Linear Algebra and its Applications 706 (2025) 55–69

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Irreducible matrix representations of quaternions

Yu Chen

Department of Mathematics, University of Torino, Torino, Italy

ARTICLE INFO

Article history:
Received 19 June 2024
Received in revised form 8 November 2024
Accepted 13 November 2024
Available online 17 November 2024
Submitted by P. Semrl

MSC:
12E15
16K20

Keywords:
Matrix representation
Quaternions
Simple algebra

ABSTRACT

We determine all irreducible real and complex matrix representations of quaternions and classify them up to equivalence. More over, we show that there is a one-to-one correspondence between the equivalence classes of the irreducible matrix representations and those of the field homomorphisms from the real numbers to the complex numbers.

1. Introduction and main theorem

The study of matrix representations for the quaternions has appeared ever since the introduction of matrix calculus by Cayley (cf. [1], p.32) in the middle of 19th century. The basic idea is to represent the quaternions by square matrices while the two fundamental operations, addition and multiplication, are preserved, which is in fact a ring homomorphism between the quaternions and a ring of square matrices. Thus a matrix representation of the quaternions by definition is a ring homomorphism from the quater-

E-mail address: yu.chen@unito.it.

https://doi.org/10.1016/j.laa.2024.11.010
0024-3795/

nion ring to a $n \times n$ matrix ring where n is the representation degree. Let $\mathbb{H}$ be the Hamiltonian quaternion ring and let $\rho : \mathbb{H} \to \mathtt{M}_n(\mathbb{R})$ be a matrix representation where $\mathtt{M}_n(\mathbb{R})$ is the $n \times n$ real matrix ring. The image $\rho(\mathbb{H})$ of $\mathbb{H}$ under ρ can be considered as a set of linear transformations on the vector space $\mathbb{R}^n$ in a natural way. Then ρ is called irreducible if there is no non-trivial proper subspace of $\mathbb{R}^n$ stabilized by $\rho(\mathbb{H})$. For instance, there is a canonical irreducible real matrix representation $\rho : \mathbb{H} \to \mathtt{M}_4(\mathbb{R})$ defined by

$$\rho(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = \begin{pmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & d & a & b \\ -d & c & -b & a \end{pmatrix}, \ \forall a, b, c, d \in \mathbb{R}.$$

This representation is an $\mathbb{R}$-algebra homomorphism. However there are many ways to represent $\mathbb{H}$ irreducibly in $\mathtt{M}_n(\mathbb{R})$ as an $\mathbb{R}$-algebra (cf. [3]), it is well-known that all those $\mathbb{R}$-algebra representations are equivalent to each other (cf. [13], Prop.(A)(i)). Yet there exists also a family of representations which are not $\mathbb{R}$-algebra homomorphisms. Consider a non-trivial field homomorphism $\sigma : \mathbb{R} \to \mathbb{C}$ which may not be the identity map on $\mathbb{R}$ (cf. [7], [16]). For each number $x \in \mathbb{R}$ denote the real and imaginary parts of $\sigma(x)$ by $\sigma(x)_r$ and $\sigma(x)_i$ respectively. Let $\rho_\sigma : \mathbb{H} \to \mathtt{M}_4(\mathbb{R})$ be a map defined by, for each $h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$ where $a, b, c, d \in \mathbb{R}$,

$$\rho_\sigma(h) = \begin{pmatrix} \sigma(a)_r - \sigma(b)_i & -\sigma(a)_i - \sigma(b)_r & -\sigma(c)_r + \sigma(d)_i & \sigma(c)_i + \sigma(d)_r \\ \sigma(a)_i + \sigma(b)_r & \sigma(a)_r - \sigma(b)_i & -\sigma(c)_i - \sigma(d)_r & -\sigma(c)_r + \sigma(d)_i \\ \sigma(c)_r + \sigma(d)_i & -\sigma(c)_i + \sigma(d)_r & \sigma(a)_r + \sigma(b)_i & -\sigma(a)_i + \sigma(b)_r \\ \sigma(c)_i - \sigma(d)_r & \sigma(c)_r + \sigma(d)_i & \sigma(a)_i - \sigma(b)_r & \sigma(a)_r + \sigma(b)_i \end{pmatrix} \tag{1}$$

It is routine to verify that ρ_σ is a well-defined irreducible matrix representation which is not an $\mathbb{R}$-algebra homomorphism unless σ is the identity map on $\mathbb{R}$ (see Eq. (13) and Prop.2.1). Evidently the family of those irreducible matrix representations of $\mathbb{H}$ is enormous, since there are infinite many homomorphisms between $\mathbb{R}$ and $\mathbb{C}$ and each of them induces a representation of $\mathbb{H}$ as above. The main purpose of this paper is to determine all irreducible real and complex matrix representations of $\mathbb{H}$.

To state the main theorem, we recall some notions. For a non-singular matrix $X \in \mathtt{GL}_n(\mathbb{R})$, denote by ι_X the inner automorphism of $\mathtt{M}_n(\mathbb{R})$ via the conjugation by X. Two representations ρ and ρ' of $\mathbb{H}$ with same degree n are said equivalent if there exists a non-singular matrix $X \in \mathtt{GL}_n(\mathbb{R})$ such that

$$\rho(h) = \iota_X \cdot \rho'(h) = X\rho'(h)X^{-1}, \ \forall h \in \mathbb{H}.$$

Denote by $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$ the set of field homomorphisms from $\mathbb{R}$ to $\mathbb{C}$. Two homomorphisms $\sigma, \sigma' \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$ are said equivalent if either $\sigma = \sigma'$ or $\sigma = \kappa\sigma'$, where κ is the complex conjugation of $\mathbb{C}$. This is clearly an equivalence relation of $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$. Our main result is as follows.

Theorem 1.

(I). *A non-trivial real matrix representation of* $\mathbb{H}$ *is irreducible if and only if its representation degree is 4.*

(II). *If* $\rho : \mathbb{H} \to \mathtt{M}_4(\mathbb{R})$ *is an irreducible representation of* $\mathbb{H}$*, there exist a* $\sigma \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$ *and a* $X \in \mathtt{GL}_4(\mathbb{R})$ *such that*

$$\rho = \iota_X \cdot \rho_\sigma \tag{2}$$

where ρ_σ *is the representation defined by (1) which is induced by* σ*. Moreover,* σ *is uniquely determined by* ρ *up to equivalence.*

(III). *Two irreducible real matrix representations* ρ *and* ρ' *of* $\mathbb{H}$ *are equivalent if and only if the field homomorphisms* σ *and* σ'*, determined by* ρ *and* ρ' *respectively, are equivalent.*

It follows from Theorem 1.(II) that an irreducible representation ρ is an $\mathbb{R}$-algebra homomorphism if and only if it is equivalent to ρ_σ where σ is the identity map on $\mathbb{R}$. Hence we obtain again the classical property that all irreducible $\mathbb{R}$-algebra matrix representations of $\mathbb{H}$ are equivalent. Meanwhile Theorem 1.(III) indicates that there exists a one-to-one correspondence between the equivalence classes of the irreducible real matrix representations of $\mathbb{H}$ and those of the field homomorphisms from $\mathbb{R}$ to $\mathbb{C}$ (see Cor. 4.1).

The proof of Theorem 1 relays on a few properties of some simple subalgebras of $\mathtt{M}_n(\mathbb{R})$, those properties are studied in the section 2. The property (I) of the theorem is proved by Proposition 2.1. A characterization of irreducible complex matrix representations of $\mathbb{H}$ (see Theorem 2) is developed in the section 3, which is essential for obtaining our main result. The proof of Theorem 1 is finalized in the section 4.

2. Preliminaries

Through out this section we fix a real matrix representation $\rho : \mathbb{H} \to \mathtt{M}_n(\mathbb{R})$ and study some related subalgebras of $\mathtt{M}_n(\mathbb{R})$ that plays an essential role for determining the representation. Denote by $\mathbb{R}[\rho(\mathbb{H})]$ the subagebra of $\mathtt{M}_n(\mathbb{R})$ generated by $\rho(\mathbb{H})$. Since the vector space $\mathbb{R}^n$ is a $\mathtt{M}_n(\mathbb{R})$-module in a natural way, it is also a $\mathbb{R}[\rho(\mathbb{H})]$-module by the restriction of coefficients.

Lemma 2.1. *A representation* $\rho : \mathbb{H} \to \mathtt{M}_n(\mathbb{R})$ *is irreducible if and only if* $\mathbb{R}^n$ *is a simple* $\mathbb{R}[\rho(\mathbb{H})]$*-module.*

Proof. This comes directly from the definitions of simple module and irreducible representation. □

Lemma 2.2. *If a subalgebra of* $\mathtt{M}_n(\mathbb{R})$ *has no zero-divisor, it is a division ring.*

Proof. Let A be a subalgebra of $\mathtt{M}_n(\mathbb{R})$ without zero-divisor. It is enough to show that every non-zero element of A is invertible with its inverse still in A. For an arbitrary non-zero matrix $X \in A$ consider its characteristic polynomial $\mathtt{det}(xI_n - X) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]$, where I_n is the $n \times n$ identity matrix. Then $(\sum_{i=1}^{n} a_i X^{i-1})X = -a_0 I_n$. Since X is not a zero-divisor, $a_0 \neq 0$ and we have

$$-\left(\sum_{i=1}^{n} a_0^{-1} a_i X^{i-1}\right) X = I_n.$$

Hence X is a non-singular matrix with its inverse $-(\sum_{i=1}^{n} a_0^{-1} a_i X^{i-1}) \in A$. □

Denote the center of $\mathbb{R}[\rho(\mathbb{H})]$ by C.

Lemma 2.3. *Let $\rho : \mathbb{H} \to \mathtt{M}_n(K)$ be an irreducible representation. The center C of $\mathbb{R}[\rho(\mathbb{H})]$ is a field.*

Proof. Since ρ is irreducible, $\mathbb{R}^n$ is a simple $\mathbb{R}[\rho(\mathbb{H})]$-module by Lemma 2.1. Let $\mathtt{End}_{\mathbb{R}[\rho(\mathbb{H})]}(\mathbb{R}^n)$ be the set of endomorphisms of $\mathbb{R}^n$ as a $\mathbb{R}[\rho(\mathbb{H})]$-module. Recall Schur's Lemma (cf. [11], p.28) which claims that the endomorphisms of a simple module form a division ring. Then C has no zero-divisor since it is contained in the division ring $\mathtt{End}_{\mathbb{R}[\rho(\mathbb{H})]}(\mathbb{R}^n)$. It follows from Lemma 2.2 that C has to be a division ring, which means that C is a field since it is a commutative ring. □

Corollary 2.1. *Let ρ and C be as above.*

(i). *ρ is an $\mathbb{R}$-algebra homomorphism if and only if $\rho(\mathbb{R}) = \mathbb{R}I_n$. In this case, $C = \mathbb{R}I_n$.*
(ii). *If ρ is not an $\mathbb{R}$-algebra homomorphism, then C is a field extension of $\mathbb{R}I_n$ with extension degree $[C : \mathbb{R}I_n] = 2$ and $C \cong \mathbb{C}$.*

Proof. The first assertion is obviously true. For the second assertion, if ρ is not an $\mathbb{R}$-algebra homomorphism, $\rho(\mathbb{R})$ is not contained in $\mathbb{R}I_n$. Then C is a non-trivial field extension of $\mathbb{R}I_n$ by Lemma 2.3. Since C is an $\mathbb{R}$-subalgebra of $\mathtt{M}_n(\mathbb{R})$, it is finite dimensional over $\mathbb{R}I_n$. Hence C is an algebraic field extension of $\mathbb{R}I_n$, which means that C has to be isomorphic to $\mathbb{C}$ since the latter is the unique non-trivial algebraic field extension over $\mathbb{R}$ up to isomorphism. Consequently $[C : \mathbb{R}I_n] = [\mathbb{C} : \mathbb{R}] = 2$. □

Lemma 2.4. *Let $\rho : \mathbb{H} \to \mathtt{M}_n(\mathbb{R})$ be an irreducible representation. Then $\mathbb{R}[\rho(\mathbb{H})]$ is a simple algebra. More over, if ρ is not an $\mathbb{R}$-algebra homomorphism, then*

$$\mathtt{dim}_{\mathbb{R}} \mathbb{R}[\rho(\mathbb{H})] = 8. \tag{3}$$

Proof. Since the center C of $\mathbb{R}[\rho(\mathbb{H})]$ is a field extension of $\rho(\mathbb{R})$ by Corollary 2.1 and $\rho(\mathbb{H})$ is a simple $\rho(\mathbb{R})$-algebra, the tensor product $C \otimes_{\rho(\mathbb{R})} \rho(\mathbb{H})$ is a simple C-algebra (cf. [11], p.226). Note that

$$\mathbb{R}[\rho(\mathbb{H})] = C[\rho(\mathbb{H})] = \left\{ \sum_{i=1}^{m} c_i \rho(h_i) | \forall c_i \in C, \forall h_i \in \mathbb{H}, m \in \mathbb{N}, 1 \leq i \leq m \right\}.$$

The map $\phi : C \otimes_{\rho(\mathbb{R})} \rho(\mathbb{H}) \to C[\rho(\mathbb{H})]$ defined by

$$\phi \left(\sum_{i=1}^{m} c_i \otimes \rho(h_i) \right) = \sum_{i=1}^{m} c_i \rho(h_i), \forall c_i \in C, \forall h_i \in \mathbb{H}, m \in \mathbb{N}, 1 \leq i \leq m$$

is a surjective C-algebra homomorphism. The simplicity of $C \otimes_{\rho(\mathbb{R})} \rho(\mathbb{H})$ implies that ϕ must be an isomorphism, which results in the simplicity of the algebra $C[\rho(\mathbb{H})]$. Moreover, if ρ is not an $\mathbb{R}$-algebra homomorphism, C is a two dimensional $\mathbb{R}$-subalgebra of $\mathbb{R}[\rho(\mathbb{H})]$ while

$$\mathtt{dim}_C C[\rho(\mathbb{H})] = \mathtt{dim}_C \ C \otimes_{\rho(\mathbb{R})} \rho(\mathbb{H}) = \mathtt{dim}_{\rho(\mathbb{R})} \rho(\mathbb{H}) = 4.$$

Therefore

$$\mathtt{dim}_{\mathbb{R}} \mathbb{R}[\rho(\mathbb{H})] = \mathtt{dim}_{\mathbb{R}} C[\rho(\mathbb{H})] = \mathtt{dim}_{\mathbb{R}} C \cdot \mathtt{dim}_C C[\rho(\mathbb{H})] = 2 \cdot 4 = 8. \quad \square$$

Corollary 2.2. *Let ρ be as above which is not an $\mathbb{R}$-algebra homomorphism. Then $\mathbb{R}[\rho(\mathbb{H})] \cong M_2(\mathbb{C})$.*

Proof. The isomorphism ϕ defined in the proof of the above lemma gives rise to an isomorphism between $\mathbb{R}[\rho(\mathbb{H})]$ and $C \otimes_{\rho(\mathbb{R})} \rho(\mathbb{H})$. Note that $\rho(\mathbb{H})$ is a quaternion algebra over the field $\rho(\mathbb{R})$ and that C is an algebraically closed field by Corollary 2.1. This implies that $C \otimes_{\rho(\mathbb{R})} \rho(\mathbb{H})$ is a quaternion algebra which is isomorphic to $\mathtt{M}_2(C)$ (cf. [15], §2.2.4). Hence we have an isomorphism between $\mathbb{R}[\rho(\mathbb{H})]$ and $M_2(\mathbb{C})$ since $\mathtt{M}_2(C)$ is isomorphic to $\mathtt{M}_2(\mathbb{C})$. There is also an alternative way to determine an isomorphism between $\mathbb{R}[\rho(\mathbb{H})]$ and $\mathtt{M}_2(\mathbb{C})$ through the algorithm of identifying the matrix ring for quaternion algebras (cf. [14], §4). $\square$

Proposition 2.1. *A matrix representation $\rho : \mathbb{H} \to \mathtt{M}_n(\mathbb{R})$ is irreducible if and only if $n = 4$.*

Proof. Suppose that ρ is irreducible. Then $\mathbb{R}[\rho(\mathbb{H})]$ is a simple subalgebra of the central simple $\mathbb{R}$-algebra $\mathtt{M}_n(\mathbb{R})$ by Lemma 2.4. Denote by $C_{\mathtt{M}_n(\mathbb{R})}(\mathbb{R}[\rho(\mathbb{H})])$ the centralizer of $\mathbb{R}[\rho(\mathbb{H})]$ in $\mathtt{M}_n(\mathbb{R})$. Recall the double centralizer theorem (cf. [11], p.232) which claims that the dimension of a central simple algebra is equal to the product of the dimension of a simple subalgebra and the dimension of its centralizer. We have

$$\mathtt{dim}_{\mathbb{R}}\mathbb{R}[\rho(\mathbb{H})] \cdot \mathtt{dim}_{\mathbb{R}} C_{\mathtt{M}_n(\mathbb{R})}(\mathbb{R}[\rho(\mathbb{H})]) = \mathtt{dim}_{\mathbb{R}}\mathtt{M}_n(\mathbb{R}) = n^2. \tag{4}$$

Since $C_{\mathtt{M}_n(\mathbb{R})}(\mathbb{R}[\rho(\mathbb{H})])$ is contained in $\mathtt{End}_{\mathbb{R}[\rho(\mathbb{H})]}\mathbb{R}^n$ which is a division ring by Schur's Lemma, it is an integral domain. Hence by Lemma 2.2, $C_{\mathtt{M}_n(\mathbb{R})}(\mathbb{R}[\rho(\mathbb{H})])$ is a division $\mathbb{R}$-algebra. Note that by Frobenius' real division algebra classification theorem (cf. [5], [10]) the existing finite-dimension division $\mathbb{R}$-algebras, up to an isomorphism, are just $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Hence the dimension of $C_{\mathtt{M}_n(\mathbb{R})}(\mathbb{R}[\rho(\mathbb{H})])$ over $\mathbb{R}$ can be just 1, 2 or 4. If ρ is an $\mathbb{R}$-algebra homomorphism, then $\mathbb{R}[\rho(\mathbb{H})] = \rho(\mathbb{H})$. In this case, the identity (4) turns to

$$\mathtt{dim}_{\mathbb{R}}\rho(\mathbb{H}) \cdot \mathtt{dim}_{\mathbb{R}} C_{\mathtt{M}_n(\mathbb{R})}(\rho(\mathbb{H})) = 4 \cdot \mathtt{dim}_{\mathbb{R}} C_{\mathtt{M}_n(\mathbb{R})}(\rho(\mathbb{H})) = n^2 \tag{5}$$

which means that the dimension of $C_{\mathtt{M}_n(\mathbb{R})}(\rho(\mathbb{H}))$ has to be either 1 or 4. However, it is obvious that $\mathtt{dim}_{\mathbb{R}} C_{\mathtt{M}_n(\mathbb{R})}(\rho(\mathbb{H})) \neq 1$ because, otherwise, the above identity implies that $n = 2$ and ρ would become an isomorphism between $\mathbb{H}$ and $\mathtt{M}_2(\mathbb{R})$ which is false. Hence

$$\mathtt{dim}_{\mathbb{R}} C_{\mathtt{M}_n(\mathbb{R})}(\rho(\mathbb{H})) = 4.$$

Then it follows again from the identity (5) that $n = 4$. Now if ρ is not an $\mathbb{R}$-algebra homomorphism, then by Lemma 2.4 the identity (4) turns to

$$8 \cdot \mathtt{dim}_{\mathbb{R}} C_{\mathtt{M}_n(\mathbb{R})}(\mathbb{R}[\rho(\mathbb{H})]) = n^2$$

In this case, the dimension of $C_{\mathtt{M}_n(\mathbb{R})}(\mathbb{R}[\rho(\mathbb{H})])$ has to be 2 and therefore $n = 4$.

On the other hand, given a representation $\rho : \mathbb{H} \to \mathtt{M}_4(\mathbb{R})$, note that $\mathbb{R}^4$ is an Artinian $\mathbb{R}[\rho(\mathbb{H})]$-module and that it contains a non-trivial simple submodule, say $V \subseteq \mathbb{R}^4$. Then V induces an irreducible subrepresentation of ρ with the representation degree equal to $\mathtt{dim}_{\mathbb{R}}V$. The above discussion on the degree of irreducible representation confirms that $\mathtt{dim}_{\mathbb{R}}V = 4$. This means that $V = \mathbb{R}^4$, therefore ρ is irreducible. □

3. Complex matrix representations of $\mathbb{H}$

The irreducible complex matrix representations of $\mathbb{H}$, apart from their own significance and applications, are essential for us to determine the real matrix representations. In this section we study those complex matrix representations by classifying them up to equivalence.

Proposition 3.1. *A complex matrix representation $\varrho : \mathbb{H} \to \mathtt{M}_n(\mathbb{C})$ of $\mathbb{H}$ is irreducible if and only if $n = 2$.*

Proof. Denote by $\mathbb{C}[\varrho(\mathbb{H})]$ the $\mathbb{C}$-subalgebra of $\mathtt{M}_n(\mathbb{C})$ generated by $\varrho(\mathbb{H})$. Note that the vector space $\mathbb{C}^n$ is a $\mathbb{C}[\varrho(\mathbb{H})]$-module in a natural way, which is a simple module if and only if ϱ is an irreducible representation. Suppose that ϱ is irreducible. It follows from Schur's Lemma

$$\mathtt{End}_{\mathbb{C}[\varrho(\mathbb{H})]}\mathbb{C}^n = \mathbb{C}I_n \tag{6}$$

Since $\varrho(\mathbb{R})$ is contained in the center of $\mathbb{C}[\varrho(\mathbb{H})]$ which, in its turn, contained in $\mathtt{End}_{\mathbb{C}[\varrho(\mathbb{H})]}\mathbb{C}^n$, we obtain that

$$\varrho(\mathbb{R}) \subseteq \mathbb{C}I_n. \tag{7}$$

This implies that $\mathbb{C} \otimes_{\varrho(\mathbb{R})} \varrho(\mathbb{H})$ is a central simple $\mathbb{C}$-algebra. Consider a map $\varphi : \mathbb{C} \otimes_{\varrho(\mathbb{R})} \varrho(\mathbb{H}) \to \mathbb{C}[\varrho(\mathbb{H})]$ defined by

$$\varphi(\sum_{i=1}^{m} c_i \otimes \varrho(h_i)) = \sum_{i=1}^{m} c_i \varrho(h_i), \forall c_i \in \mathbb{C}, \forall h_i \in \mathbb{H}, m \in \mathbb{N}, 1 \leq i \leq m$$

which is a surjective $\mathbb{C}$-algebra homomorphism. The simplicity of $\mathbb{C} \otimes_{\varrho(\mathbb{R})} \varrho(\mathbb{H})$ implies that φ is an isomorphism and therefore $\mathbb{C}[\varrho(\mathbb{H})]$ is a simple subalgebra of $\mathtt{M}_n(\mathbb{C})$. It follows from the double centralizer theorem of central simple algebra (cf. [11], p.232) that

$$\mathtt{dim}_{\mathbb{C}}\mathbb{C}[\varrho(\mathbb{H})] \cdot \mathtt{dim}_{\mathbb{C}} C_{\mathtt{M}_n(\mathbb{C})}(\mathbb{C}[\varrho(\mathbb{H})]) = \mathtt{dim}_{\mathbb{C}}\mathtt{M}_n(\mathbb{C}) = n^2, \tag{8}$$

where $C_{\mathtt{M}_n(\mathbb{C})}(\mathbb{C}[\varrho(\mathbb{H})])$ is the centralizer of $\mathbb{C}[\varrho(\mathbb{H})]$ in $\mathtt{M}_n(\mathbb{C})$. Note that

$$\mathbb{C}I_n \subseteq C_{\mathtt{M}_n(\mathbb{C})}(\mathbb{C}[\varrho(\mathbb{H})]) \subseteq \mathtt{End}_{\mathbb{C}[\varrho(\mathbb{H})]}\mathbb{C}^n.$$

Then the identity (6) implies that $\mathtt{dim}_{\mathbb{C}} C_{\mathtt{M}_n(\mathbb{C})}(\mathbb{C}[\varrho(\mathbb{H})]) = 1$. More over, the $\mathbb{C}$-algebra isomorphism φ implies that

$$\mathtt{dim}_{\mathbb{C}}\mathbb{C}[\varrho(\mathbb{H})] = \dim_{\mathbb{C}} \mathbb{C} \otimes_{\varrho(\mathbb{R})} \varrho(\mathbb{H}) = 4$$

Thus the identity (8) implies that $n = 2$.

It is obvious, on the other hand, that every non-trivial 2-dimensional complex matrix representation of $\mathbb{H}$ has to be irreducible since there exists no 1-dimensional irreducible subrepresentation for $\mathbb{H}$. □

Proposition 3.2. *Let $\varrho : \mathbb{H} \to \mathtt{M}_2(\mathbb{C})$ and $\varrho' : \mathbb{H} \to \mathtt{M}_2(\mathbb{C})$ be two complex representations of $\mathbb{H}$. Denote by $\varrho|_{\mathbb{R}}$ and $\varrho'|_{\mathbb{R}}$ the restrictions of ϱ and ϱ' on $\mathbb{R}$ respectively. If $\varrho|_{\mathbb{R}} = \varrho'|_{\mathbb{R}}$, then there exists a matrix $Z \in \mathtt{GL}_2(\mathbb{C})$ such that $\varrho' = \iota_Z \varrho$.*

Proof. Since both ϱ and ϱ' are irreducible by Proposition 3.1, it follows from (7) that both $\varrho(\mathbb{R})$ and $\varrho'(\mathbb{R})$ are contained in $\mathbb{C}I_2$. The assumption that $\varrho|_{\mathbb{R}} = \varrho'|_{\mathbb{R}}$ gives rise to a surjective $\mathbb{C}$-algebra homomorphism $\psi : \mathbb{C} \otimes_{\varrho(\mathbb{R})} \varrho(\mathbb{H}) \to \mathbb{C} \otimes_{\varrho'(\mathbb{R})} \varrho'(\mathbb{H})$ defined by

$$\psi\left(\sum_{i=1}^{m} z_i \otimes \varrho(h_i)\right) = \sum_{i=1}^{m} z_i \otimes \varrho'(h_i),\ \forall z_i \in \mathbb{C}, h_i \in \mathbb{H}, m \in \mathbb{N}, 1 \leq i \leq m$$

which has to be an isomorphism since the algebras involved are simple. Let $\varphi : \mathbb{C} \otimes_{\varrho(\mathbb{R})} \varrho(\mathbb{H}) \to \mathbb{C}[\varrho(\mathbb{H})]$ be a map defined by

$$\varphi\left(\sum_{i=1}^{m} z_i \otimes \varrho(h_i)\right) = \sum_{i=1}^{m} z_i \varrho(h_i),\ \forall z_i \in \mathbb{C}, h_i \in \mathbb{H}, m \in \mathbb{N}, 1 \leq i \leq m$$

and let $\varphi' : \mathbb{C} \otimes_{\varrho'(\mathbb{R})} \varrho'(\mathbb{H}) \to \mathbb{C}[\varrho'(\mathbb{H})]$ be a map defined by

$$\varphi'\left(\sum_{i=1}^{m} z_i \otimes \varrho'(h_i)\right) = \sum_{i=1}^{m} z_i \varrho'(h_i),\ \forall z_i \in \mathbb{C}, h_i \in \mathbb{H}, m \in \mathbb{N}, 1 \leq i \leq m.$$

Obviously both φ and φ' are $\mathbb{C}$-algebra isomorphisms. Together with ψ, they induce a map $\psi' : C[\varrho(\mathbb{H})] \to C[\varrho'(\mathbb{H})]$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{C} \otimes_{\varrho(\mathbb{R})} \varrho(\mathbb{H}) & \xrightarrow{\psi} & \mathbb{C} \otimes_{\varrho'(\mathbb{R})} \varrho'(\mathbb{H}) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ \mathbb{C}[\varrho(\mathbb{H})] & \xrightarrow{\psi'} & \mathbb{C}[\varrho'(\mathbb{H})] \end{array}$$

In particular, ψ' is a $\mathbb{C}$-algebra isomorphism since it is a composition of other tree $\mathbb{C}$-algebra isomorphisms. Recall Noether-Skolem Theorem (cf. [11], p.230) which claims that every isomorphism between two simple subalgebras of a central simple algebra comes from the restriction of an inner automorphism of the algebra. Since both $\mathbb{C}[\varrho(\mathbb{H})]$ and $\mathbb{C}[\varrho'(\mathbb{H})]$ are simple subalgebras of $\mathtt{M}_2(\mathbb{C})$, ψ' must come from an inner automorphism of $\mathtt{M}_2(\mathbb{C})$. Thus there exists a $Z \in \mathtt{GL}_2(\mathbb{C})$ such that

$$\psi'(x) = ZxZ^{-1}, \forall x \in \mathbb{C}[\varrho(\mathbb{H})].$$

Following the above commutative diagram we have

$$\varrho'(h) = \psi'\varrho(h) = Z\varrho(h)Z^{-1} = \iota_Z \cdot \varrho\ (h), \forall h \in \mathbb{H}.$$

Thus $\varrho' = \iota_Z \cdot \varrho$. □

A particular case of the complex matrix representations of $\mathbb{H}$ is where the representation is an $\mathbb{R}$-algebra homomorphism. Recall a canonical complex matrix representation $\varrho_{\mathbb{C}} : \mathbb{H} \to \mathtt{M}_2(\mathbb{C})$ defined by

$$\varrho_{\mathbb{C}}(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = \begin{pmatrix} a + b\mathbf{i} & -c - d\mathbf{i} \\ c - d\mathbf{i} & a - b\mathbf{i} \end{pmatrix},\ \forall a, b, c, d \in \mathbb{R}.$$

This is a $\mathbb{R}$-algebra matrix representation indeed. It is known that the $\mathbb{R}$-algebra representations of $\mathbb{H}$ in $\mathtt{M}_2(\mathbb{C})$ are equivalent to each other (cf. [8], p.197). As a direct consequence of the Propositions 3.1 and 3.2, we obtain this classical property again.

Corollary 3.1. *Up to equivalence, there exists a unique irreducible complex matrix representation for $\mathbb{H}$ as an $\mathbb{R}$-algebra, that is $\varrho_{\mathbb{C}}$.*

There are a lot of irreducible complex matrix representations of $\mathbb{H}$ which are not equivalent to $\varrho_{\mathbb{H}}$. For instance, consider a non-trivial endomorphism $\sigma \in \mathtt{End}(\mathbb{C})$ and define a map $\varrho_\sigma : \mathbb{H} \to \mathtt{M}_2(\mathbb{C})$ by

$$\varrho_\sigma(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = \begin{pmatrix} \sigma(a + b\mathbf{i}) & -\sigma(c + d\mathbf{i}) \\ \sigma(c - d\mathbf{i}) & \sigma(a - b\mathbf{i}) \end{pmatrix}, \ \forall a, b, c, d \in \mathbb{R}. \tag{9}$$

It is obvious that ϱ_σ is an irreducible representation which is not equivalent to $\varrho_{\mathbb{C}}$ unless the restriction of σ on $\mathbb{R}$ is the identity map of $\mathbb{R}$.

Theorem 2. *Let $\varrho : \mathbb{H} \to \mathtt{M}_2(\mathbb{C})$ be a complex matrix representation of $\mathbb{H}$, then there exist a $\sigma \in \mathtt{End}(\mathbb{C})$ and a $Z \in \mathtt{GL}_2(\mathbb{C})$ such that*

$$\varrho = \iota_Z \cdot \varrho_\sigma,$$

where ι_Z is the inner automorphism of $\mathtt{M}_2(\mathbb{C})$ via conjugation by Z.

Proof. Since ϱ is irreducible by Proposition 3.1, it follows from (7) that for each $a \in \mathbb{R}$ there is a $a_\varrho \in \mathbb{C}$ such that $\varrho(a) = a_\varrho I_2$. This induces a homomorphism $\sigma_\varrho : \mathbb{R} \to \mathbb{C}$ defined by

$$\sigma_\varrho(a) = a_\varrho, \forall a \in \mathbb{R}.$$

Extending σ_ϱ to an endomorphism σ of $\mathbb{C}$, we have for all $a \in \mathbb{R}$

$$\varrho(a) = \begin{pmatrix} \sigma(a) & 0 \\ 0 & \sigma(a) \end{pmatrix} = \varrho_\sigma(a)$$

where ϱ_σ is the matrix representation defined by (9). Then it follows from Proposition 3.2 that there exists a $Z \in \mathtt{GL}_2(\mathbb{C})$ such that $\varrho = \iota_Z \cdot \varrho_\sigma$. □

4. Proof of Theorem 1

The following property of simple algebra is needed for proving Theorem 1.

Proposition 4.1. *Let R and R' be isomorphic Artinian simple rings. Suppose that the center $C(R)$ of R is algebraically closed. Then every ring homomorphism from $C(R)$ to* *the center $C(R')$ of R' can be extended to a homomorphism from R to R'. In particular, if the homomorphism between the two centers is an isomorphism, then the extended homomorphism is an isomorphism between R and R'.*

Proof. Note that $C(R)$ and $C(R')$ are isomorphic to each other since they are the centers of two isomorphic rings. Hence $C(R')$ is algebraically closed since so is $C(R)$. It follows from Wedderburn-Artin theorem (cf. [11], P. 50) that there exist isomorphisms $\alpha : R \cong \mathtt{M}_m(C(R))$ and $\alpha' : R' \cong \mathtt{M}_m(C(R'))$ for some $m \in \mathbb{N}$. If $\beta : C(R) \to C(R')$ is a ring homomorphism, we can extend β to a homomorphism $\tilde{\beta} : \mathtt{M}_m(C(R)) \to \mathtt{M}_m(C(R'))$ defined by, for $(c_{ij}) \in \mathtt{M}_m(C(R))$,

$$\tilde{\beta}((c_{ij})) = (\beta(c_{ij})) \in \mathtt{M}_m(C(R')),\ \forall c_{ij} \in C(R),\ 1 \le i \le m, 1 \le j \le m.$$

Then the composition $\alpha'^{-1}\tilde{\beta}\alpha : R \to R'$ is an extension of β as required. Obviously if β is an isomorphism, so is $\tilde{\beta}$. □

The proof of Theorem 1.(I) is achieved by Proposition 2.1. We prove in the following the properties (II) and (III) of the theorem.

The poof of Theorem 1.(II). Suppose first that ρ is not an $\mathbb{R}$-algebra homomorphism. Note that by Corollary 2.2 $\mathbb{R}[\rho(\mathbb{H})]$ and $\mathtt{M}_2(\mathbb{C})$ are isomorphic finite-dimension central simple algebras with algebraically closed centers C and $\mathbb{C}I_2$ respectively. Obviously $\mathbb{R}I_4 \subseteq C$ and there exists a field isomorphism $\alpha : C \to \mathbb{C}I_2$ satisfying

$$\alpha(aI_4) = aI_2 \in \mathtt{M}_2(\mathbb{C}),\ \forall a \in \mathbb{R}. \tag{10}$$

It follows from Proposition 4.1, that α can be extended to a ring isomorphism $\tilde{\alpha} : \mathbb{R}[\rho(\mathbb{H})] \to \mathtt{M}_2(\mathbb{C})$. Since $\rho(\mathbb{H}) \subseteq \mathbb{R}[\rho(\mathbb{H})]$, considering ρ as a ring homomorphism from $\mathbb{H}$ to $\mathbb{R}[\rho(\mathbb{H})]$ we obtain a complex matrix representation $\tilde{\alpha} \cdot \rho : \mathbb{H} \to \mathtt{M}_2(\mathbb{C})$, which is irreducible by Proposition 3.1. Then Theorem 2 asserts that there exist a $\sigma \in \mathtt{End}(\mathbb{C})$ and a matrix $Z \in \mathtt{GL}_2(\mathbb{C})$ such that $\tilde{\alpha} \cdot \rho = \iota_Z \cdot \varrho_\sigma$. In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\varrho_\sigma} & \mathtt{M}_2(\mathbb{C}) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\iota_Z} \\ \mathbb{R}[\rho(\mathbb{H})] & \xrightarrow{\tilde{\alpha}} & \mathtt{M}_2(\mathbb{C}) \end{array} \tag{11}$$

Let $\mu : \mathbb{C} \to \mathtt{M}_2(\mathbb{R})$ be the canonical real matrix representation of $\mathbb{C}$ which is defined by

$$\mu(a + b\mathbf{i}) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \forall a, b \in \mathbb{R},$$

and let $\tilde{\mu} : \mathtt{M}_2(\mathbb{C}) \to \mathtt{M}_4(\mathbb{R})$ be an extension of μ defined by

$$\tilde{\mu}\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} \mu(z_1) & \mu(z_2) \\ \mu(z_3) & \mu(z_4) \end{pmatrix}, \forall z_i \in \mathbb{C}, 1 \leq i \leq 4.$$

Obviously $\tilde{\mu}$ is an $\mathbb{R}$-algebra homomorphism. Hence $\tilde{\mu}(\mathtt{M}_2(\mathbb{C}))$ is a subalgebra of $\mathtt{M}_4(\mathbb{R})$ which is $\mathbb{R}$-isomorphic to $\mathtt{M}_2(\mathbb{C})$. Note that $\tilde{\alpha}^{-1}(\mathtt{M}_2(\mathbb{C}))$ is also a subalgebra which is isomorphic to $\mathtt{M}_2(\mathbb{C})$ since $\tilde{\alpha}$ is an $\mathbb{R}$-algebra isomorphism by the identity (10). Thus $\tilde{\mu}(\mathtt{M}_2(\mathbb{C}))$ and $\tilde{\alpha}^{-1}(\mathtt{M}_2(\mathbb{C}))$ are isomorphic simple subalgebras of $\mathbb{M}_4(\mathbb{R})$. It follows from Noether-Skolem Theorem (cf. [11], p. 230) that there exists a non-singular matrix $Y \in \mathtt{GL}_4(\mathbb{R})$ such that $\tilde{\alpha}^{-1}(T) = Y\mu(T)Y^{-1}$ for all $T \in \mathtt{M}_2(\mathbb{C})$. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathtt{M}_2(\mathbb{C}) & = & \mathtt{M}_2(\mathbb{C}) \\ \tilde{\alpha}^{-1}\downarrow & & \downarrow\tilde{\mu} \\ \mathtt{M}_4(\mathbb{R}) & \xrightarrow{\iota_Y} & \mathtt{M}_4(\mathbb{R}) \end{array}$$

Combining this diagram with the diagram (11), we build a commutative diagram

$$\begin{array}{ccccc} \mathbb{H} & \xrightarrow{\varrho_\sigma} & \mathtt{M}_2(\mathbb{C}) & & \\ \rho\downarrow & & \downarrow\iota_Z & & \\ \mathbb{R}[\rho(\mathbb{H})] & \xrightarrow{\tilde{\alpha}} & \mathtt{M}_2(\mathbb{C}) & = & \mathtt{M}_2(\mathbb{C}) \\ \iota\downarrow & & \downarrow\tilde{\alpha}^{-1} & & \downarrow\tilde{\mu} \\ \mathtt{M}_4(\mathbb{R}) & = & \mathtt{M}_4(\mathbb{R}) & \xleftarrow[\iota_Y]{} & \mathtt{M}_4(\mathbb{R}) \end{array}$$

where ι is the immersion map. Therefore, for all $h \in \mathbb{H}$,

$$\rho(h) = \iota_Y \cdot \tilde{\mu} \cdot \iota_Z \cdot \varrho_\sigma(h) = \iota_Y \cdot \iota_{\tilde{\mu}(Z)} \cdot \tilde{\mu} \cdot \varrho_\sigma(h) = \iota_X \cdot \tilde{\mu} \cdot \varrho_\sigma(h) \tag{12}$$

where $X = Y\tilde{\mu}(Z)$. For the endomorphism σ of $\mathbb{C}$, we denote its restriction on $\mathbb{R}$ still by σ without any confusion and let ρ_σ be the real matrix representation defined by (1). Note that $\sigma(\mathbf{i}) = \pm\mathbf{i}$. It is a routine to verify that, if $\sigma(\mathbf{i}) = \mathbf{i}$,

$$\tilde{\mu} \cdot \varrho_\sigma = \rho_\sigma. \tag{13}$$

In this case, we have an equation that $\rho = \iota_X \cdot \rho_\sigma$ by (12). If $\sigma(\mathbf{i}) = -\mathbf{i}$, then

$$\tilde{\mu} \cdot \varrho_{\sigma\kappa} = \rho_\sigma$$

where κ is the complex conjugation of $\mathbb{C}$. Note that the restrictions of ϱ_σ and $\varrho_{\sigma\kappa}$ on $\mathbb{R}$ are equal, consequently there is a non-singular matrix $Z \in \mathtt{GL}_2(\mathbb{C})$ such that $\varrho_\sigma = \iota_Z \cdot \varrho_{\sigma\kappa}$ by Proposition 3.2. Then it follows from (12) that

$$\rho = \iota_X \cdot \tilde{\mu} \cdot \varrho_\sigma = \iota_X \cdot \tilde{\mu} \cdot \iota_Z \cdot \varrho_{\sigma\kappa} = \iota_{X'} \cdot \rho_\sigma$$

where $X' = X\tilde{\mu}(Z)$. Thus for both cases we obtain the identity (2).

If ρ is a $\mathbb{R}$-algebra homomorphism, then $\rho(\mathbb{H})$ and $\tilde{\mu} \cdot \varrho_{\mathbb{C}}(\mathbb{H})$ are isomorphic simple $\mathbb{R}$-subalgebras of $\mathtt{M}_4(\mathbb{R})$. In this case we obtain again by Noether-Skolem Theorem that

$$\rho = \iota_X \cdot \tilde{\mu} \cdot \varrho_{\mathbb{C}} = \iota_X \cdot \rho_\sigma$$

for a non-singular matrix $X \in \mathtt{GL}_4(\mathbb{R})$, where σ is the identity map of $\mathbb{R}$.

Finally, for each irreducible representation ρ, the uniqueness of the homomorphism $\sigma \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$ determined by ρ up to equivalence is a direct consequence of the following proposition. $\square$

Proposition 4.2. *Let σ and σ' be non-trivial homomorphisms from $\mathbb{R}$ to $\mathbb{C}$. Then ρ_σ and $\rho_{\sigma'}$ are equivalent if and only if σ and σ' are equivalent.*

Proof. Suppose that σ and σ' are equivalent. If $\sigma \neq \sigma'$, then $\sigma = \kappa\sigma'$. Hence for an arbitrary $a \in \mathbb{R}$,

$$\sigma(a)_r = \sigma'(a)_r,\ \sigma(a)_i = -\sigma'(a)_i.$$

This leads to a matrix equation

$$\rho_\sigma(a) = K\rho_{\sigma'}(a)K^{-1},\ \forall a \in \mathbb{R}$$

where

$$K = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathtt{M}_4(\mathbb{R}).$$

We also have

$$\rho_\sigma(\mathbf{i}) = K\rho_{\sigma'}(\mathbf{i})K^{-1}$$

and

$$\rho_\sigma(\mathbf{j}) = K\rho_{\sigma'}(\mathbf{j})K^{-1}.$$

Since $\mathbb{H}$ is generated by $\mathbb{R}, \mathbf{i}$ and $\mathbf{j}$, above equations imply that $\rho_\sigma = \iota_K \cdot \rho_{\sigma'}$. Thus ρ_σ and $\rho_{\sigma'}$ are equivalent.

On the other hand, if ρ_σ and $\rho_{\sigma'}$ are equivalent, there exists a non-singular matrix $X \in \mathtt{GL}_4(\mathbb{R})$ such that

$$X\rho_\sigma(h) = \rho_{\sigma'}(h)X, \ \forall h \in \mathbb{H}. \tag{14}$$

By writing $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ where $X_i \in \mathtt{M}_2(\mathbb{R})$ for $1 \le i \le 4$, we have

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} = X\rho_\sigma(\mathbf{j}) = \rho_{\sigma'}(\mathbf{j})X = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

This results in

$$X_1 = X_4, \ X_2 = -X_3.$$

Denoting by Σ_a and Σ'_a the matrices $\begin{pmatrix} \sigma(a)_r & -\sigma(a)_i \\ \sigma(a)_i & \sigma(a)_r \end{pmatrix}$ and $\begin{pmatrix} \sigma'(a)_r & -\sigma'(a)_i \\ \sigma'(a)_i & \sigma'(a)_r \end{pmatrix}$ respectively, we have by (14) that for all $a \in \mathbb{R}$

$$\begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \begin{pmatrix} \Sigma_a & 0 \\ 0 & \Sigma_a \end{pmatrix} = X\rho_\sigma(a) = \rho_{\sigma'}(a)X = \begin{pmatrix} \Sigma'_a & 0 \\ 0 & \Sigma'_a \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix}.$$

This implies that

$$X_1\Sigma_a = \Sigma'_a X_1, \tag{15}$$

and

$$X_2\Sigma_a = \Sigma'_a X_2. \tag{16}$$

Moreover, by writing $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have

$$\begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} = X\rho_\sigma(\mathbf{i}) = \rho_{\sigma'}(\mathbf{i})X = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix}.$$

This yields two matrix equations: $X_1 D = DX_1$ and $X_2 D = -DX_2$. The first equation implies that

$$X_1 = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \tag{17}$$

while the second equation implies that

$$X_2 = \begin{pmatrix} s & t \\ t & -s \end{pmatrix} \tag{18}$$

for some $x, y, s, t \in \mathbb{R}$. Note that

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} \sigma(a)_r & -\sigma(a)_i \\ \sigma(a)_i & \sigma(a)_r \end{pmatrix} = \begin{pmatrix} \sigma(a)_r & -\sigma(a)_i \\ \sigma(a)_i & \sigma(a)_r \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \forall a \in \mathbb{R}.$$

If X_1 is not a zero matrix, then by (17) it has to be invertible and the equation (15) gives rise to an identity

$$\begin{pmatrix} \sigma(a)_r & -\sigma(a)_i \\ \sigma(a)_i & \sigma(a)_r \end{pmatrix} = \begin{pmatrix} \sigma'(a)_r & -\sigma'(a)_i \\ \sigma'(a)_i & \sigma'(a)_r \end{pmatrix}, \forall a \in \mathbb{R}$$

which implies that $\sigma = \sigma'$. If X_1 is the zero-matrix, since X is non-singular, X_2 must be non-zero and therefore it has to be non-singular by (18). Hence the equation (16) becomes

$$\begin{pmatrix} s & t \\ t & -s \end{pmatrix} \begin{pmatrix} \sigma(a)_r & -\sigma(a)_i \\ \sigma(a)_i & \sigma(a)_r \end{pmatrix} \begin{pmatrix} s & t \\ t & -s \end{pmatrix}^{-1} = \begin{pmatrix} \sigma'(a)_r & -\sigma'(a)_i \\ \sigma'(a)_i & \sigma'(a)_r \end{pmatrix}, \forall a \in \mathbb{R}.$$

However, a direct matrix calculation also results in

$$\begin{pmatrix} s & t \\ t & -s \end{pmatrix} \begin{pmatrix} \sigma(a)_r & -\sigma(a)_i \\ \sigma(a)_i & \sigma(a)_r \end{pmatrix} \begin{pmatrix} s & t \\ t & -s \end{pmatrix}^{-1} = \begin{pmatrix} \kappa\sigma(a)_r & -\kappa\sigma(a)_i \\ \kappa\sigma(a)_i & \kappa\sigma(a)_r \end{pmatrix}.$$

Those two equations imply that $\sigma' = \kappa\sigma$. Thus we obtain that σ and σ' are equivalent. □

In particular, if a matrix representation ρ has different decomposition

$$\rho = \iota_X \cdot \rho_\sigma = \iota_{X'} \cdot \rho_{\sigma'}$$

then σ and σ' are equivalent by Proposition 4.2. Hence we obtain the uniqueness property of Theorem 1.(II). The property (III) of Theorem 1 is a direct consequence of Proposition 4.2. Thus we complete the proof of Theorem 1.

The following corollary is an immediate consequence of Theorem 1, it indicates how huge is the family of the irreducible real matrix representations of $\mathbb{H}$ (cf. [2], §3).

Corollary 4.1. *There exists a one-to-one correspondence between the equivalence classes of the irreducible real matrix representation of $\mathbb{H}$ and those of the field homomorphisms from $\mathbb{R}$ to $\mathbb{C}$.*

Remark. In [3] and [6] there is a concrete calculation of total 48 $\mathbb{R}$-algebra representations of $\mathbb{H}$ by computing distinct bases for quaternion subalgebras of $\mathtt{M}_4(\mathbb{R})$. Each of those bases consists of 3 ordered skew-symmetric signed permutation matrices, which are possible images of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ under a representation. A similar calculation is also applicable to each equivalence class of irreducible representations of $\mathbb{H}$ since, for an arbitrary $\sigma \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$, the imagines of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ under ρ_σ are also skew-symmetric signed permutation matrices.

We observe that the method used for obtaining Theorem 1 can be extended to the study for the matrix representations of complex quaternions with a few suitable adjustments (cf. [4], [12]), however a detailed discussion is beyond the content of this paper.

Among various applications of the matrix representations of the quaternions we mention its fundamental role in the studies of quaternion matrices and their equations (cf. [9]). Certainly a complete classification for the irreducible representations, as described by Theorem 1 and Theorem 2, provides a powerful tool for those studies.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

References

[1] A. Cayley, A memoir on the theory of matrices (1776-1886), Philos. Trans. R. Soc. Lond. 148 (1858) 17–37.
[2] Y. Chen, Matrix representations of real numbers, Linear Algebra Appl. 536 (2018) 174–185.
[3] R.W. Farebrother, J. Groß, S. Troschke, Marix representaion of quaternions, Linear Algebra Appl. 362 (2003) 251–255.
[4] C. Flaut, V. Shpakivskyi, Real matrix representations for the complex quaternions, Adv. Appl. Clifford Algebras 23 (2013) 657–671.
[5] F.G. Frobenius, Über lineare substitutionen und bilineare formen, J. Reine Angew. Math. 84 (1878) 1–63.
[6] J. Groß, G. Trenkler, S. Troschke, Quaternions: further contributions to a matrix oriented approach, Linear Algebra Appl. 326 (2001) 205–213.
[7] H. Kestelman, Automorphisms of the field of complex numbers, Proc. Lond. Math. Soc. 53 (1951) 1–12.
[8] M. Koecher, R. Remmert, Hamilton's quaternions, in: H. Ebbinghause, et al. (Eds.), Numbers, Springer-Verlag, New-York, 1991, pp. 189–220.
[9] X. Liu, Y. Zhang, Matrices over quaternion algebras, in: M. Moslehian (Ed.), Matrix and Operator Equations and Applications, Springer, Cham, 2023, pp. 139–183.
[10] R. Palais, The classification of real division algebras, Am. Math. Mon. 75 (1968) 366–368.
[11] R. Pierce, Associative Algebra, GTM, vol. 88, Springer-Verlag, New York, 1982.
[12] S. Sangwine, T. Elle, N. Bihan, Fundamental representations and algebraic properties of biquaternions or complexified quaternions, Adv. Appl. Clifford Algebras 21 (2011) 607–636.
[13] A. Trautman, Clifford algebras and their representations, in: Francoise, et al. (Eds.), Encyclopedia of Mathematical Physics, vol. 1, Elsevier, Oxford, 2006, pp. 518–530.
[14] J. Voight, Identifying the matrix ring: algorithms for quaternion algebras and quadratic forms, Quadratic and higher degree forms, Dev. Math. 31 (2013) 255–298.
[15] J. Voight, Quaternion Algebras, GTM, vol. 288, Springer, 2021.
[16] P. Yale, Automorphisms of the complex numbers, Math. Mag. 39 (1966) 135–141.