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Calculus of Variations

The nodal set of solutions to some nonlocal sublinear problems

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Abstract

We study the nodal set of stationary solutions to equations of the form $(-\Delta)^s u = \lambda_+(u_+)^{q-1} - \lambda_-(u_-)^{q-1}$ in B_1, where $\lambda_+, \lambda_- > 0$, $q \in [1, 2)$, and u_+ and u_- are respectively the positive and negative part of u. This collection of nonlinearities includes the unstable two-phase membrane problem $q = 1$ as well as sublinear equations for $1 < q < 2$. We initially prove the validity of the strong unique continuation property and the finiteness of the vanishing order, in order to implement a blow-up analysis of the nodal set. As in the local case $s = 1$, we prove that the admissible vanishing orders can not exceed the critical value $k_q = 2s/(2-q)$. Moreover, we study the regularity of the nodal set and we prove a stratification result. Ultimately, for those parameters such that $k_q < 1$, we prove a remarkable difference with the local case: solutions can only vanish with order k_q and the problem admits one dimensional solutions. Our approach is based on the validity of either a family of Almgren-type or a 2-parameter family of Weiss-type monotonicity formulas, according to the vanishing order of the solution.

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Contents

1 Introduction

The analysis of the nodal set of solutions of elliptic equations has been the subject of an intense study in the last decades, starting from the works [9, 19–21], with a special focus on the measure theoretical features of its singular part.

These works provide a fairly complete picture of the geometric structure of the nodal set in the case of solutions of linear equations and they easily extend to a wide class of superlinear equations of type $-\Delta u = f(u)$, provided that the nonlinearity is locally Lipschitz continuous, that $f(0) = 0$ and that $u \in L^\infty_{\mathrm{loc}}$. From a geometric point of view, the nodal set of a weak solution of class C^1 splits into a regular part, which is locally a C^1 graph, and a singular set which is a countable union of subsets of sufficiently smooth $(n-2)$-dimensional manifolds. Moreover these equations satisfy the strong unique continuation principle and the solutions vanish with finite integer order (see e.g. [15, 16, 21]). A similar structure also holds under weaker assumptions, that is, for weak solutions of linear equations in divergence form with Lipschitz coefficients and bounded first and zero order terms (see [19]).

Instead, the picture change drastically if we switch to semi-linear elliptic equations with non-Lipschitz nonlinearities: given $q \in [1, 2)$, let us consider for example the class of equations

$$-\Delta u = \lambda_+(u_+)^{q-1} - \lambda_-(u_-)^{q-1} \quad \text{in } B_1, \tag{1}$$

where $\lambda_+, \lambda_- > 0, q \in [1, 2)$, B_1 is the unit ball in $\mathbb{R}^n$ and $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$ are respectively the positive and negative part of u. Notice that the main feature of these equations stays in the fact that the right hand side is not locally Lipschitz continuous as function of u, and precisely has sublinear character for $q \in (1, 2)$ and discontinuous behaviour for $q = 1$. It is well known in the literature that in the case $\lambda_+, \lambda_- \leq 0$, the features of the nodal set of solutions are substantially different in comparison with the linear case since dead cores appear and no unique continuation can be expected.

However, in the unstable setting the solutions resembles some features of the linear case. Indeed, recently in [33] have been proved the validity of the unique continuation principle for every $q \in [1, 2)$ by controlling the oscillation of the Almgren-type frequency formula for solutions with a dead core. On the other hand, in [25] has been shown that the strong unique continuation principle holds for every $q \in (1, 2)$, with an alternative approach based on Carleman's estimate: in both papers it has been emphasized that the standard approaches are not applicable in the sublinear and discontinuous cases and have to be considerably adjusted. Finally, in [31] the authors investigate the geometric properties of the nodal set and the local behaviour of the solutions by proving the finiteness of the vanishing order at every point and by studying the regularity of the nodal set of any solution. More precisely, they show that the nodal set is a locally finite collection of regular codimension one manifolds up to a residual singular set having Hausdorff dimension at most $(n-2)$.

Ultimately, the main features of the nodal set are strictly related to those of the solutions to linear (or superlinear) equations, with a remarkable difference: the admissible vanishing orders can not exceed the critical value $k_q = 2/(2-q)$. Moreover, at this threshold, they proved the non-validity of any estimates of the $(n-1)$-dimensional measure of the nodal set of a solution in terms of the vanishing order.

The purpose of this paper is to study the structure of the nodal sets of nontrivial solutions to

$$(-\Delta)^s u = \lambda_+ (u_+)^{q-1} - \lambda_- (u_-)^{q-1} \quad \text{in } B_1, \tag{2}$$

where $\lambda_+, \lambda_- > 0$, $q \in [1, 2)$, $s \in (0, 1)$ and the fractional Laplacian is defined by

$$(-\Delta)^s u(x) = C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \mathrm{d}y \quad \text{with } C(n, s) = \frac{2^{2s} s \Gamma(\frac{n}{2} + s)}{\pi^{n/2} \Gamma(1-s)} .$$

This study is driven by the wish to extend the previous theory to the fractional setting emphasizing the possible difference between the two type of operators due to the nonlocal attitude of the equations. Starting from the problem of unique continuation, many result have been achieved in the study of the nodal set of solution of nonlocal elliptic equation, in particular by using local realisation of the fractional powers of the Laplacian based on the extension technique popularized by the authors in [5]. Also in this setting, the key tools in proving unique continuation in the linear case are based on the validity of an Almgren-type monotonicity formula (see [11, 12, 17]), or Carleman estimates (see e.g. [23, 24]), which are not applicable in a standard way in our case.

In a slightly different direction, researcher also analyzed the structure of the nodal sets from the geometric point of view by classifying the possible local behaviour of solution near their nodal set: recently in [28] the authors provided a stratification result for the nodal set of linear equation by applying a geometric-theoretic analysis of the nodal set of solutions to degenerate or singular equations associated to the extension operator of the fractional Laplacian. In particular, they proved the existence of two stratified singular sets where the solution either resembles a classical harmonic function or a generic polynomial: in the first case, the stratification coincides with the one of the nodal set of solutions of local elliptic equations; in the second one a stratification still occurs but the bigger stratum is contained in a countable union of $(n-1)$-dimensional $C^{1,\alpha}$ manifolds, in contrast with the local case $s = 1$ (see [28,Section 8] for more detail in this direction).

On the other hand, the picture changes considerably in the case of solution with either sublinear $q \in (1, 2)$ or discontinuous $q = 1$ nonlinearity, as in (2). Indeed, it is clear that in the case $\lambda_+, \lambda_- \leq 0$ (where the signs of the coefficients are opposite to ours), the features of the nodal set of solutions are substantially different in comparison with the linear case: dead cores appear and no unique continuation can be expected. In those scenarios one may try to describe the structure and the regularity of the free boundary $\partial\{u = 0\}$. When $q \in (1, 2)$ we refer to [36, 37] where the authors consider an Alt-Phillips type functional in the fractional setting for the case of non-negative solutions $u \geq 0$; while for the case $q = 1$ the equation is the so called two phase obstacle problem and we refer to [2, 3] and reference therein. Since in the fractional case minimisers of the two-phase obstacle problem do not change sign, we refer to [18, 26] for some general result in the one-phase setting.

In contrast, very little is known about the structure of the nodal sets in the case $\lambda_+, \lambda_- > 0$. In [1] the authors considered the unstable two-phase obstacle problem $q = 1$ and they proved that separation of phases does not occur in the unstable setting. Moreover, they characterized the local behaviour of minimisers near the free-boundary and they proved a bound on the Hausdorff dimension of the singular set.

In this paper we deal with the two phases problem (2), treating simultaneously the case $q = 1$, which we call unstable two phase membrane problem and the case $q \in (1, 2)$, a prototype of sublinear equation. Notice that our results slightly extend the classification of blow-up limit obtained for local minimisers in [1] to weak solution of (2) satisfying (4).

Statements of the main results. Let $u \in H^s(\mathbb{R}^n)$ be a weak solution of (2) in the sense of distributions. Exploiting the local realisation of the fractional Laplacian deeply explained in [5], through the paper we will developed a local analysis of solution of the extended problem in $\mathbb{R}^{n+1}_+$ (see Sect. 2 for more details). Hence, let us consider a weak solution $u \in H^{1,a}(B_1^+)$ of

$$\begin{cases} L_a u = 0 & \text{in } B_1^+ \\ -\partial_y^a u = \lambda_+(u_+)^{q-1} - \lambda_-(u_-)^{q-1} & \text{on } \partial^0 B_1^+, \end{cases} \tag{3}$$

where $a = 1 - 2s \in (-1, 1)$,

$$L_a u = \mathrm{div}(y^a \nabla u), \quad \partial_y^a u(x, 0) = \lim_{y \to 0^+} y^a \partial_y u(x, y)$$

and

$$B_r^+(X_0) = B_r(X_0) \cap \{y > 0\}, \quad \partial^0 B_r^+(X_0) = B_r(X_0) \cap \{y = 0\},$$

where $B_r(X_0)$ denote the ball of center X_0 and radius r in $\mathbb{R}^{n+1}$ (through the paper we will simply denote $B_r^+(0)$ with B_r^+). Moreover, if $X_0 \in \Sigma$, we will denote as $S_r^{n-1}(X_0)$ the boundary of $\partial^0 B_r^+(X_0)$ in Σ, that is the $(n-1)$-dimensional sphere of radius r centered at X_0. From now on, we simply write "solution" instead of "weak stationary solution", for the sake of brevity. Through the paper we will always denote with $\Gamma(u) = \{(x, 0) \colon u(x, 0) = 0\}$ the restriction of the nodal set of u on $\{y = 0\}$.

According to Definition 2.1, through the paper we will consider solutions of (3) satisfying

$$\begin{aligned} &\frac{1-n-a}{2}\int_{B_r^+(X_0)} y^a |\nabla u|^2 \,\mathrm{d}X + \frac{r}{2}\int_{\partial^+ B_r^+(X_0)} y^a |\nabla u|^2 \,\mathrm{d}\sigma = \\ &= r\int_{\partial^+ B_r^+(X_0)} y^a (\partial_r u)^2 \mathrm{d}\sigma + \int_{\partial^0 B_r^+(X_0)} \langle x - x_0, \nabla_x u\rangle (\lambda_+(u_+)^{q-1} - \lambda_-(u_-)^{q-1}) \mathrm{d}x \end{aligned} \tag{4}$$

for every $X_0 \in \partial^0 B_1^+$ and $r \in (0, \mathrm{dist}(X_0, \partial B_1))$ (see Sect. 2).

***Remark* 1.1** The validity of (4) plays a major role for the construction of monotonicity formulas and it is a necessary assumption for those point $X_0 \in \Gamma(u)$ such that $\mathcal{V}(u, X_0) = 2s/(2-q)$, in the case $q < 2(1-s)$ (see Remark 1.6). Nevertheless, it is a reasonable assumption which relax the hypothesis of being a minimal solution. Indeed, if u is a minimiser of

$$\mathcal{F}(u) = \frac{1}{2}\int_{B_1^+} y^a |\nabla u|^2 \,\mathrm{d}X - \frac{1}{q}\int_{\partial^0 B_1^+} |u|^q \mathrm{d}x, \tag{5}$$

it can be proved by taking inner variations along directions $\xi \in C_c^\infty(B_1; \mathbb{R}^{n+1})$ that u satisfies a more general class of Pohozaev-type identities (see [1] for $q = 1$). Moreover, since for $q < 2(1-s)$ there exist minimizers of (5) of type (12), one could only expect

$C^{1,\alpha}$-regularity of solutions of (3) at most for those $q \in [1, 2)$ such that $q > 2(1-s)$. For the sake of completeness, we consider this notion of solutions in order to characterize the all possible behaviours for $q \in [1, 2)$.

Inspired by [31], we introduce two different notions of vanishing order, which will be proved a posteriori to be equal. Therefore, consider the norm

$$\|u\|_{H^{1,a}(B_r^+(X_0))} = \left(\frac{1}{r^{n+a-1}} \int_{B_r^+(X_0)} y^a \, |\nabla u|^2 \, \mathrm{d}X + \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+(X_0)} y^a u^2 \mathrm{d}\sigma \right)^{1/2}.$$

Through the paper, we often use $\|\cdot\|_{X_0,r}$ to simplify the notation of the norm in $H^{1,a}(B_r^+(X_0))$.

Definition 1.2 Let $u \in H^{1,a}(B_1^+)$ be a solution of (3) satisfying (4) and $X_0 \in \Gamma(u)$. The $H^{1,a}$*-vanishing order* of u at X_0 is defined as $\mathcal{O}(u, X_0) \in \mathbb{R}^+$, with the property that

$$\limsup_{r\to 0^+} \frac{1}{r^{2k}} \, \|u\|^2_{H^{1,a}(B_r(X_0))} = \begin{cases} 0, & \text{if } 0 < k < \mathcal{O}(u, X_0) \\ +\infty, & \text{if } k > \mathcal{O}(u, X_0). \end{cases} \tag{6}$$

Moreover, if such number does not exist, i.e.

$$\limsup_{r\to 0^+} \frac{1}{r^{2k}} \, \|u\|^2_{H^{1,a}(B_r(X_0))} = 0 \quad \text{for any } k > 0,$$

we set $\mathcal{O}(u, X_0) = +\infty$.

The advantage of this formulation relays in the fact that we have better control of both the behaviour of the trace of solutions on $\partial^0 B_1^+$ and the character of the solution in the whole extended space. Instead, we recall here the classical definition of vanishing order, which will be used as well through the paper.

Definition 1.3 Let $u \in H^{1,a}(B_1^+)$ be a solution of (3) satisfying (4) and $X_0 \in \Gamma(u)$. The *vanishing order* of u at X_0 is defined as $\mathcal{V}(u, X_0) \in \mathbb{R}^+$, with the property that

$$\limsup_{r\to 0^+} \frac{1}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a u^2 = \begin{cases} 0, & \text{if } 0 < k < \mathcal{V}(u, X_0) \\ +\infty, & \text{if } k > \mathcal{V}(u, X_0). \end{cases} \tag{7}$$

By (15) we will easily deduce that $\mathcal{O}(u, X_0) \le \mathcal{V}(u, X_0)$. The following result establishes the validity of the strong unique continuation principle for every $q \in [1, 2)$, $\lambda_+ > 0$, $\lambda_- \ge 0$ and $s \in (0, 1)$

Theorem 1.4 *Let $q \in [1, 2)$, $\lambda_+ > 0$, $\lambda_- \ge 0$ and $u \in H^{1,a}(B_1^+)$ a solution of* (3) *satisfying* (4) *such that $X_0 \in \Gamma(u)$. If $\mathcal{V}(u, X_0) = +\infty$, then necessarily $u \equiv 0$; in particular, if for every $\beta > 0$ we have*

$$\lim_{|X-X_0|\to 0^+} \frac{|u(x)|}{|X - X_0|^\beta} = 0,$$

it follows that $u \equiv 0$.

This result implies the validity of the strong unique continuation principle also for the case $\lambda_- = 0$, which resembles the result in the local setting. Moreover, in the case $\lambda_+, \lambda_- > 0$, we can improve the previous result by characterizing all the admissible vanishing orders. Thus, let

$$k_q = \frac{2s}{2-q} \tag{8}$$

be the critical exponent associated to weak solutions of (3) and $\beta_q \in \mathbb{N}$ be the larger positive integer strictly smaller than k_q, that is

$$\beta_q := \begin{cases} \left\lfloor \frac{2s}{2-q} \right\rfloor, & \text{if } \frac{2s}{2-q} \notin \mathbb{N} \\ \frac{2s}{2-q} - 1 & \text{if } \frac{2s}{2-q} \in \mathbb{N}. \end{cases} \tag{9}$$

Then, the admissible vanishing orders are all the positive integers smaller or equal than β_q and the critical value k_q itself.

Theorem 1.5 *Let $q \in [1, 2)$, $\lambda_+, \lambda_- > 0$ and $u \in H^{1,a}(B_1^+)$ be a non-trivial solution of (3) satisfying (4) with $X_0 \in \Gamma(u)$. Then*

$$\mathcal{V}(u, X_0) \in \{n \in \mathbb{N} \setminus \{0\} \colon n \leq \beta_q\} \cup \left\{k_q\right\}.$$

In particular, if $k_q \leq 1$ then $\mathcal{V}(u, X_0) = k_q$.

***Remark* 1.6** In the case $s = 1$, our result recovers the case considered in [31]. Nevertheless, Theorem 1.5 reveals a deep difference between the local and nonlocal equations for small value of $s \in (0, 1)$: while the vanishing orders of solution of (1) have a universal bound $k_q = 2/(2 - q)$, which is always greater or equal than 1 for $q \in (0, 2)$ (see [31] for the sublinear case $q \in [1, 2)$, [32] for the singular case $q \in (0, 1)$), in the fractional setting this is not always true even in the sublinear case and it implies, for some values of $s \in (0, 1)$ and $q \in [1, 2)$, the occurrence of solutions which vanish only with order $k_q < 1$. As we will see, this phenomena will also affect the structure and the regularity of the nodal set.

Now, using a blow-up argument inspired by the one of [31], we proved the validity of a generalized Taylor expansion of the solutions near the nodal set: while in the linear (and superlinear) case solutions behave like homogeneous L_a-harmonic polynomials of order $k \geq 1$ symmetric with respect to $\{y = 0\}$ (see [28,Section 4] for a complete characterization of the class of symmetric L_a-harmonic polynomials $\mathfrak{s}\mathfrak{B}_k^a$), in the sublinear setting this is not necessary the case.

Theorem 1.7 *Let $q \in [1, 2)$, $\lambda_+, \lambda_- > 0$ and $u \in H^{1,a}(B_1^+)$ be a solution of (3) satisfying (4) with $X_0 \in \Gamma(u)$. Then, the following alternative holds:*

(1) if $\mathcal{V}(u, X_0) \in \{n \in \mathbb{N} \setminus \{0\} \colon n \leq \beta_q\}$, then there exists a $\mathcal{V}(u, X_0)$-homogeneous entire L_a-harmonic function $\varphi^{X_0} \in \mathfrak{s}\mathfrak{B}^a_{\mathcal{V}(u,X_0)}(\mathbb{R}^{n+1})$ symmetric with respect to $\{y = 0\}$, such that

$$u(X) = \varphi^{X_0}(X - X_0) + O(|X - X_0|^{\mathcal{V}(u,X_0)+1}); \tag{10}$$

(2) if $\mathcal{V}(u, X_0) = k_q$, then for every sequence $r_k \searrow 0^+$ we have, up to a subsequence, that

$$\frac{u(X_0 + r_k X)}{\|u\|_{X_0,r_k}} \to \overline{u} \quad \text{in } C^{0,\alpha}_{loc}(\overline{\mathbb{R}^{n+1}_+}),$$

for every $\alpha \in (0, \min(1, 2s))$, where $\overline{u}$ is a k_q-homogeneous non-trivial solution to

$$\begin{cases} L_a \overline{u} = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\partial^a_y \overline{u} = \mu \left(\lambda_+ (\overline{u}_+)^{q-1} - \lambda_- (\overline{u}_-)^{q-1}\right) & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{11}$$

for some $\mu \geq 0$. Moreover, the case $\mu = 0$ is possible if and only if $k_q \in \mathbb{N}$.

***Remark* 1.8** in the case of local diffusion $s = 1$ it is known that a growth estimate of the Laplacian of a function near its nodal set immediately implies the validity of a Taylor expansion of the function itself in terms of harmonic polynomials (see [6,Lemma 3.1]), whereas in the nonlocal setting the validity of a similar result is still unknown.

Therefore, our strategy is to take advantage of the bound on the vanishing order to ensure the validity of an asymptotic limit of the Dirichlet-to-Neumann operator ∂_y^a near the nodal set. Then, the expansion follows by a blow-up analysis based on an application of an Almgren and Monneau-type monotonicity formulas. Finally, in order to improve the convergence estimate of the remainder in the Taylor expansion (10), we apply a blow-up analysis on the difference between the function and its tangent map: this result will be crucial to prove the $C^{1,\alpha}$-regularity of the strata of the nodal set.

Thus, we think that this methodology could be use to extend the fundamental Lemma in [6,Lemma 3.1] to the fractional setting.

This result leads to a partial stratification of the nodal set and, via the dimension reduction principle due to Federer, to an estimate of the Hausdorff dimension of the nodal and singular set.

In the light of the previous results, let us define with $\mathcal{R}(u)$ and $\mathcal{S}(u)$ the regular and singular part of $\Gamma(u)$ defined by

$$\mathcal{R}(u) = \{X \in \Gamma(u)\colon\ |\nabla u|(X) \neq 0\} \quad \text{and} \quad \mathcal{S}(u) = \{X \in \Gamma(u)\colon 1 < \mathcal{V}(u, X) \leq \beta_q\}.$$

and with $\mathcal{T}(u)$ the "purely sublinear" part of the nodal set

$$\mathcal{T}(u) = \left\{X \in \Gamma(u)\colon \mathcal{V}(u, X) = k_q\right\}.$$

While in the local case $s = 1$ the sets $\mathcal{S}(u) \cup \mathcal{T}(u)$ coincides with those points with vanishing gradient, in the fractional setting $s \in (0, 1)$ this is not always the case since the critical value is not necessary greater than 1. Indeed, this decomposition of $\Gamma(u)$ seems more natural in the fractional setting: by Theorem 1.5, we already know that if $k_q > 1$ then

$$\{X \in \Gamma(u)\colon\ |\nabla u|(X) = 0\} = \mathcal{S}(u) \cup \mathcal{T}(u),$$

while if $k_q \leq 1$ we get

$$\Gamma(u) = \mathcal{T}(u).$$

Indeed we will see that, for those value of $s \in (0, 1)$ and $q \in [1, 2)$ such that $k_q > 1$, near the points of the nodal set where the function vanishes with order strictly less then k_q, the nodal set resembles the picture of the nodal set of s-harmonic functions (see [28] for a deeper analysis of the singular set $\mathcal{S}(u)$).

Theorem 1.9 *Let $q \in [1, 2)$, $\lambda_+, \lambda_- > 0$ and $u \in H^{1,a}(B_1^+)$ be a solution of* (3) *satisfying* (4). *The nodal set $\Gamma(u)$ splits as*

$$\Gamma(u) = \mathcal{R}(u) \cup \mathcal{S}(u) \cap \mathcal{T}(u),$$

where

(1) the regular part $\mathcal{R}(u)$ is locally a $C^{1,\alpha}$-regular $(n-1)$-hypersurface on $\mathbb{R}^n$;
(2) the singular part $\mathcal{S}(u)$ is an $(n-1)$-dimensional countably rectifiable set. Moreover, the decomposition

$$\mathcal{S}(u) = \bigcup_{j=0}^{n-1} \mathcal{S}_j(u)$$

holds true, where each $\mathcal{S}_j(u)$ is contained in a countable union of j-dimensional $C^{1,\alpha}$ manifolds;

(3) the sublinear part $\mathcal{T}(u)$ has Hausdorff dimension at most $(n-1)$. Moreover, for $k_q < 1$ the nodal set coincides with the sublinear stratum and the Haudorff estimate is optimal in the sense that there exists a collection of 2-dimensional k_q-homogeneous solutions such that

$$u_1(x,0) = A_1\left(x_+^{k_q} - x_-^{k_q}\right) \quad \textit{or} \quad u_2(x,0) = A_2\,|x|^{k_q} \quad \textit{for every } x \in \mathbb{R}. \tag{12}$$

The result on $\mathcal{T}(u)$ is remarkably different to its local counterpart: while for $s = 1$ the bound $(n-2)$ on the Hausdorff dimension is optimal, we believe that the result on the $(n-1)$-dimension of $\mathcal{T}(u)$ in the case $k_q < 1$ can be easily generalized to all $s \in (0,1)$ and $q \in [1,2)$, thanks to the characterization of L_a-harmonic function in [28]. Moreover, we claim that a viscosity approach, based on an improvement of flatness, could give a regularity result for those points where the blow-up limit behave like (12) (see Remark 7.1 for more detail in this direction), in the case $k_q < 1$. At the moment, we leave it as an open problem.

Structure of the paper The paper is organized as follows. In Sect. 2 we recall some preliminary results on the functional setting and the regularity of solutions. Moreover, we introduce the notions of vanishing order used through the paper. Next, in Sect. 3, we prove the validity of the weak unique continuation principle and Theorem 1.4 by using a 2-parameter Weiss-type monotonicity formula which allows, in Sect. 4, to introduce a characterisation of the threshold k_q.

Finally, in Sect. 5 we prove the first part of Theorem 1.7 and Theorem 1.9 by developing a blow-up analysis based on the validity of two Almgren-type formulas for those points with vanishing order smaller than k_q and, in Sect. 6, we complete the proof of Theorem 1.7 by applying a blow-up analysis on those points with vanishing order equal to k_q. As byproduct, we will recover Theorem 1.5.

Finally, in Sect. 7 we prove the existence of k_q-homogeneous solutions of the form (12), for those values of s and q so that $k_q < 1$. This result will lead to the Hausdorff estimate of $\mathcal{T}(u)$ in Theorem 1.9.

2 Preliminaries

In this Section we start by showing preliminary results related to the trace embedding of the $H^{1,a}$-space and the regularity of solutions to our problem.

For $a \in (-1,1), \Omega \subseteq \mathbb{R}^{n+1}_+$ let us consider $H^{1,a}(\Omega) = H^1(\Omega, y^a \mathrm{d}X)$ the weighted Sobolev space associated to the Muckenhoupt A_2 weights $\omega(X) = y^a$ with $X = (x,y)$ (see [10] for more details). Now, given a weak solution $u \in H^s(\mathbb{R}^n)$ of (2), by the Caffarelli-Silvestre extension there exists a unique function $v \in H^{1,a}(\mathbb{R}^{n+1})$ such that

$$\begin{cases} L_a v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v(x,0) = u(x) & \text{in } \mathbb{R}^n \times \{0\}\,. \end{cases}$$

with $a = 1-2s \in (-1,1)$ and $L_a v = \mathrm{div}(y^a \nabla v)$. By a trace result of Nekvinda [22], it is known that the space $H^s(\mathbb{R}^n)$ coincides with the trace on $\mathbb{R}^n \times \{0\}$ of $H^{1,a}(\mathbb{R}^{n+1})$. Thus, we obtain that, up to a normalization constant, the problem

$$\begin{cases} L_a v = 0 & \text{in } B_1^+ \\ -\partial_y^a v = \lambda_+ (v_+)^{q-1} - \lambda_- (v_-)^{q-1} & \text{on } \partial^0 B_1^+, \end{cases} \tag{13}$$

is a localized version of (2) with $u(x) = v(x, 0)$ (see also [4,Theorem 3.1]). Instead of consider general weak solution in $H^{1,a}(B_1^+)$ we need to assume the validity of some Pohozaev-type identities, which are usually associated to the concept of stationary solution with respect to domain variations.

Definition 2.1 We say that $u \in H^{1,a}(B_1^+) \cap L^\infty(B_1^+)$ is a nontrivial weak stationary solution of (13) if for every $\varphi \in C_c^\infty(B_1)$ we have

$$\int_{B_1^+} y^a \langle \nabla u, \nabla \varphi \rangle \mathrm{d}X = \int_{\partial^0 B_1^+} (\lambda_+ (u_+)^{q-1} - \lambda_- (u_-)^{q-1})\varphi \mathrm{d}x$$

and

$$\frac{1-n-a}{2} \int_{B_r^+(X_0)} y^a |\nabla u|^2 \, \mathrm{d}X + \frac{r}{2} \int_{\partial^+ B_r^+(X_0)} y^a |\nabla u|^2 \, \mathrm{d}\sigma = \\ = r \int_{\partial^+ B_r^+(X_0)} y^a (\partial_r u)^2 \mathrm{d}\sigma + \int_{\partial^0 B_r^+(X_0)} \langle x - x_0, \nabla_x u \rangle (\lambda_+ (u_+)^{q-1} - \lambda_- (u_-)^{q-1}) \mathrm{d}x \tag{14}$$

for every $X_0 \in \partial^0 B_1^+$ and $r \in (0, \mathrm{dist}(X_0, \partial B_1))$.

We remark that the existence of solutions follows by standard methods of the calculus of variations and a straightforward application of the following trace embedding.

Through the paper, for $X_0 \in \partial^0 B_1^+$ and $r \in (0, 1 - |X_0|)$, we will always consider the space $H^{1,a}(B_r^+(X_0))$ as the completion of $C^\infty(\overline{B_r^+(X_0)})$ with respect to the norm

$$\|u\|_{H^{1,a}(B_r^+(X_0))} = \left(\frac{1}{r^{n+a-1}} \int_{B_r^+(X_0)} y^a |\nabla u|^2 \, \mathrm{d}X + \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+(X_0)} y^a u^2 \mathrm{d}\sigma \right)^{1/2}.$$

By the trace embedding on $\partial^+ B_r^+(X_0)$ and the Poincaré inequality, this norm is equivalent to the classical one of [10]. From now on, we often use the notation $\|\cdot\|_{X_0,r}$ to simplify the notation of the norm in $H^{1,a}(B_r^+(X_0))$. The equivalence of this norm with the classic one is a consequence of the trace theory and the Poincaré inequality.

Lemma 2.2 *Let $u \in H^{1,a}(B^+)$ and $q \in [1, 2^\star]$, where $2^\star = 2n/(n-2s) = 2n/(n+a-1)$ is Sobolev's exponent for the fractional Laplacian. There exists a constant $C_1 = C_1(n, p, a)$ such that*

$$\left(\frac{1}{r^n} \int_{\partial^0 B_r^+} |u|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \le C_1 \|u\|_{H^{1,a}(B_r^+)}, \tag{15}$$

for every $0 < r < 1$. Namely, the space $H^{1,a}(B_r^+(X_0))$ is continuously embedded in $L^q(\partial^0 B_r^+(X_0))$, for every $r \in (0, 1)$.

Proof This result is a direct consequence of the characterization of the class of traces of $H^{1,a}(B_r^+)$ with $r \in (0, 1)$ (see [22,Theorem 2.11] for the complete theory), and the Sobolev embedding in the context of fractional Sobolev-Slobodeckij spaces $H^s(\partial^0 B_r^+)$. □

Since $\partial^0 B_r^+$ is a Lipschitz domain with bounded boundary in $\mathbb{R}^n$, the compact embedding in the fractional Sobolev spaces implies the following remark (see [8] for further details).

Lemma 2.3 *Let $q \in [1, 2^\star)$, where $2^\star = 2n/(n-2s) = 2n/(n+a-1)$. Then $H^{1,a}(B_r^+(X_0))$ is compactly embedded in $L^q(\partial^0 B_r^+(X_0))$, for every $r \in (0, 1)$.*

We conclude the Section by recalling a regularity result for solutions of (3) in the sense of Definition 2.1. As we will see through the paper, in order to develop a blow-up analysis near nodal points it is enough to prove a α-Hölder regularity for small $\alpha \in (0, 1)$, since the classification of the admissible vanishing order is obtained by monotonicity-type formulas. Therefore, by [29,Theorem 1.5] (see also [29, 30] for more regularity results), we easily deduce the following result.

Proposition 2.4 *Let $u \in H^{1,a}(B_1^+)$ be a weak solution of (3) in the sense of Definition 2.1. Then, for any compact set $K \subset B_1$ we get $u \in C^{0,\alpha}(K \cap \overline{B_1^+})$, for every $\alpha \in (0, \min\{1, 2s\})$.*

3 Strong unique continuation principle

This Section is devoted to the proof Theorem 1.4, that is the strong unique continuation principle for solution of (3). In order to achieve the main result we start our analysis by proving the weak unique continuation principle: if a solution u is identically zero in a neighborhood in $\mathbb{R}^n \times \{0\}$ of a point $X_0 \in \partial^0 B_1^+$, then necessary $u \equiv 0$ on $\partial^0 B_1^+$. Moreover, since $q \in [1, 2)$, it implies that $u \equiv 0$ on B_1^+ (see [28,Proposition 5.9]).

Our proof of the unique continuation is deeply based on the validity of an Almgren-type monotonicity formula. Indeed, let

$$F_{\lambda_+,\lambda_-}(u) = \lambda_+(u_+)^q + \lambda_-(u_-)^q,$$

then for $X_0 \in \Gamma(u)$ and $r \in (0, \mathrm{dist}(X_0, \partial^+ B_1^+))$, we introduce the functionals

$$\begin{aligned} E(X_0, u, r) &= \frac{1}{r^{n-1+a}} \left[\int_{B_r^+(X_0)} y^a |\nabla w|^2 \,\mathrm{d}X + \int_{\partial^0 B_r^+(X_0)} w \partial_y^a w \mathrm{d}x \right] \\ &= \frac{1}{r^{n+a-1}} \left[\int_{B_r^+(X_0)} y^a |\nabla u|^2 \,\mathrm{d}X - \int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x \right] \qquad (16) \\ H(X_0, u, r) &= \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+(X_0)} y^a u^2 \mathrm{d}\sigma, \end{aligned}$$

and the associated Almgren-type formula

$$N(X_0, u, r) = \frac{E(X_0, u, r)}{H(X_0, u, r)}. \qquad (17)$$

Through the paper we will often abuse the notation $E(u, r)$, $H(u, r)$ and $N(u, r)$ when it is not restrictive to assume that $X_0 = 0$. By the Gauss-Green formula we immediately obtain

$$E(X_0, u, r) = \frac{1}{r^{n+a-1}} \int_{\partial^+ B_r^+(X_0)} y^a u \partial_r u \mathrm{d}\sigma,$$

while, by differentiating the functions $r \mapsto H(X_0, u, r)$, we get

$$\begin{aligned} \frac{d}{dr} H(X_0, u, r) &= \frac{d}{dr} \left(\int_{\partial^+ B_1^+} y^a u^2 (X_0 + rx) \mathrm{d}\sigma \right) \\ &= \frac{2}{r^{n+a}} \int_{\partial^+ B_r^+} y^a u \partial_r u \mathrm{d}\sigma = \frac{2}{r} E(X_0, u, r). \qquad (18) \end{aligned}$$

In the following Proposition we compute the derivative of the denominator of the Almgren-type quotient by taking care of the sublinear term on the boundary $\partial^0 B_r^+$.

Proposition 3.1 *Let $X_0 \in \Gamma(u)$ and $r \in (0, \mathrm{dist}(X_0, \partial B_1))$. Then, it holds*

$$\frac{d}{dr}E(X_0,u,r) = \frac{2}{r^{n+a-1}}\int_{\partial^+ B_r^+(X_0)} y^a(\partial_r u)^2 \mathrm{d}\sigma + \\ + \frac{1}{r^{n+a-1}}\left[\frac{2-q}{q}\int_{S_r^{n-1}(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}\sigma - \frac{C^s_{n,q}}{qr}\int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x\right],$$

where $C^s_{n,q} = 2n - q(n-2s) > 0$.

Proof Up to translation, let us assume that $X_0 = 0$. Thus

$$\frac{d}{dr}E(u,r) = \frac{1-n-a}{r^{n+a}}\left[\int_{B_r^+} y^a\,|\nabla u|^2\,\mathrm{d}X - \int_{\partial^0 B_r^+} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x\right] + \\ + \frac{1}{r^{n+a-1}}\left[\int_{\partial^+ B_r^+} y^a\,|\nabla v|^2\,\mathrm{d}\sigma - \int_{S_r^{n-1}} F_{\lambda_+,\lambda_-}(u)\mathrm{d}\sigma\right].$$

By Definition 2.1, we need to integrate by parts only the last term in (14). More precisely, since $\nabla_x F_{\lambda_+,\lambda_-}(u) = q(-\partial^a_y u)\nabla_x u$ in $\partial^0 B_1^+$, we get

$$\begin{aligned}\int_{\partial^0 B_r^+}\langle x,\nabla_x u\rangle(-\partial^a_y u)\mathrm{d}x &= \frac{1}{q}\int_{\partial^0 B_r^+}\langle x,\nabla_x F_{\lambda_+,\lambda_-}(u)\rangle\mathrm{d}x \\ &= \frac{r}{q}\int_{S_r^{n-1}} F_{\lambda_+,\lambda_-}(u)\mathrm{d}\sigma - \frac{n}{q}\int_{\partial^0 B_r^+} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x.\end{aligned} \tag{19}$$

Summing together the previous equalities, we finally get the claimed result. We remark that the previous computations are also valid in the case $q = 1$, but require some justification. More precisely, as observed in [33,Proposition 2.7], the Gauss-Green formula holds for all vector fields $Y \in C(\overline{B_r}, \mathbb{R}^{n+1})$ with $\mathrm{div}Y \in L^1(B_r)$. In particular in (19), the Gauss-Green formula is applied to the vector field

$$Y_1 = F_{\lambda_+,\lambda_-}(u)(x,0) = (\lambda_+ u_+ + \lambda_- u_-)(x,0),$$

where

$$\mathrm{div}Y_1 = \mathrm{sign}(\lambda_+ u_+ - \lambda_- u_-)\langle x,\nabla_x u\rangle + n(\lambda_+ u_+ + \lambda_- u_-) \quad \text{a.e. in } \partial^0 B_1^+.$$

The previous quantity is absolutely integrable in $\partial^0 B_r^+$ as a direct consequence of the characterization of the class of trace of $H^{1,a}(B_r^+)$ with $r \in (0,1)$ (see [22,Theorem 2.11]). □

Combining the previous estimate, we finally get a lower bound for the derivative of the Almgren-type frequency formula.

Corollary 3.2 *Let $X_0 \in \Gamma(u)$ and $r_1, r_2 \in (0, \mathrm{dist}(X_0, \partial^+ B_1^+))$ such that $H(X_0,u,r) \neq 0$ for a.e. $r \in (r_1, r_2)$. Then*

$$\frac{d}{dr}N(X_0,u,r) \geq \frac{r\left(\dfrac{2-q}{q}\right)\displaystyle\int_{S_r^{n-1}(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}\sigma - \frac{C^s_{n,q}}{q}\int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x}{\displaystyle\int_{\partial^+ B_r^+(X_0)} y^a u^2\mathrm{d}\sigma}$$

for a.e. $r \in (r_1, r_2)$.

Proof The proof follows essentially the ideas of the similar results in literature and it is based on a straightforward combination of (18), Proposition 3.1 and the validity of the Cauchy-Schwarz inequality on $\partial^+ B_r^+(X_0)$. □

Now, we are ready to show the validity of the classical weak unique continuation principle for solution of the sub-linear nonlocal equation.

Theorem 3.3 *Let $q \in [1,2)$, $\lambda_+ > 0$, $\lambda_- \geq 0$ and $u \in H^{1,a}(B_1^+)$ be a weak solution of (3) which vanishes in a neighbourhood in $\mathbb{R}^n \times \{0\}$ of a point on $\Gamma(u)$. Then $u \equiv 0$ in $\partial^0 B_1^+$.*

Proof Let us define the vanishing set on $\mathbb{R}^n \times \{0\}$ as

$$U = \{x \in \partial^0 B_1^+ : u \equiv 0 \text{ in a neighborhood of } x\} \subset \mathbb{R}^n.$$

Since $u \not\equiv 0$ and $\partial^0 B_1^+$ is connected, we already know that U is open, non-empty and $\partial U \cap \partial^0 B_1^+ \neq \emptyset$, where ∂U is the topological boundary of U as a subset of $\mathbb{R}^n$. Let $X^* \in U$ and $r > 0$ be such that $B_r(X^*) \subset U$, then by (3) the function u satisfies

$$\begin{cases} L_a u = 0 & \text{in } B_r^+(X^*) \\ \partial_y^a u = 0 & \text{on } \partial^0 B_r^+(X^*) \\ u = 0 & \text{on } \partial^0 B_r^+(X^*). \end{cases} \tag{20}$$

By [28,Proposition 5.9], it follows that necessary $u \equiv 0$ in $B_r^+(X^*)$ and then, by the weak unique continuation principle for the L_a-operator (see [16,Theorem 1.4]) we get that $u \equiv 0$ in B_1^+. Since we already know that u is Hölder continuous, we get $u \equiv 0$ in $\partial^0 B_1^+$. □

In order to prove the strong unique continuation principle, we now introduce a 2-parameter family of Weiss-type monotonicity formulas, that will be the fundamental tool of our analysis. Indeed, inspired by [31], for $X_0 \in \Gamma(u)$ and $r \in (0, \mathrm{dist}(X_0, \partial^+ B_1^+))$ we consider the functional

$$E_t(X_0, u, r) = \frac{1}{r^{n+a-1}}\left[\int_{B_r^+(X_0)} y^a |\nabla u|^2 \,\mathrm{d}X - \frac{t}{q}\int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x\right], \tag{21}$$

and similarly we introduce the two-parameters families of functionals

$$N_t(X_0, u, r) = \frac{E_t(X_0, u, r)}{H(X_0, u, r)}, \quad W_{k,t}(X_0, u, r) = \frac{E_t(X_0, u, r)}{r^{2k}} - k\frac{H(X_0, u, r)}{r^{2k}}. \tag{22}$$

Notice that for $t = q$, we recover the functionals in (16) and their associated Almgren-type formula.

Proceeding exactly as in Proposition 3.1 and Corollary 3.2, we get

$$\frac{d}{dr}E_t(X_0, u, r) = \frac{2}{r^{n+a-1}}\int_{\partial^+ B_r^+(X_0)} y^a (\partial_r u)^2 \mathrm{d}\sigma + R_t(X_0, u, r) \tag{23}$$

where

$$R_t(X_0, u, r) = \frac{1}{r^{n+a-1}}\left[\frac{2-t}{q}\int_{S_r^{n-1}(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}\sigma - \frac{C_{n,t}^s}{qr}\int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x\right] \tag{24}$$

and $C_{n,t}^s = 2n - t(n-2s)$.

Proposition 3.4 *Let $X_0 \in \Gamma(u)$ and $r \in (0, \mathrm{dist}(X_0, \partial^+ B_1^+))$. Then we have*

$$\begin{aligned}\frac{d}{dr} W_{k,t}(X_0, u, r) = & \frac{2}{r^{n+a-1+2k}} \int_{\partial^+ B_r^+(X_0)} y^a \left(\partial_r u - \frac{k}{r} u\right)^2 \mathrm{d}\sigma + \\ & + \frac{1}{r^{n+a-1+2k}} \left[\frac{2-t}{q} \int_{S_r^{n-1}(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}\sigma \right. \\ & \left. - \frac{C_{n,t}^s - 2k(t-q)}{qr} \int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x \right].\end{aligned} \tag{25}$$

In particular, for $t = 2$ and $k \geq k_q$ the function $r \mapsto W_{k,2}(X_0, u, r)$ is monotone non-decreasing.

Proof Up to translation, let us suppose $X_0 = 0$ and $r \in (0, 1)$. A direct computation gives

$$\begin{aligned}\frac{d}{dr} W_{k,t}(u, r) = & -2k \left(\frac{E_t(u, r)}{r^{2k+1}} - k \frac{H(u, r)}{r^{2k+1}}\right) + \frac{1}{r^{2k}} \left(\frac{d}{dr} E_t(u, r) - k \frac{d}{dr} H(u, r)\right) \\ = & \frac{1}{r^{2k}} \frac{d}{dr} E_t(u, r) - 4k \frac{E_t(u, r)}{r^{2k+1}} + 2k^2 \frac{H(u, r)}{r^{2k+1}},\end{aligned} \tag{26}$$

where in the second inequality we used the estimate (18). By the Gauss-Green formula in (21) we get

$$E_t(u, r) = \frac{1}{r^{n+a-1}} \left[\int_{\partial^+ B_r^+} y^a u \partial_r u \mathrm{d}X - \frac{t-q}{q} \int_{\partial^0 B_r^+} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x\right]$$

and by taking care of the estimate (23), we finally obtain

$$\begin{aligned}\frac{d}{dr} W_{k,t}(u, r) = & \frac{2}{r^{n+a-1+2k}} \int_{\partial^+ B_r^+} y^a \left(\partial_r u - \frac{k}{r} u\right)^2 \mathrm{d}\sigma + \\ & + \frac{1}{r^{n+a-1+2k}} \left[\frac{2-t}{q} \int_{S_r^{n-1}} F_{\lambda_+,\lambda_-}(u) \mathrm{d}\sigma \right. \\ & \left. - \frac{C_{n,t}^s - 2k(t-q)}{qr} \int_{\partial^0 B_r^+} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x \right].\end{aligned}$$

Now, for $t = 2$ and $k \geq k_q$ (see (8) for the definition of the critical exponent k_q) the monotonicity follows straightforwardly by the previous computations. Indeed, we have

$$\frac{d}{dr} W_{k,t}(u, r) \geq -\frac{C_{n,t}^s - 2k(t-q)}{qr^{n+a+2k}} \int_{\partial^0 B_r^+} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x, \tag{27}$$

where $C_{n,2}^s - 2k(2-q) \leq 0$ if and only if $k \geq k_q$. □

Thus, as simple corollaries of the monotonicity result, we deduce the following results for $k \geq k_q$.

Corollary 3.5 *Let $X_0 \in \Gamma(u)$ and $k \geq k_q$. Then, there exists the limit*

$$W_{k,2}(X_0, u, 0^+) = \lim_{r \to 0^+} W_{k,2}(X_0, u, r).$$

Moreover, the map $r \mapsto W_{k_q,2}(X_0, u, r)$ is constant if and only if u is k_q-homogeneous in $\overline{\mathbb{R}^{n+1}_+}$ with respect to X_0.

Corollary 3.6 *For $X_0 \in \Gamma(u)$, there exists $k \geq k_q$ such that*

$$W_{k,2}(X_0, u, 0^+) < 0.$$

Moreover, if $W_{k_1,2}(X_0, u, 0^+) < 0$ then $W_{k_2,2}(X_0, u, 0^+) = -\infty$ for every $k_2 > k_1$.

Proof Up to translation, let us consider $X_0 = 0$ and $r \in (0, 1)$. By Theorem 3.3, since $u \not\equiv 0$ there exists $r_1 \in (0, 1)$ such that $H(u, r_1) \neq 0$. Now, there exists $k \geq k_q$ sufficiently large, such that

$$W_{k,2}(u, r_1) = \frac{E_2(u, r_1)}{r_1^{2k}} - k\frac{H(u, r_1)}{r_1^{2k}} < 0,$$

and by the monotonicity result in Proposition 3.4 we obtain $W_{k,2}(u, 0^+) \leq W_{k,2}(u, r_1) < 0$, for k sufficiently large.

Now, fixed $k_1 > 0$ such that $W_{k_1,2}(u, 0^+) < 0$, let us consider $k_2 > k_1$. Thus, for $r \in (0, 1)$

$$\begin{aligned} W_{k_2,2}(u, r) &= \frac{E_2(u, r)}{r^{2k_2}} - k_2\frac{H(u, r)}{r^{2k_2}} \\ &= \frac{1}{r^{2(k_2-k_1)}}\left[\frac{E_2(u, r)}{r^{2k_1}} - k_1\frac{H(u, r)}{r^{2k_1}}\right] - \frac{k_2 - k_1}{r^{2k_2}}H(u, r) \\ &\leq \frac{1}{r^{2(k_2-k_1)}}W_{k_1,2}(u, r), \end{aligned}$$

which implies the claimed conclusion. □

Finally, by Corollary 3.6 we are able to prove the existence of a transition exponent $\overline{k}$ for the frequency $W_{k,2}(X_0, u, 0^+)$ which characterize the possible behaviours of the Weiss frequency for every $k \geq 0$.

Corollary 3.7 *For every $X_0 \in \Gamma(u)$ such that $\mathcal{O}(u, X_0) \geq k_q$, there exists finite*

$$\overline{k} = \inf\{k > 0\colon W_{k,2}(X_0, u, 0^+) = -\infty\} \in \left[k_q, +\infty\right).$$

Moreover, the limit $W_{k,2}(X_0, u, 0^+)$ exists for every $k \geq 0$ and it satisfies

$$\begin{cases} W_{k,2}(X_0, u, 0^+) = 0 & \text{if } 0 < k < k_q \\ W_{k,2}(X_0, u, 0^+) \geq 0 & \text{if } k_q \leq k < \overline{k} \\ W_{k,2}(X_0, u, 0^+) = -\infty & \text{if } k > \overline{k}. \end{cases}$$

Proof The existence of $\overline{k} \geq 0$ follows by Corollary 3.6. Now, let us consider separately the cases $k < k_q$ and $k \geq k_q$. In the first one, since $\mathcal{O}(u, X_0) \geq k_q$, by (6) there exists $\varepsilon > 0$ such that

$$k < k_q - \varepsilon < \mathcal{O}(u, X_0),$$

and two constant $C > 0$, $r_0 > 0$, depending on ε, such that

$$\|u\|^2_{H^{1,a}(B_r^+(X_0))} \leq Cr^{2(k_q-\varepsilon)},$$

for every $r \in (0, r_0)$. By definition of the Weiss-type formula, we get

$$\begin{aligned} \left|W_{k,2}(X_0, u, r)\right| &\leq C\frac{1}{r^{2k}}\left((1 + k)\,\|u\|^2_{H^{1,a}(B_r(X_0))} + \frac{2}{q}r^{1-a}\,\|u\|^q_{H^{1,a}(B_r(X_0))}\right) \\ &\leq C\frac{1}{r^{2k}}\left(r^{2\alpha} + r^{2s+q\alpha}\right), \end{aligned}$$

with $\alpha = k_q - \varepsilon$. Finally, since $q \in [1, 2)$, we get

$$\left|W_{k,2}(X_0, u, r)\right| \le Cr^{2(k_q - k - \varepsilon)} + Cr^{2(k_q - k - \frac{q\varepsilon}{2})}$$

which leads to the claimed result ad $r \to 0^+$. In particular, this estimate suggests that $\overline{k} \ge k_q$.

Instead, in the case $k > k_q$ the existence of a non-negative limit for $k < \overline{k}$ follows by the monotonicity result in Proposition 3.4 and by Corollary 3.6. □

The previous result emphasizes an hidden relation between the notion of $H^{1,a}$-vanishing order and the transition exponent $\overline{k}$ defined in Corollary 3.7, which will be deeply examined in Sect. 4. Finally, we can prove the main result of the Section.

Proof of Theorem 1.4 By contradiction, suppose that $u \not\equiv 0$ on $\partial^0 B_1^+$ and $\mathcal{O}(u, X_0) = +\infty$, i.e.

$$\limsup_{r \to 0^+} \frac{1}{r^{2\beta}} \|u\|^2_{H^{1,a}(B_r(X_0))} = 0, \quad \text{for any } \beta > 0.$$

In particular, given $\overline{k} > 0$ as in Corollary 3.7, let us fix $k > \overline{k}$ and $\beta = 2k/q$. Thus, there exists $r_0 > 0$ and $C > 0$ such that

$$\frac{1}{r^{n+a-1}} \int_{B_r^+(X_0)} y^a |\nabla u|^2 \, \mathrm{d}X + \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+(X_0)} y^a u^2 \mathrm{d}\sigma \le Cr^{\frac{4}{q}k} \quad \text{for every } r \in (0, r_0). \tag{28}$$

On one side, since $2k/q > k$ for $q \in [1, 2)$, by the previous inequality we easily have

$$H(X_0, u, r) \le Cr^{2k} \quad \text{for every } r \in (0, r_0)$$

while, by an integration by parts, fixed $\Lambda = \max\{\lambda_+, \lambda_-\}$ we get

$$\begin{aligned} \frac{1}{r^{n+a-1}} \int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x &\le \frac{\Lambda}{r^{n+a-1}} \int_{\partial^0 B_r^+(X_0)} |u|^q \, \mathrm{d}x \\ &\le Cr^{1-a} \|u\|^q_{H^{1,a}(B_r(X_0))} \le Cr^{2k}, \end{aligned}$$

where in the second inequality we use Lemma 2.2 and in the last one (28). Finally, collecting the previous estimate, for every $r \in (0, r_0)$ we have

$$\begin{aligned} W_{k,2}(X_0, u, r) &\ge -\frac{1}{r^{n+a-1+2k}} \frac{2}{q} \int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x - \frac{k}{r^{2k}} H(X_0, u, r) \\ &\ge -\left(\frac{2}{q} + k\right) C, \end{aligned}$$

and in particular $W_{k,2}(X_0, u, 0^+) > -\infty$, in contradiction with the fact that, being $k > \overline{k}$, by Corollary 3.7 we must have $W_{k,2}(X_0, u, 0^+) = -\infty$ for any $k > \overline{k}$. □

4 The transition exponent for the Weiss-type formula

In this Section we develop a finer analysis of the transition exponent $\overline{k}$ for the Weiss-type monotonicity formula $W_{k,2}$ in the case $\mathcal{O}(u, X_0) \ge k_q$. The main result of the Section is a characterization of $\overline{k}$ in terms of the critical exponent k_q and the $H^{1,a}$-vanishing order, which allows to prove an upper bound for the admissible vanishing orders $\mathcal{O}(u, X_0)$ of u at X_0 (see Proposition 4.7).

First, we start by proving the following partial characterization of the transition exponent $\overline{k}$ of $W_{k,2}$ (see (3.7)) in terms of k_q.

Proposition 4.1 *For every $X_0 \in \Gamma(u)$ such that $\mathcal{O}(u, X_0) \geq k_q$, we have*

$$\overline{k} = \inf\left\{k > 0 \colon W_{k,2}(X_0, u, 0^+) = -\infty\right\} = k_q.$$

Moreover, combining the previous estimate with Corollary 3.7 we deduce that $W_{k,2}(X_0, u, 0^+)$ exists for every $k \geq 0$ and

$$\begin{cases} W_{k,2}(X_0, u, 0^+) = 0 & \text{if } 0 < k < k_q \\ W_{k,2}(X_0, u, 0^+) = -\infty & \text{if } k > k_q. \end{cases} \tag{29}$$

Following the strategy presented in [31], this result will be a consequence of the following Lemmata in which we assume that $\overline{k} > k_q$.

***Remark* 4.2** Since we never use the assumption $\mathcal{O}(u, X_0) \geq k_q$, we highlight that the following results are still true near nodal point with $\mathcal{O}(u, X_0) < k_q$.

Lemma 4.3 *Let $X_0 \in \Gamma(u)$ and assume that $\overline{k} \geq k_q$. Then*

$$E_2(X_0, u, r) \geq 0 \quad \text{and} \quad H(X_0, u, r) > 0,$$

for every $r \in (0, \text{dist}(X_0, \partial^+ B_1^+))$. Moreover, if $k > \overline{k}$, we deduce

$$\limsup_{r \to 0^+} N_2(X_0, u, r) \leq k \quad \text{and} \quad \liminf_{r \to 0^+} \frac{H(X_0, u, r)}{r^{2k}} = +\infty.$$

Proof Fixed $k \in (k_q, \overline{k}]$, we already know by Proposition 3.4 that $r \mapsto W_{k,2}(X_0, u, r)$ is monotone non-decreasing and, by Corollary 3.7, that $W_{k,2}(X_0, u, 0^+) \geq 0$. Hence, for every $r \in (0, \text{dist}(X_0, \partial B^+))$ we get

$$0 \leq W_{k,2}(X_0, u, r) \leq \frac{1}{r^{2k}} E_2(X_0, u, r).$$

Moreover, since $W_{k,q}(X_0, u, r) \geq W_{k,2}(X_0, u, r) \geq 0$ for every $r \in (0, \text{dist}(X_0, \partial B^+))$, by (18) we deduce

$$\frac{d}{dr} \frac{H(X_0, u, r)}{r^{2k}} = \frac{2}{r} W_{k,q}(X_0, u, r) \geq 0. \tag{30}$$

Finally, if $H(X_0, u, r_1) = 0$ for some $r_1 > 0$, by the monotonicity of (30), we deduce that $u \equiv 0$ in $B^+_{r_1}(X_0)$, in contradiction with Theorem 3.3.

Hence, collecting the previous inequality, we get $N_2(X_0, u, r) \geq 0$ and in particular, since $k > \overline{k}$ we get

$$-\infty = W_{k,2}(X_0, u, 0^+) = \lim_{r \to 0^+} \frac{H(X_0, u, r)}{r^{2k}} (N_2(X_0, u, r) - k).$$

Also, since $H(X_0, u, r)/r^{2k} \geq 0$, we finally deduce

$$-k \leq \liminf_{r \to 0^+} (N_2(X_0, u, r) - k) \leq \limsup_{r \to 0^+} (N_2(X_0, u, r) - k) \leq 0,$$

which implies the desired claim. □

As a consequence, for every $t \in (0, 2)$ the associated Almgren-type formula $N_t(X_0, u, r)$ is non-negative for every $r \in (0, \mathrm{dist}(X_0, \partial^+ B+))$.

Since in this Section we are proceeding by assuming by contradiction that $\overline{k} > k_q$, consider t the medium point between k_q and $\overline{k}$.

Lemma 4.4 *Let $X_0 \in \Gamma(u)$ and assume that $\overline{k} > k_q$. Given*

$$\widetilde{k} = \frac{1}{2}\left(k_q + \overline{k}\right) \quad \textit{and} \quad \tilde{t} = \frac{2n + 2kq}{2k + n - 2s} \in (q, 2),$$

then the map $r \mapsto W_{k,\tilde{t}}(X_0, u, r)$ is monotone non-decreasing in $(0, \mathrm{dist}(X_0, \partial^+ B^+))$, for every $k \geq \widetilde{k}$.

Proof The proof is a direct corollary of Proposition 3.4. More precisely, since $q \in [1, 2)$ and $k \geq \widetilde{k} > k_q$ we get that

$$\tilde{t} = \frac{2n + 2kq}{2k + n - 2s} \longleftrightarrow C^s_{n,\tilde{t}} - 2k(\tilde{t} - q) = 0,$$

which implies, by (27), the claimed result. □

Therefore, under the absurd assumption $\overline{k} > k_q$, we can prove that the transition exponent $\overline{k}$ associated to $W_{k,2}(X_0, u, 0^+)$ coincides with the transition exponent associated to the frequency $W_{k,\tilde{t}}(X_0, u, 0^+)$.

Lemma 4.5 *Assume that $\overline{k} > k_q$, then*

$$\overline{k} = \inf\{k \geq \tilde{k} \colon W_{k,\tilde{t}}(X_0, u, 0^+) = -\infty\}. \tag{31}$$

In particular, for every $k > \overline{k}$ we get

$$\limsup_{r \to 0^+} N_{\tilde{t}}(X_0, u, r) \leq k.$$

Proof Following the reasoning in Corollary 3.6, we can immediately deduce the existence of $k \geq \tilde{k}$ such that $W_{k,\tilde{t}}(X_0, u, 0^+) < 0$. Hence, we can reasonably define the quantity

$$\overline{\overline{k}} = \inf\{k \geq \tilde{k} \colon W_{k,\tilde{t}}(X_0, u, 0^+) = -\infty\},$$

for which

$$\begin{cases} W_{k,\tilde{t}}(X_0, u, 0^+) \geq 0 & \text{if } \overline{k} \leq k < \overline{\overline{k}} \\ W_{k,\tilde{t}}(X_0, u, 0^+) = -\infty & \text{if } k > \overline{\overline{k}}. \end{cases}$$

Since $\tilde{t} < 2$, we fist have $W_{k,\tilde{t}}(X_0, u, r) \geq W_{k,2}(X_0, u, r)$ for every $0 < r < R$ and $k > 0$. Now, on one side $W_{k,\tilde{t}}(X_0, u, 0^+) = -\infty$ implies $W_{k,2}(X_0, u, 0^+) = -\infty$ and hence $\overline{\overline{k}} \geq \overline{k}$. So, let us suppose by contradiction that $\overline{\overline{k}} > \overline{k}$, hence there exists $k \in (\overline{k}, \overline{\overline{k}})$ such that $W_{k,\tilde{t}}(X_0, u, 0^+) \geq 0$.

By the monotonicity result in Lemma 4.4 we get $W_{k,\tilde{t}}(X_0, u, r) \geq 0$ for $r > 0$ and, since $\tilde{t} \in (q, 2)$, we deduce

$$W_{k,q}(X_0, u, r) \geq W_{k,\tilde{t}}(X_0, u, r) \geq 0, \tag{32}$$

for every $r \in (0, \text{dist}(X_0, \partial^+ B^+))$. Finally, recalling the relation in (30), by (32) it follows that $r \mapsto r^{-2k} H(X_0, u, r)$ is monotone non-decreasing and in particular there exists finite

$$\lim_{r \to 0^+} \frac{H(X_0, u, r)}{r^{2k}} \in (0, +\infty),$$

which contradicts Lemma 4.3. □

Lemma 4.6 *Let $X_0 \in \Gamma(u)$ and $\overline{k} \geq k_q$. There exists a sequence $(r_n)_n$ such that $r_i \to 0^+$ and*

$$\frac{1}{r_n^{n+a-1+2\overline{k}}} \int_{\partial^0 B^+_{r_n}(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x \to 0.$$

Proof Let $k \in [k_q, \overline{k})$, by Corollary 3.4 and Corollary 3.7 we have $W_{k,2}(X_0, u, r) \geq 0$ for every $r \in (0, \text{dist}(X_0, \partial^+ B^+))$. Since for any fixed radius $r > 0$ the function $k \mapsto W_{k,2}(X_0, u, r)$ is continuous, we infer as $k \to \overline{k}^-$ that $W_{\overline{k},2}(X_0, u, r) \geq 0$, which implies by continuity that $W_{\overline{k},2}(X_0, u, 0^+) \geq 0$.

Thus, for any $\overline{r} \in (0, \text{dist}(X_0, \partial^+ B^+))$ we get

$$0 \leq \int_0^{\overline{r}} \frac{d}{dr} W_{\overline{k},2}(X_0, u, s) \mathrm{d}s = W_{\overline{k},2}(X_0, u, \overline{r}) - W_{\overline{k},2}(X_0, u, 0^+) < +\infty.$$

On the other side, by (27) we deduce

$$\int_0^{\overline{r}} \frac{1}{s} \left(\frac{1}{s^{n+a-1+2\overline{k}}} \int_{\partial^0 B^+_s(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x \right) \mathrm{d}s < +\infty, \tag{33}$$

which implies, combined with the non-integrability of $s \mapsto s^{-1}$ in 0, that if

$$\liminf_{r \mapsto 0^+} \frac{1}{r^{n+a-1+2\overline{k}}} \int_{\partial^0 B^+_r(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x > 0,$$

then (33) would not be true. Thus, this implication suggests that the previous liminf has to be null. □

Proof of Proposition 4.1 The proof is based on a blow-up argument: given $X_0 \in \Gamma(u)$ assume that $\overline{k} > k_q$ and let $(r_n)_n$ be the sequence introduced in Lemma 4.6. Therefore, consider the blow-up sequence

$$u_n(X) = \frac{u(X_0 + r_n X)}{\sqrt{H(X_0, u, r_n)}} \quad \text{for } X \in B^+_{R/r_n}$$

where $R = \text{dist}(X_0, \partial^+ B^+)$. Thanks to Lemma 4.3, we have $H(X_0, u, r_n) > 0$ and $E_2(X_0, u, r_n) \geq 0$, which lead to

$$\int_{\partial^+ B^+_1} y^a u_n^2 \mathrm{d}\sigma = 1 \quad \text{and} \quad \int_{B^+_1} y^a |\nabla u_n|^2 \mathrm{d}X = \frac{\dfrac{1}{r_n^{n+a-1}} \displaystyle\int_{B^+_{r_n}(X_0)} y^a |\nabla u|^2 \mathrm{d}X}{\dfrac{1}{r_n^{n+a}} \displaystyle\int_{\partial^+ B^+_{r_n}(X_0)} y^a u^2 \mathrm{d}\sigma}.$$

On the other hand by Lemma 4.3 we deduce

$$\int_{B^+_{r_n}(X_0)} y^a |\nabla u|^2 \mathrm{d}X \geq \frac{2}{q} \int_{\partial^0 B^+_{r_n}(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x,$$

which implies, since $\tilde{t} < 2$ that

$$\begin{aligned}
&\frac{1}{r_n^{n+a-1}} \int_{B^+_{r_n}(X_0)} y^a |\nabla u|^2 \, \mathrm{d}X \\
&\quad \le \frac{2}{2-\tilde{t}} \frac{1}{r_n^{n+a-1}} \left(\int_{B^+_{r_n}(X_0)} y^a |\nabla u|^2 \, \mathrm{d}X - \frac{\tilde{t}}{q} \int_{\partial^0 B^+_{r_n}(X_0)} F_{\lambda_+,\lambda_-}(u) \mathrm{d}x \right) \\
&\quad \le \frac{2}{2-\tilde{t}} E_{\tilde{t}}(X_0, u, r_n).
\end{aligned}$$

As a consequence of the previous estimates and Lemma 4.5, we get

$$\int_{B_1^+} y^a |\nabla u_n|^2 \, \mathrm{d}X \le \frac{2}{2-\tilde{t}} N_{\tilde{t}}(X_0, u, r_n) \le C.$$

Since the sequence $(u_n)_n$ is uniformly bounded in $H^{1,a}(B_1^+)$, the compactness of the Sobolev embedding implies that $(u_n)_n$ converges weakly in $H^{1,a}(B^1_+)$ and strongly in $L^{2,a}(\partial^+ B_1^+)$ to a function $\overline{u} \in H^{1,a}(B^1_+)$.

Moreover, since by [22,Theorem 2.11] the space of the trace of functions in $H^{1,a}(B^+)$ on the set $\partial^0 B^+$ coincides with the Sobolev-Slobodeckij space $H^s(\partial^0 B^+)$, by the Riesz–Frechet–Kolmogorov Theorem, the trace operator

$$H^{1,a}(B_1^+) \hookrightarrow\hookrightarrow L^p(\partial^0 B_1^+)$$

is well defined an compact for every $p \in [1,2]$ (see Lemma 2.3). Hence, since $q \in [1,2)$, we get

$$\int_{\partial^+ B_1^+} y^a \overline{u}^2 \mathrm{d}\sigma = 1 \quad \text{and} \quad \lim_{n\to\infty} \int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(u_n)\mathrm{d}x = \int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(\overline{u})\mathrm{d}x. \tag{34}$$

Since the first equality implies that $\overline{u} \not\equiv 0$ on $\partial^+ B_1^+$, we deduce by the trace embedding that $\overline{u} \not\equiv 0$ on the whole B_1^+. On the other side, we get

$$\int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(u_n)\mathrm{d}x = \left(\frac{r^{(a-1+2\overline{k})/q}}{\sqrt{H(X_0,u,r_n)}} \right)^q \frac{1}{r_n^{n+a-1+2\overline{k}}} \int_{\partial^0 B^+_{r_n}(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x.$$

By direct computation, since we are assuming $\overline{k} > k_q$, we have $2(2\overline{k}+a-1)/q > 2\overline{k}$ and, for n sufficiently large, it implies

$$\int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(u_n)\mathrm{d}x \le \left(\frac{H(X_0,u,r_n)}{r_n^{2\overline{k}}} \right)^{-q/2} \frac{1}{r_n^{n+a-1+2\overline{k}}} \int_{\partial^0 B^+_{r_n}(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x$$

where, by Lemma 4.3 and Lemma 4.6, the right hand side goes to 0 as $n \mapsto +\infty$. By (34) we infer that

$$\int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(\overline{u})\mathrm{d}x = 0 \longleftrightarrow \overline{u} \equiv 0 \ \text{ on } \ \partial^0 B_1^+.$$

On the other hand, since $(u_n)_n$ is uniformly bounded in $H^{1,a}(B_1^+)$ and $u_n \rightharpoonup \overline{u}$ weakly in $H^{1,a}$, from

$$-\partial_y^a u_n = \left(\frac{r_n^{2\overline{k}}}{H(X_0,u,r_n)} \right)^{\frac{2-q}{2}} \left(\lambda_+(u_n)_+^{q-1} - \lambda_-(u_n)_-^{q-1} \right) \quad \text{on} \quad \frac{\partial^0 B_R^+ - X_0}{r_n},$$

we deduce that the limit function $\overline{u} \in H^{1,a}_{\text{loc}}(\overline{R^{n+1}_+})$ is a weak solution of

$$\begin{cases} L_a\overline{u} = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \partial^a_y \overline{u} = 0 & \text{on } \mathbb{R}^n \times \{0\} \\ \overline{u} = 0 & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{35}$$

such that $\overline{u} \not\equiv 0$ on $\mathbb{R}^{n+1}_+$. The contradiction follows immediately by the unique continuation principle for the traces of L_a-harmonic functions (see [28, Proposition 5.9]). □

The following result completes the previous characterization in the case $\mathcal{O}(u, X_0) \geq k_q$ by relating the critical exponent k_q and the transition exponent $\overline{k}$ to the $H^{1,a}$-vanishing order of u at X_0. More precisely, it implies that the solutions of (3) can vanish with order less or equal than k_q.

Proposition 4.7 *Let u be a solution of (3) and $X_0 \in \Gamma(u)$ such that $\mathcal{O}(u, X_0) \geq k_q$. Then, the vanishing order $\mathcal{O}(u, X_0)$ is characterized by*

$$\mathcal{O}(u, X_0) = \inf\left\{ k > 0\colon W_{k,2}(X_0, u, 0^+) = -\infty \right\} = k_q.$$

Furthermore, we get

$$\begin{cases} W_{k,2}(X_0, u, 0^+) = 0 & \text{if } 0 < k < \mathcal{O}(u, X_0) \\ W_{k,2}(X_0, u, 0^+) = -\infty & \text{if } k > \mathcal{O}(u, X_0). \end{cases}$$

Proof The proof of this result follows the one of its local counterpart in [31]. For the sake of simplicity, let us denote with $\|\cdot\|_{H^{1,a}(B^+_r(X_0))} = \|\cdot\|_{X_0,r}$. Now, fixed $X_0 \in \Gamma(u)$, let us prove that

$$\liminf_{r\to 0^+} \frac{\|u\|^2_{X_0,r_n}}{r^{2\overline{k}}} > 0, \tag{36}$$

where $\overline{k} = k_q$. After that, the result will follow by Proposition 4.1 and (29). By contradiction, let us suppose there exists a sequence $r_n \to 0^+$ such that

$$\lim_{n\to\infty} \frac{\|u\|^2_{X_0,r_n}}{r_n^{2\overline{k}}} = 0. \tag{37}$$

Then, consider the blow-up sequence associated to the $H^{1,a}$-norm, defined as

$$u_r(X) = \frac{u(X_0 + rX)}{\|u\|_{X_0,r}}, \quad \text{such that } \|u_n\|_{0,1} = 1. \tag{38}$$

As we deduce in the proof of Proposition 4.1, since the blow-up sequence $(u_n)_n$ is uniformly bounded in $H^{1,a}(B^+_1)$, the compactness of the Sobolev embedding implies that $(u_n)_n$ converges weakly in $H^{1,a}(B^+_1)$ and strongly in $L^{2,a}(\partial^+ B^+_1)$ to a function $\overline{u} \in H^{1,a}(B^+_1)$. Similarly, the traces on $\partial^0 B^+_1$ converge strongly in $L^q(\partial^0 B^+_1)$ to the trace of $\overline{u}$, for every $q \in [1, 2)$. In particular,

$$\lim_{n\to\infty} W_{\overline{k},2}(X_0, u, r_n) = \lim_{n\to\infty} \left[\frac{\|u\|^2_{X_0,r_n}}{r_n^{2\overline{k}}} \left(\int_{B^+_1} y^a\, |\nabla u_n|^2\, \mathrm{d}X - \overline{k} \int_{\partial^+ B^+_1} y^a u_n^2 \mathrm{d}\sigma \right) + \right.$$
$$\left. -\frac{2}{q} \left(\frac{\|u\|^2_{X_0,r_n}}{r_n^{2\overline{k}}} \right)^{q/2} \int_{\partial^0 B^+_1} F_{\lambda_+,\lambda_-}(u_n)\mathrm{d}x \right] = 0.$$

Thus, since the limit $W_{\overline{k},2}(X_0, u, 0^+)$ exists, by the monotonicity result in Proposition 3.4, we get that $W_{\overline{k},2}(X_0, u, r) \geq 0$ and $W_{\overline{k},q}(X_0, u, r) \geq 0$ for every $r \in (0, \mathrm{dist}(X_0, \partial^+ B_1^+))$ and $q < 2$.

First, by Lemma 4.3, we know that $E_2(X_0, u, r) \geq 0$ and $H(X_0, u, r) > 0$ for every $r \in (0, \mathrm{dist}(X_0, \partial^+ B_1^+))$ and, for every $k > \overline{k}$ we get

$$\liminf_{r\to 0^+} \frac{H(X_0, u, r)}{r^{2k}} = +\infty. \tag{39}$$

Now, let us compute the same limit in the case $\overline{k}$. Since the function $r \mapsto H(X_0, u, r)/r^{2\overline{k}}$ is monotone non-decreasing, there exists the limit as $r \to 0^+$ and, by (37), we get

$$0 \leq \frac{H(X_0, u, r_n)}{r_n^{2\overline{k}}} \leq \frac{\|u\|^2_{X_0, r_n}}{r_n^{2\overline{k}}} \to 0$$

which implies

$$\lim_{r\to 0^+} \frac{H(X_0, u, r)}{r^{2\overline{k}}} = 0. \tag{40}$$

In order to reach a contradiction, we need to prove that the blow-up limit satisfies $\overline{u} \equiv 0$, in contradiction with the normalization (38) (see the conclusion of the Section). □

Lemma 4.8 *Fixed $X_0 \in \Gamma(u)$ and $\overline{k} = k_q$ let us suppose that (40) holds true. Then, we get*

$$\liminf_{r\to 0^+} \frac{W_{\overline{k},q}(X_0, u, r) r^{2\overline{k}}}{H(X_0, u, r)} = 0 \quad \textit{and} \quad \lim_{r\to 0^+} \frac{W_{\overline{k},2}(X_0, u, r) r^{2\overline{k}}}{H(X_0, u, r)} = 0. \tag{41}$$

Proof Let us consider first the limit associated to the case $t = q$ and, by contradiction, assume that $\varepsilon > 0$ and $r_0 \in (0, \mathrm{dist}(X_0, \partial^+ B_1^+))$ such that

$$\frac{W_{\overline{k},q}(X_0, u, r) r^{2\overline{k}}}{H(X_0, u, r)} \geq \varepsilon \quad \text{for every } r \in (0, r_0).$$

By (18), we deduce that

$$\frac{d}{dr} \log\left(\frac{H(X_0, u, r)}{r^{2\overline{k}}}\right) = \frac{2}{r} \frac{W_{\overline{k},q}(X_0, u, r) r^{2\overline{k}}}{H(X_0, u, r)} \geq \frac{2\varepsilon}{r},$$

and integrating by parts the previous inequality between $r \in (0, r_0)$ and r_0 we get

$$\frac{H(X_0, u, r)}{r^{2\overline{k}+2\varepsilon}} \leq \frac{H(X_0, u, r_0)}{r_0^{2\overline{k}+2\varepsilon}} < \infty \quad \text{for every } r \in (0, r_0).$$

In particular

$$\limsup_{r\to 0^+} \frac{H(X_0, u, r)}{r^{2\overline{k}+2\varepsilon}} < +\infty,$$

in contradiction with (39) with $k = \overline{k} + \varepsilon$.
Now, for $t = 2$ and $\overline{k} = k_q$ we already know by Proposition 3.4 that

$$\frac{d}{dr} W_{\overline{k},2}(X_0, u, r) = \frac{2}{r^{n+a-1+2\overline{k}}} \int_{\partial^+ B_r^+(X_0)} y^a \left(\partial_r u - \frac{\overline{k}}{r} u\right)^2 \mathrm{d}\sigma.$$

In the remaining part of the proof, for the sake of simplicity we omit the dependence with respect to u and X_0. Hence, combining the previous derivative with (30) we get

$$\left(\frac{H(r)}{r^{2\overline{k}}}\right)^2 \frac{d}{dr}\left(\frac{r^{2\overline{k}}W_{\overline{k},2}(r)}{H(r)}\right) = \frac{H(r)}{r^{2\overline{k}}}\frac{d}{dr}W_{\overline{k},2}(r) - \frac{2}{r}W_{\overline{k},2}(r)W_{\overline{k},q}(r),$$

and since $0 \le W_{\overline{k},2}(r) \le W_{\overline{k},q}(r)$ we infer that

$$\begin{aligned}
&\left(\frac{H(r)}{r^{2\overline{k}}}\right)^2 \frac{d}{dr}\left(\frac{r^{2\overline{k}}W_{\overline{k},2}(r)}{H(r)}\right) \ge \\
&\ge \frac{2}{r^{2n+2a-1+4\overline{k}}}\int_{\partial^+ B_r^+} y^a u^2 \mathrm{d}\sigma \int_{\partial^+ B_r^+} y^a\left(\partial_r u - \frac{\overline{k}}{r}u\right)^2 \mathrm{d}\sigma - \frac{2}{r}\left(W_{\overline{k},q}(r)\right)^2 \\
&\ge \frac{2}{r^{2n+2a-1+4\overline{k}}}\left[\int_{\partial^+ B_r^+} y^a u^2 \mathrm{d}\sigma \int_{\partial^+ B_r^+} y^a(\partial_r u)^2 \mathrm{d}\sigma - \left(\int_{\partial^+ B_r^+} y^a u \partial_r u \mathrm{d}\sigma\right)^2\right],
\end{aligned}$$

which is non-negative by the Cauchy-Schwarz inequality. Since $H(r) > 0$ and $0 \le W_{\overline{k},2}(r) \le W_{\overline{k},q}(r)$, the previous part of the proof yields that the second limit in (41) exists and is equal to zero. □

Conclusion of the proof of Proposition 4.7 Since $\|u\|^2_{X_0,r} \ge H(X_0,u,r)$, by Lemma 4.8, there exists a sequence $r_m \to 0^+$ such that

$$\lim_{m\to\infty} \frac{r_m^{2\overline{k}}W_{\overline{k},q}(X_0,u,r_m)}{\|u\|^2_{X_0,r_m}} = \lim_{m\to\infty} \frac{r_m^{2\overline{k}}W_{\overline{k},2}(X_0,u,r_m)}{\|u\|^2_{X_0,r_m}} = 0. \tag{42}$$

Now, let u_m be the blow-up subsequence of (38) associated to the sequence $(r_m)_m$ which converges to a limit function $\overline{u}$. First, by (41) we infer

$$\begin{aligned}
0 \le \left(\frac{r_m^{2\overline{k}}}{H(r_m)}\right)^{\frac{2-q}{2}} \int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(u_m)\mathrm{d}x &\le \frac{r_m^{2s}\,\|u\|^q_{X_0,r_m}}{H(r_m)}\int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(u_m)\mathrm{d}x \\
&= \frac{1}{r_m^{n+a-1}H(r_m)}\int_{\partial^0 B_{r_m}^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x \\
&= \frac{q}{2-q}\frac{r_m^{2\overline{k}}\left(W_{k,q}(r_m) - W_{k,2}(r_m)\right)}{H(r_m)} \to 0^+,
\end{aligned}$$

which implies, combined with the strong convergence in $L^q(\partial^0 B_1^+)$ and (40), that $\overline{u} \equiv 0$ on $\partial^0 B_1^+$. On the other side, by (42) we deduce that

$$\begin{aligned}
0 &\le \frac{r_m^{2s}}{\|u\|^{2-q}_{X_0,r_m}}\int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(u_m)\mathrm{d}x \\
&= \frac{1}{r_m^{n+a-1}\,\|u\|^2_{X_0,r_m}}\int_{\partial^0 B_{r_m}^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x \\
&= \frac{q}{2-q}\frac{r_m^{2\overline{k}}\left(W_{k,q}(X_0,u,r_m) - W_{k,2}(X_0,u,r_m)\right)}{\|u\|^2_{X_0,r_m}} \to 0^+,
\end{aligned}$$

as $m \to +\infty$. Therefore, collecting the previous results we get

$$\begin{aligned}
0 &= \lim_{m\to\infty} \frac{r_m^{2\overline{k}} W_{\overline{k},2}(X_0, u, r_m)}{\|u\|^2_{X_0,r_m}} \\
&= \lim_{m\to\infty} \left(\int_{B_1^+} y^a \, |\nabla u_m|^2 \, \mathrm{d}X - \frac{2r_m^{1-a}}{q\, \|u\|^{2-q}_{X_0,r_m}} \int_{\partial^0 B_1^+} F_{\lambda_+,\lambda_-}(u_m)\mathrm{d}x - \overline{k} \int_{\partial^+ B_1^+} y^a u_m^2 \mathrm{d}\sigma \right) \\
&= \lim_{m\to\infty} \left(\int_{B_1^+} y^a \, |\nabla u_m|^2 \, \mathrm{d}X - \overline{k} \int_{\partial^+ B_1^+} y^a u_m^2 \mathrm{d}\sigma \right),
\end{aligned}$$

which implies that $\|u_m\|^2_{0,1} \to (\overline{k}+1)\, \|\overline{u}\|^2_{L^{2,a}(\partial^+ B_1^+)}$. Since by (38) the normalization implies $\|u_m\|_{0,1} = 1$ for every m, we immediately deduce that $\overline{u} \not\equiv 0$ in B_1^+. Therefore, the conclusion follows as in the proof of Proposition 4.1. □

5 Blow-up analysis for $\mathcal{O}(u, X_0) < k_q$

In this Section we initiate the blow-up analysis of the nodal set starting from those points with vanishing order smaller than the critical value $k_q = 2s/(2-q)$. The main strategy is to develop a blow-up argument based on the validity of Almgren-type and Weiss-type monotonicity formulas, which provide a Taylor expansion of the solutions near the nodal set in terms of L_a-harmonic polynomials symmetric with respect to $\{y = 0\}$.

We initiate the analysis by introducing an Almgren-type monotonicity formula. More precisely, by using the upper bound on the $H^{1,a}$-vanishing order of u, we prove the validity of a monotonicity result for functional $N(X_0, u, r) = N_q(X_0, u, r)$ introduced in (17).

Proposition 5.1 *Let $K \subset\subset \partial^0 B_1^+$ and suppose there exists $\delta > 0$ such that*

$$\mathcal{O}(u, X_0) \le k_q - \delta \quad \textit{for every } X_0 \in \Gamma(u) \cap K. \tag{43}$$

Then there exists $r_0 > 0$ such that for every $X_0 \in \Gamma(u) \cap K$

$$r \mapsto e^{\tilde{C} r^\alpha} (N(X_0, u, r) + 1)$$

is monotone non-decreasing for $r \in (0, \min(r_0, \mathrm{dist}(K, \partial^0 B^+)))$, for some constant $\alpha = \alpha(\delta, n, s, q)$ and $\tilde{C} = \tilde{C}(\delta, n, s, q)$. Moreover, for every $X_0 \in \Gamma(u)$ such that $\mathcal{O}(u, X_0) < k_q$ there exits the limit

$$N(X_0, u, 0^+) = \lim_{r\to 0^+} e^{\tilde{C} r^\alpha} \left(N(X_0, u, r) + 1\right) - 1$$

and the map $X_0 \mapsto N(X_0, u, 0^+)$ is upper semi-continuous on $\Gamma(u)$.

Proof Let $K \subset\subset \partial^0 B_1^+$ and $\alpha > 0$ to be made precise later. Let $X_0 \in K$ and, for the sake of simplicity, we omit the dependence of the functionals with respect to u and X_0. By

Corollary 3.2, we easily get

$$\begin{aligned}\frac{d}{dr}\log(N(r)+1) &\geq \frac{1}{E(r)+H(r)}\frac{1}{r^{n+a-1}}\left[\frac{2-q}{q}\int_{S_r^{n-1}(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}\sigma\right.\\ &\qquad \left.-\frac{C^s_{n,q}}{qr}\int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x\right]\\ &\geq -\frac{C^s_{n,q}}{q(E(r)+H(r))}\frac{1}{r^{n+a}}\int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x\end{aligned} \tag{44}$$

with $C^s_{n,q} = 2n - q(n-2s)$. On the other hand, by Lemma 2.2

$$\begin{aligned}E(r)+H(r) &\geq \|u\|^q_{X_0,r}\left(\|u\|^{2-q}_{X_0,r} - C_1 r^{2s}\right)\\ &\geq \frac{C}{r^n}\left(\|u\|^{2-q}_{X_0,r} - C_1 r^{2s}\right)\int_{\partial^0 B_r^+(X_0)} |u|^q\,\mathrm{d}x.\end{aligned} \tag{45}$$

Now, we want to show that there exists $\alpha, r_0, C_2 > 0$ such that

$$\frac{\|u\|^{2-q}_{X_0,r}}{r^{2s}} - C_1 > C_2\frac{1}{r^\alpha}, \tag{46}$$

for every $r \in (0, r_0)$. Then, combining the previous inequality with (44) and (45), we will get

$$\frac{d}{dr}\log(N(r)+1) \geq -\frac{\tilde{C}}{r\left(\dfrac{\|u\|^{2-q}_{X_0,r}}{r^{2s}} - C_1\right)} \geq -\tilde{C}r^{\alpha-1},$$

as we claimed. First, by (43), let us choose $\alpha = \delta/2$ and consider

$$k_2 = k_q - \alpha \geq \mathcal{O}(u, X_0),$$

for every $X_0 \in \Gamma(u) \cap K$. Indeed, by the definition of $H^{1,a}$-vanishing order, there exists $r_2 > 0$ and $C_2 > 0$ such that, for every $r \in (0, r_2)$

$$\|u\|_{X_0,r} \geq C_2 r^{k_2} \longleftrightarrow \frac{\|u\|^{2-q}_{X_0,r}}{r^{2s}} \geq C_2 r^{(2-q)k_2-2s} = C_2 r^{-\alpha}. \tag{47}$$

Since $\delta = \delta(K)$, the constant C_2, α and r_2 depend only on the choice of the compact K. Finally, the upper semi-continuity follows by a standard argument. □

Using this monotonicity result we can prove the equivalence between the notion of $H^{1,a}$-vanishing order $\mathcal{O}(u, X_0)$ and the one introduced in Definition 1.3.

Corollary 5.2 *Let $X_0 \in \Gamma(u)$ be such that $\mathcal{O}(u, X_0) < k_q$. Then*

$$\mathcal{O}(u, X_0) = \mathcal{V}(u, X_0).$$

Proof Suppose by contradiction that $\mathcal{O}(u, X_0) < \mathcal{V}(u, X_0)$ and consider $k \in (\mathcal{O}(u, X_0), \mathcal{V}(u, X_0))$. Let us write

$$k = \frac{2s-\alpha}{2-q},$$

for some $\alpha > 0$. Now, let $r \in (0, \mathrm{dist}(X_0, \partial^0 B_1^+))$, by (45) we get

$$\|u\|_{X_0,r}^q \left(\|u\|_{X_0,r}^{2-q} - C_1 r^{2s}\right) \le E(X_0, u, r) + H(X_0, u, r) \\ = H(X_0, u, r)(N(X_0, u, r) + 1) \tag{48}$$

which implies

$$\frac{\|u\|_{X_0,r}^2}{r^{2k}} \le \left[\frac{N(X_0, u, r) + 1}{\|u\|_{X_0,r}^{2-q} - C_1 r^{2s}} r^{k(2-q)}\right]^{2/q} \left(\frac{H(X_0, u, r)}{r^{2k}}\right)^{2/q}. \tag{49}$$

As in (47), in the proof of Proposition 5.1, there exists $r_0 > 0$ and $C_0 > 0$ such that

$$\|u\|_{X_0,r}^{2-q} - C_1 r^{2s} \ge C_0 r^{2s-\alpha} = C_0 r^{k(2-q)},$$

for every $r \in (0, r_0)$. With a slight abuse of notations, it is not restrictive to assume that r_0 corresponds to the radius introduced in Proposition 5.1.

Finally, by the monotonicity result, fixed $R = \min\{r_0, \mathrm{dist}(X_0, \partial^0 B^+)\}$ we deduce, for every $r \in (0, R)$, that

$$\begin{aligned} \frac{\|u\|_{X_0,r}^2}{r^{2k}} &\le C\,[(N(X_0, u, r) + 1)]^{2/q} \left(\frac{H(X_0, u, r)}{r^{2k}}\right)^{2/q} \\ &\le C\left[e^{\tilde{C}R}(N(X_0, u, R) + 1)\right]^{2/q} \left(\frac{H(X_0, u, r)}{r^{2k}}\right)^{2/q} \end{aligned}$$

where $C > 0$ depends only on C_0. Thus, by Definition 1.3 we get that $\mathcal{O}(u, X_0) \ge \mathcal{V}(u, X_0)$ that, in combination with the opposite inequality, implies the desired result. □

Similarly, we show that in the case $\mathcal{O}(u, X_0) = \mathcal{V}(u, X_0) < k_q$, the possible vanishing orders correspond to the possible limits of the Almgren-type frequency formula. For the sake of completeness, we report the proof of this result which is deeply based on the validity of the Almgren-type monotonicity result.

Corollary 5.3 *Let $X_0 \in \Gamma(u)$ be such that $\mathcal{V}(u, X_0) < k_q$. Then $\mathcal{V}(u, X_0) = N(X_0, u, 0^+)$.*

Proof By (16) and Definition 1.3, we claim that

$$\limsup_{r\to 0^+} \frac{H(X_0, u, r)}{r^{2k}} = \begin{cases} 0, & \text{if } 0 < k < N(X_0, u, 0^+) \\ +\infty, & \text{if } k > N(X_0, u, 0^+). \end{cases}$$

It is not restrictive to assume that $X_0 = 0$ and $r \in (0, R)$, for some $R > 0$ that will be choose later. By definition of $r \mapsto H(0, u, r) = H(u, r)$ we immediately get for every $r \in (0, R)$ that

$$\frac{d}{dr} \log H(u, r) = \frac{2}{r} N(u, r) \tag{50}$$

and in particular for every $k > 0$, by Proposition 5.1, there exists $\alpha, \tilde{C} > 0$ such that

$$\left(\frac{H(u, R)}{R^{2\overline{N}}}\right) r^{2(\overline{N}-k)} \le \frac{H(u, r)}{r^{2k}} \le \left(\frac{H(u, R)}{R^{2\underline{N}}}\right) r^{2(\underline{N}-k)}, \tag{51}$$

with

$$\underline{N} = e^{-\tilde{C}R^\alpha}(N(u, 0^+) + 1) - 1 \quad \text{and} \quad \overline{N} = e^{\tilde{C}R^\alpha}(N(u, R) + 1) - 1.$$

Suppose first $\mathcal{V}(u,0) < N(u,0^+)$, so there exists $\varepsilon > 0$ such that $k := N(u,0^+) - \varepsilon > \mathcal{V}(u,0)$. Let $R > 0$ be such that

$$(1 - e^{-\tilde{C}R^\alpha})(N(u,0^+) + 1) < \frac{\varepsilon}{2},$$

where $\tilde{C}, \alpha > 0$ are introduced in Proposition 5.1. Thus, we get $\underline{N} - k > \varepsilon/2$ and consequently by (51)

$$\frac{H(u,r)}{r^{2k}} \leq \left(\frac{H(u,R)}{R^{2\underline{N}}}\right) r^{2(\underline{N}-k)} < C_2 r^\varepsilon,$$

for some constant $C_2 > 0$ depending only on $R > 0$. The absurd follows immediately since $k > \mathcal{V}(u,0)$, namely

$$+\infty = \limsup_{r\to 0^+} \frac{H(u,r)}{r^{2k}} \leq \left(\frac{H(u,R)}{R^{2\underline{N}}}\right) r^{2(\underline{N}-k)} < C_2 \lim_{r\to 0^+} r^\varepsilon = 0.$$

Similarly, if $\mathcal{V}(u,0) > N(u,0^+)$ consider $k = N(u,0^+) + \varepsilon$, with $\varepsilon > 0$ sufficiency small so that $\mathcal{V}(u,0) > k$. By the monotonicity result Proposition 5.1, let $R > 0$ be such that

$$e^{\tilde{C}R^\alpha}(N(u,R) + 1) - (N(u,0^+) + 1) < \frac{\varepsilon}{2}.$$

Hence, since $\overline{N} - k < -\varepsilon/2$, we get by (51)

$$\frac{H(u,r)}{r^{2k}} \geq \left(\frac{H(u,R)}{R^{2\overline{N}}}\right) r^{2(\overline{N}-k)} \geq C_2 r^{-\varepsilon}$$

for some constant $C_2 > 0$ depending only on $R > 0$. The contradiction follows by Definition 1.3. □

In particular, from the previous equivalences, for those points satisfying $\mathcal{O}(u, X_0) < k_q$, it holds

$$\mathcal{O}(u, X_0) = \mathcal{V}(u, X_0) = N(X_0, u, 0^+).$$

Moreover, for every $k_1 < N(X_0, u, 0^+) < k_2$ there exist $C_1, C_2 > 0$ such that

$$C_2 r^{2k_2} \leq \|u\|^2_{X_0,r} \leq C_1 r^{2k_1}, \tag{52}$$

for $r \in (0, R)$, for some $R > 0$ sufficiently small.

Finally, we can introduce the following notion of stratum of the nodal set.

Definition 5.4 Let $k < k_q$ we define

$$\Gamma_k(u) := \{X_0 \in \Gamma(u) \colon \mathcal{O}(u, X_0) = k\}.$$

While in the local case, in [31, 33] the authors proved the existence of a generalized Taylor expansion of the solution near the nodal set by applying an iteration argument based on the results of [6], we apply a blow-up analysis in order to understand how the solutions behave near the nodal set $\Gamma(u)$.

Hence, given $X_0 \in \Gamma(u)$, for any $r_k \downarrow 0^+$, we define as normalized blow-up sequence

$$u_k(X) = \frac{u(X_0 + r_k X)}{\sqrt{H(X_0, u, r_k)}} \quad \text{for } X \in B^+_{X_0, r_k} = \frac{B^+_1 - X_0}{r_k},$$

such that

$$\begin{cases} -L_a u_k = 0 & \text{in } B^+_{X_0,r_k} \\ -\partial^a_y u_k = \left(\dfrac{r_k^{k_q}}{\sqrt{H(X_0,u,r_k)}}\right)^{2-q} \left[\lambda_+(u_k)_+^{q-1} - \lambda_-(u_k)_-^{q-1}\right] & \text{on } \partial^0 B^+_{X_0,r_k}. \end{cases} \tag{53}$$

Let us introduce the notation

$$0 < \alpha_k = \left(\frac{r_k^{k_q}}{\sqrt{H(X_0,u,r_k)}}\right)^{2-q} < +\infty,$$

Since we are assuming $\mathcal{O}(u, X_0) < k_q$, the sequence $(\alpha_k)_k$ is bounded and converges to 0 as $k \to \infty$.

Theorem 5.5 *Let $X_0 \in \Gamma(u)$ be such that $\mathcal{O}(u, X_0) < k_q$ and u_k be a normalized blow-up sequence centered in X_0 and associated with some $r_k \downarrow 0^+$. Then, there exists $p \in H^{1,a}_{loc}(\overline{\mathbb{R}^{n+1}_+})$ such that, up to a subsequence, $u_k \to p$ in $C^{0,\alpha}_{loc}(\overline{\mathbb{R}^{n+1}_+})$ for every $\alpha \in (0,1)$ and strongly in $H^{1,a}_{loc}(\overline{\mathbb{R}^{n+1}_+})$. In particular, the blow-up limit satisfy*

$$\begin{cases} -L_a p = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\partial^a_y p = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{54}$$

The proof will be presented in a series of lemmata.

Lemma 5.6 *Let $X_0 \in \Gamma(u)$ such that $\mathcal{O}(u, X_0) < k_q$. For any given $R > 0$, we have*

$$\|u_k\|_{0,R} \le C$$

where $C > 0$ is a constant independent on $k > 0$. Moreover $u_k \to p$ strongly in $H^{1,a}(B^+_R)$ for every $R > 0$, for some $p \in H^{1,a}_{loc}(\overline{\mathbb{R}^{n+1}_+})$ such that $\|p\|_{L^{2,a}(\partial^+ B^+)} = 1$.

Proof Set $\rho_k^2 = H(X_0,u,r_k)$, then by the definition of u_k, (50) and Proposition 5.1 we obtain

$$\begin{aligned} \int_{\partial^+ B^+_R} y^a u_k^2 \mathrm{d}\sigma &= \frac{1}{\rho_k^2 r_k^{n+a}} \int_{\partial B_{Rr_k}(X_0)} y^a u^2 \mathrm{d}\sigma \\ &= R^{n+a} \frac{H(X_0,u,Rr_k)}{H(X_0,u,r_k)} \\ &\le R^{n+a} \left(\frac{Rr_k}{r_k}\right)^{2\tilde{C}} \end{aligned}$$

which gives us $\|u_k\|^2_{L^{2,a}(\partial^+ B^+_R)} \le C(R) R^{n+a}$. Instead, inspired by Corollary 5.2, let

$$k = \frac{2s-\alpha}{2-q},$$

for some $\alpha > 0$, then

$$\begin{aligned} \|u_k\|^q_{0,R} &= \frac{1}{\rho_k^q} \|u\|^q_{X_0,r_k R} \\ &\le C\left[e^{\tilde{C}R}(N(X_0,u,R)+1)\right]^{2/q} \frac{H(X_0,u,Rr_k)}{H(X_0,u,R)} \rho_k^{2-q} (Rr_k)^{\alpha-2s} \\ &\le C(R)\rho_k^{2-q}(Rr_k)^{\alpha-2s} \end{aligned}$$

and by Lemma 2.2 and Proposition 5.1, we infer

$$\begin{aligned}
&\frac{1}{R^{n+a-1}}\int_{B_R^+} y^a\,|\nabla u_k|^2\,\mathrm{d}X\\
&= E(X_0,u,Rr_k) + \left(\frac{r_k^{k_q}}{\sqrt{H(X_0,u,r_k)}}\right)^{2-q}\frac{1}{R^{n+a-1}}\int_{\partial^0 B_R^+} F_{\lambda_+,\lambda_-}(u_k)\mathrm{d}x\\
&\leq CN(X_0,u,Rr_k)\frac{1}{R^{n+a}}\int_{\partial^+ B_R^+} y^a u_k^2\mathrm{d}\sigma + R^{1-a}\frac{r_k^{2s}}{\rho_k^{2-q}}\|u_k\|_R^q\\
&\leq C(R)N(X_0,u,Rr_k) + C(R)R^{1-a}\frac{r_k^{2s}}{\rho_k^{2-q}}\rho_k^{2-q}(Rr_k)^{\alpha-2s}\\
&\leq C(R)(1+R^\alpha),
\end{aligned}$$

which finally implies the uniform bound.

Thus, up to a subsequence, we have proved the existence of a non trivial function $p \in H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ such that $\|p\|_{L^{2,a}(\partial B_1^+)} = 1$ and $u_k \rightharpoonup p$ in $H^{1,a}(B_R^+)$ for every $R > 0$. Moreover, since u_k is uniformly bounded in $H^{1,a}(B_R^+)$, we get that, up to a subsequence, $u_k \to p$ strongly in $L^2(B_R^+)$ and in $L^p(\partial^0 B_R^+)$, for every $p \in [1, 2^*)$.

On the other hand, we omit the details of the strong convergence in $H^{1,a}(B_R^+)$ since it simply follows by testing (53) with $(u_k - p)\eta$, where $\eta \in C^\infty_c(B_R)$ is an arbitrary cut-off function, and passing then to the limit. □

So far we have proved the existence of a nontrivial solution $p \in H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}) \cap L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ of (54) such that, up to a subsequence, we have $u_k \to p$ strongly in $H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$. With the following result we complete the compactness result by showing the uniform convergence in $C^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ for $\alpha \in (0,1)$.

Lemma 5.7 *For every $R > 0$ there exists $C > 0$, independent of k, such that*

$$[u_k]_{C^{0,\alpha}(\overline{B_R^+})} = \sup_{X_1,X_2\in\overline{B_R^+}} \frac{|u(X_1)-u(X_2)|}{|X_1-X_2|^\alpha} \leq C$$

for every $\alpha \in (0,1)$.

Proof The proof follows essentially the ideas of the similar results in [28, 34, 35]: the critical exponent $\alpha = 1$ is related to a Liouville type theorem for L_a-harmonic function in $\mathbb{R}^{n+1}$ symmetric with respect to the characteristic manifold $\{y = 0\}$, as given in [29]. □

As a first Corollary we deduce that the possible vanishing orders of u in the case $\mathcal{O}(u, X_0) < k_q$ are completely classified as the possible vanishing orders of L_a-harmonic function even with respect to $\{y = 0\}$ (see [28] for a complete analysis of the nodal set of L_a-harmonic functions).

Corollary 5.8 *Let $X_0 \in \Gamma(u)$ be such that $k = \mathcal{O}(u, X_0) < k_q$. Then $k \in 1 + \mathbb{N}$ and every blow-up limit centered at X_0 is a k-homogeneous solution of (54).*

Proof Let $k = \mathcal{O}(u, X_0) < k_q$. By Theorem 5.5 we already know that given $(u_j)_j$ a normalized blow-up sequence centered in X_0 and associated to some $r_j \to 0^+$, it converges strongly in $H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ and uniformly on every compact set of $\overline{\mathbb{R}^{n+1}_+}$ to some $p \in H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ such that

$$\begin{cases} -L_a p = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\partial^a_y p = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

On the other hand, by Corollary 5.2 and Corollary 5.3 we get $N(X_0, u, 0^+) = k$. Moreover, by the strong convergence of $(u_j)_j$, we have

$$N(0, p, r) = \lim_{j\to\infty} N(0, u_j, r) = \lim_{j\to\infty} N(X_0, u, rr_j) = N(X_0, u, 0^+) \quad \text{for every } r > 0,$$

where

$$N(0, p, r) = \frac{r \displaystyle\int_{B_r^+} y^a |\nabla p|^2 \, \mathrm{d}X}{\displaystyle\int_{\partial^+ B_r^+} y^a p^2 \mathrm{d}\sigma}.$$

Since p is a global L_a-harmonic function even with respect to $\{y = 0\}$, by [28,Lemma 4.7] we deduce that p is k-homogeneous in $\mathbb{R}^{n+1}_+$ with $k = 1 + \mathbb{N}$. □

Inspired by the notations introduced by [28], we denote with $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ the class of L_a-harmonic polynomial symmetric with respect to $\{y = 0\}$ and homogeneous of order k.

In order to conclude the local analysis near the points of the nodal set such that $\mathcal{O}(u, X_0) < k_q$ we need to better understand the Taylor expansion of the function u near nodal points. Inspired by quite standard techniques (see [17, 18] for similar results in the context of obstacle type problems with weights) we start by introducing the following Weiss-type monotonicity formula.

Proposition 5.9 *Let $X_0 \in \Gamma(u)$ be such that $k = \mathcal{O}(u, X_0) < k_q$. Given $\delta = 2s - (2-q)k > 0$, there exist $R_1 > 0$ and $C_2 > 0$ such that*

$$r \mapsto W_k(X_0, u, r) + C_2(n, s, q, \Lambda, k) r^{\delta-\varepsilon}$$

is monotone non-decreasing, for every $r \in (0, \min\{R_1, \mathrm{dist}(X_0, \partial^+ B_1^+)\})$ and $\varepsilon < \delta$. In particular, we get

$$W_k(X_0, u, 0^+) = \lim_{r\to 0^+} W_k(X_0, u, r) = 0. \tag{55}$$

Proof For $k > 0$, by Proposition 3.4 and Lemma 2.2, we get

$$\frac{d}{dr} W_k(X_0, u, r) \geq -\frac{C^s_{n,2}\Lambda}{q r^{n+a+2k}} \int_{\partial^0 B_r^+} |u|^q \, \mathrm{d}x \geq -C r^{2s-1} \frac{\|u\|^q_{X_0,r}}{r^{2k}}$$

where $C = C(n, q, s, \Lambda)$. By definition of $H^{1,a}$-vanishing order, for every $k_1 < \mathcal{O}(u, X_0)$ there exists $R_1, C_1 > 0$ such that

$$\|u\|^2_{X_0,r} \leq C r^{2k} \longrightarrow \frac{d}{dr} W_k(X_0, u, r) \geq -C_1 r^{2s-1-2k+qk_1},$$

for every $r < R_1$. Since $k < k_q$, there exist $\delta > 0$ such that

$$k = \frac{2s - \delta}{2 - q}.$$

Thus, for every $\varepsilon < \delta$, if we take $k_1 = k - \varepsilon/q$ we get that $r \mapsto W_k(X_0, u, r) + C_2 r^{\delta-\varepsilon}$ is monotone non-decreasing, where C_2 does not depend on $\varepsilon > 0$.

Finally, since by Corollary 5.2 and Corollary 5.3 we have $k = \mathcal{O}(u, X_0) = N(X_0, u, 0^+)$, we get

$$W_k(X_0, u, 0^+) = \lim_{r \to 0^+} \frac{H(X_0, u, r)}{r^{2k}} (N(X_0, u, r) - k) = 0.$$

□

The next monotonicity formulas is a Monneau-type formula, which will allow to prove uniqueness of the blow-up near nodal points satisfying $\mathcal{O}(u, X_0) < k_q$.

Proposition 5.10 *Let $X_0 \in \Gamma(u)$ such that $k = \mathcal{O}(u, X_0) < k_q$. Given $\delta = 2s-(2-q)k > 0$, there exist $R_1 > 0$ and $C_2 > 0$ such that, for every homogenous L_a-harmonic polynomial $p \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$, the map*

$$r \mapsto \frac{H(X_0, u - p_{X_0}, r)}{r^{2k}} = \frac{1}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a \left(u - p_{X_0}\right)^2 \mathrm{d}\sigma$$

satisfies

$$\frac{d}{dr} \frac{H(X_0, u - p_{X_0}, r)}{r^{2k}} \geq -C(1 + \|p_{X_0}\|_{L^\infty(B_1^+)}) r^{-1+\delta-\varepsilon},$$

for every $r \in (0, \min\{R_1, \mathrm{dist}(X_0, \partial^+ B_1^+)\}$ and $\varepsilon < \delta$, with $p_{X_0}(X) = p(X - X_0)$.

Proof The strategy is inspired by known result for the thin-obstacle problem (see [17]) and for the study of the nodal set of L_a-harmonic functions (see [28]). First, since $k = \mathcal{O}(u, X_0)$ we already know $W_k(X_0, u, 0^+) = 0$. Now, let $w = u - p_{X_0}$, then on one hand we have

$$\begin{aligned}\frac{d}{dr}\left(\frac{1}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a w^2 \mathrm{d}\sigma\right) &= \frac{2}{r^{n+a+1+2k}} \int_{\partial^+ B_r^+(X_0)} y^a w(\langle X - X_0, \nabla w\rangle - kw)\mathrm{d}\sigma \\ &= \frac{2}{r} W_k(X_0, w, r).\end{aligned}$$

On the other hand, looking at the expression of the k-Weiss functional, we have

$$\begin{aligned}W_k(X_0, u, r) &= W_k(X_0, w + p_{X_0}, r) \\ &= \frac{1}{r^{n+a-1+2k}} \left(\int_{B_r^+(X_0)} y^a (|\nabla w|^2 + 2\langle \nabla w, \nabla p\rangle)\mathrm{d}X \right. \\ &\quad \left. - \frac{k}{r} \int_{\partial^+ B_r^+(X_0)} y^a (w^2 + 2wp)\mathrm{d}\sigma\right) \\ &\quad + \frac{1}{r^{n+a-1+2k}} \int_{\partial^+ B_r^+(X_0)} (w + p_{X_0}) \partial^a_y (w + p_{X_0})\mathrm{d}x \\ &= W_k(X_0, w, r) + \frac{1}{r^{n+a-1+2k}} \int_{\partial^0 B_r^+(X_0)} p_{X_0} \partial^a_y w \mathrm{d}x + \\ &\quad + \frac{2}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a w(\langle \nabla p_{X_0}, X - X_0\rangle - kp)\mathrm{d}\sigma \\ &= W_k(X_0, u - p_{X_0}, r) + \frac{1}{r^{n+a-1+2k}} \int_{\partial^0 B_r^+(X_0)} p_{X_0} \partial^a_y u \mathrm{d}x,\end{aligned}$$

where $C = C(\lambda_+, \lambda_-)$ and in the second equality we used the k-homogeneity of $p_{X_0} \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$. Hence we finally infer

$$\begin{aligned}\frac{d}{dr}\frac{H(X_0, u - p_{X_0}, r)}{r^{2k}} &= \frac{2}{r} W_k(X_0, u - p_{X_0}, r)\\ &\geq \frac{2}{r} W_k(X_0, u, r) + \frac{2C}{r^{n+a+2k}} \int_{\partial^0 B_r^+(X_0)} p_{X_0} |u|^{q-2} u \mathrm{d}x.\end{aligned}$$

On one side by Proposition 5.9 we have

$$W_k(X_0, u, r) = W_k(X_0, u, r) - W_k(X_0, u, 0^+) \geq -C_2(n, s, q, \Lambda, k) r^{\delta-\varepsilon},$$

with $\delta = 2s - (2-q)k > 0$ and $\varepsilon < \delta$. On the other, under the notations of Proposition 5.9, for every $\varepsilon \in (0, \delta)$ let us introduce

$$k_1 = k - \frac{\varepsilon}{q-1} < k = \frac{2s - \delta}{2 - q}.$$

Then, by (52) we infer the existence of $R > 0$ sufficiently small such that

$$\begin{aligned}&\left| \frac{C}{r^{n+a+2k}} \int_{\partial^0 B_r^+(X_0)} p_{X_0} |u|^{q-2} u \mathrm{d}x \right|\\ &\leq \left\| p_{X_0} \right\|_{L^\infty(B_r^+)} \frac{C}{r^{n+a+2k}} \left(\int_{\partial^0 B_r^+(X_0)} |u|^q \,\mathrm{d}x \right)^{(q-1)/q} |B_r|^{1/q}\\ &\leq \left\| p_{X_0} \right\|_{L^\infty(B^+)} \frac{C}{r^{a+k}} \|u\|^{q-1}_{H^{1,a}(B_r^+)}\\ &\leq C \left\| p_{X_0} \right\|_{L^\infty(B^+)} r^{2s-1-k+(q-1)k_1}\\ &\leq C \left\| p_{X_0} \right\|_{L^\infty(B^+)} r^{-1+\delta-\varepsilon},\end{aligned} \tag{56}$$

for $r \in (0, R)$. Hence, there exist $R_1 > 0$ and $C = C(n, s, q, \Lambda, k)$ such that

$$r \mapsto \frac{H(X_0, u - p_{X_0}, r)}{r^{2k}} + C(1 + \left\| p_{X_0} \right\|_{L^\infty(B^+)}) r^{\delta-\varepsilon},$$

is monotone nondecreasing $r \in (0, \min\{R_1, \mathrm{dist}(X_0, \partial^+ B_1^+)\})$ and $\varepsilon < \delta$. □

For the sake of simplicity, we will use through the paper the following notation for the previous monotonicity formula

$$M(X_0, u, p_{X_0}, r) = \frac{H(X_0, u - p_{X_0}, r)}{r^{2k}}.$$

Starting from these results, we will improve our knowledge of the blow-up convergence by proving the existence of a unique non trivial blow-up limit at every point of the nodal set $\Gamma(u)$, which will be called the tangent map φ^{X_0} of u at X_0.

Lemma 5.11 *Let $X_0 \in \Gamma(u)$ be such that $k = \mathcal{O}(u, X_0) < k_q$. Then, there exists $r_0 > 0$ and $C > 0$ such that*

$$H(X_0, u, r) \leq C r^{2k} \quad \textit{for } r \in (0, r_0).$$

Proof Let $k = \mathcal{O}(u, X_0)$ and $\delta = k_q - k$. By (50) and Proposition 5.1, there exist $r_0 > 0$, $\alpha = \alpha(\delta, n, s, q)$ and $\tilde{C} = \tilde{C}(\delta, n, s, q)$ such that

$$\begin{aligned}\frac{d}{d\rho}\log\left(\frac{H(X_0, u, \rho)}{\rho^{2k}}\right) &= \frac{2}{\rho}\left(N(X_0, u, \rho) - k\right)\\ &= \frac{2}{\rho}\left(e^{-\tilde{C}\rho^\alpha} e^{\tilde{C}\rho^\alpha}(N(X_0, u, \rho) + 1) - 1 - k\right)\\ &\geq 2(k+1)\frac{e^{-\tilde{C}\rho^\alpha} - 1}{\rho},\end{aligned}$$

for every $\rho \in (0, r_0)$. Thus, given $r < r_0$ and integrating between r and r_0 we get

$$\frac{H(X_0, u, r)}{r^{2k}} \leq \frac{H(X_0, u, r_0)}{r_0^{2k}} \exp\left(2(k+1)\int_0^{r_0}\frac{e^{-\tilde{C}\rho^\alpha} - 1}{\rho}\mathrm{d}\rho\right) \leq C,$$

as we claimed. □

Lemma 5.12 *Let $X_0 \in \Gamma(u)$ be such that $k = \mathcal{O}(u, X_0) < k_q$. Then, there exists $C > 0$ such that*

$$\sup_{\partial B_r(X_0)} |u(X)| \geq C r^k \quad \text{for } 0 < r < R$$

where $R = 1 - \mathrm{dist}(X_0, \partial^0 B_1)$.

Proof Since the proof is an adaptation of a similar result for the thin-obstacle problem (see [17,Lemma 2.8.1]), we briefly sketch the main ideas.

Fix $X_0 \in \Gamma(u)$ and suppose by contradiction that given a decreasing sequence $r_j \downarrow 0$ we have

$$\lim_{j\to\infty}\frac{H(X_0, u, r_j)^{1/2}}{r_j^k} = \lim_{j\to\infty}\left(\frac{1}{r_j^{n+a+2k}}\int_{\partial^+ B^+_{r_j}(X_0)} y^a u^2 \,\mathrm{d}\sigma\right)^{1/2} = 0.$$

Thus, for $r_j \leq R = \min(r_0, \mathrm{dist}(X_0, \partial^0 B^+))$, consider the blow-up sequence

$$u_j(X) = \frac{u(X_0 + r_j X)}{\rho_j} \quad \text{where } \rho_j = H(X_0, u, r_j)^{1/2},$$

centered in $X_0 \in \Gamma(u)$. By Theorem 5.5 and Corollary 5.8 the sequence $(u_j)_j$ converges, up to a subsequence, strongly in $H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ and uniformly on every compact set of $\mathbb{R}^{n+1}_+$ to some L_a-harmonic homogenous polynomial p of degree k symmetric with respect to $\{y = 0\}$ such that $H(0, p, 1) = 1$.

Therefore, under the notations in Proposition 5.10, by taking p_{X_0} as above we get

$$\begin{aligned}
&M(X_0, u, p_{X_0}, 0^+)\\
&= \lim_{r\to 0} \frac{1}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a (u - p_{X_0})^2 \, d\sigma + C(1 + \|p_{X_0}\|_{L^\infty(B^+)}) r^{\delta-\varepsilon}\\
&= \lim_{r\to 0} \int_{\partial^+ B_1^+} y^a \Big(\frac{u(X_0 + rX)}{r^k} - p(X)\Big)^2 \, d\sigma + C(1 + \|p_{X_0}\|_{L^\infty(B^+)}) r^{\delta-\varepsilon}\\
&= \int_{\partial^+ B_1^+} y^a p^2 \, d\sigma\\
&= \frac{1}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a p_{X_0}^2 \, d\sigma,
\end{aligned}$$

where in the third equality we used the assumption on the growth of u. By the monotonicity result of Proposition 5.10, we obtain

$$\begin{aligned}
&\frac{1}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a (u - p_{X_0})^2 \, d\sigma + C(1 + \|p_{X_0}\|_{L^\infty(B^+)}) r^{\delta-\varepsilon}\\
&\quad \geq \frac{1}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a p_{X_0}^2 \, d\sigma
\end{aligned}$$

and similarly

$$\frac{1}{r^{n+a+2k}} \int_{\partial^+ B_r^+(X_0)} y^a (u^2 - 2u p_{X_0}) \, d\sigma + C(1 + \|p_{X_0}\|_{L^\infty(B^+)}) r^{\delta-\varepsilon} \geq 0.$$

On the other hand, rescaling the previous inequality and using the notion of blow-up sequence u_k defined as above, we obtain

$$\int_{\partial^+ B_1^+} y^a \left(\frac{H(X_0, u, r_j)^{1/2}}{r_j^k} u_j^2 - 2u_j p \right) d\sigma \geq -C(1 + \|p_{X_0}\|_{L^\infty(B^+)}) \frac{r_j^{k+\delta-\varepsilon}}{H(X_0, u, r_j)^{1/2}}.$$

Since $\mathcal{V}(u, X_0) = \mathcal{O}(u, X_0) = k$, by Definition 1.3 we get

$$\limsup_{j\to\infty} \frac{H(X_0, u, r_j)}{r_j^{2(k+\delta-\varepsilon)}} = +\infty,$$

and consequently, passing to the limit as $j \to \infty$ in the previous inequality, we obtain

$$\int_{\partial^+ B_1^+} y^a p^2 \, d\sigma \leq 0$$

in contradiction with $p \not\equiv 0$. □

Theorem 5.13 *Let $X_0 \in \Gamma(u)$ be such that $k = \mathcal{O}(u, X_0) < k_q$. Then there exists a unique nonzero $\varphi^{X_0} \in \mathfrak{sB}_k^a(\mathbb{R}^{n+1})$ blow-up limit such that*

$$u_{X_0,r}(X) = \frac{u(X_0 + rX)}{r^k} \longrightarrow p(X). \tag{57}$$

Moreover, we define as tangent map of u at X_0 the unique $\varphi^{X_0} \in \mathfrak{sB}_k^a(\mathbb{R}^{n+1})$ that satisfies (57).

Proof The proof is a straightforward consequence of the Weiss and Monneau monotonicity formulas. Indeed, up to a subsequence $r_j \to 0^+$, we have that $u_{X_0,r_j} \to p$ in $C^{0,\alpha}_{\mathrm{loc}}$ for every $\alpha \in (0,1)$. The existence of such limit follows directly from the growth estimate of Lemma 5.11 and, by Lemma 5.12, we have p is not identically zero. Now, by Proposition 5.9, for any $r > 0$ we have

$$W_k(0, p, r) = \lim_{j\to\infty} W_k(0, u_{X_0,r_j}, r) = \lim_{j\to\infty} W_k(X_0, u, rr_j) = W_k(X_0, u, 0^+) = 0,$$

where

$$W_k(0, p, r) = \frac{1}{r^{2k}}\left[\frac{1}{r^{n+a-1}}\int_{B_r^+} y^a\,|\nabla p|^2\,\mathrm{d}X - k\frac{1}{r^{n+a-1}}\int_{\partial^+ B_r^+} y^a p^2 \mathrm{d}\sigma\right].$$

In particular, by [28,Proposition 5.2] it implies that the p is k-homogeneous L_a-harmonic function even with respect to $\{y = 0\}$, that is $p \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$. Now, by Proposition 5.10, the limit of the Monneau-type formula exists and can be computed by

$$\begin{aligned} M(X_0, u, p_{X_0}, 0^+) &= \lim_{j\to\infty} M(X_0, u, p_{X_0}, r_j) \\ &= \lim_{j\to\infty} M(0, u_{X_0,r_j}, p, 1) \\ &= \lim_{j\to\infty} \int_{\partial^+ B_1^+} y^a (u_{X_0,r_j} - p)^2\, d\sigma = 0. \end{aligned}$$

Now, suppose by contradiction that for any other subsequence $r_i \to 0^+$ we have that $(u_{X_0,r_i})_i$ converges to another blow-up limit $q \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ with $q \not\equiv p$, then

$$\begin{aligned} &M(X_0, u, p_{X_0}, 0^+) \\ &\quad= \lim_{i\to\infty} M(X_0, u, p_{X_0}, r_i) = \lim_{i\to\infty}\int_{\partial^+ B_1^+} y^a (u_{r_i} - p)^2\, d\sigma = \int_{\partial^+ B_1^+} y^a (q-p)^2\, d\sigma, \end{aligned}$$

which contradicts $M(X_0, u, p_{X_0}, 0^+) = 0$. □

Thanks to the uniqueness and the non-degeneracy of the blow-up limit, we can also construct the generalized Taylor expansion of the solution on the nodal set.

Theorem 5.14 *Let $X_0 \in \Gamma(u)$ be such that $k = \mathcal{O}(u, X_0) < k_q$ and φ^{X_0} be the tangent map of u at X_0. Then*

$$u(X) = \varphi^{X_0}(X - X_0) + o(|X - X_0|^k). \tag{58}$$

Moreover, the map $X_0 \mapsto \varphi^{X_0}$ from $\Gamma_k(u)$ to $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ is continuous.

Proof The proof is inspired by a similar result for the thin-obstacle problem (see [17,Lemma 2.8.1] for more details) based on the validity of the Monneau monotonicity formula. Indeed, fixed $X_0 \in \Gamma(u)$ let φ^{X_0} be the unique tangent map of u at X_0 defined by Theorem 5.13. Therefore, given $\varepsilon > 0$, Proposition 5.10 implies the existence of $r_\varepsilon = r_\varepsilon(X_0)$, $\delta_\varepsilon = \delta_\varepsilon(X_0)$ such that

$$M(X_1, u, \varphi^{X_0}, r) < 2\varepsilon + C r_\varepsilon^{\delta/2} \quad \text{for } r \in (0, r_\varepsilon)$$

and for $X_1 \in \Gamma_k(u) \cap \Sigma \cap B_{\delta_\varepsilon}(X_0)$. Thus, by rescaling the monotonicity formula in X_1, we can conclude

$$\begin{aligned} M(X_1, u, \varphi^{X_0}, 0^+) &= \lim_{r\to 0} M(X_1, u, \varphi^{X_0}, r) \\ &= \lim_{r\to 0} \int_{\partial^+ B_1^+} y^a \left(\frac{u(X_1 + rX)}{r^k} - \varphi^{X_0}(X) \right)^2 d\sigma \\ &= \int_{\partial^+ B_1^+} y^a \left(\varphi^{X_1} - \varphi^{X_0} \right)^2 d\sigma \le 2\varepsilon + C r_\varepsilon^{\delta/2}. \end{aligned}$$

□

Finally, we improve the convergence rate $o(|X - X_0|^k)$ of the previous generalized Taylor's expansion into a quantitative bound of the form $O(|X - X_0|^{k+\delta})$ for some $\delta > 0$.

This result is obtained by proving the validity of an Almgren-type monotonicity result for the difference between the solution u and its tangent map φ^{X_0} at X_0. Since $\mathcal{O}(u, x_0) < k_q$, this analysis resembles the starting point of the iteration argument already used in the case of L_a-harmonic function in [28] to obtain higher order Taylor expansion near nodal points. Notice that this methodology has been used in the last years to study the stratification of obstacle type problems (see [14] for a finer analysis of higher order iterations in the context of thin-obstacle problems).

Theorem 5.15 *Let $X_0 \in \Gamma(u)$ be such that $\mathcal{O}(u, X_0) < k_q$ and set*

$$w(X) = u(X) - \varphi^{X_0}(X - X_0),$$

where φ^{X_0} is the tangent map of u at X_0. Then, there exist r_0 and an absolutely continuous map $\Psi(r)$ satisfying

$$0 \le \Psi(r) \le C r^\alpha, \quad \text{for } r \in (0, r_0)$$

and some $\alpha > 0$, such that

$$r \mapsto e^{\tilde{C}\Psi(r)}(N(X_0, w, r) + 1)$$

is monotone non-decreasing for $r \in (0, r_0)$. Consequently, there exists the limit

$$N(X_0, w, r) = \lim_{r\to 0^+} N(X_0, w, r).$$

Proof For the sake of simplicity, it is not restrictive to assume that $X_0 = 0$. Set $k = \mathcal{O}(u, 0) < k_q$, then by Lemma 5.11, Lemma 5.12 and Theorem 5.13 there exists $C_1, C_2 > 0$ such that

$$C_1 r^k \le \|u\|_{0,r} \le C_2 r^k \quad \text{for } r \in (0, \overline{r}). \tag{59}$$

Now, given $\varphi \in \mathfrak{sB}_k^a(\mathbb{R}^{n+1})$ the unique tangent map of u at $0 \in \Gamma(u)$, let us consider the difference $w = u - \varphi \in H^{1,a}(B_r^+)$ which solves

$$\begin{cases} L_a w = 0 & \text{in } B_r^+ \\ -\partial_y^a w = \lambda_+(w+\varphi)_+^{q-1} - \lambda_-(w+\varphi)_-^{q-1} & \text{on } \partial^0 B_r^+. \end{cases} \tag{60}$$

Thus, following the same computation of the last Section, we easily deduce by an integration by parts (see the proof of Proposition 3.1) that

$$\frac{d}{dr}E(w, r) = \frac{2}{r^{n-1+a}} \int_{\partial^+ B_r^+} y^a (\partial_r w)^2 \mathrm{d}\sigma + R(w, r), \quad \frac{d}{dr}H(w, r) = \frac{2}{r}E(w, r),$$

where E, H are defined by (16) and

$$R(w,r)=\frac{1-n-a}{r^{n+a}}\int_{\partial^0 B_r^+} w\partial_y^a w \mathrm{d}x+\frac{1}{r^{n-1+a}}\int_{S_r^{n-1}} w\partial_y^a w \mathrm{d}\sigma$$
$$-\frac{2}{r^{n+a}}\int_{\partial^0 B_r^+}\partial_y^a w\langle\nabla w, x\rangle \mathrm{d}x.$$

Therefore, by the Cauchy-Schwarz inequality on $\partial^+ B_r^+$, the associated Almgren-type formula (17) satisfies

$$\frac{d}{dr}\log(N(w,r)+1)\geq\frac{R(w,r)}{E(w,r)+H(w,r)}.$$

On one hand, we have

$$R(w,r)=\frac{2-q}{q}\frac{1}{r^{n+a-1}}\int_{S_r^{n-1}} F_{\lambda_+,\lambda_-}(w+\varphi)\mathrm{d}\sigma-\frac{C_{n,q}^s}{qr^{n+a}}\int_{\partial^0 B_r^+} F_{\lambda_+,\lambda_-}(w+\varphi)\mathrm{d}x+$$
$$+\frac{2s-n-2}{r^{n+a}}\int_{\partial^0 B_r^+}\varphi f(w+\varphi)\mathrm{d}x+\frac{1}{r^{n+a-1}}\int_{S_r^{n-1}}\varphi f(w+\varphi)\mathrm{d}\sigma$$

where $f(t)=\lambda_+ t_+^{q-1}-\lambda_- t_-^{q-1}$. On the other hand, by Lemma 2.2 and (56) we get

$$E(w,r)+H(w,r)\geq\|u\|_{0,r}^2-Cr^{2s}\|u\|_{0,r}^q+\frac{1}{r^{n-1+a}}\int_{\partial^0 B_r^+}\varphi\partial_y^a u\mathrm{d}x$$
$$\geq\|u\|_{0,r}^2-Cr^{2s}\left(\|u\|_{0,r}^q+\|\varphi\|_{L^\infty(B_1)}r^k\|u\|_{0,r}^{q-1}\right)$$

In order to estimate the last remainder $R(w,r)$ we need the introduce the auxiliary function

$$\psi(r)=r\left(\frac{1}{r^n}\int_{\partial^0 B_r^+}|u|^q\,\mathrm{d}x\right)^h$$

for $h\in(0,1)$ to be chosen later. A direct computation yields the identity

$$\psi'(r)=\frac{\psi(r)}{r}\left(hn+1+hr\frac{\int_{S_r^{n-1}}|u|^q\,\mathrm{d}\sigma}{\int_{\partial^0 B_r^+}|u|^q\,\mathrm{d}x}\right)$$

which implies, by Lemma 2.2, that

$$\frac{1}{r^{n-1}}\int_{S_r^{n-1}}|u|^q\,\mathrm{d}\sigma\leq\frac{\psi'(r)}{h}\|u\|_{0,r}^{q(1-h)}.$$

Finally, we get

$$\left|\frac{1}{r^{n+a-1}}\int_{S_r^{n-1}}\varphi f(w+\varphi)\mathrm{d}\sigma\right|\leq\frac{r^{2s-1}}{h}\|\varphi\|_{L^\infty(B_1)}r^k\psi'(r)\|u\|_{0,r}^{(q-1)(1-h)}$$

and consequently

$$R(w,r)\geq -Cr^{2s-1}\left(\|u\|_{0,r}^q+\|\varphi\|_{L^\infty(B_1)}r^k\|u\|_{0,r}^{q-1}\right.$$
$$\left.+\|\varphi\|_{L^\infty(B_1)}r^k\psi'(r)\|u\|_{0,r}^{(q-1)(1-h)}\right)\tag{61}$$

for some $h \in (0, 1)$. By (59) and (46), there exists $\alpha > 0$ such that

$$\begin{aligned} \frac{d}{dr}\log(N(w,r)+1) &\geq -\frac{C}{r}\frac{r^{2s+kq}\left(1+\psi'(r)r^{kh(1-q)}\right)}{r^{2s-\alpha}\,\|u\|_{0,r}^{q}} \\ &\geq -\frac{C}{r}r^{\alpha}\left(1+\psi'(r)r^{kh(1-q)}\right). \end{aligned} \tag{62}$$

Hence, let

$$\Psi(r) = \int_0^r r^{\alpha-1}(1+\psi'(t)t^{kh(1-q)})\mathrm{d}t.$$

Then, by Lemma 2.2 we first deduce $0 \leq \psi(r) \leq Cr^{1+kqh}$ and then

$$\begin{aligned} 0 \leq \Psi(r) &= \int_0^r t^{\alpha-1}\left(1+\psi'(t)t^{kh(1-q)}\right)\mathrm{d}t \\ &= \frac{r^{\alpha}}{\alpha} + \left[\psi(r)r^{kh(1-q)+\alpha-1}\right]_0^r - \int_0^r \psi(t)\frac{t^{kh(1-q)+\alpha-2}}{kh(1-q)+\alpha-1}\mathrm{d}t \\ &\leq \frac{r^{\alpha}}{\alpha} + r^{\alpha+kh} + C\int_0^r t^{kh+\alpha-1}\mathrm{d}t \\ &\leq Cr^{\alpha}, \end{aligned}$$

for r sufficiently small. Therefore, we deduce that the function

$$r \mapsto e^{C\Psi(r)}(N(w,r)+1) \tag{63}$$

is absolutely continuous and increasing for $r \in (r_1, r_2)$, for some $0 < r_1 < r_2$. Following a standard argument, the modified Almgren-type formula (63) can be defined for all $r \in (0, r_2)$, and it can be extended for $r = 0$ by taking its limit for $r \to 0^+$. □

***Remark* 5.16** Notice that, under the notations of Theorem 6.1, all the computations up to (62) still hold in the critical case $\mathcal{O}(u, 0) = k_q$ with $k_q \in \mathbb{N}$ (that is $\mu = 0$ with μ defined by (68)). Indeed, in Sect. 6 we will prove that if $k_q \in \mathbb{N}$ the blow-up limit p is an homogeneous L_a-harmonic function symmetric with respect to $\{y = 0\}$, and the function $w = u - p$ still satisfies (60). However, in that context the computations will lead to

$$\frac{d}{dr}\log(N(w,r_k)+1) \geq -\frac{C}{r_k}\frac{\alpha_k\left(1+\alpha_k^{1/(2-q)}(1+\psi' r^{kh(q-1)})\right)}{1-C\alpha_k\left(1+\alpha_k^{1/(2-q)}\right)} \quad \text{with } \alpha_k = \left(\frac{r_k^{k_q}}{\|u\|_{H^{1,a}(B_{r_k})}}\right)^{2-q}.$$

By the dichotomy (67), even if $\mu = 0$ yields to $\alpha_k \to 0^+$, this is not enough to ensure the integrability of the right hand side of the previous inequality. As remarked in [31], is possible that a sophisticated Fourier expansion finally lead to uniqueness: indeed it will imply that $r_k \mapsto \alpha_k(r_k)$ is Dini-continuous, which will be enough to ensure the validity of an Almgren-type monotonicity result.

As a simple corollary of the monotonicity result in Proposition 5.10 for the Monneau-type formula, we easily deduce a lower bound for the Almgren-type formula evaluated on w.

Corollary 5.17 *Let $X_0 \in \Gamma(u)$ be such that $\mathcal{O}(u, X_0) < k_q$. Then $N(X_0, u - \varphi^{X_0}, 0^+) \geq \mathcal{O}(u, X_0)$.*

In order to improve the growth order of the remainder in (58), we start by proving a blow-up argument based on the validity of the previous Almgren-type monotonicity formula. Hence, given $X_0 \in \Gamma(u)$ and $w \in H^{1,a}(B_r^+)$ as in Theorem 5.15, we consider the normalized blow-up sequence $(w_k)_k$ centered in X_0 associated to some $r_k \downarrow 0^+$ (see (5) for the definition of normalized blow-up sequence), such that

$$\begin{cases} -L_a w_k = 0 & \text{in } B^+_{X_0,r_k} \\ -\partial^a_y w_k = \alpha_k \left[\lambda_+ \left(\beta_k w_k + \varphi^{X_0}\right)_+^{q-1} - \lambda_- \left(\beta_k w_k + \beta_k \varphi^{X_0}\right)_-^{q-1}\right] & \text{on } \partial^0 B^+_{X_0,r_k}. \end{cases}$$

with

$$\alpha_k = \frac{r_k^{2s+\mathcal{O}(u,X_0)(q-1)}}{\sqrt{H(X_0,w,r_k)}} \quad \text{and} \quad \beta_k = \frac{\sqrt{H(X_0,w,r_k)}}{r_k^{\mathcal{O}(u,X_0)}}.$$

Lemma 5.18 *Under the previous notations, let $0 < k_1 \le k_2$ be such that $\mathcal{O}(u, X_0) \le k_1 < k_2$. Then, if $k_1 \le N(X_0, u - \varphi^{X_0}, 0^+) \le k_2$ we infer*

$$\beta_k \to 0^+ \quad \textit{and} \quad 0 \le \alpha_k \le C r^{(2-q)\left(k_q - \mathcal{O}(u,X_0)\right)},$$

for some $C > 0$ and k sufficiently large.

Proof First, by Proposition 5.10 we already know that $\beta_k \to 0^+$. Now, let $k_1, k_2 > 0$ be such that $\mathcal{O}(u, X_0) \le k_1 < k_2$ and $k_1 \le N(X_0, u - \varphi^{X_0}, 0^+) \le k_2$. By (18) and Theorem 5.15 we have that if $k_1 \le N(X_0, u - \varphi^{X_0}, 0^+) \le k_2$ then there exits $C_1, C_2, \overline{r} > 0$ such that

$$C_1 r^{k_1} \le \sqrt{H(X_0, u - \varphi^{X_0}, r)} \le C_2 r^{k_2},$$

for every $r \in (0, \overline{r})$. Thus

$$\alpha_k \le C r_k^{2s-k_1+\mathcal{O}(u,X_0)(q-1)},$$

for k sufficiently large such that $r_k \le \overline{r}$. Finally, by Corollary 5.17, if $k_1 = \mathcal{O}(u, X_0) < k_q$ we get

$$\alpha_k \le C r^{(2-q)\left(k_q - \mathcal{O}(u,X_0)\right)},$$

as we claimed. □

In the following Proposition we finally compute the vanishing order of $u - \varphi^{X_0}$ in terms of $\mathcal{O}(u, X_0)$.

Proposition 5.19 *Let $X_0 \in \Gamma(u)$ be such that $\mathcal{O}(u, X_0) < k_q$. Then*

$$N(X_0, u - \varphi^{X_0}, 0^+) = \mathcal{O}(u, X_0) + \delta, \quad \textit{for some } \delta \in \mathbb{N}, \delta \ge 1.$$

Proof Let $w = u - \varphi^{X_0}$ and $(w_k)_k$ be the normalized blow-up sequence centered at X_0 and associated to some $r_k \to 0^+$. As we did in Lemma 5.6, exploiting the normalization with respect to the $L^{2,a}(\partial^+ B_1^+)$-norm and the validity of the Almgren-type monotonicity formula, it is easy to see that $(w_k)_k$ is uniformly bounded in $H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ and it converges, up to subsequence, to some $p \in H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}) \cap L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ such that $\|p\|_{L^{2,a}(\partial^+ B_1^+)} = 1$.

On the other hand, since $\mathcal{O}(u, X_0) < k_q$, by Lemma 5.18 we get that both $(\alpha_k)_k$ and $(\beta_k)_k$ approach zero, as k goes to infinity. Therefore, following the same contradiction argument of Lemma 5.7, the sequence $(w_k)_k$ is uniformly bounded in $C^{0,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ for every $\alpha \in (0,1)$ and it converges uniformly on every compact set to some $p \in \mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$. Indeed, by the strong convergence of $(w_k)_k$, we get

$$N(0,p,r) = \lim_{k\to\infty} N(0,w_k,r) = \lim_{k\to\infty} N(X_0,w,rr_k) = N(X_0,w,0^+) \quad \text{for every } r>0,$$

where

$$N(0,p,r) = \frac{r\displaystyle\int_{B_r^+} y^a\,|\nabla p|^2\,\mathrm{d}X}{\displaystyle\int_{\partial^+B_r^+} y^a p^2\mathrm{d}\sigma}.$$

Therefore, p is an homogeneous L_a-harmonic function even with respect to $\{y=0\}$ of order $N(0,p,1)$. By [28,Lemma 4.7] we first get that $N(0,p,1)\in\mathbb{N}$ while by Theorem 5.13 we deduce that $N(0,p,1) > \mathcal{O}(u,X_0)$. Since $N(0,p,1) = N(X_0,w,0^+)$ we finally get the claimed result. □

Thanks to this classification, we can improve the growth order of the remainder in (58).

Corollary 5.20 *Let $X_0 \in \Gamma(u)$ be such that $k = \mathcal{O}(u,X_0) < k_q$ and φ^{X_0} be the tangent map of u at X_0. Then*

$$u(X) = \varphi^{X_0}(X - X_0) + O(|X - X_0|^{k+1}),$$

Moreover, the map $X_0 \mapsto \varphi^{X_0}$ from $\Gamma_k(u)$ to $\mathfrak{s}\mathfrak{B}^a_k(\mathbb{R}^{n+1})$ is Hölder continuous.

Having established Theorem 5.14 and Proposition 5.19, we can finally show the validity of the first part of Theorem 1.7 and Theorem 1.9.

Proof of Theorem 1.7 Let us consider the case $\mathcal{V}(u,X_0) < k_q$. By Corollary 5.2 and Corollary 5.3 we already know that

$$\mathcal{O}(u,X_0) = \mathcal{V}(u,X_0) = N(X_0,u,0^+).$$

Therefore the results of this Section hold true also for the case $\mathcal{V}(u,X_0) < k_q$. In particular, by Corollary 5.8, we know that $\mathcal{V}(u,X_0)$ must be a positive integer and, by Theorem 5.14 and Proposition 5.19, it follows the validity of expansion (10). □

Finally, by applying a variant of the classical Federer's dimension reduction principle [7,Theorem 8.5] (for the classical result we refer to [27,Appendix A]), and the Whitney's extension theorem (we refer to [13] and the reference therein) we can easily estimate the Hausdorff dimension of the singular strata.

Proof of Theorem 1.9 First, since $\Gamma(u) = \mathcal{T}(u)$ for those values of $s\in(0,1)$ and $q\in[1,2)$ such that $k_q \le 1$, let us concentrate on the opposite case. Seeing that on $\mathcal{R}(u)\cup\mathcal{S}(u)$ all the notions of vanishing order coincide, that is

$$\mathcal{O}(u,X_0) = \mathcal{V}(u,X_0) = N(X_0,u,0^+) < k_q,$$

we can easily adapt the general approach of [28] by using the validity of the Almgren-type monotonicity formula. More precisely, by a straightforward application of Corollary 5.20 and the implicit function theorem, we already deduce that

$$\mathcal{R}(u) = \left\{X \in \Gamma(u) \colon N(X_0,u,0^+) = 1\right\},$$

which is relatively open in $\Gamma(u)$ and it is a $(n-1)$-dimensional regular set of class $C^{1,\alpha}$. Moreover, by the upper semi-continuity of $X_0 \mapsto N(X_0, u, 0^+)$, the proof of the Hausdorff estimate

$$\dim_{\mathcal{H}} \mathcal{S}(u) \leq n-1$$

follows the one of [28,Theorem 6.3]).

On the other hand, it is possible to apply step by step the proof of [28,Theorem 7.7] and [28,Theorem 7.8] (using Corollary 5.20 instead of [28,Theorem 5.12] and the generalized formulation of the Whitney's extension theorem in [13] for $C^{m,\omega}$-functions), obtaining the desired result for the stratification of the singular set. The crucial idea is that the Whitney's extension allows to study the structure of the nodal set just by using the generalized Taylor expansion (10) without the high-order differentiability of the function itself. □

6 Blow-up analysis for $\mathcal{O}(u, X_0) = k_q$

The previous analysis terminates the study of the nodal set in those points where the local behaviour of the solutions resemble the one of the s-harmonic functions. In this Section we will complete our study by considering the threshold case $\mathcal{O}(u, X_0) = k_q$. The following result is the second part of Theorem 1.7.

Theorem 6.1 *Let $q \in [1,2)$, $\lambda_+, \lambda_- > 0$ and $u \in H^{1,a}(B_1)$, $u \not\equiv 0$ be a solution of (3).*

If $X_0 \in \Gamma(u)$ satisfies $\mathcal{O}(u, X_0) = k_q$, then for every sequence $r_k \to 0^+$ we have, up to a subsequence, that

$$\frac{u(X_0 + r_k X)}{\|u\|_{X_0, r_k}} \to \overline{u} \quad in\ C^{0,\alpha}_{loc}(\overline{\mathbb{R}^{n+1}_+}),$$

for every $\alpha \in (0, \min(1, 2s))$, where $\overline{u}$ is a k_q-homogeneous non-trivial solution to

$$\begin{cases} L_a \overline{u} = 0 & in\ \mathbb{R}^{n+1}_+ \\ -\partial^a_y \overline{u} = \mu \left(\lambda_+ (\overline{u}_+)^{q-1} - \lambda_- (\overline{u}_-)^{q-1}\right) & on\ \mathbb{R}^n \times \{0\}, \end{cases} \tag{64}$$

for some $\mu \geq 0$. Moreover, the case $\mu = 0$ is possible if and only if $k_q \in \mathbb{N}$.

The proof will be presented in a series of lemmata. Given $X_0 \in \Gamma(u)$ such that $\mathcal{O}(u, X_0) = k_q$ and $r_k \to 0^+$, we consider normalized blow-up sequence

$$u_k(X) = \frac{u(X_0 + r_k X)}{\|u\|_{X_0, r_k}} \quad \text{with } X \in B^+_{X_0, r_k} = \frac{B^+_1 - X_0}{r_k}, \tag{65}$$

for $0 < r_k < R < \mathrm{dist}(X_0, \partial B_1)$. Thus $\|u_k\|_{0,1} = 1$ and

$$\begin{cases} -L_a u_k = 0 & \text{in } B^+_{X_0, r_k} \\ -\partial^a_y u_k = \left(\dfrac{r_k^{k_q}}{\|u\|_{X_0, r_k}}\right)^{2-q} \left[\lambda_+ (u_k)^{q-1}_+ - \lambda_- (u_k)^{q-1}_-\right] & \text{on } \partial^0 B^+_{X_0, r_k}. \end{cases}$$

By Proposition 4.7 (in particular by (36)), there exists $C > 0$ such that

$$0 < \alpha_k = \left(\frac{r_k^{k_q}}{\|u\|_{X_0, r_k}}\right)^{2-q} \leq C,$$

for every $r_k < R$. As we pointed out in the previous Sections, the $H^{1,a}$-normalization seems to be more suitable for the critical case $\mathcal{O}(u, X_0) = k_q$ and it overcomes the lack of monotonicity of the Almgren-type formula. The following is a compactness result for the blow-up sequence.

Lemma 6.2 *For every $R > 0$, there exists $k_R > 0$ such that, for every $k > k_R$, the sequence $(u_k)_k$ is uniformly bounded in $H^{1,a}(B_R^+)$ and, up to a subsequence, it converges strongly in $L^{2,a}(B_R^+)$ and $H^{1,a}(B_R^+)$.*

Proof The convergence of the sequence $(u_k)_k$ with respect to the strong topology in $H^{1,a}(B_R^+)$ is a straightforward consequence of the uniform bound in $H^{1,a}(B_R^+)$. Indeed, suppose there exists $k_R > 0$ such that, for every $k > k_R$ the sequence is uniformly bounded in $H^{1,a}(B_R^+)$, then it implies that up to a subsequence $(u_k)_k$ weakly converges in $H^{1,a}(B_R^+)$ and strongly in $L^{2,a}(B_R^+)$. Moreover, by trace embedding, the traces of $(u_k)_k$ on Σ strongly converge in $L^p(\partial^0 B_R^+)$, for every $p \in [1, 2^\star)$.

Finally, by testing the equation against $(u_k - u)\eta$, where $\eta \in C_c^\infty(B_R)$, we easily deduce the validity of the strong convergence by passing to the limit as $k \to +\infty$. More precisely,

$$\int_{B_R^+} y^a \eta \langle \nabla u_k, \nabla (u_k - u)\rangle \mathrm{d}X = -\int_{B_R^+} y^a (u_k - u)\langle \nabla u_k, \nabla \eta\rangle \mathrm{d}X + \\ + \alpha_k \int_{\partial^0 B_R^+} \eta (u_k - u)(\lambda_+ (u_k)_+^{q-1} - \lambda_- (u_k)_-^{q-1})\mathrm{d}x.$$

Since $(u_k)_k$ is uniformly bounded in $H^{1,a}(B_R)$ and it converges strongly in $L^{2,a}(B_R^+)$, the first term in the right hand side tends to 0 as $k \to \infty$. Similarly, since α_k is bounded and $u_k \to u$ strongly in $L^p(\partial^0 B_R^+)$ for $p \in [1, 2^\star)$, the second term vanishes too. Finally, regarding the left hand side, by the weak convergence we get

$$\int_{B_R^+} y^a \eta \langle \nabla u_k, \nabla (u_k - u)\rangle \mathrm{d}X = \int_{B_R^+} y^a \eta \left(|\nabla u_k|^2 - |\nabla u|^2\right) \mathrm{d}X + o(1)$$

as k goes to $+\infty$, which leads to the claimed result.

Hence, it remains to prove the validity of a uniform bounds in $H^{1,a}$. By definition of $(u_k)_k$, since

$$\|u_k\|_{0,R} = \frac{\|u\|_{X_0, r_k R}}{\|u\|_{X_0, r_k}}$$

the first part of the result follows if there exists $k_R, C_R > 0$ such that

$$\|u\|_{X_0, r_k R} \le C_R \|u\|_{X_0, r_k}, \quad \text{for every } k \ge k_R.$$

Thus, suppose by contradiction that, up to a subsequence, for $r_k \searrow 0$ it results

$$\frac{\|u\|_{X_0, r_k R}}{\|u\|_{X_0, r_k}} \to +\infty.$$

We claim, in such case, that

$$\frac{\|u\|_{X_0, r_k R}}{(r_k R)^{k_q}} \to +\infty \tag{66}$$

as $k \to \infty$. If not, by (36), we would have that

$$\|u\|_{X_0, r_k R} \le C (r_k R)^{k_q} \le C R^{k_q} \|u\|_{X_0, r_k},$$

against the absurd hypothesis. Thus, by Lemma 2.2, we get for every $r > 0$ that

$$\frac{1}{r^{n+a-1}}\int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x \le C\Lambda r^{2s}\,\|u\|^q_{X_0,r} = C\Lambda\left(\frac{r^{k_q}}{\|u\|_{X_0,r}}\right)^{2-q}\|u\|^2_{X_0,r},$$

where $\Lambda = \max\{\lambda_+,\lambda_-\}$, which implies

$$0 \le \frac{1}{(r_k R)^{n+a-1}\,\|u\|^2_{X_0,r_kR}}\int_{\partial^0 B^+_{r_kR}(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x \le C\Lambda\left(\frac{(r_kR)^{k_q}}{\|u\|_{X_0,r_kR}}\right)^{2-q} \longrightarrow 0,$$

as $k \to \infty$. On the other hand, by combining

$$\begin{aligned} W_{k,t}(X_0,u,r) = \frac{\|u\|^2_{X_0,r}}{r^{2k}}\Bigg[&1-(k+1)\frac{H(X_0,u,r)}{\|u\|^2_{X_0,r}} \\ &-\frac{t}{q}\frac{1}{r^{n+a-1}\,\|u\|^2_{X_0,r}}\int_{\partial^0 B_r^+(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x\Bigg], \end{aligned}$$

with the monotonicity of $r \mapsto W_{k_q,2}(X_0,u,r)$, we deduce that

$$\begin{aligned} C &\ge W_{k_q,2}(X_0,u,r_kR) \\ &\ge \frac{\|u\|^2_{X_0,(r_kR)}}{(r_kR)^{2k_q}}\Bigg[1-(k_q+1)\frac{H(X_0,u,r_kR)}{\|u\|^2_{X_0,r_kR}} \\ &-\frac{2}{q}\frac{1}{(r_kR)^{n+a-1}\,\|u\|^2_{X_0,r_kR}}\int_{\partial^0 B^+_{r_kR}(X_0)} F_{\lambda_+,\lambda_-}(u)\mathrm{d}x\Bigg] \\ &\ge \frac{\|u\|^2_{X_0,(r_kR)}}{(r_kR)^{2k_q}}\left[\frac{3}{4}-(k_q+1)\frac{H(X_0,u,r_kR)}{\|u\|^2_{X_0,r_kR}}\right] \end{aligned}$$

for k sufficiently large. Together with (66), it implies

$$\frac{H(X_0,u,r_kR)}{\|u\|^2_{X_0,r_kR}} \ge \frac{1}{2(k_q+1)}$$

as k sufficiently large. Therefore, if we consider the new sequence

$$v_k(X) = \frac{u(X_0+r_kRX)}{\|u\|_{X_0,r_kR}},$$

since it is uniformly bounded in $H^{1,a}(B_1)$ and it satisfies

$$\begin{cases} -L_a v_k = 0 & \text{in } B^+_{X_0,r_kR} \\ -\partial^a_y v_k = \left(\dfrac{(r_kR)^{k_q}}{\|u\|_{X_0,r_kR}}\right)^{2-q}\left[\lambda_+(v_k)_+^{q-1}-\lambda_-(v_k)_-^{q-1}\right] & \text{on } \partial^0 B^+_{X_0,r_kR}, \end{cases}$$

we deduce from the first part of the proof that, up to a subsequence, it converges strongly in $L^{2,a}(B_1)$, $L^{2,a}(\partial B_1)$ and in $H^{1,a}(B_1)$ to a function $\overline{v} \in H^{1,a}(B_1)$. Moreover, by (66), it solves

$$\begin{cases} -L_a\overline{v} = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\partial^a_y\overline{v} = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Now, the strong convergence in $L^{2,a}(\partial B_1)$ implies

$$H(0,\overline{v},1)=\lim_{k\to\infty}H(0,v_k,1)=\lim_{k\to\infty}\frac{H(X_0,u,r_kR)}{\|u\|^2_{X_0,r_kR}}\geq\frac{1}{2(k_q+1)},$$

that is $\overline{v}\not\equiv 0$ on $\partial^0 B_1^+$. On the other hand, by the absurd assumption, we have

$$\|\overline{v}\|_{0,1/R}=\lim_{k\to\infty}\|v_k\|_{0,1/R}=\lim_{k\to\infty}\frac{\|u\|_{X_0,r_k}}{\|u\|_{X_0,r_kR}}=0,$$

which implies that $\overline{v}\equiv 0$ on $\partial^0 B^+_{1/R}$. The contradiction follows by the unique continuation property for L_a-harmonic function even with respect to $\{y=0\}$ (see [11] for the classic unique continuation theorem of L_a-harmonic functions). □

Lemma 6.3 *Under the previous notations, the sequence $(u_k)_k$ is uniformly bounded in $C^{0,\alpha}_{loc}(\overline{\mathbb{R}^{n+1}_+})$ for every $\alpha\in(0,\min(1,k_q))$. Moreover, up to a subsequence, it converges uniformly on every compact set of $\overline{\mathbb{R}^{n+1}_+}$.*

Proof The proof follows essentially the ideas of the similar results in [28, 34, 35] and the result of Proposition 2.4. □

So far we have proved the strong convergence of the blow-up sequence $(u_k)_k$ in $H^{1,a}_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$ and uniformly on every compact set, to a function $\overline{u}\in H^{1,a}_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})\cap L^\infty_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$. The next step is to prove the homogeneity of the blow-up limit and the complete characterization of the possible limits.

Conclusion of the proof of Theorem 6.1 Since by Proposition 4.7 there exists $C>0$ such that $\alpha_k\in(0,C)$, up to a subsequence, we have either

$$\frac{\|u\|_{X_0,r_k}}{r_k^{k_q}}\to l\in(0,+\infty)\quad\text{or}\quad\frac{\|u\|_{X_0,r_k}}{r_k^{k_q}}\to+\infty. \tag{67}$$

First, suppose that the limit l is finite. By Lemma 6.2, together with a diagonal argument, we get that $u_k\to\overline{u}$ strongly in $H^{1,a}_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$ and uniformly on every compact set. It is also clear that the limit $\overline{u}$ solves (64) with

$$\mu=l^{-2/k_q} \tag{68}$$

and $\overline{u}\not\equiv 0$ since, by strong $H^{1,a}(B_1^+)$-convergence, we have $\|\overline{u}\|_{0,1}=1$. Now, since it remains to prove that $\overline{u}$ is homogeneous, we start by considering the Weiss type formula $W_{k_q,2}(0,u_k,R)$, that is

$$W_{k_q,2}(0,u_k,R)=\frac{r_k^{2k_q}}{\|u\|^2_{X_0,r_k}}W_{k_q,2}(X_0,u,r_kR). \tag{69}$$

Indeed, passing to the limit as $k\to\infty$, we deduce by the uniform convergence that

$$W_{k_q,2}(0,\overline{u},R)=\lim_{k\to\infty}\frac{r_k^{2k_q}}{\|u\|^2_{X_0,r_k}}W_{k_q,2}(X_0,u,r_kR)=\frac{1}{l^2}W_{k_q,2}(X_0,u,0^+),$$

for any $R>0$, namely the map $R\mapsto W_{k_q,2}(0,\overline{u},R)$ is constant. Therefore, by Corollary 3.5, it follows that $\overline{u}$ is k_q-homogeneous.

Let us deal with the second case in (67). By following the same arguments of the case $l \in (0, +\infty)$ up to the validity of a Weiss-type monotonicity result, we already know that, up to a subsequence, $(u_k)_k$ converges uniformly on every compact set, to a function $\overline{u} \in H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}) \cap L^{\infty}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ which satisfies

$$\begin{cases} -L_a \overline{u} = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\partial^a_y \overline{u} = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{70}$$

Now, even if (69) still holds true, we can not conclude that $\overline{u}$ is k_q-homogeneous as before. Instead, by (69) and the monotonicity of $R \mapsto W_{k_q,2}(X_0, u, R)$, we get

$$W_{k_q,2}(0, u_k, R) \leq \frac{r_k^{2k_q}}{\|u\|^2_{X_0,r_k}} W_{k_q,2}(X_0, u, R_0)$$

with $R_0 \in (0, \mathrm{dist}(X_0, \partial B_1))$ arbitrarily chosen and k sufficiently large. By the previous estimate, we have

$$\begin{aligned} \frac{1}{R^{n+a-1}} \int_{B_R^+} y^a |\nabla u_k|^2 \,\mathrm{d}X \leq \frac{k_q}{R^{n+a}} \int_{B_R^+} y^a u_k^2 \mathrm{d}X + \frac{(r_k R)^{2k_q}}{\|u\|^2_{X_0,r_k}} W_{k_q,2}(X_0, u, R_0) + \\ + \frac{\alpha_k}{R^{n+a-1}} \int_{\partial^0 B_R^+} F_{\lambda_+,\lambda_-}(u_k) \mathrm{d}x, \end{aligned}$$

where the terms in the right hand side go to zero since $\alpha_k \to 0^+$ and $\|u\|_{X_0,r_k} / r_k^{k_q} \to +\infty$. Finally, passing to the limit as $k \to \infty$, we get

$$\frac{1}{R^{n+a-1}} \int_{B_R^+} y^a |\nabla \overline{u}|^2 \,\mathrm{d}X \leq k_q \frac{1}{R^{n+a}} \int_{B_R^+} y^a \overline{u}^2 \mathrm{d}X, \tag{71}$$

for every $R > 0$. On the other hand, since $\mathcal{O}(u, X_0) = k_q$, we get

$$\limsup_{r \to 0^+} \frac{1}{r^{2\alpha}} \|u\|^2_{H^{1,a}(B_r(X_0))} = \begin{cases} 0, & \text{if } 0 < \alpha < k_q \\ +\infty, & \text{if } \alpha > k_q. \end{cases}$$

By Lemma 6.2 and (71), for every $\alpha > 0$ we have

$$\begin{aligned} \frac{1}{R^{2\alpha}} \|\overline{u}\|^2_{H^{1,a}(B_R)} &\leq \frac{1 + k_q}{R^{2\alpha}} \frac{1}{R^{n+a}} \int_{B_R^+} y^a \overline{u}^2 \mathrm{d}X \\ &= \lim_{k \to \infty} \frac{1 + k_q}{R^{2\alpha}} \frac{1}{(R r_k)^{n+a}} \int_{B^+_{R r_k}(X_0)} y^a u^2 \mathrm{d}X \\ &= (1 + k_q) \lim_{k \to \infty} \frac{H(X_0, u, R r_k)}{(R r_k)^{2\alpha}} r_k^{2\alpha} \\ &\leq (1 + k_q) r_0^{2\alpha} \limsup_{k \to \infty} \frac{1}{(R r_k)^{2\alpha}} \|u\|^2_{X_0, r_k R}, \end{aligned}$$

which yields that $\mathcal{O}(\overline{u}, 0) \geq k_q$. Since we already know that $\overline{u}$ is a weak solution of (70), by [28,Lemma 4.7] we get that

$$\frac{1}{R^{n+a-1}} \int_{B_R^+} y^a |\nabla \overline{u}|^2 \mathrm{d}X \geq k_q \frac{1}{R^{n+a}} \int_{B_R^+} y^a \overline{u}^2 \mathrm{d}X,$$

which implies with (71) that $k_q \in 1 + \mathbb{N}$ and that $\overline{u}$ is k_q-homogeneous in $\mathbb{R}^{n+1}_+$. □

By having established the compactness of the blow-up sequence for those point satisfying $\mathcal{O}(u, X_0) = k_q$, we can finally prove the equivalence between the two notion of vanishing order.

Corollary 6.4 *For every $X_0 \in \Gamma(u)$, we have $\mathcal{O}(u, X_0) = \mathcal{V}(u, X_0)$.*

Proof Since we already proved in Corollary 5.2 the previous equivalence for the case $\mathcal{O}(u, X_0) < k_q$, let us focus on the case $\mathcal{O}(u, X_0) = k_q$ and let us prove that

$$0 < \liminf_{r\to 0^+} \frac{H(X_0, u, r)}{\|u\|^2_{X_0,r}} \leq 1.$$

Since the upper estimate follows by the definition of the norm in $H^{1,a}(B_r(X_0))$, suppose by contradiction that there exists $r_k \to 0^+$ such that

$$\frac{H(X_0, u, r)}{\|u\|^2_{X_0,r}} \to 0^+. \tag{72}$$

Since $\mathcal{O}(u, X_0) = k_q$, the normalized blow-up sequence

$$u_k(X) = \frac{u(X_0 + r_k X)}{\|u\|_{X_0,r_k}}$$

converges, up to a subsequence, to an homogenous non-trivial solution $\overline{u}$ of (64) in $\mathbb{R}^{n+1}$. On the other hand, by (72) we get

$$\int_{\partial^+ B_1^+} y^a \overline{u}^2 = \lim_{k\to\infty} \int_{\partial^+ B_1^+} y^a u_k^2 = \lim_{k\to\infty} \frac{H(X_0, u, r_k)}{\|u\|^2_{X_0,r_k}} \to 0.$$

By homogeneity, it implies that $\overline{u} \equiv 0$ on $\mathbb{R}^{n+1}$, a contradiction. □

Up to the previous Corollary, we knew that Theorem 1.5 was valid for the $H^{1,a}$-vanishing order. Finally, we can complete the proof in terms of the classic vanishing order $\mathcal{V}(u, X_0)$.

Proof of Theorem 1.5 By Proposition 4.7 we already know that the maximum admissible $H^{1,a}$-vanishing order is equal to $k_q = k_q$. If $\mathcal{O}(u, X_0) < k_q$, by Corollary 5.2 and Corollary 5.3 we already know that

$$\mathcal{O}(u, X_0) = \mathcal{V}(u, X_0) = N(X_0, u, 0^+).$$

Therefore by Corollary 5.8 we know that $\mathcal{V}(u, X_0)$ must be a positive integer.

If instead $\mathcal{O}(u, X_0) = k_q$, by Corollary 6.4 we finally deduce that $\mathcal{V}(u, X_0) = k_q$, as we claimed. □

7 One-dimensional k_q-homogeneous solution

By Theorem 1.5 we already know that for those values of $s \in (0, 1)$, $q \in [1, 2)$ such that $k_q \leq 1$ it holds $\Gamma(u) = \mathcal{T}(u)$ with

$$\mathcal{T}(u) = \left\{X \in \Gamma(u) \colon \mathcal{V}(u, X) = k_q\right\}.$$

In this Section, we prove the existence of k_q-homogeneous solutions of (64) whose traces on $\mathbb{R}^n \times \{0\}$ are one-dimensional, for those values of the parameters s and q such that $k_q < 1$.

Thanks to the Federer's reduction principle, this result allows to control the Hausdorff dimension of $\mathcal{T}(u)$ and to prove that the nodal set is a collection of point with vanishing order k_q and Hausdorff dimension less or equal than $(n-1)$, in contrast with the case $s=1$.

***Remark* 7.1** The classification of k_q-homogeneous solutions depending only on two-variables (x_1, y) is the starting point for a complete understanding of the regularity of the sublinear set $\mathcal{T}(u)$. Indeed, we claim that a possible improvement of flatness approach, via a viscosity formulation of the sublinear set $\mathcal{T}(u)$, could give a complete picture of the biggest stratum of $\mathcal{T}(u)$. Moreover, we think that this strategy can be easily extended to the case $k_q > 1, k_q \notin \mathbb{N}$ by taking care of the classification of L_a-harmonic polynomial in [28].

Theorem 7.2 *For every $s \in (0,1)$, $q \in [1,2)$ and $\lambda_+, \lambda_- > 0$ such that $k_q < 1$, there exists a k_q-homogeneous function u such that $u(0,0)=0$ and*

$$\begin{cases} -L_a u = 0 & \text{in } \mathbb{R}^2_+ \\ -\partial^a_y u = \lambda_+(u_+)^{q-1} - \lambda_-(u_-)^{q-1} & \text{on } \mathbb{R}\times\{0\}. \end{cases} \tag{73}$$

In particular, by exploiting the homogeneity of u, the previous problem is equivalent to consider

$$\begin{cases} -(\sin^a(\theta)\varphi')' = \mu \sin^a(\theta)\varphi & \text{in } (0,\pi) \\ -\partial^a_\theta \varphi(0) = \lambda_+(\varphi_+(0))^{q-1} - \lambda_-(\varphi_-(0))^{q-1} \\ -\partial^a_\theta \varphi(\pi) = \lambda_+(\varphi_+(\pi))^{q-1} - \lambda_-(\varphi_-(\pi))^{q-1}, \end{cases} \tag{74}$$

with $\mu = k_q\left(k_q + 1 - 2s\right)$ and $u(X) = |X|^k \varphi(X\,|X|^{-1})$.

In order to simplify the proofs, we will first address the case $\lambda_+ = \lambda_-$ by proving existence of solutions of (73) whose traces on $\mathbb{R}\times\{0\}$ are either of the form

$$u(x,0) = A_1\left(x_+^{k_q} - x_-^{k_q}\right) \quad \text{or} \quad u(x,0) = A_2\,|x|^{k_q}\,,$$

for some positive constants A_1, A_2 depending only on s, q and λ_+. Indeed, since these prototypes of solution are either symmetric of antisymmetric with respect to x, the construction of Theorem 7.4 will imply the existence of solution of (73) for every $\lambda_+, \lambda_- > 0$.

In the following Lemma we prove a sufficient condition for the existence of non-trivial solutions to (74) in $(0,T)$ for some $T \in (0,\pi)$, in the case $\lambda_+ = \lambda_-$.

Lemma 7.3 *Given $T \in (0,\pi)$ and*

$$X = \left\{u \in H^{1,a}\left((0,T)\right) : u(T) = 0\right\},$$

let us consider the mixed Dirichlet-Neumann eigenvalue associated to $(0,T)$

$$\lambda_M(T) = \min\left\{\frac{\int_0^T \sin^a(\theta)(u')^2}{\int_0^T \sin^a(\theta)u^2} : u \in X\setminus\{0\}, \partial^a_\theta u(0) = 0\right\}.$$

Then, if $\mu = k_q\left(k_q+1-2s\right) < \lambda_M(T)$ there exists an unique positive function $\varphi \in X$ such that

$$\begin{cases} -(\sin^a(\theta)\varphi')' = \mu \sin^a(\theta)\varphi & \text{in } (0,T) \\ -\partial^a_\theta \varphi(0) = \lambda_+(\varphi_+(0))^{q-1}. \end{cases} \tag{75}$$

Proof Under the previous notations, let us consider the minimization problem $\min_{\varphi \in X} J(\varphi)$ with

$$J(u) = \frac{1}{2}\int_0^T \sin^a(\theta)\left((u')^2 - k_q\left(k_q+1-2s\right)u^2\right)\mathrm{d}\theta - \frac{F_{\lambda_+,0}(u)(0)}{q}.$$

Since $q \in [1,2)$, for every $u \in X$ there exists $\bar{t} > 0$ small enough such that $J(tu) < 0$ for every $t \in (0, \bar{t})$.

Notice that critical point of J in X are solution of (75), i.e. for every $\phi \in X$ we get

$$\begin{aligned}
\mathrm{d}J(u)[\phi] &= \int_0^T \sin^a(\theta)\left(u'\phi' - k_q\left(k_q+1-2s\right)u\phi\right)\mathrm{d}\theta - \left(\lambda_+(u_+(0))^{q-1}\phi_+(0)\right)\\
&= -\int_0^T \left((\sin^a(\theta)u')' + k_q\left(k_q+1-2s\right)u\right)\phi \mathrm{d}\theta - \partial_\theta^a u(0)\phi(0)\\
&\quad - \left(\lambda_+(u_+(0))^{q-1}\phi_+(0)\right).
\end{aligned}$$

By the Sobolev embedding, for every $n > 2s$, $q \in [1,2)$ it holds

$$\begin{aligned}
\int_{S^{n-1}} g^q \mathrm{d}\sigma_x \le \tilde{C}\left|\partial^0 B^+\right|^{\frac{2n-(n-2s)q}{2n}} &\left(\int_{S^n_+} \sin^a(\theta)\left|\nabla_S g\right|^2 \mathrm{d}\sigma_X\right.\\
&\left. +(k_q^2+n+2k_q-2s)\int_{S^n_+} \sin^a(\theta) g^2 \mathrm{d}\sigma_X\right)^{q/2}
\end{aligned}$$

with

$$\begin{aligned}
\tilde{C} &= \frac{(n+k_q q)(C_{n,s}N_s)^{q/2}}{(n+2k_q-2s)^{q/2}}, \quad N_s = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)},\\
C_{n,s} &= \frac{2^{-2s}}{\pi^s}\left(\frac{\Gamma(\frac{n-2s}{2})}{\Gamma(\frac{n+2s}{2})}\right)\left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{\frac{2s}{n}}.
\end{aligned}$$

Thus, for $n = 1$ and $\mu = k_q\left(k_q+1-2s\right)$ we get

$$\begin{aligned}
J(u) \ge{}& \frac{1}{2}\int_0^T \sin^a(\theta)\left((u')^2 - \mu u^2\right)\mathrm{d}\theta +\\
&- \frac{\lambda_+}{q}C\left(\int_0^T \sin^a(\theta)(u')^2 \mathrm{d}\theta + (k_q^2+1+2k_q-2s)\int_0^T \sin^a(\theta)u^2 \mathrm{d}\theta\right)^{q/2}
\end{aligned}$$

with

$$C = \left(\frac{\Gamma(s)}{\Gamma(1-s)}\right)^{q/2}\frac{1}{\pi^{qs}}\left(\frac{\Gamma(\frac{1-2s}{2})}{\Gamma(\frac{1+2s}{2})}\right)^{q/2}\frac{1+k_q q}{(1+2k_q-2s)^{q/2}}2^{1-(1-s)q}.$$

Moreover, since by the Poincaré inequality in X we have

$$\int_0^T \sin^a(\theta)u^2 \mathrm{d}\theta \le C_p\int_0^T \sin^a(\theta)(u')^2 \mathrm{d}\theta,$$

for some positive constant C_p, we get

$$J(u) \geq \frac{1}{2}\left(1 - C_p\mu\right)\int_0^T \sin^a(\theta)(u')^2 \mathrm{d}\theta + \\ - \frac{\Lambda((k_q^2 + 1 + 2k_q - 2s)C_p + 1)^{q/2}}{q} C\left(\int_0^T \sin^a(\theta)(u')^2 \mathrm{d}\theta\right)^{q/2}.$$

Finally, since

$$\frac{1}{C_p} = \min_{u\in X} \frac{\displaystyle\int_0^T \sin^a(\theta)(u')^2 \mathrm{d}\theta}{\displaystyle\int_0^T \sin^a(\theta)u^2 \mathrm{d}\theta} = \lambda_M(T),$$

it follows that $C_p\mu < 1$, which implies that J is bounded from below and coercive. Since X is weakly closed, the direct method of the calculus of variations implies the existence of a minimizer u which solves (75). Moreover, we can prove that u is positive: indeed, since if u is a minimizer the same holds also for $|u|$, we can already suppose that $u \geq 0$. Now the strong maximum principle implies that either $u > 0$ or $u \equiv 0$, but the latter options can be easily ruled out observing that $J(u) < 0$.

Finally, if we suppose there exists two different solutions φ_1, φ_2 of (75), it is straightforward to see that there exists a linear combination $w = \varphi_1 - C\varphi_2$, with $C > 0$ such that $\varphi_1^{q-1}(0) = C\varphi_2^{q-1}(0)$ and

$$-\partial_\theta^a w(0) = -\partial_\theta^a \varphi_1(0) + C\partial_\theta^a \varphi_2(0) = \lambda_+(\varphi_1(0)^{q-1} - C\varphi_2(0)^{q-1}) = 0. \tag{76}$$

Moreover

$$\begin{cases} -(\sin^a(\theta)w')' = \mu \sin^a(\theta)w & \text{in } (0, T) \\ w(T) = 0, \ \partial_\theta^a w(0) = 0. \end{cases}$$

Necessary w must vanishes identically in $(0, T)$: indeed, if not either the function is strictly positive in $(0, T)$ or it changes sign in $(0, T)$, both in contradiction with the assumption $\mu < \lambda_M(T)$. Hence, $\varphi_1 \equiv C\varphi_2$ in $[0, T]$, which contradicts the definition of C. □

Theorem 7.4 *Let $k_q < 1$, then for every $\lambda_+ > 0$ there exist only two k_q-homogeneous solutions $u_1, u_2 \in H^{1,a}(\mathbb{R}^2_+)$ of*

$$\begin{cases} -L_a u = 0 & \text{in } \mathbb{R}^2_+ \\ -\partial_y^a u = \lambda_+ |u|^{q-2} u & \text{on } \mathbb{R}\times\{0\}, \end{cases}$$

such that

$$u_1(x, 0) = A_1\left(x_+^{k_q} - x_-^{k_q}\right) \quad or \quad u_2(x, 0) = A_2 |x|^{k_q}, \tag{77}$$

for some positive constants A_1, A_2 depending only on s, q and λ_+.

Proof Notice first that the condition $k_q < 1$ immediately implies $s \in (0, 1/2)$. Since we plan to prove the existence of a k_q-homogeneous function, it is obvious that its trace must be of the form (77). Moreover, if we suppose by contradiction that there exist two solutions u and v with the same type of traces (either like $u_1(\cdot, 0)$ or $u_2(\cdot, 0)$) then, it must exist a constant

$C > 0$ such that $u_\pm^{q-1}(x,0) = Cv_\pm^{q-1}(x,0)$ in $\mathbb{R}$. Consequently, the function $w = u - Cv$ is a k_q-homogeneous solution of

$$\begin{cases} L_a w = 0 & \text{in} \quad \mathbb{R}^2_+ \\ -\partial^a_y w = 0 & \text{on} \quad \mathbb{R} \times \{0\}. \end{cases}$$

By the classification of [28,Lemma 4.7] we already know that either $k_q \in 1 + \mathbb{N}$ or $w \equiv 0$. Since $k_q < 1$, necessary $w \equiv 0$, in contradiction with the choice of $C > 0$.

In order to construct two functions with these features, let us consider the symmetric and antisymmetric solution of the eigenvalue problem associated the traces on S^1 of u.

Hence, for the antisymmetric case, fixed $T = \pi/2$, by Lemma 7.3 there exists $\varphi \in H^{1,a}(0, \pi/2)$ such that $\varphi(\pi/2) = 0$ and

$$\begin{cases} -(\sin^a(\theta)\varphi')' = \mu \sin^a(\theta)\varphi & \text{in} \quad (0, \pi/2) \\ -\partial^a_\theta \varphi(0) = \lambda_+(\varphi_+(0))^{q-1}. \end{cases}$$

Hence, we define

$$\varphi_1(\theta) = \begin{cases} \varphi(\theta) & \text{if } \theta \in (0, \pi/2) \\ -\varphi(\pi - \theta) & \text{if } \theta \in (\pi/2, \pi) \end{cases},$$

an antisymmetric solution of (74) with $\lambda_+ = \lambda_-$. On the other hand, let us consider the symmetric eigenfunction ϕ defined as

$$\begin{cases} -(\sin^a(\theta)\phi')' = \lambda_1(T)\sin^a(\theta)\phi & \text{in } (T, \pi - T) \\ \phi > 0 & \text{in } (T, \pi - T) \\ \phi(T) = 0 = \phi(\pi - T), \end{cases} \tag{78}$$

for $T \in (0, \pi/2)$, where $\lambda_1(T)$ is the fist eigenvalue associated to $(T, \pi - T)$. By monotonicity of the eigenvalue with respect to the set inclusion, we already know that $T \mapsto \lambda_1(T)$ is increasing and it satisfies

$$\lim_{T \to 0^+} \lambda_1(T) = 2s \quad \text{and} \quad \lambda_1(\arctan(\sqrt{2(1-s)})) = 2.$$

Thus, since $s < 1/2$, there exists $T^* \in (0, \arctan(\sqrt{2(1-s)}))$ such that $\lambda_1(T^*) = k_q$. Furthermore, by applying Lemma 7.3 with $T = T^*$, there exists a function $\psi \in H^{1,a}(0, T^*)$ such that $\psi(T^*) = 0$ and

$$\begin{cases} -(\sin^a(\theta)\psi')' = \mu \sin^a(\theta)\psi & \text{in} \quad (0, T^*) \\ -\partial^a_\theta \psi(0) = \lambda_+(\psi_+(0))^{q-1}. \end{cases}$$

Finally, let $C > 0$ be such that $-C\phi'(T^*) = \psi'(T^*)$, then if we define

$$\varphi_2(\theta) = \begin{cases} \psi(\theta) & \text{if } \theta \in (0, T) \\ -C\phi(\theta) & \text{if } \theta \in (T, \pi - T) \\ \psi(\pi - \theta) & \text{if } \theta \in (\pi - T, \pi) \end{cases},$$

we get a symmetric solution of (74) with Thus, the solutions u_i are defined as the homogeneous extension of φ_i in $\overline{\mathbb{R}^{n+1}_+}$

$$u_i(X) = |X|^{k_q}\, \varphi_i\left(\frac{X}{|X|}\right),$$

which gives the claimed result. □

Finally, by applying the Federer's reduction principle in the form of [7,Theorem 8.5], we can conclude the proof of Theorem 1.9 as a byproduct of the results of this Section.

Proof *(Conclusion of the proof of Theorem 1.9)* Let us consider the class of functions $\mathcal{F}$ defined as

$$\mathcal{F} = \left\{ u \in L^{\infty}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_{+}}) \setminus \{0\} \,\middle|\, \begin{array}{l} u \text{ solves (73) in } B_r(X_0), \text{ for some } r \in \mathbb{R},\ X_0 \in \mathbb{R}^n \times \{0\} \\ \text{for some } \lambda_+, \lambda_-, \mu > 0 \end{array} \right\}.$$

endowed with the topology associated to the uniform convergence and

$$\overline{\mathcal{S}} \colon u \mapsto \mathcal{T}(u).$$

We already know that $\mathcal{F}$ is close under rescaling, translation and normalization. Moreover, by Theorem 6.1 the hypothesis of the existence of a blow-up limit in $\mathcal{F}$ is satisfied, as well as the singular set assumption. Thus, the Federer's reduction principle [7,Theorem 8.5] is applicable and it implies the existence of an integer $d \in [0, n]$ such that

$$\dim_{\mathcal{H}} \mathcal{T}(u) \leq d,$$

for every function $u \in \mathcal{F}$. Suppose by contradiction that $d = n$, this would implies the existence of $\varphi \in \mathcal{F}$ such that $\overline{\mathcal{S}}(\varphi) = \mathbb{R}^n$, that is $\varphi \equiv 0$ on $\mathbb{R}^n$. Thus $\varphi \equiv 0$ on the whole $\mathbb{R}^{n+1}$, which contradicts the fact the $0 \notin \mathcal{F}$. Actually, since Theorem 7.2 ensures the existence of a $(n-1)$-linear subspace $E \subset \mathbb{R}^n$ and a k_q-homogeneous function $\varphi \in \mathcal{F}$ such that $\overline{\mathcal{S}}(\varphi) = E$, we get $d = n - 1$. □

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