z)$, contradiction. $\square$

Remark 2.5.2. When we consider functions which are defined on more than one degree, we can find UTI function that are constant on some degrees and injective on others. For example, consider f that maps computable reals to $000\ldots$ and every non-computable real y to $1^\frown y$. This function is uniformly order preserving on every degree: let $v(i)$ be the program that checks whether the first bit of the oracle is 0, and in that case outputs 0 on every input, otherwise acts as program i on the oracle with the first bit chopped off. If $x, y \in 2^\omega$ are both computable or both non-computable, and $x \geq_T y$ via i, it is clear that $f(x) \geq_T f(y)$ via $v(i)$. Hence, f is uniformly Turing invariant via $(i, j) \mapsto (v(i), v(j))$. Moreover, it is constant on a degree and non-constant on all other degrees.

Remark 2.5.3. Also observe that, although there is no distinction between being UTI on every degree and being UTI everywhere, this function is an example of a function that is UOP on every degree but not everywhere: if it were, then for every non-computable y, one could decide whether φ_i^y is computable or not by checking whether $\varphi_{v(i)}^{1^\frown y}(0)$ is 0 or 1, where v is a computable uniformity function in the order-preserving sense (which would exist by Lemma 2.2.2). In other words, we would have a function computable function r that maps every index i such that φ_i^y is total to 1, if φ_i^y is computable, to 0 otherwise. Let c be a computable function such that

$$\varphi^y_{c(i)}(n) = \begin{cases} y(n) & i \notin y'[n] \\ 0 & \text{otherwise,} \end{cases}$$

where $y'[n]$ denotes the approximation of y' at stage n, so that $\varphi^y_{c(i)}$ equals y if $i \notin y'$, otherwise it is eventually constantly 0. Since y is not computable, we get

$$r(c(i)) = \begin{cases} 1 & i \in y' \\ 0 & \text{otherwise,} \end{cases}$$

whence we get $y' \leq_T 0$, contradiction.

The proof of Lemma 2.5.1 does not tell us that if f is UTI and injective on every degree, then it is injective everywhere, whereas the proof of Theorem 2.1.2 does, and also tells us that the left inverse is provided by a Turing reduction.

Proposition 2.5.4. *If A is a Turing-invariant subset of 2^ω and $f : A \to 2^\omega$ is a UTI function which is non-constant on every Turing degree $\subseteq A$, then f is injective and there is a Turing reduction providing a left inverse of it, i.e. there is $i \in \omega$ such that*

$$\varphi_i^{f(x)} = x$$

for all $x \in \text{dom}(f)$. Moreover, the index i of this Turing reduction can be effectively produced from an index of a computable uniformity function of f.

Proof. The proof is essentially the same as that of Theorem 2.1.2. By Lemma 2.2.2, take a computable uniformity function u for f, and let φ_e be the projection on the first coordinate of u, so that, for all $x, y \in A$,

$$x \equiv y \text{ via } (i,j) \implies f(x) \geq_T f(y) \text{ via } \varphi_e(i,j).$$

Let $a, b, m \in \omega$ be such that

$$x \equiv_T 0^\frown x \text{ via } (a,m) \qquad x \equiv_T 1^\frown x \text{ via } (b,m)$$

and let r be a computable function such that, for all $n \in \omega$,

$$x \equiv_T x(n)^\frown x \text{ via } (r(n), m).$$

Note that a, b, m and r do not depend on anything else. Now, for all $x \in A$, $f(x)$ can compute x as follows. First, it searches the first $k \in \omega$ such that $f(0^\frown x)$ differs from $f(1^\frown x)$ on the k-th digit; $f(x)$ can compute $f(0^\frown x)$ and $f(1^\frown x)$ as $\varphi^{f(x)}_{\varphi_e(a,m)}$ and $\varphi^{f(x)}_{\varphi_e(b,m)}$ respectively. If $f(0^\frown x)(k) = i_0$ and $f(1^\frown x)(k) = i_1$, let t be a function sending i_0 to 0 and i_1 to 1. Then, notice that $n \mapsto t\big(f(x(n)^\frown x)(k)\big)$ coincides with x and $f(x(n)^\frown x) = \varphi_e^{f(x)}(r(n), m)$ So this algorithm computes x from $f(x)$ and its code is clearly computable from e. □

When working with UTI functions in the usual global setting, that is, on pointed perfect sets (see Appendix B for the definition), it is trivial to refine a cone on which $f(x) \geq_T x$ to a uniformly pointed perfect set on which $f(x) \geq_T x$ via the same i, i.e. on which f admits a computable left inverse. With uniformly Turing invariant functions, we now know that no refinement is needed. Lemma 5.4.9 will also guarantee that this holds not only for cones and, more generally, for all Turing-invariant sets, but for any Turing-invariant subset A of a uniformly pointed perfect set: if f is UTI on A and such that $f(x) \geq_T x$ on A, f has computable left inverse on A.

Chapter 3

The structure of single Turing degrees

Although Turing degrees are usually viewed as the "atoms" of the main structure investigated in computability theory, namely $(\mathcal{D}, \leq_T)$, Turing reductions provide each Turing degree with a quite interesting structure, as witnessed by the fact that a deep result like uniform Martin's conjecture part I can be deduced from the study of such structures. Thus, in this chapter, we collect some results and questions about Turing degrees regarded as structures themselves.

3.1 Homomorphisms between Turing degrees

Given $A \subseteq 2^\omega$, we call "Turing reducibility via" on A the following two-sorted relation:

$$\{ (e, x, y) \in \omega \times A \times A \mid y \leq_T x \text{ via } e \} .$$

Turing equivalence via on A is defined analogously.

Even though, single Turing degrees are trivial structures when equipped with Turing reducibility or equivalence, they are *not* trivial when endowed with Turing reducibility *via* or Turing equivalence *via*. So, for instance, we might want to understand, if the complexity of $[x]_T$ as a structure depends on the computational complexity of $[x]_T$ as a Turing degree, or how the structure on $[x]_T$ relates to the structure on a different $[y]_T$.

Definition 3.1.1. An **embedding** of $[x]_T$ into $[y]_T$ with respect to Turing reducibility via is a pair of functions (f, u), with $f : [x]_T \to [y]_T$ and $u : \omega \to \omega$ such that

$$\begin{aligned} i = j &\iff u(i) = u(j) \\ y = z &\iff f(y) = f(z) \\ y \leq_T z \text{ via } i &\iff f(y) \leq_T f(z) \text{ via } u(i), \end{aligned}$$

for all $i, j \in \omega$ and $y, z \in [x]_T$. In other words, we view $([x]_T, \omega)$ and $([y]_T, \omega)$ as two-sorted structures with Turing reducibility via as the only relation on them, and embeddings are functions between the domains (one for each sort) preserving the truth of atomic formulas in both ways.

An embedding of $[x]_T$ into $[y]_T$ with respect to Turing *equivalence* via is defined analogously: is a pair of injective functions (f, u) which preserve Turing *equivalence* via in both directions.

Theorem 3.1.2. *For all Turing degrees $[x]_T$ and $[y]_T$, the following are equivalent:*

1. *the structure on $[x]_T$ is embeddable in the structure on $[y]_T$, when the structure is given by Turing reducibility via;*
2. *the structure on $[x]_T$ is embeddable in the structure on $[y]_T$, when the structure is given by Turing equivalence via;*
3. *there exists a non-constant UTI function $f : [x]_T \to [y]_T$;*
4. *$x \leq_T y$.*

Proof. *1* $\implies$ *2*: if (f, u) is an embedding with respect to Turing reducibility via, then $(f, u \times u)$ is an embedding with respect to Turing equivalence via, where $u \times u$ is the map $(i, j) \mapsto (u(i), u(j))$.

2 $\implies$ *3*: if (f, u) is an embedding of $[x]_T$ into $[y]_T$ with respect to Turing equivalence via, then f is an injective (hence non-constant) uniformly Turing invariant function from $[x]_T$ to $[y]_T$.

3 $\implies$ *4*: we get $x \leq_T y$ from Theorem 2.1.2.

4 $\implies$ *1*: define $f : [x]_T \to [y]_T, z \mapsto z \oplus y$. Observe that f is injective and its range is indeed included in $[y]_T$ because $x \leq_T y$. It is easy to see that there is an injective $u : \omega \to \omega$ such that, for all $z_1, z_2, z_3 \in 2^\omega$ and all $i \in \omega$, we have

$$z_1 \leq_T z_2 \text{ via } i \iff (z_1 \oplus z_3) \leq_T (z_2 \oplus z_3) \text{ via } u(i).$$

Thus, (f, u) is an embedding of $[x]_T$ into $[y]_T$ with respect to Turing reducibility via. □

The failure of analog of local uniform Martin's conjecture part I for similar reducibilities such as $\leq_A$, as we observed in section 2.3, leaves open questions like the following:

Question 3.1.3. *Are there pairs of distinct arithmetic degrees that are bi-embeddable with respect to arithmetic reducibility via?*

3.2 Reducing Turing reducibility to computable reducibility

Recall that, given two binary relations R and S on sets X and Y respectively, a homomorphism from R to S is a function $f : X \to Y$ such that

$$x \, R \, y \implies f(x) \, S \, f(y), \quad \forall x, y \in X.$$

Furthermore, such f is a reduction if

$$x \, R \, y \iff f(x) \, S \, f(y), \quad \forall x, y \in X.$$

In the first chapter, we talked about Borel homomorphisms and reductions between equivalence relations on standard Borel spaces, whereas now we are going to talk about computable homomorphisms and reductions between equivalence relations on ω. When $X = Y = \omega$ and there is a computable reduction from R to S, one says that R is **computably reducible** to S, and writes $R \leq_c S$. When $R \leq_c S$ and $S \leq_c R$, one says that R and S are **computably bi-reducible**, and writes $R \sim_c S$.

Similarly to Borel reducibility, computable reducibility is a well-established tool to compare the complexity of equivalence relations (on ω instead of 2^ω), and thus measure the difficulty of the classification problems those equivalence relations embody (see, for example, [CHM11]).

Computable reducibility and bi-reducibility are themselves a Borel quasi-order and a Borel equivalence relation respectively, whether they are considered on the Polish space $2^{\omega\times\omega}$ of all binary relations, or they are considered on the closed subset (hence, Polish space itself) $\mathrm{ER} \subseteq 2^{\omega\times\omega}$ of all equivalence relations on ω. For the rest of the paper, we refer to $\leq_c$ and $\sim_c$ as being defined on ER.

As we are going to show, essentially the same argument that led to Theorem 2.1.2 entails that $\leq_T$ is Borel reducible to $\leq_c$, and hence $\equiv_T$ is Borel reducible to $\sim_c$.

Definition 3.2.1. For $x \in 2^\omega$, define $\approx_T^x$ to be the equivalence relation on ω such that $i \approx_T^x j$ if and only if φ_i^x and φ_j^x are the same partial function.

Theorem 3.2.2. *The map $x \mapsto \approx_T^x$ is a Borel reduction from $\leq_T$ to $\leq_c$ (and hence from $\equiv_T$ to $\sim_c$).*

Proof. Suppose that $x \leq_T y$, say, via k. Then,

$$i \approx_T^x j \iff i \approx_T^{\varphi_k^y} j \iff (ik) \approx_T^y (jk),$$

where ik denotes the composition of computer programs as in the proof of Lemma 2.2.2. Thus, $\approx_T^x$ is computably reducible to $\approx_T^y$ via the map $i \mapsto ik$.

Vice versa, suppose $(\approx_T^x) \leq_c (\approx_T^y)$ as witnessed by the computable reduction v. We exploit the same idea as in Theorem 2.1.2. Choose any two $a, b \in \omega$ such that $a \not\approx_T^x b$ and hence, since v is a reduction, $v(a) \not\approx_T^y v(b)$. This means there is some k such that $\varphi_{v(a)}^y(k) \not\simeq \varphi_{v(b)}^y(k)$. Thus, $\varphi_{v(a)}^y(k)$ and $\varphi_{v(b)}^y(k)$ cannot be both undefined, so suppose, for example, that $\varphi_{v(a)}^y(k)$ is defined and equals, say, m; then, whether is defined or not, $\varphi_{v(b)}^y(k)$ does not equal m. Take now a computable function r such that, for all n,

$$\varphi_{r(2n)}^x = \begin{cases} \varphi_a^x & \text{if } x(n) = 1 \\ \varphi_b^x & \text{if } x(n) = 0 \end{cases} \qquad \varphi_{r(2n+1)}^x = \begin{cases} \varphi_b^x & \text{if } x(n) = 1 \\ \varphi_a^x & \text{if } x(n) = 0. \end{cases}$$

Using the fact that v is a reduction (in particular, a homomorphism), we get

$$\varphi^y_{v(r(2n))} = \begin{cases} \varphi^y_{v(a)} & \text{if } x(n) = 1 \\ \varphi^y_{v(b)} & \text{if } x(n) = 0 \end{cases} \qquad \varphi^y_{v(r(2n+1))} = \begin{cases} \varphi^y_{v(b)} & \text{if } x(n) = 1 \\ \varphi^y_{v(a)} & \text{if } x(n) = 0. \end{cases}$$

Now, to know if $x(n)$ equals 1 or 0, it suffices for y to parallel compute $\varphi^y_{v(r(2n))}(k)$ and $\varphi^y_{v(r(2n+1))}(k)$ and wait for m to come out as the output of either computation. If m comes from $\varphi^y_{v(r(2n))}(k)$, then $x(n) = 1$, otherwise, if it comes from $\varphi^y_{v(r(2n+1))}(k)$, then $x(n) = 0$.

Since v and r are computable and there exists a universal oracle Turing machine, the function $n \mapsto \varphi^y_{v(r(n))}$ is computable in y, and hence the procedure above describes a program that computes x from y.

Thus, we have

$$x \leq_T y \iff (\approx^x_T) \leq_c (\approx^y_T)$$

and the Borelness of the map is clear. □

Remark 3.2.3. The connection between Theorem 2.1.2 and Theorem 3.2.2 is the following. If we examine the proof of the former, we can observe that it holds not only when $f : [x]_T \to 2^\omega$ is UTI, but it suffices that f admits a *computable* function u such that, for all y,

$$x \equiv_T y \text{ via } (i, j) \implies f(x) \equiv_T f(y) \text{ via } u(i, j).$$

So f need not be UTI as the above formula holds for a fixed x, but not necessarily for an arbitrary one.

On the other hand, consider the set of partial, unary, computable-in-x functions $\{ \varphi^x_e \mid e \in \omega \}$: a homomorphism v from $\approx^x_T$ to $\approx^y_T$ defines a function

$$f : \{ \varphi^x_e \mid e \in \omega \} \to \{ \varphi^y_e \mid e \in \omega \}$$

by

$$f(\varphi^x_e) = \varphi^y_{v(e)}, \quad \forall e \in \omega$$

and vice versa. Then, v is a computable variance function for f in (x, y), in the sense of Definition 2.4.4. Moreover, f is also *injective* when v is not only a homomorphism, but also a *reduction*. Thus, we can view the proof that $(\approx^x_T) \leq_c (\approx^y_T)$ implies $x \leq_T y$ as the argument for Theorem 2.1.2 applied to such f.

Remark 3.2.4. The reduction provided by Theorem 3.2.2 is uniform in both directions in a computable way. Let us be more clear. If E and F are equivalence relations on ω, let us say that $E \leq_c F$ via i if φ_i is a reduction from E to F. Then, the proof of Theorem 3.2.2 gives us two computable functions $u, v : \omega \to \omega$ such that $x \leq_T y$ via i if and only if $(\approx^x_T) \leq_c (\approx^y_T)$ via $u(i)$ and $(\approx^x_T) \leq_c (\approx^y_T)$ via j if and only if $x \leq_T y$ via $v(j)$. Note that, in the analogy between Theorem 2.1.2 and Theorem 3.2.2, the existence and computability of the function v is the analog of Proposition 2.5.4.

Remark 3.2.5. In [CHM11], the authors indicated a way to turn an equivalence relation E on 2^ω to an equivalence relation E^{ce} on ω, defined by

$$i\, E^{ce}\, j \iff W_i\, E\, W_j,$$

where W_i denotes the i-th computably enumerable set, i.e. $\mathrm{dom}(\varphi_i)$. In particular, they studied $=^{ce}$. Of course, the same process can be done relative to any oracle $x \in 2^\omega$: we could define

$$i\, E^{ce,x}\, j \iff W_i^x\, E\, W_j^x.$$

Then, it is easy to see that $(=^{ce,x}) \sim_c (\approx_T^x)$ for all x, so the map $x \mapsto (=^{ce,x})$ is an equivalent Borel reduction from $\leq_T$ to $\leq_c$.

It would be interesting to understand the behavior of the map T that takes a countable Borel equivalence relation E to the Borel equivalence relation $T(E)$ that makes the map $x \mapsto E^{ce,x}$ a reduction from $T(E)$ to $\sim_c$. Our result just tells us that $T(=_{2^\omega}) = \equiv_T$.

In the theory of countable Borel equivalence relations, a fundamental result by Adams and Kechris revealed the intricacy of the structure of $\leq_B$ on countable Borel equivalence relations.

Theorem 3.2.6 (Adams-Kechris, [AS00])**.** *The partial order of Borel sets under inclusion can be embedded in the quasi-order of Borel reducibility of countable Borel equivalence relations, i.e., there is a map $A \mapsto E_A$ from the Borel subsets of $\mathbb{R}$ to countable Borel equivalence relations such that $A \subseteq B \iff E_A \leq_B E_B$. In particular it follows that any Borel partial order can be embedded in the quasi-order of Borel reducibility of countable Borel equivalence relations.*

This theorem disclosed at once many features of $\leq_B$ on countable Borel equivalence relations, like — for instance — that it features antichains of size $2^{\aleph_0}$ and chains of size $\aleph_1$.

Theorem 3.2.2 can be viewed as something similar for the theory of equivalence relations on ω. Indeed, we know from computability theory that there are many orders that we can embed into the Turing degrees.

Theorem 3.2.7 (Sacks, [Sac61])**.** *Every partial order of cardinality $\leq \aleph_1$ in which every downward cone is countable can be embedded into the Turing degrees.*

Corollary 3.2.8. *Let* ER *be the set of equivalence relations on ω. Every partial order of cardinality $\leq \aleph_1$ in which every downward cone is countable can be embedded into* $(\mathrm{ER}/{\sim_c}, \leq_c)$.

We also know that there are antichains of Turing degrees of size $2^{\aleph_0}$ (for example, that given by minimal Turing degrees).

Corollary 3.2.9. *There are $2^{\aleph_0}$ equivalence relations on ω that are mutually $\leq_c$-incomparable.*

In [ABM21], the authors give further evidence of the intricacy of the structure $(\mathrm{ER}, \leq_c)$, proving that its first-order theory is computably isomorphic to second-order arithmetic, just like the theory of $(2^\omega, \leq_T)$ (a celebrated result of Simpson, [Sim77]).

Another outstanding recent result was found by Patrick Lutz and Benny Siskind [LS], who proved part I of Martin's conjecture for order-preserving functions. Their proof gives part I of Martin's conjecture for *Borel* order-preserving functions assuming Projective Determinacy PD. As a consequence, at least assuming PD, the order-preserving version of Kechris' Conjecture 1.3.2 is false, i.e. $\leq_T$ is not a universal countable Borel quasi-order. This does not disprove Kechris' original conjecture, but might indicate the insight behind it be not so sound. However, we have $(\leq_T) \leq_B (\leq_c)$, so $\leq_c$ can still be universal.

Question 3.2.10. *Is $\leq_c$ a universal countable Borel quasi-order, and hence $\sim_c$ a universal countable Borel equivalence relation?*

3.3 The theory of single Turing degrees

Viewing each Turing degree as a two-sorted structure arises many questions about their theory, i.e. the set of true first-order sentences they satisfy. In particular, it would be nice to understand what is the theory that *almost all* of them — in the sense of Martin's measure — satisfy.

Proposition 3.3.1 (observed by Montalbán)**.** *Let $\mathcal{L}$ be either the language of Turing reducibility via, or of Turing equivalence via. Then, there is a cone of Turing degrees that have the same $\mathcal{L}$-theory.*

Proof. Of course, the set of first-order sentences in $\mathcal{L}$ can be easily enumerated, therefore the theory of a Turing degree can be easily identified with an element of 2^ω. Let $t : 2^\omega \to 2^\omega$ be the function taking x to the $\mathcal{L}$-theory of $[x]_T$. It is easy to see that $t(x) \leq_T x^{(\omega)}$ via a fixed i, for all x; in particular, t is Borel. Moreover, clearly,

$$x \equiv_T y \implies t(x) = t(y),$$

so, by Fact 2.1.3 and Borel Determinacy, we deduce that t is constant on a cone. □

There is a number of questions we can ask about this almost certain theory.

Question 3.3.2. *Is the almost certain theory of single Turing degrees decidable? If not, what is its Turing degree?*

Question 3.3.3. *Are all Turing degrees elementary equivalent, with the language of Turing equivalence via?*

We did not ask the previous question with respect to the language of Turing *reducibility* via because we already know the answer would be negative.

Proposition 3.3.4. *The sentence*

$$\exists i\ \exists x\ \forall y\ (x \leq_T y \text{ via } i)$$

is true in $[0]_T$ *and false in every other Turing degree.*

Proof. The truth of the sentence in $[0]_T$ is trivial. Now assume it is true in $[z]_T$, and let us prove that $[z]_T = [0]_T$. So suppose that we have $i \in \omega$ and $x \equiv_T z$ such that $x = \varphi_i^y$, for all $y \equiv_T z$. Then we can compute x without oracles: for arbitrary n, run $\varphi_i^\sigma(n)$ for k steps, varying the finite string σ and $k \in \omega$ until we find an output. This output will actually be found and will be $x(n)$. Indeed, it is obvious that such σ and k must exist as, for instance, $\varphi_i^z(n)$ converges (and equals $x(n)$), and vice versa if $\varphi_i^\sigma(n) = j$, then $j = \varphi_i^{\sigma^\frown z}(n) = x(n)$. □

Question 3.3.5. *Are all non-zero Turing degrees elementary equivalent, with the language of Turing reducibility via?*

Proposition 3.3.6. *All sentences in the language of Turing reducibility via with parameters and without quantification over reals have the same truth value in all Turing degrees, and their truth value is decidable.*

Proof. If we have such a sentence θ, we have a finite number of reals to take into account (those that appear as parameters in θ), say $x_1, \ldots, x_n$, and we have to decide whether a sentence about reducibilities between them is true or false. Of course, we can assume that the x_k's are pairwise distinct. We show how to decide the truth value of θ with an algorithm that does not depend on the Turing degree we are working with. The key point is that the possible relations that a Turing reducibility can induce on $x_1, \ldots, x_n$ are all of the $(n+1)^n$ conceivable possibilities. Each x_k can be sent by Turing reduction i to one of the x_j's, or to none of them. To see that each possible combination is induced by an actual Turing reduction, we can use an argument similar to the one for Lemma 2.5.1. For all $j, k \in \{1, \ldots, n\}$, fix a Turing reduction $i_{j,k}$ taking x_j to x_k. Fix ℓ large enough to have $x_1 \restriction \ell, \ldots, x_n \restriction \ell$ pairwise distinct. Then we can build a Turing reduction that maps oracles starting with $x_j \restriction \ell$ to the desired x_k, or to a fresh element of the Turing degree.

This tells us that quantifiers over indices of reductions only range in this set of $(n+1)^n$ possible configurations, so the truth value of sentence (3.3.1) can be decided by brute force. □

Corollary 3.3.7. *All sentences in the language of Turing reducibility via where quantifiers over reals are to the left of quantifiers over indices have the same truth value in all Turing degrees, and their truth value is decidable.*

Proof. Let η be a sentence as in the hypotheses, and let *theta* be η without the initial quantifiers over reals. Let $x_1, \ldots, x_n$ be the variables of the sort 'real' that appear in θ (which are, thus, all free). Observe that the formula

$$\theta \wedge \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \tag{3.3.1}$$

is either true for all assignments or false for all assignments to the variables $x_1, \ldots, x_n$, and this truth value can be decided by the algorithm in the last proof. Observe also that θ is equivalent to the disjunction of all formulas of the form

$$\theta \wedge \bigwedge_{(i,j)\in A} x_i \neq x_j \wedge \bigwedge_{(i,j)\in I_n^2 \setminus A} x_i = x_j, \tag{3.3.2}$$

where $I_n = \{1, \ldots, n\}$ and A is a symmetric, non-reflexive subset of ${I_n}^2$. Each formula of the form (3.3.2) can be transformed into an equivalent to a formula of the form (3.3.1) exploiting the equalities to diminish the number of real parameters (or could result immediately false if we get by transitivity of equality that $x_i = x_j$ but $(i, j) \in A$). If (3.3.2) turns out to be false, then we can just remove it from the disjunction, whereas if it turns out to be true we can substitute it with

$$\bigwedge_{(i,j)\in A} x_i \neq x_j \wedge \bigwedge_{(i,j)\in I_n^2 \setminus A} x_i = x_j.$$

This algorithm transforms θ into a quantifier-free formula in the empty language[1], hence transforms η into a sentence in the empty language. □

Remark 3.3.8. The previous two results are also valid for the language of Turing *equivalence* via, as formulas in this language can be seen as particular formulas in the language of Turing reducibility via.

[1]The only atomic formulas are equalities.

Chapter 4

Part II, locally?

Since we showed that part I of Martin's conjecture for UTI functions follows from a local obstruction — namely that no Turing degree $[x]_T$ can be mapped in a non-constant UTI way into a degree $\not\geq_T [x]_T$ —, it comes natural to ask whether something similar happens with *part II*. The implication from *4* to *3* in Theorem 3.1.2 tells us there is no such strong local obstruction for part II of uniform Martin's conjecture. For instance, the reason for the non-existence of UTI functions which are globally in-between the identity function and the Turing jump cannot be the non-existence of such intermediate functions on single degrees, as there do exist injective uniformly Turing invariant functions from any Turing degree $[x]_T$ into any degree $[y]_T$. However, there is still a chance that a more subtle local obstruction exists that — possibly using a more complex argument than just applying Turing Determinacy and Fact 2.1.3 — implies at least part of Steel's Theorem 1.2.2, such as the non-existence of uniformly Turing invariant functions globally strictly in-between an iteration of the jump and the successive iteration.

In this chapter, we shall explore this possibility. But before we get into that, recall that Steel's Theorem 1.2.2 had a forefather, namely Lachlan's Theorem 1.2.1. Thus, we start this chapter by asking ourselves whether the latter result follows from a local obstruction. We are going to prove that it does, and in a strong way as part I of uniform Martin's conjecture does. This is joint work with Patrick Lutz.

4.1 Lachlan's theorem, locally

Theorem 4.1.1 (joint with Patrick Lutz)**.** *If the c.e. operator*

$$W_e : z \mapsto W_e^z$$

is continuous at x, then $W_e^x \leq_T x \oplus 0'$. On the other hand, if W_e is discontinuous at x and uniformly Turing invariant on $[x]_T$, then $W_e^x \equiv_T x'$.

Proof. First, note that, for all $m \in \omega$, we have $W_e^x(m) = 1$ iff there is a finite initial segment σ of x s.t. $W_e^\sigma(m) = 1$, and hence $W_e^z(m) = 1$ for all z extending

σ. We write $\sigma \prec z$ to say that z extends σ. Thus, W_e is continuous at x iff

$$\forall m\Big(W_e^x(m) = 0 \iff \exists\sigma \prec x\forall\tau\big(W_e^{\sigma^\frown\tau}(m) = 0\big)\Big). \tag{4.1.1}$$

Suppose first that W_e is continuous at x. Since the property $\forall\tau\big(W_e^{\sigma^\frown\tau}(m) = 0\big)$ is decidable in $0'$, we have in this case that W_e^x is co-c.e. in $x \oplus 0'$. Of course, W_e^x is also c.e. in x and hence $W_e^x \leq_T x \oplus 0'$.

Now, let's suppose that W_e is discontinuous at x and uniformly Turing invariant on $[x]_T$. Note that the right-to-left implication in (4.1.1) always holds, so in this case, we must have some $m \in \omega$ such that $W_e^x(m) = 0$ and for all $\sigma \prec x$ there is some τ such that $W_e^{\sigma^\frown\tau}(m) = 1$. Note that, there's in fact an effective procedure that, given such a σ, finds a suitable τ. Let's denote τ_σ the τ found in this way.

So, if we denote by $x'[l]$ the l-th approximation of the jump of x, we can define a computable r such that

$$\varphi^x_{r(n)} = \begin{cases} x & n \notin x' \\ (x \restriction l)^\frown\tau_{x\restriction l}{}^\frown x & \text{if } l \text{ is the least such that } n \in x'[l]. \end{cases}$$

Let's call $\varphi^x_{r(n)} = x_n$. There's also a computable s such that $\varphi^{x_n}_{s(n)} = x$ for all n, because to compute the k-th digit of x from x_n we could first check if $n \in (x_n)'[k]$: if no, we just output the k-th digit of x_n; if yes, we take the first l such that $n \in (x_n)'[l]$. Then we know that $x_n \restriction l = x \restriction l$. So we can compute $\tau_{x\restriction l}$ and we know that $x_n = (x \restriction l)^\frown\tau_{x\restriction l}{}^\frown x$, so we now can output $x(k) = x_n(k + l + |\tau_{x\restriction l}|)$.

Thus, we have $x \equiv_T x_n$ via $(r(n), s(n))$ with r and s computable and $W_e^{x_n}(m) = 1$ iff $x'(n) = 1$. So the argument in the proof of Theorem 2.1.2 gives us $W_e^x \geq_T x'$. We shall recall it here for the reader's convenience. We can choose a computable downward variance function v of W_e in x, so that we have, for all n, $W_e^x \geq_T W_e^{x_n}$ via $v(r(n), s(n))$. Since $n \mapsto v(r(n), s(n))$ is computable, W_e^x can compute $n \mapsto W_e^{x_n}(m) = x'(n)$. □

Notice that, in the previous theorem, we work with the hypothesis of continuity / discontinuity in x of W_e, viewed as an operator from 2^ω to 2^ω, but we also aim to work with operators defined on single Turing degrees. However, there is no possibility of confusion, in the light of the following:

Remark 4.1.2. Given a c.e. operator $W_e : z \mapsto W_e^z$, a dense subset D of 2^ω (such as a Turing degree) and $x \in D$, it is equivalent that W_e is continuous at x or that $W_e \restriction D$ is continuous at x.

The left-to-right implication is trivial. For the other one, we automatically have $W_e^x(m) = 1 \iff \exists\sigma \prec x\big(W_e^\sigma(m) = 1\big)$, for all m. Moreover, the continuity of $W_e \restriction D$ at x gives us that $W_e^x(m) = 0$ iff there is $\sigma \prec x$ such that, for all τ that extend σ and that can be extended by some $z \in D$, $W_e^\tau(m) = 0$. But *every* τ can be extended by some $z \in D$ for D is dense, so this gives the continuity of W_e at x.

Applying Theorem 4.1.1 to uniformly Turing invariant c.e. operators on degrees $\geq_T 0'$ we get a local version of Lachlan's theorem.

Theorem 4.1.3 (joint with Patrick Lutz)**.** *Suppose that $x \geq_T 0'$ and the c.e. operator on $[x]_T$*

$$W_e : [x]_T \to 2^\omega$$
$$z \longmapsto W_e^z$$

is uniformly Turing invariant. If this operator is continuous at x, then either it is constantly equal to a c.e. set, or $W_e^x \equiv_T x$. On the other hand, if it is discontinuous at x, then $W_e^x \equiv_T x'$.

Proof. The last statement is given by Theorem 4.1.1. On the other hand, if c.e. operator W_e is continuous at x, Theorem 4.1.1 gives us $W_e^x \leq_T x \oplus 0'$, but since we are now assuming $x \geq_T 0'$, we have $W_e^x \leq_T x$. Since $z \mapsto W_e^z$ is uniformly Turing invariant on $[x]_T$, by Theorem 2.1.2 we know that it's either constant on $[x]_T$ or $W_e^x \geq_T x$. In the latter case, $W_e^x \equiv_T x$. In the former one, $W_e^x(m) = 1$ iff there is some $z \equiv_T x$ and some initial segment σ of z such that $W_e^\sigma(m) = 1$. Thus, since every Turing degree is dense in 2^ω, we have that $[x]_T \ni z \mapsto W_e^z$ is constantly equal to the c.e. set

$$\{\, m \in \omega \mid \exists \sigma \in 2^{<\omega} : W_e^\sigma(m) = 1 \,\} . \qquad \square$$

Remark 4.1.4. We stated Theorem 4.1.1 and Theorem 4.1.3 for UTI c.e. operators for aesthetic reasons, but we could have stated them with the (strictly[1]) weaker hypothesis that the c.e. operator admits a computable downward variance function in x.

Remark 4.1.5. Observe that, as a consequence of the results above, a uniformly degree invariant c.e. operator on a degree $[x]_T$ greater than $0'$ is equivalently continuous at *one* or at *each* point of $[x]_T$, as this is equivalent to have $W_e^x \leq_T x$. This is not true for degrees that are not above $0'$, for example for 1-generic degrees. Indeed, if x is 1-generic, the Turing jump is continuous at x but not at $x \oplus x$, even though clearly $x \equiv_T x \oplus x$.

Clearly, we can obtain Lachlan's Theorem 1.2.1 by our result in the same way we got uniform Martin's conjecture part I from Theorem 2.1.2: if $W_e : x \mapsto W_e^x$ is uniformly degree invariant, consider the set of Turing degrees above $0'$ on which W_e is continuous: by Turing determinacy, this set has to either be disjoint from or include a cone. In the former case, $W_e^x \equiv_T x'$ on a cone, by Theorem 4.1.3. Otherwise, applying Turing determinacy again, there is either a cone on which $W_e^x \equiv_T x$, or there is a cone of degrees on each of which W_e

[1] We shall say more about this in Chapter 5: we are going to show that being UTI and admitting a computable downward variance function are globally equivalent properties, although we are exhibiting in Proposition 5.1.2 (computable, hence c.e.) functions on single degrees that admit computable variance functions (hence *downward* variance functions) but are not UTI.

is constant, which means that W_e is constant on a cone by Fact 2.1.3. This new proof of Lachlan's theorem requires the same determinacy assumptions as Lachlan's, but it arguably sheds a new light on the reason why his result holds.

Despite we've been able to trace back the dichotomy of Lachlan's theorem to a local dichotomy that takes place in every Turing degree in the cone above $0'$, we are left with the curiosity of knowing what behaviors are possible for uniformly degree invariant c.e. operators restricted to degrees outside that cone.

Question 4.1.6. *Are there $e \in \omega$ and $x \in 2^\omega$ such that W_e is continuous on the whole Turing degree of x but $W_e^x \not\leq_T x$?*

4.2 The whole part II, locally?

Now let us explore the possibility that not only Lachlan's theorem, but the whole part II of uniform Martin's conjecture is implied by local phenomena. The first problem is that it is not immediate to figure out a property about arbitrary UTI functions on single degrees that could imply a statement about the set of global UTI functions being pre-wellordered by $\leq_M$ and the Turing jump providing the successor. For UTI c.e. operators on degrees above $0'$ we got the dichotomy "either below the identity or above the jump" because we exploited properties of c.e. operators, but for arbitrary UTI functions on single degrees, as we said, there is no such dichotomy as shown by Theorem 3.1.2: the UTI image of a degree $[x]_T$ can be inside any degree $\geq_T [x]_T$. However, Theorem 4.1.3 furnishes us an idea of how we can rephrase the dichotomy "either below the identity or above the jump" to make it plausible also locally, that is: "either continuous or above the jump". In fact, in Theorem 3.1.2, we used continuous functions of the kind $z \mapsto z \oplus y$ in order to exhibit a UTI function from any $[x]_T$ to any $[y]_T$ in the cone above $[x]_T$. Furthermore, it is well-known that, if $A \subseteq 2^\omega$, it is equivalent for a function $f : A \to 2^\omega$ to be continuous or to admit $i \in \omega$ and $p \in 2^\omega$ such that, for all $x \in A$:

$$f(x) \leq_T x \oplus p \text{ via } i.$$

This means it is equivalent that $f \leq_M \mathrm{id}_{2^\omega}$ or that f is globally continuous, i.e. continuous on a uniformly pointed perfect set.[2] This leads us to our first conjecture.

Conjecture 4.2.1. *If $f : [x]_T \to [y]_T$ is uniformly Turing invariant, then either f is continuous or $f(x) \geq_T x'$.*

Let us try to use this idea to get a local conjecture that would imply part II of uniform Martin's conjecture. A consequence of Slaman and Steel's work is that the restriction of Steel's Conjecture 1.2.4 to Borel functions is equivalent to the following:

[2]See Appendix B for reference on what we mean by 'globally' and on uniformly pointed perfect sets; implication left-to-right uses Corollary $B.1.2$ to get an $i \in \omega$ and a uniformly pointed perfect set P such that $f(x) \leq_T x$ via i on P.

Conjecture 4.2.2 (Borel Steel's conjecture)**.** *If $f : 2^\omega \to 2^\omega$ is a Borel TI function, then either the degree of $f(x)$ is constant on a cone, or there exists $\alpha < \omega_1$ such that $f(x) \equiv_T x^{(\alpha)}$ on a cone.*

Remark 4.2.3. In [MSS16, Theorem 1.8], the authors report as an easy consequence of Slaman and Steel's work the equivalence between Borel Martin's conjecture, Borel Steel's conjecture and Conjecture 4.2.2. However, while the equivalence between the latter two is indeed easy to deduce from uniform Martin's conjecture (and Becker's work, see section 4.3), we take the opportunity to point out that the equivalence with Borel Martin's conjecture is not that easy to derive, and there is actually no known proof for it.

In order to make a local version of Conjecture 4.2.2, it might be convenient to rephrase it as a dichotomy at each $\alpha < \omega_1$.

Conjecture 4.2.4 (Borel Steel's conjecture — alternative version)**.** *Fix α with $1 \leq \alpha < \omega_1$. Then, for every Borel TI function $f : 2^\omega \to 2^\omega$ such that the degree of $f(x)$ is not constant on a cone, either there is $\beta < \alpha$ such that $f(x) \leq_T x^{(\beta)}$ on a cone, or $f(x) \geq_T x^{(\alpha)}$ on a cone.*

Of course, the statement that every non-constant Borel UTI $f : [x]_T \to 2^\omega$ either satisfies $f(x) \leq_T x^{(\beta)}$ for some $\beta < \alpha$ or $f(x) \geq_T x^{(\alpha)}$ is false for every $\alpha < \omega_1$: by Theorem 3.1.2 we know that, for every $y \geq_T x$, there is a non-constant continuous UTI function $f : [x]_T \to [y]_T$. However, the property $f(x) \leq_T x^{(\beta)}$ on a cone is equivalent to the condition that f is globally Baire class β, as this means that exist $i \in \omega$ such that$f(x) \leq_T (x \oplus z)^{(\alpha)}$ via i globally.[3]

Recall that the α-th jump of x is not necessarily defined if $\alpha \geq \omega_1^x$; however, given any $\alpha < \omega_1$, it suffices to consider the cone above a real coding a well-order of ω of order-type α to get a cone of reals x such that $\alpha < \omega_1^x$, and hence $x^{(\alpha)}$ is well defined. Putting everything together, we can make the following attempt for a local conjecture.

Attempted Conjecture 4.2.5. *Fix α with $1 \leq \alpha < \omega_1$. Then, for all x in a cone, every non-constant Borel UTI function $f : [x]_T \to 2^\omega$ satisfies one of the following:*

- *f is Baire class β for some $\beta < \alpha$, or*
- *$f(x) \geq_T x^{(\alpha)}$.*

Unfortunately, this conjecture is true but this doesn't help in any way to prove the previous conjectures. The reason why it is true is actually trivial:

Remark 4.2.6 (Lutz). Every function f on a countable domain $\{ x_n \}_n \subset \omega^\omega$ is Baire class 1. Indeed, such an f is the pointwise limit of $(f_n)_n$, where f_n is the

[3]See Appendix B to recall what we mean by 'globally'

continuous function defined as follows. For all n, let s_n be a natural number such that $x_0 \restriction s_n, \ldots, x_n \restriction s_n$ are all different. Then define,

$$f_n(x) = \begin{cases} f(x_m) & \text{if } x \in N_{x_m \restriction s_n} \text{ for some } m \leq n, \\ \text{your favorite real} & \text{otherwise.} \end{cases}$$

Then of course each f_n is continuous and $f_n \longrightarrow f$ pointwise as $f_n(x_m) = f(x_m)$ for all m and $n \geq m$.

This Remark tells us we cannot expect to easily deduce the global Borel Steel's conjecture from such a triviality. However, Conjecture 4.2.1 is still open, and we are about to show that it does imply the global conjecture easily.

Proposition 4.2.7. *($\mathsf{ZF} + \mathsf{TD}$) Suppose $f : 2^\omega \to 2^\omega$ is a Turing invariant function such that, for every degree $[x]_T$, either f is continuous on $[x]_T$ or $f(x) \geq_T x'$. Then, there is cone C such that either f is continuous on C or $f(x) \geq_T x'$ for all $x \in C$.*

Proof. First, use TD to get a cone C such that either f is continuous on every degree $\subset C$, or $f(x) \geq_T x'$ on C. Of course, in the latter case we are done, whereas in the former one, we do not know whether f is continuous on C. For any continuous function g from a subset of 2^ω to 2^ω, define p_g to be a real that codes the function from $2^{<\omega}$ to $2^{<\omega}$

$$\sigma \mapsto \text{the longest } \tau \text{ not longer than } \sigma \text{ such that } g(N_\sigma) \subseteq N_\tau.$$

Then, it's easy to see there is $i \in \omega$ (independent of g) such that, for all x in the domain of g,

$$g(x) \leq_T x \oplus p_g \text{ via } i.$$

Going back to our f, in the case $f \restriction [x]_T$ is continuous for all x in a cone C, use Fact 2.1.3 to get a cone D on which the function

$$\begin{aligned} &C \to 2^\omega \\ &x \mapsto p_{f \restriction [x]_T} \end{aligned}$$

is constant. Then, we have $p \in 2^\omega$ and $i \in \omega$ such that $f(x) \leq_T x \oplus p$ via i for all $x \in D$, and this clearly implies f is continuous on D. $\square$

Remark 4.2.8. We just proved that if f is Baire class 0 (i.e. continuous) on every degree in a cone, then is Baire class 0 on a cone, and we also pointed out that every function is Baire class 1 on any Turing degree, whereas of course there are plenty of them which are not Baire class 1 globally. This shows that, even though for all $\alpha < \omega_1$ we have $i_\alpha \in \omega$ such that

$$f \text{ is Baire class } \alpha \iff \exists p \in 2^\omega, \forall x \in \operatorname{dom}(f)\colon f(x) \leq_T (x \oplus p)^{(\alpha)} \text{ via } i_\alpha,$$

there is a canonical way of choosing p in the left-to-right direction if and only if $\alpha = 0$. So, for example, $f(x) = x''$ is not Baire class 1 globally, so for every degree $[y]_T$, we do have a $p \in 2^\omega$ such that $x'' \leq_T (x \oplus p)'$ via i_1, but we do not have, under TD, a *function* that maps every degree to such a p. Indeed, in the construction in Remark 4.2.6 we had to choose an enumeration of the countable domain, that is, we essentially need to choose a representative of the degree in order to perform the construction of Remark 4.2.6 on that degree. Thus, a vague insight for a better conjecture than Attempted Conjecture 4.2.5 would be: either we can find a parameter witnessing $f.[x]_T \to 2^\omega$ is Baire class β with $\beta < \alpha$ without having to choose a representative inside $[x]_T$, or $f(x) \geq_T x^{(\alpha)}$.

If we want to deduce Conjecture 4.2.4 from a local fact in the fashion of Proposition 4.2.7 we might conjecture that such local fact be

Attempted Conjecture 4.2.9. *Fix α with $1 \leq \alpha < \omega_1$. Then, for all x in a cone, every non-constant Borel UTI function $f : [x]_T \to 2^\omega$ satisfies one of the following:*

- *there exist $\beta < \alpha$, $p \in 2^\omega$ and $i \in \omega$ such that*
$$f(y) \leq_T y^{(\beta)} \oplus p \text{ via } i \tag{4.2.1}$$
for all $y \in [x]_T$, or
- $f(x) \geq_T x^{(\alpha)}$.

Observe that the difference between this and Attempted Conjecture 4.2.5 is that (4.2.1) is a stronger requirement than asking f to be Baire class β, as the latter would translate to: there exist $p \in 2^\omega$ and $i \in \omega$ such that, for all $y \in [x]_T$,
$$f(y) \leq_T (y \oplus p)^{(\beta)} \text{ via } i.$$
However, this attempt is doomed to failure, too.

Proposition 4.2.10 (joint with Patrick Lutz)**.** *For all $z \in 2^\omega$, there exists $x \geq_T z$ such that, for no $i \in \omega$ and $p \in 2^\omega$ we have*
$$(y \oplus x)' \leq_T y' \oplus p \text{ via } i \tag{4.2.2}$$
for all $y \in [x]_T$.

Proof. First, we prove the result for z computable, and then we show how the proof can be relativized. Let x be 1-generic.[4] Suppose for contradiction that there are some $i \in \omega$ and $p \in 2^\omega$ such that we have (4.2.2) for all $y \in [x]_T$. For y 1-generic, we can compute y' uniformly in $y \oplus 0'$, and hence $y' \oplus p$ uniformly in $y \oplus 0' \oplus p$. So we have, for a suitable $j \in \omega$ and for $q = p \oplus 0'$, that $(y \oplus x)' = \varphi_j^{y \oplus q}$ for all 1-generic $y \in [x]_T$. There is some n such that the n-th bit of $(y \oplus x)'$ is 1 if $x = y$ and 0 otherwise. Since $\varphi_j^{x \oplus q}(n) = (x \oplus x)'(n) = 1$, there is some $k \in \omega$ (the *use* of the computation $\varphi_j^{x \oplus q}(n)$) such that $\varphi_j^{y \oplus q}(n)$

[4] For the notion of 1-genericity, we refer the reader to textbooks on computability theory such as [Soa16].

converges and equals 1 for all y extending $x \upharpoonright k$. But it is easy to find a 1-generic $y \equiv_T x$ so that $x \upharpoonright k = y \upharpoonright k$ but $x \neq y$. For this y we have a contradiction, because $(y \oplus x)'(n) = \varphi_j^{y \oplus q}(n) = 1$, but also $(y \oplus x)'(n) = 0$, because x and y are not equal. To relativize this proof in the cone above z, take x 1-generic relative to z, and use $x \oplus z$ everywhere instead of x. □

This tells us how to build a counterexample to Attempted Conjecture 4.2.9 with $\alpha \geq 2$ in every cone: given any $z \in 2^\omega$, it tells us we can find $x \geq_T z$ and a uniformly Turing invariant $f : [x]_T \to [x']_T$ which is not of the form $y \mapsto \varphi_i^{y \oplus p}$ nor $y \mapsto \varphi_i^{y' \oplus p}$, and hence not of the form $y \mapsto \varphi_i^{y^{(\beta)} \oplus p}$ for any β, but also $f(x)$ is not Turing reducible to $x^{(\alpha)}$ for any countable ordinal $\alpha \geq 2$, as $f(x) \equiv_T x'$. Thus, whereas Attempted Conjecture 4.2.5 is useless as it is trivially true for $\alpha \geq 2$, Attempted Conjecture 4.2.9 is useless as it is blatantly false for $\alpha \geq 2$. For $\alpha = 1$, however, they both coincide with Conjecture 4.2.1, which, on the contrary, seems reasonable and would be useful as shown by Proposition 4.2.7. In section 4.3, we try to propose some other idea to detect some Martin's-conjecture-part-II-like phenomenon for UTI functions at local level, whereas in section 4.4 we shall present some partial results towards Conjecture 4.2.1.

4.3 Becker's theorem, locally?

In [Bec88], Becker reproved part II of uniform Martin's conjecture in a particularly perspicuous way: he used the descriptive set-theoretic notion of "reasonable pointclass" and proved that, under AD, every UTI $f >_M \mathrm{id}_{2^\omega}$ is Turing equivalent on a cone to a Γ-jump operator

$$J_\Gamma : x \mapsto \text{a universal } \Gamma(x) \text{ subset of } \mathbb{N}$$

for some reasonable pointclass Γ. Reasonable pointclasses are indeed lightface pointclasses that can be relativized to arbitrary $x \in 2^\omega$ and admit universal sets. For example, the Turing jump $x \mapsto x'$ is a Σ^0_1-jump operator, the relativization of Kleene's $\mathcal{O}$, $x \mapsto \mathcal{O}^x$, is a Π^1_1-jump operator, and so on.

Part II of uniform Martin's conjecture then follows from the link between the ordering $\leq_M$ on pointclass jump operators and Wadge reducibility $\leq_W$ on 2^ω. Kihara and Montalbán's [KM18] proved a version of Becker's result for Turing to many-one uniformly invariant functions, pushing even further this connection.

Thus, we might ask whether these results arise locally. In fact, Becker's theorem tells us that, up to Turing equivalence on a cone, there exist no other UTI functions besides constant functions, identity function and pointclass jump operators (under AD), so it is natural to ask whether any UTI functions that have nothing to do with constant functions, identity function and pointclass jump operators can exist locally.

Question 4.3.1. *Fix a Turing degree $[x]_T$, and consider the smallest family $\mathcal{J}_x$ of functions $f : [x]_T \to 2^\omega$ that contains*

- *all constant functions from* $[x]_T$ *to* 2^ω
- $\mathrm{id}_{[x]_T}$
- *all pointclass jump operators defined on* $[x]_T$

and closed under joins and Turing functionals. Do all UTI functions from $[x]_T$ *to* 2^ω *belong to* $\mathcal{J}_x$*?*

4.4 Partial results on the "continuous or above the jump" dichotomy

We have already seen a partial result towards Conjecture 4.2.1, namely Theorem 4.1.3, which proves true the conjecture for c.e. operators. We shall prove it here for other special cases. In subsection 4.4 we are proving a special case of the "continuous or above the jump" dichotomy, namely a "*constant* or above the jump" dichotomy under certain hypotheses. Before we do that, we remark an interesting topological analogy.

A topological analogy

Theorem 4.4.1 (Carroy, [Car13])**.** *If* $f : 2^\omega \to 2^\omega$ *is a Borel function, then either its range is countable,*[5] *or there are continuous maps* $\sigma : 2^\omega \to 2^\omega$ *and* $\tau : \mathrm{ran}(f) \to 2^\omega$ *such that* $\mathrm{id}_{2^\omega} = \tau \circ f \circ \sigma$*.*

Theorem 4.4.2 (Solecki dichotomy for Borel functions, [PS12])**.** *Let* X *be an analytic subset of a Polish space, and* Y *be a separable metrizable space. For a Borel function* $f : X \to Y$*, either the domain of* f *can be partitioned into countably many pieces such that* f *is continuous on each piece, or there are topological embeddings* $\varphi : (\omega+1)^\omega \to X$ *and* $\psi : \omega^\omega \to Y$ *such that* $\psi \circ P = f \circ \varphi$*, where* P *is the Pawlikowski function.*

The Pawlikowski function P applied to $x \in (\omega+1)^\omega$ is defined as

$$P(x)(n) = \begin{cases} x(n)+1 & x(n) < \omega \\ 0 & x(n) = \omega. \end{cases}$$

Recall that $(\omega+1)^\omega$ is endowed with the product topology and that $\omega+1$ is endowed with the usual topology generated by intervals of the form $[\alpha;\beta)$, so that every point of $(\omega+1)^\omega$ is isolated except ω, which is a limit point. The Pawlikowski function is thus a discontinuous function, and is related to the Turing jump. Indeed, every real x can be mapped to

$$\tilde{x}(\langle e,s\rangle) = \begin{cases} 1 & \text{if } \varphi_e^x(e) \text{ converges in less than } s \text{ steps,} \\ 0 & \text{otherwise} \end{cases}$$

[5] Notice this means we can partition the domain of f into countably many sets such that f is constant on each set. This will entail that f is constant on a uniformly pointed perfect set (see Appendix B).

and then $\tilde{x}$ can be mapped continuously to $\delta(\tilde{x}) \in (\omega+1)^\omega$, where

$$\delta(\tilde{x})(e) = \begin{cases} \min\{\, s<\omega \mid \tilde{x}(\langle e,s\rangle)=1\,\} & \text{if this set is non-empty,}\\ \omega & \text{otherwise.}\end{cases}$$

Now it is easy to see that $P(\delta(\tilde{x}))$ is another way of writing the Turing jump of x in ω^ω: instead of writing 1 when $\varphi_e^x(e)$ converges, we write $s+1$ where s is the number of steps that it takes to converge).

This is why we view Theorem 4.4.2 a topological analog of the "continuous or above the jump" dichotomy. In fact, following a suggestion of Carroy, Kihara showed in an unpublished work that it is possible use Solecki dichotomy to easily derive the following:

Theorem 4.4.3 (Kihara). *If $f : 2^\omega \to 2^\omega$ is Borel and order-preserving, then either f is continuous on a uniformly pointed perfect set (and hence $f \leq_M \mathrm{id}_{2^\omega}$) or $f \geq_T^\triangledown (\mathrm{id}_{2^\omega})'$, where*

$$g \leq_T^\triangledown g \iff \exists p \in 2^\omega, \forall x \geq_T p : g(x) \leq_T h(x) \oplus p.$$

Analogously, Theorem 4.4.1 can be viewed as a topological analog of the "constant or above the identity" dichotomy, and indeed one can use that to prove the following, in the same fashion as Kihara's result.

Proposition 4.4.4. *If $f : 2^\omega \to 2^\omega$ is Borel and order-preserving, then either f is constant on a uniformly pointed perfect set, or $f \geq_T^\triangledown \mathrm{id}_{2^\omega}$.*

The Turing to many-one case

In [KM18], Montalbán and Kihara proved an interesting version of uniform Martin's conjecture for Turing to many-one uniformly invariant functions. Recall that we say $x \in 2^\omega$ is many-one reducible to $y \in 2^\omega$ via i (denoted $x \leq_m y$ via i) if $x = y \circ \varphi_i$, and as usual $x \equiv_m y$ via (i, j) if $x \geq_m y$ via i and $x \leq_m y$ via j. Then a function $f : \omega^\omega \to 2^\omega$ is called Turing to many-one uniformly invariant (or just **Turing to many-one UI**) if there is a function $u : \omega^2 \to \omega^2$ such that, for all $x, y \in \omega^\omega$ and $(i, j) \in \omega^2$,

$$x \equiv_T y \text{ via } (i,j) \implies f(x) \equiv_m f(y) \text{ via } u(i,j).$$

Montalbán and Kihara considered the pre-order on the set of Turing to many-one uniformly invariant functions given by

$$f \leq_m^\triangledown g \iff \exists p \in \omega^\omega, \forall x \geq_T p : f(x) \leq_m^p g(x),$$

where $\leq_m^p$ denotes many-one reducibility relative to oracle p.[6] They proved that $\leq_m^\triangledown$ induces, on the set of $\equiv_m^\triangledown$-classes of Turing to many-one uniformly

[6] That is, if $x, y \in 2^\omega$, we have $x \leq_m^p y$ if and only if there is a computable-in-p function w such that $x = y \circ w$.

invariant functions, a semi-well-order which is isomorphic to Wadge degrees of subsets of ω^ω ordered by Wadge reducibility. A particular consequence of this is the following. Let TJ denote the Turing jump operator, and $\overline{(\cdot)}$ denote complementation when applied to subsets of ω (or elements of 2^ω), and *pointwise* complementation when applied to functions into 2^ω.

Theorem 4.4.5 (Kihara and Montalbán, [KM18]). *If $f : \omega^\omega \to 2^\omega$ is Turing to many-one uniformly invariant, then either f is constant on a cone, or $f \geq_m^\nabla TJ$, or $f \geq_m^\nabla \overline{TJ}$.*

So, of course, we asked ourselves if we could prove this fact locally. We have some results in this direction, which can also be regarded as partial results towards Conjecture 4.2.1, but we need to take into account partial functions.

Check Appendix A to see definition and notation on non-deterministic Turing reducibility. We just recall here that we denote by $\omega^{\subseteq\omega}$ denote the set of all partial functions from ω to ω. For shortness, we abbreviate 'non-deterministically Turing' as 'NTuring'. Our result holds both for deterministic and non-deterministic Turing equivalence, so we merge the two statements into a single one using shorthand '(N)Turing equivalence' and '$\equiv_{(N)T}$'.

Theorem 4.4.6. *Let $p \in \omega^{\subseteq\omega}$. If $f : \{ q \in \omega^{\subseteq\omega} \mid q \equiv_{(N)T} p \} \to 2^\omega$ is (N)Turing to many-one uniformly invariant and non-constant, then either $f(q) \geq_m q'$ for all $q \in \operatorname{dom}(f)$ or $f(q) \geq_m \overline{q'}$ for all $q \in \operatorname{dom}(f)$, where q' denotes the (N)Turing jump of q.*[7]

We start noticing that we can prove the computable uniformity function lemma also for Turing to many-one uniformly invariant functions, with the observation — which will come in handy in a while — that we may suppose it to range only into indices of *total* computable functions.

Lemma 4.4.7. *If f is (N)Turing to many-one uniformly invariant on a Turing invariant set, be it a subset of $\omega^{\subseteq\omega}$, ω^ω, or 2^ω, f admits a computable uniformity function u such that, for all i, j, both coordinates of $u(i, j)$ are indices of total functions.*

Proof. We just need to recall the proof of the computable uniformity function lemma 2.2.2 and check that, applied to our hypotheses, gives us our thesis. There are $(a_1, b_1), \ldots, (a_n, b_n)$ "generating" the Turing bi-reductions, meaning that every (i, j) has a canonical word in the letters $(a_1, b_1), \ldots, (a_n, b_n)$ that always acts like (i, j). We take any computable function v of f and we define a computable uniformity function u setting $u(i, j)$ as the canonical word for (i, j), but with letters $(a_1, b_1), \ldots, (a_n, b_n)$ replaced by $v(a_1, b_1), \ldots, v(a_n, b_n)$.

[7] The definition of $q' \in 2^\omega$ is the usual one: $q'(n) = 1 \iff \varphi_n^q(n)$ converges, but $\varphi_n^{(\cdot)}$ will denote either the deterministic or non-deterministic n-th Turing reduction depending if we are considering the deterministic or non-deterministic statement.

Since on the codomain we have the many-one action, the operation between the letters $v(a_1, b_1), \dots, v(a_n, b_n)$ is now the operation $*$ defined by

$$(k_1, l_1) * (k_2, l_2) = (c(k_1, k_2), c(l_2, l_1)),$$

where $\varphi_{c(k_1,k_2)} = \varphi_{k_2} \circ \varphi_{k_1}$. It is clear that both coordinates of each $v(a_i, b_i)$ have to be indices of total functions, since there surely exist x and q in the domain of f such that $p \equiv_T q$ via (a_i, b_i), and so $f(p) \equiv_m f(q)$ via $v(a_i, b_i)$. Thus, both coordinates of $u(i, j)$ are indices of total functions, too, as they index a composition of total functions. □

Proof of Theorem 4.4.6. First, we remark that we just need to prove $f(q) \geq_m q'$ or $f(x) \geq_m \overline{q'}$ for *one* $q \in \mathrm{dom}(f)$: indeed, if we have, for instance, $f(q) \geq_m q'$ for some $q \in \mathrm{dom}(f)$, we also have

$$\big(q \equiv_{(N)T} s \text{ and } f(q) \geq_m q'\big) \implies f(s) \equiv_m f(q) \geq_m q' \equiv_m s'.$$

Let v be the projection on the first coordinate of a computable uniformity function of f. First, let us assume that p is computable, so that the empty function $\emptyset$ belongs to the domain of f. Then, there must be $q \in \mathrm{dom}(f)$ such that $f(q) \neq f(\emptyset)$. Then, we have a computable function $r : \omega \to \omega$ such that

$$p_n := \varphi^p_{r(n)} = \begin{cases} q & \text{if } p'(n) = 1 \\ \emptyset & \text{if } p'(n) = 0. \end{cases}$$

Here, $\varphi_i^{(\cdot)}$ denotes the i-th deterministic or non-deterministic Turing reduction, and p' is the jump defined based on deterministic or non-deterministic Turing reductions, according to the statement we are proving. Also, since p is computable, there is a program e that does not query the oracle and computes p, so $p \equiv_{(N)T} p_n$ via $(r(n), e)$, for all n. Thus, $\bigoplus_n f(p_n)$ coincides with $\bigoplus_n \big(f(p) \circ \varphi_{v(r(n),e)}\big)$, which also coincides with

$$f(p) \circ \big(\bigoplus_n \varphi_{v(r(n),e)}\big). \tag{4.4.1}$$

Let $k \in \omega$ be such that $f(q)(k) \neq f(\emptyset)(k)$. The careful reader has surely noticed that core of the argument is the same as in Theorem 2.1.2: we shall find either p' or $\overline{p'}$ on the k-th row of (4.4.1). If $f(q)(k) = 1$, then $f(\emptyset)(k) = 0$ and so the function $m \mapsto f(p_m)(k)$, that is the function

$$m \mapsto \left(f(p) \circ \bigoplus_n \varphi_{v(r(n),e)}(\langle k, m\rangle)\right), \tag{4.4.2}$$

is exactly p'; this entails $f(p) \geq_m p'$. Otherwise, if $f(q)(k) = 0$, then (4.4.2) is $\overline{p'}$, so that $f(p) \geq_m \overline{p'}$.

Now, let us assume p is not computable. We claim there are $q, s \in \mathrm{dom}(f)$ that are incompatible[8] and such that $f(q) \neq f(s)$. Indeed, pick $q \in \mathrm{dom}(f)$

[8] This means there exists $n \in \mathrm{dom}(q) \cap \mathrm{dom}(s)$ such that $q(n) \neq s(n)$.

such that $f(p) \neq f(q)$. Let s be in $\mathrm{dom}(f)$ and incompatible with both p and q: for instance, choose $m_0 \in \mathrm{dom}(p)$ and $m_1 \in \mathrm{dom}(q)$ with $m_0 \neq m_1$ (we can find them because now elements of $\mathrm{dom}(f)$ are non-computable), and set

$$s(n) \simeq \begin{cases} p(n) & \text{if } n \notin \{ m_0, m_1 \} \\ p(m_0) + 1 & \text{if } n = m_0 \\ q(m_0) + 1 & \text{if } n = m_1. \end{cases}$$

Since $f(p) \neq f(q)$, either $f(p) \neq f(s)$ or $f(q) \neq f(s)$, so we are done.

Without loss of generality, let us assume p and s are incompatible and have a different image via f. We claim that, for all $q \in \mathrm{dom}(f)$ incompatible with p, we have $f(q) \neq f(p)$. Indeed, if $f(q) = f(p)$ we could use an argument as in the injectivity lemma 2.5.1 to find a contradiction. Let us show this. Since p and q are incompatible, i.e. there is m in the intersection of their domains such that $p(m) \neq q(m)$, both deterministic and non-deterministic Turing reductions can distinguish between them in a finite time. So we can find i such that $\varphi_i^q = p$ and $\varphi_i^p = s$. Moreover, since s is incompatible with p, we can find in the same way j such that $\varphi_j^s = p$ and $\varphi_j^p = q$. Then, the contradiction is the same as in the injectivity lemma: $f(q) \geq_m f(p)$ via $v(i, j)$ and $f(p) \geq_m f(s)$ via $v(i, j)$ but $f(q) = f(p) \neq f(s)$.

Now fix $m \in \mathrm{dom}(p)$, and let $\tilde{p}$ denote p with the m-th digit replaced by $p(m)+1$, and $\hat{p}$ denote p with the m-th digit undefined. By our previous claim, $f(p) \neq f(\tilde{p})$. This entails either $f(p) \neq f(\hat{p})$ or $f(\tilde{p}) \neq f(\hat{p})$. Without loss of generality, we can assume that $f(p) \neq f(\hat{p})$. Now, we can repeat the same argument to prove either $f(p) \geq_m p'$ or $f(p) \geq_m \overline{p'}$: there is a computable r with which we can define

$$p_n \coloneqq \varphi_{r(n)}^p = \begin{cases} p & \text{if } p'(n) = 1 \\ \hat{p} & \text{otherwise.} \end{cases}$$

Indeed, the program indexed by $r(n)$ can do the following: just output the i-th digit of the oracle p for every input $i \neq m$, while for $i = m$ first run $\varphi_n^p(n)$ before outputting $p(m)$. Of course, there is also a natural number e such that $p \equiv_{(N)T} p_n$ via $(r(n), e)$ for all n. If k is such that $f(p)(k) \neq f(\hat{p})(k)$, then (4.4.2) is either p' or $\overline{p'}$. □

The connection between NTuring and enumeration reducibility we mentioned in Appendix A enables us to rephrase Theorem 4.4.6 in terms of enumeration degrees.

Corollary 4.4.8. *Fix $A \subseteq \omega$, and let $f : [A]_e \to 2^\omega$ be enumeration to many-one uniformly invariant and non constant. Then, either $f(B) \geq_m (S_B)'$ for all $B \in [A]_e$, or $f(B) \geq_m \overline{(S_B)'}$ for all $B \in [A]_e$, where $(S_B)'$ denotes the NTuring jump of S_B.*[9]

[9] S_B denotes the semi-characteristic function of B, see Appendix A.

Proof. From Appendix A, we know that the map $p \mapsto \mathrm{gr}(p)$ is NTuring to enumeration UI and maps $[S_A]_{NT}$ into $[\mathrm{gr}(S_A)]_e = [A]_e$. Call $\tilde{f}$ the composition $f \circ \mathrm{gr}$, which is thus a NTuring to many-one UI function from $[S_A]_{NT}$ into 2^ω. From (A.2.1) of Appendix A, we know $B \equiv_e \mathrm{gr}(S_B)$, so that, for all $B \in [A]_e$, we have

$$f(B) \equiv_m \tilde{f}(S_B).$$

Thus, if we manage to apply Theorem 4.4.6 to $\tilde{f}$, we are done. We just need to prove that $\tilde{f}$ is non-constant. For this, notice that $B \equiv_e \mathrm{gr}(S_B)$ via a fixed (i, j), so that $f(B) \equiv_m \tilde{f}(S_B)$ via $u(i, j)$ for all $B \in [A]_e$, if u is a uniformity function of f. Then, choose B such that $f(A) \neq f(B)$ and note that it would be contradictory if $\tilde{f}(S_A) = \tilde{f}(S_B)$. □

We would like to improve the statement of Theorem 4.4.6 in several ways: first of all, proving it for functions defined on the set of *total* functions Turing equivalent to x. A nice way to do that would be to prove that every Turing to many-one UI function f on a Turing-invariant subset A of ω^ω can be extended to a NTuring to many-one UI function $\tilde{f}$ on the $\equiv_{NT}$-closure of $A \subset \omega^{\subseteq\omega}$: this would match our insight that all Turing to many-one UI functions can be gotten as explained in section 4.3, but — and this is the only difference with UTI functions — *without employing the identity function*.

Another (perhaps) improvable aspect of our proof is that it does not lead to an analog of Proposition 2.5.4 with many-one reductions, although — as an empirical observation — we always have one single many-one reduction providing the left inverse of every Turing to many-one UI function we know.

Remark 4.4.9. An interesting feature of our proof of Theorem 4.4.6 is that it does not merely use the computability of a downward variance functions like the proofs of Theorem 2.1.2 or Theorem 4.1.3, but it actually needs a uniformity function in the case $\mathrm{dom}(f) \neq \{ q \in \omega^{\subseteq\omega} \mid q \equiv_T \emptyset \}$ in order to carry out the argument in Gura's injectivity lemma 2.5.1 fashion.

If we want to get a similar statement that just uses a computable downward variance function, we can take inspiration from the argument to prove the thesis in the case $\mathrm{dom}(f) = \{ q \in \omega^{\subseteq\omega} \mid q \equiv_T \emptyset \}$.[10]

Proposition 4.4.10. *Let $p \in \omega^{\subseteq\omega}$ and let v be a variance function in p, with respect to (N)Turing reducibility via (on $\omega^{\subseteq\omega}$) and Turing reducibility via (on 2^ω), of the function $f : \{ q \in \omega^{\subseteq\omega} \mid q \leq_T p \} \to 2^\omega$. Then either f is constant or $f(p) \oplus v \geq_T p'$, where p' denotes the (N)Turing jump of p.*

Proof. Let $q \in \mathrm{dom}(f)$ be such that $f(q) \neq f(\emptyset)$. We have a computable function $r : \omega \to \omega$ such that

$$p_n \coloneqq \varphi^p_{r(n)} = \begin{cases} q & \text{if } p'(n) = 1 \\ \emptyset & \text{if } p'(n) = 0. \end{cases}$$

We carry out the usual argument. □

[10]Note that $\{ q \in \omega^{\subseteq\omega} \mid q \equiv_T \emptyset \} = \{ q \in \omega^{\subseteq\omega} \mid q \equiv_{NT} \emptyset \}$.

We point out that this statement is a generalization of Rice's theorem. In fact, index sets relative to $x \in 2^\omega$ (i.e. $\approx_T^x$-invariant sets, where $\approx_T^x$ is as in Definition 3.2.1) can be viewed essentially as variance functions (in the order preserving sense) of 2-valued functions.

Theorem 4.4.11 (Rice, relativized version)**.** *Fix $x \in 2^\omega$. Let $A \subseteq \omega$ be an index set relative to x. Then, either A is trivial (i.e. $A = \emptyset$ or $A = \omega$), or A is undecidable in x, and in particular $A \geq_T x'$.*

Proof from Proposition 4.4.10. Define $f : \{\, q \in \omega^{\subseteq\omega} \mid q \leq_T x \,\} \to 2^\omega$ by

$$f(\varphi_i^x) = \begin{cases} 111\dots & \text{if } i \in A, \\ 000\dots & \text{otherwise.} \end{cases}$$

Notice this is a well-posed definition because A is $\approx_T^x$-invariant. Now, choose $i, j \in \omega$ be such that $\varphi_i^y = y$ for all y and $\varphi_j^{111\dots} = 000\dots$. We can define a variance function (in the order preserving sense) v of f in x as follows: if $i \in A$, then and let $v : \omega \to \omega$ map A to i and $\omega \setminus A$ to j, otherwise let v map A to j and $\omega \setminus A$ to i. Of course, $v \equiv_T A$. Observe that f is constant if and only if A is trivial. If this is not the case, then Proposition 4.4.10 gives us $v \geq_T x'$. □

For those who are annoyed by partial functions and rather fancy a statement about Turing to many-one UI functions defined on a *usual* Turing degree, we can state the following result. We need to ask for a variance function v that makes f satisfy — instead of just the usual *forward* implication — a *two-sided* implication. A little bit like in Definition 3.1.1, but we do not need v to be a uniformity function.

Proposition 4.4.12. *Let $x, y \in 2^\omega$. Suppose $f : [x]_T \to [y]_m$ and $v : \omega^2 \to \omega^2$ are such that*

$$x \equiv_T z \text{ via } (i,j) \iff f(x) \equiv_m f(z) \text{ via } u(i,j)$$

for all $z \in [x]_T$ and $(i,j) \in \omega^2$. Then $y \oplus v \geq_T x'$.

Proof. Consider a computable $r : \omega \to \omega$ such that

$$x_n := \varphi_{r(n)}^x = \begin{cases} x & \text{if } x'(n) = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Also consider e such that $\varphi_e^p = p$ for all $p \in \omega^{\subseteq\omega}$, so that $x'(n) = 1$ iff $x \equiv_T x$ via $(r(n), e)$ iff $f(x) \equiv_m f(x)$ via $v(r(n), e)$. Then, if $s(n)$ and $t(n)$ are the first and the second component of $v(r(n), e)$ respectively, we have that s and t are computable and

$$x'(n) = 1 \iff \big(f(x) = f(x) \circ \varphi_{s(n)} \text{ and } f(x) = f(x) \circ \varphi_{t(n)}\big).$$

By the previous remark, also for those n such that $x'(n) = 0$, we can assume that $\varphi_{s(n)}$ and $\varphi_{t(n)}$ are total. (This is the only point in which the hypothesis that f is Turing to *many-one* uniformly invariant; in fact the proof also works if many-one is substituted with truth table, for example).

So from $f(x) \oplus v$ we can compute w_0 and w_1, where

$$w_0 = \bigoplus_n f(x) \circ \varphi_{s(n)} \qquad w_1 = \bigoplus_n f(x) \circ \varphi_{t(n)},$$

and enumerate the n's such that either the n-th column of w_0 or the n-th column of w_1 differ from $f(x)$. In this way we enumerate $\overline{x'}$ from $f(x) \oplus v$.

On the other hand, notice f is not constant. Indeed, note that $f(x) \equiv_m f(x)$ via $v(e,e)$, so if $f(x) = f(z)$ then we would have $x \equiv_T z$ via (e,e), that is $x = z$. Hence, f is non-constant and we can apply Theorem 2.4.3 to conclude that $f(x) \oplus v \geq_T x$, and thus that x' is enumerable from $f(x) \oplus v$. So, since both x' and $\overline{x'}$ are enumerable from $f(x) \oplus v$, we conclude $f(x) \oplus v \geq_T x'$ by Post's theorem. □

Those who are annoyed by partial functions may have noticed that we avoided their cause of annoyance only in the statement, but not in proof. In the latter, in fact, we dealt with the *empty* function even though it was not in the domain of our function, but we have been able to tell that the indices leading to it would behave well thanks to the two-sided implication we had in the hypotheses.

Chapter 5

Weakened uniformity assumptions

In this chapter, we also deal with the classic global case of Martin's conjecture.We examine various weakenings of uniformity that are enough to prove Martin's conjecture and in many cases we prove they coincide globally with uniformity, although they might differ from each other locally.

If we look closely to the proof of part II and I of uniform Martin's conjecture in [Ste82] and [SS88], as well as our own proof of part I of uniform Martin's conjecture locally, we can notice a number of other hypotheses can be assumed alternatively to uniform degree invariance in order to carry out the argument. In this chapter, we delve into some of these alternatives of uniformity that still enable to prove Martin's conjecture with the same techniques as in the uniform case. The first example is one that we already pointed out, namely having a computable variance function.

5.1 Admitting computable variance functions

We say that function $f : 2^\omega \to 2^\omega$ satisfies a property "globally" if f satisfies it on a uniformly pointed perfect set (see Appendix B). As usual in the Martin's conjecture context, when we are interested in *global* properties of functions, we assume Determinacy.

The first thing we point out about the property of having a computable variance function in x is that it can be a proper weakening of uniformity only locally, as it globally coincides with it.

Proposition 5.1.1. *Under* AD*, the following are equivalent for any Turing invariant function* $f : 2^\omega \to 2^\omega$*:*

1. *f is UTI on a uniformly pointed perfect set;*
2. *there is a uniformly pointed perfect set P such that, for all $x \in P$, there is a computable variance function of $f \restriction P$ in x.*

The proof will be shown at page 61, as it relies on some properties that will be developed in section 5.4. Indeed, we could give the proof of *2* $\implies$ *1* right now, but for *1* $\implies$ *2* we need a generalization of Lemma 2.2.2 to uniformly pointed perfect sets — namely Lemma 5.4.9 — we still don't have. Anyways, we now show that admitting a computable variance function actually *is* a proper weakening of uniformity from the point of view of single degrees.

Proposition 5.1.2. *For all x and y in 2^ω with $x \leq_T y$, there exists a function $f : [x]_T \to [y]_T$ that admits a computable variance function in x but is not uniformly Turing invariant.*

Proof. Define $f(z) = z(0)^\frown y$. Thus, if $z \leq_T x$ via e, then $f(z) = \varphi_e^x(0)^\frown y$ and $f(x) = x(0)^\frown y$. Since x is computable in y, it is easy to see there is a computable function v such that, for all $e \in \omega$ for which $\varphi_e^x \in 2^\omega$:

$$\varphi_e^x(0)^\frown y \leq_T x(0)^\frown y \text{ via } v(e).$$

Then, v is a computable variance function of f in x in the order-preserving sense. Obviously, $\varphi_e^x(0)^\frown y \geq_T x(0)^\frown y$ via a constant function, so f admits a computable variance function in x in the usual, Turing invariant sense as well. By Proposition 5.2.2, it also admits a computable variance function in every point in $[x]_T$. Finally, f is not uniformly Turing invariant by Lemma 2.5.1, as it is neither injective, nor constant. □

Note that, if there is a computable variance function for $f : [x]_T \to [y]_T$, then we have to have $x \leq_T y$ by Theorem 2.4.3, so we found counterexamples everywhere we could. However, it would be interesting to know if we could find counterexamples without using Lemma 2.5.1.

Question 5.1.3. *If $f : [x]_T \to [y]_T$ admits a computable variance function and is injective, must it be uniformly Turing invariant?*

Observe that in uniformity arguments such as in Theorem 2.1.2 or Theorem 4.1.3 (although not in Theorem 4.4.6, as we pointed out in Remark 4.4.9), the computability-in-$f(x)$ of the variance function would suffice. Actually, we do not even need the computability-in-$f(x)$ of the whole variance function, but just of a downward variance function.

5.2 Computable downward variance functions

The computability-in-$f(x)$ of a downward variance function is enough for the uniformity arguments we mentioned earlier. Moreover, this is equivalent to the computability *tout court* of a downward variance function.

Proposition 5.2.1. *Let $A \subseteq 2^\omega$, $f : A \to 2^\omega$ and $x \in A$. If there is a computable-in-$f(x)$ downward variance function of f in x, then there is a computable one, too.*

Proof. Suppose $e \in \omega$ is such that, for all $y \in A$,

$$x \equiv_T y \text{ via } (i,j) \implies f(x) \geq_T f(y) \text{ via } \varphi_e^{f(x)}(\langle i,j\rangle).$$

The consequent means

$$\varphi^{f(x)}_{\varphi_e^{f(x)}(\langle i,j\rangle)} = f(y).$$

But note that there is $d \in \omega$ such that, for all $a, b \in \omega$ and $z \in 2^\omega$:

$$\varphi^z_{\varphi^z_a(b)} = \varphi^z_{\varphi_d(\langle a,b\rangle)}.$$

So let $v : \omega^2 \to \omega$ be the function that takes (i,j) to $\varphi_d(\langle e, \langle i,j\rangle\rangle)$ and note that v is computable and satisfies

$$x \equiv_T y \text{ via } (i,j) \implies f(x) \geq_T f(y) \text{ via } v(i,j).$$ □

Note that the same proof does not work for upward variance functions, and thus for variance functions, either. Now, let us show that the existence of a computable variance function in one point x implies the existence of computable variance functions in every point in the degree of x; this works for any kind of variance function (*tout court*, downward or upward) and for computability relative to any oracle.

Proposition 5.2.2. *Let $A \subseteq 2^\omega$, $f : A \to 2^\omega$ Turing invariant, $x, y \in A$ with $x \equiv_T y$ and $p \in 2^\omega$. If there is a computable-in-p downward variance / upward variance / variance function of f in x, then there is also a computable-in-p downward variance / upward variance / variance function of f in y.*

Proof. Let us prove the downward variance function case. The upward case is analogous and the variance *tout court* one descends from these two. So let v be a computable-in-p downward variance function of f in x. Choose (k,l) such that $x \equiv_T y$ via (k,l). Since f is Turing invariant, we can also pick $e \in \omega$ such that $f(y) \geq_T f(x)$ via e. If $y \equiv_T z$ via (i,j), then $x \equiv_T z$ via $(i *_T k, l *_T j)$, where $*_T$ is a computable binary operation that gives, for all $w \in 2^\omega$ and $m, n \in \omega$,

$$\varphi_m^{\varphi_n^w} = \varphi^w_{m *_T n}.$$

Hence, $f(x) \geq_T f(z)$ via $v(i *_T k, l *_T j)$ and $f(y) \geq_T f(z)$ via $v(i *_T k, l *_T j) *_T e$. Thus, the map

$$(i,j) \mapsto v(i *_T k, l *_T j) *_T e$$

is a computable-in-p downward variance function of f in y. □

This is why we are just going to say that $f : [x]_T \to 2^\omega$ "admits computable variance functions" without specifying if we mean in one point or in all points. There are a couple of interesting things that we can deduce from this proof. The first one is that the mass problem of finding a downward variance / upward variance / variance function of f in x is Medvedev reducible (and, hence, equivalent) to that of finding one in y.[1]

The second thing we can observe is that we can cook up an index for a computable-in-p downward variance function in y from an index of a computable-in-p downward variance function in x and an index e that computes $f(x)$ from $f(y)$. In the case p is computable, this tells the following: if we think that x is fixed, the problem of finding a function w that maps (k, l) to an index of a computable downward variance function of f in y, where $x \equiv_T y$ via (k, l), is a mass problem that can be Medvedev reduced to the problem of finding an *upward*-variance function of f in x (needed to get the index e that computes $f(x)$ from $f(y)$).

These mass problems are actually Medvedev equivalent: if we have that $w(k, l)$ is an index of a computable downward variance function in y if $x \equiv_T y$ via (k, l), then we have that

$$x \equiv_T y \text{ via } (k, l) \implies f(y) \geq_T f(x) \text{ via } \varphi_{w(k,l)}(\langle l, k \rangle).$$

In conclusion, if there is a computable downward variance function in x, then there are computable downward variance functions in all $y \in A \cap [x]_T$; furthermore, upward variance functions can be as easy as a function representing how an index of a computable downward variance changes when we vary $y \in A \cap [x]_T$ can be, and vice versa. In the light of this, we ask the following:

Question 5.2.3. *Are there $x, y \in 2^\omega$ and $f : [x]_T \to [y]_T$ such that f admits a computable downward variance function, but no computable upward variance function (hence, no computable variance function* tout court*)?*

Note that the role of downward and upward variance functions can be switched in the observation above, but not, for instance, in Proposition 5.2.1. Moreover, while having a computable downward variance function is enough to carry out uniformity arguments such as that of Theorem 2.1.2, the computability of an upward variance function doesn't seem to be useful for that scope. However, *globally*, these notions again coincide with uniformity.

Remark 5.2.4. If $v : \omega^2 \to \omega$ that is a downward / upward variance function of $f : A \to 2^\omega$ in all $x \in A$, then $\tilde{v}$ defined by $\tilde{v}(i, j) = v(j, i)$ is an upward / downward variance function of f in all $x \in A$; hence f is uniformly Turing invariant on A.

Thus, if P is a pointed perfect set and $f : P \to 2^\omega$ is such that, for all $x \in P$, there is a computable downward / upward variance function of f in x, then under AD f is UTI on a pointed perfect set.

The second part of the remark is proved proceeding as in Proposition 5.1.1 and taking a pointed perfect set P_1 on which f admits the *same* downward /

[1] Recall that any $A \subseteq \omega^\omega$ is called a mass problem, and, given mass problems A and B, A is said to be **Medvedev reducible** (or *strongly* reducible) to B — denoted $A \leq_s B$ — if there is an $e \in \omega$ such that, for all $x \in B$, $\varphi_e^x \in A$. If A and B are Medvedev reducible to each other, they're said Medvedev equivalent.

upward variance function, and then conclUTIng that f is UTI on P_1 by the first part of the remark.

When a function v is downward variance function of f in every point of the domain of f, we say v is a **downward uniformity function** of f. We now introduce another notion.

5.3 Preserving uniform Turing reducibility

Definition 5.3.1. If $f : A \to 2^\omega$ with $A \subseteq 2^\omega$, we say f is uniform reducibility preserving if, for all $x \in A$ and all $(x_n)_n \in A^\omega$, we have

$$x \geq_T \bigoplus_n x_n \implies f(x) \geq_T \bigoplus_n f(x_n). \tag{5.3.1}$$

Note that an equivalent way of phrasing this is:

$$\big(\exists i \in \omega, \forall n \in \omega : \ x \geq_T x_n \text{ via } \varphi_i(n)\big) \implies \big(\exists e \in \omega, \forall n \in \omega : \ f(x) \geq_T f(x_n) \text{ via } \varphi_e(n)\big).$$

So the reason for this name is that, if we define the "relation" of **uniform Turing reducibility** as

$$(\leq_{uT}) = \left\{ \big((x_n)_n, x\big) \in (2^\omega)^\omega \times 2^\omega \;\middle|\; \exists e : \forall n : x_n \leq_T x \text{ via } \varphi_e(n) \right\},$$

we can say that uniform reducibility preserving functions are the homomorphisms of $\leq_{uT}$. For this reason, we also call them **$\leq_{uT}$-preserving** functions.

Remark 5.3.2. Notice that all uniform reducibility preserving functions are order-preserving. On the other hand, all uniformly order-preserving functions are uniform reducibility preserving because of Lemma 2.2.2. Indeed, if f has a computable variance function (in the order-preserving sense) v in x and $x \geq_T x_n$ via $r(n)$ with r computable, then $f(x) \geq_T f(x_n)$ via $v(r(n))$.

Thus, uniformity reducibility preservance is a weakening of order-preserving uniformity and it provides the core of the arguments used to prove the order-preserving uniform Martin's conjecture. However, we want to introduce a similar weakening of *degree-invariant* uniformity.

One possibility could be weakening (5.3.1) and asking that, for all $x, x_0, x_1, \ldots$ in the domain of f:

$$x \equiv_T \bigoplus_n x_n \implies f(x) \geq_T \bigoplus_n f(x_n). \tag{5.3.2}$$

For Turing invariant functions, this is of course equivalent to saying that

$$f\left(\bigoplus_n x_n\right) \geq_T \bigoplus_n f(x_n),$$

for all sequences $(x_n)_n$ of points of the domain of f such that $\bigoplus_n x_n$ is also in the domain of f. From (5.3.2) we say that this is equivalent to saying that, if $x \geq_{uT} (x_n)_n$ and $\bigoplus_n x_n \geq_T x$, then $f(x) \geq_{uT} (f(x_n))_n$. This is implied by f being UOP but it's not immediately implied by f being UTI; moreover, there's something weaker than this that still makes the arguments work. So we define **uniform Turing equivalence** as

$$(\equiv_{uT}) = \left\{ \left((x_n)_n, x\right) \in (2^\omega)^\omega \times 2^\omega \;\middle|\; \exists i,j : \forall n : x_n \equiv_T x \text{ via } \left(\varphi_i(n), \varphi_j(n)\right) \right\}$$

and we define $(\equiv_{uT}, \leq_{uT})$**-preserving** functions as the homomorphisms from $\equiv_{uT}$ to $\leq_{uT}$, that is, functions that map sequences that are uniformly Turing equivalent to x, to sequences that are uniformly Turing reducible to $f(x)$.

Notice that if $(x_n)_n$ is uniformly Turing equivalent to x, then it's also uniformly Turing equivalent to all y is the same Turing degree as x, that is also the Turing degree of all x_n's. So we can think of $\equiv_{uT}$ as a unary relation on $(2^\omega)^\omega$, saying that a sequence is included in a Turing degree and is uniformly Turing equivalent to any of its elements. An analogous observation holds for $\leq_{uT}$, $(\equiv_{uT}, \leq_{uT})$-preservance only has to do with the sequences: it's about sending uniformly equivalent sequences to uniformly reducible sequences.

This property is the core for Steel's and Slaman and Steel's arguments to prove uniform Martin's conjecture, as well as the arguments presented in this thesis to prove Theorem 2.1.2 and Theorem 4.1.3. Indeed, the main feature of all of them is that, to prove that $f(x)$ computes a given real z, one builds up a sequence $(x_n)_n \equiv_{uT} x$ such that, for some computable function r, $z(n) = f(x_n)(r(n))$ for all n. Then, one only needs the $(\equiv_{uT}, \leq_{uT})$-preservance of f to conclude $f(x) \geq_{uT} (f(x_n))_n$ and hence $f(x) \geq_T z$.

Remark 5.3.3. $(\equiv_{uT}, \leq_{uT})$-preserving function are automatically Turing invariant and, if f has a computable downward variance function in all points of its domain, then f is $(\equiv_{uT}, \leq_{uT})$-preserving. In fact, we have the following. Say that $x \geq_{uT} (x_n)_n$ via e if, for all n, $x \geq_T x_n$ via $\varphi_e(n)$, and $(x_n)_n \equiv_{uT} x$ via (i,j) if, for all n, $x_n \equiv_T x$ via $(\varphi_i(n), \varphi_j(n))$. Then f has a computable downward variance function in x if and only if there is a computable map $r : \omega^2 \to \omega$ such that

$$(x_n)_n \equiv_{uT} x \text{ via } (i,j) \implies \left(f(x_n)\right)_n \leq_{uT} f(x) \text{ via } r(i,j).$$

One verse is obvious, while for the other one: let $i \mapsto \mathrm{cost}(i)$ be a computable function sending i to an index of the function constantly equal to i. Then $y \equiv_T x$ via (i,j) implies $(y)_n \equiv_{uT} x$ via $(\mathrm{cost}(i), \mathrm{cost}(j))$ and $(f(y))_n \leq_{uT} f(x)$ via $r(\mathrm{cost}(i), \mathrm{cost}(j))$ implies $f(y) \leq_T f(x)$ via $\varphi_{r(\mathrm{cost}(i),\mathrm{cost}(j))}(0)$.

Notably, whereas for the previous weakenings of uniformity we were able to establish the coincidence — at least globally — with uniform Turing invariance, we are not able to do that for $(\equiv_{uT}, \leq_{uT})$-preservance, which is the core of uniformity arguments.

Notice that if we strengthen $(\equiv_{uT}, \leq_{uT})$-preservance asking that, for all $x \in \mathrm{dom}(f)$ and all $(i,j) \in \omega^2$, there exists $e \in \omega$ such that, for all $(x_n)_n \in$

$\mathrm{dom}(f)^\omega$

$$\{\, n\in\omega \mid x \equiv_T x_n \text{ via } (\varphi_i(n),\varphi_j(n)) \,\} \subseteq \{\, n\in\omega \mid f(x) \geq_T f(x_n) \text{ via } \varphi_e(n) \,\}$$

(whereas $(\equiv_{uT}, \leq_{uT})$-preservance only asks the previous inclusion to hold when the left-hand set is ω), then we get a property that is equivalent to having a computable downward variance function in every point. Indeed, given $x \in \mathrm{dom}(f)$, if we consider (i,j) such that $\varphi_i(\langle k,l\rangle) = k$ and $\varphi_j(\langle k,l\rangle) = l$, then the e given by this property would be an index for a computable downward variance function in x.

Remark 5.3.4. $(\leq_{uT}, \leq_{uT})$-preservance for functions from $\omega^{\subseteq\omega}$ to $\omega^{\subseteq\omega}$ coincides with admitting computable variance functions in the order-preserving sense, i.e. with respect to Turing *reducibility* via. Indeed, in $\omega^{\subseteq\omega}$ we clearly have $x \geq_{uT} (\varphi_n^x)_n$, so $f(x) \geq_{uT} \left(\varphi_n^{f(x)}\right)_n$ would mean there is a computable variance function of f in x in the order-preserving sense.

5.4 Variance functions with non-FPF degrees

We now consider functions such that, for all x in the domain, there is a downward variance function v in x that does not compute a function without fixed points relative to x, that is: for all total $h \leq_T v$, there is $e \in \omega$ such that

$$\varphi^x_{h(e)} = \varphi^x_e.$$

Even enough is not enough to carry the usual uniform invariance arguments, we are considering this notion because we are going to prove that it coincides globally with uniform invariance.

First of all, let us note that, in the above hypothesis on f, we are essentially asking the degree of v not to be FPF relative to x, which means the following:

Definition 5.4.1. A function $h : \omega \to \omega$ is called FPF (fixed point free) relative to a parameter x if, for all $e \in \omega$,

$$\varphi^x_{h(e)} \neq \varphi^x_e.$$

A Turing degree is called FPF relative to x if it contains an FPF relative to x function.

Remark 5.4.2. It is equivalent, for all $x, y \in \omega^\omega$, that y *computes* a FPF relative to x function, or that y *is Turing equivalent* to such a function. Indeed, if y computes an FPF rel. to x function h, then it can also compute $\tilde{h}$, where, for all n, $\tilde{h}(n)$ is a program is obtained by $h(n)$ adding in the beginning some fake instructions that code $y(n)$ but don't change the behavior of the program. Thus, $\tilde{h}$ is FPF rel. to x as well, and $\tilde{h} \equiv_T y$. Hence, a Turing degree is FPF rel. to x equivalently if it contains a FPF rel. to x function or if its every element computes such a function.

FPF degrees are a well established subject in literature. For example, FPF rel. to x degrees above x coincide with DNR rel. to x degrees above x (see [Joc+89, Lemma 4.2]). A Turing degree is DNR relative to x if it computes a diagonally non-recursive-in-x function, that is, a function h such that $h(e) \neq \varphi_e^x(e)$, for all e. However, since we don't want to assume that the downward variance function in x has to compute x, we do not have the equivalence between these two notions in our setting.

The proof of that the property of admitting downward variance functions in x of non-FPF degree rel. to x coincides globally with uniform Turing invariance uses AD and relies on a game that we are going to define. We shall need some facts about uniformly pointed perfect sets.

Generators of Turing bi-reductions on uniformly pointed perfect sets

In the proof of Lemma 2.2.2 we essentially used the fact that Turing reductions have a property that could be described as being effectively generated by finitely many generators, meaning the following:

Definition 5.4.3. Let $A \subseteq 2^\omega$ and $G \subseteq \omega^2$. We say that Turing bi-reductions on A are **(effectively) generated** by G if there is an (effective) procedure that maps every $(i,j) \in \omega^2$ to a word

$$\big((a_{j_0}, b_{j_0}), \ldots, (a_{j_m}, b_{j_m})\big)$$

with every letter $(a_{j_k}, b_{j_k}) \in G$ and with the property that, for all $x, y \in A$,

$$\begin{aligned} x \equiv_T y \text{ via } (k,l) \iff & \text{ there are } z_0, \ldots, z_{m+1} \in A \text{ with } z_0 = x,\ z_{m+1} = y \\ & \text{ and } \forall k \leq m : z_k \equiv_T z_{k+1} \text{ via } (a_{j_k}, b_{j_k}). \end{aligned}$$

Let us explain our choice for the nomenclature in this definition. If we denote by Φ_i the i-th Turing reduction, so that Φ_i is a partial function from 2^ω to 2^ω such that $\Phi_i(x) = y$ if and only if $x \geq_T y$ via i, we denote by $\tilde{\Phi}_{i,j}$ the partial function from 2^ω to 2^ω such that $\tilde{\Phi}_{i,j}(x) = y$ if and only if $x \equiv_T y$ via (i,j), and we call $\tilde{\Phi}_{i,j}$ the (i,j)-th Turing bi-reduction. Note that $\{\, \tilde{\Phi}_{i,j} \,\}_{(i,j)}$ is a family of partial Borel functions generating Turing equivalence, as well as a monoid[2] under composition. Then, Turing bi-reductions on A are generated by G in the sense of Definition 5.4.3 if and only if $\{\, \tilde{\Phi}_{i,j} \restriction A \,\}_{(i,j)\in\omega^2}$ is generated as a monoid by $\{\, \tilde{\Phi}_{i,j} \restriction A \,\}_{(i,j)\in G}$. The fundamental detail is that all of $z_0,\ldots,z_{m+1}$ in Definition 5.4.3 belong to A. In that case, $\tilde{\Phi}_{k,l}$ equals

$$\tilde{\Phi}_{a_{j_m}, b_{j_m}} \circ \cdots \circ \tilde{\Phi}_{a_{j_0}, b_{j_0}}.$$

Definition 5.4.3 can be trivially generalized to any generating family of partial Borel functions.

[2] One almost has inverses, except that $\tilde{\Phi}_{i,j} \circ \tilde{\Phi}_{j,i} = \tilde{\Phi}_{j,i} \circ \tilde{\Phi}_{i,j}$ equals the identity function on the domain of $\tilde{\Phi}_{i,j}$, which is not the neutral element of the monoid unless $\tilde{\Phi}_{i,j}$ is total.

Definition 5.4.4. Let $\{\psi_i\}_i$ be a generating family of partial functions[3] on a set X, and let $A \subseteq X$ and $G \subseteq \omega^2$. We say that the family $\{\psi_i\}_i$ on A is **(effectively) generated** by G if there is a(n effective) procedure that maps every $(i,j) \in \omega^2$ to a word

$$\big((a_{j_0}, b_{j_0}), \ldots, (a_{j_m}, b_{j_m})\big)$$

with every letter $(a_{j_k}, b_{j_k}) \in G$ and with the property that, for all $x, y \in A$,

$$x \; E_{\{\psi_i\}} \; y \text{ via } (k,l) \iff \begin{array}{l}\text{there are } z_0, \ldots, z_{m+1} \in A \text{ with } z_0 = x,\ z_{m+1} = y \\ \text{and } \forall k \leq m : z_k \; E_{\{\psi_i\}} \; z_{k+1} \text{ via } (a_{j_k}, b_{j_k}).\end{array}$$

We say that a generating family is **finitely (effectively) generated** if it is (effectively) generated by a finite set. We now need a generalization of what we proved in Lemma 2.2.2: what we proved there is that Turing reductions are finitely effectively generated on 2^ω, now we are proving that they are *on every uniformly pointed perfect set.*

Lemma 5.4.5. *There are computable functions $a_0, \ldots, a_4$ and $b_0, \ldots, b_4$ such that, given a uniformly pointed perfect set P with index i, Turing bi-reductions on P are effectively generated by $\big(a_0(i), b_0(i)\big), \ldots, \big(a_4(i), b_4(i)\big)$.*

Proof. The idea of the proof is as in that of Lemma 2.2.2. This time, we cannot simply append 0's and 1's in front of reals to code programs, nor simply erase initial digits to get rid of those codes, because we don't want to get out of the uniformly pointed perfect set. However, when we have x on a uniformly pointed perfect set P, with an index i of P we can have x compute the tree of P, and once we have that, we can perform h and h^{-1}. So, if b, c and m are in the proof of Lemma 2.2.2, we can define $b(i)$ so that

$$\begin{aligned}\varphi^x_{b(i)} &= h(\varphi_b^{h^{-1}(x)}) = h(0^\frown h^{-1}(x)),\\ \varphi^x_{c(i)} &= h(\varphi_c^{h^{-1}(x)}) = h(1^\frown h^{-1}(x)),\\ \varphi^x_{m(i)} &= h(\varphi_m^{h^{-1}(x)}) = h(h^{-1}(x)^-).\end{aligned}$$

Finally, let $d(i)$ be the program that, with oracle $x \in P$ (where P is a uniformly pointed perfect set with index i), looks for $k, l \in \omega$ such that $x = \varphi^y_{b(i)^k c(i) b(i)^l c(i)}$, and then outputs $\varphi^{\varphi^y_k}_{b(i)^l c(i) b(i)^k c(i)}$. We are using the same ideas of the proof of Lemma 2.2.2, but we are moving on an arbitrary uniformly pointed perfect set rather than on 2^ω; notice that if $P = 2^\omega$, then $h = h^{-1} = \mathrm{id}_{2^\omega}$ and $b(i)$, $c(i)$, $m(i)$ and $d(i)$ coincide respectively with b, c, m and d of that proof. So, analogously to there, here we have, for all $x, y \in P$, $x \equiv_T y$ via (k,l) if and only if

$$x \equiv_T y \text{ via } (m,c)(m,b)^k(m,c)(m,b)^l(d,d)(b,m)^k(b,c)(b,m)^l(c,m),$$

[3] See Definition 1.3.6; we do not need the partial functions to be Borel here, but we do adopt Marks' convention that there is a computable operation on indices providing the index for the composition.

where, for brevity, we wrote b instead of $b(i)$, c instead of $c(i)$, etc. This shows that any uniformly pointed perfect with index i is effectively generated by $\big(b(i), m(i)\big), \big(m(i), b(i)\big), \big(c(i), m(i)\big), \big(m(i), c(i)\big), \big(d(i), d(i)\big)$. □

Lemma 5.4.5 asserts that the generating family of Turing bi-reductions is computably finitely effectively generated on uniformly pointed perfect sets, in the sense of the next definition.

Definition 5.4.6. We say that a generating family of functions $\{\,\psi_i\,\}$ is **computably finitely (effectively) generated on uniformly pointed perfect sets** if there exist finitely many computable functions $i_0, \ldots, i_n$ and $j_0, \ldots, j_n$ such that, for every $e \in \omega$ and every uniformly pointed perfect set P with index e, we have that, on P, $\{\,\psi_i\,\}$ is (effectively) generated by

$$\big(i_0(e), j_0(e)\big), \ldots, \big(i_n(e), j_n(e)\big).$$

Definition 5.4.7. We say that a Turing invariant function $f : A \to 2^\omega$, with $A \subseteq 2^\omega$, **preserves** a pair of indices (i, j) if there exists $e \in \omega$ such that, for all $x, y \in A$,

$$\forall x, y \in A : \Big(x \equiv_T y \text{ via } (i, j) \implies f(x) \geq_T f(y) \text{ via } e\Big). \tag{5.4.1}$$

Then, a function is UTI when it preserves all indices. The second half of the argument in the proof of Lemma 2.2.2 is synthesized by the following:

Remark 5.4.8. Note that, if Turing bi-reductions on $A \subseteq 2^\omega$ are generated by G and $f : A \to 2^\omega$ preserves all members of G, then f is uniformly Turing invariant. Moreover, there is an algorithm that, using an oracle $p \oplus q$, where

- p is a function that takes every $(i, j) \in \omega^2$ to a word $\big((a_{j_0}, b_{j_0}), \ldots, (a_{j_m}, b_{j_m})\big)$ with each letter (a_{j_k}, b_{j_k}) in G satisfying the properties described in Definition 5.4.3;
- q is a function that takes every $(i, j) \in G$ to an e via which (5.4.1) holds;

computes a uniformity function for f.

Therefore, if Turing bi-reductions are *finitely* and *effectively* generated on A, then there is a computable function p as above, and every function q as above is computable since it is finite. Hence, there is a computable uniformity function for f.

Note that if $\{\,\psi_i\,\}$ is finitely (effectively) generated on A and $B \subseteq A$ is closed under $E_{\{\,\psi_i\,\}}$, then $\{\,\psi_i\,\}$ is also finitely (effectively) generated on B. Thus, from Lemma 5.4.5 we deduce that Turing bi-reductions are finitely effectively generated on every Turing-invariant subset of a uniformly pointed perfect set. Thus, we have found the following generalization of Lemma 2.2.2.

Lemma 5.4.9. *If P is a uniformly pointed perfect set, A is a subset of it that is closed under $\equiv_T$ and $f : A \to 2^\omega$ is UTI, then there is a computable uniformity function of f.*

This was the missing ingredient to prove Proposition 5.1.1.

Proof of Proposition 5.1.1. *2* $\implies$ *1* follows from the fact that, under AD, any function from 2^ω to ω is constant on a pointed perfect set. Thus, f will be uniformly invariant on any pointed perfect set on which the map

$$p(x) = \text{the first } e \text{ such that } \varphi_e \text{ is a variance function of } f \restriction P \text{ in } x$$

is constant. The converse implication is the reason why we give this proof only now. Indeed, now we can exploit Lemma 5.4.9. All we need to do is to refine the pointed perfect set P on which f is UTI to a uniformly pointed perfect set $U \subseteq P$ (using Fact B.1.1), notice that f is *a fortiori* UTI on U and then use Lemma 5.4.9. □

Definition 5.4.10. We call **uniformly pointed perfect degree** the intersection between a uniformly pointed perfect set and a Turing degree.

Remark 5.4.11. Lemma 5.4.9 guarantees that all theorems about UTI functions on one Turing degree proved in the previous chapters, such as Theorem 2.1.2 and Theorem 4.1.3, hold for UTI functions on one uniformly pointed perfect degree.

In the paragraphs culminating in Question 2.3.4, we pointed out the difficulty of generalizing the computable uniformity function lemma to the arithmetic case, but we also stressed there is no problem in generalizing it to the arithmetic order-preserving case. Also here, we can notice we have the same problem: the family of arithmetic reductions is computably finitely effectively generated, so we can prove Lemma 5.4.9 for uniformly arithmetically order-preserving functions, but we ask:

Question 5.4.12. *Is the family of arithmetic bi-reductions computably finitely effectively generated on uniformly pointed perfect sets?*

Since every function from a $\leq_T$-cofinal subset of 2^ω is clearly constant on a $\leq_T$-cofinal set, by Fact B.1.1 every such function is — under AD — constant on a uniformly pointed perfect set. For this reason, if f is a Turing invariant function, given any finite set I of pair of indices, f preserves I on a uniformly pointed perfect set P_I. We also know that every uniformly pointed perfect set P is generated by a finite set of pair of indices J_P that can be computed from an index of P. If $J_{P_I} = I$, then f is UTI on the uniformly pointed perfect set P_I. This is the idea that leads to the game we are defining next.

A game to prove uniformity

The game we are going to introduce is a variation of the game that produces a uniformly pointed perfect set inside a cofinal set as in Fact B.1.1, but with extra-conditions that insure that the function f preserves the generators of Turing bi-reductions on that uniformly pointed perfect set. Hence, let us review the original game.

Proof of Fact B.1.1. Consider the game

$$\begin{array}{llllllll} \mathbf{I} & i & & x(0) & & x(1) & \dots \\ \mathbf{II} & & y(0) & & y(1) & & \dots \end{array}$$

where **I** wins if and only if $x \in A$ and $x \geq_T y$ via i. If τ is a strategy for **II**, then pick $x \in A$ with $x \geq_T \tau$. Note that there is a computable function r such that, for all $i \in \omega$ x computes $\tau(i^\frown x)$ (that is, the response given by τ to $i^\frown x$) via $r(i)$. Now, if $\bar{i}$ is a fixed point relative to x of r (in the sense of the Fixed Point Theorem, that is: $\varphi^x_{r(\bar{i})} = \varphi^x_{\bar{i}}$), then **I** wins against τ playing $\bar{i}^\frown x$.

Hence, by AD, **I** has a winning strategy σ for this game. If i is the first move given by σ, then the set

$$U = \{\, x \in 2^\omega \mid \exists y \in 2^\omega : i^\frown x = \sigma(y \oplus \sigma) \,\}$$

is a uniformly pointed perfect set included in A: it's closed because it is the continuous image of a compact in a Hausdorff space; σ is injective (because $\sigma(y) \geq_T y$ via i and $\sigma(z) \geq_T z$ via i, so if $\sigma(y) = \sigma(z)$, then $y = z$), thus U does not contain isolated points, as if it did have one, then $\sigma(\cdot \oplus \sigma)$ would be constant on a basic open set. Thus, U is perfect. Moreover, every $x \in U$ can compute σ using i, and with σ one can compute the tree of U. So there's in fact a computable function s such that, for all winning strategies σ of a game of this kind, $s(\sigma(\emptyset))$ is an index for the set U constructed using this σ. □

Definition 5.4.13 (The game $G_{f,P}$)**.** Given a uniformly pointed perfect set P and a TI function $f : A \to 2^\omega$ with $P \subseteq A \subseteq 2^\omega$, let $G_{f,P}$ be the game

$$\begin{array}{llllllll} \mathbf{I} & \langle i, e_0, \dots, e_4 \rangle & & x(0) & & x(1) & \dots \\ \mathbf{II} & & y(0) & & y(1) & & \dots \end{array}$$

in which the conditions for **I** to win are the following two:

1. $x \in P$ and $x \geq_T y$ via i;
2. for all $z \in P$ and for all $k \leq 4$:

$$x \equiv_T z \text{ via } (a_k \circ s(i), b_k \circ s(i)) \implies f(x) \geq_T f(z) \text{ via } e_k,$$

   where s is a computable function described in the following:

The game is built so that if **I** has a winning strategy σ and we let $\langle i, \vec{e} \rangle = \sigma(\emptyset)$, then the set

$$U_\sigma = \{\, x \in 2^\omega \mid \exists z \in 2^\omega : \langle i, \vec{e} \rangle^\frown x = \sigma(z \oplus \sigma) \,\}$$

is a uniformly pointed perfect set whose index can be computed from i: indeed, any $x \in U_\sigma$ computes $z \oplus \sigma$ via i, for some $z \in 2^\omega$, and once σ is computed, all of U_σ can be computed. Thus, an index of U_σ can be effectively found starting from i, where i is the first component of $\sigma(\emptyset) = \langle i, e_0, \dots, e_4 \rangle$. The function s in player **I**'s second winning condition is any computable function providing this effective procedure, that is mapping every $i \in \omega$ to an index of U_σ, where σ is any winning strategy for **I** such that, for some $e_0, \dots, e_4 \in \omega$, $\sigma(\emptyset) = \langle i, e_0, \dots, e_4 \rangle$. Therefore, by Lemma 5.4.5, the generators of Turing bi-reductions on U_σ will be

$$\big(a_0 \circ s(i), b_0 \circ s(i)\big), \dots, \big(a_4 \circ s(i), b_4 \circ s(i)\big),$$

which we shall denote more shortly by

$$g_0(i), \dots, g_4(i).$$

Hence, if there is a winning strategy σ for **I**, the first winning condition guarantees that U_σ is a uniformly pointed perfect set, while the role of second winning condition is guaranteeing that that the generators of Turing bi-reductions on U_σ are preserved by f. This makes f UTI on U_σ by Remark 5.4.8 if **I** has a winning strategy.

Proposition 5.4.14. *(*AD *not required). Let $P \subseteq 2^\omega$ be a uniformly pointed perfect set and f be a function defined on a superset of P. If* **I** *has a winning strategy for $G_{f,P}$, then f is UTI on a u.p.p. set $U \subseteq P$. Vice versa, if f is UTI on P, then* **I** *has a winning strategy for $G_{f,P}$.*

Proof. We have just proved the former statement. For the latter, suppose f is UTI on P and let u be a downward uniformity function of $f \restriction P$. Let $h : 2^\omega \to P$ denote the standard homeomorphism between 2^ω and P. Let $i \in \omega$ be s.t. $h(y) \geq_T y$ via i for all $y \in 2^\omega$. Then a winning strategy σ for **I** is the following: the first move is

$$\langle i, u(g_0(i)), \dots, u(g_4(i)) \rangle$$

and then σ responds to y by playing $h(y)$ (it is indeed immediate that the first k bits of y determine the first k digit of $h(y)$). ☐

Proposition 5.4.15. *Let $P \subseteq 2^\omega$ be a uniformly pointed perfect set and f any function defined on a superset of P. If for cofinally many $x \in P$ there is a downward variance function v of $f \restriction P$ in x whose Turing degree is not FPF relative to x, then player* **II** *doesn't have a winning strategy in $G_{f,P}$.*

Proof. Suppose σ is a strategy for **II**. Choose $x \in P$ such that $x \geq_T \sigma$ and there is a downward variance function v of $f \restriction P$ in x as in the hypothesis. Given $i, e_0, \dots, e_4 \in \omega$, $\sigma(\langle i, \vec{e} \rangle ^\frown x)$ is computable from x via $r(i, \vec{e})$, where r is a suitable computable function. Let $e_k(i)$ be $v(g_k(i))$, so that player **I** can guarantee her second winning condition is satisfied if she plays $\langle i, \vec{e}(i) \rangle ^\frown x$, for whatever $i \in \omega$. Since $i \mapsto r(i, \vec{e}(i))$ is computable from v, it must have a fixed point relative to x: there must exist j such that $\varphi^x_j = \varphi^x_{r(j, \vec{e}(j))} = \sigma(\langle j, \vec{e}(j) \rangle ^\frown x)$. In other words, $x \geq_T \sigma(\langle j, \vec{e}(j) \rangle ^\frown x)$ via j, thus **I**'s first winning condition is also satisfied. Hence, σ is not winning. ☐

Application of the game

Now we can get the main result of this section as an immediate corollary of the previous propositions.

Theorem 5.4.16. *Assume* AD *and let* $f : 2^\omega \to 2^\omega$ *be a Turing invariant function. The following are equivalent:*

1. *there is a uniformly pointed perfect set on which* f *is UTI;*
2. *there is a uniformly pointed perfect set* P *such that, for all* $x \in P$*, there is a downward variance function* v *of* $f \restriction P$ *in* x *whose Turing degree is not FPF relative to* x.

Notice that this is a strengthening of the more immediate Proposition 5.1.1, which is used to prove *1* $\implies$ *2*, while of course *2* $\implies$ *1* follows from the previous results on the game $G_{f,P}$. It is easy to see that the previous result remains true if replace 'downward variance function' with 'upward variance function' in its statement, as it suffices to use a modified version of $G_{f,P}$ in which we replace '$f(x) \geq_T f(z)$ via e_k' with '$f(x) \leq_T f(z)$ via e_k' in **I**'s second winning condition. In fact, Theorem 5.4.16 remains true when Turing bi-reductions are replaced, in the codomain of f, by any generating family of partial functions (see footnote 3).

Theorem 5.4.17. *Assume* AD. *Let* $\{\psi_i\}$ *be a generating family of partial functions on a set* X*, and* $\{\theta_i\}$ *be a generating family of partial functions on* 2^ω *which — like the family of Turing reductions — is computably finitely effectively generated on uniformly pointed perfect sets. Let* $f : 2^\omega \to X$ *be a homomorphism from* $E_{\{\theta_i\}}$ *to* $E_{\{\psi_i\}}$*. Then, the following are equivalent:*

1. *there is a uniformly pointed perfect set* P *such that* $f \restriction P$ *is a uniform homomorphism from* $E_{\{\theta_i\}}$ *to* $E_{\{\psi_i\}}$*;*
2. *there is a uniformly pointed perfect set* P *such that, for all* $x \in P$*, there is a computable variance function*[4] *of* $f \restriction P$ *in* x*;*
3. *there is a uniformly pointed perfect set* P *such that, for all* $x \in P$*, there is a computable-in-*$f(x)$ *variance function of* $f \restriction P$ *in* x*;*
4. *there is a uniformly pointed perfect set* P *such that, for all* $x \in P$*, there is a downward or upward variance function* v *of* $f \restriction P$ *in* x *such that* $[v]_T$ *is not FPF relative to* x.

Proof. *1* $\implies$ *2* is based on Lemma 5.4.9, which only uses the fact that the family of Turing reductions is finitely effectively generated on every uniformly pointed perfect set and that there is a computable operation on indices providing an index for the composition, and we assume this in the definition of generating family of functions.

[4] Of course, with respect to $\{\theta_i\}$ and $\{\psi_i\}$.

2 $\implies$ *3* is trivial. Note that *3* $\implies$ *2* follows from Proposition 5.2.1 and and *2* $\implies$ *1* follows from Remark 5.2.4. Hence, *1*, *2* and *3* are equivalent.

Another trivial implication is *2* $\implies$ *4*. Lastly, for *4* $\implies$ *1* first notice that we can refine P to either a uniformly pointed perfect set (which we still call P with an abuse of notation) in which all *downward* variance functions have the desired property, or to a uniformly pointed perfect set in which all *upward* variance functions do. Say, for instance, we are in the first case, i.e. all downward variance functions v of $f \upharpoonright P$ in x are such that $[v]_T$ is not FPF relative to x, for all $x \in P$. Then, we only need to define a variation of $G_{f,P}$ in which the second winning condition for **I** is

$$x \; E_{\{\,\theta_i\,\}} \; z \text{ via } \big(c_k(s(i)), d_k(s(i))\big) \implies \psi_k(f(x)) = f(z),$$

for all $k \leq n$ and all $z \in P$. Then, it's easy to check that the analogs of Proposition 5.4.14 and Proposition 5.4.15 hold. □

Remark 5.4.18. If, in the statement of Theorem 5.4.17, $\{\,\theta_i\,\}$ is just computably finitely generated on uniformly pointed perfect sets, not necessarily effectively[5] then we lose implication *1* $\implies$ *2*, but still that *1* implies *4*. On the other hand, if we have that $\{\,\theta_i\,\}$ is just finitely effectively generated on every uniformly pointed perfect set, then we lose implication *2* $\implies$ *4*, but we still have the equivalence between *1*, *2* and *3*.

Remark 5.4.19. Note that all items in Theorem 5.4.17 are saying that f globally satisfies some nice property (see Definition B.3.1 and the following part of Appendix B), so Proposition B.3.2 furnishes further equivalent items to those in Theorem 5.4.17.

Thus, Theorem 5.4.17 gives a characterization of global uniformity not only in the Turing case, but also for other kinds of uniformity that have been found interesting in literature: uniformity for Turing- to many-one- invariant functions (see [KM18]), and uniformity for arithmetically order-preserving functions (see [MSS16])[6].

Observe that refining the domain of a function enlarges the set of possible variance functions for it. Also in cases different from the Turing one, global uniformity is characterized by the possibility of refining the domain of the function to a uniformly pointed perfect set which makes it possible to find, in its every point, variance functions that are computable, or not too impossible to compute. So we ask the following:

[5]This means there exists a computable function that, on input e, outputs a finite number of indices that generate $\{\,\theta_i\,\}$ on any uniformly pointed perfect set P with index e, but there might not be an effective way to witness that these indices generate $\{\,\theta_i\,\}$ on P, i.e. no computable function to associate every (i, j) to a word in these indices so that $\tilde{\theta}_{i,j}$ is the same as the composition of the bi-reductions indexed by those letters.

[6][MSS16] presents many uniformly arithmetically order-preserving functions mentions, but refers to them just as uniformly arithmetically invariant functions. Note that we do not know if Theorem 5.4.17 also holds for UAI functions, as we do not have an answer to Question 5.4.12.

Question 5.4.20. *Suppose $f : 2^\omega \to 2^\omega$ is Turing invariant and, for cofinally many Turing degrees $[x]_T$, there exists a uniformly pointed perfect set P such that f restricted to $[x]_T \cap P$ admits computable variance functions. Must f be uniformly Turing invariant on a uniformly pointed perfect set?*

In other words, if a function is UTI on cofinally many uniformly pointed perfect degrees, must it be globally UTI? A positive answer to this question would imply that global uniformity can be retraced from local information, and interestingly would imply there is a Π^1_2 statement equivalent to Steel's conjecture. The latter — like all propositions asserting that all TI functions globally satisfy a nice property — originally is Π^1_3 statement, so discovering it's actually Π^1_2 would have the interesting consequence that, by Shoenfield's absoluteness, its truth value cannot be changed by forcing. Arguably, then, the enigma whether Steel's conjecture were true or false would, at least, have to have an answer.

A question related to Question 5.4.20 is:

Question 5.4.21. *Is there a function $f : [x]_T \to [y]_T$, with $x \leq_T y$, that does not admit computable variance functions on any uniformly pointed perfect degree $\subseteq [x]_T$?*

Notice that we can harmlessly replace 'admits computable variance functions' with 'is UTI' in Question 5.4.20, while it is not clear whether the same change would be harmless for the previous question. So we also ask:

Question 5.4.22. *Is there a function $f : [x]_T \to [y]_T$, with $x \leq_T y$, that is not uniformly Turing invariant on any uniformly pointed perfect degree $\subseteq [x]_T$?*

Chapter 6

Measuring the complexity of variance

In this chapter, we briefly outline a topic that might be interesting to investigate, which was inspired by the observations we made after Proposition 5.2.2.

6.1 Variance functionals

We shall call **functional** any map from $A \subseteq 2^\omega$ to $\mathcal{P}(\omega^\omega)$. If A is a non-empty subset of 2^ω and $f : A \to 2^\omega$ is a Turing invariant function, we define the **variance functional** of f as

$$V_f : A \to \mathcal{P}(\omega^\omega)$$

that takes x to the set of variance functions of f in x.[1] In [DS97], Downey and Shore introduced a similar object, that we shall redub $R_f : A \to 2^\omega$, taking x to

$$\{\, \langle i, j, k, l \rangle \mid \varphi_i^x \in A,\ x \equiv_T \varphi_i^x \text{ via } (i, j) \text{ and } f(x) \equiv_T f(\varphi_i^x) \text{ via } (k, l) \,\}\,.$$

What they proved is that R_f is uniformly Turing invariant, $R_f(x) \geq_T f(x)''$ and, if A is a pointed perfect set and $\mathrm{id} \leq_M f \leq_M \mathrm{id}'$, then $R_f(x) \equiv_T f(x)''$, hence f'' is UTI and so is either id'' or id''' by Steel's uniform Martin's conjecture part II.

As we said, for us the motivation to introduce V_f are the observations we made after Proposition 5.2.2. See footnote 1 at page 54 to recall the definition of Medvedev reducibility $\leq_s$, and notice that the observation right after that footnote actually proves the following:

Proposition 6.1.1. *Variance functionals of TI functions are Turing to Medvedev uniformly invariant.*

[1] For the sake of notation, we identify a variance function $v : \omega^2 \to \omega^2$ with $\tilde{v} : \omega \to \omega$, $\langle i, j \rangle \mapsto \langle v(i, j) \rangle$.

Proof. Fix $f : A \to 2^\omega$ and $x, y \in A$ with $x \equiv_T y$ via (k, l). The argument of Proposition 5.2.2 illustrates a program that, using any downward variance function v of f in x as oracle, and given input k, l and e such that $x \equiv_T y$ via (k, l) and $f(x) \leq_T f(y)$ via e, computes a downward variance function $\varphi^v_{t(k,l,e)}$ of f in y. Now, if we have a variance function w of f in x, the projection v on the second component of w is a downward variance function of f in x, while we can get e as the second component of $w(i, j)$. This gives us a downward variance function $\varphi^w_{s(k,l)}$ of f in y, and analogously one could get an upward one. Essentially, there is a computable function u such that, for all $x, y \in A$, if $x \equiv_T y$ via (k, l), then

$$w \in V_f(x) \implies \varphi^w_{u(k,l)} \in V_f(y),$$

that is, $V_f(x) \geq_s V_f(y)$ via $u(k, l)$. Thus, u is a computable downward uniformity function of V_f. □

From the previous argument we can tell that variance functionals are uniformly invariant from Turing to a stronger form of Medvedev equivalence, that is, Medvedev equivalence via total Turing functionals.

We can also introduce the downward variance functional of f, denoted D_f, as the functional sending $x \in \mathrm{dom}(f)$ to the set of all downward variance functionals of f in x. Note that downward variance functionals are Turing to Medvedev invariant, but not necessarily uniformly, for what we said about e in the previous proof. However, notice that D_f is uniformly invariant if f is UTI.

Question 6.1.2. *If the downward variance functional D_f of a TI $f : 2^\omega \to 2^\omega$ is uniformly Turing to Medvedev invariant,*[2] *must f be UTI, at least on a uniformly pointed perfect set?*

Remark 6.1.3. The set $V_f(x)$ is always cofinal with respect to $\leq_T$. Indeed, given $v \in V_f(x)$ and $p \in \omega^\omega$, we can consider the variance function $w \in V_f(x)$ that, given $\langle i, j\rangle$, outputs $\langle k, l\rangle$ which is almost $v(\langle i, j\rangle)$, except that $p(\langle i, j\rangle)$ is coded inside of the instructions of the programs in a recognizable way that is makes no practical effect on the computation (we could think of a comment in the first line of the source code containing $p(\langle i, j\rangle)$ many 1s).

Remark 6.1.4. For all $\langle i, j\rangle \in \omega$, the set $\{\, v(\langle i, j\rangle) \mid v \in V_f(x) \,\}$ is closed under the symmetric and transitive relation $\sim^{f(x)}$ defined on ω by

$$\langle k, l\rangle \sim^{f(x)} \langle m, n\rangle \iff \begin{cases} \varphi_k^{f(x)} = \varphi_m^{f(x)} \\ \varphi_l^{\varphi_k^{f(x)}} = \varphi_n^{\varphi_m^{f(x)}} \end{cases}.$$

Moreover, $\{\, v(\langle i, j\rangle) \mid v \in V_f(x) \,\}$ equals ω if there is no $y \in A$ such that $x \equiv_T y$ via (i, j).

[2]We might also ask this question with the stronger hypothesis that D_f is uniformly invariant from Turing to Medvedev via total Turing functionals.

Definition 6.1.5. If $F : A \to \mathcal{P}(\omega^\omega)$ and $G : B \to \mathcal{P}(\omega^\omega)$ are Turing to Medvedev invariant, with $A \cap B$ cofinal with respect to $\leq_T$, then we write $F \leq_s^\nabla G$ if $F(x) \leq_s G(x)$ for $\leq_T$-cofinally many $x \in A \cap B$.[3]

A rephrasing of Lemma 5.4.9 in terms of $\leq_s^\nabla$ is that, for every UTI f defined on a uniformly pointed perfect set P, V_f is $\leq_s^\nabla$ the functional taking every real to $\{ 000 \dots \}$. Another way of putting it is the following:

Proposition 6.1.6. *Let P be a uniformly pointed perfect set and $f : P \to 2^\omega$ be Turing invariant. Then f is UTI on P if and only if $[V_f]_{\equiv_s^\nabla}$ is minimum, that is, $V_f \leq_s^\nabla G$ for every functional G.*

This simple result suggests we can view the $\equiv_s^\nabla$-degree of V_f as a measure of how far f is from being UTI. Further evidence is given by the following:

Remark 6.1.7. If we have Turing invariant $f, g : A \to 2^\omega$ such that $f(x) \equiv_T g(x)$ via a fixed pair of indices for all $x \in A \subseteq 2^\omega$, then $V_f(x) \equiv_s V_g(x)$ (via a fixed pair of indices as well) for all $x \in A$.

The intuitive interpretation of this might be that, if $f(x) \equiv_T g(x)$ via a fixed pair of indices, then f is as far from being UTI as g, so V_f and V_g have the same $\equiv_s^\nabla$-degree. On the other hand, if $f(x) <_T g(x)$ via a fixed index, it need not be that f is more close to being UTI than g, so the $\equiv_s^\nabla$-degree of V_f need not be below that of V_g.

With an abuse of notation, we still denote by f the functional sending x to $\{ f(x) \}$. Then, rephrasing the equivalence of items 1, 2 and 3 of Theorem 5.4.17 in terms of $\leq_s^\nabla$, we get the following:

Proposition 6.1.8. *For every Turing invariant $f : P \to 2^\omega$, where P is a uniformly pointed perfect set, the following are equivalent:*

1. *there is a uniformly pointed perfect set Q on which f is UTI;*
2. *there is a uniformly pointed perfect set Q such that $V_{f\restriction Q} \leq_s^\nabla \mathrm{id}_{2^\omega}$;*
3. *there is a uniformly pointed perfect set Q such that $V_{f\restriction Q} \leq_s^\nabla f$.*

Note that $V_f \leq_s^\nabla f \implies V_f \leq_s^\nabla G$ for every functional G (although of course f is not $\leq_s^\nabla G$ in general), while we do not have (a proof of) $V_f \leq_s^\nabla \mathrm{id}_{2^\omega} \implies V_f \leq_s G$ but just of $V_f \leq_s^\nabla \mathrm{id}_{2^\omega} \implies V_{f\restriction Q} \leq_s G$ for some uniformly pointed perfect $Q \subseteq \mathrm{dom}(f)$.

Anyways, this phenomenon suggests the presence of a “watershed” for variance functionals. So it would be interesting to know whether they satisfy a dichotomy in the spirit of part I of (uniform) Martin’s conjecture, and if — also in this case — this is retrievable locally.

[3]Equivalently, for sufficiently large $x \in A \cap B$, i.e. on the intersection of a cone and $A \cap B$. This equivalence follows from Martin’s cone theorem and the fact F and G are Turing to Medvedev invariant on $A \cap B$.

Chapter 7

Conclusions and future directions

Looking at questions related to uniformity from a local point of view gave us some new results, such as Theorem 3.2.2, as well as known results with a simpler proof, like in the case of uniform Martin's conjecture part I or Lachlan's theorem on uniform Sacks question.

However, the local approach might be useless for uniform Martin's conjecture part II, as it might appear from what we said in section 4.2. Yet, it might also be the case that we just need to employ sharper tools. For instance, let us consider the toy example of the odometer on 3^ω and 2^ω respectively, which we considered in section 1.4. The impossibility of a uniform Borel reduction from the former to the latter proven in Theorem 1.4.6 might be the consequence of the necessity to *choose* a representative in each class in order to uniform, non-constant reduction from a class on 3^ω to 2^ω. One should find adequate tools to give a rigorous formalization to this insight, and then maybe one could use these tools to detect uniform Martin's conjecture part II locally. For example, given two Turing degrees $\boldsymbol{x}$ and $\boldsymbol{y}$ with $\boldsymbol{x} \leq_T \boldsymbol{y}$, we needed to choose a representative $y \in \boldsymbol{y}$ in order to build the non-constant UTI function $f : \boldsymbol{x} \to \boldsymbol{y}$, $z \mapsto z \oplus y$, like we did in Theorem 3.1.2. But we do not need to choose one if $\boldsymbol{y}$ is an iteration of the jump of $\boldsymbol{x}$. So looking for an adequate framework for such kind of arguments might be a compelling line of work for the future.

Another possible intriguing development is the study of variance functionals, as shown in Chapter 6, their possible behaviors, dichotomies and so on, since they can be regarded as a way to measure the non-uniformity of TI functions. It might be particular interesting to look at the properties of the map $P \mapsto [V_{f \restriction P}]_{\equiv_s^\triangledown}$, where P varies among all uniformly pointed perfect sets $P \subseteq \mathrm{dom}(f)$. Variance functionals might also be very interesting to analyze at local level, and for example one could look at the map $D \mapsto [V_{f \restriction D}]_{\equiv_s}$, where this time D varies among all uniformly pointed perfect degrees included in a fixed Turing degree in the domain of f.

Uniform Turing invariance on uniformly pointed perfect degrees is indeed — in our opinion — another topic worth investigating; in particular, we point out the three questions with which we concluded Chapter 5.

Appendix A

Non-deterministic Turing reducibility

A.1 Computing with partial oracles

We denote by $\omega^{\subseteq\omega}$ denote the set of all partial functions from ω to ω. For $p, q \in \omega^{\subseteq\omega}$, say as usual that $p \leq_T q$ via i if $p = \varphi_i^q$, $p \leq_T q$ if $p \leq_T q$ via i for some $i \in \omega$, and $p \equiv_T q$ if $p \leq_T q$ and $q \leq_T p$. We keep the notation $[x]_T$ denoting the usual Turing degree of x, that is $\{\, z \in 2^\omega \mid z \equiv_T x \,\}$.

We have something to specify here, as it may not be clear what happens when we run an oracle Turing machine with a partial function as oracle. What happens is that, when the oracle is queried about something outside its domain, no answer will ever be given and the computation diverges. To emphasize that this is the way the oracle Turing machine works with partial oracles, we add the word **deterministic** in front of 'oracle Turing machine', or 'Turing reduction', etc.

In fact, there is a different notion, called **non-deterministic** Turing reducibility, in which partial oracles behave as follows. We can imagine the oracle might take whatever time it needs to answer the queries, with the possibility of never answering if and only if the query is about an undefined digit; however, while waiting for the oracle's answers, the non-deterministic oracle Turing machine can still perform other tasks, so the computation need not diverge. However, in order to say that $p \in \omega^{\subseteq\omega}$ non-deterministically Turing reduces $q \in \omega^{\subseteq\omega}$, we require that the possible output on input i is always $q(i)$, regardless of the time it takes for oracle p to answer the queries. Moreover, if no output is found, then $q(i)$ must be undefined.

There is an equivalent, classic way of presenting non-deterministic Turing reducibility: the machine does loop if it queries the oracle outside of the domain, but it can also follow multiple computation paths and halt as soon as one converges. The program of the machine can include instructions that makes the machine "duplicate" itself into two copies, one performing the program with a certain parameter set to 0, the other with that parameter set to 1. So for example the machine can duplicate itself into two copies, one querying the oracle about $2i$, where i is the input, the other querying the oracle about $2i+1$. Perhaps, one the two copies will halt and the other will not. What we require in order to say that $p \in \omega^{\subseteq\omega}$ non-deterministically Turing reduces $q \in \omega^{\subseteq\omega}$ is that, if there is a convergent computation path of the non-deterministic machine with oracle p on input i, then every convergent computation path converges to the same output, namely $p(i)$. On the other hand, if all computation paths diverge, then $p(i)$ must be undefined. This presentation of non-deterministic Turing reducibility is the right one, and ours differs to it from a complexity theory perspective, but the two are equivalent from our computability theoretic perspective in which the duration of computation doesn't really make a difference. For more reference on non-deterministic Turing reducibility, see [Coo04].

If $p, q \in \omega^{\subseteq\omega}$, the fact that that p is non-deterministically Turing reducible to q is written as $p \leq_{NT} q$, and non-deterministic Turing equivalence, denote by $\equiv_{NT}$, is the equivalence relation generated by the quasi-order $\leq_{NT}$ as usual. By $[x]_{NT}$ we denote the set of all partial functions that are non-deterministically Turing equivalent to x, even though $x \in 2^\omega$. So

$$[x]_{NT} = \{\, p \in \omega^{\subseteq\omega} \mid p \equiv_{NT} x \,\}.$$

We use the shorthand '**NTuring**' for 'non-deterministically Turing'. To stress the difference between Turing and NTuring reducibility, the former is called deterministic Turing reducibility. In the light of what we said, it is easy to notice that, for all $x, y \in \omega^\omega$, $x \leq_T y$ if and only if $x \leq_{NT} y$ but, for partial function p and q, $p \leq_T q$ clearly implies $p \leq_{NT} q$ but the converse is false in general. For example, if we denote by S_A the semi-characteristic function of $A \subseteq \omega$, that is

$$S_A(n) = \begin{cases} 1 & \text{if } n \in A, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

we have that $S_{A\oplus\overline{A}}$ can deterministically compute only computable sets of natural numbers, whereas it can non-deterministically compute A. Indeed, if a deterministic (i.e. standard) oracle Turing machine halts on a certain input with oracle $S_{A\oplus\overline{A}}$, it means it only queried about digits of the oracle that equal 1, so it would give the same output if the oracle were the constant function $n \mapsto 1$. On the other hand, to non-deterministically compute A from oracle $S_{A\oplus\overline{A}}$, given input i we can ask the oracle two questions: "Does the $2i$-th digit of the oracle equal 1?" and "Does the $(2i+1)$-th digit of the oracle equal 1?". If we receive a positive answer from the first question, we output 1, whereas if we receive a positive answer from the second question, we output 0.

Of course, we can index in a standard way both deterministic and non-deterministic oracle Turing machines, and we denote both the deterministic and non-deterministic i-th Turing reduction by $\varphi_i^{(\cdot)}$, but we have to always make clear which notion we are using when dealing with non-total oracles. For instance, we can give the usual definition of Turing jump of a partial function

p using either notion, but it makes a huge difference which notion we use if p is non-total. Notice, for example, that — by what we said above — for all $A \subseteq \omega$, the deterministic jump of $S_{A\oplus\overline{A}}$ is many-one equivalent to $0'$, whereas its non-deterministic jump is many-one equivalent to A'.

A.2 Connection with enumeration reducibility

If A and B are sets of natural numbers, we say A is enumeration reducible to B via $i \in \omega$, written $A \leq_e B$ via i, if

$$A = \{\, n \in \omega \mid \text{there is a finite } F \subseteq B \text{ such that } \langle F, n\rangle \in W_i \,\},$$

where $\langle F, n\rangle$ denotes the natural number coding the pair (F, n), where F is a finite set of natural numbers and n is a single natural number. Of course, we say A is enumeration reducible to B, and we write $A \leq_e B$, if $A \leq_e B$ via i for some i. Intuitively, $A \leq_e B$ means it is possible to effectively produce an enumeration of A from an enumeration of B.

We refer to textbooks as [Coo04] for a fuller treatment of enumeration reducibility. The point we want to make here is that enumeration reducibility is almost the same as non-deterministic Turing reducibility. Indeed, the map

$$\begin{aligned} \mathcal{P}(\omega) &\to \omega^{\subseteq\omega} \\ A &\mapsto S_A \end{aligned}$$

is such that $A \leq_e B \iff S_A \leq_{NT} S_B$, and there is a function $u : \omega \to \omega$ such that $A \leq_e B$ via i if and only if $S_A \leq_{NT} S_B$ via $u(i)$. On the other hand, if, given $p \in \omega^{\subseteq\omega}$, we define $\mathrm{gr}(p) = \{\, \langle n, k\rangle \mid n \in \mathrm{dom}(p), p(n) = k \,\}$, then the map

$$\begin{aligned} \omega^{\subseteq\omega} &\to \mathcal{P}(\omega) \\ p &\mapsto \mathrm{gr}(p) \end{aligned}$$

is such that $p \leq_{NT} q \iff \mathrm{gr}(p) \leq_e \mathrm{gr}(q)$, and there is a function $v : \omega \to \omega$ such that $p \leq_{NT} q$ via i if and only if $\mathrm{gr}(p) \leq_e \mathrm{gr}(q)$ via $v(i)$. Furthermore, there are (i, j) and (k, l) in ω^2 such that

$$\mathrm{gr}(S_A) \equiv_e A \text{ via } (i, j), \qquad S_{\mathrm{gr}(p)} \equiv_{NT} p \text{ via } (k, l) \tag{A.2.1}$$

for all $A \subseteq \omega$ and $p \in \omega^{\subseteq\omega}$.

Appendix B

Uniformly pointed perfect sets and global properties

We use uniformly pointed perfect sets to define what it means for a function to satisfy a (nice) property *globally*. In the context of Martin's conjecture, both *uniformly* pointed perfect sets and pointed perfect sets *tout court* are used for this purpose, and lead to an equivalent notion, but we shall work only with the former for convenience.

B.1 Uniformly pointed perfect sets

A set $P \subseteq 2^\omega$ is called a pointed perfect set if it is perfect, i.e. closed with no isolated points, and the tree of P is computable from any $x \in P$. Moreover, it's called **uniformly pointed perfect** if it is perfect and there is a single index i via which every $x \in P$ computes the tree of P. One such index i is called **index** of P.

Like all perfect subsets of the Cantor space, pointed perfect sets are homeomorphic to 2^ω itself. But an extra feature of them is that, given a pointed perfect set $P = [T]$, the most obvious homeomorphism $h : 2^\omega \to P$ satisfies $h(x) \equiv_T x \oplus T$ for all $x \in 2^\omega$, which implies that h is order-preserving (i.e. preserves $\leq_T$) and $h(x) \equiv_T x$ on the cone above T. Furthermore, if P is uniformly pointed, there is a single pair of indices via which $h(x) \equiv_T x \oplus T$ for all x, so h is also uniformly order-preserving. This is one of the reasons why we prefer working with uniformly pointed perfect set rather than pointed perfect sets *tout court*.

A variant of celebrated Martin's cone theorem is that, under AD, every set $A \subseteq 2^\omega$ which is cofinal with respect to $\leq_T$ contains a uniformly pointed perfect set (see [Mar12]). We present the proof of it at page 62, as its argument provides the core of the argument presented in that section.

Fact B.1.1. *($\mathsf{ZF} + \mathsf{AD}$) If $A \subseteq 2^\omega$ is cofinal with respect to $\leq_T$, then A contains a uniformly pointed perfect set.*

Corollary B.1.2. *($\mathsf{ZF}+\mathsf{AD}$) Suppose $A \subseteq 2^\omega$ is cofinal with respect to $\leq_T$ and π is any function from A to ω. Then, π is constant on a uniformly pointed perfect set.*

Proof. If, for all $i \in \omega$, $\pi^{-1}(i)$ was uncofinal, say disjoint from the cone above x_i, then $A = \bigcup_i \pi^{-1}(i)$ would be disjoint from the cone above $\bigoplus_i x_i$, which is against the hypothesis.[1] Thus, there is at least one $i \in \omega$ such that $\pi^{-1}(i)$ is cofinal with respect to $\leq_T$. By Fact B.1.1, there is uniformly pointed perfect set $U \subseteq \pi^{-1}(i)$, so of course π is constant on U. □

B.2 Uniform invariance on uniformly pointed perfect sets

Uniformly pointed perfect sets are useful when considering UTI functions globally because of the following:

Proposition B.2.1. *For every Turing invariant $f : 2^\omega \to 2^\omega$, the following are equivalent (over $\mathsf{ZF}+\mathsf{AD}$):*

1. *f is UTI on a uniformly pointed perfect set;*
2. *there exists $g : 2^\omega \to 2^\omega$ such that g is UTI on 2^ω (equivalently, UTI on a cone) and $f(x) \equiv_T g(x)$ on a cone;*
3. *for all uniformly pointed perfect set P there is a uniformly pointed perfect set $Q \subseteq P$ such that f is UTI on Q.*

The equivalence between *1* and *2* can be found in [MSS16, footnote 1], whereas the equivalence between those and *3* is due to us. Although being very easy to proof, it is at first sight quite surprising to us that the uniformity on a pointed perfect set P implies uniformity in all sufficiently refined pointed perfect sets, even if disjoint from P. Before we give a proof, we point out that this feature of the property of "being UTI" is shared with a large class of nice properties, in the sense of the following definition.

B.3 Properties that hold globally

Definition B.3.1. Let us call **nice** a property for TI functions if it is preserved both by pre-composition with UOP functions which are $\equiv_M$-equivalent to id_{2^ω}, by post-composition with Turing bi-reductions and by restriction to uniformly pointed perfect sets; that is, if the set $\mathcal{N}$ of TI functions from a subset of 2^ω to 2^ω satisfying this property is such that:

1. if $f \in \mathcal{N}$, $f : A \to 2^\omega$, and $h : 2^\omega \to A$ is UOP and s.t. $h \equiv_M \mathrm{id}_{2^\omega}$, then $f \circ h \in \mathcal{N}$;

[1] We should perhaps stress the fact that we are using $\mathsf{CC}_\mathbb{R}$ to choose the x_i's, and we can do that thanks to Theorem 2.1.4.

2. if $f, g : A \to 2^\omega$ are such that $f(x) \equiv_T g(x)$ via (i, j) for all $x \in A$ and $f \in \mathcal{N}$, then $g \in \mathcal{N}$;

3. if f has property $\mathcal{N}$ and U is a uniformly pointed perfect set, then $f \restriction U$ has property $\mathcal{N}$.

Examples of nice properties are "being UOP", "being UTI", "being continuous", and in general "being Baire class α" for all $\alpha < \omega_1$. Let us prove the generalization of Proposition $B.2.1$ to nice properties.

Proposition B.3.2. *Let $\mathcal{N}$ be a nice property. Then, for every Turing invariant $f : 2^\omega \to 2^\omega$, the following are equivalent (over $\mathsf{ZF} + \mathsf{AD}$):*

1. *the restriction of f to a uniformly pointed perfect set has property $\mathcal{N}$;*

2. *there exists $g : 2^\omega \to 2^\omega$ such that g has property $\mathcal{N}$ (equiv. the restriction of g on a cone has property $\mathcal{N}$) and $f(x) \equiv_T g(x)$ on a cone;*

3. *for all uniformly pointed perfect set P there is a uniformly pointed perfect set $Q \subseteq P$ such that $f \restriction Q$ has property $\mathcal{N}$.*

Proof. Of course, *3* $\implies$ *1* is immediate. For *1* $\implies$ *2*: if $f : U \to 2^\omega$ has property $\mathcal{N}$, U is uniformly pointed perfect set and $h : 2^\omega \to U$ is its standard homeomorphism with the Cantor space, then h is UOP and $h(x) \equiv_T x$ on a cone, so that $(f \circ h)(x) \equiv_T f(x)$ on a cone because f is TI, and $f \circ h$ has property $\mathcal{N}$ by condition 1 of nice properties.

Now let's prove *2* $\implies$ *3*. Choose a cone on which g has $\mathcal{N}$ and a cone on which $f(x) \equiv_T g(x)$, and then pick a cone C within the intersection of these cones. Given any uniformly pointed perfect set P, we have that $P \cap C$ is cofinal with respect to $\leq_T$, so by Corollary $B.1.2$ there is a uniformly pointed perfect set $Q \subseteq P \cap C$ and a pair of indices (i, j) such that $f(x) \equiv_T g(x)$ via (i, j) for all $x \in Q$. Then $f \restriction Q$ by conditions 2 and 3 of nice properties. □

This gives us a solid notion on which we can base our definition of "globally satisfying a nice property".

Definition B.3.3. We say that a TI function $f : 2^\omega \to 2^\omega$ **globally** satisfies a nice property $\mathcal{N}$ to mean that the restriction of f to some uniformly pointed perfect set satisfies $\mathcal{N}$, or if either of the equivalent clauses established by Proposition $B.3.2$ holds.

Remark B.3.4. We can define an analogous notion of nice properties holding globally for arithmetically invariant functions instead of Turing invariant ones. In general, for every generating family of partial functions $\{ \psi_i \}$ on 2^ω such that

- $\{ \psi_i \}$ generates a quasi-order $\leq_{\{ \psi_i \}}$ which is coarser than $\leq_T$;
- the $E_{\{ \psi_i \}}$-closure of a $\leq_T$-cone is a $\leq_{\{ \psi_i \}}$-cone;

we can modify the notion of nice property for $E_{\{\psi_i\}}$-invariant functions just turning condition 2 of Definition B.3.1 to

2. if $f, g : A \to 2^\omega$ are such that $f(x)E_{\{\psi_i\}}g(x)$ via (i, j) for all $x \in A$ and $f \in \mathcal{N}$, then $g \in \mathcal{N}$;

and Proposition B.3.2 will hold for nice properties of $E_{\{\psi_i\}}$-invariant functions just by changing item *2* to

2. there exists $g : 2^\omega \to 2^\omega$ such that g has property $\mathcal{N}$ (equiv. the restriction of g on a cone has property $\mathcal{N}$) and $f(x)E_{\{\psi_i\}}g(x)$ on a cone;

where "cone" can equivalently denote a Turing cone or a $\leq_{\{\psi_i\}}$-cone, since the set on which $f(x)E_{\{\psi_i\}}g(x)$ is $E_{\{\psi_i\}}$-invariant.

Acknowledgments

According to the point of view, this can be considered the most pointless or the most meaningful part of this dissertation. For the academics who read this thesis because of the math, the content of this couple of pages might be of no interest, or possibly of relative interest. On the other hand, for those who read this just because they care for me and for what I do, and do not understand the math, this will be the only significant couple of pages. For this reason, contrary to common practice, I will start by dedicating these words to them.

Carrying out this PhD was a beautiful experience I will always remember joyfully. I think that when I'm happy, it is never for only one reason, so I have to thank several circumstances for having had such a happy time recently. In particular, I have to thank many people for creating such favorable circumstances around me, even when the outer circumstances were not that good. First of all, my family: my mom Rossana, my dad Paolo, my brother Aurelio, my uncle Stefano and my grandma Dina, for they raised me the person I am and — despite my many faults — I like being, and they keep providing me with all the love and caring that one could desire from their family.

Who made the last period incredibly splendid was Federica. Before I met her, I was actually struggling with my PhD. I felt alone, stuck, run out of ideas and fed up with the situation. I didn't even know what to desire about my future. She entered my life bringing — and having myself revive — enthusiasm, curiosity, depth, fun, love. She made even quarantine feel marvelous, so deep thanks go to her. Thanks also to her family, Monica, Gregorio, Francesco, Federica and Anna for welcoming me so dearly. Thanks to her wonderful friends, who are now friends of mine as well: Gianluca, Ilaria, Anita, Marco, Luís, Maria, Patrick and Debora, among others.

Thanks to Mario, Marco and particularly Simone for their longlife friendship. Thanks to Francesco and Azdine for the same reason. Thanks to Andrea, Mauricio, Giulia, and all the wonderful group of friends I had in Berkeley. Thanks to Fernando, my dear friend, fellow traveller, and companion of PhD in Logic. Thanks to all my dear friends, colleagues of PhD and office mates: Eugenio, Tsvetlin, Claudio, Carola, Riccardo, Salvatore, Francesca and Fabio. A special thank to Eugenio and Carola, who really helped me a lot more than once.

Thanks to Patrick Lutz, whom I met in Berkeley and became a friend, a collaborator, and who was extremely welcoming in Berkeley. Thanks to the

Institute for Mathematical Sciences of the National University of Singapore for having me give a talk and hosting me for a whole month of conferences in 2019, which gave me the opportunity to meet — besides young talented mathematicians like Patrick — many celebrated giants of mathematics, such as Ted Slaman, Antonio Montalbán, Andrew Marks, among others. I had the pleasure to visit Ted and Antonio in Berkeley. I am extremely grateful to Antonio for inviting me, mentoring me and supporting me for my whole stay in Berkeley, and for helping me find a postdoc position when I thought I wanted to pursue one. For this reason, he introduced me to Andreas Weiermann, whom I would also like to thank for his kindness, patience and support he had over a series of emails. Andrew Marks was also very kind in patiently answering my emails and solving my doubts, since the very first times as a PhD student till recently, so many thanks go to him as well.

Last, but not least, a special thank go to the Logic Group of Torino, which provided a stimulating environment for my education, with the many Logic classes it made efforts to supply in my Bachelor's, Master's and Phd, and for all the conferences and talks they have been organizing. A special thank go to Alessandro Andretta for all the teachings and the helpful conversations. An even bigger thank go to my supervisor Luca Motto Ros, for all the time he devoted to me teaching, advising, brainstorming, supporting and patiently monitoring the making of this thesis.

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