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Gaussian quadrature rules with an exponential weight on the real semiaxis

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We consider some 'truncated' Gaussian rules based on the zeros of the orthonormal polynomials w.r.t. the weight function $w(x) = e^{-x^{-\alpha}-x^{\beta}}$ with $x \in (0, +\infty)$, $\alpha > 0$ and $\beta > 1$. We show that these formulas are stable and converge with the order of the best polynomial approximation in suitable function spaces. Moreover, we apply these results to the related Lagrange interpolation process in weighted L^2 spaces. Finally, some numerical tests are shown.

Keywords: Gaussian quadrature rules; orthogonal polynomials; Lagrange interpolation.

1. Introduction

This paper deals with the weighted polynomial approximation of functions defined on the real semiaxis $(0, +\infty)$ which can increase exponentially at the endpoint 0 and $+\infty$. In particular, we consider Gaussian rules for computing integrals of the form

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x, \quad w(x) = e^{-x^{-\alpha}-x^{\beta}}, \quad \alpha > 0,\ \beta > 1.$$

This topic has received no attention in the literature, as far as we know. Since the weight function w is nonclassical, from the numerical point of view, a recent progress in symbolic computation and variable–precision arithmetic enables to overcome the numerical instability in the procedure for generating the recursion coefficients of the corresponding orthogonal polynomials.

We study the behaviour of the Gaussian rule in several spaces of continuous functions with weighted uniform metric, proving that the formula converges with the order of the best weighted polynomial approximation, with proper assumptions (see Proposition 3.1) and with geometric rate for infinitely differentiable functions (see Theorem 3.2). Nevertheless, in Theorem 3.3 we show that the error of this formula does not converge with the optimal rate (i.e., with the order of the best weighted polynomial approximation) in weighted L^1-Sobolev spaces, in analogy with the case of different exponential weights (see, for instance, Della Vecchia & Mastroianni, 2003; Mastroianni & Monegato, 2003; Mastroianni & Notarangelo, 2010; De Bonis *et al.*, 2012) and in contrast with the case of 'doubling' weights on bounded intervals. This difficulty also implies that the sequence of the related Lagrange operators $\{L_m(w)\}$ cannot be uniformly bounded in weighted L^2-Sobolev spaces.

To overcome this problem, we suggest a 'truncated' Gaussian rule (see Definition 3.8). This formula converges with the same order of the ordinary Gaussian rule for continuous functions (see Proposition 3.4 and Theorem 3.5). Moreover, it converges with the order of the best polynomial approximation for functions belonging to weighted L^1-Sobolev spaces, as shown in Theorem 3.6.

As an application of these results, we define a modified Lagrange operator and prove its uniform boundedness in weighted L^2-Sobolev spaces, under proper assumptions (see Theorems 4.3 and 4.4).

All results in this paper are new and the estimates cannot be improved for the considered classes of functions.

The paper is structured as follows. In Section 2, we recall some basic facts about weighted polynomial approximation with the weight w. Our main results concerning Gaussian rules and Lagrange interpolation are stated in Sections 3 and 4, respectively, and proved in Section 5. Section 6 deals with the computation of the Mhaskar–Rahmanov–Saff numbers (MRS numbers) related to w, and Section 7 with numerical construction of quadrature rules. Finally, in Section 8 we give some numerical example.

2. Basic facts and preliminary results

In the sequel $c, \mathcal{C}$ stands for positive constants that can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ when $\mathcal{C}$ is independent of $a,b,\ldots$. Furthermore, $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leqslant \mathcal{C}$. Finally, $\mathcal{P}_m$ denotes the set of all algebraic polynomials of degree at most m.

Let us consider the weight function

$$w(x) = e^{-x^{-\alpha}-x^{\beta}}, \quad \alpha > 0, \ \ \beta > 1, \ \ x \in \mathbb{R}_+ = (0, +\infty) \tag{2.1}$$

that can be seen as a combination of a Pollaczeck-type weight $e^{-x^{-\alpha}}$ and a Laguerre-type weight $e^{-x^{\beta}}$. Since the weight w is nonsymmetric, in analogy with the Laguerre case, we have two MRS numbers, namely $\varepsilon_t = \varepsilon_t(w)$ and $a_t = a_t(w)$, defined by

$$t = \frac{1}{\pi}\int_{\varepsilon_t}^{a_t} \frac{xQ'(x)}{\sqrt{(a_t - x)(x-\varepsilon_t)}}\,\mathrm{d}x \tag{2.2}$$

and

$$0 = \frac{1}{\pi}\int_{\varepsilon_t}^{a_t} \frac{Q'(x)}{\sqrt{(a_t - x)(x-\varepsilon_t)}}\,\mathrm{d}x, \tag{2.3}$$

where $Q'(x) = -\alpha x^{-\alpha-1} + \beta x^{\beta-1}$. From the definition it follows that ε_t is a decreasing function and a_t is an increasing function of t, with

$$\lim_{t\to+\infty} \varepsilon_t = 0, \qquad \lim_{t\to+\infty} a_t = +\infty.$$

By definition, we have

$$\varepsilon_t(w^{\lambda}) = \varepsilon_{t/\lambda}(w) \quad \text{and} \quad a_t(w^{\lambda}) = a_{t/\lambda}(w).$$

In Section 6, we will give a method to compute ε_t and a_t. For the moment, we recall the equivalences

$$\varepsilon_t = \varepsilon_t(w) \sim \left(\frac{\sqrt{a_t}}{t}\right)^{1/(\alpha+1/2)} \quad \text{and} \quad a_t = a_t(w) \sim t^{1/\beta}, \tag{2.4}$$

where the constants in '$\sim$' are independent of t.

Associated with the MRS numbers are the following restricted range inequalities. Letting $0 < p \leqslant +\infty$, for any $P_m \in \mathcal{P}_m$, we have (see Levin & Lubinsky, 2001, pp. 95–96; Mastroianni *et al.*, 2013)

$$\|P_m w\|_\infty = \max_{x\in[\varepsilon_m, a_m]} |P_m(x)w(x)|, \tag{2.5}$$

and

$$\int_0^{+\infty} |P_m w|^p(x)\,\mathrm{d}x \leqslant \mathcal{C} \int_{\varepsilon_m}^{a_m} |P_m w|^p(x)\,\mathrm{d}x, \tag{2.6}$$

where $\mathcal{C}$ is independent of m and P_m. On the other hand, we have

$$\|P_m w\|_{L^p(\mathbb{R}_+\setminus[\varepsilon_{sm}, a_{sm}])} \leqslant \mathcal{C}\, e^{-cm^\nu} \|P_m w\|_p, \quad s > 1, \tag{2.7}$$

where

$$\nu = \left(1 - \frac{1}{2\beta}\right)\frac{2\alpha}{2\alpha+1}, \tag{2.8}$$

and $\mathcal{C}$ and c are independent of m and P_m.

2.1 *Orthonormal polynomials*

Let us denote by $\{p_m(w)\}_{m\in\mathbb{N}}$ the sequence of the orthonormal polynomials defined by

$$p_m(w,x) = \gamma_m x^m + \text{ lower degree terms}, \quad \gamma_m = \gamma_m(w) > 0$$

and

$$\int_0^{+\infty} p_m(w,x)p_n(w,x)w(x)\,\mathrm{d}x = \delta_{m,n}.$$

The zeros of $p_m(w)$ lie in the MRS interval associated with $\sqrt{w}$. To be more precise, we have (see Levin & Lubinsky, 2001, pp. 312–324)

$$\tilde{\varepsilon}_m < x_1 < x_2 < \cdots < x_m < \tilde{a}_m,$$

with $\tilde{\varepsilon}_m = \varepsilon_m(\sqrt{w}) = \varepsilon_{2m}(w)$ and $\tilde{a}_m = a_m(\sqrt{w}) = a_{2m}(w)$.

$$x_1 - \tilde{\varepsilon}_m \sim \delta_m, \quad \delta_m \sim \left(\frac{\sqrt{a_m}}{m}\right)^{(2/3)((2\alpha+3)/(2\alpha+1))} \sim m^{-(2/3)((2\alpha+3)/(2\alpha+1))(1-1/(2\beta))} \tag{2.9}$$

and

$$\tilde{a}_m - x_m \sim a_m m^{-2/3} \sim m^{1/\beta-2/3}, \tag{2.10}$$

where the constants in '$\sim$' are independent of m.

The distance between two consecutive zeros $\Delta x_k = x_{k+1} - x_k$ can be estimated by

$$\Delta x_k \sim \Psi_m(x_k), \quad k = 1, \ldots, m-1, \tag{2.11}$$

where

$$\Psi_m(x_k) = \frac{\tilde{a}_m x_k}{m\sqrt{(x_k - \tilde{\varepsilon}_m + \delta_m)(\tilde{a}_m - x_k + \tilde{a}_m m^{-2/3})}} \tag{2.12}$$

and the constants in '$\sim$' are independent of k and m.

The mth Christoffel function

$$\lambda_m(w,x) = \left(\sum_{k=0}^{m-1} p_m^2(w,x)\right)^{-1}$$

satisfies (see Levin & Lubinsky, 2001, p. 257)

$$\lambda_m(w,x) \sim \Psi_m(x)w(x), \quad x \in [\tilde{\varepsilon}_m, \tilde{a}_m], \tag{2.13}$$

where Ψ_m is given by (2.12) and the constants in '$\sim$' are independent of m. Then we define the Christoffel numbers, setting

$$\lambda_k(w) = \lambda_m(w, x_k), \quad k = 1, \ldots, m.$$

The weight w, introduced in this section, is a crucial tool for the polynomial approximation of functions having domain $\mathbb{R}_+$ and exponential growth for $x \to 0$ and $x \to +\infty$. To this aim we recall the definition of some function spaces, in which the convergence of the best polynomial approximation has been studied (see Mastroianni & Notarangelo, 2013b). In these function spaces we are going to study the behaviour of the Gaussian rules.

2.2 *Function spaces*

We denote by $L^p(A)$, $A \subset \mathbb{R}_+$ and $1 \leqslant p < +\infty$, the collection of all measurable functions f such that $\|f\|^p_{L^p(A)} = \int_A |f(x)|^p \, dx < +\infty$. For $A = \mathbb{R}_+$ we write $\|f\|_p = \|f\|^p_{L^p(\mathbb{R}_+)}$.

Letting w be given by (2.1), $x \in \mathbb{R}_+$, we introduce the weight function

$$u(x) = (1+x)^\delta w^a(x), \quad 0 < a \leqslant 1, \;\; \delta \geqslant 0. \tag{2.14}$$

From (2.5) to (2.7), we can easily deduce the analogous polynomial inequalities with the weight w replaced by u. In fact, letting $n = m/a + \lceil \delta \rceil$, for any $P_m \in \mathcal{P}_m$, with $0 < p \leqslant +\infty$, we have (see Mastroianni *et al.*, 2013)

$$\|P_m u\|_p \leqslant \mathcal{C}\|P_m u\|_{L^p[\varepsilon_n, a_n]}, \tag{2.15}$$

where $\mathcal{C} \neq \mathcal{C}(m, P_m)$, ε_n and $a_n = a_n(w)$ are the MRS numbers related to the weight w.

Analogously, from inequality (2.7) we can deduce

$$\|P_m u\|_{L^p(\mathbb{R}_+\setminus[\varepsilon_{sn},a_{sn}])} \leqslant \mathcal{C}\, e^{-cm^{\nu}} \|P_m u\|_p, \quad s>1, \tag{2.16}$$

where $n \sim m$, $\mathcal{C} \neq \mathcal{C}(m, P_m)$, $c \neq c(m, P_m)$ and ν is defined by (2.8).

Let us now introduce some function spaces. For $1 \leqslant p < +\infty$, we will write $f \in L^p_u$ if $fu \in L^p$, with

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_0^{+\infty} |fu|^p(x)\,\mathrm{d}x\right)^{1/p} < +\infty.$$

Whereas, for $p = +\infty$, we define

$$L^\infty_u = C_u = \left\{ f \in C^0(\mathbb{R}_+) : \lim_{x\to 0^+} f(x)u(x) = 0 = \lim_{x\to+\infty} f(x)u(x) \right\},$$

with the norm

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x\in\mathbb{R}_+} |f(x)u(x)|.$$

Here, $C^0(\mathbb{R}_+)$ denotes the set of all continuous functions on $\mathbb{R}_+ = (0, +\infty)$,

The Sobolev-type spaces are given by

$$W^p_r(u) = \{f \in L^p_u : f^{(r-1)} \in AC(\mathbb{R}_+),\ \|f^{(r)}\varphi^r u\|_p < +\infty\}, \quad 1 \leqslant r \in \mathbb{Z}^+,$$

where $1 \leqslant p \leqslant +\infty$, $\varphi(x) := \sqrt{x}$ and $AC(\mathbb{R}_+)$ denotes the set of all functions which are absolutely continuous on every closed subset of $(0, +\infty)$. We equip these spaces with the norm

$$\|f\|_{W^p_r(u)} = \|fu\|_p + \|f^{(r)}\varphi^r u\|_p.$$

The following main part of the rth modulus of smoothness has been introduced in Mastroianni & Notarangelo (2013b). For $f \in L^p_u$, $1 \leqslant p \leqslant \infty$, $r \geqslant 1$ and $t > 0$ sufficiently small, we set

$$\Omega^r_\varphi(f,t)_{u,p} = \sup_{0<h\leqslant t} \|\Delta^r_{h\varphi}(f)u\|_{L^p(\mathcal{I}_h(c))}, \quad \mathcal{I}_h(c) = \left[h^{1/(\alpha+1/2)}, \frac{c}{h^{1/(\beta-1/2)}}\right],$$

where $c > 1$ is a fixed constant, and

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x + (r-i)h\varphi(x)), \quad \varphi(x) = \sqrt{x}.$$

Then we define the complete rth modulus of smoothness by

$$\omega^r_\varphi(f,t)_{u,p} = \Omega^r_\varphi(f,t)_{u,p} + \inf_{q\in\mathcal{P}_{r-1}} \|(f-q)u\|_{L^p(0,t^{1/(\alpha+1/2)}]} + \inf_{q\in\mathcal{P}_{r-1}} \|(f-q)u\|_{L^p[ct^{-1/(\beta-1/2)},+\infty)},$$

with $c > 1$ a fixed constant. This modulus is equivalent to the following K-functional:

$$K(f,t^r)_{u,p} = \inf_{g\in W^p_r(u)} \{\|(f-g)u\|_p + t^r\|g^{(r)}\varphi^r u\|_p\},$$

namely, for any $f \in L^p_u$, $1 \leqslant p \leqslant \infty$, we have $\omega^r_\varphi(f,t)_{u,p} \sim K(f,t^r)_{u,p}$, where the constants in '$\sim$' are independent of f and t.

By means of the rth modulus of smoothness, for $1 \leqslant p \leqslant +\infty$, we can define the Zygmund-type spaces

$$Z_s^p(u) := Z_{s,r}^p(u) = \left\{ f \in L_u^p : \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} < +\infty,\ r > s \right\}, \quad s \in \mathbb{R}_+,$$

with the norm

$$\|f\|_{Z_s^p(u)} = \|f\|_{L_u^p} + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}.$$

In the sequel, we will denote these subspaces briefly by $Z_s^p(u)$. Moreover, we remark that $\omega_\varphi^r(f,t)_{u,p} \sim \Omega_\varphi^r(f,t)_{u,p}$ for any $Z_s^p(u)$ (see Mastroianni & Notarangelo, 2013b).

Let us denote by $E_m(f)_{u,p} = \inf_{P\in\mathcal{P}_m} \|(f-P)u\|_p$ the error of best polynomial approximation of a function $f \in L_u^p$, $1 \leqslant p \leqslant +\infty$.

The order of convergence of $E_m(f)_{u,p}$ can be estimated using the previous modulus of smoothness. In fact, for any $f \in L_u^p$, $1 \leqslant p \leqslant +\infty$, and $m > r \geqslant 1$, the following Jackson, weak Jackson and Stechkin inequalities

$$E_m(f)_{u,p} \leqslant \mathcal{C}\omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p}, \tag{2.17}$$

$$E_m(f)_{u,p} \leqslant \mathcal{C} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,\mathrm{d}t, \quad r < m \tag{2.18}$$

and

$$\omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \sum_{i=0}^m \left(\frac{i}{\sqrt{a_i}}\right)^r \frac{E_i(f)_{u,p}}{i},$$

hold with $a_m \sim m^{1/\beta}$ and $\mathcal{C} \neq \mathcal{C}(f,m)$ (see Mastroianni & Notarangelo, 2013b). From the previous inequalities, for any $f \in W_r^p(u)$, $1 \leqslant p \leqslant +\infty$, we obtain

$$E_m(f)_{u,p} \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f^{(r)}\varphi^r u\|_p, \quad \mathcal{C} \neq \mathcal{C}(m,f). \tag{2.19}$$

Whereas, for any $f \in Z_s^p(u)$, $1 \leqslant p \leqslant +\infty$, we obtain

$$E_m(f)_{u,p} \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}, \quad r > s,\ \mathcal{C} \neq \mathcal{C}(m,f). \tag{2.20}$$

3. Gaussian formulas

The Gaussian rule related to the weight $w(x) = e^{-x^{-\alpha}-x^\beta}$ can be defined by the equality

$$\int_0^{+\infty} P_{2m-1}(x)w(x)\,\mathrm{d}x = \sum_{k=1}^m \lambda_k(w)P_{2m-1}(x_k), \tag{3.1}$$

where x_k are the zeros of $p_m(w)$, $\lambda_k(w)$ are the Christoffel numbers, which holds for any polynomial $P_{2m-1} \in \mathcal{P}_{2m-1}$.

Thus, the error of the Gaussian rule for any continuous function f is given by

$$e_m(f) = \int_0^{+\infty} f(x)w(x)\,\mathrm{d}x - \sum_{k=1}^{m} \lambda_k(w)f(x_k).$$

If we assume $f \in C_u$, then we can write

$$\left|\sum_{k=1}^{m} \lambda_k(w)f(x_k)\right| \leqslant \|fu\|_\infty \sum_{k=1}^{m} \frac{\lambda_k(w)}{u(x_k)} \leqslant \mathcal{C}\|fu\|_\infty \int_0^{+\infty} \frac{w(x)}{u(x)}\,\mathrm{d}x$$

and the next proposition easily follows.

PROPOSITION 3.1 If $w/u \in L^1$, then, for any $f \in C_u$, we have

$$|e_m(f)| \leqslant \mathcal{C}E_{2m-1}(f)_{u,\infty}, \tag{3.2}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

This proposition generalizes a result due to Uspensky (1928), who first proved the convergence of Gaussian rules on unbounded intervals related to Laguerre and Hermite weights (see also Mastroianni & Milovanović, 2008, pp. 341–345; Mastroianni & Notarangelo, 2010).

Note that the assumption $w/u \in L^1$ in Proposition 3.1 is fulfilled if $a = 1$ and $\delta > 1$, or if $a < 1$ and δ is arbitrary. The error estimate (3.2) implies the convergence of the Gaussian rule for any $f \in C_u$. For smoother function, for instance $f \in W_r^\infty(u)$, by (3.2) and (2.19), we obtain

$$|e_m(f) \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f^{(r)}\varphi^r u\|_\infty,$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$ and $a_m \sim m^{1/\beta}$.

Thus, a natural question is to establish the degree of convergence of $e_m(f)$ if the function f is infinitely differentiable, i.e., $f \in C^\infty(\mathbb{R}_+)$. We recall that Aljarrah (1980, 1983) proved estimates of $e_m(f)$ related to Hermite or Freud weights for analytic functions in some domains of the complex plane containing the quadrature nodes. For precise estimates, considering the same class of functions and different weights, we refer to Lubinsky (1983). Here, we consider the case of infinitely differentiable functions on $\mathbb{R}_+$, with the condition that $(f^{(m)}u)(x)$ is uniformly bounded w.r.t. m and x. We note that the derivatives of the function can increase exponentially for $x \to 0$ and $x \to +\infty$.

THEOREM 3.2 Let $u(x) = (1+x)^\delta w^a(x)$, with $0 < a < 1$ and δ arbitrary. For any infinitely differentiable function f, if $K(f) := \sup_m \|f^{(m)}u\|_\infty < +\infty$, we have

$$\lim_m \sqrt[m]{\frac{|e_m(f)|}{K(f)}} = 0. \tag{3.3}$$

In Section 5, we will give a simple proof of inequality (3.3) that can be applied in different contexts.

Now, in order to study the behaviour of the Gaussian rule in Sobolev spaces $W_r^1(w)$, it is natural to investigate whether estimates of the form

$$|e_m(f)| \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1, \quad \mathcal{C} \neq \mathcal{C}(m,f), \ f \in W_1^1(w), \tag{3.4}$$

hold true.

We recall that, as shown in Mastroianni & Notarangelo (2013b), for the error of best approximation we have

$$E_m(f)_{w,1} \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1, \quad \mathcal{C} \neq \mathcal{C}(m,f), \ f \in W_1^1(w).$$

On the other hand, inequality (3.4) holds, mutatis mutandis, for Gaussian rules on bounded intervals related to Jacobi weights. But, as for many exponential weights (see Della Vecchia & Mastroianni, 2003; Mastroianni & Monegato, 2003; De Bonis *et al.*, 2012), inequality (3.4) is false in the sense of the following theorem.

THEOREM 3.3 Let $w(x) = e^{-x^{-\alpha}-x^{\beta}}$, $\alpha > 0$ and $\beta > 1$. Then, for any $f \in W_1^1(w)$, we have

$$|e_m(f)| \leqslant \mathcal{C}m^{1/3}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1, \tag{3.5}$$

where $\mathcal{C}$ is independent of m and f. Moreover, for a sufficiently large m (say $m \geqslant m_0$), there exists a function f_m, with $0 < \|f_m'\varphi w\|_1 < +\infty$, and a constant $\mathcal{C} \neq \mathcal{C}(m,f_m)$ such that

$$|e_m(f_m)| \geqslant \mathcal{C}m^{1/3}\frac{\sqrt{a_m}}{m}\|f_m'\varphi w\|_1. \tag{3.6}$$

Nevertheless, estimates of the form (3.4) are required in different contexts. So, in order to obtain this kind of error estimates, using also an idea from Mastroianni & Monegato (2003), we are going to modify the Gaussian rule.

With $\theta \in (0,1)$ fixed, we define two indexes $j_1 = j_1(m)$ and $j_2 = j_2(m)$ as

$$x_{j_1} = \max_{1\leqslant k\leqslant m}\{x_k : x_k \leqslant \tilde{\varepsilon}_{\theta m}\} \quad \text{and} \quad x_{j_2} = \min_{1\leqslant k\leqslant m}\{x_k : x_k \geqslant \tilde{a}_{\theta m}\}, \tag{3.7}$$

respectively, and set

$$Z_{\theta,m} = \{x_k = x_{m,k}(w) : x_{j_1} \leqslant x_k \leqslant x_{j_2}\}.$$

For the sake of completeness, if $\{x_k : x_k \leqslant \tilde{\varepsilon}_{\theta m}\}$ or $\{x_k : x_k \geqslant \tilde{a}_{\theta m}\}$ are empty, we set $x_{j_1} = x_1$ or $x_{j_2} = x_m$, respectively.

Furthermore, for a sufficiently large N, let $\mathcal{P}_N^*$ denote the collection of all polynomials of degree at most N, vanishing at the zeros of $p_m(w)$ which are smaller than x_{j_1} or larger than x_{j_2}, i.e.,

$$\mathcal{P}_N^* = \{P \in \mathcal{P}_N : P(x_i) = 0, \ x_i \notin Z_{\theta,m}\}.$$

Naturally, $p_m(w) \in \mathcal{P}_N^*$, for $N \geqslant m$ and $\theta \in (0,1)$ arbitrary.

Now, in analogy with (3.1), we define the new Gaussian rule, by means of the equality

$$\int_0^{+\infty} Q_{2m-1}(x)w(x)\,\mathrm{d}x = \sum_{k=1}^{m} \lambda_k(w)Q_{2m-1}(x_k) = \sum_{k=j_1}^{j_2} \lambda_k(w)Q_{2m-1}(x_k),$$

which holds for every $Q_{2m-1} \in \mathcal{P}^*_{2m-1}$.

Then, for any continuous function f, the 'truncated' Gaussian rule is defined as

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x = \sum_{k=j_1}^{j_2} \lambda_k(w)f(x_k) + e^*_m(f), \tag{3.8}$$

whose error $e^*_m(f)$ is the difference between the integral and the quadrature sum.

Compared with the Gaussian rule (3.1), in the formula (3.8) the terms of the quadrature sum corresponding to the zeros which are 'close' to the MRS numbers are dropped. From the numerical point of view, this fact has two consequences. First, it avoids overflow phenomena (taking into account that, in general, the function f is exponentially increasing at the endpoints of $\mathbb{R}_+$). Moreover, it produces a computational saving, which is evident in the numerical treatment of linear functional equations.

We are now going to study the behaviour $e^*_m(f)$ in C_u and $W^1_r(w)$. We will see that the errors $e_m(f)$ and $e^*_m(f)$ have essentially the same behaviour in C_u, but not in $W^1_r(w)$, since $e^*_m(f)$ satisfies (3.4), while $e_m(f)$ does not.

The behaviour of $e^*_m(f)$ in C_u is given by the following proposition.

PROPOSITION 3.4 Assume that $w/u \in L^1$. Then, for any $f \in C_u$, we obtain

$$|e^*_m(f)| \leqslant \mathcal{C}\{E_M(f)_{u,\infty} + e^{-cm^\nu}\|fu\|_\infty\}, \tag{3.9}$$

where $M = \lfloor(\theta/(\theta+1))m\rfloor$, $\theta \in (0,1)$, ν is given by (2.8), $\mathcal{C} \neq \mathcal{C}(m,f)$ and $c \neq c(m,f)$.

In particular, if $f \in W^\infty_r(u)$, inequality (3.9) becomes

$$|e^*_m(f)| \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W^\infty_r(u)}. \tag{3.10}$$

For smoother functions, the analogue of Theorem 3.2 is given by the following statement.

THEOREM 3.5 If the weight u and the function f satisfy the assumption of Theorem 3.2, then, for any $0 < \mu < \alpha(1 - 1/(2\beta))/(\alpha + 1/2)$ fixed, we obtain

$$\lim_m \left(\frac{|e^*_m(f)|}{\|fu\|_\infty + \|f^{(m)}u\|_\infty}\right)^{1/m^\mu} = 0, \tag{3.11}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

For functions $f \in W^1_1(w)$ or $f \in Z^1_s(w)$, $1 < s \in \mathbb{R}_+$, the following theorem states the required estimates.

THEOREM 3.6 For any $f \in W^1_1(w)$, we have

$$|e^*_m(f)| \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1 + \mathcal{C}\,e^{-cm^\nu}\|fw\|_1. \tag{3.12}$$

Moreover, for any $f \in Z_s^1(w)$, with $s > 1$, we obtain

$$|e_m^*(f)| \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\int_0^{\sqrt{a_m}/m}\frac{\Omega_\varphi^r(f,t)_{w,1}}{t^2}\,\mathrm{d}t + \mathcal{C}\,e^{-cm^\nu}\|fw\|_1, \tag{3.13}$$

where $r > s > 1$. In both cases, $\mathcal{C}$ and c do not depend on m and f, and ν is given by (2.8).

In conclusion, inequality (3.12) is the required estimate and, by (3.13), it can be generalized as

$$|e_m^*(f)| \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \|f\|_{Z_s^1(w)}, \quad \mathcal{C} \neq \mathcal{C}(m,f),$$

for $f \in Z_s^1(w)$, $s > 1$. In particular, if s is an integer number, recalling (3.12), the Zygmund norm can be replaced by the Sobolev norm.

Finally, we emphasize that the previous estimate cannot be improved, since, in these function spaces, $e_m^*(f)$ converges to 0 with the order of the best polynomial approximation.

4. Lagrange interpolation in $L^2_{\sqrt{w}}$

Here, we want to apply the results in Section 3 to estimate the error of the Lagrange interpolation process based on the zeros of $p_m(w)$, with $w(x) = e^{-x^{-\alpha}-x^\beta}$, $\alpha > 0$ and $\beta > 1$. If $f \in C^0(\mathbb{R}_+)$, then the Lagrange polynomial interpolating f at the zeros of $p_m(w)$ is defined by

$$L_m(w,f,x) = \sum_{k=1}^m l_k(w,x)f(x_k), \quad l_k(w,x) = \frac{p_m(w,x)}{p_m'(w,x_k)(x-x_k)},$$

and we are going to study the error $\|L_m(w,f)\sqrt{w}\|_2$ for different function classes.

Since

$$\|L_m(w,f)\sqrt{w}\|_2^2 = \sum_{k=1}^m \frac{\lambda_k(w)}{w(x_k)}(f\sqrt{w})^2(x_k) \tag{4.1}$$

and we are dealing with an unbounded interval, we cannot expect an analogue of the theorem by Erdős & Turán (1937). On the other hand, if $f \in C_u$, with $u(x) = (1+x)^\delta\sqrt{w(x)})$, $\delta > 1/2$, it is easily seen that

$$\|[f - L_m(w,f)]\sqrt{w}\|_2 \leqslant \mathcal{C}E_{m-1}(f)_{u,\infty}, \quad \mathcal{C} \neq \mathcal{C}(m,f).$$

Nevertheless, as for the Gaussian formula, if $f \in W_1^2(\sqrt{w})$, then $L_m(w,f)$ does not have an optimal behaviour, i.e., an estimate of the form

$$\|[f - L_m(w,f)]\sqrt{w}\|_2 \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi\sqrt{w}\|_2, \quad \mathcal{C} \neq \mathcal{C}(m,f),$$

does not hold. To this aim, for any $f \in C^0(\mathbb{R}_+)$, we introduce the following ‘truncated’ Lagrange polynomial

$$L_m^*(w,f,x) = \sum_{k=j_1}^{j_2} l_k(w,x)f(x_k),$$

where j_1, j_2 are given by (3.7).

Naturally, in general, $L_m^*(w,P) \neq P$ for arbitrary polynomials $P \in \mathcal{P}_{m-1}$ (e.g., $L_m(w,\mathbf{1}) \neq \mathbf{1}$). But $L_m^*(w,Q) = Q$ for any $Q \in \mathcal{P}_{m-1}^*$ and $L_m^*(w,f) \in \mathcal{P}_{m-1}^*$ for any $f \in C^0(\mathbb{R}_+)$. So, the operator $L_m^*(w)$ is a projector from $C^0(\mathbb{R}_+)$ into $\mathcal{P}_{m-1}^*$.

Moreover, considering the weight

$$\sigma(x) = (1+x)^\delta \sqrt{w(x)}, \quad \delta > 0, \tag{4.2}$$

we can show that every function $f \in L_\sigma^p$ can be approximated by polynomials of $\mathcal{P}_m^*$. To this aim, we define

$$\tilde{E}_m(f)_{\sigma,p} = \inf_{P \in \mathcal{P}_m^*} \|(f-P)\sigma\|_p, \quad 1 \leqslant p \leqslant +\infty.$$

Lemma 4.1 For any $f \in L_\sigma^p$, where σ is given by (4.2) and $1 \leqslant p \leqslant +\infty$, we have

$$\tilde{E}_m(f)_{\sigma,p} \leqslant \mathcal{C}\{E_M(f)_{\sigma,p} + e^{-cm^\nu}\|f\sigma\|_p\},$$

where $M = \lfloor (\theta/(\theta+1))m \rfloor$, $\theta \in (0,1)$, ν is given by (2.8), $\mathcal{C} \neq \mathcal{C}(m,f)$ and $c \neq c(m,f)$.

As an immediate consequence of the previous lemma and equality (4.1), we get the following proposition.

Proposition 4.2 For any $f \in C_\sigma$, with σ as (4.2), $\delta > 1/2$, we have

$$\|[f - L_m^*(w,f)]\sqrt{w}\|_2 \leqslant \mathcal{C}\{E_M(f)_{\sigma,\infty} + e^{-cm^\nu}\|f\sigma\|_\infty\},$$

where $M = \lfloor (\theta/(\theta+1))m \rfloor$, $\theta \in (0,1)$, ν is given by (2.8), $\mathcal{C} \neq \mathcal{C}(m,f)$ and $c \neq c(m,f)$.

We are going to study the behaviour of the sequence $\{L_m^*(w)\}_m$ in the Sobolev spaces $W_r^2(\sqrt{w})$, which is interesting in different contexts.

To this regard, we observe that, since no results concerning the sequence of the Fourier sum $\{S_m(w)\}_m$ are known, we cannot deduce the behaviour of $\{L_m^*(w)\}_m$ from that of $\{S_m(w)\}_m$. Therefore, we need a different approach.

The following theorem describes the behaviour of the operator $L_m^*(w)$ in different function spaces.

Theorem 4.3 Assume that $f \in L^2_{\sqrt{w}}$ and

$$\int_0^1 \frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^{3/2}} \mathrm{d}t < +\infty, \quad r \geqslant 1, \tag{4.3}$$

then we have

$$\|[f - L_m^*(w,f)]\sqrt{w}\|_2 \leqslant \mathcal{C}\left\{\left(\frac{\sqrt{a_m}}{m}\right)^{1/2} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^{3/2}} \mathrm{d}t + e^{-cm^\nu}\|f\sqrt{w}\|_2\right\}, \tag{4.4}$$

where ν is given by (2.8) and the constants $\mathcal{C}, c$ are independent of m and f.

Note that the assumption (4.3) implies $f \in C^0(\mathbb{R}_+)$ (see Mastroianni & Notarangelo, 2013c).

The error estimate (4.4) has interesting consequences.

Firstly, if

$$\sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^s}\,\mathrm{d}t < +\infty, \qquad r > s > 1/2,$$

i.e., $f \in Z_s^2(\sqrt{w})$, then the order of convergence of the process is $\mathcal{O}((\sqrt{a_m}/m)^s)$. If $f \in W_r^2(\sqrt{w})$, $r \geqslant 1$ is integer, we have

$$\|[f - L_m^*(w,f)]\sqrt{w}\|_2 \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r^2(\sqrt{w})}. \tag{4.5}$$

This means that the process converges with the error of the best approximation for the considered classes of functions.

Secondly, we are now able to show the uniform boundedness of the sequence $\{L_m^*(w)\}$ in the Sobolev spaces.

THEOREM 4.4 With the previous notation, for any $f \in W_r^2(\sqrt{w})$, $r \geqslant 1$, we have

$$\sup_m \|L_m^*(w,f)\|_{W_r^2(\sqrt{w})} \leqslant \mathcal{C}\|f\|_{W_r^2(\sqrt{w})}, \quad \mathcal{C} \neq \mathcal{C}(f). \tag{4.6}$$

Moreover, for any $f \in W_s^2(\sqrt{w})$, $s > r$, we have

$$\|f - L_m^*(w,f)\|_{W_r^2(\sqrt{w})} \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{s-r} \|f\|_{W_s^2(\sqrt{w})}, \quad \mathcal{C} \neq \mathcal{C}(m,f). \tag{4.7}$$

REMARK 4.5 In all the estimates for $e_m^*(f)$ and $(f - L_m^*(w,f))$, a constant $\mathcal{C} \neq \mathcal{C}(m,f)$ appears. We have not pointed out the dependence on the parameter $\theta \in (0,1)$, since θ is fixed. Nevertheless, it is useful to observe that $\mathcal{C} = \mathcal{C}(\theta) = \mathcal{O}((\theta/(1-\theta))^2)$. So, it is clear that the parameter θ cannot assume the value 0 or 1 and the 'truncation' is necessary in this sense (see Proposition 5.1 for more details).

5. Proofs

First of all, we recall some polynomial inequalities, proved in Mastroianni *et al.* (2013) (see also Levin & Lubinsky, 2001). Let u be the weight in (2.14) and $1 \leqslant p \leqslant +\infty$. Then, for any $P_m \in \mathcal{P}_m$, we have

$$\|P_m^{(r)}\varphi u\|_p \leqslant \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^r \|P_m u\|_p, \quad r \geqslant 1, \tag{5.1}$$

$$\|P_m u\|_q \leqslant \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^{(1/p-1/q)((2\alpha+2)/(2\alpha+1))} \|P_m u\|_p, \quad p < q, \tag{5.2}$$

where $a_m = a_m(w) \sim m^{1/\beta}$ and $\mathcal{C} \neq \mathcal{C}(m, P_m)$.

Let us now prove Theorem 3.2, since Proposition 3.1 easily follows using the arguments in Section 3.

Proof of Theorem 3.2. If f is an infinitely differentiable function, then it is well known that

$$e_m(f) = \frac{1}{(2m)!\gamma_m^2}\int_0^{+\infty} f^{(2m)}(\xi_x)p_m^2(w,x)w(x)\,\mathrm{d}x,$$

where γ_m is the leading coefficient of $p_m(w)$ and ξ_x belongs to the smaller interval containing $x, x_1, \ldots, x_m$. It follows that

$$e_m(f) \leqslant \frac{\|f^{(2m)}u\|_\infty}{(2m)!\gamma_m^2} \int_0^{+\infty} p_m^2(w,x)w(x)u^{-1}(\xi_x)\,\mathrm{d}\,x. \tag{5.3}$$

In order to estimate the integral at the right-hand side, we recall that the zeros of $p_m(w)$ are located as follows:

$$\tilde{\varepsilon}_m < x_1 < x_2 < \cdots < x_m < \tilde{a}_m,$$

with $\tilde{\varepsilon}_m = \varepsilon_{2m} = \varepsilon_{2m}(w)$ and $\tilde{a}_m = a_{2m} = a_{2m}(w)$. Hence, if $x \in (0, \varepsilon_{2m})$, we get $\xi_x \geqslant x$ and since u^{-1} is decreasing, we have $u^{-1}(\xi_x) \leqslant u^{-1}(x)$. Using the restricted range inequality (2.15) related to the weight w/u, with $n = 2m/(1-a) + \lceil|\delta|\rceil$, we obtain

$$\begin{aligned}\int_0^{\varepsilon_{2m}} p_m^2(w,x)w(x)u^{-1}(\xi_x)\,\mathrm{d}\,x &\leqslant \int_0^{\varepsilon_{2m}} p_m^2(w,x)w(x)u^{-1}(x)\,\mathrm{d}\,x\\ &\leqslant \mathcal{C}\int_{\varepsilon_n}^{a_n} p_m^2(w,x)w(x)u^{-1}(x)\,\mathrm{d}\,x\\ &\leqslant \mathcal{C}u^{-1}(a_n).\end{aligned} \tag{5.4}$$

Analogously, one can show that

$$\int_{a_{2m}}^{+\infty} p_m^2(w,x)w(x)u^{-1}(\xi_x)\,\mathrm{d}\,x \leqslant \mathcal{C}u^{-1}(a_n). \tag{5.5}$$

Finally, it is easily seen that

$$\int_{\varepsilon_{2m}}^{a_{2m}} p_m^2(w,x)w(x)u^{-1}(\xi_x)\,\mathrm{d}\,x \leqslant u^{-1}(a_{2m}) \leqslant u^{-1}(a_n). \tag{5.6}$$

Combining (5.3) with (5.6), we obtain

$$\sqrt[2m]{\frac{|e_m(f)|}{K(f)}} \leqslant \mathcal{C}\left[\frac{u^{-1}(a_n)}{(2m)!\gamma_m^2}\right]^{1/2m}, \tag{5.7}$$

where $K(f) = \sup_m \|f^m u\|_\infty$. Since (see Levin & Lubinsky, 2001, p. 25)

$$\gamma_m(w) = \frac{1}{\sqrt{2\pi}}\left(\frac{4}{\tilde{a}_m + \tilde{\varepsilon}_m}\right)^{m+1/2} \exp\left(\frac{1}{\pi}\int_{\tilde{\varepsilon}_m}^{\tilde{a}_m} \frac{Q(x)}{\sqrt{(\tilde{a}_m - x)(x - \tilde{\varepsilon}_m)}}\,\mathrm{d}x\right)(1 + o(1)),$$

where $Q(x) = \frac{1}{2}(1/x^\alpha + x^\beta)$, we have

$$\left(\frac{1}{\gamma_m^2}\right)^{1/2m} = \mathcal{O}(a_m), \quad (u^{-1}(a_n))^{1/2m} = \mathcal{O}(e^{a_n^\beta/(2m)}) = \mathcal{O}(1), \quad \left(\frac{1}{(2m)!}\right)^{1/2m} = \mathcal{O}(1/m).$$

Hence, the term at the right-hand side of (5.7) is $\mathcal{O}(a_m/m)$ as $m \to +\infty$ and our claim follows. □

PROPOSITION 5.1 Let w be the weight in (2.1), $\tilde{\varepsilon}_m$ and $\tilde{a}_m$ the MRS numbers related to $\sqrt{w}$, and $\theta \in (0,1)$ be fixed. For any $x \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]$ the function

$$\Psi_m(x) = \frac{\tilde{a}_m x}{m\sqrt{(x - \tilde{\varepsilon}_m + \delta_m)(\tilde{a}_m - x + \tilde{a}_m m^{-2/3})}}$$

satisfies

$$\mathcal{C}\frac{\sqrt{a_m}}{m}\sqrt{x} \leqslant \Psi_m(x) \leqslant \mathcal{C}\left(\frac{\theta}{1-\theta}\right)^2 \frac{\sqrt{a_m}}{m}\sqrt{x},$$

where the constants $\mathcal{C}$ are independent of θ and m.

Proof. From Levin & Lubinsky (2001, p. 73), we can easily deduce

$$1 - \frac{\tilde{a}_{\theta m}}{\tilde{a}_m} \geqslant \mathcal{C}\left(\frac{1}{\theta} - 1\right)^2 \quad \text{and} \quad \frac{\tilde{\varepsilon}_{\theta m}}{\tilde{\varepsilon}_m} - 1 \geqslant \mathcal{C}\tilde{\varepsilon}_{\theta m}\left(\frac{1}{\theta} - 1\right)^2,$$

whence the proposition follows. □

Proof of Theorem 3.3. Let us first prove inequality (3.5). By the Peano theorem, we have

$$e_m(f) = \int_0^{+\infty} e_m(\Gamma_t) f'(t)\,\mathrm{d}t, \quad \Gamma_t(x) = (x-t)_+^0 = \begin{cases} 1, & x > t, \\ 0, & x \leqslant t, \end{cases} \tag{5.8}$$

with

$$e_m(\Gamma_t) = \int_0^{+\infty} \Gamma_t(x) w(x)\,\mathrm{d}x - \sum_{k=1}^{m} \lambda_k(w)\Gamma_t(x_k).$$

It is easily seen that (see Freud, 1971, p. 105)

$$e_m(\Gamma_t) \begin{cases} = -\displaystyle\int_0^t w(x)\,\mathrm{d}x, & 0 < t \leqslant x_1, \\ \leqslant \lambda_m(w,t), & x_1 \leqslant t \leqslant x_m, \\ = \displaystyle\int_t^{+\infty} w(x)\,\mathrm{d}x, & t \geqslant x_m. \end{cases}$$

For $0 < t \leqslant x_1$, since $x_1 \sim \tilde{\varepsilon}_m$ and by (2.4), we have

$$\int_0^t w(x)\,\mathrm{d}x \leqslant \mathcal{C}\int_0^t e^{-x^{-\alpha}}\,\mathrm{d}x \leqslant \mathcal{C}t^{\alpha+1}\int_0^t \mathrm{d}(e^{-x^{-\alpha}}) \leqslant \mathcal{C}\tilde{\varepsilon}_m^{\alpha+1/2}\sqrt{t}\,e^{-t^{-\alpha}} \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\varphi(t)w(t). \tag{5.9}$$

Analogously, for $t \geqslant x_m$, since $x_m \sim \tilde{a}_m$ and again by (2.4), we obtain

$$\int_0^t w(x)\,\mathrm{d}x \leqslant \mathcal{C}\int_t^{+\infty} e^{-x^{\beta}}\,\mathrm{d}x \leqslant \mathcal{C}t^{1-\beta}\int_t^{+\infty} \mathrm{d}(e^{-x^{\beta}}) \leqslant \mathcal{C}\tilde{a}_m^{1/2-\beta}\sqrt{t}\,e^{-t^{\beta}} \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\varphi(t)w(t). \tag{5.10}$$

Now, let $x_1 \leqslant t \leqslant x_m$ and $\theta \in (0,1)$ be fixed. From (2.13) and Proposition 5.1 we deduce

$$\lambda_m(w,t) \sim \begin{cases} \dfrac{\sqrt{a_m}}{m}\sqrt{\dfrac{t}{t-\tilde{\varepsilon}_m}}\varphi(t)w(t), & t \in [x_1, \tilde{\varepsilon}_{\theta m}], \\ \dfrac{\sqrt{a_m}}{m}\varphi(t)w(t), & t \in [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}], \\ \dfrac{\sqrt{a_m}}{m}\sqrt{\dfrac{\tilde{a}_m}{\tilde{a}_m - t}}\varphi(t)w(t), & t \in [\tilde{a}_{\theta m}, x_m], \end{cases}$$

where, for $t \in [x_1, \tilde{\varepsilon}_{\theta m}]$,

$$\sqrt{\frac{t}{t-\tilde{\varepsilon}_m}} \leqslant \sqrt{\frac{\tilde{\varepsilon}_{\theta m}}{x_1 - \tilde{\varepsilon}_m}} \sim m^{\frac{1}{3}(1-1/2\beta)(2\alpha/(2\alpha+1))}$$

and, for $t \in [\tilde{a}_{\theta m}, x_m]$,

$$\sqrt{\frac{\tilde{a}_m}{\tilde{a}_m - t}} \leqslant \sqrt{\frac{\tilde{a}_m}{\tilde{a}_m - x_m}} \sim m^{1/3}$$

by (2.9), (2.10) and (2.4). Whence, by (5.8–5.10), we obtain inequality (3.5).

In order to prove (3.6), we consider the function

$$f_m(x) = \begin{cases} 0, & 0 \leqslant x \leqslant y_m, \\ x - y_m, & y_m \leqslant x \leqslant x_m, \\ \dfrac{\sqrt{a_m}}{m}\sqrt{x_m}, & x \geqslant x_m, \end{cases}$$

where $y_m := x_m - (\sqrt{a_m}/m)\sqrt{x_m}$. Of course, $f_m \in W_1^1(w)$ and

$$0 < \|f_m'\varphi w\|_1 = \int_{y_m}^{x_m} \varphi(x)w(x)\,\mathrm{d}x < +\infty.$$

Using the expression of the Peano remainder, we obtain

$$e_m(f_m) = \int_{y_m}^{x_m}\left[\int_t^{+\infty} w(x)\,\mathrm{d}x - \lambda_m(w)\right] f_m'(t)\,\mathrm{d}t.$$

By (2.13) and (2.10), we have

$$\lambda_m(w) \sim \Delta x_{m-1} w(x_m) \sim \Delta x_{m-1} w(x_m) \geqslant \mathcal{C} m^{1/3}\frac{\sqrt{a_m}}{m}\varphi(t)w(t), \quad t \in (y_m, x_m),$$

since $w(t) \sim w(x_m)$ and $\varphi(t) \sim \varphi(x_m)$ for $|x_m - t| \leqslant (\sqrt{a_m}/m)\sqrt{x_m}$ (see Mastroianni & Notarangelo, 2013b). On the other hand, we have already proved that

$$\int_t^{+\infty} w(x)\,\mathrm{d}x \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\varphi(t)w(t).$$

Hence, for a sufficiently large m, we obtain

$$|e_m(f_m)| = \int_{y_m}^{x_m} \left[\lambda_m(x) - \int_t^{+\infty} w(x)\,\mathrm{d}x\right] |f_m'(t)|\,\mathrm{d}t \geqslant \mathcal{C} m^{1/3} \frac{\sqrt{a_m}}{m} \|f_m' \varphi w\|_1,$$

i.e., (3.6). □

From the proof, it seems to be clear that the extra factor $m^{1/3}$ in (3.5) is due to the formula (2.10), i.e., the distance between two consecutive zeros of $p_m(w)$ close to $\tilde{a}_m$. So, taking also into account (2.9), in order to obtain estimates of the form (3.4), the 'truncation' of the terms related to the zeros closest to $\tilde{a}_m$ and $\tilde{\varepsilon}_m$ seems to be necessary.

Proof of Proposition 3.4. In order to prove inequality (3.9), let $P \in \mathcal{P}_M$, $M = \lfloor (2\theta/(\theta+1))m \rfloor$ be the polynomial of best approximation of $f \in C_u$, where $w/u \in L^1$. So, we can write

$$e_m^*(f) = e_m^*(f - P) + e_m^*(P).$$

For the second term at the right-hand side, by (2.13) and (2.16), we have

$$\begin{aligned} e_m^*(P) &= \int_0^{+\infty} P(x) w(x)\,\mathrm{d}x - \sum_{k=j_1}^{j_2} \lambda_k(w) P(x_k) = \sum_{k=1}^{j_1} \lambda_k(w) P(x_k) + \sum_{k=j_2}^{m} \lambda_k(w) P(x_k) \\ &\leqslant \mathcal{C} \max_{x \in \mathbb{R}_+ \setminus [\tilde{\varepsilon}_{\theta m}, \tilde{a}_{\theta m}]} |P(x) u(x)| \int_0^{+\infty} \frac{w(x)}{u(x)}\,\mathrm{d}x \leqslant \mathcal{C}\, e^{-cm^\nu} \|Pu\|_\infty \leqslant \mathcal{C}\, e^{-cm^\nu} \|fu\|_\infty, \end{aligned}$$

where ν is given by (2.8).

While, for the first term we obtain

$$\begin{aligned} |e_m^*(f - P)| &\leqslant \int_0^{+\infty} |f - P|(x) w(x)\,\mathrm{d}x + \sum_{k=j_1}^{j_2} \lambda_k(w) |f - P|(x_k) \\ &\leqslant \mathcal{C} \|(f - P)u\|_\infty \int_0^{+\infty} \frac{w(x)}{u(x)}\,\mathrm{d}x \\ &\leqslant \mathcal{C} E_M(f)_{u,\infty}, \end{aligned}$$

and inequality (3.9) follows.

Finally, to prove inequality (3.10), it suffices to apply (3.9) and (2.19). □

Proof of Theorem 3.5. We can write the error of the truncated Gaussian rule as

$$e_m^*(f) = e_m(f) + \sum_{k=1}^{j_1} \lambda_k(w) f(x_k) + \sum_{k=j_2}^{m} \lambda_k(w) f(x_k), \tag{5.11}$$

where the error of the complete Gaussian rule

$$e_m(f) = \int_0^{+\infty} f(x)w(x)\,\mathrm{d}x - \sum_{k=1}^{m} \lambda_k(w)f(x_k) = \frac{1}{(2m)!\gamma_m^2} \int_0^{+\infty} f^{(2m)}(\xi_x)p_m^2(w,x)w(x)\,\mathrm{d}x$$

has been estimated in the proof of Theorem 3.2.

For the first sum at the right-hand side of (5.11), we have

$$\sum_{k=1}^{j_1} \lambda_k(w)|f(x_k)| \leqslant \mathcal{C}\|fu\|_\infty \sum_{k=1}^{j_1} \Delta x_k(w)\frac{w(x_k)}{u(x_k)} \leqslant \mathcal{C}\|fu\|_\infty \int_{\tilde{\varepsilon}_m}^{\tilde{\varepsilon}_{\bar{\theta}m}} \frac{w(x)}{u(x)}\,\mathrm{d}x$$

$$\leqslant \mathcal{C}\|fu\|_\infty \int_{\tilde{\varepsilon}_m}^{\tilde{\varepsilon}_{\bar{\theta}m}} e^{-(1-a)x^{-\alpha}}\,\mathrm{d}x,$$

where $0 < \theta < \bar{\theta} < 1$. Now, the integrand function in the last integral is decreasing and it is bounded by

$$e^{-(1-a)/\tilde{\varepsilon}_m^{\alpha}} \leqslant \mathcal{C}\, e^{-c(1-a)m^{(1-1/2\beta)(\alpha/(\alpha+1/2))}}.$$

Then, for any $\mu \in (0,1)$, we obtain

$$\lim_m \left(\sum_{k=1}^{j_1} \lambda_k(w)f(x_k)\right)^{1/m^\mu} = 0.$$

We can proceed in an analogous way to show that

$$\lim_m \left(\sum_{k=j_2}^{m} \lambda_k(w)f(x_k)\right)^{1/m^\mu} = 0.$$

Hence, taking also into account the proof of Theorem 3.2, we get (3.11). □

In order to prove Theorem 3.6, we need the following proposition.

PROPOSITION 5.2 Let u be the weight in (2.14) and $P_m \in \mathcal{P}_m$ be a polynomial of quasi best approximation for $f \in L_u^p$, $1 \leqslant p \leqslant +\infty$, i.e.,

$$\|(f - P_m)u\|_p \leqslant \mathcal{C}E_m(f)_{u,p}.$$

Then, if $f \in W_r^p(u)$, $r \geqslant 1$, we have

$$\|(f - P_m)^{(r)}\varphi^r u\|_p \leqslant \mathcal{C}\|f^{(r)}\varphi^r u\|_p. \tag{5.12}$$

Moreover, for any $f \in Z_s^p(u)$, $s > 1$, we have

$$\|(f - P_m)'\varphi u\|_p \leqslant \mathcal{C} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^2}\,\mathrm{d}t, \qquad r \geqslant 2. \tag{5.13}$$

In both cases $\mathcal{C} \neq \mathcal{C}(m,f)$.

Proof. For any $f \in W_r^p(u)$, we can write

$$\|(f - P_m)^{(r)}\varphi^r u\|_p \leqslant \|f^{(r)}\varphi^r u\|_p + \|P_m^{(r)}\varphi^r u\|_p.$$

Recalling also that (see Mastroianni & Notarangelo, 2013b)

$$\|P_m^{(r)}\varphi^r u\|_p \leqslant \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^r \omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} \leqslant \mathcal{C}\|f^{(r)}\varphi^r u\|_p,$$

inequality (5.12) follows.

Now, let $f \in Z_s^p(u)$ and $\{P_m\}_m$ be polynomials of quasi best approximation for $f \in L_u^p$. Then the equality

$$f - P_m = \sum_{k=0}^{+\infty}(P_{2^{k+1}m} - P_{2^k m})$$

holds a.e. in $\mathbb{R}_+$.

Moreover, using the Bernstein inequality (5.1) and the Jackson inequality (2.17), we obtain

$$\begin{aligned}
\|(P_{2^{k+1}m} - P_{2^k m})'\varphi u\|_p &\leqslant \mathcal{C}\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}}\omega_\varphi^r\left(f, \frac{\sqrt{a_{2^k m}}}{2^k m}\right)_{u,p} \\
&\leqslant \mathcal{C}\omega_\varphi^r\left(f, \frac{\sqrt{a_{2^k m}}}{2^k m}\right)_{u,p}\int_{\sqrt{a_{2^{k+1}m}}/(2^{k+1}m)}^{\sqrt{a_{2^k m}}/(2^k m)}\frac{\mathrm{d}t}{t^2} \\
&\leqslant \mathcal{C}\int_{\sqrt{a_{2^{k+1}m}}/(2^{k+1}m)}^{\sqrt{a_{2^k m}}/(2^k m)}\frac{\omega_\varphi^r(f,t)_{u,p}}{t^2}\,\mathrm{d}t.
\end{aligned}$$

Whence, summing on $k \geqslant 0$, we obtain

$$\|(f - P_m)'\varphi u\|_p \leqslant \sum_{k=0}^{+\infty}\|(P_{2^{k+1}m} - P_{2^k m})'\varphi u\|_p,$$

where the series at the right-hand side converges, and then

$$\|(f - P_m)'\varphi u\|_p \leqslant \mathcal{C}\int_0^{\sqrt{a_m}/m}\frac{\omega_\varphi^r(f,t)_{u,p}}{t^2}\,\mathrm{d}t, \qquad r > s > 1,$$

and then (5.13), taking also into account that $\omega_\varphi^r(f,t)_{u,p} \sim \Omega_\varphi^r(f,t)_{u,p}$ for $f \in Z_s^p(u)$. □

Proof of Theorem 3.6. In order to prove inequality (3.12), we first note that

$$\left|\sum_{k=j_1}^{j_2}\lambda_k(w)f(x_k)\right| \leqslant \mathcal{C}\|fw\|_{L^1[x_{j_1},x_{j_2}]} + \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_{L^1[x_{j_1},x_{j_2}]}, \tag{5.14}$$

for any $f \in W_1^1(w)$. In fact, for $j_1 \leqslant k \leqslant j_2 - 1$, we obtain

$$\Delta x_k|f(x_k)|w(x_k) \leqslant \int_{x_k}^{x_{k+1}}|f(x)|w(x)\,\mathrm{d}x + \mathcal{C}\frac{\sqrt{a_m}}{m}\int_{x_k}^{x_{k+1}}|f'(x)|\varphi(x)w(x)\,\mathrm{d}x$$

and

$$\Delta x_{j_2}|f(x_{j_2})|w(x_{j_2}) \leqslant \int_{x_{j_2-1}}^{x_{j_2}} |f(x)|w(x)\,\mathrm{d}x + \mathcal{C}\frac{\sqrt{a_m}}{m}\int_{x_{j_2-1}}^{x_{j_2}} |f'(x)|\varphi(x)w(x)\,\mathrm{d}x,$$

since $w(x) \sim w(y)$ and $\varphi(x) \sim \varphi(y)$ for $x, y \in [x_k, x_{k+1}]$, $j_1 \leqslant k \leqslant j_2 - 1$, i.e., by (2.11), $|x-y| \leqslant \mathcal{C}\sqrt{a_m x_k}/m$ (see Mastroianni & Notarangelo, 2013b). Summing up, on $j_1 \leqslant k \leqslant j_2 - 1$, by (2.13), inequality (5.14) follows.

Let us now prove (3.12), with $f \in W_1^1(w)$. Letting $P \in \mathcal{P}_M$ be the polynomial of best approximation of $f \in L_w^1$, we can write

$$e_m^*(f) = e_m^*(f-P) + e_m^*(P). \tag{5.15}$$

For the second term at the right-hand side, using arguments similar to those in the proof of Proposition 3.4 and the Nikolskii inequality (5.2), we obtain

$$|e_m^*(P)| \leqslant \mathcal{C}\,e^{-cm^\nu}\|Pw\|_\infty \leqslant \mathcal{C}\,e^{-cm^\nu}\|Pw\|_1 \leqslant \mathcal{C}\,e^{-cm^\nu}\|fw\|_1.$$

For the first term in (5.15), using (5.14), we obtain

$$\begin{aligned} |e_m^*(f-P)| &\leqslant \mathcal{C}\|(f-P)w\|_1 + \mathcal{C}\frac{\sqrt{a_m}}{m}\|(f-P)'\varphi w\|_1 \\ &\leqslant \mathcal{C}E_M(f)_{w,1} + \mathcal{C}\frac{\sqrt{a_m}}{m}\|(f-P)'\varphi w\|_1. \end{aligned}$$

Now, for the first term at the right-hand side by the Favard theorem (see Mastroianni & Notarangelo, 2013b), we have

$$E_M(f)_{w,1} \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}E_M(f')_{\varphi w,1} \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1.$$

For the second term at the right-hand side, we use (5.12) and inequality (3.12) follows.

Let us consider the case $f \in Z_s^1(w)$ and prove inequality (3.13). We can proceed in analogy with the first part of this proof, taking into account that, by the weak Jackson inequality (2.18), we obtain

$$E_M(f)_{w,1} \leqslant \mathcal{C}\frac{\sqrt{a_m}}{m}\int_0^{\sqrt{a_m}/m}\frac{\Omega_\varphi^r(f,t)_{w,1}}{t^2}\,\mathrm{d}t.$$

For the second term we can use (5.13) and the proof is complete. □

Proof of Lemma 4.1. Let us consider the Lagrange polynomial $L_m(w,F)$, interpolating a continuous function F at the zeros $x_1,\dots,x_m$. By using the relations (see Levin & Lubinsky, 2001, pp. 22–23)

$$\sup_{x\in\mathbb{R}_+}|p_m(w,x)\sqrt{w(x)}\sqrt[4]{|(x-\tilde{\varepsilon}_m)(\tilde{a}_m-x)|}| \sim 1,$$

$$\sup_{x\in\mathbb{R}_+}|p_m(w,x)\sqrt{w(x)}| \sim m^{\frac{1}{6}(1-1/2\beta)((2\alpha+3)/(2\alpha+1))},$$

$$\frac{1}{|p_m'(w,x_k)|\sqrt{w(x_k)}} \sim \Delta x_k\sqrt[4]{(x_k-\tilde{\varepsilon}_m)(\tilde{a}_m-x_k)},$$

we can easily deduce

$$\|L_m(w,F)\sigma\|_\infty \leqslant \mathcal{C}m^\tau \|F\sigma\|_\infty, \quad F \in C_\sigma, \tag{5.16}$$

for some $\tau > 0$, where $\mathcal{C} \neq \mathcal{C}(m,f)$.

Now, let P_M, $M = \lfloor(\theta/(\theta+1))m\rfloor$, be the polynomial of best approximation for $f \in L^p_\sigma$ and $Q_{m-1} = \mathcal{L}^*_m(w,P_M) \in \mathcal{P}^*_{m-1}$. Hence, we obtain

$$\begin{aligned}\|(f-Q_{m-1})\sigma\|_p &\leqslant \|(f-P_M)\sigma\|_p + \|(P_M - Q_{m-1})\sigma\|_p \\ &= \|(f-P_M)\sigma\|_p + \|[L_m(\sigma,P_M) - \mathcal{L}^*_m(\sigma,P_M)]\sigma\|_p \\ &= E_M(f)_{\sigma,p} + \left\| \sum_{x_k \notin Z_{\theta,m}} \ell_k(w) P_M(x_k)\sigma \right\|_p,\end{aligned}$$

where $\ell_k(w)$ are the fundamental Lagrange polynomials based on the zeros $x_1,\ldots,x_m$.

For the second summand at the right-hand side, by using inequalities (5.16) and (2.16), we have

$$\left\| \sum_{x_k \notin Z_{\theta,m}} \ell_k(w) P_M(x_k)\sigma \right\|_p \leqslant \mathcal{C}m^\tau \|P_M\sigma\|_{L^\infty\{x \notin [\tilde{\varepsilon}_{\theta m},\tilde{a}_{\theta m}]\}} \leqslant \mathcal{C}\, e^{-cM^\nu} \|P_M\sigma\|_\infty.$$

Hence, using the Nikolskii inequality (5.2), the proof is complete. □

In order to prove Theorem 4.3, we will need the following lemma.

LEMMA 5.3 For any $f \in C(\mathbb{R}_+) \cap L^2_{\sqrt{w}}$, we have

$$\left(\sum_{k=j_1}^{j_2} \lambda_k(w) f^2(x_k)\right)^{1/2} \leqslant \mathcal{C}\left[\|f\sqrt{w}\|_2 + \left(\frac{\sqrt{a_m}}{m}\right)^{1/2} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi(f,t)_{\sqrt{w},2}}{t^{3/2}}\,\mathrm{d}t\right],$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

Proof. Inequalities analogous to the previous one have been proved in several contexts with different weight function (see, e.g., Mastroianni & Russo, 1999; Mastroianni & Vértesi, 2006; Mastroianni & Notarangelo, 2013a). For the reader's convenience, we will show the main steps of the proof.

Let $A = [a, a+\delta]$. For any continuous function on A, the inequality

$$\delta^{1/2} \max_{x\in A} |f(x)| \leqslant \mathcal{C}\left[\|f\|_{L^2(A)} + \delta^{1/2}\int_0^\delta \frac{\omega(f,t)_{L^2(A)}}{t^{1+1/2}}\,\mathrm{d}t\right],$$

holds (Ivanov, 1986), where ω denotes the ordinary modulus of smoothness.

So, with $j_1 \leqslant k \leqslant j_2$, $A = I_k = [x_k, x_{k+1}]$ and $\delta = \Delta x_k \sim (\sqrt{a_m}/m)\sqrt{x_k}$, we obtain

$$(\Delta x_k)^{1/2}|f(x_k)| \leqslant \mathcal{C}\left[\|f\|_{L^2(I_k)} + (\Delta x_k)^{1/2}\int_0^{(\sqrt{a_m}/m)\sqrt{x_k}} \frac{\omega(f,t)_{L^2(I_k)}}{t^{1+1/2}}\,\mathrm{d}t\right].$$

Taking into account that $w(x) \sim w(x_k)$ if $x \in I_k$ (see Mastroianni & Notarangelo, 2013b), making some simple computation, for $j_1 \leqslant k \leqslant j_2$, we obtain

$$\Delta x_k |f(x_k)\sqrt{w(x_k)}|^2 \leqslant \mathcal{C}\left[\|f\sqrt{w}\|^2_{L^2(I_k)} + \left(\frac{\sqrt{a_m}}{m}\right)\int_0^{(\sqrt{a_m}/m)\sqrt{x_k}} \frac{\omega(f,t\sqrt{x_k})_{\sqrt{w},I_k}}{t^{3/2}}\,\mathrm{d}t\right],$$

where

$$\omega(f,t\sqrt{x_k})_{\sqrt{w},I_k} = \sup_{0<h\leqslant t\sqrt{x_k}} \|\Delta_h(f)\sqrt{w}\|_{L^2(I_k)}.$$

Moreover, for any $g \in W_1^2(\sqrt{w})$, we have

$$\begin{aligned}\omega(f,t\sqrt{x_k})_{\sqrt{w},I_k} &\leqslant \mathcal{C}[\|(f-g)\sqrt{w}\|_{L^2(I_k)} + t\sqrt{x_k}\|g'\sqrt{w}\|_{L^2(I_k)}]\\ &\leqslant \mathcal{C}[\|(f-g)\sqrt{w}\|_{L^2(I_k)} + t\|g'\varphi\sqrt{w}\|_{L^2(I_k)}]\\ &=: A_k(t)\end{aligned}$$

and then

$$\Delta x_k |f(x_k)\sqrt{w(x_k)}|^2 \leqslant \mathcal{C}\left[\|f\sqrt{w}\|^2_{L^2(I_k)} + \left(\frac{\sqrt{a_m}}{m}\right)\int_0^{(\sqrt{a_m}/m)\sqrt{x_k}} \frac{A_k(t)}{t^{3/2}}\,\mathrm{d}t\right].$$

Summing up, on $j_1 \leqslant k \leqslant j_2$ and using the Buniakovski inequality (see Hardy *et al.*, 1952, Theorem 201, p. 148)

$$\left(\sum_k \left(\int f_k(x)\,\mathrm{d}x\right)^p\right)^{1/p} \leqslant \int \left(\sum_k f_k^p(x)\right)^{1/p} \mathrm{d}x, \quad 1 < p < +\infty,$$

we obtain

$$\left(\sum_{k=j_1}^{j_2} \Delta x_k |f(x_k)\sqrt{w(x_k)}|^2\right)^{1/2} \leqslant \mathcal{C}\left\{\|f\sqrt{w}\|_{L^2[x_{j_1},x_{j_2+1}]} + \left(\frac{\sqrt{a_m}}{m}\right)^{1/2}\int_0^{\sqrt{a_m}/m} \frac{(\sum_{k=j_1}^{j_2} A_k(t)^2)^{1/2}}{t^{3/2}}\,\mathrm{d}t\right\}.$$

Since

$$\left(\sum_{k=j_1}^{j_2} A_k(t)^2\right)^{1/2} \leqslant \mathcal{C}[\|(f-g)\sqrt{w}\|_{L^2[x_{j_1},x_{j_2+1}]} + t\|g'\varphi\sqrt{w}\|_{L^2[x_{j_1},x_{j_2+1}]}]$$

and, for $0 < h \leqslant t \leqslant \sqrt{a_m}/m$, we have $[x_{j_1}, x_{j_2+1}] \subset [h^{1/(\alpha+1/2)}, ch^{-1/(\beta-1/2)}]$, taking the infimum on g, we obtain

$$\left(\sum_{k=j_1}^{j_2} A_k(t)^2\right)^{1/2} \leqslant \mathcal{C}K(f,t)_{\sqrt{w},2} \sim \Omega_\varphi(f,t)_{\sqrt{w},2}.$$

Hence, our claim follows, taking also into account that $\lambda_k(w) \sim \Delta x_k w(x_k)$, $j_1 \leqslant k \leqslant j_2$. □

Proof of Theorem 4.3. Let $P \in \mathcal{P}^*_{m-1}$ be the polynomial of best approximation for $f \in L^2_{\sqrt{w}}$. We can write

$$\|[f - L^*_m(w,f)]\sqrt{w}\|_2 \leqslant \|(f-P)\sqrt{w}\|_2 + \|L^*_m(w,f-P)\sqrt{w}\|_2.$$

For the first term at the right-hand side, by Lemma 4.1 and inequality (2.18), we have

$$\begin{aligned}\|(f-P)\sqrt{w}\|_2 &\leqslant \mathcal{C}\{E_M(f)_{\sqrt{w},2} + e^{-cm^\nu}\|f\sqrt{w}\|_2\}\\ &\leqslant \mathcal{C}\left\{\left(\frac{\sqrt{a_m}}{m}\right)^{1/2}\int_0^{\sqrt{a_m}/m}\frac{\Omega^r_\varphi(f,t)_{\sqrt{w},2}}{t^{3/2}}\,\mathrm{d}t + e^{-cm^\nu}\|f\sqrt{w}\|_2\right\}.\end{aligned}$$

For the second term, we obtain

$$\begin{aligned}\|L^*_m(w,f-P)\sqrt{w}\|_2 &= \left(\sum_{k=j_1}^{j_2}\lambda_k(w)(f-P)^2(x_k)\right)^{1/2}\\ &\leqslant \mathcal{C}\left[\|(f-P)\sqrt{w}\|_2 + \left(\frac{\sqrt{a_m}}{m}\right)^{1/2}\int_0^{\sqrt{a_m}/m}\frac{\Omega_\varphi(f-P,t)_{\sqrt{w},2}}{t^{3/2}}\,\mathrm{d}t\right].\end{aligned}$$

Hence, it remains to estimate the last term. Since, using arguments similar to those in Mastroianni & Russo (1999, p. 280), one can show that

$$\int_0^{\sqrt{a_m}/m}\frac{\Omega_\varphi(f-P,t)_{\sqrt{w},2}}{t^{3/2}}\,\mathrm{d}t \leqslant \mathcal{C}\int_0^{\sqrt{a_m}/m}\frac{\Omega^r_\varphi(f,t)_{\sqrt{w},2}}{t^{3/2}}\,\mathrm{d}t,$$

we obtain

$$\|L^*_m(w,f-P)\sqrt{w}\|_2 \leqslant \mathcal{C}\left\{\left(\frac{\sqrt{a_m}}{m}\right)^{1/2}\int_0^{\sqrt{a_m}/m}\frac{\Omega^r_\varphi(f,t)_{\sqrt{w},2}}{t^{3/2}}\,\mathrm{d}t + e^{-cm^\nu}\|f\sqrt{w}\|_2\right\},$$

and inequality (4.4) follows. □

Proof of Theorem 4.4. Let us first prove (4.6). We can write

$$\|L^*_m(w,f)\|_{W^2_r(\sqrt{w})} \leqslant \|f\|_{W^2_r(\sqrt{w})} + \|f - L^*_m(w,f)\|_{W^2_r(\sqrt{w})}.$$

Hence, we have to estimate the last term at the right-hand side. By definition, we have

$$\|f - L^*_m(w,f)\|_{W^2_r(\sqrt{w})} = \|[f - L^*_m(w,f)]\sqrt{w}\|_2 + \|[f - L^*_m(w,f)]^{(r)}\varphi^r\sqrt{w}\|_2.$$

The first summand at the right-hand side can be estimated by (4.5). For the second term, letting P_m be the polynomial of best approximation of f, we obtain

$$\begin{aligned}\|[f - L^*_m(w,f)]^{(r)}\varphi^r\sqrt{w}\|_2 &\leqslant \|[f-P_m]^{(r)}\varphi^r\sqrt{w}\|_2 + \|[P_m - L^*_m(w,f)]^{(r)}\varphi^r\sqrt{w}\|_2\\ &\leqslant \mathcal{C}\|f^{(r)}\varphi^r\sqrt{w}\|_2 + \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^r\{\|[P_m - f]\sqrt{w}\|_2 + \|[f - L^*_m(w,f)]\sqrt{w}\|_2\}\\ &\leqslant \mathcal{C}\|f\|_{W^2_r(\sqrt{w})},\end{aligned}$$

using Proposition 5.2, the Bernstein inequality (5.1) the Jackson inequality (2.17) and the error bound (4.5).

In order to prove (4.7), we can proceed in a similar way. We omit the details. □

6. Computation of the Mhaskar–Rahmanov–Saff numbers

In this section, we discuss a method for computing the MRS numbers $\varepsilon_t = \varepsilon_t(w)$ and $a_t = a_t(w)$ for the weight function (2.1). These numbers are defined by (2.2) and (2.3), where $Q'(x) = -\alpha x^{-\alpha-1} + \beta x^{\beta-1}$.

After the linear transformation

$$x = \frac{\varepsilon_t + a_t}{2} + \frac{a_t - \varepsilon_t}{2}\xi, \quad -1 < \xi < 1,$$

i.e., $x = s(1 + A\xi)$, where

$$\frac{a_t - \varepsilon_t}{a_t + \varepsilon_t} = A, \quad \frac{a_t + \varepsilon_t}{2} = s,$$

we obtain $(a_t - x)(x - \varepsilon_t) = A^2 s^2 (1 - \xi^2)$, and then (2.2) and (2.3) reduce to

$$t = \beta s^{\beta} \psi(A;\beta) - \alpha s^{-\alpha} \psi(A;-\alpha) \tag{6.1}$$

and

$$0 = \beta s^{\beta-1} \psi(A;\beta - 1) - \alpha s^{-\alpha-1} \psi(A;-\alpha - 1), \tag{6.2}$$

respectively, where we introduced the function $\psi : (-1,1) \to \mathbb{R}$ by

$$\psi(A;\gamma) = \frac{1}{\pi} \int_{-1}^{1} \frac{(1 + A\xi)^{\gamma}}{\sqrt{1-\xi^2}}\, \mathrm{d}\xi,$$

with an arbitrary parameter $\gamma \in \mathbb{R}$. This function can be expressed in terms of the hypergeometric function

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

as follows:

$$\psi(A;\gamma) = {}_2F_1\left(\frac{1-\gamma}{2}, -\frac{\gamma}{2}; 1; A^2\right).$$

Here, $(a)_k$ denotes Pochhammer's symbol that is defined by

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad (a)_0 = 1,$$

where Γ is the Euler gamma function.

For given α and β, from (6.2) we obtain

$$s = \left(\frac{\alpha}{\beta} \cdot \frac{\psi(A; -\alpha - 1)}{\psi(A; \beta - 1)}\right)^{1/(\alpha+\beta)}, \tag{6.3}$$

and then, using (6.1) we obtain the following equation for finding A for a given t,

$$G(A; \alpha, \beta) + G(A; -\beta, -\alpha) = t, \tag{6.4}$$

where

$$G(A; \alpha, \beta) = \beta \left(\frac{\alpha}{\beta} \cdot \frac{\psi(A; -\alpha - 1)}{\psi(A; \beta - 1)}\right)^{\beta/(\alpha+\beta)} \psi(A; \beta).$$

Finally, the MRS numbers are given by $a_t = s(1 + a)$ and $\varepsilon_t = s(1 - a)$, where $A = a$ is the unique solution of the nonlinear equation (6.4) in the interval $(0, 1)$ and s is given by (6.3). The corresponding MATHEMATICA code can be given in the following form:

```
MRSNumbers[t_,alpha_,beta_]:= Module[{psi,funG,al,be,ga,A,a,s},
 psi[A_,ga_]:= Hypergeometric2F1[(1-ga)/2,-ga/2,1,A^2];
 funG[A_,al_,be_]:=be(al/be psi[A,-al-1]/psi[A,be-1])^(be/(al+be))
 psi[A,be];
 a=A/. FindRoot[funG[A,alpha,beta]+funG[A,-beta,-alpha]==t,
 {A,1-10^(-6)}];
 s=(alpha/beta psi[a,-alpha-1]/psi[a,beta-1])^(1/(alpha+beta));
Return[{s(1+a),s(1-a),a,s}]];
```

As a starting value for solving the nonlinear equation (6.4) we can take some value very close to 1, for example, $1 - 10^{-6}$.

The MRS numbers a_t and ε_t, for $\alpha = 1$ and $\beta = 1.2$, 1.4 and 1.9 are displayed in Fig. 1. The cases when $\beta = 1.5$ and $\alpha = 0.8$, 1 and 1.5 are presented in Fig. 2. Note that the graphs of a_t then almost coincide.

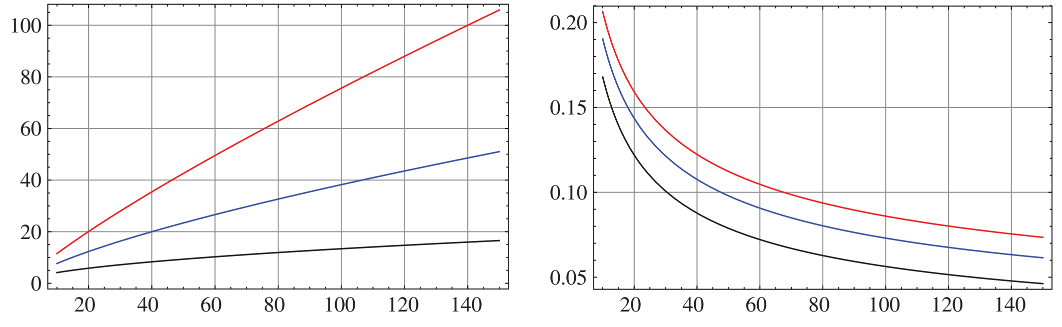

FIG. 1. The MRS numbers a_t (left) and ε_t (right) for $\alpha = 1$ and $\beta = 1.2$ (red line), $\beta = 1.4$ (blue line) and $\beta = 1.9$ (black line).

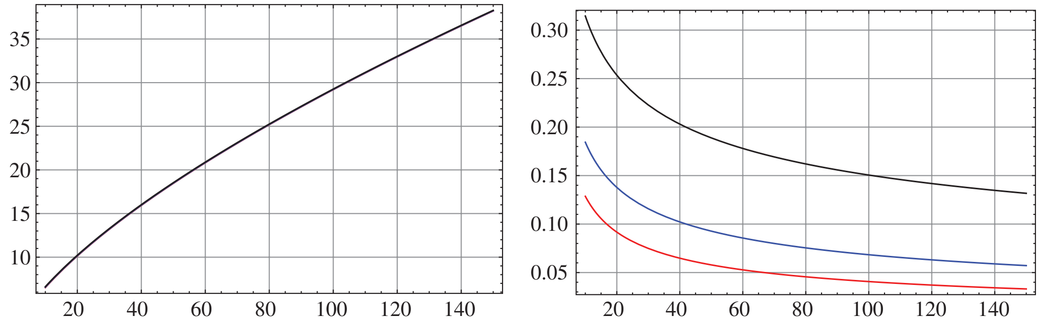

FIG. 2. The MRS numbers a_t (left) and ε_t (right) for $\beta = 1.5$ and $\alpha = 0.8$ (red line), $\alpha = 1$ (blue line) and $\alpha = 1.5$ (black line).

7. Numerical construction of quadrature rules

In this section, we consider numerical construction of the Gaussian quadrature formulae with respect to the weight function $w(x) = w^{(\alpha,\beta)}(x) = e^{-x^{-\alpha}-x^{\beta}}$ on $\mathbb{R}_+$. Thus, we want to construct the parameters of m-point Gaussian quadrature,

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x = \sum_{k=1}^{m} \lambda_k f(x_k) + e_m(f), \tag{7.1}$$

the nodes $x_k = x_{m,k}(w)$ and Christoffel numbers $\lambda_k = \lambda_{m,k}(w)$, for an arbitrary $m \leqslant n$. In such a procedure, we need the moments

$$\mu_k = \mu_k^{(\alpha,\beta)} = \int_0^{+\infty} x^k w^{(\alpha,\beta)}(x)\,\mathrm{d}x, \quad k = 0, 1, \ldots, 2n-1, \tag{7.2}$$

in order to construct the recursive coefficients $\alpha_k = \alpha_k(w^{(\alpha,\beta)})$ and $\beta_k = \beta_k(w^{(\alpha,\beta)})$, $k \leqslant n-1$, in the three-term recurrence relation for the corresponding (monic) orthogonal polynomials $\pi_k(x) = \pi_k(w^{(\alpha,\beta)}; x)$,

$$\pi_{k+1}(x) = (x - \alpha_k)\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0, 1, \ldots, n-1, \tag{7.3}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$. In that way, we have access to all polynomials $\pi_k(w^{(\alpha,\beta)}; x)$ of degree at most n and a possibility for constructing Gaussian rules for every $m \leqslant n$ points. Usually, the Chebyshev method of moments (or modified moments) is not applicable in a standard machine arithmetic for a sufficiently large m, since the process is ill-conditioned, especially for the weights on the infinite intervals as in our case. Then, a construction of recursive coefficients must be carefully realized by an application of the discretized Stieltjes–Gautschi procedure (Gautschi, 1982). However, recent progress in symbolic computation and variable–precision arithmetic now makes it possible to generate the coefficients α_k and β_k in the three-term recurrence relation (7.3) directly by using the standard method of moments in sufficiently high precision. Respectively, symbolic/variable–precision software for orthogonal polynomials is available (Gautschi's package `SOPQ` in MATLAB (see Gautschi, 2004, 2006) and the MATHEMATICA package `OrthogonalPolynomials` (see Cvetković & Milovanović, 2004; Milovanović & Cvetković, 2011)). Thus, all that is required is a procedure for (symbolic) calculation of the moments in variable–precision arithmetic.

Thus, in order to overcome the numerical instability in the procedure for generating the recursion coefficients α_k and β_k in the MATHEMATICA package `OrthogonalPolynomials`,

```
{alpha,beta}=aChebyshevAlgorithm[moments, WorkingPrecision-> WP],
```
(7.4)

we must put *WP* to be sufficiently large, so that the relative errors in these coefficients satisfy

$$\left|\frac{\Delta\alpha_k}{\alpha_k}\right| < \varepsilon, \quad \left|\frac{\Delta\beta_k}{\beta_k}\right| < \varepsilon, \quad k = 0, 1, \ldots, n-1, \tag{7.5}$$

where ε is the required accuracy. The list of moments (`moments`) contains $2n$ elements and it can be given in a symbolic form.

In the case $\alpha = \beta$, i.e., $w(x) = w^{(\alpha,\alpha)} = e^{-x^{-\alpha}-x^{\alpha}}$ on $\mathbb{R}_+$, we can calculate the moments (7.2) in the form

$$\mu_k = \mu_k^{(\alpha,\alpha)} = \int_0^{+\infty} x^k w(x)\,\mathrm{d}x = \frac{2}{\alpha} K_{(k+1)/\alpha}(2), \quad k = 0, 1, 2, \ldots, \tag{7.6}$$

where $K_r(z)$ is the modified Bessel function of the second kind. In the MATHEMATICA Package this function is implemented as `BesselK[r,z]`, and its value can be evaluated with an arbitrary precision.

The case $\alpha \neq \beta$ is more complicated, especially for symbolic computations, but in some cases for integer (or rational) values of parameters, the moments can be expressed in terms of the Meijer G-function. In a standard case, the Meijer G-function is defined as (see Bateman & Erdélyi, 1981, p. 207)

$$\begin{aligned} G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) &\equiv G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_n; a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m; b_{m+1}, \ldots, b_q \end{matrix}\right) \\ &= \frac{1}{2\pi\mathrm{i}} \int_L \frac{\prod_{\nu=1}^m \Gamma(b_\nu - s) \prod_{\nu=1}^n \Gamma(1 - a_\nu + s)}{\prod_{\nu=m+1}^q \Gamma(1 - b_\nu + s) \prod_{\nu=n+1}^p \Gamma(a_\nu - s)} z^s \mathrm{d}s, \end{aligned}$$

where an empty product is interpreted as 1, $1 \leqslant m \leqslant q$, $1 \leqslant n \leqslant p$, and parameters a_ν and b_ν are such that no pole of $\Gamma(b_\nu - s)$, $\nu = 1, \ldots, m$, coincides with any pole of $\Gamma(1 - b_\mu + s)$, $\mu = 1, \ldots, n$. Roughly speaking, the contour L separates the poles of functions $\Gamma(b_1 - s), \ldots, \Gamma(b_m - s)$ from the poles of $\Gamma(1 - a_1 + s), \ldots, \Gamma(1 - a_n + s)$. A discussion on three different paths of integration is given in Bateman & Erdélyi (1981, p. 207). An alternative equivalent definition of the Meijer G-function can be done in terms of inverse Mellin transform (see Prudnikov *et al.*, 1990, p. 793). The Meijer G-function is a very general function which reduces to simpler special functions in many common cases. In MATHEMATICA, the Meijer G-function is implemented as

```
MeijerG[{{a1,...,an},{an1,,...,ap}},{{b1,...,bm},{bm1,...,bq}},z]
```

and it is suitable for both symbolic and numerical manipulation and its value can be evaluated with an arbitrary precision. In many special cases, `MeijerG` is automatically converted to other functions.

For only some specific values of α and β, we mention here the corresponding moments $\mu_k^{(\alpha,\beta)}$ expressed in terms of the Meijer G-function. For example,

$$\mu_k^{(1,2)} = \frac{1}{2^{k+2}\sqrt{\pi}} G_{2,4}^{3,1}\left(\frac{1}{4} \,\middle|\, \begin{matrix} -;- \\ -\dfrac{k+1}{2}, -\dfrac{k}{2}, 0; - \end{matrix}\right), \quad k \geqslant 0 \tag{7.7}$$

and

$$\mu_k^{(2,1)} = \frac{2^k}{\sqrt{\pi}} G_{2,4}^{3,1}\left(\frac{1}{4}\left|\begin{matrix} -;- \\ 0, \dfrac{k+1}{2}, \dfrac{k+2}{2}; - \end{matrix}\right.\right), \quad k \geqslant 0.$$

For $\alpha = 1$ and $\beta = 3$, we have

$$\mu_k^{(1,3)} = \frac{1}{2 \cdot 3^{k+3/2}\pi} G_{2,5}^{4,1}\left(\frac{1}{27}\left|\begin{matrix} -;- \\ -\dfrac{k+1}{3}, -\dfrac{k}{3}, -\dfrac{k-1}{3}, 0; - \end{matrix}\right.\right), \quad k \geqslant 0,$$

while for $\alpha = 3$ and $\beta = 1$,

$$\mu_k^{(3,1)} = \frac{3^{k+1/2}}{2\pi} G_{2,5}^{4,1}\left(\frac{1}{27}\left|\begin{matrix} -;- \\ 0, \dfrac{k+1}{3}, \dfrac{k+2}{3}, \dfrac{k+3}{3}; - \end{matrix}\right.\right), \quad k \geqslant 0.$$

In the case of rational parameters, $\alpha = 1/2$ and $\beta = 3/2$, the moments are

$$\mu_k^{(1/2,3/2)} = \frac{1}{3^{2k+5/2}\pi} G_{2,5}^{4,1}\left(\frac{1}{27}\left|\begin{matrix} -;- \\ -\dfrac{2k+2}{3}, -\dfrac{2k+1}{3}, -\dfrac{2k}{3}, 0; - \end{matrix}\right.\right), \quad k \geqslant 0,$$

and for $\alpha = 1/3$ and $\beta = 3/2$, we have $z = 1/1549681956 = 1/(4 \cdot 9^9)$ and $b_\nu = b_\nu^{(k)} = -(3k+4-\nu)/9$, $\nu = 1, \dots, 9$, $b_{10} = 0$, $b_{11} = \frac{1}{2}$, so that

$$\mu_k^{(1/3,3/2)} = \frac{1}{16 \cdot 3^{6k+6}\pi^{9/2}} G_{2,12}^{11,1}\left(\frac{1}{4 \cdot 9^9}\left|\begin{matrix} -;- \\ b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, 0, \frac{1}{2}; - \end{matrix}\right.\right).$$

8. Numerical examples

In this section, we first consider the numerical construction of Gaussian quadratures with respect to the weight function $w(x) = w^{(2,2)}(x) = e^{-x^{-2}-x^2}$ on $\mathbb{R}_+$, and then apply them to calculating some integrals.

In order to generate quadratures for $m \leqslant n = 300$, we need the first 600 moments $\mu_k = \mu_k^{(2,2)}$, given by (7.6) or in the MATHEMATICA package `OrthogonalPolynomials` (see Cvetković & Milovanović, 2004; Milovanović & Cvetković, 2011) with the command

```
moments=Table[BesselK[(k+1)/2,2], {k,0,600}].
```

Suppose that we need a very high precision in quadrature parameters, e.g., 70 decimal digits. Then, in order to overcome the numerical instability in the procedure for generating the recursion coefficients α_k and β_k, we must put `WP = 400` in (7.4). It gives all recursive coefficients for $k < n = 300$, with relative errors $\varepsilon < 10^{-72}$ (see (7.5)), which is easy to check by taking some bigger `WorkingPrecision`, for example, `WP = 500`. As we can see, the calculation of the recursive coefficients is a very sensitive process, which here, in the worst case, causes a loss of about 328 decimal digits! Note that in the case `WP = 500`, the all recursive coefficients are determined with $\varepsilon < 10^{-172}$. But, when we need, for example, only the first 100 coefficients, with $\varepsilon < 10^{-52}$, then it is enough to put `WP = 150`. In that case, the loss is about 98 decimal digits.

TABLE 1 *Relative errors in Gaussian sums for* $m = 5(5)60$

m	$r_m(f_1)$	$r_m(f_2)$
5	2.36(−6)	1.18(−10)
10	1.96(−10)	5.77(−19)
15	4.68(−14)	9.58(−24)
20	2.14(−17)	8.39(−30)
25	1.55(−20)	1.05(−34)
30	1.58(−23)	4.45(−40)
35	2.11(−26)	4.52(−45)
40	3.56(−29)	2.71(−49)
45	7.26(−32)	1.09(−53)
50	1.75(−34)	4.92(−58)
55	4.89(−37)	2.74(−62)
60	1.55(−39)	1.95(−66)

EXAMPLE 8.1 We apply Gaussian quadratures to calculating the integrals $I(f) = \int_0^{+\infty} f(x)\, e^{-1/x^2-x^2}\, \mathrm{d}x$ for

$$f(x) = f_1(x) = \cosh\left(\frac{1}{x+1}\right)\cosh(x-1) \quad \text{and} \quad f(x) = f_2(x) = \arctan\left(\frac{1+x}{4}\right).$$

The values of these integrals can be evaluated with a high precision using MATHEMATICA function `NIntegrate`,

$$I(f_1) = 0.14567508123417523466238503493352795784627835 3\ldots,$$
$$I(f_2) = 0.05919060160521161205909757688728518192042075978791293950109922933439 4\ldots.$$

Relative errors in the corresponding Gaussian sums $Q_m(f)$, given by $r_m(f) = |(Q_m(f) - I)/I(f)|$, are presented in Table 1 for $m = 5(5)60$. Numbers in parentheses indicate decimal exponents, for example $2.36(-6) = 2.36 \times 10^{-6}$. The convergence for both smooth functions is very fast.

EXAMPLE 8.2 We apply now the same quadratures to calculating the corresponding integral for the function $f(x) = |\cos x|^{5/4}$, with $w(x) = e^{-x^{-2}-x^2}$. In the second column of Table 2, we present the relative errors of the Gaussian rule for $m = 10(10)50$ and $m = 100(50)300$. We note that $f \in Z^{\infty}_{5/4}(\sqrt{w})$. So, by Proposition 3.1 and (2.20), the results in Table 2 are concordant with the theoretical order of convergence, that is $m^{-15/16}$.

In the third and fourth column of Table 2, we show the indices j_1 and j_2 (see definition (3.7)) and relative errors obtained for the 'truncated' Gaussian rule with $\theta = 1/10$ and $\theta = 1/20$, respectively. Finally, we compare these results with a different kind of truncation: in the Gaussian rule, we omit the terms with index k such that $\lambda_k(w)|f(x_k)| < \varepsilon$, being ε the precision to be achieved in the computations. In the last column, we show the indices and the relative errors for this kind of truncated rule, choosing $\varepsilon = 10^{-5}$.

For calculating the relative errors, we used as exact value

$$I = \int_0^{+\infty} |\cos x|^{5/4} e^{-x^{-2}-x^2}\, \mathrm{d}x = 0.04552779434634736\ldots$$

TABLE 2 *Relative errors in quadrature sums for $m = 10(10)50$ and $m = 100(50)300$*

	Gaussian rule	$\theta = 1/10$		$\theta = 1/20$		$\varepsilon = 10^{-5}$	
m	$r_m(f)$	(j_1, j_2)	$r_m(j_1, j_2)$	(j_1, j_2)	$r_m(j_1, j_2)$	(j_1, j_2)	$r_m(j_1, j_2)$
10	3.70(−3)	(1, 7)	4.54(−3)	(1, 6)	1.65(−2)	(1, 8)	3.71(−3)
20	1.74(−3)	(2, 12)	2.58(−3)	(2, 10)	1.34(−2)	(1, 13)	1.76(−3)
30	2.07(−3)	(2, 17)	2.01(−3)	(3, 15)	4.37(−4)	(2, 17)	2.01(−3)
40	1.89(−3)	(3, 23)	1.97(−3)	(4, 19)	4.08(−3)	(3, 20)	2.14(−3)
50	1.29(−4)	(3, 28)	1.21(−4)	(5, 23)	2.11(−3)	(4, 24)	1.45(−4)
100	3.51(−4)	(5, 55)	3.51(−4)	(7, 45)	2.86(−4)	(7, 38)	1.37(−5)
150	1.92(−4)	(7, 82)	1.92(−4)	(9, 68)	1.83(−4)	(11, 51)	3.10(−4)
200	3.21(−6)	(8, 109)	3.21(−6)	(11, 90)	7.58(−7)	(14, 62)	6.45(−4)
250	8.04(−5)	(10, 136)	8.04(−5)	(13, 112)	7.94(−5)	(18, 73)	7.79(−4)
300	1.03(−4)	(11, 163)	1.03(−4)	(15, 134)	1.04(−4)	(21, 83)	1.04(−3)

obtained by using MATHEMATICA function `NIntegrate` and the following decomposition

$$
\begin{aligned}
I &= I_0 + \sum_{k=0}^{+\infty} I_k \\
&= \int_0^{\pi/2} (\cos x)^{5/4} w(x)\,\mathrm{d}x + \sum_{k=0}^{+\infty} \int_0^{\pi} (-1)^k \left(\cos\left(t + (2k-1)\frac{\pi}{2}\right)\right)^{5/4} w\left(t + (2k-1)\frac{\pi}{2}\right)\,\mathrm{d}t,
\end{aligned}
$$

i.e.,

$$
I = \int_0^{\pi/2} (\cos x)^{5/4} w(x)\,\mathrm{d}x + \sum_{k=0}^{+\infty} \int_0^{\pi} (\sin t)^{5/4} w\left(t + (2k-1)\frac{\pi}{2}\right)\,\mathrm{d}t,
$$

where

$$
\begin{aligned}
I_0 &= 0.0422224454817570724303\ldots, \\
I_1 &= 0.00330333952732946 2572\ldots, \\
I_2 &= 1.447179253891697823900\ldots \times 10^{-12}, \\
I_3 &= 3.561804871271085508749\ldots \times 10^{-30}, \quad \text{etc.}
\end{aligned}
$$

EXAMPLE 8.3 Finally, we consider the integral $I(\gamma) = \int_0^{+\infty} e^{-1/x-x^2}\cos(\gamma x)\,\mathrm{d}x$, with an oscillatory function $f(x) = \cos(\gamma x)$. For $\gamma = 20,\ 40$ and 50, the values of $I(\gamma)$ (with 50 decimal digits in mantissa) are

$$
\begin{aligned}
I(20) &= 1.3434119769068606998768292975416538163974512371710\ldots \times 10^{-4}, \\
I(40) &= -1.1557245733888179431855415988485607276183305580618\ldots \times 10^{-5}, \\
I(50) &= 7.6826375869578254631463184451896978291202126322514\ldots \times 10^{-7}.
\end{aligned}
$$

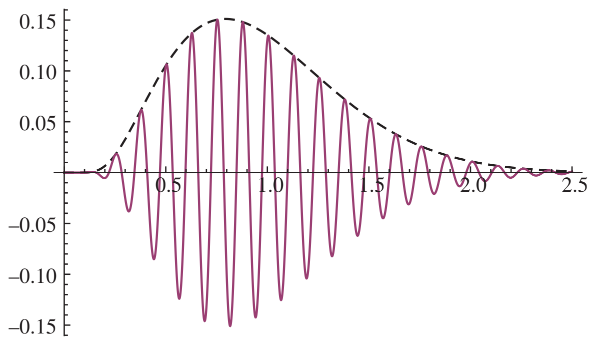

FIG. 3. Graphs of the weight function $w^{(1,2)}(x)$, with parameters $\alpha = 1$ and $\beta = 2$ (dashed line), and the integrand $w^{(1,2)}(x)\cos 50x$ (solid violet line).

TABLE 3 *Maximal relative errors in recursion coefficients α_k and β_k, $k \leqslant 150$, obtained by* (7.4), *with the* `WorkingPrecision` $\to$ *WP, and the corresponding running time*

WP	$\max_k \lvert \Delta\alpha_k/\alpha_k \rvert$	$\max_k \lvert \Delta\beta_k/\beta_k \rvert$	Running time
50	2.08(3)	1.74(7)	1′ 19″
100	5.72(2)	1.31(6)	1′ 24″
150	2.10(1)	1.58(3)	1′ 35″
200	3.83(−40)	2.02(−40)	1′ 49″
250	1.33(−89)	6.52(−90)	2′ 10″
300	2.33(−139)	1.25(−139)	2′ 43″
350	1.01(−189)	5.65(−190)	3′ 17″
400	1.55(−239)	8.63(−240)	3′ 58″
450	4.88(−289)	2.66(−289)	4′ 58″
500			6′ 07″

In Fig. 3, we present the graphs of the weight function $w^{(1,2)}(x) = e^{-1/x-x^2}$ and the integrand $w^{(1,2)}(x)\cos 50x$.

In order to construct the m-point Gaussian quadrature rule (7.1) for calculating oscillatory integrals $I(\gamma)$, with respect to the weight function $w^{(1,2)}$, for each $m \leqslant n = 150$, we need the moments $\mu_k^{(1,2)}$, $k \leqslant 300$, which are given in terms of Meijer G-function in symbolic form (7.7). This function can be evaluated with an arbitrary precision in MATHEMATICA. Taking the `Working Precision` in (7.4), as $WP = 50(50)500$, the maximal relative errors in recursion coefficients, as well as the corresponding running times, are given in Table 3. The running time is done in the fourth column, and it is expressed in minutes and seconds. The running time is evaluated by the function `Timing` in MATHEMATICA and it includes only CPU time spent in the MATHEMATICA kernel. Such a way may give different results on different occasions within a session, because of the use of internal system caches. In order to generate worst-case timing results independent of previous computations, we used the command `ClearSystemCache[]`. All computations were performed in MATHEMATICA, Ver. 8.0.4, on MacBook Pro Retina, OS X 10.8.2.

TABLE 4 *Relative errors in Gaussian approximations for integrals $I(\gamma)$ for $\gamma = 20, 40, 50, 60$*

m	$\gamma = 20$	$\gamma = 40$	$\gamma = 50$	$\gamma = 60$
40	6.07(−4)			
50	1.88(−8)			
60	1.15(−13)	1.10(3)		
70	5.46(−20)	1.42(2)		
80	9.72(−27)	1.49(0)		
90	1.55(−33)	1.16(−1)		
100	7.41(−41)	1.06(−3)	1.08(3)	
110	7.18(−49)	4.50(−6)	9.73(1)	
120	5.63(−59)	2.86(−8)	1.82(0)	
130	5.29(−66)	1.83(−11)	2.74(−1)	5.60(2)
140	9.57(−76)	2.91(−15)	6.88(−3)	5.41(1)
150	1.70(−84)	2.61(−18)	8.76(−5)	7.44(0)

Thus, if we want to construct Gaussian quadratures (7.1), for $m \leqslant n = 150$ with a high precision, for example `Precision->85`, in order to use them for oscillatory integrals, we use the recursion coefficients obtained for $WP = 250$. In that case, we obtain the Gaussian parameters by the procedure of Golub & Welsch (1969), realized in the MATHEMATICA Package `OrthogonalPolynomials` as

```
aGaussianNodesWeights[m,alpha,beta,WorkingPrecision->90,Precision->85].
```

It means that we can compute the parameters (nodes and weights) in all m-point Gaussian formulae for $m \leqslant n = 150$ with the same precision, because the Golub–Welsch algorithm is well conditioned. The running time for constructing 15 Gaussian formulas for $m = 10(10)150$ is about 7 s. An application of such formulas to the oscillatory integrals $I(\gamma)$, for $\gamma = 20, 40, 50, 60$, gives approximations, with relative errors presented in Table 4. As we can see, these formulas (for $m \leqslant 150$) practically cannot be applied for calculating integrals $I(\gamma)$ for $\gamma > 50$.

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