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Abstract

We construct the local and nonlocal conserved densities for the Camassa-Holm equation by solving a suitable Riccati equation. We also define a KP extension for the local Camassa-Holm hierarchy.

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1 Introduction

Since its introduction in [1] as a model for shallow water waves, the Camassa–Holm equation

$$u_t - u_{xxt} = 6u_x u - 4u_{xx}u - 2uu_{xxx} \tag{1}$$

has been the subject of a great number of papers. In particular, some of them ([9],[4],[7], just to cite a few) have been devoted to investigate its bi–Hamiltonian structure and its relation with other non linear partial differential equations living on the same phase space, namely the Korteweg de Vries (KdV) equation, the Hunter–Saxton equation [3], and the Harry Dym (HD) equation [5]. It is well-known how to use the bi–Hamiltonian structure of the KdV equation to construct the whole corresponding hierarchy (see, e.g., [8]). This structure has been used in [2] to relate in a geometrical way the KdV hierarchy with the celebrated Kadomtsev–Petviashvili (KP) hierarchy. (In this paper by "KP hierarchy" we mean the Sato form, in terms of pseudodifferential operators, of the KP hierarchy. This is a hierarchy of (1+1) evolution equations for infinitely many fields). The same results have been found in [11] for the HD equation, recovering in this way the KP extension of the Harry Dym hierarchy presented in [10].

The aim of this paper is to develop an analogous theory for the CH equation as well. More precisely, we show how the bi–Hamiltonian approach to the CH allows us to write a Riccati equation whose solutions give rise to the conserved densities of the CH equation. As it is well-known, there are two hierarchies of such conserved quantities. The first one contains the Camassa–Holm equation, and its Hamiltonian densities are nonlocal, while the other one is formed by local densities. (To the best of our knowledge, and according to [6], there is no sound proof of the existence of an infinite number of members of the nonlocal hierarchy. In this paper we give such a proof). Moreover we show that the local CH hierarchy can be embedded in a wider hierarchy in an infinite number of fields, exactly like the KdV equation can be included in the KP hierarchy.

The paper is organized as follows. In the next section we use the bi–Hamiltonian method to find a Riccati equation that allows us to construct the two above mentioned hierarchies of conserved quantities. In the third and last section we show — using the technique of the Faà di Bruno polynomials — how the local CH hierarchy can be framed in a more general hierarchy which plays the same role that the KP hierarchy does in the KdV case.

2 The conserved densities

It is well known that the CH equation (1) is a bi–Hamiltonian system on the infinite dimensional space $C^\infty(S^1, \mathbb{R})$ of C^∞-functions from the circle to the real line, with respect to the Poisson tensors

$$P_0 = \frac{1}{2}\partial_x - \frac{1}{2}\partial_x^3 , \qquad P_1 = -(m\partial_x + \partial_x m) ,$$

where m is defined by $m = (1 - \partial_x^2)u$. In this section we construct the positive and negative CH hierarchies using the so-called method of the Casimir of the Poisson pencil. This means that we look for Casimirs of the Poisson pencil $P_\lambda = P_0 - \lambda P_1$, i.e., for functionals $H(\lambda)$ whose differentials $v(\lambda)$ belong to the kernel of P_λ. The coefficients of the Laurent expansion of any such Casimir $H(\lambda) = \sum_{k=-\infty}^{+\infty} H_k \lambda^{-k}$ provide indeed a set of functionals satisfying the Lenard-Magri recursion relation

$$P_0 dH_k = P_1 dH_{k+1}$$

and thus in involution with respect to both Poisson structures (see, e.g., [8]).

The key point to determine $v(\lambda)$ is to notice that

$$vP_\lambda v = v\left(\frac{1}{2}v_x - \frac{1}{2}v_{xxx} + 2\lambda m v_x + \lambda m_x v\right) = \left(\frac{1}{4}v^2 - \frac{1}{2}v_{xx}v + \frac{1}{4}v_x^2 + \lambda m v^2\right)_x . \quad (2)$$

This result is a direct consequence of the fact that $\int vP_\lambda v$ is the antisymmetric action of the 2–tensor P_λ on the pair (v, v). The condition $P_\lambda v(\lambda) = 0$ is thus equivalent to

$$\frac{1}{4}v^2 - \frac{1}{2}v_{xx}v + \frac{1}{4}v_x^2 + \lambda m v^2 = f(m, \lambda) , \quad (3)$$

where $f(m, \lambda)$ satisfies $f_x = 0$. It turns out that v is an exact 1-form if f does not depend on m. Without loss of generality, we can put $f(\lambda) = \frac{\lambda}{4}$. Then equation (3) can be put in the useful Riccati form

$$h_x + h^2 = \frac{1}{4} + z^2 m \quad (4)$$

through the transformation

$$-v_x + 2vh = z , \quad (5)$$

where $z^2 = \lambda$. Let us verify that $v(\lambda)$ is an exact 1–form:

$$\begin{aligned} \langle v, \dot{m}\rangle &\overset{(4)}{=} \int_{S^1} vz^{-2}(\dot{h}_x + 2h\dot{h})\, dx \\ &\overset{(5)}{=} \frac{1}{z^2}\int_{S^1} (-v_x + 2vh)\, \dot{h}\, dx \\ &= \frac{1}{z}\frac{d}{dt}\int_{S^1} h\, dx , \end{aligned} \quad (6)$$

so that the potential of $v(\lambda)$ is given by $H = z^{-1}\int_{S^1} h\, dx$. In the next two subsections we will solve the Riccati equation (4) using a formal development for the function h. We will find a first solution of the form $h(z) = \sum_{i=-1}^{\infty} h_i z^{-i}$, where the h_i are functions of m and its x-derivatives. This solution gives rise to the local (often called also negative) CH hierarchy, starting from a Casimir of P_1. The second solution has the form $k(z) = \sum_{i=0}^{\infty} k_i z^{-i}$, where the k_i are now nonlocal densities. They constitutes the usual (nonlocal, positive) CH hierarchy, which the CH equation belongs to. In other words, we will deal with two possible different choices of the essential singularity of the solutions of the Riccati equation (4).

2.1 The local CH hierarchy

In this case the Laurent expansion of the density of Hamiltonians is

$$h(z) = h_{-1}z + h_0 + \frac{h_1}{z} + \frac{h_2}{z^2} + \ldots = h_{-1}z + h_0 + \sum_{i=1}^{\infty}\frac{h_i}{z^i} , \quad (7)$$

where the maximum degree of the positive term is established by (4).
If we substitute this expansion in (4) we get

$$\sum_{i=-2}^{\infty}\left(\left(h_{ix}+\sum_{j=-1}^{i+1}h_{i-j}h_j\right)\frac{1}{z^i}\right)=\frac{1}{4}+z^2m\ . \tag{8}$$

Then the obtained system can be solved in a purely algebraic way by equating the terms with the same degree in z:

$$\begin{array}{lll}
z^2 & h_{-1}^2=m & h_{-1}=\sqrt{m} \\
z^1 & h_{-1x}+2h_0h_{-1}=0 & h_0=\left(\ln(m^{-\frac{1}{4}})\right)_x \\
z^0 & h_{0x}+h_0^2+2h_1h_{-1}=\frac{1}{4} & h_1=\frac{1}{8\sqrt{m}}-\frac{1}{8}\frac{m_{xx}}{\sqrt{m^3}}+\frac{5}{32}\frac{m_x^2}{\sqrt{m^5}} \\
z^{-1} & h_{1x}+2h_0h_1+2h_2h_{-1}=0 & h_2=\left(-\frac{1}{16m}+\frac{1}{16}\frac{m_{xx}}{m^2}+\frac{5}{64}\frac{m_x^2}{m^3}\right)_x
\end{array}$$

and so on. It can be shown that the even densities are x-derivatives, so that $H=z^{-1}\int_{S^1}h\,dx$ is actually a Laurent series in λ:

$$H(\lambda)=\sum_{i=0}^{+\infty}H_{2i}\lambda^{-i}\ .$$

If j is even, we call X_j the Hamiltonian vector field associated with $H_j=\int_{S^1}h_{j-1}\,dx$ by means of the Poisson operator P_0. By construction, X_j is the Hamiltonian vector field associated with H_{j+2} by means of P_1. The first nontrivial equation of the hierarchy, corresponding to X_0, is

$$\begin{aligned}
\frac{\partial m}{\partial t_0}=P_0dH_0 &= (\partial_x-\partial_x^3)\frac{1}{4\sqrt{m}} \\
&= -\frac{1}{8}\frac{m_x}{\sqrt{m^3}}-\frac{1}{8}\frac{m_{xxx}}{\sqrt{m^3}}+\frac{9}{16}\frac{m_{xx}m_x}{\sqrt{m^5}}-\frac{15}{32}\frac{m_x^3}{\sqrt{m^7}} \\
&= P_1dH_2\ .
\end{aligned} \tag{9}$$

2.2 The nonlocal CH hierarchy

In this case the Laurent expansion of the solution $k(z)$ of the Riccati equations (4) is

$$k(z)=k_0+k_{-1}z+k_{-2}z^2+k_{-3}z^3+\ldots=\sum_{i=0}^{+\infty}k_{-i}z^i\ . \tag{10}$$

Substituting it in (4), we obtain

$$\sum_{i=0}^{\infty}\left(\left(k_{-ix}+\sum_{j=0}^{i}k_{-j}k_{j-i}\right)z^i\right)=\frac{1}{4}+z^2m\ . \tag{11}$$

Exactly as before, we can find a recursive solution of (11) comparing the terms of the same degree on both side of the equation. However, in the present case this requires to solve at any step a differential equation. This fact is responsible for the presence of nonlocal quantities. The first equation to be considered is that related to the coefficient of z^0,

$$k_{0x} + k_0^2 = \frac{1}{4} \ .$$

It is easily checked that the only periodic solutions of this equation are the constant solutions $k_0 = \pm\frac{1}{2}$. Let us choose the positive constant solution. Next, the coefficient of z^1 gives

$$k_{-1x} + 2k_0 k_{-1} = 0$$

or, using $k_0 = \frac{1}{2}$,

$$(1 + \partial_x)k_{-1} = 0.$$

This linear equation is solved by $k_{-1} = c\exp(-x)$, where c is a constant. But again among them the only solution which lies in $C^\infty(S^1, \mathbb{R})$ is the trivial one: $h_{-1} = 0$. More generally, the operator $1 + \partial_x$ is invertible in $C^\infty(S^1, \mathbb{R})$. The unique solution of

$$(1 + \partial_x)k = f(x)$$

of period 1 is indeed explicitly given by

$$k(x) = \int_0^x e^{y-x} f(y)\, dy + \frac{1}{e-1}\int_0^1 e^{y-x} f(y)\, dy \ .$$

From $k_{-1} = 0$ one can immediately show that $k_{-2n-1} = 0$ for all $n \geq 0$. In fact, k_{-2n-1} appears for the first time in the coefficient of z^{n+1},

$$k_{-2n-1x} + k_{-2n-1} + 2\sum_{i=1}^{n} k_{-2i} k_{-2(n-i)-1} = 0 \ ,$$

and this allows us to prove by recursion that all the odd terms in the Laurent series $k(z)$ are zero. Using the remaining equations,

$$\begin{array}{ll} z^2 & k_{-2x} + 2k_0 k_{-2} + k_{-1}^2 = m \\ z^4 & k_{-4x} + 2k_0 k_{-4} + 2k_{-1}k_{-3} + k_{-2}^2 = 0 \\ z^6 & h_{-6x} + 2k_0 k_{-6} + 2k_{-1}k_{-5} + k_{-3}^2 + 2k_{-2}k_{-4} = 0 \\ \dots & \dots \end{array}$$

it is now simple to find the even terms:

$$\begin{aligned} &k_{-2} = (1+\partial_x)^{-1} m \\ &k_{-4} = -(1+\partial_x)^{-1}\left((1+\partial_x)^{-1} m\right)^2 \\ &k_{-6} = 2(1+\partial_x)^{-1}\Big((1+\partial_x)^{-1} \cdot \left((1+\partial_x)^{-1} m\right)^2 (1+\partial_x)^{-1} m\Big) \\ &\dots = \dots \end{aligned}$$

Thanks to the invertibility of the operator $1 + \partial_x$ in the space of C^∞ periodic functions, we can conclude that there is an infinite sequence of (increasingly nonlocal) densities k_{-2i}, giving rise to a set of functionals in involution with respect to both Poisson brackets. More precisely, let $K(\lambda) = \int_{S^1} k\, dx$ be the Casimir of the Poisson pencil constructed with k.

Then $K(\lambda) = \frac{1}{4} + \sum_{j=1}^{\infty} K_{-2j}\lambda^j$, with $K_{-2j} = \int_{S^1} k_{-2j}\,dx$. We call X_j the Hamiltonian vector field associated with K_j by means of the Poisson operator P_1. By construction, X_j is the Hamiltonian vector field associated with K_{j-2} by means of P_0.
Since $k_{-2x} + k_{-2} = m$, we have that $K_{-2} = \int_{S^1} k_{-2}\,dx = \int_{S^1} m\,dx$, so that the first equation of the hierarchy is $m_{t_{-2}} = P_1 dK_{-2} = -m_x$. In order to write the second vector field, we recall that $k_{-4x} + k_{-4} + k_{-2}^2 = 0$ and therefore

$$K_{-4} = \int_{S^1} k_{-4}\,dx = -\int_{S^1} k_{-2}^2\,dx = -\int_{S^1} \left((1+\partial_x)^{-1}m\right)^2\,dx\ .$$

This functional becomes local after the usual change of variables $m = u - u_{xx}$, that is invertible in the space of C^∞ periodic functions because it is the composition of $1+\partial_x$ and $1-\partial_x$. Its inverse is explicitly given by

$$u(x) = \int_0^x m(y)\sinh(y-x)\,dy + \frac{1}{2\sinh\frac{1}{2}}\int_0^1 m(y)\cosh(y-x-\frac{1}{2})\,dy\ .$$

In terms of u we have that $K_{-4} = -\int_{S^1} (u^2 + {u_x}^2)\,dx$, so that

$$\begin{aligned}\frac{\partial m}{\partial t_{-4}} = P_1 dK_{-4} &= -(m\partial_x + \partial_x m)(1-\partial_x^2)^{-1}(-2u + 2u_{xx}) \\ &= 4mu_x + 2m_x u\ ,\end{aligned} \tag{12}$$

that is, with $t_{-4} = t$,

$$u_t - u_{xxt} = 6u_x u - 4u_{xx}u_x - 2uu_{xxx}\ , \tag{13}$$

which is the standard Camassa-Holm equation with null critical velocity term. The next symmetry of the hierarchy is related to the Hamiltonian $K_{-6} = \int_{S^1} k_{-6} = 2\int_{S^1}(u^3 + uu_x^2)\,dx$, which, using (5) to compute dK_{-6}, gives

$$\frac{\partial m}{\partial t_{-6}} = P_1 dK_{-6} = -(m\partial_x + \partial_x m)(1-\partial_x^2)^{-1}(6u^2 - 2u_x^2 - 4uu_{xx}).$$

We remark that, due to the Lenard-Magri recursion relations, the equation (13) can be obtained also as $\frac{\partial m}{\partial t_{-4}} = P_0 dK_{-6}$.

3 KP extension of the local CH hierarchy

In [2] it has been shown how to generate, from a bi–Hamiltonian viewpoint, the KP hierarchy starting from the KdV hierarchy. The same procedure has been performed in [11], where the KP extension of the Harry Dym hierarchy (already found in [10]) has been recovered. The idea is quite simple and can be successfully applied to the local CH hierarchy, as we do in this section.

In Subsection 2.1 we have found a map $m \mapsto h(m)$, where $h(m)$ is the unique solution of the Riccati equation (4) with the asymptotic expansion (7). Since the coefficients of $h(m)$ are the densities of the Hamiltonians of the hierarchy, the time derivatives of $h(m)$ must be an x-derivative, that is,

$$\frac{\partial h}{\partial t_s} = \partial_x H^{(s)}\ , \qquad s = 0, 2, \ldots,$$

for suitable currents $H^{(s)}$. They can be explicitly constructed after noticing that the vector fields of the hierarchy are not only bi–Hamiltonian, but they are Hamiltonian with respect to every Poisson structure of the pencil. Indeed, one can immediately see that

$$\frac{\partial h}{\partial t_{2s}} = P_\lambda(\lambda^s dH(\lambda))_{\rm reg} = P_\lambda(\sum_{r=0}^{s} dH_{2r}\lambda^{s-r}) \ , \tag{14}$$

where "reg" stands for the regular part in the Laurent expansion. Moreover, using (4) the Poisson pencil P_λ can be factorized in the following way:

$$P_\lambda = -\frac{1}{2}(\partial_x + 2h)\partial_x(\partial_x - 2h).$$

Substituting it in the derivative with respect to t_{2s} of (4), we obtain

$$\lambda^{-1}(\partial_x + 2h)h_{t_{2s}} = -\frac{1}{2}(\partial_x + 2h)\partial_x(\partial_x - 2h)(\lambda^s dH(\lambda))_{\rm reg}.$$

From the previous equation, due to the particular form of the z–development of h, it follows that the continuity equation

$$\frac{\partial h}{\partial t_{2s}} = \partial_x\Big(-\frac{\lambda}{2}(\partial_x - 2h)(\lambda^s dH(\lambda))_{\rm reg}\Big) \tag{15}$$

holds true. Therefore, the currents $H^{(2s)}$ we are looking for are given by

$$H^{(2s)} = -\frac{\lambda}{2}(\partial_x - 2h)(\lambda^s dH(\lambda))_{\rm reg} = -\frac{\lambda}{2}(\lambda^s v(\lambda)_x)_{\rm reg} + h\lambda(\lambda^s v(\lambda))_{\rm reg} \ . \tag{16}$$

The next step is to realize that the currents $H^{(s)}$, where s is even, can be obtained directly from a Laurent series h of the form (7), without using the Riccati equation (4). We start with the preliminary consideration that they can be written in two equivalent ways:

$$H^{(s)} = \sum_{i=0}^{s}\Big(-\frac{1}{2}v_{i,x}(z^{s-i+2}) + v_i(z^{s-i+2}h)\Big) \tag{17}$$

and

$$\begin{aligned} H^{(s)} &= -\frac{1}{2}(\partial_x - 2h)\Big(z^{s+2}dH(z) - z^2(z^s dH(z))_{\rm sing}\Big) = \\ &= \frac{1}{2}z^{s+2}(-v_x + 2hv) - \frac{z^2}{2}(\partial_x - 2h)(z^s v)_{\rm sing} \\ &\overset{(5)}{=} \frac{1}{2}z^{s+3} - \frac{z^2}{2}(\partial_x - 2h)(z^s v)_{\rm sing} \ , \end{aligned} \tag{18}$$

where with "sing" we means the singular part of the expansion in z. Equation (18), using (4), gives the regular asymptotic behavior of the currents, that is, $(H^{(s)})_{reg} = \frac{1}{2}z^{s+3} + O(z)$. On the other hand, equation (17) implies that the currents belong to a particular vector space H_h, which can be constructed using only the Hamiltonian density h. It is defined as the linear span over the functions $C^\infty(S^1, \mathbb{R})$ of the Faà di Bruno polynomials $h^{(n)} = (\partial_x + h)^n z^2$, with $n \geq 0$, $h^{(0)} = z^2$.

Proposition 3.1 *The currents $H^{(2s)}$, with $s \geq 0$, are elements of H_h.*

Proof Thanks to the representation (17), it suffices to show that z^{2i} and $z^{2i}h$ are elements of H_h for all $i \geq 1$. First of all, $z^2 = h^{(0)} \in H_h$ and $z^2 h = h^{(1)} \in H_h$ by definition of H_h. Moreover, the Riccati equation multiplied by z^2,

$$z^2(h_x + h^2) = \frac{z^2}{4} + z^4 m \ , \tag{19}$$

shows that $z^4 = \frac{1}{m}(h^{(2)} - \frac{1}{4}h^{(0)}) \in H_h$. Now, acting with $(\partial_x + h)$ on both sides of (19), we can show that $z^4 h \in H_h$. More generally, acting with $(\partial_x + h)^n$, we prove that $z^2(H_h) \subset H_h$, and this concludes the proof.

□

At this point it is not difficult to see that the current $H^{(s)}$ can be characterized in a unique way by the following properties:

1. $H^{(s)} = \frac{1}{2}z^{s+3} + O(z)$;
2. $H^{(s)} \in H_h$.

Therefore, we can assume that h is an arbitrary Laurent series of the form

$$h(z) = h_{-1}z + \sum_{i=0}^{+\infty} \frac{h_i}{z^i} \ , \tag{20}$$

where the coefficients h_i are not constrained by the Riccati equation, and we can define the currents $H^{(s)}$, for all $s \geq 0$, imposing the two above-mentioned properties. Then we define the s-th equation of the *local KP–CH hierarchy* as

$$\frac{\partial h}{\partial t_s} = \partial_x H^{(s)} \ , \qquad s \geq 0 \ . \tag{21}$$

It is an evolution equation in an infinite number of fields given by the coefficients h_{-1},h_0, $h_1, \ldots$ of h.

In order to write these equations one has to compute the first Faà di Bruno polynomials,

$$\begin{aligned}
h^{(0)} &= z^2 \\
h^{(1)} &= h_{-1}z^3 + h_0 z^2 + h_1 z + \ldots \\
h^{(2)} &= (h_{-1}^2)z^4 + (h_{-1x} + 2h_{-1}h_0)z^3 + (h_{0x} + h_0^2 + 2h_{-1}h_1)z^2 \\
&\quad +(h_{1x} + 2h_{-1}h_2 + 2h_0 h_1)z + \ldots \\
h^{(3)} &= \ldots
\end{aligned}$$

Then the first currents are given by

$$\begin{aligned}
H^{(0)} &= \frac{1}{2h_{-1}}h^{(1)} - \frac{h_0}{2h_{-1}}h^{(0)} = \frac{1}{2}z^3 + \frac{h_1}{2h_{-1}}z + \ldots \\
H^{(1)} &= \frac{1}{2h_{-1}^2}h^{(2)} - \left(\frac{h_{-1x}}{2h_{-1}^3} + \frac{h_0}{h_{-1}^2}\right)h^{(1)}
\end{aligned}$$

$$
\begin{aligned}
& -\left(\frac{h_{0x}}{2h_{-1}^2}+\frac{h_0^2}{2h_{-1}^2}+\frac{h_1}{h_{-1}}-\frac{h_{-1x}h_0}{2h_{-1}^3}-\frac{h_{-1}h_0^2}{h_{-1}^3}\right)h^{(0)} \\
= \; & \frac{1}{2}z^4+\left(\frac{h_{-1x}}{2h_{-1}^2}+\frac{h_2}{h_{-1}}+\frac{h_1h_0}{h_{-1}^2}-\frac{h_{-1x}h_1}{2h_{-1}^3}-\frac{h_0h_1}{h_{-1}^2}\right)z+\dots \\
H^{(2)} = \; & \dots
\end{aligned}
$$

To recover the local CH hierarchy, one has to impose on h the constraint given by the Riccati equation (4). It entails that all the fields h_i can be written in terms of m and its x-derivatives. Thus the local KP–CH hierarchy (21) reduces to the local CH hierarchy. This reduction can be interpreted, like in the KdV and HD cases, as a stationary reduction. Indeed, the Riccati equation and the very definition of the currents imply that the current $H^{(1)}$ is equal to $\frac{z^4}{2}$ and therefore that t_1 is a stationary time. From the proposition 3.1 it also follows that all the odd times are stationary.

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