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Game-theoretic models for auctions in electricity markets

By

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Declaration

I hereby declare that the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

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January 26, 2024

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To my parents

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Abstract

In electricity markets, a significant challenge arises from the inability to store electricity on a substantial scale, leading to the dispatching problem, i.e., the continuous management of energy flows within the transmission network. Consequently, these markets adopt a specific structure: most electricity is sold in advance through the day-ahead market, that typically operates as a uniform-price auction, while lower volumes are traded in the ancillary services market to address congestion, reserve margins, and real-time balancing, typically employing a pay-as-bid auction.

In this thesis, we introduce and analyze the pay-as-bid auction game, a supply function model with discriminatory pricing and asymmetric firms. In contrast to uniform-price auctions, where electricity is rewarded at the market-clearing price, in pay-as-bid auctions, participants are remunerated based on their bid prices. The model involves strategies represented as functions relating price to quantity, resembling Supply Function Equilibria (SFE) game models, but with pay-as-bid remuneration and without accounting for demand uncertainty.

First, we observe that pure-strategy Nash equilibria do not generally exist when the strategy space includes all non-decreasing continuous functions. To address this, the strategy space is restricted to K-Lipschitz supply functions, ensuring the existence of pure-strategy Nash equilibria. Also, a characterization is given: Nash equilibria are piece-wise affine functions with slope K. More precisely, we show that Nash equilibria of the pay-as-bid auction game can be fully characterized in terms of Nash equilibria of a restricted game with continuous scalar actions. Importantly, we do not assume a parametric model from the outset, but instead we establish the optimality of a parametric game within the domain of all K-Lipschitz supply functions. The second main contribution of the thesis is focused on pay-as-bid auction games with affine demand and quadratic costs. In this scenario, we provide a comprehensive characterization of all Nash equilibria of the game and we show that they all lead

to the same market-clearing price and utilities for all agents. Additionally, we derive a concise closed-form expression for the unique market-clearing price at Nash equilibrium as K tends towards infinity. Based on this, we demonstrate that the pay-as-bid auction game lies between the Bertrand and Cournot oligopoly models and results in a lower market-clearing price compared to Supply Function equilibria. In the last part of the dissertation, we present a preliminary study using data collected from Italian electricity markets. In the day-ahead market data analysis, we observe that the market-clearing price estimates in the pay-as-bid auction game closely match those derived from SFE game models. However, the differences become more noticeable as the number of agents decreases. Additionally, the study explores ancillary services markets, observing that submitted offer bids resemble piece-wise affine supply functions.

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Chapter 1

Introduction

1.1 Motivation

Over the past decades, the electricity industry has undergone significant transformations as it has shifted towards deregulation and competition [2, 3]. The liberalization of electricity markets brings both opportunities and challenges when it comes to enhancing the stability and efficiency of the power grid [4]. The move towards competition in the electricity sector has prompted researchers to create decision and analysis support models that are suited to the changing market conditions [5–7].

These changes have resulted in the establishment of wholesale electricity markets in many regions [8]. In this new environment, the operation of power generation units no longer relies on centralized procedures dictated by the state or utilities. Instead, decentralized decisions made by generation companies, driven by their goal of maximizing profits, dictate the actual operation. All companies engage in competition to offer generation services at a price determined by the market, influenced by the interaction of all participants and the demand.

Consequently, electricity companies are now exposed to greater risks, and their demand for suitable decision-support models has increased significantly. Likewise, regulatory agencies also require analysis-support models to monitor and supervise market behavior. Traditional electrical operation models are ill-suited for these new circumstances since they did not account for market behavior, which now acts as the main driving force behind operational decisions. As a result, a new and highly engaging area of research has emerged within the electrical industry [9–12] and

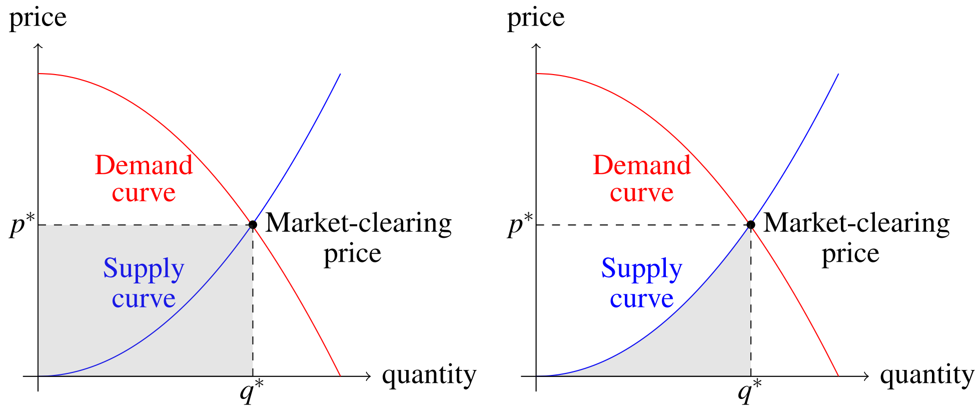


Fig. 1.1 On the left, the total uniform-price remuneration, on the right, the total pay-as-bid remuneration (gray-shaded regions).

the control community [13–19]. Numerous publications demonstrate the extensive efforts made by the research community to develop electricity market models that are tailored to the new competitive context [20–24].

Electricity markets have a unique characteristic that sets them apart from traditional financial markets and commodity markets. Unlike other commodities, electricity cannot be easily stored on a large scale. Therefore, in wholesale electricity markets, the production and consumption of electricity must be in balance at all times. This poses significant challenges when designing electricity market auctions [25, 26, 8]. In order to achieve balance between electricity supply and demand at all times, several markets have been developed, including intra-day, real-time, and control reserve markets [27–31]. These markets play a vital role in ensuring the equilibrium between supply and demand, which has become even more critical with the integration of volatile renewable energy sources into the grid [32, 33].

Wholesale electricity auctions in oligopolistic markets follow two pricing rules: the uniform price rule and the pay-as-bid rule [25, 26]. In both uniform-price and pay-as-bid auctions, participants submit bids indicating the quantity of electricity they can supply and the corresponding price. The system operator selects the lowest bids to balance supply and demand, with the market clearing price set by the last winning producer. In uniform-price auctions, all winning producers receive the market clearing price, regardless of their individual bids. In pay-as-bid auctions, however, winning producers are paid based on their bid prices. Fig. 1.1 illustrates the difference between the two mechanisms. The intersection of aggregate demand

and supply represents the market equilibrium. The total remuneration, represented by the gray shaded region, captures the difference between the prices received by winning producers.

In the context of electricity markets, the debate between the uniform price auction and the pay-as-bid auction emerged in the late 1990s and early 2000s, and in particular after the temporary adoption in England and Wales of discriminatory pricing in day-ahead markets [25], as well as after the ensuing debate in California [34, 35]. Understanding which type of auction is better is still an open question [36].

Currently, day-ahead markets are usually structured as uniform-price auctions, while pay-as-bid auctions are often employed in ancillary servises markets and balancing markets [8]. Due to the high electricity prices experienced in recent years, the EU Agency for the Cooperation of Energy Regulators (ACER) [37] is considering alternative price formation models to replace uniform-price auctions in day-ahead markets, reopening the debate. One potential alternative is indeed the use of pay-as-bid auctions, which aim to ensure reliable and affordable electricity while decoupling prices from marginal technologies like gas and coal, especially with the high penetration of renewables.

1.2 Related literature

From a game-theoretic point of view, appropriate models for studying wholesale markets for electricity are Cournot-based models [38–40], auction models [41, 42, 17, 43] and Supply Function Equilibrium (SFE) game models [1, 44–50]. With this last approach, instead of setting their price bids (Bertrand competition, [51, 52]) or quantities (Cournot competition, [53, 52]), firms bid a supply function relating the price to the quantity. SFE models were first introduced in [1], and then applied to electricity markets in [44]. The authors observed that, in the absence of uncertainty, there exists an infinite number of Nash equilibria. However, when the demand is uncertain and a firm faces a range of possible residual demand curves, it generally expects higher profits by expressing its decisions in terms of a supply function that specifies the price at which it offers different quantities to the market. Unlike traditional equilibrium models that rely on solving algebraic equations, calculating an SFE involves solving a set of differential equations. This poses limitations on the numerical tractability of SFE models and, therefore, researchers have focused on the linear SFE model, in which the demand is affine, marginal costs are linear or affine and SFE can be obtained in terms of linear or affine supply functions [48]. Another way to reduce the number of Nash equilibria is to consider SFE parametrized models [50, 54]. Comparisons between SFE models and Cournot-based models can be found in [55–57].

While there is a vaste amount of literature directed to the study of SFE outcomes in uniform-price auctions [1, 44–50], the behavior of SFE models is less clear when discriminatory prices are considered. In [58], a SFE game model with uncertain demand and pay-as-bid remuneration is proposed and analyzed. The authors assume symmetric continuous marginal costs and perfectly inelastic demand. Existence of an equilibrium is ensured if the hazard rate of the perfectly inelastic demand is monotonically decreasing and sellers have non-decreasing marginal costs. They also compare the equilibrium outcome with uniform-price auctions and conclude that average prices are weakly lower in the discriminatory auction. In [59], SFE in discriminatory auctions are compared to uniform-price auctions when suppliers have capacity constraints. The authors formulate a supply function equilibrium model with inelastic time-varying demand and with constant symmetric marginal costs. They show that payments made to the suppliers in the unique equilibrium of the discriminatory auction can be less than the payments in the uniform-price auction, depending on which uniform-price auction equilibrium is selected. It is worth noting that these models assume perfectly inelastic demand and symmetric costs. To the best of our knowledge, there is a noticeable lack of models that incorporate supply functions and adopt more general assumptions. Additionally, finding SFE in these models often involves solving differential equations, making them less practical for real-world applications.

The uniform price auction and the pay-as-bid auction are studied also with discrete-step supply offers. A first comprehensive analysis of the two auction formats is presented in [60], where the authors analyze the two auction rules under two polar market structures (perfectly competitive and monopolistic supply), with demand uncertainty, and find that under perfect competition there is a trade-off between efficiency and the level of consumer surplus. In [61], a two-player static auction game is considered with a big player with market power and small player. The authors observe that, both under elastic and inelastic demand, the total payment of consumers would be smaller under pay-as-bid pricing for the two-player game; however, under pay-as-bid the equilibrium is a mixed-strategy equilibrium, which is presumably undesirable from an operational point of view. In [62], uniform auctions result in higher average prices than discriminatory auctions, but the ranking in terms of productive efficiency is ambiguous.

In [63] and [64], Nash equilibria are fully characterized for a model of a pay-as-bid electricity market based on a multi-leader-common-follower model of a pay-as-bid electricity market in which the producers provide the regulator with either linear or quadratic bids. The authors describe necessary and sufficient conditions for their existence as well as providing explicit formulas of such equilibria in the market. In [65], the pay-as-bid (or discriminatory) auction is studied in the context of treasury securities and commodities. The authors prove the uniqueness of its pure-strategy Bayesian Nash equilibrium and establish a tractable representation of equilibrium bids. They show that supply transparency and full disclosure are optimal in pay-as-bid, though not necessarily in uniform-price; pay-as-bid is revenue dominant and might be welfare dominant; and, under assumptions commonly imposed in empirical work, the two formats are revenue and welfare equivalent. In [66], a perfect competition model with a continuum of generation technologies and uncertain elastic demand is developed. They compare pay-as-bid and uniform-price auctions, to investigate their long-term efficiency. The findings show that pay-as-bid auctions generally lead to more competitive behavior and lower prices compared to uniform-price auctions. The aggregate capacity remains unchanged under reasonable assumptions. However, when including a continuum of generation technologies, the study reveals that pay-as-bid auctions result in an inefficient generation mix. This inefficiency occurs due to consumers' willingness to pay exceeding the marginal cost, leading to distorted long-run investment incentives for producers.

In [67] the two auction formats are compared using a multi-agent approach, where each adaptive agent represents a generator who develops bid prices based on Q-Learning algorithm. In the experimental results, the authors observe that the pay-as-bid auction indeed results in lower market prices and price volatility, as expected. In [68], a methodology for supporting the trading decisions of producers in a pay-as-bid auction for power systems reserve is proposed. One bidder is assumed to behave strategically and the behavior of the remaining is summarized in a probability distribution of the market price and a reaction function to price dumping by the strategic bidder. Taking the characteristics of Germany markets into account, the methodology is applied using exemplary data and it is shown that the methodology helps to manage existing price uncertainties.

1.3 Contributions

In this thesis, we propose and analyze a supply function model with pay-as-bid remuneration and asymmetric firms, called pay-as-bid auction game.

In our model, strategies are functions relating price to the quantity, in the spirit of Supply Function Equilibria game models. In contrast to SFE, we consider pay-as-bid remuneration and we do not consider uncertainty in the demand. A crucial issue of the supply function model is that the strategy set is an infinite dimensional space. We first observe that, if the strategy space includes all non-decreasing continuous functions that are zero in zero, pure strategy Nash equilibria do not exist in general. This observation stands in stark contrast to the findings of SFE game models, wherein it was noted that in the absence of uncertainty, the number of Nash equilibria becomes infinite. This fundamental disparity arises due to the use of pay-as-bid remuneration, where the trajectory of the supply function plays a crucial role.

Our main result is to prove that, by restricting the strategy space to K-Lipschitz supply functions, existence of pure-strategy Nash equilibria is guaranteed and a characterization can be given. Indeed, under this assumption, the complexity of the problem can be dramatically reduced, since best responses can be characterized as piece-wise affine functions with slope K. The main properties of the pay-as-bid auction game with K-Lipschitz supply functions, that we call K-Lipschitz (K-L) pay-as-bid (PAB) auction game, can be then investigated through a restricted game with continuous scalar actions. This result yields a fundamental simplification and paves the way to a thorough analysis of our model. It distinctly differs from the SFE game models approach in that it operates deterministically. It is essential to note that we do not assume a parametric model outright; instead, we have discovered and established the optimality of a parametric game within the domain of all K-Lipschitz supply functions. Indeed, Nash equilibria of the K-L PAB auction game can be fully characterized in terms of Nash equilibria of the restricted game. We then study Nash equilibria of the restricted game, proving in particular that utilities are quasi-concave in the strategies of the agents. Existence of Nash equilibria is then guaranteed by classical results.

Our second main contribution is on K-L PAB auction games with affine demand and quadratic costs. In this case, we prove uniqueness of Nash equilibria of the restricted game and we give a complete characterization of all Nash equilibria of the

K-L PAB auction game. There is an infinite number of Nash equilibria but they all give rise to the same market-clearing price and to the same utilities for all the agents. Finally, we investigate the equilibrium outcome of the K-L PAB auction game in the scenario where K tends towards infinity. This case is of utmost significance as increasing the value of K widens the strategy space of K-Lipschitz supply functions, thereby relaxing our initial assumption. We first observe that, as K approaches infinity, the equilibrium strategies converge towards step functions that are zero up to the market-clearing price, gradually approximating the behavior observed when studying the general case. Then, we derive a concise closed-form expression for the unique market-clearing price and compute the sold quantities and the utilities for every firm. Based on this closed form expression, we show that the pay-as-bid auction game ranks intermediate between Betrand and Cournot oligopoly models and gives a lower market-clearing price than Supply Function equilibria.

The increasing difference property of the utilities of the restricted game is briefly discussed. The game is not supermodular in general, although we observe that, when the demand is affine or quadratic, the utilities of the restricted game satisfy the increasing difference property in a relevant subset of the strategy space. Current work includes a deeper analysis towards this direction.

In addition to our theoretical analysis, we conduct a preliminary study using data collected from Italian electricity markets. In the day-ahead market data analysis, we observe that our estimations of market-clearing prices in the K-Lipschitz pay-as-bid auction game closely align with those derived from Supply Function Equilibria (SFE) game models, although the difference becomes more pronounced as the number of agents decreases. Furthermore, our attention shifted to ancillary services markets, where we observe that the aggregation of submitted offer bids results in piece-wise affine supply functions. The ongoing research involves delving deeper into these aspects for a more comprehensive exploration.

1.4 Organization of the dissertation

The dissertation is organized as follows. In Chapter 2 we introduce the preliminary notions on game theory and oligopoly models that are needed for the dissertation. Specifically, we provide basic notions on games in Section 2.2, and define supermodular games in Section 2.2.1 and quadratic games in Section 2.2.2. We then introduce three oligopoly models in Section 2.3: we describe the Cournot competition in Section 2.3.1, the Bertrand competition in Section 2.3.2 and the Supply Function Equilibria game model in 2.3.3.

In Chapter 3, we present our main results.

- In Section 3.2, we give the definition of the pay-as-bid auction game and we prove that in its general form the game does not admit Nash equilibria. We conclude the section defining the K-L PAB auction game.

- Section 3.3 is devoted to the presentation and the proof of the main result: existence and characterization of (pure strategy) Nash equilibria for the K-L PAB auction game. More precisely, in Section 3.3.1, we establish that when the strategy set of all agents is confined to K-Lipschitz supply functions, not only best responses do exist, but their structure is relatively straightforward to determine, i.e., they are piecewise affine functions with slope K that can be described by a single scalar value for each agent. Consequently, Nash equilibria of the original K-L PAB auction game correspond to those of a finite-dimensional game, wherein bidders are required to choose these scalar parameters. In Section 3.3.2, we delve into the analysis of this finite-dimensional game, demonstrating that the utility functions are continuous and quasi-concave with respect to these parameters. We prove the existence of Nash equilibria as a consequence of these properties. Finally, in Section 3.3.3, we present our main result and provide illustrative examples to enhance the understanding of the findings.

- Section 3.4 is devoted to a comprehensive study of the K-L PAB auction game with affine demand and quadratic costs. We begin the study by observing, in Section 3.4.1, that Nash equilibria can be fully characterized by studying a subset of the strategy space where utilities are differentiable. In Section 3.3.1, we prove uniqueness of Nash equilibria of the restricted game and we compute it explicitly. We then prove as a consequence that all Nash equilibria of the K-L PAB auction game give the same market-clearing price and the same utilities for all agents, thus observing that the equilibrium outcome is unique. In Section 3.4.3, we compute the resulting market-clearing price, utilities and sold quantities for K that goes to infinity. In Section 3.4.4, we use the derived closed-form expression to establish that the market-clearing price of the pay-as-bid auction game at Nash equilibrium occupies an intermediate position between the Bertrand and Cournot oligopoly models. Furthermore, it yields a lower market-clearing price compared to the Supply Function equilibria.

- In Section 3.5, we make some considerations on the supermodularity properties of the game and we introduce some current work.

The contents of this chapter are based on the conference publication [69] co-authored with Prof. Giacomo Como and Prof. Fabio Fagnani, as well as on unpublished material that we plan to submit for journal publication.

Chapter 4 delves into a preliminary exploratory analysis of the data gathered from the Italian wholesale electricity market. In Section 4.2, we provide a concise description of the current structure of the Italian wholesale electricity market and elaborate on the data used for our analysis. Section 4.3 focuses on the Italian day-ahead market, which currently operate as a uniform-price auction, and makes some comparison on market-clearing prices resulting in SFE and in Nash equilibria of the K-L PAB auction game for K that approaches infinity. Finally, in Section 4.4, we discuss the structure of ancillary services markets and we observe that the interpolation of submitted offer bids leads to piece-wise affine supply functions.

In order to maintain a consistent presentation, we have chosen not to include all the products of the PhD research in this manuscript. Here are brief summaries of the works that are not included:

- [70] focuses on Nash equilibria in games where both coordinating and anti-coordinating agents coexist and interact through an all-to-all network, possibly with different thresholds. While analyzing games with only one type of agents (coordinating or anti-coordinating) and even with heterogeneities is feasible, the simultaneous presence of both types of agents complicates the analysis, and the existence of Nash equilibria is not guaranteed. The main result of this work establishes a verifiable condition on the threshold distributions that characterizes the existence of Nash equilibria in such mixed games. When this condition is satisfied, an explicit algorithm is provided to determine the complete set of such equilibria. Additionally, for the special case when only one type of agents is present, the results allow for an explicit computation of the cardinality of Nash equilibria.

- [71] delves into network games featuring both coordinating and anti-coordinating players. It first presents graph-theoretic conditions for the existence of pure-strategy Nash equilibria in mixed network coordination/anti-coordination games of any size. For cases where these conditions are met, the study examines the asymptotic behavior of best-response dynamics and provides sufficient conditions for finite-time convergence to the set of Nash equilibria. The results are based on an extension and refinement of the notion of network cohesiveness and introduce the concept of network indecomposibility.

- [72] explores the robustness of binary-action heterogeneous network coordination games incorporating an external field representing the biases of different players towards one action over the other. The study establishes necessary and sufficient conditions for the global stability of consensus equilibria under best response-type dynamics, considering constant or time-varying values of the external field. The research applies these results to the analysis of mixed network coordination and anti-coordination games and identifies sufficient conditions for the existence and global stability of pure strategy Nash equilibria. The findings are applicable to general weighted directed interaction networks and rely on supermodularity properties of coordination games to characterize conditions for a novel notion of robust improvement and best response paths.

While these works are excluded from the present manuscript, they contribute valuable insights to the broader research context.

1.5 Notation

Throughout, $\mathbb{R}_+$ will stand for the set of nonnegative reals. For a non-empty interval $\mathcal{I} \subseteq \mathbb{R}$, we shall denote by $\mathcal{C}^0(\mathcal{I})$ and $\mathcal{C}^k(\mathcal{I})$, respectively, the sets of continuous and k-times continuously differentiable functions $f : \mathcal{I} \to \mathbb{R}$.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we introduce the notation, theoretical concepts and tools that will be used throughout this thesis. More precisely, the chapter is divided into two parts.

In Section 2.2, we present notions of game theory and we recall the definition and the basic properties of two remarkable classes of games: supermodular games, in Section 2.2.1, and quadratic games, in Section 2.2.2.

In Section 2.3, we describe two standard oligopoly game models, i.e., the Cournot competition in Section 2.3.1 and the Bertrand competition in Section 2.3.2. We conclude by presenting in details the Supply Function Equilibria game model in Section 2.3.3.

2.2 Notions on games

In this section, we introduce the basic concepts of (noncooperative) game theory, a branch of mathematics that deals with competitive environments where different decision-makers interact strategically.

We refer to the decision-makers as agents (or players). Each agent has a set of possible strategies and chooses her strategy with the aim of maximizing a utility function that depends on both her strategy and the strategies of the other agents.

Formally, a *(strategic form) game* is defined as a triple $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ where

- $\mathcal{N} = \{1,\dots,n\}$ is the finite set of agents (or players),

while, for each agent i in $\mathcal{N}$,

- $\mathcal{A}_i$ is the set of strategies (or actions), and
- $u_i : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathbb{R}$ is the utility function (a.k.a., reward or payoff function).

The strategy of an agent i in $\mathcal{N}$ is denoted with $x_i \in \mathcal{A}_i$ and all strategies are gathered in a vector $x \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ that is called strategy profile or configuration. Throughout, we shall denote with $\mathcal{X} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ the configuration space. The utility function $u_i : \mathcal{X} \to \mathbb{R}$ of an agent i in $\mathcal{N}$ returns the utility $u_i(x)$ that i gets when a strategy profile x in $\mathcal{X}$ is chosen by the agents, namely, when each agent j is playing the strategy $x_j \in \mathcal{A}_j$.

Remark 2.1. *Throughout the thesis, we mostly consider games where the strategy space is the same for all the agents, i.e.,* $\mathcal{A}_i = \mathcal{A}$ *for all* i *in* $\mathcal{N}$*. In this case, we use the notation* $\mathcal{U} = (\mathcal{N}, \mathcal{A}, \{u_i\}_{i\in\mathcal{N}})$.

Let $x_{-i} = x_{|\mathcal{N}\setminus\{i\}}$ be the vector obtained from the strategy profile x by removing its i-th entry. Then, with a slight abuse of notation, we can write

$$u_i(x_i, x_{-i}) = u_i(x) \tag{2.1}$$

for the utility received by agent i when she chooses to play strategy x_i, and the rest of the agents choose to play x_{-i}.

As anticipated, the main assumption in game theory is that every agent i in $\mathcal{N}$ is rational and choses her strategy x_i from the strategy set $\mathcal{A}_i$ so as to maximize her own utility $u_i(x_i, x_{-i})$. Since the utility depends not only on agent i's strategy x_i but also on the strategies of the other agents x_{-i}, it is reasonable to introduce the (set-valued) best response (BR) function

$$\mathcal{B}_i(x_{-i}) = \underset{x_i\in\mathcal{A}_i}{\operatorname{argmax}}\, u_i(x_i, x_{-i}) \tag{2.2}$$

Assuming that agent i in $\mathcal{N}$ is aware of the strategies played by the rest of the agents and these are not changing, the best response function returns the rational choice in the given situation, i.e., the strategy (or strategies) x_i in $\mathcal{A}_i$ that maximizes the utility function given that the other agents play x_{-i}. Notice that $\mathcal{B}_i(x_{-i})$ can be an empty set when $\mathcal{A}_i$ is not finite. We can now give the definition of (pure strategy) Nash equilibrium.

Definition 2.1 (Pure strategy Nash equilibrium)**.** *A (pure strategy) Nash equilibrium (NE) for the game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ is a strategy configuration $x^* \in \mathcal{X}$ such that*

$$x_i^* \in \mathcal{B}_i(x_{-i}^*), \quad \forall i \in \mathcal{N}.$$

The Nash equilibrium is strict if, moreover, $|\mathcal{B}_i(x_{-i}^*)| = 1$ for every agent i in $\mathcal{N}$.

Then, a Nash equilibrium is a strategy profile such that no agent has any incentive to *unilaterally* deviate from her strategy. Indeed, the utility she is getting with that strategy is the best possible given the strategy chosen by the other agents. We remark that it is not at all guaranteed that coordinated deviations of multiple agents from their strategies in a Nash equilibrium could not lead to a higher utility for these agents. Throughout, we shall denote with $\mathcal{X}^*$ the set of Nash equilibria of a game, which can be empty or include one or more elements.

Existence and uniqueness of Nash equilibria are not guaranteed in general. For games with continuous strategy domains, that is, when the strategy set $\mathcal{A}_i \subseteq \mathbb{R}^q$ for all i in $\mathcal{N}$, there is a well-known result for existence of Nash equilibria (see [73], pp. 19-20), which is an application of Kakutani's fixed-point theorem [74] to game theory following insights proposed in [75–77].

Theorem 2.1. *Consider a game $(\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ such that for each $i \in \mathcal{N}$:*

- *$\mathcal{A}_i \subseteq \mathbb{R}^q$ is nonempty, compact, and convex;*
- *$u_i(x)$ is continuous in x and concave in x_i for all x_{-i}.*

Then, the set of Nash equilibria $\mathcal{X}^*$ is non-empty.

Remark 2.2. *Quasi-concavity of $u_i(x_i, x_{-i})$ in x_i is sufficient. We recall that a function $f : X \to \mathbb{R}$ is quasi-concave if, for all $c \in \mathbb{R}$, the upper level set $P_c = \{x \in X \mid f(x) \geq c\}$ is convex. If $X \subseteq \mathbb{R}$ and f is continuous, quasi-concavity can be proved* *by showing that f is either monotonic, increasing or decreasing, or bimodal, first increasing and then decreasing in* $[0,\hat{p}]$ *(see Appendix A).*

We conclude this section introducing an important game-theoretic learning process, the best response dynamics. We start with the definition of the *asynchronous best response dynamics*, where agents in a strategic form game get randomly activated one at a time and switch to a best response strategy. Consider a strategic-form game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$. The continuous-time asynchronous best response dynamics is a Markov chain $X(t)$ with state space $\mathcal{X}$, where every agent $i \in \mathcal{N}$ is equipped with an independent rate-1 Poisson clock. When her clock ticks at time t, agent i updates her strategy to some y_i chosen from the strategy set $\mathcal{A}_i$ with conditional probability distribution that is uniform over the best response set (assuming that such a set is finite)

$$\mathcal{B}_i(X_{-i}(t)) = \operatorname*{argmax}_{x_i\in\mathcal{A}_i} u_i(x_i, X_{-i}(t)) \tag{2.3}$$

In particular, when the best response is unique, agent i updates her strategy to such best response strategy. Hence, the continuous-time asynchronous best response dynamics is a continuous-time Markov chain $X(t)$ with state space coinciding with the configuration space $\mathcal{X}$ of the game and transition rate matrix Λ as follows: $\Lambda_{xy} = 0$ for every two configurations $x, y \in \mathcal{X}$ that differ in more than one entry, and

$$\Lambda_{xy} = \begin{cases} |Bi(x_{-i})|^{-1} & \text{if } y_i \in \mathcal{B}_i(x_{-i}) \\ 0 & \text{if } y_i \notin \mathcal{B}_i(x_{-i}) \end{cases} \tag{2.4}$$

for every two configurations $x, y \in \mathcal{X}$ differing in entry i only, i.e., such that $x_i \neq y_i$ and $x_{-i} = y_{-i}$.

One could also consider the *discrete-time synchronous best response dynamics*, where all agents update time to their unique best response at the same time. In such a case, the update rule (2.4) can be written as a discrete-time dynamics for each time $t \in \mathbb{N}$ as:

$$x_i(t+1) = \mathcal{B}_i(x_{-i}(t)) \tag{2.5}$$

Notice that, if the best response in unique, this is a deterministic dynamics. A natural question that rises is under what assumptions the dynamics (2.3) (and (2.5)), starting from a certain initial state $x_0 \in \mathcal{X}$, converges to the set of Nash equilibria of $\mathcal{X}^*$.

We now introduce two important classes of games: supermodular games and quadratic games. These two classes of games have useful properties that are analyzed in this section and will be used in the following chapters.

2.2.1 Supermodular games

Supermodular games are an important class of games [78–82]. They are characterized by “strategic complementarities”, that is, when one agent increases the strategy, the others have an incentive in doing the same. These games enjoy very useful properties in terms of existence and structure of their Nash equilibria.

In the general setting, the definition of supermodular games requires that the strategy sets $\mathcal{A}_i$ are partially ordered sets satisfying the properties of being a lattice [78]. In this section, we simply let $\mathcal{A}_i \subseteq \mathbb{R}$ for all $i \in \mathcal{N}$. Then, we consider the component-wise partial order $\leq$ on the strategy configuration space $\mathcal{X}_{-i}$, formally defined by $x_{-i} \leq x'_{-i}$ if and only if $x_j \leq x'_j$ for all $j \in \mathcal{N} \setminus \{i\}$.

We start with the definition of the increasing difference property, which is the fundamental feature of supermodular games.

Definition 2.2 (Increasing difference property). *A game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ satisfies the increasing difference property if, for all $x'_i \geq x_i$ and $x'_{-i} \geq x_{-i}$, it holds*

$$u_i(x'_i, x'_{-i}) - u_i(x_i, x'_{-i}) \geq u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i}) \,. \tag{2.6}$$

Notice that the increasing difference property formalizes the so called strategic complements effect: the increase of one agent’s strategy makes more profitable for the others also to increase theirs. For games with continuous strategies and sufficiently regular utility functions, this property can be checked evaluating second derivatives, as presented in the following proposition.

Proposition 2.1. *Consider a game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ where $\mathcal{A}_i \subseteq \mathbb{R}$ and u_i is twice continuously differentiable for all i in $\mathcal{N}$. Then, the game has the increasing difference property if and only if, for all $i, j \in \mathcal{N}$, $i \neq j$, it holds*

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j}(x) \geq 0 \qquad \forall x \in \mathcal{X} \,.$$

We can now give the definition of supermodular games.

Definition 2.3. *A game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ is supermodular if for all $i \in \mathcal{N}$*

- *$\mathcal{A}_i$ is a compact set of $\mathbb{R}$ for all $i \in \mathcal{N}$;*
- *$u_i(x_i, x_{-i})$ is upper semi-continuous in x_i and continuous in x_{-i};*
- *the game satisfies the increasing difference property.*

Most of the properties of supermodular games follows from the following key fact.

Proposition 2.2. *For a supermodular game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ the following facts hold:*

- *for every $i \in \mathcal{N}$ and x_{-i}, the best response set $\mathcal{B}_i(x_{-i})$ has a maximum and a miminum element denoted, respectively, with $\mathcal{B}_i^+(x_{-i})$ and $\mathcal{B}_i^-(x_{-i})$;*
- *$\mathcal{B}_i^+(x_{-i})$ and $\mathcal{B}_i^-(x_{-i})$ are monotone non-decreasing in x_{-i}.*

The above proposition can be used to prove the following fundamental statement, gathering key properties that these games feature.

Theorem 2.2. *For a supermodular game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$, the following facts hold:*

- *the set of pure Nash equilibria $\mathcal{X}^*$ is always non-empty;*
- *there exist two pure Nash equilibria $\underline{x}^* \leq \bar{x}^*$ such that for every pure Nash equilibrium x^* we have that $\underline{x}^* \leq x^* \leq \bar{x}^*$*

The main idea behind the proof of Theorem 2.2 is that, if we define the *synchronous largest best response dynamics* and the *synchronous smallest best response dynamics* respectively as

$$\begin{cases} \bar{x}(t+1) = \mathcal{B}^+(\bar{x}(t)) \\ \bar{x}(0) = \sup \mathcal{X} \end{cases} \qquad \begin{cases} \underline{x}(t+1) = \mathcal{B}^-(\underline{x}(t)) \\ \underline{x}(0) = \inf \mathcal{X} \end{cases} \tag{2.7}$$

then, they converge to $\bar{x}^*$ and $\underline{x}^*$, respectively. This follows from the non-decreasing monotonic property of $\mathcal{B}^+(\bar{x}(t))$ and $\mathcal{B}^-(\underline{x}(t))$ (Proposition 2.2). The statement can be then proved by showing that $\bar{x}^*$ and $\underline{x}^*$ are Nash equilibria and they are respectively the highest and the lowest.

Notice that, this implies that the two systems in (2.7) are both possible algorithms to compute Nash equilibria in supermodular games. Also, observe that, if $\underline{x} = \bar{x}$, the supermodular game admits a unique Nash equilibrium. In this case, convergence to the unique Nash equilibrium is guaranteed.

Corollary 2.1. *Suppose a supermodular game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ admits a unique pure strategy Nash equilibrium x^*. Then, the synchronous best response dynamics converges to x^* with probability 1.*

We conclude this section by presenting the following theorem. For games with continuous strategies and sufficiently regular utility functions, also uniqueness can be checked evaluating second derivatives.

Theorem 2.3. *Consider a game $\mathcal{U} = (\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ where, for all i in $\mathcal{N}$, $\mathcal{A}_i$ are compact intervals and $u_i \in C^2(\mathcal{X})$ are such that*

- $\frac{\partial^2 u_i}{\partial x_i \partial x_j}(x) \geq 0$ *for all* $x \in \mathcal{X}$
- $-\frac{\partial^2 u_i}{\partial x_i^2}(x) \geq \sum_{j\neq i} \frac{\partial^2 u_i}{\partial x_i \partial x_j}(x)$ *for all* $x \in \mathcal{X}$.

Then, the game is supermodular and has a unique Nash equilibrium.

In the following, we introduce another class of games, i.e., quadratic games. As we shall see, quadratic games are supermodular under some assumptions.

2.2.2 Quadratic games

Quadratic games are a family of games that have drained a lot of attention in the recent past and appeared in various applicative scenarios from social and economic sciences [83–87].

In the general form, quadratic games have continuous strategy sets, i.e., $\mathcal{A}_i \subseteq \mathbb{R}$, for all i, and the utilities are given by

$$u_i(x) = \rho_i(x_i) + \delta x_i \sum_j W_{ij} x_j + h_i(x_{-i}) \tag{2.8}$$

where ρ_i is a quadratic function. These games model situations when agents are involved in some common activity: x_i is the level of activity of agent i and $\rho_i(x_i)$ utility in the absence of social interactions. Social interactions are captured by δW_{ij}, where δ captures the importance of social interaction and W_{ij} quantifies the strength of the influence of j on i. Finally, the function h_i captures pure externalities, that is, spillovers that do not affect best responses.

Notice that, if $W_{ij} \geq 0$ for every i, j in $\mathcal{N}$, quadratic games are network games [84]. Indeed, let us model networks as finite directed weighted graphs $\mathcal{G} = (\mathcal{N}, \mathcal{E}, W)$, with set of nodes $\mathcal{N}$, set of directed links $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$, and weight matrix W in $\mathbb{R}_+^{\mathcal{V} \times \mathcal{V}}$, whose entries are such that $W_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. Then, the quadratic game with utility in (2.8) can be defined as a game on the graph $\mathcal{G}$ where the agent set of the game coincides with the node set of the graph and each agent i plays a two-player game with every neighbor, i.e., with all j in $\mathcal{N}$ such that $W_{ij} > 0$.

If $\rho_i(x_i) = b_i x_i - \frac{1}{2} x_i^2$, we find the canonical quadratic game with utilities, for i in $\mathcal{N}$,

$$u_i(x_i, x_{-i}) = \underbrace{(b_i + \delta \sum_{j \neq i} W_{ij} x_j)}_{\text{returns from own strategy}} \; x_i - \underbrace{\frac{1}{2} x_i^2}_{\substack{\text{private costs}\\ \text{of own strategy}}} + \underbrace{h_i(x_{-i})}_{\text{pure externalities}} .$$

where $x_i \in \mathbb{R}^+$ is the level activity of agent i, b_i is the marginal individual reward from level of activity x_i, $\frac{1}{2} x_i^2$ is the cost for providing a level of activity x_i and $W_{ij} \geq 0$ is the benefit from peer interaction.

Notice that the best response of agent i is always a singleton and is linear in x_j for all j in $\mathcal{N} \setminus \{i\}$. More precisely, it is given by

$$\mathcal{B}_i(x_{-i}) = b_i + \delta \sum_j W_{ij} x_j \tag{2.9}$$

Then, if they exist, Nash equilibria are the solutions of the system

$$x = b + \delta W x .$$

If $\delta\rho(W) < 1$, the game possess a unique Nash equilibrium given by

$$x^* = (I - \delta W)^{-1} b \tag{2.10}$$

where $x_i^* = \sum_k \delta^k \sum_j (W^k)_{ij} a_j$. Also, if we consider the synchronous best response dynamics defined in (2.5) in Section 2.2, that is, if we imagine that agents update their strategy simultaneously to their current best response in (2.9), we obtain the dynamical system:

$$x(t+1) = \delta W x(t) + b\,.$$

If $\delta\rho(W) < 1$, the evolution $x(t)$ converges to the unique Nash equilibrium $x^* = \delta W x^* + b$. Notice that this also an algorithm to compute the Nash equilibrium.

Remark 2.3. *Notice that, if $\delta W_{ij} \geq 0$ for every i, j, the game has the increasing difference property in* (2.6). *In this case, the quadratic game is supermodular if the strategy sets $\mathcal{A}_i$ are compact for all i in $\mathcal{N}$ and ρ_i and h_i are continuous. On the other hand, if we consider $\mathcal{A}_i = [0, r_i]$ where $r_i \geq 0$ for all i in $\mathcal{N}$, we have a* constrained quadratic game. *Existence of a Nash equilibrium is still guaranteed by Theorem A.1 (and by the supermodular property) but the best response set takes the form*

$$\mathcal{B}_i(x_{-i}) = \min\{\max\{\delta W_{ij} x_j + a_i, 0\}, r_i\}$$

Hence, it is in general no more guaranteed that Nash equilibria can be found as a solution of (2.10).

2.3 Game models of oligopoly

In this section, we introduce three models of "oligopoly" (competition between a small number of sellers, [88]).

We first present two famous examples of games that are often used in microeconomic theory [52]. These models were first studied in the 19th century, prior to the formalization of the concept of Nash equilibrium for strategic games in general. The main difference between these two models lies in the strategy used by the firms to compete with each other. The first model, proposed by the economist Cournot in 1838 [53], involves a "quantity offering" game, while the second model, the Bertrand competition [51], is a "price offering" game.

We conclude this section by presenting the Supply Function equilibria game model [1], where every producer chooses as her strategy a supply function returning the quantity $S(p)$ that it is willing to sell at a minimum unit price p. Such game is a generalization of both models where firms can either set a fixed quantity (Cournot model) or set a fixed price (Bertrand model).

2.3.1 Cournot competition

Cournot's model in economics is a model of oligopoly where firms produce homogeneous products and compete in *quantities* [53]. Suppose that there are $n \geq 2$ firms producing a homogenous good for the same market. The cost of firm $i \in \{1, \ldots, n\}$ to produce a quantity x_i of the good is $C_i(x_i)$, where $C_i : [0, +\infty) \to \mathbb{R}$ is an increasing function of the quantity x_i. All the quantity is sold at the same price, which is determined according to the *inverse demand function* $P : [0, +\infty) \to \mathbb{R}$ that returns the marginal price for the total output $q = \sum_{j \in \mathcal{N}} x_j$. A standard assumption in microeconomics is that P is decreasing in the quantity q. The goal of each firm is to choose the quantity x_i that they will produce so as to maximize the profit, which equals the revenue minus the cost.

The Cournot competition can be then defined as a game $\mathcal{U} = (\mathcal{N}, \mathcal{A}, \{u_i\}_{i \in \mathcal{N}})$ where the set of agents $\mathcal{N} = \{1, \ldots, n\}$ coincides with the set af all the firms, the strategy is the quantity that the firm produces, i.e., $x_i \in \mathcal{A} = [0, \infty)$, and, for every agent i in $\mathcal{N}$, the utility function is given by

$$u_i(x_1, x_2, \ldots, x_n) = x_i P\left(\sum_{j \in \mathcal{N}} x_j\right) - C_i(x_i)\,. \tag{2.11}$$

Depending on the choice of the costs C_i for i in $\mathcal{N}$ and the inverse demand P, we can have different outcomes.

We shall begin by observing that, when $P(q)$ is a continuous concave non-increasing inverse demand function with $P(0) > 0$ and $P(\bar{q}) = 0$ for some $\bar{q} > 0$ and the production costs $C_i(x_i)$ are continuous and convex for all agents i in $\mathcal{N}$, the utility function in (2.11) is continuous in x and concave in x_i for all i and x_{-i}. Also, the strategy space can be restricted w.l.o.g. to $\mathcal{A} = [0, \bar{q}]$, which is nonempty, compact and convex. Then, we can apply Theorem 2.1 and conclude that, under these assumptions, the Cournot competition always admits pure strategy Nash equilibria, as stated in the following Corollary.

Corollary 2.2. *Let P in $\mathcal{C}^0(\mathbb{R}_+)$ be non-increasing and concave with $P(0) > 0$ and $P(\bar{q}) = 0$ for some $\bar{q} > 0$, and let C_i in $\mathcal{C}^0(\mathbb{R}_+)$ be convex for all i in $\mathcal{N}$. Then, the Cournot competition admits at least one Nash equilibrium.*

Also, we can observe that the Cournot competition with $n = 2$ agents (Cournot duopoly) is supermodular under one further assumption.

Remark 2.4 (Cournot Duopoly)**.** *Consider the Cournot competition with $n = 2$ agents (Cournot duopoly). Let us assume that the price function $P(q)$ is twice continuously differentiable and such that*

$$P'(q) + x_i P''(q) \leq 0$$

which formalizes the reasonable assumption that firm i's marginal revenue decreases in x_{-i}. Let us now re-parameterize the game by introducing the new variables $z_1 = x_1$ and $z_2 = -x_2$, so that $q = z_1 - z_2$. With this choice we have that $\mathcal{A}_i = \mathbb{R}^+$ for all $i = 1, 2$ and

$$\begin{aligned} \frac{\partial^2 u_1}{\partial z_1 \partial z_2} &= -(p'(q) + z_1 p''(q)) \geq 0 \\ \frac{\partial^2 u_2}{\partial z_1 \partial z_2} &= -p'(q) + z_2 p''(q) = -(p'(q) + q_2 p''(q)) \geq 0 \,. \end{aligned} \tag{2.12}$$

Hence, the game is supermodular by Proposition 2.1.

In general, submodular two agent games can be made supermodular by reversing the order on one of the strategies so that they also exhibit the useful properties of supermodular games. This trick does not work, however, for more than two-agent games, which may exhibit dramatically different properties than the supermodular ones.

Let us conclude with some examples where we compute explicitly the Nash equilibrium. In the first example, we study the Cournot competition when the costs are linear and the price is affine, while in the second example we consider quadratic costs. This second example is fundamental as we will recall it later when comparing the Cournot competition to the pay-as-bid auction game, presented and studied in the next chapter.

Example 2.1 (Linear costs and affine price)**.** *Let us consider two firms, that is, $n = 2$, with homogeneous linear cost functions $C_i(x_i) = cx_i$ for $i = 1,2$ and $c > 0$. We shall study Nash equilibria in the Cournot competition when, for $N > 0$ and $\gamma > 0$, the inverse demand function is given by*

$$P(q) = \frac{1}{\gamma}[N-q]_+ = \max\{(N-q)/\gamma, 0\}\,.$$

According to (2.11)*, the utility of a firm $i = 1,2$ is*

$$u_i(x_1,x_2) = x_i\frac{[N-x_1-x_2]_+}{\gamma} - cx_i\,.$$

Let $i = 1,2$ and $x_{-i} \in [0,\infty)$. The best response function of agent i is given by

$$\mathcal{B}_i(x_{-i}) = \frac{1}{2}[N-\gamma c - x_{-i}]_+\,.$$

If we impose the condition that $x_1 \in B_1(x_2)$ and $x_2 \in B_2(x_1)$, we find the unique symmetric Nash equilibrium

$$x_1^* = x_2^* = \frac{1}{3}[N-\gamma c]_+\,.$$

Notice that the resulting price p^c is lower than the monopoly price, indeed,

$$c < p^c = \frac{1}{3}[N-2\gamma c]_+ < \frac{1}{2}[N-\gamma c]_+ = p^*\,.$$

Also, we remark that, if $K > c$, the equilibrium profits are strictly positive, i.e., $u_i(x_i,x_{-i}) > 0$.

Example 2.2 (Quadratic costs and affine price)**.** *Let us consider the same example as before, but in this case, the two firms have quadratic cost functions $C_i(x_i) = \frac{1}{2}c_i x_i^2$ with $c > 0$ for $i = 1,2$. The utility of a firm $i = 1,2$ is then*

$$u_i(x_1,x_2) = x_i\frac{[N-x_1-x_2]_+}{\gamma} - \frac{1}{2}c_i x_i^2\,.$$

Let $i = 1,2$ and $x_{-i} \in [0,\infty)$. The best response function of agent i is given by

$$\mathcal{B}_i(x_{-i}) = \frac{[N-x_{-i}]_+}{2+c_i\gamma}\,.$$

If we impose the condition that $x_1 \in B_1(x_2)$ and $x_2 \in B_2(x_1)$, we find the unique Nash equilibrium

$$x_i^* = \frac{N(1+c_i\gamma)}{2c_1\gamma + 2c_2\gamma + c_1c_2\gamma^2 + 3}, \quad i = 1,2.$$

If $c_1 = c_2$, the unique symmetric Nash equilibrium is given by

$$x_1^* = x_2^* = \frac{N}{3+\gamma c}.$$

The resulting equilibrium price is

$$p_{C,2}^* = D(x_1^* + x_2^*) = \frac{N(1+\gamma c)}{\gamma(3+\gamma c)} = N\left(\gamma + \frac{2\gamma}{1+\gamma c}\right)^{-1}. \tag{2.13}$$

Similarly, if we consider $n > 0$ agents, having symmetric costs functions $C_i(x_i) = \frac{1}{2}cx_i^2$ for all i in $\mathcal{N}$, we find the best response, for all x_{-i},

$$\mathcal{B}(x_{-i}) = \frac{[N - \sum_{j\neq i} x_j]_+}{2+c\gamma}$$

If we look for symmetric equilibria, we find that x^* is a Nash equilibrium if it satisfies

$$x^* = \frac{[N-(n-1)x^*]_+}{2+c\gamma} \quad \Leftrightarrow \quad x^* = \frac{N}{1+n+c\gamma}$$

leading to the equilibrium price

$$p_C^* = N\left(\gamma + \frac{n\gamma}{1+\gamma c}\right)^{-1} \tag{2.14}$$

2.3.2 Bertrand competition

As presented in the previous section, in the Cournot competition, each firm chooses the quantity to produce and the market price is determined based on the demand for the total output produced. On the other hand, in the Bertrand model, firms set the *price* and produce enough output to meet the market demand at that price, considering the prices chosen by all other firms [51]. This model aims to address the same questions as the Cournot competition, but provides some different answers.

The setting is similar for both models: we consider $n \geq 2$ firms producing a quantity q_i of homogenous good at cost $C_i(q_i)$, where C_i is increasing in q_i. The *demand function* $D(p)$ determines the total demand given the price p. If different prices are set, consumers will buy from the firm with the lowest price, which will produce enough output to meet the market demand, while all the other firms get a zero utility. If multiple firms charge identical prices, they split the market equally.

Formally, the set of agents $\mathcal{N} = \{1, \dots, n\}$ is made of n firms equipped with cost functions C_i for all i in $\mathcal{N}$ and their strategy is the price $x_i \in \mathcal{A} = [0, \infty)$. Let D be a decreasing function of price p. Then, the Bertrand competition is a game $\mathcal{U} = (\mathcal{N}, \mathcal{A}, \{u_i\}_{i \in \mathcal{N}})$ with utilities, for i in $\mathcal{N}$,

$$u_i(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_i > p^*(x) \\ \frac{D(p^*(x))}{k(x)} p^*(x) - C_i\left(\frac{D(p^*(x))}{k(x)}\right) & \text{if } x_i = p^*(x) \,. \end{cases} \tag{2.15}$$

where $p^*(x) = \min_i x_i$ denotes the minimum offered price and $k(x) = |\operatorname{argmin}_i x_i|$ the number of firms bidding that price. We remark that Theorem 2.1 cannot be applied to the Bertrand competition as the utility function in (2.15) is discontinuous. Nash equilibria can indeed might fail to exist, as shown in the following example.

Example 2.3 (Linear costs and affine demand)**.** *Let $n = 2$ and let $C_i(q_i) = cq_i$ with $c > 0$ for all $i = 1, 2$. For $N > 0$ and $\gamma > 0$, we shall consider the affine demand function*

$$D(p) = [N - \gamma p]_+ \,.$$

Then, the utilities in (2.15) *become, for i in $\mathcal{N}$,*

$$u_i(x_1, x_2) = \begin{cases} 0 & \text{if } x_i > x_{-i} \,, \\ (x_i - c)[N - \gamma x_i]_+/2 & \text{if } x_i = x_{-i} \,, \\ (x_i - c)[N - \gamma x_i]_+ & \text{if } x_i < x_{-i} \,. \end{cases}$$

Notice that there is a unique symmetric Nash equilibrium

$$x_1^* = x_2^* = c \,,$$

where the equilibrium price is $p^b = c$ and firms make zero utility. Since the outcome is not in line with real-life observations, it is known as the "Bertrand Paradox".

Notice that if the costs are heterogeneous, i.e., $C_i(q_i) = c_i q_i$ with $c_1 < c_2$, the Bertrand competition does not admit Nash equilibria. Indeed, consider any strategy configuration $x = (x_1, x_2)$. If $x_i < c_i$ for some $i = 1,2$, the utility of agent i is negative and therefore x is not a Nash equilibrium. On the other hand, when $x_2 \geq c_2 > c_1$, agent 1 has an incentive in bidding x_1 in (c_1, x_2) in order to capture all the market demand, but, for any x_1 in (c_1, x_2), there exists $\tilde{x}_1$ in (x_1, x_2) giving a higher utility. Then, the best response set of agent 1 is empty when $x_2 \geq c_2 > c_1$. Therefore, under these assumptions, the game admits no Nash equilibria.

In Example 2.3, we observed that, when the demand is affine and the costs are linear, there is a unique symmetric Nash equilibrium for homogeneous cost where both firms receive zero utility, while, when costs are heterogeneous, the Bertrand competition does not admit Nash equilibria. Nash equilibria also fail to exist when we introduce capacity constraints.

Symmetric equilibria might exist for heterogeneous costs in the Bertrand competition, for instance, when the costs are quadratic instead of linear, as shown in the following example.

Example 2.4 (Duopoly with quadratic costs and affine demand)**.** *Let $n = 2$ and $C_i(q) = \frac{1}{2} c_i q^2$ with $c_i > 0$ for $i = 1,2$. We shall consider the same demand as before, i.e., $D(p) = N - \gamma p$ with $N \geq 0$ and $\gamma > 0$. The utilities of the Bertrand competition are then computed as follows*

$$u_i(x_i, x_{-i}) = \begin{cases} 0 & \text{if } x_i > x_{-i}\,, \\ x_i[N - \gamma x_i]_+/2 - c_i([N - \gamma x_i]_+)^2/8 & \text{if } x_i = x_{-i}\,, \\ x_i[N - \gamma x_i]_+ - c_i([N - \gamma x_i]_+)^2/2 & \text{if } x_i < x_{-i}\,. \end{cases}$$

In the symmetric Nash equilibrium (if it exists) both agents choose identical price level, denoted by x. Let us assume that $N - \gamma x \geq 0$. Then, the utility of agent i is

$$u_i(x, x) = \frac{(N - \gamma x)x}{2} - \frac{1}{8} c_i (N - \gamma x)^2\,. \tag{2.16}$$

If agent i reduces her price slightly below x by playing some $x_i < x$, she would capture the entire market demand, earning the utility

$$u_i(x_i, x) = (N - \gamma x_i)x_i - \frac{1}{2} c_i (N - \gamma x_i)^2 \tag{2.17}$$

Agent i has not incentive in undercutting below x if and only if the utility in (2.16) *is greater or equal to the utility in* (2.17) *with* $x_i = x$*, that is,*

$$\frac{(N-\gamma x)x}{2} - \frac{1}{8}c_i(N-\gamma x)^2 \geq (N-\gamma x)x - \frac{1}{2}c_i(N-\gamma x)^2$$

or, equivalently,

$$\frac{1}{8}(N-\gamma x)\,(3Nc_i - (3\gamma c_i + 4)x) \geq 0\,.$$

Since $N - \gamma x \geq 0$*, it follows that, none of the agents will have any incentive in changing their strategy for*

$$x \leq \frac{3Nc_i}{3\gamma c_i + 4} \quad \forall i \in \{1,2\}\,. \tag{2.18}$$

Moreover, in a Nash equilibrium, utilities of firms must be greater or equal to zero, that is,

$$\frac{(N-\gamma x)x}{2} - c_i\frac{(N-\gamma x)^2}{8} \geq 0 \quad \Leftrightarrow \quad \frac{1}{8}(N-\gamma x)((4+\gamma c_i)x - Nc_i) \geq 0\,,$$

thus finding

$$x \geq \frac{Nc_i}{4+\gamma c_i} \quad \forall i \in \{1,2\}\,. \tag{2.19}$$

Let

$$x_m = \max\left\{\frac{Nc_1}{4+\gamma c_1}, \frac{Nc_2}{4+\gamma c_2}\right\} = \frac{Nc_M}{4+\gamma c_M}$$
$$x_M = \min\left\{\frac{3Nc_1}{3\gamma c_1 + 4}, \frac{3Nc_2}{3\gamma c_2 + 4}\right\} = \frac{3Nc_m}{3\gamma c_m + 4}\,,$$

where $c_m = \min\{c_1, c_2\}$ *and* $c_M = \max\{c_1, c_2\}$*. If* $x_m \leq x_M$*, then any strategy configuration* $x_1^* = x_2^* = x^*$ *where* x^* *in* $[x_m, x_M]$ *is a symmetric Nash equilibrium of the game. Notice that, if* $c = c_1 = c_2$*, we have that*

$$\frac{Nc}{4+\gamma c} \leq \frac{3N}{4+3\gamma c} \quad \Leftrightarrow \quad Nc \geq 0\,,$$

which is always satisfied for $c \geq 0$ *and* $N \geq 0$*. Therefore, in the symmetric case, there exists an infinite number of Nash equilibria, i.e., every strategy configuration*

$x_1^* = x_2^* = p_{B,\,2}^*$ *where*

$$p_{B,\,2}^* \in \left[N\left(\gamma + \frac{4}{c}\right)^{-1}, N\left(\gamma + \frac{\gamma}{1+\gamma c}\right)^{-1} \right]. \tag{2.20}$$

In this last example, we will study symmetric Nash equilibria when $n > 2$ when the costs are quadratic and symmetric with $c_i = c > 0$ for all i in $\mathcal{N}$ and affine demand. Similarly to the previous case, we find a continuum of Nash equilibria. This example will be recalled in the next chapter when comparing the Bertrand competition to the pay-as-bid auction game.

Example 2.5 (Symmetric quadratic costs and affine demand)**.** *Let $n > 2$ and $C_i(q) = \frac{1}{2}c_i q^2$ with $c_i = c > 0$ for all i in $\mathcal{N}$. We shall consider the same demand as before, i.e., $D(p) = N - \gamma p$ with $N \geq 0$ and $\gamma > 0$. We follow the same reasoning as Example 2.4 and we look for symmetric Nash equilibria. In this case, for every price $x > 0$, the symmetric strategy configuration $(x,\dots,x)$ gives the utility*

$$u_i(x,\dots,x) = \frac{x(N-\gamma x)}{n} - \frac{c(N-\gamma x)^2}{2n^2}$$

As before, agent i can win all the market quantity by bidding $x_i < x$, thus obtaining

$$u_i(x_i, x_{-i} = (x,\dots,x)) = x(N-\gamma x) - \frac{c(N-\gamma x)^2}{2}.$$

Then, a necessary condition for $(x,\dots,x)$ to be a Nash equilibrium is given by

$$\frac{x(N-\gamma x)}{n} - \frac{c(N-\gamma x)^2}{2n^2} \geq x(N-\gamma x) - \frac{c(N-\gamma x)^2}{2}$$

thus finding

$$x \leq \frac{N}{\frac{2n}{(n+1)c} + \gamma}. \tag{2.21}$$

On the other hand, the utility of every agent i in $\mathcal{N}$ in a Nash equilibrium must be non-negative, thus obtaining the condition

$$x \geq \frac{N}{\frac{2n}{c} + \gamma}. \tag{2.22}$$

Therefore, similarly to the previous case, if we combine (2.21) *and* (2.22)*, we find a continuum of Nash equilibria, that is, for every* α *in* $\left[0, \frac{n^2}{1+n}\right]$

$$x^*(\alpha) = N\left(\gamma + \frac{2(n-\alpha)}{c}\right)^{-1} \tag{2.23}$$

defines a symmetric Nash equilibrium of the game. Since strategies in the Bertrand competition are prices, we also have that, for every α *in* $\left[0, \frac{n^2}{1+n}\right]$*, the equilibrium price is given by*

$$p_B^*(\alpha) = x^*(\alpha) = N\left(\gamma + \frac{2(n-\alpha)}{c}\right)^{-1}. \tag{2.24}$$

Finally, notice that, by definition of the condition (2.22)*, in the Nash equilibrium* $x^*(0)$*, we have that the utility of every agent is zero, i.e.,*

$$u_i(x^*(0), \ldots, x^*(0)) = 0$$

for all i *in* $\mathcal{N}$*.*

Notice that the same reasoning can be easily generalized to the asymmetric case when $c_i \geq 0$ *for all* i *in* $\mathcal{N}$*.*

2.3.3 Supply Function Equilibria (SFE)

The Supply Function equilibria (SFE) game model was first proposed by Klemperer and Meyer [1] in 1989. Unlike the Cournot competition, where firms choose the quantity to produce, and the Bertrand competition, where firms set the price, in the SFE game model firms' strategies are *supply functions* of price. For each marginal price p, the supply curve S_i returns the quantity $S_i(p)$ that firm i is willing to produce for such price. Once all the supply curves are submitted to the market, the *equilibrium marginal price* p^* is determined as the intersection of the demand curve and the sum of all the supplies. At the end of the game, firm i sells a quantity $S_i(p^*)$ at the *market-clearing price* p^* and her utility is given by profit minus costs. Therefore, SFE model uniform price auctions, as all firms are remunerated at the same price. In the following, we will present the model in details.

The general setting is similar to the one considered in the Cournot and Bertrand competition, although slightly more restrictive. In the first version, the authors considered

1. an agent set $\mathcal{N} = \{1,2\}$, with $n = 2$ identical firm with symmetric cost functions C in $\mathcal{C}^2(\mathbb{R}_+)$ with $C'(q) \geq 0$ and $C''(q) \geq 0$ for all $q \geq 0$;
2. an aggregate demand function D in $\mathcal{C}^2(\mathbb{R}_+)$ with $D'(p) < 0$ and $D''(p) \leq 0$ for all p in $(0, \hat{p})$, where $\hat{p}$ is defined as the price such that $D(\hat{p}) = 0$.

As anticipated, the strategy space is $\mathcal{A} = \mathcal{C}^2([0, \hat{p}])$, i.e., the strategy of each firm is a supply function of price. We will denote the strategy of firm i in $\mathcal{N}$ with $S_i \in \mathcal{A}$ to remark that we are dealing with functions. For a strategy configuration $S = (S_1, S_2)$, the utility of an agent $i \in \mathcal{N}$ in the SFE game model without uncertainty is given by

$$u_i(S_1, S_2) = p^* S_i(p^*) - C(S_i(p^*)) \, ,$$

where $p^* \in [0, \hat{p})$ is the unique *market-clearing price* satisfying

$$D(p^*) = S_1(p^*) + S_2(p^*) \, ,$$

i.e., the market-clearing price at which total demand equals total supply. We remark that under the previous assumptions, such price always exists unique.

The first fundamental observation is that, without uncertainty, there exists an infinite number of Nash equilibria. More precisely, any strictly positive quantity pair (q_1, q_2) satisfying $p^* = D^{-1}(q_1 + q_2) \geq C'(q_i)$ for all $i = 1,2$ can be supported by an infinite number of Nash equilibria $S = (S_1, S_2)$ satisfying $S_1(p^*) = q_1$ and $S_2(p^*) = q_2$. Analogously, any price $p^* \in [0, \hat{p}]$ satisfying $p^* \geq C'(q_i)$ can be supported by an infinite number of Nash equilibria leading to such marginal market-clearing price. In words, without uncertainty, it is always possible to construct a supply function equilibrium giving the desired outcome. This is observed in the following claim, proved in [1].

Proposition 2.3. *Let (q_1, q_2), $q_1, q_2 \geq 0$ be such that $p^* = D^{-1}(q_1 + q_2) \geq C'(q_i)$ for all $i = 1,2$. Then, there exists an infinite number of Nash equilibria of the SFE game model without uncertainty such that $S_1(p^*) = q_1$ and $S_2(p^*) = q_2$.*

Remark 2.5. *The idea behind the proof of Proposition 2.3 is the following. Consider any (q_1, q_2) and let $p^* = D^{-1}(q_1 + q_2)$. In order to prove the claim, we shall show* *that it is possible to construct $S_1(\cdot)$ and $S_2(\cdot)$ such that $q_1 = S_1(p^*)$, $q_2 = S_2(p^*)$ and (S_1, S_2) is a Nash equilibrium of the SFE game model. We then fix $S_2(p)$ and we compute the optimal strategy for firm* 1*. The profit-maximizing price is then given by*

$$\max_p p[\underbrace{D(p) - S_2(p)}_{\text{residual demand curve}}] - C(\underbrace{D(p) - S_2(p)}_{\text{residual demand curve}}).$$

The previous optimization problem leads to the first-order condition:

$$\underbrace{D(p) - S_2(p)}_{D(p^*)-q_2=q_1} + [p - C'(\underbrace{D(p) - S_2(p)}_{D(p^*)-q_2=q_1})][D'(p) - S_2^{'}(p)] = 0.$$

Then, p^* satisfies the first-order condition if:

$$S_2^{'}(p^*) = \frac{q_1}{p^* - C'(q_1)} + D'(p^*), \tag{2.25}$$

Notice that, without any restriction on S_i for $i = 1, 2$, it is always possible to find a supply function satisfying this condition. Also, this is only a local condition, so there are indeed an infinite number of supply functions satisfying (2.25)*. Indeed, p^* is a local profit-maximum for $i = 1, 2$ along i's residual demand if*

$$S_j^{'}(p^*) = \frac{q_i}{p^* - C'(q_i)} + D'(p^*), \qquad S_j^{''}(p^*) \geq 0, \qquad j \neq i.$$

To complete our construction, it remains only to extend $S_1(\cdot)$ and $S_2(\cdot)$ over the whole domain of prices $[0, \hat{p})$ in such a way that

(i) p^* is a global profit-maximum for each firm and

(ii) p^* is the only market-clearing price.

This is always possible in many possible ways (see [1]). An example is shown in Fig. 2.1.

We briefly comment on this result. The multiplicity of equilibria in supply functions stems from the fact that the slope of $S_j(\cdot)$ through (p^*, q_j) ensures that (p^*, q_i) is the point along i's residual demand where i's marginal revenue equals its marginal cost. Since in the absence of exogenous uncertainty, i's residual demand is certain, then, as long as global second-order conditions are satisfied, any supply

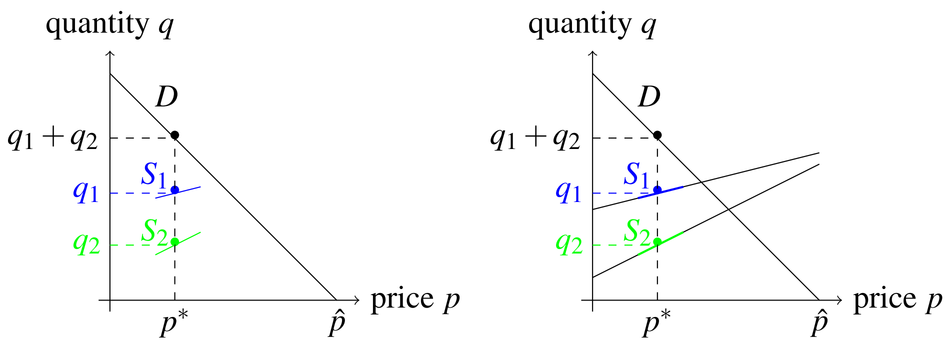


Fig. 2.1 Graphical representation of the proof of Proposition 2.3.

function for i that intersects its residual demand just once, at (p^*, q_i), is an optimal response to $S_j(\cdot)$. Thus, i is willing to choose a supply function with a slope through (p^*, q_j) that ensures that (p^*, q_j) is profit-maximizing along j's residual demand. Given this, j is willing to choose $S_j(\cdot)$.

Such issue is solved when introducing *uncertainty* in the model. In the SFE model under uncertainty, the model and the assumptions are the same, except for the *demand* that now is affected by an exogenous shock ε, which is a scalar random variable with density $f > 0$ on $[\underline{\varepsilon}, \bar{\varepsilon}]$. The demand function is then given by $D(p,\varepsilon)$ with the further assumption that $D_p < 0$, $D_{pp} \leq 0$, and $D_\varepsilon > 0$.

In this setting, the authors look for ex-post Nash equilibria (S_1, S_2), i.e., strategy configuration that are Nash equilibria of the game $\forall \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$. More precisely, they impose the ex-post optimal adjustment to the shock, thus finding the condition

$$\max_p p[D(p,\varepsilon) - S_2(p)] - C(D(p,\varepsilon) - S_2(p)), \qquad \forall \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$$

By looking for only symmetric Nash Equilibria and by assuming $D_{p\varepsilon} = 0$, the authors obtain the first-order condition :

$$S'(p) = \frac{S}{p - C'(S)} + D_p(p) \equiv f(p,S), \qquad \forall p \in [\underline{p}, p^*]. \tag{2.26}$$

Studying solutions of the first-order condition and verifying second-order conditions, they found different results that are gathered in following claim, where $e(Q,p)$ satisfy $Q = D(p, e(Q,p))$.

Proposition 2.4. *If ε has full support, i.e., $\underline{\varepsilon} = e(0,0)$, $\bar{\varepsilon} = \infty$, then*

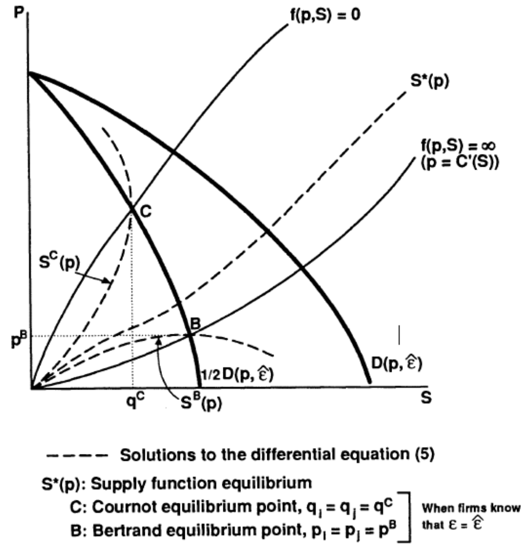


FIGURE 2.—Comparison of supply function equilibrium with Cournot and Bertrand equilibria.

Fig. 2.2 Comparison with Cournot and Bertrand (see [1])

- $S(\cdot)$ *is a symmetric SFE tracing through ex post optimal points* $\Leftrightarrow$ *for all* $p \geq 0$, $S(\cdot)$ *satisfies* (2.26) *and* $0 < S'(p) < \infty$*;*
- *there exists a SFE tracing through ex post optimal points (either single trajectory or a connected set of trajectories);*
- *if* $\underline{\varepsilon} = e(0,0)$, *there exists no asymmetric supply function equilibria.*

Notice that the role of uncertainty is that the first-order conditions must hold at *every* price for which some realization of ε clears the market. In this setting, they also observed that price, quantity, and profits in any SFE are intermediate between the Cournot and Bertrand equilibrium levels, for any realized value of ε (Fig. 2.2).

SFE can be then found as solutions of the differential equation in (2.26). It is not always possible to solve (2.26) and this is one of the main drawbacks of SFE. The differential equation can be easily solved for the case of quadratic costs and affine demand, as shown in the following example.

Example 2.6 (Quadratic costs and affine demand)**.** *Let*

1. $D(p,\varepsilon) = -\gamma p + \varepsilon$ *where* $\gamma > 0$ *and* ε *has (full) support* $[0,\infty)$ *(Fig. 2.3)*
2. $C(q) = cq^2/2$.

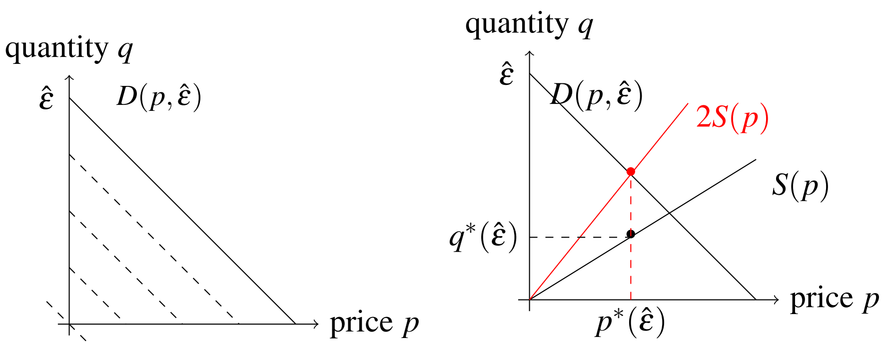


Fig. 2.3 On the left, example of linear demand with an exogenous shock ε. On the right, example of symmetric Supply Function Equilibrium.

The non-autonomous differential equation is then given by

$$S'(p) = \frac{S}{p-cS} - \gamma = \frac{(1+\gamma c)S - \gamma p}{p-cS}.$$

and the unique solution with positive slope at the origin is

$$S(p) = \frac{1}{2}\left(-\gamma + \sqrt{\gamma^2 + \frac{4\gamma}{c}}\right)p. \tag{2.27}$$

Since $S'(p) > 0$ for all p this is the unique SFE. The resulting market-clearing price is

$$p^* = N\left(\sqrt{\gamma^2 + \frac{4\gamma}{c}}\right)^{-1}.$$

To sum up, the goal of the authors was to propose a richer model of competition in oligopoly and to resolve competing predictions of different models (Bertrand and Cournot). With the presence of uncertainty, the authors obtain conditions for existence, characterization and (stronger) sufficient conditions for uniqueness (see [1]). They also observed that their model predicts intermediate outcomes between ex-post Cournot and Bertrand and provide an explicit solution for the linear case.

Generalizations of such model to the case of $n > 2$ agents, nonidentical cost functions, production capacity contraints and networked systems were later proposed in the literature [45–47, 49]. Applications mostly involve linear supply functions [48].

We remark that the generalization to non-symmetric costs is not straightforward, also for linear supply functions, as this implies solving different differential equations (not all firms compete for all prices), thus causing issues with continuity and differentiability. Anyway, in the special case when linear marginal costs are linear (instead of affine), these issues are not present and we can compute explicitly the unique linear SFE, as presented in the following example.

Example 2.7. *Consider a the demand in Example 2.6, and the costs* $C_i(q) = \frac{1}{2}c_i q^2$ *with* $c_i > 0$ *for all* i *in* $\mathcal{N}$*. Then, linear supply functions of the form* $S_i(p) = \beta_i p$ *are SFE if and only if, for all* i*,* $\beta_i \geq 0$ *and*

$$\beta_i = (1 - c_i\beta_i)\left(\gamma + \sum_{j\neq i}\beta_j\right). \tag{2.28}$$

This is proved in [48]. The resulting equilibrium price in this case is

$$p^*_{SFE} = N\left(\gamma + \sum_i \beta_i\right)^{-1}. \tag{2.29}$$

We remark that this example will be recalled in the next chapter to make comparisons between the SFE and Nash equilibria of the pay-as-bid auction game.

Another way to overcome the issue of an infinite number of Nash equilibria without uncertainty is to parameterize supply functions [50], although this is a fundamental restriction of the model. As we shall see in the next chapter, the issue of an infinite number of Nash equilibria is not present with pay-as-bid remuneration as the whole path of the supply function matters in discriminatory pricing. On the other hand, Nash equilibria might fail to exist. In the next chapter, we will propose a model of supply function game with pay-as-bid remuneration and without uncertainty, for which we provide a complete analysis.

Chapter 3

Pay-as-bid auction game

3.1 Introduction

In this chapter, we present our principal findings.

In Section 3.2, we introduce the pay-as-bid auction game and demonstrate that, in its general form, the game does not admit Nash equilibria. To address this, we define the K-Lipschitz (K-L) pay-as-bid (PAB) auction game, which forms the foundation for our subsequent analysis.

Section 3.3 focuses on the primary result: the existence and characterization of Nash equilibria for the K-L PAB auction game. In Section 3.3.1, we establish that, when all agents are confined to selecting K-L supply functions as their strategies, not only do best responses exist, but their structure is relatively straightforward, represented by piecewise affine functions with a slope of K that can be described by a single scalar value for each agent. Consequently, Nash equilibria of the original K-Lipschitz PAB auction game correspond to equilibria of a finite-dimensional game, where bidders must choose these scalar parameters. In Section 3.3.2, we analyze this finite-dimensional game, demonstrating that the utility functions are continuous and quasi-concave with respect to these parameters, thereby proving the existence of Nash equilibria. In Section 3.3.3, we state our main result and enhance understanding with illustrative examples.

Section 3.4 is dedicated to a comprehensive investigation of the K-L PAB auction game under affine demand and quadratic costs. In Section 3.4.1, we observe that

Nash equilibria can be fully characterized by studying a subset of the strategy space where utilities are differentiable. In Section 3.3.1, we prove the uniqueness of Nash equilibria for the restricted game and explicitly compute them. We then demonstrate that all Nash equilibria of the K-L PAB auction game yield the same market-clearing price and utilities for all agents, thereby establishing the uniqueness of the equilibrium outcome. In Section 3.4.3, we analyze the market-clearing price, utilities, and sold quantities as K approaches infinity. In Section 3.4.4, we use the derived closed-form expression to establish that the market-clearing price of the pay-as-bid auction game at Nash equilibrium falls between the Bertrand and Cournot oligopoly models. Additionally, it results in a lower market-clearing price compared to Supply Function equilibria.

In Section 3.5, we study the supermodularity properties of the game and discuss ongoing work in this area.

3.2 Model and problem formulation

In this section, we present the problem setting and we give the definition of the pay-as-bid (PAB) auction game. We then make some remarks on the choice of the strategy space and we define the K-Lipschitz (K-L) pay-as-bid (PAB) auction game, which is the main object of our analysis.

The general setting is the same in all the chapter. More precisely, we will consider a system made of:

- an agent set $\mathcal{N} = \{1, \ldots, n\}$ of n firms equipped with asymmetric *cost functions* $C_i(q)$, for i in $\mathcal{N}$, where q denotes the sold quantity. We assume that the cost function C_i in $\mathcal{C}^2(\mathbb{R}_+)$ is non-decreasing and convex for every firm i in $\mathcal{N}$;
- an aggregate *demand function* D in $\mathcal{C}^2(\mathbb{R}_+)$, which returns the quantity $D(p)$ that consumers are willing to buy at a (maximum) unit price p. We assume that $D(p)$ is strictly decreasing and concave and we define $\hat{p}$ as the price such that $D(\hat{p}) = 0$.

The *strategy* of an agent i in $\mathcal{N}$ is a supply function belonging to a predetermined nonempty subset $\mathcal{A}$ of the set of non-decreasing continuous functions that are 0 in 0, i.e.,

$$\mathcal{A} \subseteq \mathcal{F} = \{S \in \mathcal{C}^0([0,\hat{p}])\,, S(0) = 0\,, S \textit{ non-decreasing}\}\,. \tag{3.1}$$

The supply function S_i returns the quantity $q = S_i(p)$ that the agent is willing to produce at (minimum) unit price p. Strategies of the pay-as-bid auction game are then denoted with S_i, for all i in $\mathcal{N}$, to highlight the infinite dimensional strategy space. According to Section 2.2, a strategy configuration is given by $S = (S_1,\dots,S_n)$, while $S_{-i} = \{S_j\}_{j\neq i}$ gathers all strategies but the strategy of agent i.

Given a demand function $D(p)$ and a strategy configuration S, the *market-clearing price* is determined as the price that matches total demand and total supply, that is, p^* in $[0,\hat{p}]$ satisfying

$$D(p^*) = \sum_{i=1}^{n} S_i(p^*)\,. \tag{3.2}$$

We remark that $p^* = p^*(S)$ is a function of the strategy configuration S. The market-clearing price p^* captures implicitly the strategical interaction among firms.

Remark 3.1. *Existence and uniqueness of a market-clearing price p^* in $[0,\hat{p}]$ for every demand D and strategy configuration S are guaranteed by the assumptions of a strictly decreasing continuous demand function and increasing continuous supply functions satisfying $S_i(0) = 0$ for all i. The market-clearing price determines the total quantity that will be sold by each agent in the auction, that is, $q_i^* = S_i(p^*)$ for every i in $\mathcal{N}$. An example of market-clearing price is depicted on the left of Fig.3.1.*

We define the following class of games based on the pay-as-bid remuneration.

Definition 3.1 (Pay-as-bid auction game)**.** *For a given $\mathcal{A} \subseteq \mathcal{F}$, the pay-as-bid (PAB) auction is a game with agent set $\mathcal{N}$, strategy space $\mathcal{A}$ and utilities, for every i in $\mathcal{N}$,*

$$u_i(S_i,S_{-i}) := p^* S_i(p^*) - \int_0^{p^*} S_i(p)\,\mathrm{d}p - C_i(S_i(p^*))\,, \tag{3.3}$$

where $p^* := p^*(S_i,S_{-i})$ is the unique market-clearing price satisfying (3.2).

Throughout, we shall denote the PAB auction game with $\mathcal{U} = (\mathcal{N},\mathcal{A},\{u_i\}_{i\in\mathcal{N}})$.

Remark 3.2. *The interpretation of the utility function with pay-as-bid remuneration is the following. Agent i sells the quantity $S_i(p^*)$ at the bid price and the final utility*

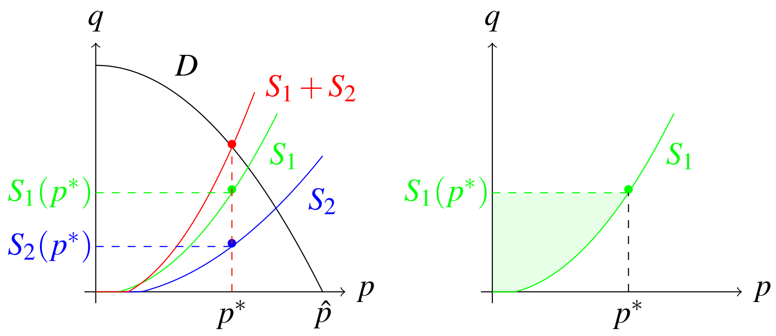


Fig. 3.1 The market-clearing price (on the left) and the pay-as-bid remuneration (on the right).

is given by the total revenue minus the production cost. The first two terms in (3.3) *quantify the revenue with the pay-as-bid remuneration. Indeed, notice that, when S_i is differentiable*

$$\int_0^{p^*} p\, S_i'(p)\, \mathrm{d}p = p^* S_i(p^*) - \int_0^{p^*} S_i(p)\, \mathrm{d}p\,.$$

Also, if S_i is invertible, the total revenue in the PAB auction game equals the integral from 0 *to $S_i(p^*)$ of the inverse of S_i, that is, the price function $p_i(q) := S_i^{-1}(q)$ of agent i. The price function assigns to each quantity the marginal price at which agents are willing to sell such quantity for. Therefore, its integral from* 0 *to q_i^* determines the total pay-as-bid remuneration for agent i for a quantity q_i^*. By considering the formula in (3.3), we do not need to make any assumption on S_i.*

An example of remuneration of the PAB auction game is depicted on the right of Figure 3.1. When the supply function is S_1 and the market-clearing price is p^*, the total revenue for agent 1 *coincides with the green area (the utility is then given by revenue minus costs).*

Remark 3.3. *Recall that total remuneration in uniform-price auctions is calculated as $p^* S_i(p^*)$. Therefore, the only difference between the two auction formats lies in the integral term. We remark that this is a fundamental difference as, in this way, we take into account the trajectory of the supply function.*

Throughout the analysis, we shall focus on existence and characterization of (pure-strategy) Nash equilibria of the PAB auction game.

Our first observation is that, when supply functions can be generic non-increasing continuous functions, that is, when $\mathcal{A} = \mathcal{F}$ as in (3.1), the PAB auction game does not admit Nash equilibria in general. In Proposition 3.1, we prove that, for every $i \in \mathcal{N}$ and other agents' strategies S_{-i}, the best-response is either $S^0 \equiv 0$ or does not exist. Then, the only Nash equilibrium that can exist is $S^* = [S^0, \dots, S^0]$, which is not particularly interesting, as all agents are selling a zero amount of quantity.

Proposition 3.1. *Consider the PAB auction game with strategy space $\mathcal{A} = \mathcal{F}$ as in (3.1). Then, for every i in $\mathcal{N}$ and S_{-i} in $\mathcal{A}^{\mathcal{N}\setminus\{i\}}$, $\mathcal{B}_i(S_{-i}) = \emptyset$ or $B_i(S_{-i}) = S^0$.*

Proof. Let us fix S_{-i} in $\mathcal{A}^{\mathcal{N}\setminus i}$. We shall prove that for every feasible supply function $S_i \neq S^0$ in $\mathcal{A}$, there exists another feasible supply function $\tilde{S}_i$ in $\mathcal{A}$ yielding to the same market-clearing price and a strictly higher utility. Formally, let $S_i \neq S^0$ be any non-decreasing continuous function yielding to an market-clearing price $p^* \neq 0$. We shall then define

$$\tilde{S}_i(p) := S_i\left(\frac{p^2}{p^*}\right).$$

Observe that $\tilde{S}_i(0) = S_i(0)$ and $\tilde{S}_i(p^*) = S_i(p^*)$. Also $S_i(p) \geq \tilde{S}_i(p)$ for all p in $[0, p^*]$. More precisely, we have that $S_i(p) = \tilde{S}_i(p)$ for all $p \in [0, p^*]$ if and only if S_i is constantly equal to 0. Therefore, if $S_i(p) \neq 0$ for some $p \in (0, \hat{p})$, then there must exist $p_0 \in (0, p^*)$ such that $S_i(p_0) > \tilde{S}_i(p_0)$. For continuity, this implies that $\exists \varepsilon > 0$ such that $S_i(p) > \tilde{S}_i(p)$ for all $p \in (p_0 - \varepsilon, p_0 + \varepsilon)$. We then obtain that

$$\int_0^{p^*} \tilde{S}_i(p)\,\mathrm{d}p < \int_0^{p^*} S_i(p)\,\mathrm{d}p,$$

while the other terms in the utility in (3.3) remain constant. Consequently, we find that $u_i(\tilde{S}_i, S_{-i}) > u_i(S_i, S_{-i})$. Then, the best response is either S^0 or does not exist. This concludes the proof. □

Remark 3.4. *The proof of Proposition 3.1 suggests that best responses would exist if one could use step functions. However, enlarging the strategy space to discontinuous functions would lead to a number of different technical difficulties. For instance, one has to solve some technical problems in the definition of the game. Indeed, the existence of a unique market-clearing price p^* as the unique solution of* (3.2) *is not guaranteed anymore. Anyway, even when we technically solve such problem, Nash equilibria might fail to exist.*

Thus, the main issue is that, without any particular restriction on the strategy space, the best response can be a step function, leading to an empty best response set. We solve this problem by restricting the strategy space of the agents to the space of K-Lipschitz supply functions, for a fixed $K > 0$. We recall that a function $S : [0,\hat{p}] \to [0,\infty)$ is *K-Lipschitz* for $K > 0$ if

$$|S(x) - S(y)| \leq K|x-y|\,, \quad \forall x,y \in [0,\hat{p}]\,, x \neq y\,.$$

We can now give the formal definition of the K-Lipschitz pay-as-bid auction game.

Definition 3.2 (K-Lipschitz pay-as-bid auction game)**.** *Let $K > 0$. The K-Lipschitz (K-L) pay-as-bid (PAB) auction game is a PAB auction game as in 3.1 with strategy space $\mathcal{A} = \mathcal{A}_K$ where*

$$\mathcal{A}_K := \{S \in \mathcal{F} : S \text{ is } K\text{-Lipschitz}\}\,. \tag{3.4}$$

From now on, the K-L PAB auction game, denoted with $\mathcal{U}_K = (\mathcal{N}, \mathcal{A}_K, \{u_i\}_{i\in\mathcal{N}})$, will be the main object of our analysis.

Observe that the equilibrium outcomes of the K-L PAB auction game are influenced by several key factors. These include the number of participating firms, each with their respective cost functions denoted with C_i for i in $\mathcal{N}$, and the demand function represented by D, as well as the price at which the demand becomes zero, denoted by $\hat{p}$. Additionally, the parameter $K > 0$ plays a significant role in the model. While the former factors are inherent parameters of the system, the K-Lipschitz assumption is a technical constraint. Consequently, it can be viewed as a parameter that the market maker manipulates to influence the market's outcome or as a factor that determines how much the K-L PAB auction game approximates the real system, becoming less restrictive as K approaches infinity. We will delve into this second scenario in more detail and discuss the limit as K goes to infinity for the case when the demand is affine and the costs are quadratic, as discussed in Section 3.4.3. Finally, two fundamental features of the model are the infinite dimensional strategy space and the pay-as-bid remuneration. Therefore, the main role of the market maker is to choose the auction format. Therefore, comparisons between this auction format and other oligopoly models are discussed in Section 3.4.4.

3.3 Existence and characterization of Nash equilibria

In this section, we will present and prove our main result, i.e., existence and characterization of (pure strategy) Nash equilibria of the K-L PAB auction game. Our study is structured as follows. In Section 3.3.1, we will prove that, if the strategy set of all agents is restricted to K-Lipschitz supply functions, not only best responses do exist, but it is rather simple to determine their structure. They are a subset of piecewise affine functions that can be parametrized by a single scalar value for every agent. This in particular implies that Nash equilibria of the original PAB auction game correspond to those of a finite dimensional game whereby the bidders have to choose such scalar parameter. In Section 3.3.2, we will then study such finite-dimensional game, in particular showing that the utility functions are continuous and quasi-concave in such parameters and proving existence of Nash equilibria as a consequence. In Section 3.3.3, we present our main result and we discuss some examples.

3.3.1 Characterization of best responses

In the following we shall prove that, if we restrict the strategy space to K-Lipschitz supply functions, for some $K > 0$, best responses can be characterized up to the market-clearing price p^* as piecewise affine functions.

Proposition 3.2 (Affine best-response)**.** *Consider the K-L PAB auction game $\mathcal{U}_K$ for some $K > 0$. For every agent i in $\mathcal{N}$ and strategies S_i in $\mathcal{A}$ and S_{-i} in $\mathcal{A}^{\mathcal{N}\setminus\{i\}}$, let $p^* = p^*(S_i, S_{-i})$ denote the unique market-clearing price satisfying (3.2). Then, for all p in $[0, p^*]$,*

$$S_i \in \mathcal{B}_i(S_{-i}) \quad \Rightarrow \quad S_i(p) = K[p - x_i]_+ \tag{3.5}$$

for some x_i in $[0, \hat{p}]$.

Proof. Let i in $\mathcal{N}$ and S_{-i} in $\mathcal{A}^{\mathcal{N}\setminus\{i\}}$. Consider a generic function S_i in $\mathcal{A}_K$ and let p^* denote the market-clearing price satisfying (3.2). We shall prove the statement by construction, that is, we provide a strategy $\tilde{S}_i$ in $\mathcal{A}_K$ of the form in (3.5) that gives a greater or equal utility than S_i. We then observe that $\tilde{S}_i$ gives the same utility as S_i if and only if $S_i \equiv \tilde{S}_i$ for all p in $[0, p^*]$.

The construction of $\tilde{S}_i$ of the form in (3.5) is made as follows. We want to construct a piecewise affine function with slope K that crosses the same p^* as S_i. Therefore, we define $\tilde{S}_i(p) = K[p - x_i]_+$ with

$$x_i := \frac{\sum_{j \neq i} S_j(p^*) - D(p^*)}{K} + p^* .$$

An example is shown in Fig. 3.2 for S_1 and S_2 as in Fig. 3.1 and $K = 3$. We remark that $\tilde{S}_i$ belongs to $\mathcal{A}_K$.

We shall now prove that $S_i(p) \geq \tilde{S}_i(p)$ for all p in $[0, p^*]$. First, notice that, for p in $[0, x_i]$, the inequality is trivial since $S_i(p) \geq \tilde{S}_i(p) = 0$. Let p in $(x_i, p^*]$. Then, the inequality is satisfied as

$$\begin{aligned} S_i(p^*) - S_i(p) = |S_i(p^*) - S_i(p)| &\overset{(1)}{\leq} K|p^* - p| = K(p^* - p) \\ &= K(p^* - x_i) - K(p - x_i) \\ &= \tilde{S}_i(p^*) - \tilde{S}_i(p) \\ &= S_i(p^*) - \tilde{S}_i(p) \end{aligned}$$

where (1) is guaranteed by the K-Lipschitz property of S_i.

To sum up, we observed that $S_i(p) \geq \tilde{S}_i(p)$ for all p in $[0, p^*]$ and $S_i(p^*) = \tilde{S}_i(p^*)$. Therefore,

$$\begin{aligned} u_i(S_i, S_{-i}) &= p^* S_i(p^*) - \int_0^{p^*} S_i(p) dp - C_i(S_i(p^*)) \\ &\leq p^* \tilde{S}_i(p^*) - \int_0^{p^*} \tilde{S}_i(p) dp - C_i(\tilde{S}_i(p^*)) \\ &= u_i(\tilde{S}_i, S_{-i}) . \end{aligned}$$

We remark that, for all p in $[0, p^*]$,

$$\int_0^{p^*} S_i(p)\,\mathrm{d}p = \int_0^{p^*} \tilde{S}_i(p)\,\mathrm{d}p \quad \Leftrightarrow \quad S_i(p) = \tilde{S}_i(p) ,$$

which implies that $u_i(S_i, S_{-i}) = u_i(\tilde{S}_i, S_{-i})$ if and only if $S_i \equiv \tilde{S}_i$ on $[0, p^*]$. Otherwise, $u_i(S_i, S_{-i}) < u_i(\tilde{S}_i, S_{-i})$. This concludes the proof. □

Remark 3.5. *Proposition 3.2 is illustrated in Fig. 3.2. Consider two generic supply functions S_1 and S_2 as in Fig. 3.1. Notice that when playing $\tilde{S}_1(p) = K[p - x_1]_+$ for x_1 as in figure, agent* 1 *receives a higher utility than the one obtained by playing*

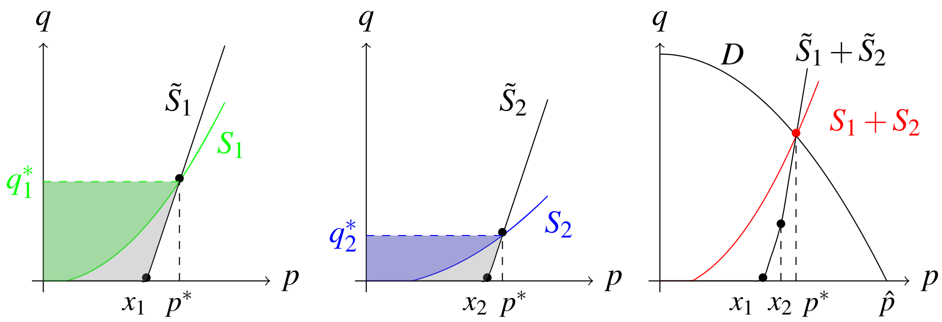


Fig. 3.2 Explanation of Proposition 3.2 (see Remark).

S_1. Indeed, the remuneration increases (colored areas), while the market-clearing price does not change, thus yielding to the same sold quantity. The same happens for agent 2 *when playing $\tilde{S}_2(p) = K[p - x_2]_+$ instead of S_2. Then, for any strategy S_i, it is possible to construct another supply $\tilde{S}_i$ as in* (3.5) *yielding to a higher utility. Thus, best responses must have such form up to the price p^*.*

Proposition 3.2 yields a fundamental simplification in our problem. Indeed, according to (3.5), every Nash equilibrium S^*, for all p in $[0, p^*]$, necessarily exhibits the form

$$S_i^*(p) = K[p - x_i]_+$$

for suitable values x_i in $[0, \hat{p}]$. In particular, this yields a complexity reduction from an infinite-dimensional strategy space to a finite-dimensional one. In order to find Nash equilibria, we can indeed further restrict the strategy space to considering just functions as in (3.5), which are parametrized by just one parameter, that is, $x_i \in [0, \hat{p}]$, for i in $\mathcal{N}$.

We shall give the definition of restricted K-L PAB auction game, which is defined in the same setting of the K-L PAB auction game. Let C_i in $\mathcal{C}^2(\mathbb{R}_+)$, for i in $\mathcal{N}$, be non-decreasing and convex, D in $\mathcal{C}^2(\mathbb{R}_+)$ be strictly decreasing and concave and let $\hat{p}$ be the unique price at which $D(\hat{p}) = 0$.

Definition 3.3 (Restricted K-L PAB auction game)**.** *Let $K > 0$. The restricted (K-L PAB auction) game $\mathcal{U}_r = (\mathcal{N}, \mathcal{A}_r, \{u_i^r\}_{i \in \mathcal{N}})$ has agent set $\mathcal{N}$, strategy space $\mathcal{A}_r = [0, \hat{p}]$ and utility functions, for i in $\mathcal{N}$,*

$$u_i^r(x_i, x_{-i}) := p^* K[p^* - x_i]_+ - \frac{K}{2}\left([p^* - x_i]_+\right)^2 - C_i\left(K[p^* - x_i]_+\right) \qquad (3.6)$$

where $p^* = p^*(x_i, x_{-i})$ is the unique market-clearing price satisfying

$$D(p^*) = K \sum_{i=1}^{n} [p^* - x_i]_+ \,. \tag{3.7}$$

Remark 3.6. *Observe that, without loss of generality, we can study Nash equilibria of the restricted game for the case when $K = 1$ and then generalize the results to the general case. Indeed, for $K \neq 1$, $K > 0$, the restricted game $\tilde{\mathcal{U}}_r$ with demand function $\tilde{D}(p) := D(p)/K$, cost functions $\tilde{C}_i(q) := C_i(Kq)/K$ and $\tilde{K} = 1$ gives rise to the same Nash equilibria of the game $\mathcal{U}_r$ with $K \neq 1$. Utilities of the game $\mathcal{U}_r$ can be computed through the formula $u_i^r(x_i, x_{-i}) = K\tilde{u}_i^r(x_i, x_{-i})$. This remark will be recalled several times in the dissertation, as it simplifies the notation in the proofs.*

We can now state the following corollary that formalizes what previously observed, which is a direct consequence of Proposition 3.2.

Corollary 3.1. *Consider the K-L PAB auction game, for some $K > 0$, and the restricted game with the same K. The following two facts hold:*

- *if S^* is a Nash equilibrium of the K-L PAB auction game and p^* is the marginal market-clearing price satisfying* (3.2)*, then there exists a Nash equilibrium x^* of the restricted game such that*

  $$S_i^*(p) = K[p - x_i^*]_+$$

  for all p in $[0, p^*]$.

- *If x^* is a Nash equilibrium of the restricted game and p^* is the marginal market-clearing price satisfying* (3.7)*, then any strategy configuration $S^* \in \mathcal{A}_K \times \cdots \times \mathcal{A}_K$ such that*

  $$S_i^*(p) = K[p - x_i^*]_+$$

  for all $p \in [0, p^*]$ is a Nash equilibrium of the K-L PAB auction game.

Remark 3.7. *We remark that there is an infinite number of Nash equilbria of the K-L PAB auction game associated to a Nash equilibrium x^* of the restricted game.*

Indeed, any strategy configuration S^* defined, for all i in $\mathcal{N}$, as

$$S_i^*(p) = \begin{cases} K[p - x_i^*]_+ & \text{if } p \in [0, p^*] \\ S_i(p) & \text{otherwise} \end{cases}$$

where S_i is nondecreasing and such that $S_i(p^*) = K[p^* - x_i^*]$ is a Nash equilibrium of the K-L PAB auction game.

Corollary 3.1 proves that it is sufficient to study the restricted game $\mathcal{U}_r$ to determine existence and characterization of Nash equilibria of the game $\mathcal{U}$. In the next sections, we shall prove that the game $\mathcal{U}_r$ admits at least one Nash equilibrium.

3.3.2 Existence of Nash equilibria of the restricted game

In this section, we shall prove that the restricted PAB auction game $\mathcal{U}_r$ admits at least one Nash equilibrium. Thanks to Corollary 3.1, also the K-L PAB auction game will thus have Nash equilibria.

More precisely, we prove existence of Nash equilibria by showing that u_i^r is quasi-concave in $x_i \in [0, \hat{p}]$ for every choice of $x_{-i} \in [0, \hat{p}]^{\mathcal{N} \setminus \{i\}}$. This technical step is fundamental to apply Theorem 2.1 in Section 2.2 that provides sufficient conditions for the existence of Nash equilibria in games with continuous strategy spaces.

Proposition 3.3. *For every agent i, the utility function u_i^r in* (3.6) *is quasi-concave in $x_i \in [0, \hat{p}]$ for every choice of $x_{-i} \in [0, \hat{p}]^{\mathcal{N} \setminus \{i\}}$.*

Proof. Following the insights from Remark 3.6, we set $K = 1$ in all the proof. We will prove quasi-concavity showing that u_i^r is either monotonic, increasing or decreasing, or bimodal, first increasing and then decreasing in $[0, \hat{p}]$ (see Appendix A). We assume that i is fixed as well as x_{-i}, and we set

$$u : [0, \hat{p}] \to \mathbb{R}, \quad u(x) = u_i^r(x, x_{-i}) \tag{3.8}$$

The main difficulty to prove our claims lies in the implicit way the utility function is defined. Below, we exploit formulas (3.6) to obtain information on the regularity and the derivatives of the function u. We first define $\varphi : [0, \hat{p}] \to [0, \hat{p}]$ such that for every x in $[0,\hat{p}]$, $\varphi(x)$ is the unique market-clearing price when $x_i = x$. Namely, $\varphi(x)$ is the unique solution of

$$D(\varphi(x)) = [\varphi(x) - x]_+ + \sum_{j \neq i} [\varphi(x) - x_j]_+ \tag{3.9}$$

It follows from (3.6) that

$$u(x) = \varphi(x)[\varphi(x) - x]_+ - \frac{1}{2}([\varphi(x) - x]_+)^2 - C_i([\varphi(x) - x]_+)\,,$$

We now let $\bar{x} \in [0,\hat{p}]$ to be the price for which the residual demand equals zero, that is, $D(\bar{x}) - \sum_{j \neq i}[\bar{x} - x_j]_+ = 0$. From the fact that the market-clearing price is uniquely defined, it follows that if $x \geq \bar{x}$, necessarily $\varphi(x) = \bar{x}$ and thus, in particular, $\varphi(x) \leq x$. Conversely, if $x \leq \bar{x}$, we have that $\varphi(x) \geq x$. Therefore, we can rewrite u as follows:

$$u(x) = \begin{cases} \frac{1}{2}\varphi^2(x) - \frac{1}{2}x^2 - C_i(\varphi(x) - x) & \text{if } 0 \leq x \leq \bar{x}\,, \\ -C_i(0) & \text{if } \bar{x} < x \leq \hat{p}\,. \end{cases} \tag{3.10}$$

We now study the behavior of φ and then of u on $[0,\bar{x}]$. We indicate with $f : [0,\hat{p}] \to \mathbb{R}$ the function given by

$$f(z) = z + \sum_{j \neq i}[z - x_j]_+ - D(z) \tag{3.11}$$

and we notice that, from (3.9), we have that

$$x = f(\varphi(x)) \quad \forall x \leq \bar{x}$$

As f is strictly increasing and thus invertible, we have that $\varphi(x) = f^{-1}(x)$ for x in $[0,\bar{x}]$. Notice that f is continuous and piecewise $\mathcal{C}^2$ with lack of derivatives at points $\{x_j \ \ j \neq i\}$ and moreover that $f'(z) > 0$ wherever the derivative exists. Basic calculus shows that $\varphi(x)$ is also continuous and $\mathcal{C}^2$ on $[0,\bar{x}]$ except at points in

$$\mathcal{D} = \{x_j = \varphi(x_j) \ \ j \neq i, x_j \in [0,\bar{x}]\}$$

Moreover, if $x \notin \mathcal{D}$, we have that

$$\begin{aligned} 0 < \varphi'(x) &= \frac{1}{f'(\varphi(x))} \\ &= \frac{1}{1+\sum_{j\neq i} \mathbb{1}_{\{\varphi(x)>x_j\}} - D'(\varphi(x))} < 1\,. \end{aligned} \tag{3.12}$$

$$\varphi''(x) = \frac{D''(\varphi(x))\varphi'(x)}{(f'(\varphi(x)))^2} \le 0\,. \tag{3.13}$$

because of the standing assumption on D.

If $x \notin \mathcal{D}$ we can compute

$$\begin{aligned} u'(x) =& \varphi(x)\varphi'(x) - x - C_i'(\varphi(x)-x)(\varphi'(x)-1) \\ =& (\varphi(x) - C_i'(\varphi(x)-x))(\varphi'(x)-1) + \varphi(x) - x\,. \end{aligned} \tag{3.14}$$

We now show that in any interval not intersecting $\mathcal{D}$, the derivative of u either does not change sign or changes sign once from positive to negative. This follows from the fact that, if $x^* \in]0,\bar{x}[\setminus\mathcal{D}$ is such that $u'(x^*) = 0$, it follows from (3.14) and from (3.12) that

$$\varphi(x^*) - C_i'(\varphi(x^*)-x^*) = \underbrace{(\varphi(x^*)-x^*)}_{\ge 0} / \underbrace{(1-\varphi'(x^*))}_{>0} \ge 0 \tag{3.15}$$

and, therefore,

$$\begin{aligned} u''(x^*) = & \underbrace{\varphi'(x^*)^2 - 1}_{<0} - \underbrace{C_i''(\varphi(x^*)-x^*)}_{\ge 0}(\varphi'(x^*)-1)^2 \\ & + \underbrace{(\varphi(x^*) - C_i'(\varphi(x^*)-x^*))}_{\ge 0}\underbrace{\varphi''(x^*)}_{\le 0} < 0\,. \end{aligned} \tag{3.16}$$

We are left with studying the behavior of u at points in $\mathcal{D}$. Notice first that because of (3.12),

$$0 \le \varphi_+'(x_j) \le \varphi_-'(x_j) < 1$$

This relation together with the fact that $\varphi(x) \geq x$ for every $x \in [0,\bar{x}]$ allows to conclude from the expression (3.14) that

$$\begin{aligned} \varphi(x_j) - C_i'(\varphi(x_j) - x_j) \geq 0 \quad &\Rightarrow \quad u_-'(x_j) \geq u_+'(x_j) \\ \varphi(x_j) - C_i'(\varphi(x_j) - x_j) \leq 0 \quad &\Rightarrow \quad 0 \leq u_-'(x_j) \leq u_+'(x_j)\,. \end{aligned} \tag{3.17}$$

for every $x_j \in \mathcal{D}$. In the first case, in case of a change of sign, necessarily it will be from positive to negative. In the second case the sign remains positive. This completes the proof. □

Proposition 3.3 and Theorem 2.1 in Section 2.2 yield the following result.

Corollary 3.2. *The restricted game $\mathcal{U}_r$ admits pure strategy Nash equilibria.*

According to Corollary 3.1, existence of Nash equilibria of the restricted game guarantees existence of Nash equilibria of the K-L PAB auction game. This observations leads to our main result, which is stated in the next section.

3.3.3 Nash equilibria of the K-L PAB auction game

In this section, we state our main result, that is, existence of Nash equilibria of the K-L PAB auction game. Also, a characterization is given: in Nash equilibria, the strategies of the agents are piecewise affine functions with slope K. We conclude the section by presenting an example of K-L PAB auction game and by comparing Nash equilibria for different values of $K > 0$.

Our main result is stated in Theorem 3.1.

Theorem 3.1. *The K-L PAB auction game admits at least one Nash equilibrium S^* with the form, for every agent i in $\mathcal{N}$:*

$$S_i^*(p) = K[p - x_i]_+\,, \qquad \forall p \in [0, p^*]$$

for some x_i in $[0,\hat{p}]$.

Proof. Theorem 3.1 is a direct consequence of Corollary 3.1 and Corollary 3.2. Indeed, in Corollary 3.1, we prove that Nash equilibria of the K-L PAB auction game can be fully characterized starting from Nash equilibria of the restricted game, while, in Corollary 3.2, we show that existence of Nash equilibria the restricted game Nash equilibria is guaranteed. □

In Theorem 3.1, we proved that, by confining the strategy space to K-Lipschitz supply functions, existence of Nash equilibria of the pay-as-bid auction game is guaranteed and we can provide a characterization of these equilibria. More precisely, Nash equilibria are piece-wise affine functions with slope K. We remark that we did not assume a parametric model, but we show the optimality of a parametric game encompassing all K-Lipschitz supply functions.

In the following, we present an example of the K-L PAB auction game and we compare Nash equilibria for different values of $K > 0$.

Example 3.1. *Let us consider the following setting. There are $n = 4$ agents partecipating in the auction game and their costs functions are:*

$$C_1(q) = \frac{1}{4}q^2, \quad C_2(q) = \frac{1}{2}q^2,$$
$$C_3(q) = q^2, \quad C_4(q) = \frac{3}{2}q^2.$$

The aggregate demand function is given by $D(p) = 100 - p - p^2$ and, therefore, $\hat{p} \approx 9.51$ (recall that, by definition, $D(\hat{p}) = 0$).

We now consider the PAB auction game with strategy space $\mathcal{A}_K$ for $K = 5$. Theorem 3.1 guarantees that there exists at least one Nash equilibrium $S^* = (S_1^*, S_2^*, S_3^*, S_4^*)$ of the form

$$S_1^*(p) = 5[p - x_1^*]_+ \quad S_2^*(p) = 5[p - x_2^*]_+$$
$$S_3^*(p) = 5[p - x_3^*]_+ \quad S_4^*(p) = 5[p - x_4^*]_+$$

for some $x_i \in [0, \hat{p}]$ with $i = 1, \ldots, 4$. The configuration S^* is indeed a Nash equilibrium for $x_1^* = 5.96$, $x_2^* = 6.92$, $x_3^* = 7.49$ and $x_4^* = 7.69$. In Figure 3.3, we see the aggregate demand and suppy when agents bid the Nash equilibrium supply functions. The market-clearing price is then $p^* \approx 8.25$ and the utilities are $u_1(S^*) = 48.56$, $u_2(S^*) = 28.32$, $u_3(S^*) = 15.46$ and $u_4(S^*) = 8.07$.

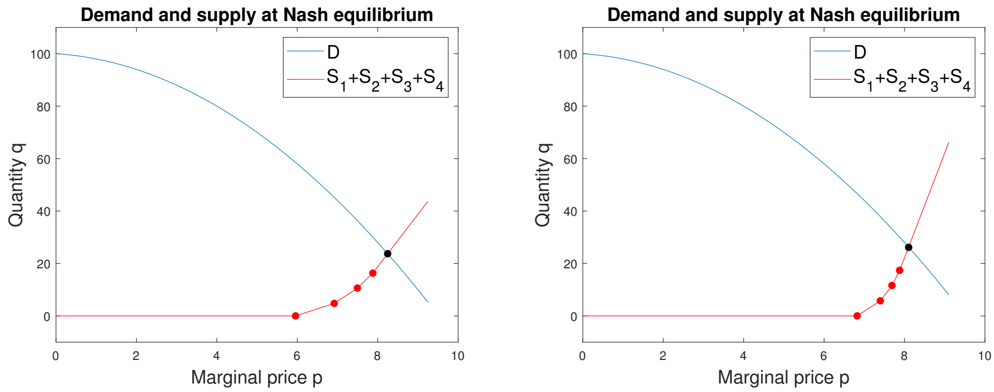


Fig. 3.3 Aggregate demand and supply curves at Nash equilibrium for the setting in Section 4 and $K = 5$ (on the left) and $K = 10$ (on the right).

Let us now study the PAB auction game when $K = 10$. In this case, we find the Nash equilibrium:

$$S_1^*(p) = 10[p - x_1^*]_+ \quad S_2^*(p) = 10[p - x_2^*]_+$$
$$S_3^*(p) = 10[p - x_3^*]_+ \quad S_4^*(p) = 10[p - x_4^*]_+$$

with $x_1^* = 6.82$, $x_2^* = 7.4$, $x_3^* = 7.69$ and $x_4^* = 7.88$. In Figure 3.3, we see the aggregate demand and suppy when agents bid the Nash equilibrium supply functions. The market-clearing price, in this second case, is $p^* \approx 8.1$ while the utilities are $u_1(S^*) = 54.57$, $u_2(S^*) = 29.78$, $u_3(S^*) = 15.54$ and $u_4(S^*) = 7.87$.

Notice that, for a higher K, the market-clearing price decreases, while utilities increase. This observation will be discussed in detail in the next section for the special case of affine demand and quadratic costs.

3.4 Characterization of Nash equilibria with affine demand and quadratic costs

In this section, we will investigate uniqueness and characterization of Nash equilibria of the K-L PAB auction game in the linear case, that is, when the demand is affine, i.e.,

$$D(p) = N - \gamma p \tag{3.18}$$

with $N \geq 0$ and $\gamma > 0$, and the costs are quadratic (linear marginal costs), that is, for every agent $i \in \mathcal{N}$,

$$C_i(q) = \frac{1}{2} c_i q^2 \tag{3.19}$$

with $c_i > 0$. According to Corollary 3.1 and all the considerations we made in the previous section, we can characterize Nash equilibria of the K-L PAB auction game by characterizing Nash equilibria of the restricted game. Therefore, in the following sections, we focus on the characterization of Nash equilibria of the restricted game and we then apply the results to the K-L PAB auction game.

Our study is structured as follows. We first characterize Nash equilibria in a subset of the strategy configuration set, where the utilities are differentiable. We will then derive a closed form expression of the unique Nash equilibrium of the restricted game in this subset and prove uniqueness of Nash equilibria in the whole game. Starting from the unique Nash equilibrium of the restricted game, we can determine explicitly all Nash equilibria of the K-L PAB auction game, which are all characterized by the same market-clearing price and the same utilities for all agents. For this unique equilibrium outcome, we will compute explicitly the limit of the market-clearing price as K goes to infinity and compare the outcome of our model to the equlibrium outcome of standard oligopoly models and the Supply Function equilibrium.

3.4.1 Considerations on the strategy configuration space

As previously observed in Section 3.3.2, difficulties in the study of Nash equilibria of the restricted game raise from the definition of the market-clearing price in (3.7) and from the non-differentiability of utility functions due to the presence of the positive part. In this section, we will prove that we can restrict the configuration space of the restricted game without affecting the search of Nash equilibria. More precisely, we will prove that we can characterize the set of Nash equilibria of the restricted game by first characterizing those belonging to a subset of the configuration space, where differentiability of the utility functions is guaranteed.

Recall that the strategy space of the restricted game is $\mathcal{A}_r = [0, \hat{p}]$ where $\hat{p}$ is the price at which the demand is zero and the configuration space of all the agents is

$\mathcal{X}_r := \mathcal{A}_r^n = [0, \hat{p}]^n$. We shall define the following subset of the configuration space:

$$\tilde{\mathcal{X}}_r = \{x \in \mathcal{X}_r \mid x_i \leq p^*, \forall i \in \mathcal{N}\} \subseteq \mathcal{X}_r. \tag{3.20}$$

In words, a strategy configuration x belongs to $\tilde{\mathcal{X}}_r$ if all agents are playing a strategy less or equal to the resulting market-clearing price. We remark that the market-clearing price p^* depends on the strategy of all the agents. Therefore, the constraint on one strategy depends on all the other strategies.

Remark 3.8. *Observe that, for $x \in \tilde{\mathcal{X}}_r$, it holds that p^* satisfies* (3.7) *if and only if*

$$D(p^*) = K \sum_{i=1}^{n} (p^* - x_i). \tag{3.21}$$

Therefore, the set $\tilde{\mathcal{X}}_r$ gathers all the strategies for which the market-clearing price p^* can be computed according to (3.21)*, that is, omitting the positive part.*

In the following Proposition, we prove that for each strategy x in $\mathcal{X}_r \setminus \tilde{\mathcal{X}}_r$ we can find a strategy $\tilde{x}$ in $\tilde{\mathcal{X}}_r$ leading to the same market-clearing price and the same utility for all agents.

Proposition 3.4. *Consider the restricted game $\mathcal{U}_r$. Then, for every strategy configuration $x \in \mathcal{X}_r$ leading to a market-clearing price p^*, the strategy configuration $\tilde{x} \in \tilde{\mathcal{X}}_r$ defined, for all i in $\mathcal{N}$, as*

$$\tilde{x}_i = \begin{cases} x_i & \text{if } x_i \leq p^* \\ p^* & \text{otherwise}, \end{cases} \tag{3.22}$$

yields the same market-clearing price p^* and is such that $u_i^r(\tilde{x}) = u_i^r(x)$ for all $i \in \mathcal{N}$.

Proof. Notice that $\tilde{x} \in \tilde{\mathcal{X}}_r$ by definition. Also, the strategy $\tilde{x}$ leads to the same market-clearing price as the strategy x since

$$D(p^*) = K \sum_{i=1}^{n} [p^* - x_i]_+ = K \sum_{i=1}^{n} [p^* - \tilde{x}_i]_+ .$$

More in general, it holds that $[p^* - x_i]_+ = [p^* - \tilde{x}_i]_+$ for all $i \in \mathcal{N}$. Therefore, we find that, for all $i \in \mathcal{N}$,

$$u_i^r(\tilde{x}) = u_i^r(x).$$

This concludes the proof. □

Remark 3.9. *The idea behind Proposition 3.4 is the following. All agents i in $\mathcal{N}$ that are playing a strategy $x_i \geq p^*$ are out of the market, as they sell a zero quantity up to the price x_i and such price is greater than the market-clearing price p^*. Therefore, any strategy $x_i > p^*$ can be replaced with the strategy $x_i = p^*$ without affecting the strategy outcome.*

We now want to expand this observation to the characterization of Nash equilibria. According to Proposition 3.4, we have that, if x^* is a Nash equilibrium of the restricted game $\mathcal{U}_r$ and p^* is the resulting market-clearing price, then one of the following holds:

(i) either $x^* \in \tilde{\mathcal{X}}_r$, or

(ii) there exists a Nash equilibrium $\tilde{x}^* \in \tilde{\mathcal{X}}_r$ of the form in (3.22) giving the same market-clearing price and the same utilities for all the agents.

Then, we can determine the whole set of Nash equilibria of the restricted game by studying Nash equilibria in $\tilde{\mathcal{X}}_r$.

In the following, we provide a necessary and sufficient condition for the set of Nash equilibria $\mathcal{X}_r^*$ to be a subset of $\tilde{\mathcal{X}}_r$.

Proposition 3.5. *Let $\mathcal{X}_r^*$ denote the set of Nash equilibria of the restricted game $\mathcal{U}_r$ and let $\tilde{\mathcal{X}}_r$ be defined in* (3.20). *Then,*

$$\mathcal{X}_r^* \subseteq \tilde{\mathcal{X}}_r \,.$$

if and only if, for all x^* in $\mathcal{X}_r^* \cap \tilde{\mathcal{X}}_r$, $x_i^* < p^*$ for all i in $\mathcal{N}$.

Proof. We shall prove both implications by contradiction.

($\Rightarrow$) Assume that there exists a Nash equilibrium x^* in $\mathcal{X}_r^* \cap \tilde{\mathcal{X}}_r$ in which some agent i in $\mathcal{N}$ plays $x_i^* = p^*$. Then every configuration x satisfying $x_i > p^*$ and $x_j = x_j^*$ for $j \neq i$ is a Nash equilibrium of the game. These configurations do not belong to $\tilde{\mathcal{X}}_r$.

($\Leftarrow$) Similarly, assume that there exists a Nash equilibrium $x^* \in \mathcal{X}_r^* \setminus \tilde{\mathcal{X}}_r$. Then, by definition of $\tilde{\mathcal{X}}_r$, there must exist an agent i in $\mathcal{N}$ playing $x_i^* > p^*$. Also, according to Proposition 3.4, the strategy $\tilde{x}^*$ in (3.22) gives the same equilibrium outcome as x^*. Then, we have found a Nash equilibrium $\tilde{x}^*$ in $\mathcal{X}_r^* \cap \tilde{\mathcal{X}}_r$ in which the agent i plays $x_i = p^*$ for some i in $\mathcal{N}$, which contradicts the assumption. This concludes the proof. □

In the following section, we shall investigate uniqueness and characterization of Nash equilibria of the restricted game with affine demand and quadratic costs in the subset $\tilde{\mathcal{X}}_r$ and then generalize the result to the whole game.

3.4.2 Uniqueness and characterization of Nash equilibria

In this Section, we characterize all Nash equilibria of the K-L PAB auction game for the case when the costs are quadratic and the demand is affine. More precisely, we prove uniqueness of the equilibrium outcome, meaning that all Nash equilibria result in the same utilities for all the agents and a unique market-clearing price, and we provide a closed form expression for all the Nash equilibria of the game. We start by proving uniqueness of Nash equilibria of the restricted game and by computing a closed form expression of the Nash equilibrium x^*.

According to Proposition 3.4, we can investigate uniqueness and characterization of Nash equilibria of the restricted game in the configuration space $\tilde{\mathcal{X}}_r$. Observe that, for the demand in (3.18), we can compute explicitly the market-clearing price in (3.7), defined in (3.20). In particular, if we consider a strategy configuration $x \in \tilde{\mathcal{X}}_r$, we can also omit the positive part, thus obtaining the simple expression

$$p^* = \frac{x_i + \sum_{j \neq i} x_j + N/K}{\gamma/K + n}. \tag{3.23}$$

If we substitute the market-clearing price in (3.23) in the utility defined in (3.8), after some algebraic computations, we obtain that the utility of an agent i in $\mathcal{N}$ is given by

$$u_i^r(x_i, x_{-i}) = d_i \left(a_i(\tilde{N} + \sum_{j \neq i} x_j)x_i - \frac{1}{2}x_i^2 + h_i(x_{-i}) \right) \tag{3.24}$$

where

$$a_i = \frac{1 + \tilde{c}_i(\tilde{\gamma} + n - 1)}{(\tilde{\gamma} + n - 1)(\tilde{\gamma} + n + 1 + \tilde{c}_i(\tilde{\gamma} + n - 1))}, \tag{3.25}$$

and

$$d_i = \frac{(\tilde{\gamma}+n-1)(\tilde{\gamma}+n+1+\tilde{c}_i(\tilde{\gamma}+n-1))K}{(\tilde{\gamma}+n)^2}, \, h_i(x_{-i}) = \frac{(\sum_{j\neq i} x_j + \tilde{N})^2(1-\tilde{c}_i)}{2d_i(\tilde{\gamma}+n)^2}.$$

where we applied the change of variables $\tilde{\gamma} = \gamma/K$, $\tilde{N} = N/K$ and $\tilde{c}_i = Kc_i$ for all i in $\mathcal{N}$ to simplify the notation (see Remark 3.6).

Equation (3.24) provides a closed-form expression for the utility functions of the restricted game for any strategy configuration x in $\tilde{\mathcal{X}}_r$. We remark that utilities are now expressed in an explicit form and they are differentiable as there are no positive parts. Also, notice that the utility functions are quadratic functions, as observed in the following remark.

Remark 3.10. *Recall that, according to Section 2.2.2, quadratic games have the form*

$$u_i(x_i,x_{-i}) = \underbrace{(b_i + \delta \sum_{j\neq i} W_{ij}x_j)\; x_i}_{\text{returns from own action}} - \underbrace{\frac{1}{2}x_i^2}_{\substack{\text{private costs}\\ \text{of own action}}} + \underbrace{h_i(x_{-i})}_{\text{pure externalities}}$$

for all agents i in $\mathcal{N}$. Then, according to (3.24)*, the utility of an agent i in $\mathcal{N}$ in the restricted game when x is in $\tilde{\mathcal{X}}_r$ is proportional to the utility of a quadratic game with $b_i = \tilde{N}a_i$, $\delta = 1$ and $W_{ij} = a_i$ for all $j \neq i$ and $W_{ij} = 0$ for $j = i$. We recall that, if $\delta\rho(W) < 1$, the unique Nash equilibrium of a quadratic game with strategy space $\mathcal{A} = \mathbb{R}$ can be found through the formula*

$$x^* = (I - \delta W)^{-1} b \tag{3.26}$$

In the following, we prove that the unique Nash equilibrium of the restricted game is indeed given by the unique x^* solving (3.26)*. We remark that proving that $I - \delta W$ is non singular is not sufficient to show uniqueness and characterization of the Nash equilibrium of the restricted game. Indeed, we must also prove that x^* solving* (3.26) *belongs to $\tilde{\mathcal{X}}_r$. Also, to prove uniqueness, according to Proposition 3.5, we must prove that x^* solving* (3.26) *satisfies $x_i < p^*$ for all i in $\mathcal{N}$.*

In the following theorem, we prove that the restricted game with affine demand in (3.18) and quadratic costs as in (3.19) admits a unique Nash equilibrium, for which we provide a closed form expression.

Theorem 3.2. *Consider the restricted game for $K > 0$ with n agents, affine demand as in* (3.18) *and quadratic costs as in* (3.19) *for some given $N, \gamma > 0$ and $c_i > 0$ for $i \in \mathcal{N}$. Then,*

(i) there exists a unique Nash equilibrium x^* in $\mathcal{X}_r$;

(ii) the unique Nash equilibrium is given by

$$x^* = \frac{N\alpha}{K(1 - \|\alpha\|_1)} . \tag{3.27}$$

where, for all i in $\mathcal{N}$,

$$\alpha_i = \frac{1 + Kc_i(n - 1 + \gamma/K)}{(n + \gamma/K)(n + \gamma/K + Kc_i(n - 1 + \gamma/K))} . \tag{3.28}$$

(iii) the resulting market-clearing price at the Nash equilibrium x^* is

$$p^* = \frac{N}{(Kn + \gamma)(1 - \|\alpha\|_1)} . \tag{3.29}$$

Proof. We remark that utility u_i^r is differentiable for $x \in \mathrm{int}\tilde{\mathcal{X}}_r$ and quasi-concave in x_i. Starting from the utility in (3.24), we can easily find that, for all i in $\mathcal{N}$, x_i^* is a stationary point of $u_i^r(x_i, x_{-i}^*)$ in $\mathrm{int}\,\tilde{\mathcal{X}}_r$ if and only if

$$x_i^* = a_i \left(\tilde{N} + \sum_{j \neq i} x_j^* \right)$$

with a_i as in (3.25). Thus, we obtain that Nash equilibria x^* in $\mathrm{int}\,\tilde{\mathcal{X}}_r$ must satisfy the linear system $Ax^* = \tilde{N}a$ with A and a given by

$$A = \begin{bmatrix} 1 & -a_1 & -a_1 & \dots & -a_1 \\ -a_2 & 1 & -a_2 & \dots & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ -a_n & -a_n & -a_n & \dots & 1 \end{bmatrix}, \qquad a = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} \tag{3.30}$$

Notice that, since $\tilde{\gamma} > 0$, a_i is increasing in $\tilde{c}_i$. Therefore, we have that

$$0 < a_i \leq \lim_{\tilde{c}_i \to \infty} a_i = \frac{1}{\tilde{\gamma} + n - 1} < \frac{1}{n - 1} .$$

This proves that A is strictly diagonally dominant. Therefore, A is non-singular and the system admits exactly one solution. Furthermore, we can rewrite the matrix A in the form

$$A = [a + \mathbb{1}] - a\mathbb{1}^T ,$$

where $[a+\mathbb{1}]$ denotes the diagonal matrix with diagonal $a+\mathbb{1}$. Then, by applying the Sherman–Morrison formula [89, 90]

$$(M + uv^T)^{-1} = M^{-1} - \frac{M^{-1}uv^TM^{-1}}{1+u^TM^{-1}v} ,$$

we shall find

$$\begin{aligned} A^{-1} &= ([a+\mathbb{1}] - a\mathbb{1}^T)^{-1} = [a+\mathbb{1}]^{-1} + \frac{[a+\mathbb{1}]^{-1}a\mathbb{1}^T[a+\mathbb{1}]^{-1}}{1-\mathbb{1}^T[a+\mathbb{1}]^{-1}a} = \\ &= \frac{[a+\mathbb{1}]^{-1}}{1-\sum_j \frac{a_j}{1+a_j}} \left(\left(1 - \sum_j \frac{a_j}{1+a_j}\right)\mathrm{Id} + \begin{bmatrix} a_1/(1+a_1) & \dots & a_1/(1+a_n) \\ \dots & \dots & \dots \\ a_n/(1+a_1) & \dots & a_n/(1+a_n) \end{bmatrix} \right) . \end{aligned}$$

Notice that

$$\left(\left(1 - \sum_j \frac{a_j}{1+a_j}\right)\mathrm{Id} + \begin{bmatrix} a_1/(1+a_1) & \dots & a_1/(1+a_n) \\ \dots & \dots & \dots \\ a_n/(1+a_1) & \dots & a_n/(1+a_n) \end{bmatrix} \right) \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix} .$$

Therefore, the unique solution of the system is given by

$$x^* = A^{-1}\tilde{N}a = \frac{\tilde{N}[a+\mathbb{1}]^{-1}}{1-\sum_j \frac{a_j}{1+a_j}} \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix} = \frac{\tilde{N}}{1-\sum_j \frac{a_j}{1+a_j}} \begin{bmatrix} a_1/(1+a_1) \\ \dots \\ a_n/(1+a_n) \end{bmatrix} = \frac{\tilde{N}\alpha}{1-\|\alpha\|_1} ,$$

where, for all i in $\mathcal{N}$,

$$\alpha_i = \frac{a_i}{1+a_i} = \frac{1+\tilde{c}_i(n-1+\tilde{\gamma})}{(n+\tilde{\gamma})(n+\tilde{\gamma}+\tilde{c}_i(n-1+\tilde{\gamma}))} ,$$

We can find the same expressions as in (3.27) and (3.28) by substituting $\tilde{\gamma} = \gamma/K$, $\tilde{N} = N/K$ and $\tilde{c}_i = Kc_i$ for all i in $\mathcal{N}$ at the end. If we substitute x^* in (3.23), we find the market-clearing price

$$p^* = \frac{\tilde{N}}{(n+\tilde{\gamma})\left(1-\sum_j \alpha_j\right)}$$

The strategy configuration x^* in (3.27) is a Nash equilibrium of the restricted game if it satisfies the initial constraint, that is, if $x^* \in \tilde{\mathcal{X}}_r$. Then, to conclude, we need to check that $x_i^* \in [0, p^*]$ for all i in $\mathcal{N}$. Notice that α_i is strictly increasing in $\tilde{c}_i$ and therefore it holds

$$0 < \alpha_i < \lim_{\tilde{c}_i \to +\infty} \alpha_i = \frac{1}{n+\tilde{\gamma}} < \frac{1}{n}.$$

Then, we have that $0 < \alpha_i < \frac{1}{n+\tilde{\gamma}}$ and $\sum_j \alpha_j < 1$, which proves that

$$0 < x_i^* = \frac{\tilde{N}\alpha_i}{1-\sum_j \alpha_j} < p^* = \frac{N}{(n+\tilde{\gamma})\left(1-\sum_j \alpha_j\right)}$$

Then, we proved that $x_i^* \in (0, p^*)$ for all i in $\mathcal{N}$ and, therefore, $x^* \in \operatorname{int}\tilde{\mathcal{X}}_r$. Then, x^* in (3.27) is a Nash equilibrium of the game and it is the unique Nash equilibrium in $\operatorname{int}\tilde{\mathcal{X}}_r$. We remark that there cannot be Nash equilibria on the boundary of $\tilde{\mathcal{X}}_r$, as the utility is a quadratic concave function with a stationary point x^* with $x_i^* \in (0, p^*)$ for all i. Then, according to Proposition 3.5 , the strategy configuration x^* is the unique Nash equilibrium of the game.

□

In Theorem 3.2, we prove that, when the demand is affine and the costs are quadratic, the restricted game has a unique Nash equilibrium that takes the form in (3.23). The characterization of Nash equilibria of the restricted game permits to characterize Nash equilibria of th K-L PAB auction game as well. The following result is a direct consequence of Theorem 3.2 and Corollary 3.1.

Corollary 3.3. *Consider the K-L PAB auction game, for $K > 0$, with n agents, affine demand as in* (3.18) *and quadratic costs as in* (3.19) *for some given $N \geq 0$ and $\gamma > 0$ and $c_i > 0$ for $i \in \mathcal{N}$. Then,*

(i) all Nash equilibria of the game yield the same market-clearing price and same utilities for all agents.

(ii) A strategy configuration S^* is a Nash equilibrium if and only if

$$S_i^*(p) = \begin{cases} K[p - x_i^*]_+ & \text{if } p \leq p^* \\ S_i(p) & \text{otherwise} \end{cases}$$

where S_i is any non-decresing function satisfying $S_i(p^*) = K[p^* - x_i^*]$ and x^* is given by (3.27)*, i.e.,*

$$x^* = \frac{N\alpha}{K(1 - \|\alpha\|_1)},$$

with, for all i in $\mathcal{N}$,

$$\alpha_i = \frac{1 + Kc_i(n - 1 + \gamma/K)}{(n + \gamma/K)(n + \gamma/K + Kc_i(n - 1 + \gamma/K))}.$$

(iii) The resulting equilibrim price in all Nash equilibria is, as defined in (3.29)*,*

$$p^* = \frac{N}{(Kn + \gamma)(1 - \|\alpha\|_1)}$$

We conclude this section with an example where we compare Nash equilibria for different values of $K > 0$.

Example 3.2. *Let us consider the following setting. There are $n = 4$ agents parteci-pating in the K-L PAB auction game and their costs functions are:*

$$C_1(q) = \frac{1}{4}q^2, \quad C_2(q) = \frac{1}{2}q^2,$$
$$C_3(q) = q^2, \quad C_4(q) = 2q^2.$$

The aggregate demand function is given by

$$D(p) = 100 - 10p$$

and, therefore, $\hat{p} = 10$ (recall that, by definition, $D(\hat{p}) = 0$). Notice that this is a K-L PAB auction game with affine demand and quadratic costs with $c_1 = 1/2$, $c_2 = 1$, $c_3 = 2$, $c_4 = 4$, $N = 100$ and $\gamma = 10$.

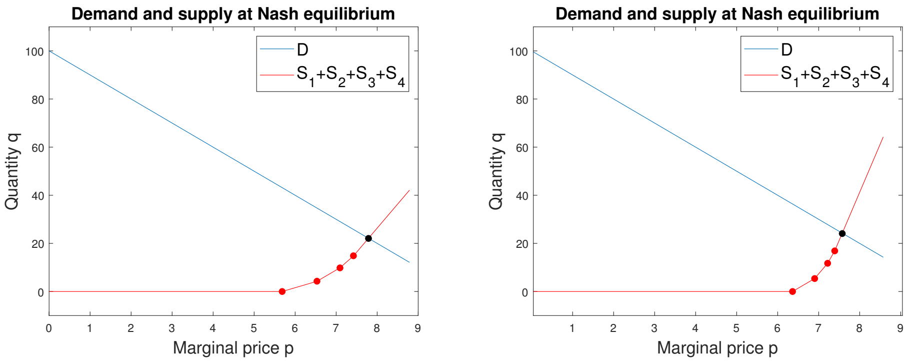


Fig. 3.4 Aggregate demand and supply curves at Nash equilibrium for $K = 5$ and $K = 10$.

We now consider the K-L PAB auction game with $K = 5$*. Theorem 3.2 guarantees that every Nash equilibrium* $S^* = (S_1^*, \dots, S_4^*)$ *takes the form, for* $p \in [0, p^*]$*,*

$$S_i^*(p) = 5[p - x_i^*]_+$$

for all i in $\mathcal{N}$*, where* $x^* = (x_1^*, \dots, x_4^*)$ *is given by* (3.27)*. By applying* (3.28)*, we find* $\alpha_1 \approx 0.1216$*,* $\alpha_2 \approx 0.1398$*,* $\alpha_3 \approx 0.1518$ *and* $\alpha_4 \approx 0.1588$ *and* $\|\alpha\|_1 \approx 0.572$*. Then, the unique Nash equilibrium of the restricted game* x^* *in* (3.27) *is such that* $x_1^* \approx 5.68$*,* $x_2^* \approx 6.53$*,* $x_3^* \approx 7.09$ *and* $x_4^* \approx 7.42$*. Let us denote with* p_K^* *the resulting market-clearing price depending on K. The market-clearing price for* $K = 5$*, which can be computed through the formula* (3.29)*, is*

$$p_5^* \approx 7.79.$$

The utilities at all Nash equilibria are $u_1(S^*) \approx 43.2$*,* $u_2(S^*) \approx 25.25$*,* $u_3(S^*) = 13.78$ *and* $u_4(S^*) = 7.22$*. In Figure 3.4, we see the aggregate demand and suppy when agents bid the Nash equilibrium supply functions.*

Let us now study the K-L PAB auction game in the same setting for $K = 10$*. In this case, we find that all Nash equilibria take the form, for* $p \in [0, p^*]$*,*

$$S_i^*(p) = 10[p - x_i^*]_+$$

with $x_1^* \approx 6.36$*,* $x_2^* \approx 6.9$*,* $x_3^* \approx 7.22$ *and* $x_4^* \approx 7.39$*. More precisely, according to* (3.27)*, we have that* $\alpha_1 \approx 0.168$*,* $\alpha_2 \approx 0.1822$*,* $\alpha_3 \approx 0.1906$ *and* $\alpha_4 \approx 0.1952$ *and* $\|\alpha\|_1 \approx 0.736$*. In Figure 3.4, we see the aggregate demand and suppy when agents bid the Nash equilibrium supply functions. The market-clearing price, in this second*

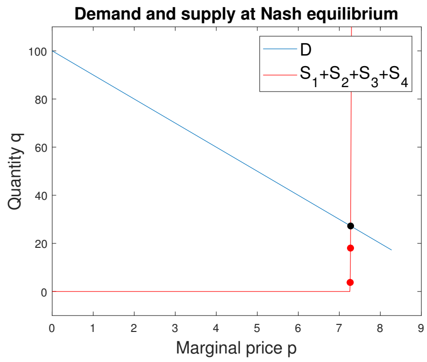


Fig. 3.5 Aggregate demand and supply curves at Nash equilibrium for $K = 1000$.

case, is

$$p^*_{10} \approx 7.57,$$

while the utilities are $u_1(S^*) \approx 47.74$*,* $u_2(S^*) \approx 26.07$*,* $u_3(S^*) \approx 13.66$ *and* $u_4(S^*) \approx 7$.

Notice that, with a higher value of K, the market-clearing price decreases, while utilities increase for firms with lower costs and decrease for firms with higher costs.

Let us consider $K = 1000$*. In this case, we find the Nash equilibrium:*

$$S^*_i(p) = 1000[p - x^*_i]_+$$

for all i in $\mathcal{N}$ *with* $x^*_1 \approx 7.261$*,* $x^*_2 \approx 7.269$*,* $x^*_2 \approx 7.272$ *and* $x^*_4 \approx 7.274$*. In this case, we find* $\alpha_1 \approx 0.2489$*,* $\alpha_2 \approx 0.2491$*,* $\alpha_3 \approx 0.2493$ *and* $\alpha_4 \approx 0.2493$ *and* $\|\alpha\|_1 \approx 0.736$*. In Figure 3.5, we see the aggregate demand and suppy when agents bid the Nash equilibrium supply functions. The market-clearing price, in this second case, is*

$$p^*_{1000} \approx 7.276$$

while the utilities are $u_1(S^*) \approx 52.84$*,* $u_2(S^*) \approx 26.45$*,* $u_3(S^*) \approx 13.23$ *and* $u_4(S^*) \approx 6.62$*.*

We can observe that, as K increases, all x^*_i *approach the market-clearing price* p^**, while utilities are different among agents (due to the heterogenous costs).*

Example 3.2 leads to a discussion on the value of K. Indeed, our model requires K to be fixed, but for any K we obtain different equilibrium outcomes. Recall that, according to Proposition 3.1 and the following remark, best responses would exist in the general setting if one could use step functions. Also, as K increases, we are enlarging the strategy space. Therefore, a fundamental example that requires a deeper study is the case when K approaches infinity. In the following section, we will compute explicitly the resulting market-clearing price when K goes to infinity. This limit permits a comparison with equilibrium outcome in standard oligopoly models, such as Cournot and Bertrand, and in the Supply Function euqilibrium.

3.4.3 Limit as K goes to infinity

In this section, we study the equilibrium outcome of the K-L PAB auction game when the demand is affine, the costs are quadratic and K approaches infinity. More precisely, we will find a simple closed form expression for the unique market-clearing price p^*_∞ resulting from any Nash equilibrium S^* when K goes to infinity. We also compute, for every agent i in $\mathcal{N}$, the sold quantity $q^*_{i,\infty}$ and the utility $u_{i,\infty}$. This case is fundamental as, by increasing the value of K, we are enlarging the strategy space $\mathcal{A}_K$ of K-Lipschitz supply functions, thus lightening our assumption.

According to Corollary 3.3, the market-clearing price p^* is the same in all Nash equilibria and it is given by (3.29). In the following proposition, we compute the limit of p^* in (3.29) as K goes to infinity. The resulting market-clearing price, denoted with p^*_∞, is the limit price at Nash equilibrium that we reach enlarging the strategy space of K-Lipschitz functions.

Proposition 3.6. *Let p^* be defined as in* (3.29) *for some $N \geq 0$, $\gamma > 0$ and $c_i > 0$ for i in $\mathcal{N}$. Then,*

$$p^*_\infty := \lim_{K \to +\infty} p^*(K) = N \left(\sum_i \frac{1}{c_i} + \gamma \right)^{-1}. \tag{3.31}$$

Proof. Recall that p^* is defined in (3.29) as

$$p^*(K) = \frac{N}{(Kn+\gamma)(1-\sum_{i=1}^n \alpha_i(K))}$$

where we recall that α_i is defined for all i in $\mathcal{N}$ in (3.28). We first observe that, for all i in $\mathcal{N}$, it holds that $\lim_{K\to\infty} \alpha_i(K) = \frac{1}{n}$ and therefore $\lim_{K\to\infty}(1-\sum_{i=1}^N \alpha_i) = 0$. Then, it is reasonable to compute the first-order Taylor expansion of α_i in $1/K$ for $K \to +\infty$. Let i in $\mathcal{N}$. For $K \to \infty$, after some algebraic computations, we shall obtain

$$\alpha_i(K) = \frac{1}{n}\left(\frac{1+l_i\frac{1}{K}}{1+m_i\frac{1}{K}+o(\frac{1}{K})}\right)$$

with $l_i = (1+\gamma c_i)/(c_i(n-1)$ and $m_i = (n+2c_i\gamma - c_i\gamma/n)/(c_i/(n-1))$. Then, for $K \to \infty$, we have

$$\begin{aligned}
\alpha_i(K) &= \frac{1}{n}\left(1+l_i\frac{1}{K}\right)\left(1+m_i\frac{1}{K}+o\left(\frac{1}{K}\right)\right)^{-1} = \\
&= \frac{1}{n}\left(1+l_i\frac{1}{K}\right)\left(1-m_i\frac{1}{K}+o\left(\frac{1}{K}\right)\right) = \\
&= \frac{1}{n}\left(1+(l_i-m_i)\frac{1}{K}+o\left(\frac{1}{K}\right)\right)
\end{aligned}$$

Thus, leading to,

$$\alpha_i(K) = \frac{1}{n} - \left(\frac{1}{nc_i}+\frac{\gamma}{n^2}\right)\frac{1}{K}+o\left(\frac{1}{K}\right) \tag{3.32}$$

for all i in $\mathcal{N}$. If we substitute this expression in (3.29) we find

$$\begin{aligned}
p^*(K) &= \frac{N\frac{1}{K}}{(n+\gamma/K)\left(\sum_i\left(\frac{1}{nc_i}+\frac{\gamma}{n^2}\right)\frac{1}{K}+o\left(\frac{1}{K}\right)\right)} = \\
&= \frac{N\frac{1}{K}}{\left(\sum_i\frac{1}{c_i}+\gamma\right)\frac{1}{K}+o(\frac{1}{K})}
\end{aligned}$$

The computation of the limit of p^* for $K \to \infty$ is now straightforward, leading to the expression in (3.31). This concludes the proof.

□

In Proposition 3.6, we obtained a simple closed form expression for the market-clearing price p^*_∞ resulting at any Nash equilibrium S^* when K approaches infinity. Notice that the market-clearing price is directly proportional to the total quantity $N > 0$ that consumers are willing to buy for zero price, while it decreases with in the parameter $\gamma > 0$ that captures the sensitivity of the demand to the price. It also increases with the parameters of the costs $c_i > 0$, for all i in $\mathcal{N}$.

Similarly, we can also compute sold quantities and utilities at Nash equilibrium for the limit case when K goes to infinity. For every i in $\mathcal{N}$, let us denote the quantity sold by agent i at any Nash equilibrium S^* with

$$q_i^*(K) := S_i^*(p^*(K)) = K(p^*(K) - x_i^*(K)) \tag{3.33}$$

where $p^*$is defined in (3.29) and x^* is defined in (3.27). In the following proposition, we compute the limit of $q_i^*(K)$ and $u_i(x^*(K))$ for K that goes to infinity.

Proposition 3.7. *For every i in $\mathcal{N}$, let q_i^*be defined as in* (3.33)*. Then,*

$$q_{i,\infty}^* = \lim_{K\to+\infty} q_i^*(K) = \frac{1}{c_i} p_\infty^* = \frac{N}{c_i}\left(\sum_i \frac{1}{c_i} + \gamma\right)^{-1} \tag{3.34}$$

and

$$u_{i,\infty}^* = \lim_{K\to\infty} u_i(x^*(K)) = \frac{1}{2c_i}(p_\infty^*)^2 = \frac{N^2}{2c_i}\left(\sum_j \frac{1}{c_j} + \gamma\right)^{-2} \tag{3.35}$$

Proof. Observe that

$$\begin{aligned}
S_i^*(p^*) &= K\left(\frac{N}{(Kn+\gamma)(1-\sum_{i=1}^n \alpha_i(K))} - \frac{N\alpha_i(K)}{K(1-\sum_j \alpha_j(K))}\right) = \\
&= K\left(1 - n\alpha_i(K) - \gamma\alpha_i(K)\frac{1}{K}\right)\frac{N}{(Kn+\gamma)(1-\sum_{i=1}^n \alpha_i(K))} = \\
&= K\left(1 - n\alpha_i(K) - \gamma\alpha_i(K)\frac{1}{K}\right) p^*(K)
\end{aligned}$$

In the proof of Proposition 3.6, we observe that, for $K \to \infty$,

$$\alpha_i(K) = \frac{1}{n} - \left(\frac{1}{nc_i} + \frac{\gamma}{n^2}\right)\frac{1}{K} + o\left(\frac{1}{K}\right)$$

Therefore, for $K \to \infty$,

$$S_i^*(p^*) = \left(\frac{1}{c_i} + o(1)\right) p^*(K)$$

which implies

$$\lim_{K\to\infty} S_i^*(p^*) = \frac{1}{c_i} p_\infty^*$$

Then, the utility of agent i is given by

$$\lim_{K\to\infty} u_i(x^*(K)) = \frac{1}{c_i}(p^*_\infty)^2 - \frac{1}{2c_i}(p^*_\infty)^2 = \frac{1}{2c_i}(p^*_\infty)^2 = \frac{N^2}{2c_i}\left(\sum_j \frac{1}{c_j} + \gamma\right)^{-2}$$

□

We shall conclude this section with the following two fundamental remarks.

Remark 3.11. *We remark that, for $K \to \infty$, we have that $\lim_{K\to\infty} x_i^*(K) = p^*_\infty$ for all agents i in $\mathcal{N}$, where x^* is the unique equilibrium of the restricted game. This observation implies that, as K approaches infinity, Nash equilibria of the K-L PAB auction game converge to a set of step functions that are zero up to the market-clearing price p^*_∞. This was also observed in Example 3.2. Therefore, by increasing the value of K, we are estimating the behavior of the PAB auction game with strategy space $\mathcal{A} = \mathcal{F}$ (recall the proof of Proposition 3.1 and Remark 3.4).*

Remark 3.12. *In Proposition 3.2, we find that, as K goes to infinity, the total sold quantity at Nash equilibrium in the K-L PAB auction game is given by $q^*_{i,\infty} = \frac{1}{c_i} p^*_\infty$, for all i in $\mathcal{N}$. Notice that we find the same sold quantity if we consider uniform-price remuneration and supply bids $S_i(p) = \frac{1}{c_i} p$ for all i in $\mathcal{N}$. Also, observe that, in this setting, marginal costs are given by $C_i'(q) = c_i q$ (thus implying $q = \frac{1}{c_i} p$). If we combine all these remarks, we find that we observe the same outcome as truthful bidding in uniform-price auctions.*

3.4.4 Comparative statics

In this section, we aim to compare the market-clearing price at Nash equilibrium in the K-L PAB auction when K goes to infinity with market-clearing prices at Nash equilibria of other oligopoly models. Throughout, we will consider a market with n agents, an aggregate affine demand as in (3.18) and quadratic costs as in (3.19) for some given $N, \gamma > 0$ and $c_i > 0$ for all i.

We shall first focus on the symmetric case when $c_i = c > 0$ for all i in $\mathcal{N}$. Then, according to Examples 2.2 and 2.5, and Corollary 3.3, we have that:

- in the Cournot model, where the strategy is the quantity that firms want to produce, the market-clearing price is given by (2.14), i.e.,

$$p^*_{\mathrm{C}} = N\left(\gamma + \frac{n\gamma}{1+\gamma c}\right)^{-1};$$

- in the Bertrand model, where the strategy is the marginal price at which firms want to produce, there is a continuum of Nash equilibria where, according to (2.24),

$$p^*_{\mathrm{B}}(\alpha) = N\left(\gamma + \frac{2(n-\alpha)}{c}\right)^{-1} \quad \forall \alpha \in \left[0, \frac{n^2}{1+n}\right];$$

- in the K-L PAB auction game, as K goes to infinity, the market-clearing price at Nash equilibrium is given by (3.31), thus finding for symmetric costs:

$$p^*_{\infty} = N\left(\gamma + \frac{n}{c}\right)^{-1}.$$

Few algebraic computations lead to the following fundamental remark.

Remark 3.13. *For any $N \geq 0$, $\gamma > 0$ and $c > 0$, it holds that*

$$p^*_B(0) < p^*_{\infty} < p^*_C \tag{3.36}$$

Then, the market-clearing price at Nash equilibrium in the K-L PAB auction as K goes to infinity lies intermediate between the minimum market-clearing price in Bertrand equilibria and the unique market-clearing price at Nash equilibrium in Cournot competition.

Let us now consider asymmetric costs $C_i(q) = \frac{1}{2}c_i q^2$, with $c_i > 0$, for $i = 1,\dots,n$ and $D(p) = N - \gamma p$, with $N \geq 0$ and $\gamma > 0$. Then, we have that, according to Example 2.7, i.e.,

- in the Supply Function Equilibria (SFE) game model, the market-clearing price is given by (2.29), that is,

$$p^*_{\mathrm{SFE}} = N\left(\gamma + \sum_i \beta_i\right)^{-1}$$

  where, for all i, $\beta_i \geq 0$ and satisfy (2.28), i.e., $\beta_i = (1 - c_i\beta_i)\left(\gamma + \sum_{j\neq i}\beta_j\right)$.

- in the K-L PAB auction game for $K \to \infty$, the market-clearing price is given by (3.31), i.e.,

$$p_\infty^* = N\left(\gamma + \sum_i \frac{1}{c_i}\right)^{-1}.$$

Then, we have the following fundamental remark.

Remark 3.14. *Observe that, by definition of β_i in* (2.28)*, $\beta_i \geq 0$ if and only if $1 - c_i\beta_i \geq 0$, thus implying $\beta_i \leq 1/c_i$. Then, it holds*

$$p_\infty^* < p_{SFE}^*$$

Thus, the market-clearing price at Nash equilibrium in the K-L PAB auction as K goes to infinity is always strictly lower than the market-clearing price in the SFE.

3.5 Discussion on supermodularity

In this section, we discuss the supermodularity property of the restricted game. The first fundamental observation is that supermodularity of the restricted game $\mathcal{U}_r$ is not guaranteed in the general case, as shown in the following example.

Example 3.3. *Let us consider a game with two agents, that is, $n = 2$, and let $D(p) = N - \gamma p$ with $N = 100$ and $\gamma = 1$. Let us also assume that agent 1 has quadratic costs, that is, $C_1(q) = \frac{1}{2}c_1 q^2$ with $c_1 = 1$. We shall discuss the supermodular property of the restricted game $\mathcal{U}_r$ with $K = 1$.*

Let $x_2 = 0$ and $x_1 = \frac{N}{\gamma+1} = 50$. Observe that, for such values of x_1 and x_2, we find

$$N - \gamma p^* = [p^* - x_1]_+ + [p^* - x_2]_+ \Leftrightarrow p^* = \frac{N}{\gamma+1} = 50\,.$$

Therefore, agent 1 does not sell any quantity and her utility is $u_1(x_1, x_2) = -C_1(0) = 0$. The same holds if she increases her strategy. For instance, for $x_1' = 50.2$, we find $u_1(x_1', x_2) = -C_1(0) = 0$.

On the other hand, the utility of agent 1 *changes when agent* 2 *increases her strategy. Let* $x_2' = 1$*. Then, we find*

$$\begin{aligned} N - \gamma p^* &= [p^* - x_1]_+ + [p^* - x_2']_+ \\ \Leftrightarrow \quad p^* &= \frac{N + x_1 + x_2'}{\gamma + 2} = 50.3\bar{3} \end{aligned}$$

Observe that $p^* > x_1$ *and therefore agent* 1 *sells a quantity* $q_1 = p^* - x_1$*. The utility is then given by*

$$u_1(x_1, x_2') = \frac{(p^*)^2}{2} - \frac{(x_1)^2}{2} - \frac{1}{2}(p^* - x_1)^2 = 50/3 = 16.\bar{6}\,.$$

Similarly, for $x_1' = 50.2$ *and* $x_2' = 1$*, we find* $p^* \approx 50.4$ *and* $u_1(x_1', x_2') \approx 10.04$*.*

The game is supermodular if the utilities u_1 *and* u_2 *satisfy (2.6) for all* $x_1' \geq x_1$ *and* $x_2' \geq x_2$*. According to our previous computations, for* $x_1' = 50.2 \geq x_1 = 50$ *and* $x_2' = 1 \geq x_2 = 0$*, we obtain*

$$\begin{aligned} u_1(x_1', x_2') - u_1(x_1, x_2') &\approx -6.63 \\ &\not\geq u_1(x_1', x_2) - u_1(x_1, x_2) = 0\,. \end{aligned}$$

Therefore, the game is not supermodular.

Example 3.3 shows that the restricted game $\mathcal{U}_r$ can fail to be supermodular also in the case with two agents, affine demand and quadratic costs. On the other hand, we observe that, if the demand is affine, the utilities have the increasing difference property in the subset $\tilde{\mathcal{X}}_r$ in (3.20), as defined in the previous section. This is proved in the following in Proposition.

Proposition 3.8. *Let us consider the demand in* (3.18)*. Then, the utility* u_i^r *satisfy the increasing difference property in* (2.6) *for all* $x_i' \geq x_i$ *and* $x_{-i}' \geq x_{-i}$ *satisfying*

$$(x_i, x_{-i}), (x_i', x_{-i}), (x_i, x_{-i}'), (x_i', x_{-i}') \in \tilde{\mathcal{X}}_r\,. \tag{3.37}$$

Proof. Let $K = 1$. According to Remark 3.6, if the increasing difference property holds for $K = 1$, then it holds for every $K > 0$. Also, let us define the function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ as

$$F(x, y) = D(y) - \sum_{i=1}^{n} (y - x_i)\,. \tag{3.38}$$

Then, the market-clearing price in the set $\tilde{\mathcal{X}}_r$ can be found through the implicit function $y : \mathbb{R}^n \to \mathbb{R}$ whose graph $(x, y(x))$ is precisely the set of all $F(x,y) = 0$. For any $x \in \tilde{\mathcal{X}}_r$, we can then rewrite the utility of an agent i in (3.6) in the form

$$u_i^r(x_i, y) = \frac{1}{2}y^2 - \frac{1}{2}x_i^2 - C_i(y - x_i)\,.$$

We remark that we can write the utility in this way, that is, omitting the positive parts, only if we consider strategies x in the subset $\tilde{\mathcal{X}}_r$. Also, observe that $u_i^r(x_i, y) = u_i^r(x_i, x_{-i})$ for all i. Let us then compute the partial derivative of u_i^r in x_i, thus obtaining

$$\frac{\partial u_i^r}{\partial x_i}(x_i, y) = y\frac{\partial y}{\partial x_i} - x_i - C_i'(y - x_i)\left(\frac{\partial y}{\partial x_i} - 1\right). \tag{3.39}$$

According to the implicit function theorem and the definition of F in (3.38), we have that

$$\frac{\partial y}{\partial x_i} = -\frac{\partial F}{\partial x_i}(x,y)\left(\frac{\partial F}{\partial y}(x,y)\right)^{-1} = \frac{1}{n - D'(y)} \overset{(1)}{=} \frac{1}{n+\gamma}, \tag{3.40}$$

here (1) holds true since, for the affine demand $D(y) = N - \gamma y$, we have that $D'(y) = -\gamma > 0$. Also, observe that $\frac{\partial y}{\partial x_i} > 0$, thus implying that y is increasing in all its entries. Combining (3.39) and (3.40), we obtain

$$\frac{\partial u_i^r}{\partial x_i}(x_i, y) = \frac{y}{n+\gamma} - x_i + C_i'(y - x_i)\frac{n-1+\gamma}{n+\gamma}\,.$$

It is now straightforward to observe that $\frac{\partial u_i^r}{\partial x_i}$ is increasing in y (recall that, by assumption, C_i is convex). Since y is increasing in x_{-i}, this proves that the utility have the increasing difference property in the interior of $\tilde{\mathcal{X}}_r$ when the demand is affine. □

Remark 3.15. *In the proof of Proposition 3.8, we compute the partial derivative of u_i^r with respect to x_i and we observe that, if the demand is affine, it is increasing in the market-clearing price, which is increasing in x_{-i}. The same proof can be generalized to a quadratic demand $D(p) = N - \frac{1}{2}\gamma p^2$, but it does not hold anymore when the demand is for istance $D(p) = N - \frac{1}{3}\gamma p^3$. In this case, the utilities do not satisfy the increasing difference property in the subset $\tilde{\mathcal{X}}_r$.*

In Proposition 2.2 and in Remark 3.15, we show that the utilities of the restricted game satisfy the increasing difference property in the subset $\tilde{\mathcal{X}}_r$ when the demand is affine or is quadratic. Also, in Remark 3.15, we observe that this is no longer true when the demand is cubic.

These observations are both relevant. On one side, it is fundamental to observe that the game is not supermodular and that utilities do not satisfy the increasing difference property in general in the subset $\tilde{\mathcal{X}}_r$. Therefore, the result on existence of Nash equilibria of the restricted game obtained in Proposition 3.3 is non-trivial. On the other hand, we would like to exploit the observation on the increasing difference property for the special cases of affine and quadratic demand. The game in $\tilde{\mathcal{X}}_r$ is a constrained game, therefore properties of supermodular gams do not apply directly to our case. Anyway, our conjecture is that we might use insights from supermodularity to compute and characterize Nash equilibria also for general costs. Current work includes a deeper analysis in this direction.

Chapter 4

Preliminary case study: the Italian wholesale electricity market

4.1 Introduction

This chapter is devoted to a preliminary exploratory analysis of the data from the Italian wholesale electricity market.

It is structured as follows. In Section 4.2, we briefly describe the current structure of the Italian wholesale electricity market and we describe the data used in the analysis. In Section 4.3, we focus on day-ahead markets, currently structured as uniform-price auctions. We first compute the aggregate demand and supply curves based on submitted bids and the market-clearing price, accordingly. We then propose a method to estimate, using the data, the parameters of our model, i.e., the affine demand and the quadratic production costs. We then compute the market-clearing price in SFE and in the K-L PAB auction game for K that goes to infinity and we compare the outcomes. In Section 4.4, we make some remarks on the current structure of ancillary services markets. We also observe that, if we interpolate submitted offer bids in ancillary services markets, we obtain piece-wise affine supply functions.

4.2 Structure of the Italian electricity market

The wholesale electricity market is a marketplace where electricity is traded between electricity generators, wholesalers, and large consumers, such as industrial users or electricity retailers. It is the primary arena for buying and selling electricity in bulk quantities. In the wholesale electricity market, electricity is bought and sold through various trading mechanisms, including auctions, bilateral contracts, and power exchanges.

The Italian wholesale electricity market was established following the liberalization of the electric sector in 1999 through legislative decree D. Lgs. n. 79/1999, commonly referred to as the "Decreto Bersani." This decree aimed to promote competition in the production and wholesale activities of electricity while ensuring maximum transparency and efficiency of ancillary services. The implementation of the "Decreto Bersani" brought about significant changes, including the creation of two distinct entities that are vital players in the Italian energy sector:

- GME (Gestore dei Mercati Energetici) is responsible for organizing and managing the Italian wholesale electricity market. It collects offers from market participants, evaluates them, and communicates the results.
- Terna S.p.A. is the Transmission System Operator (TSO) in Italy. It is in charge of managing the energy transmission process and ensuring the safety and stability of the National Power System.

These components collectively form the backbone of the Italian electricity market, ensuring smooth operations and efficient trading.

The electricity exchange consists of two parts: the spot market, called Mercato a Pronti, and the forward market, called Mercato a Termine. Here, we focus on the spot market, which handles short-term energy trading. It is organized as a centralized and auction-based market, where electricity is traded through the following main markets.

- The *Day-Ahead Market* (MGP - Mercato del Giorno Prima) is the main trading platform where electricity is bought and sold for delivery on the following day. It is managed by GME. Market participants, including generators, traders, and large consumers, submit their bids and offers indicating the quantity of electricity they are willing to buy or sell and the price at which they are willing to transact. The market clears through a uniform price auction mechanism, with the highest bids for buying and the lowest offers for selling being matched until the demand and supply are balanced. The market-clearing price, known as the marginal price, is determined by the intersection of the demand and supply curves and represents the price at which all accepted offers are settled.

- The *Intraday Market* (MI - Mercato Intraday) allows market participants to adjust their positions closer to real-time to account for changes in supply and demand. It enables participants to trade electricity for delivery within a few hours up until 45 minutes before the start of the trading interval. The intraday market provides flexibility and helps balance the system by allowing participants to adjust their portfolios and optimize their positions based on real-time conditions. Intraday markets are managed by the GME through a uniform price auction mechanism.

- The *Balancing Market* (MSD - Mercato di Servizi di Dispacciamento) ensures the security of the Power System by balancing energy and voltage profiles. Offers are submitted to the GME, but they are accepted based on Terna requirements. Terna aims to provide the necessary resources for secondary power reserve, tertiary power reserve, and congestion management. MSD requires specific prerequisites for participating Production Units, including constraints on maximum capacity, technical minimums, ramping rates, start-up times, and the use of programmable energy sources. In MSD, units are remunerated at their bid prices through a pay-as-bid system, unlike MGP and MI, where the market-clearing price determines the outcome. Terna imposes convexity constraints on the bids in MSD to ensure their feasibility.

Our data encompasses information from the electricity markets MGP, MI, and MSD. It includes details of all the bids *submitted* in these markets of every hour of each day in the period 2014-2018. In this preliminary analysis, we will show examples from December 2018. In the following, we list the selected variables that we will take into consideration in our analysis:

- **DATE**: Date of the bid

- **HOUR**: Hour of the bid

- **UNIT_REFERENCE**: Coded name of the Production Unit submitting the bid.
- **PURPOSE**: For Production Units, the variable indicating the bid's purpose will be "OFF" when the intent is to sell energy or "BID" if the unit wants to buy energy. For Consumption Units, the reverse applies. In short, "OFF" means the bid is for an upward service, while "BID" means it is for a downward service.
- **STATUS**: This variable can have only two values: "ACC" (accepted) or "REJ" (rejected). Other STATUS labels ("REP" - replaced, "REV" - revoked, "INC" - inconsistent, "SUB" - submitted) are not considered in this analysis.
- **QUANTITY**: Quantity of electricity offered for the specific hour, expressed in MWh (Megawatt-hours).
- **AWARDED_QUANTITY**: The real quantity of energy to be exchanged by the Production Unit (PU) for an accepted bid, and subsequently remunerated [MWh].
- **PRICE**: The original price of the bid, expressed in €/MWh (Euros per Megawatt-hour).
- **AWARDED_PRICE**: The price at which an accepted bid is remunerated [€/MWh].

In MGP, the AWARDED_PRICE corresponds to the value of the Unique National Price (PUN) or the zonal price, and, for a given hour and zone, it will be the same for every bid of every production unit. In MSD, the AWARDED_PRICE corresponds to the submitted price (pay-as-bid auction mechanism).

We shall remark that the Italian Power System, known as "Sistema Elettrico Nazionale" (SEN), is a complex network involving production, transmission, and distribution of electric energy. It is divided into six geographical zones, each with variations in energy volumes and generation technologies. Southern regions rely heavily on wind-powered plants, while the North has a higher concentration of hydroelectric power plants, particularly in the Northern Alps. Additionally, Italy ranks seventh globally in geothermal installed capacity, with all its production plants located in the Center-North, primarily in Tuscany. The Italian Power System is also interconnected with neighboring countries through eight virtual zones, including

France, Switzerland, Austria, Slovenia, BSP, Corsica, Corsica AC, and Greece. These zones and their transmission limits are taken into consideration also in day-ahead markets. In this preliminary study, we do not take into consideration the network parameters but this is definitely a subject for further work.

4.3 Market clearing prices in Italian the day-ahead-market (MGP)

In this section, we aim to provide an estimate of the market clearing price of the Italian day-ahead market, called Mercato del Giorno Prima (MGP), and to compare this estimate with the market-clearing price obtained with the K-L PAB auction game when K goes to infinity. To this aim, we shall use as benchmark the Supply Function Equilibria game model presented in 2.3.3.

4.3.1 Demand and supply bids

In the Italian day-ahead market (MGP) both producers and consumers can participate. As anticipated, for each datetime, i.e., for each hour of each day, producers (resp. consumers) can submit multiple pairs of the form (q, p), meaning that they are willing to sell (resp. buy) the quantity q for the minimum (resp. maximum) price p.

Since we have all submitted offers and bids, we can compute the aggregate demand and offer curve for each datetime t. More precisely, let $O_t = \{1, \dots, n_t^o\}$ and $B_t = \{1, \dots, n_t^b\}$ where n_t^o denotes the total number of offers at datetime t and n_{t}^b is the total number of bids. Then, we denote with $\{q_{i,t}^o\}_{i \in O}$, $\{p_{i,t}^o\}_{i \in O}$, $\{q_{i,t}^b\}_{i \in B}$, $\{p_{i,t}^b\}_{i \in B}$ the sets gathering all submitted offer quantities and prices and bid quantities and prices. The aggregate demand curve at datetime t is then given by

$$D_t(p) := \sum_{i \in O_t,\, p_{i,t}^o \geq p} q_{i,t}^o \,. \tag{4.1}$$

while the aggregate supply curve is

$$S_t(p) := \sum_{i \in B_t,\, p_{i,t}^b \leq p} q_{i,t}^b \,. \tag{4.2}$$

Notice that the demand and supply curve are not continuous. In the following, we shall compute the market-clearing price as

$$p_t^* = \min\{p : D_t(p) \le S_t(p)\}.$$

Example 4.1. *Let us consider the datetime t = 2018-12-01 12:00:00. In the upper plot of Figure 4.1, we can see the aggregate demand and supply defined in* (4.1) *and* (4.2)*, respectively. Observe that many bid prices are equal to* 3000 *€/MWh. This is the maximum price that consumers can bid in the auction. If a consumer submits a bid with price* 3000 *€/MWh it means that she will buy the bid quantity of electricity at any market-clearing price. Similarly, there are many offers with the price* 0 *€/MWh. This means that producers will produce the offered quantity regardless of the market-clearing price. We remark that this behavior can be observed only in uniform price auctions, since their remuneration is not the bid price. These bids and offers are not competitive in the market. They only change the total quantity offered/demanded in the market. In the lower part of Figure 4.1, we plot the aggregate demand and supply curves excluding these offers and bids.*

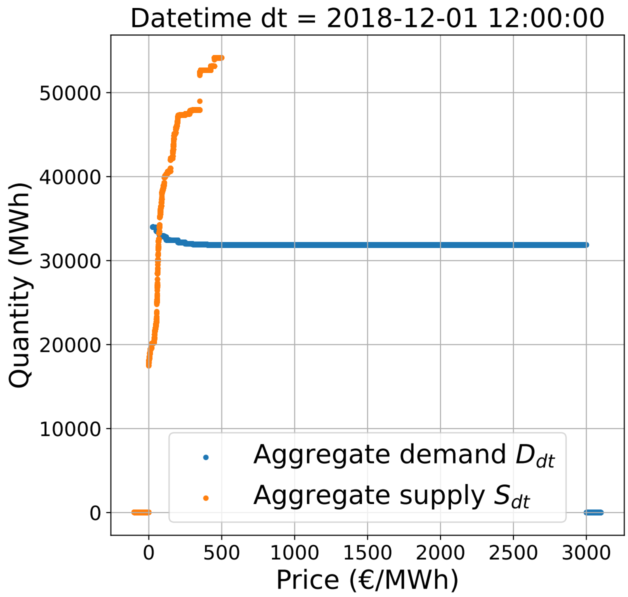
In the following, we will compare the market-clearing price resulting from the intersection of total demand and total supply with the market-clearing price at SFE and at Nash equilibrium of the K-L PAB auction game when K goes to infinity.

4.3.2 Model and parameter estimation

In this section, we will briefly recall the models presented in the previous chapters. We will then explain the proposed method to estimate the parameters and the market-clearing prices at Nash equilibrium.

Recall that, in both the K-L PAB auction game model and the SFE game model, the agent set is given by $\mathcal{N} = \{1, \dots, n\}$ where n is the number of producers competing in the market. The aggregate demand curve is assumed to be given and known to all producers and that it takes the affine form in (3.18) in Section 3.4. Therefore, for each datetime t, we aim to find an estimation for

$$D_t(p) = N_t - \gamma_t p$$



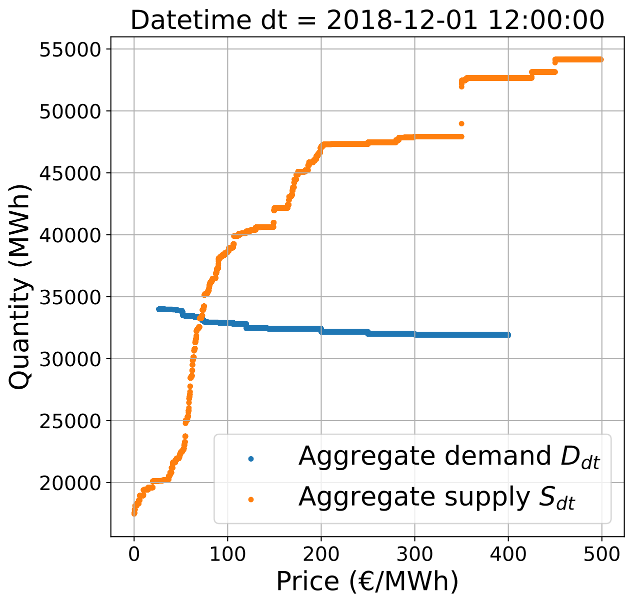


Fig. 4.1 Aggregate demand and supply for the datetime 2018-12-01 12:00:00

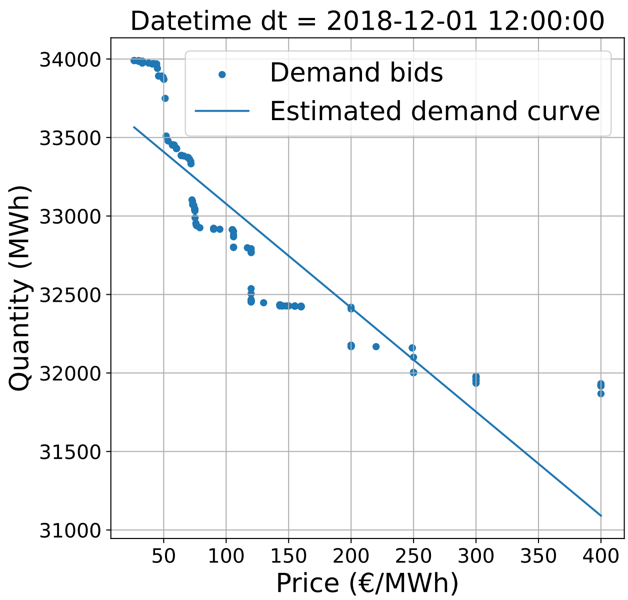


Fig. 4.2 Estimated affine demand curve

with $N_t \geq 0$ and $\gamma_t > 0$. Also, recall that production costs are assumed to be quadratic cost functions of the form in (3.19) in Section 3.4, i.e.,

$$C_i(q) = \frac{1}{2} c_i q^2$$

with $c_i > 0$ for all i in $\mathcal{N}$.

Therefore, the input parameters of the model are, for each datetime t, the total demand for zero price $N_t \geq 0$, the sensitivity to the price $\gamma_t > 0$, the number of producers n_t that are actively participating in the auction and their cost parameters $c_i > 0$ for all i in $\mathcal{N}$ that we assume to be independent from t.

We estimate the parameters $N_t \geq 0$ and $\gamma_t > 0$ through a simple linear regression on the dataset $\{p^b_{i,t}, D_t(q^b_{i,t})\}_{i \in B_t}$. Since the price 3000 €/MWh is meaningless, we exclude all consumers bidding this price from the estimation, that is, for all i in B_t, we consider only the pairs $(p^b_{i,t}, D_t(q^b_{i,t}))$ where $i \in B_t$ and $p^b_{i,t} < 3000$ €/MWh. These bids then only contribute for the estimate of the parameter N_t but they do not influence the parameter γt. In Figure 4.2, we see the obtained estimated curve for the datetime t = 2018/12/01 12:00:00. In this case, we obtain $N_t = 33739.6$ MWh and $\gamma_t = 6.6$ MWh^2/€.

	TR01_UNIT_REFERENCE_NO	max_mgp_qty	TR01_AWARDED_PRICE_NO	c
853	UP_LEVANTE_3	335.700	75.00	0.402145
596	UP_DI2513_NORD_C	62.024	64.62	1.875339
1180	UP_S.FIORANO_1	267.441	74.49	0.501352
974	UP_NCTLVRNFRR_1	785.000	48.06	0.110201
742	UP_FOIANO2_1	68.135	48.09	1.270448
505	UP_DI2055_NORD_C	346.831	68.79	0.357010
929	UP_MONCALRPW_2	378.400	58.98	0.280560
987	UP_NPWRBRNDSI_9	349.000	58.54	0.301926
1126	UP_PRTMPDCLCL_3	76.000	142.84	3.383053
992	UP_NPWRLVORNO_7	100.000	96.66	1.739880

Fig. 4.3 Samples of cost parameters.

For this preliminary study, we estimate cost parameters in a data-based manner. Recall that we have to estimate one parameter, i.e., $c_i \geq 0$ for each producer i in $\mathcal{N}$. We estimate these parameter in three steps. First, we compute the maximum sold quantity in the current month, denoted with $q_i^{\max}$, to find an estimate of the *maximum capacity* of each production unit. Then, we compute the minimum marginal price, denoted with $p_i^{\min}$, at which the maximum quantity was sold in the month. Finally, assuming that the marginal cost of producing the maximum quantity is the minimum accepted price, we impose that the cost function crosses such point, i.e.,

$$C_i(q_i^{\max}) = \frac{1}{2} c_i (q_i^{\max})^2 = q_i^{\max} p_i^{\min} .$$

We remark that, if firms always have positive utility, this is an overestimation of the costs of the firm. On the other hand, costs change in time due to the changes in fuel costs. Therefore, the minimum at which the total quantity was sold in all the month could be close to the actual value. In Fig. 4.3, we sample 10 production units and show their cost estimation.

Finally, we need to select the production units that are actively participating in the market. On December 1^{st}, 2018, at 12:00, there are 1211 production units submitting bids in MGP. Among them, there are 597 producers submitting non zero-offers. Many of these producers are very small or submit some offers that are very low and that are not significant in determining the market-clearing price. Also, there

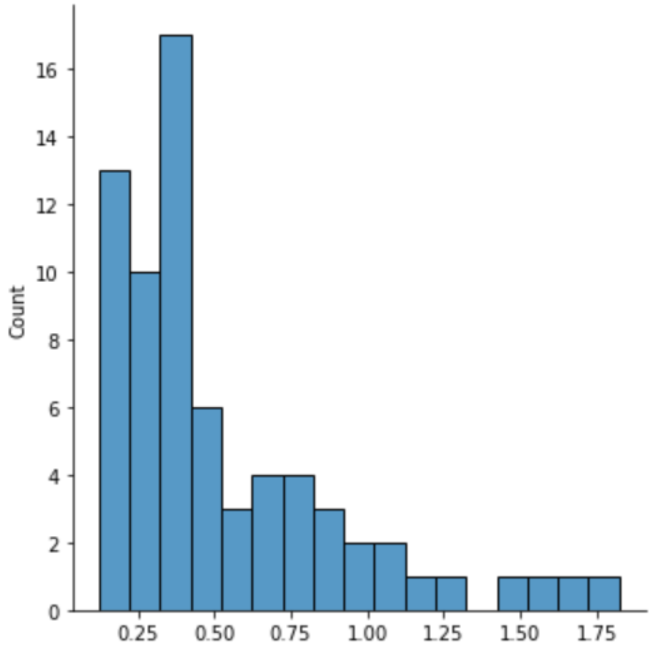


Fig. 4.4 Distribution of cost parameters.

are production units with very small capacities. In our analysis, we selected firms satisfying the following conditions:

1. the submitted offered price is higher than the lowest price of the month and lower than the highest price of the month;

2. their capacity, estimated as maximum sold quantity in the month, is higher than a minimum quantity value, for example, 50 MWh.

We include in the set $\mathcal{N}_t = \{1, \dots, n_t\}$ only the firms satisfying these two conditions. In Fig. 4.4, we show the distribution of the cost parameters for the 71 firms selected on December 1st, 2018, at 12:00:00.

Now that we have an estimate of the game parameters, we can compute the market-clearing prices with the two different auction mechanisms. According to Section 2.3.3, if we consider the Supply Function equilibria game model, we find that linear supply functions of the form:

$$S_{i,t}(p) = \beta_{i,t} p \quad \forall i \in \mathcal{N}$$

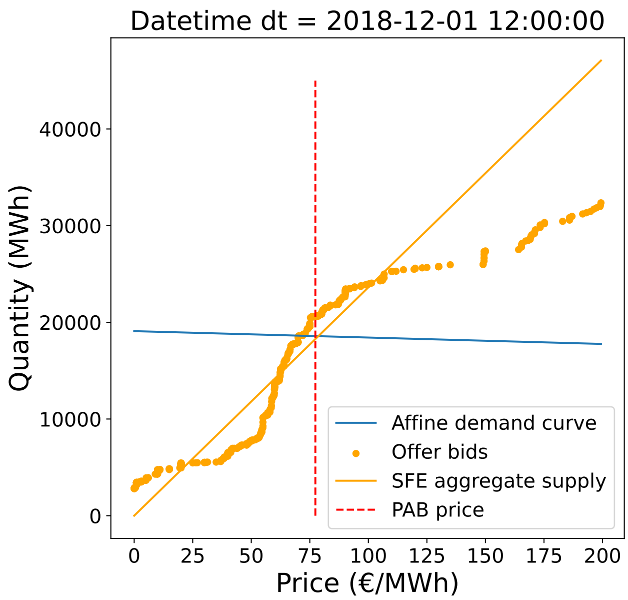


Fig. 4.5 Comparison among data market-clearing price, SFE resulting price and PAB resulting price.

are Nash equilibria if and only if β_i, for all i in $\mathcal{N}$, satisfy

$$\beta_{i,t} = \frac{\gamma_t + \sum_{i \neq j} \beta_{j,t}}{1 + c_i\left(\gamma_t + \sum_{i \neq j} \beta_{j,t}\right)}.$$

The resulting market-clearing $p^*_{\mathrm{SFE},i}$ will be:

$$p^*_{\mathrm{SFE},t} = \frac{N_t}{\sum_i \beta_{i,t} + \gamma_t}.$$

If we consider the K-L PAB auction game model, with K that goes to infinity, we find the market-clearing

$$p^*_{\infty,t} = \frac{N_t}{\sum_i 1/c_i + \gamma_t}.$$

Below, we present a specific example.

Example 4.2. *If we apply our study for the datetime t =2018-12-01 12:00:00, we obtain the prices* $p^*_t = 71.4$, $p^*_{SFE,t} = 82.2$ *and* $p^*_{\infty,t} = 80.6$. *In Fig. 4.5, we plot the market-clearing price that we observed on that datetime, the aggregate supply curve at SFE and the corresponding market-clearing price and the market-clearing price*

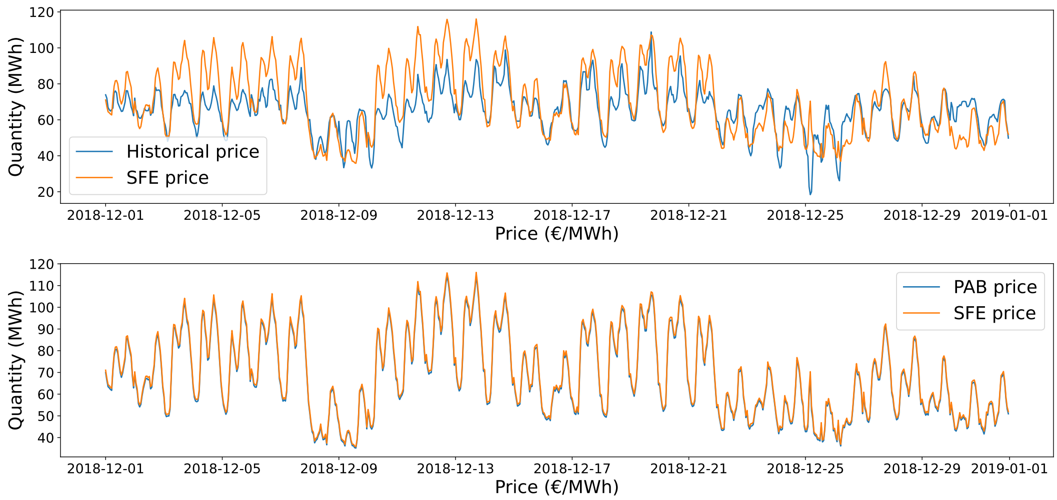


Fig. 4.6 In the upper panel, market-clearing price estimation with SFE game model for December 2018. In the lower panel, comparison between SFE and PAB market-clearing prices.

with a K-L PAB auction when K goes to infinity. Observe that the pay-as-bid auction gives a market-clearing price very close to the SFE market-clearing price.

4.3.3 Preliminary analysis and current work

In this section, we show the estimates that we obtain for the market-clearing prices at Nash equilibria in the SFE game model and in the K-L PAB auction game when K goes to infinity. We apply the previous reasoning to all December 2018. We recall that the auction takes place at every hour of every day.

In the upper panel of Fig. 4.6, we show the market-clearing price at Nash equilibrium in the SFE game model and the market-clearing price observed from data. In this simulation, we set, when selecting the firms, the minimum quantity to 100 MWh. Observe that, sometimes, we are overestimating the price. This is probably due to the rough estimation of costs. In the lower panel of Fig. 4.6, we observe that market-clearing price at SFE is very close to the market-clearing price at Nash equilibrium in the K-L PAB auction when K goes to infinity. This would suggest that, with full information, there is not a big difference between uniform-price auctions and pay-as-bid auctions.

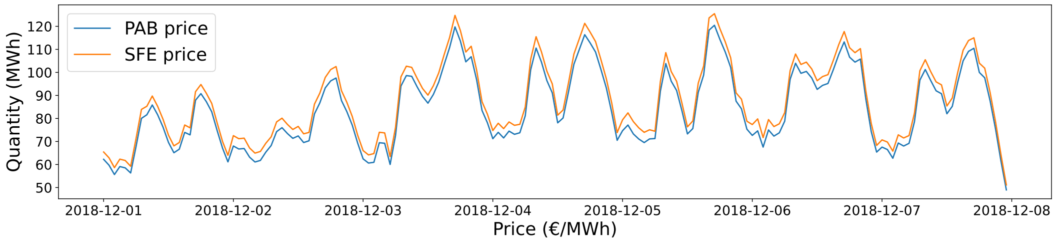


Fig. 4.7 Market-clearing price estimate with minimum quantity 600 MWh.

In Fig, 4.7, we show the market-clearing price at SFE and when we select only producers with minimum quantity 600 MWh. In this case, we have an average of 15 firms selected, i.e., a smaller number of producers. This is not a good estimation of the resulting market-clearing price in the day-ahead market. Anyway, we show the simulation with aim of pointing out that we see some more differences in the uniform price remuneration versus the pay-as-bid remuneration when we reduce the number of agents. In general, we observe that the difference between the two market-clearing prices increase when considering a lower number of agents, or by increasing differences in the costs.

Based on these very preliminary observations, we are currently working on two main directions.

On one hand, we would like to provide better estimates of the parameters, especially those related to the production costs. We would like to include in our analysis a clustering among production units that takes into consideration their technology. Also, in the model, production costs and total demand are assume to be common knowledge, although in reality they are not. Therefore, we would like to use past data to predict the parameters. More precisely, we plan to use timeseries to predict demand parameters. Also, we have already tried to estimate cost parameters based on the data from the previous months and we are observing similar results.

On the other hand, we would like to improve our model to make it more realistic. More precisely, we made the two main following observations:

- we can find a better estimates of the production costs by considering quadratic cost functions of the form

$$C_i(q) = a_i q + c_i q^2$$

with $a_i, c_i \geq 0$ for all i in $\mathcal{N}$. We believe that it is possible to characterize Nash equilibria also in this setting, although the generalization is not straightforward.

- In order to have a wider range of costs and observe more differences among the different auctions, we must take into consideration capacity constraints. Further work includes a deeper analysis in this direction.

Finally, fundamental steps are to introduce uncertainty in the demand and consider the network in the model.

4.4 Supply functions in ancillary services markets (MSD)

In this section, we would like to make some brief considerations on Italian ancillary services markets.

As anticipated, MSD has some peculiarities compared to the other markets, such as using a "Pay-as-bid" mechanism for determining the awarded price, and it includes additional variables SCOPE (type of service offered) and ADJUSTED_PRICE (price corrected by TSO to meet market constraints). There are four main types of bids:

- AS + OFF: Bids to switch on energy production, from zero to the minimal operating power of the PU.
- AS + BID: Bids to switch off energy production, from the minimal operating power of the PU to zero.
- GR + OFF: Step bid to increase the amount of electricity produced, covering the entire operating range from the current operating point to the maximal power.
- GR + BID: Step bid to decrease the amount of electricity produced, covering the entire operating range from the current operating point to the minimal power.

The "AS" bids relate to the overall functioning of the PU, allowing Terna to control whether the unit is producing electricity or not in each hour. The "GR" bids provide flexibility for reserve-related problems.

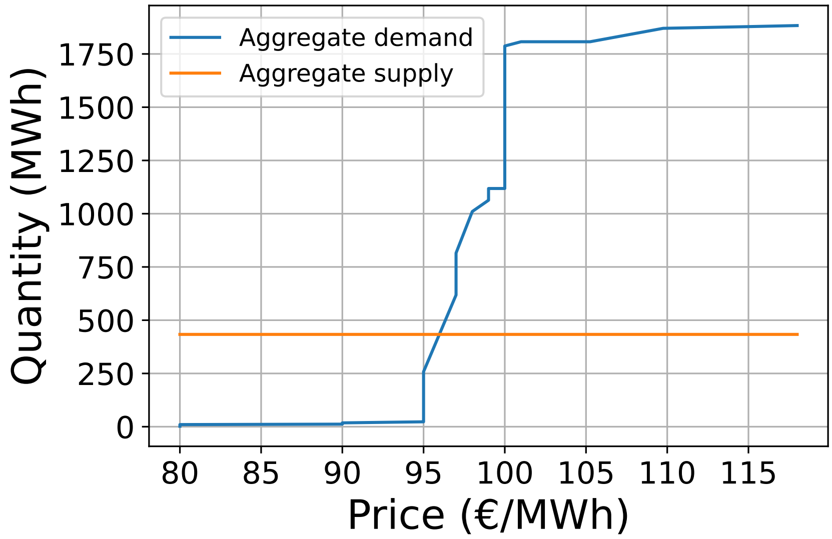


Fig. 4.8 Aggregate demand and supply curves for Example 4.3

Terna imposes convexity constraints on the bids in MSD to ensure their feasibility. These constraints require that downward step bids are considered before "AS+BID" offers and that upward step bids are considered before "GR+OFF" offers, following the PU's operating range.

MSD is the most complex market due to the unpredictability of issues that drive its existence and the difference in the rewarding system for operators. Terna establishes the requirements for this market to ensure the stability and secure operation of the Power System. Offers and bids are therefore accepted according to needs of the Transmission System.

We remark that Terna's strategy in accepting the bids is unknown as well as the real Italian transmission network. Indeed, we observed that bids are usually not accepted in merit order and, in particular, merit order is usually not even respected in the same grid point. Therefore, we are currently creating some clusters based on offered services, technologies and capacities in order to determine which production units are actually competing for that demand. We shall briefly present one example.

Example 4.3. *In this example, we consider data for the datetime t = 2018-12-20 08:00:00 and we focus on offers from production units in the North of Italy. In this cluster, we considered only production units that did not sell anything in the day-ahead market ("ignition" service) and that sold in MSD an overall quantity greater that* 500 *MWh in the month of December (many production units never sell*

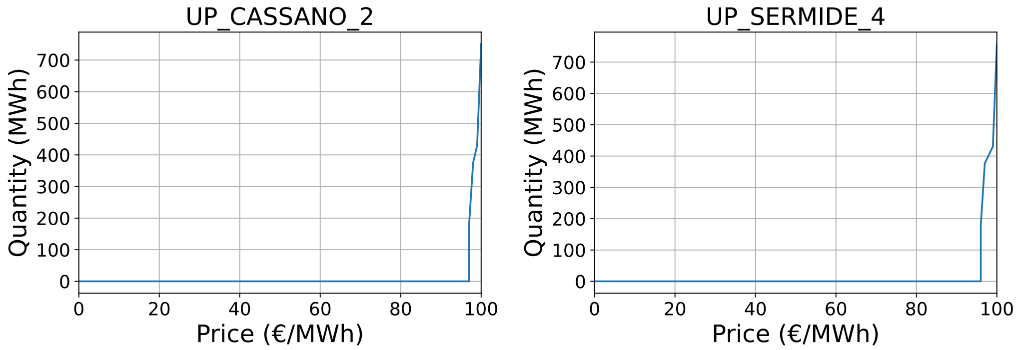


Fig. 4.9 Piece-wise affine supply functions for Example 4.3.

in MSD). In this way, we were able to see some merit order. In Fig. 4.8, we computed the total demand and supply curve. Notice that the demand is constantly equal to the total sold quantity. We used this approximation as sensitivity to price is very low in ancillary services market as the service must be provided in some way. The market-clearing price computed in this way is 96*, while the maximum accepted price among firms is* 97*. These values are very closed compared to those that we would obtain by computing total supply regardless of the cluster. Observe that submitted bids for this cluster do take the form of piece-wise affine supply functions (see Fig. 4.9). The two supply curves for production unit* 1 *and* 2 *are computed based on the bids* $q_1 = [180.0, 375.0, 430.0, 751.0]$*,* $p_1 = [97.0, 98.0, 99.0, 100.0]$ *and* $q_2 = [180.0, 377.0, 430.0, 758.0]$*,* $p_2 = [96.0, 97.0, 99.0, 100.0]$*, respectively.*

Also, we remark that, in MSD, quantities can be sold in or bought by producers as they can rearrange the quantities sold in MGP. Therefore, further work includes a generalization of our model in a way to include upward and downward bids. More precisely, we would like to study a concatenation of a uniform price auction and a pay-as-bid auction modeled as a two stage game.

Chapter 5

Conclusion and Further Research

5.1 Conclusion

In this dissertation, we present and examine the pay-as-bid auction game, a supply function model with pay-as-bid remuneration and asymmetric firms. In our model, strategies are functions relating price to quantity, similar to Supply Function Equilibria (SFE) game models. However, unlike traditional SFE models, we incorporate pay-as-bid remuneration and do not consider demand uncertainty. It is worth noting that pay-as-bid auctions have received relatively less attention in the literature compared to uniform-price auctions, particularly within the context of supply function models, despite having several applications in electricity markets, as for instance in ancillary service markets. The limited existing results establish existence of Nash equilibria under certain conditions on the demand distribution and involve solving a set of differential equations. In contrast, our fundamental finding is that, by limiting the strategy space to K-Lipschitz supply functions, pure-strategy Nash equilibria exist and they can be represented as piece-wise affine functions with slope K. More precisely, Nash equilibria of the K-Lipschitz (K-L) pay-as-bid (PAB) auction game can be fully characterized starting from Nash equilibria of a restricted game with continuous scalar actions. This result paves the way to our comprehensive analysis of the game. To the best of our knowledge, there are no similar findings available in existing literature.

The theoretical analysis of the pay-as-bid auction game is carried out in Chapter 3. In the first part of the chapter, we deal with the game in its general form. Our main contributions are the following:

- we introduce the pay-as-bid auction game and reveal that in its general form, Nash equilibria do not exist;
- we show that, when all agents are constrained to choose K-Lipschitz supply functions as their strategies, not only do best responses exist, but they have a straightforward structure represented by piecewise affine functions with slope K. Best responses can be then characterized by a single scalar value per agent;
- we demonstrate, as a consequence, that Nash equilibria of the original K-L PAB auction game correspond to equilibria of a finite-dimensional game, where bidders must choose these scalar parameters;
- we analyze this finite-dimensional game, demonstrating that the utility functions are continuous and quasi-concave with respect to these parameters, thus proving the existence of Nash equilibria of the restricted game;
- we prove existence and characterization of Nash equilibria for the K-L PAB auction game, as a consequence.

In the second part of the chapter, we focus on the pay-as-bid auction game with affine demand and quadratic costs. Our main contributions are the following:

- we prove the uniqueness of Nash equilibria for the restricted game and explicitly compute them, revealing that all Nash equilibria of the K-L PAB auction game yield the same market-clearing price and utilities for all agents;
- we compute the market-clearing price, utilities, and sold quantities as K tends towards infinity;
- we use the derived closed-form expression to demonstrate that the market-clearing price of the pay-as-bid auction game at Nash equilibrium lies between the Bertrand and Cournot oligopoly models and it is lower compared to the market-clearing price in Supply Function equilibria.

In the last part of the chapter, we make the important observation that the restricted game is not supermodular in general, even in the simplest scenario involving two agents, affine demand, and quadratic costs. This finding is significant as it demonstrates that our result on the existence of Nash equilibria of the restricted game is non-trivial. However, we notice that, when the demand is affine or quadratic, the utilities satisfy the increasing difference property within a relevant subset of the strategy space. The ongoing research involves conducting a more thorough analysis in this specific direction.

Alongside our theoretical analysis, we also conduct a preliminary exploratory analysis of data from the Italian wholesale electricity market. In Chapter 4, we make the following preliminary observations:

- when examining data from day-ahead markets, we find that the market-clearing price estimates in the pay-as-bid auction game closely align with those obtained from SFE game models. However, as the number of agents decreases, the differences between the two models become more apparent;
- analyzing data from ancillary services markets, we observe that the submitted offer bids exhibit characteristics similar to piece-wise affine supply functions.

5.2 Ongoing work

Some current work is introduced in the dissertation. In particular, in Section 3.5, we observe that, when the demand is affine or quadratic, the utilities of the restricted game satisfy the increasing difference property in a relevant subset of the strategy space, where utilities are differentiable. Also, in Section 3.4.1, we prove that the analysis of Nash equilibria can be done in this subset. Although properties of supermodular games do not apply directly to constrained games, our conjecture is that we might use insights from supermodularity to compute and characterize Nash equilibria also for general costs.

We also aim to validate the model using data from the Italian electricity market, improving the preliminary analysis presented in Chapter 4. Specifically, we seek to enhance the accuracy of production cost estimates and extend the analysis of day-ahead markets to a broader framework. Despite the inherent complexity of ancillary services markets, we are determined to refine our analysis in this domain. One specific area of improvement is the clustering of production units, which will offer deeper insights into the strategic behavior of both agents and the Transmission System Operator (Terna). Our preliminary analysis indicates that, currently, production units are not behaving strategically due to uncertainties in demand and a lack of knowledge regarding Terna's bid acceptance strategy.

Finally, current work includes incorporating the network structure of electricity markets into our model, which is essential given the crucial role of the network in the transmission and distribution of electricity. The problem of incorporating the network structure in electricity wholesale markets is addressed in [39] within the context of the Cournot competition. We are currently working on exploiting the network properties and the role of the market maker in the equilibrium outcome of networked Cournot competitions [40, 91]. We also aim to extend the same framework to the pay-as-bid auction game. In the extended model, we contemplate a network of interconnected markets, where edges have capacity constraints. Producers can offer their supply functions in a single market, while a market maker determines the flows among the markets with the goal of optimizing a function of social welfare. Our ongoing research is directed towards identifying conditions related to demand and costs under which utilities are concave or quasi-concave, thereby establishing the existence of Nash equilibria.

5.3 Future research

Future research endeavors will be dedicated to refine our model to better align with the complexities and realities of electricity markets. To achieve this, the following specific goals have been identified:

- generalizing the analysis of Section 3.4 to encompass affine marginal costs, allowing for a more realistic representation of production costs in electricity markets;
- addressing the complexities arising from capacity constraints, as they play a crucial role in determining the feasibility and efficiency of market operations;

- accounting for demand uncertainty, a pervasive aspect of real-world electricity markets. By exploring the significance of uncertainty in demand, we can better assess the robustness and resilience of the proposed models and strategies.

Other main directions for future research include:

- relaxing the assumption on K-Lipschitz supply functions and exploring more general conditions on the strategy set that ensure the existence of best responses and Nash equilibria. This could involve investigating less restrictive bounds or incorporating discontinuous functions. Optimal control approaches and variational inequalities might be useful tools for studying this problem;
- identifying more general conditions for the uniqueness of Nash equilibria, which will provide valuable insights into the stability and convergence properties of the auction game model;
- studying the dynamics of the game. Understanding the evolution and behavior of the game over time can be crucial in capturing real-world market dynamics;
- investigating the combination of a uniform-price auction and a pay-as-bid auction as a two-stage game. Drawing inspiration from the structure of existing electricity markets, understanding the interplay between these auction mechanisms can offer valuable insights into their combined impact and potential benefits. This exploration can contribute to designing more efficient and effective auction models for the electricity market.

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Appendix A

Existence and uniqueness of Nash equilibria in continuous games

A function $f : X \to \mathbb{R}$ is *quasi-concave* if, for all $c \in \mathbb{R}$, the upper level set $P_c = \{x \in X \mid f(x) \geq c\}$ is convex. If $X \subseteq \mathbb{R}$, then a quasi-concave function f can be characterized in the following way.

Proposition A.1. *Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$. Then, f is quasi-concave on $[a,b] \subseteq X$ if and only if one of the following conditions holds:*

- *f is nondecreasing;*
- *f is nonincreasing;*
- *there exists $x^* \in (a,b)$ such that f is nondecreasing on $[a,x^*]$ and nonincreasing on $(x^*,b]$;*
- *there exists $x^* \in (a,b)$ such that f is nondecreasing on $[a,x^*)$ and nonincreasing on $[x^*,b]$.*

Proof. Suppose that f satisfies one of the conditions and let $c \in [a,b]$ and $x_1, x_2 \in P_c$. Since $f(x_1) \geq c$ and $f(x_2) \geq c$, we have that $f(x) \geq c$ for every $x \in [x_1, x_2]$. Thus $x \in P_c$, so that P_c is convex and hence f is quasi-concave.

If f does not satisfy any of the conditions then we can find x_1, x_2, and x_3 in $[a,b]$ such that $x_1 < x_2 < x_3$ and $f(x_2) < \min\{f(x_1), f(x_3)\}$. Then the upper level set P_c

for $c = \min\{f(x_1), f(x_3)\}$ includes x_1 and x_3, but not x_2, and hence is not convex, so that f is not quasi-concave. □

To show that a game has a Nash equilibrium it suffices to show that there is a profile x^* of actions such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*) \text{ for all } i \in \mathcal{N}\,. \tag{A.1}$$

Define the set-valued function $\mathcal{B} : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}^{\mathcal{N}}$ by

$$\mathcal{B}(x) = \times_{i\in\mathcal{N}} \mathcal{B}_i(x_{-i})\,. \tag{A.2}$$

Then (A.1) can be written in vector form as $x^* \in \mathcal{B}(x^*)$. Fixed point theorems give conditions on $\mathcal{B}$ under which there indeed exists a value of x^* for which $x^* \in \mathcal{B}(x^*)$. The fixed point theorem that we use is the following (due to [74]).

Lemma A.1 (Kakutani's fixed point theorem)**.** *Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which*

- *for all $x \in X$ the set $f(x)$ is nonempty and convex;*
- *the graph of f is closed (i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$, and $y_n \to y$, we have $y \in f(x)$).*

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

We can now state the fundamental result in game theory on existence of pure-strategy Nash equilibria (see [73], pp. 19-20), which is an application of Kakutani's fixed-point theorem to game theory following insights proposed in [75], [76], and [77].

Theorem A.1. *The strategic game $(\mathcal{N}, \mathcal{A}, \{u_i\}_{i\in\mathcal{N}})$ has a Nash equilibrium if for all $i \in \mathcal{N}$*

- *$\mathcal{A}$ is a nonempty compact convex subset of an Euclidian space*

and the utility function $u_i(x_i, x_{-i})$ is

- *continuous in x_{-i},*
- *continuous and quasi-concave in x_i.*