POLITECNICO DI TORINO

UNIVERSITÀ DEGLI STUDI DI TORINO

Doctoral Dissertation

Doctoral Program in Pure and Applied Mathematics (32^{th}cycle)

Ulrich and Instanton Bundles on some Low-Dimensional Varieties

By

Vincenzo Antonelli

******

Supervisor:

Prof. Gianfranco Casnati, Advisor

Prof. Francesco Malaspina, Co-Advisor

Doctoral Examination Committee:

Prof. Enrique Arrondo, Universidad Complutense de Madrid

Prof. Cinzia Casagrande, Università di Torino

Prof. Rosa Maria Miró-Roig, Universitat de Barcelona

Università di Torino - Politecnico di Torino

21st January 2020

Declaration

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

Vincenzo Antonelli
21st January 2020

* This dissertation is presented in partial fulfillment of the requirements for the degree of ***Philosophiæ Diploma*** **(PhD degree)** in **Pure and Applied Mathematics**.

Contents

Chapter 1

Introduction

The study of vector bundles supported on a smooth projective variety is a powerful tool to understand its geometry.

Let $X \subset \mathbb{P}^N$ be a projective variety. It is naturally endowed with the very ample line bundle $\mathcal{O}_X(h) = \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1)$. Let us denote by R_X the coordinate ring of X. We say that X is arithmetically Cohen-Macaulay (aCM for short) if $H^i(\mathcal{I}_X(th)) = 0$ for $t \in \mathbb{Z}$ and $1 \leq i \leq \dim(X)$, where $\mathcal{I}_X$ is the ideal sheaf of X.

A coherent sheaf E over a variety X is called aCM if is locally Cohen-Macaulay and all the twists of its intermediate cohomology groups vanish, i.e. $H^i(X, E(th)) = 0$ for $t \in \mathbb{Z}$ and $1 \leq i \leq \dim(X) - 1$. When X is aCM, aCM sheaves correspond to *Maximally Cohen-Macaulay* modules over R_X through the correspondence which associate to a sheaf E its R_X-module of global section $\bigoplus_t H^0(X, E(th))$.

The study of aCM sheaves supported on X gives us a measure of the complexity of the variety itself. For example the projective space $\mathbb{P}^N$ is characterized by Horrock's theorem by the property that a vector bundle is aCM if and only if it is totally decomposable [60]. This can be interpreted as the idea that “simple" varieties should support “simple" category of aCM sheaves. Starting from Horrock's result, the characterization of vector bundles without intermediate cohomology has been an interesting topic in algebraic geometry (see for example [72] and [21] for smooth quadrics, [11] for the Grassmannian of lines in $\mathbb{P}^4$ and [10] for Fano threefolds with $b_2 = 1$). Following these lines, a cornerstone result was the classification of aCM varieties of *finite representation type*, i.e. those varieties that support only a finite number of indecomposable aCM vector bundles. It turned out that they fall into a very short list: three or less reduced points on $\mathbb{P}^2$, projective spaces, smooth quadric hypersurfaces $X \subseteq \mathbb{P}^n$, cubic scrolls in $\mathbb{P}^4$, any Veronese surface in $\mathbb{P}^5$ or rational normal curves, see [46].

For the rest of aCM varieties, an interesting problem is to give a criterium to split them in a finer classification. An approach was offered by representation theory. An aCM variety $X \subseteq \mathbb{P}^n$ is called of *tame representation type* if for each r the indecomposable aCM sheaves of rank r form a finite number of families of dimension at most one. On the other hand, X will be called of *wild representation type* if there exists l-dimensional families of non-isomorphic indecomposable aCM sheaves for arbitrary large l. In [51] Faenzi and Malaspina deal with the tame case. They showed that, apart from smooth elliptic curves, the only aCM varieties of tame representation type are the quartic surface scrolls. In [52] D. Faenzi and J. Pons-Llopis proved that a reduced non-degenerate closed aCM subscheme $X \subset \mathbb{P}^n$ of dimension $m \geq 1$ is of wild type unless is either one of the finite representation cases listed above, or it is a smooth elliptic curve or a smooth rational surface scroll of degree 4, completing the list of non-CM-wild varieties in a broad sense. For more details the interested reader can see [80].

In the first part of this thesis we will study a particular family of aCM sheaves, namely the Ulrich ones. They are defined to be the aCM sheaves whose corresponding module $H^0(E)$ has the maximum number of generators.

Ulrich bundles were introduced in a purely algebraic context in 1984. The study of these objects began in [91] by Ulrich. In the paper he investigated criteria for a local Cohen-Macaulay ring to be Gorenstein. One criteria involved finitely generated modules M of positive rank over a local Cohen-Macaulay ring R. In this case, the minimal numbers of generators $\nu(M)$ of M is bounded above by the product of the multiplicity of R and the rank of M. One necessary condition for R to be Gorenstein is the existence of a finitely generated R- module M of positive rank such that the minimal number of generators is big enough. Because of this Ulrich raised the following question:

- Let R be a local Cohen-Macaulay ring with positive dimension and infinite residue class field. Does there exist a Cohen-Macaulay R-module M with positive rank such that $\nu(M) = e(R)\,\mathrm{rk}(M)$?

Few years later Brennan, Herzog and Ulrich investigated these modules, which they called “Maximally Generated Cohen-Macaulay Modules" (MGMCM in short) in [19]. In particular, they proved that a homogeneous, two-dimensional Cohen-Macaulay domain R with infinite residue class field admits an MGMCM module. The term “Ulrich module" as a synonym for a MGMCM was coined by Backelin and Herzog in [13]. In that article, they proved the existence of Ulrich modules on hypersurface rings.

Beauville in [16] made a systematic exposition of the relationship between Ulrich bundles supported on a hypersurface X and the existence of a determinantal or pfaffian representation of X. Ulrich bundles have been thoroughly studied in recent years because of this relation with the determinantal representation of the variety but also because of their relationship with Chow forms and Clifford algebras. In fact in [55] it has been proved that rank r Ulrich bundles are in one-to-one correspondence with the equivalence classes of matrix representation of the generalized Clifford algebra C_f of f.

In [47] Eisenbud and Schreyer characterized Ulrich sheaves E on a n-dimensional aCM variety $X \subseteq \mathbb{P}^N$ with respect to a very ample line bundle $\mathcal{O}_X(h)$. Let E be an initialized (i.e. $H^0(X,E(-h)) = 0$ but $H^0(X,E) \neq 0$) aCM coherent sheaf on X, then the following are equivalent

(i) E is Ulrich.

(ii) E admits a linear $\mathcal{O}_{\mathbb{P}^N}$-resolution of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-N+n)^{a_{N-n}} \to \cdots \to \mathcal{O}_{\mathbb{P}^N}(-1)^{a_1} \to \mathcal{O}_{\mathbb{P}^N}^{a_0} \to E \to 0.$$

If this occurs then $a_0 = \deg(X)\,\mathrm{rk}(E)$ and $a_i = \binom{N-n}{i} a_0$ for all i.

(iii) $h^i(X,E(-ih)) = h^j(X,E(-(j+1)h)) = 0$ for each $i > 0$ and $j < n$.

(iv) For some (resp. all) finite linear projections $\pi : X \to \mathbb{P}^n$, the sheaf $\pi_* E$ is the trivial sheaf $\mathcal{O}_{\mathbb{P}^n}^t$ for some t.

In particular, initialized Ulrich sheaves are 0-regular and therefore they are globally generated.

Ulrich sheaves carry many interesting properties and they are the simplest bundles from the cohomological point of view. Furthermore an Ulrich sheaf is locally free on the complement of the singular locus of the variety X. In [47] the authors raised the following questions:

- Is every variety $X \subset \mathbb{P}^N$ the support of an Ulrich sheaf? If so, what is the smallest possible rank for such a sheaf?

The answers to these questions are still unknown in general, although there are several scattered results. The survey [17] is a standard reference for an introduction to Ulrich bundles. In this paper Beauville also obtains some results on rank one and two Ulrich bundles on surfaces and threefolds.

Casanellas and Hartshorne proved in [22] that a smooth cubic surface $F \subset \mathbb{P}^3$ is endowed with families of arbitrary large dimension of non-isomorphic, indecomposable initialized Ulrich bundles. In cases like this , we will say that the variety is *Ulrich-wild.*

For what concerns other projective surfaces, Ulrich bundles on $K3$ surfaces have been thoroughly studied in the past years. In [7] Aprodu, Farkas and Ortega showed the existence of families of stable, even rank Ulrich bundles on $K3$ surfaces with Picard number one and satisfying a mild generality condition. This result somehow generalizes the paper [35] by E. Coskun, Kulkarni and Mustopa where they focused on Ulrich bundles over smooth quartic surfaces in $\mathbb{P}^3$. In [31] Casnati and Galluzzi proved the existence of non-special rank two Ulrich bundles on non-degenerate $K3$ surfaces of degree greater than two. Finally Faenzi in [50] proved the existence of rank two special Ulrich bundles on any $K3$ and for any polarization by deforming an aCM vector bundle over the surface.

Moreover there have also been results on other surfaces and higher dimensional varieties and we briefly list some of them in what follows. In [8] Aprodu, Huh, Malaspina and Pons-Llopis characterized Ulrich bundles on smooth rational normal scroll as the bundles admitting a special type of filtration. In [24] Casnati proved the existence of special Ulrich bundles on non-special surfaces with $p_g = q = 0$ and he studied when such varieties are Ulrich-wild. In [26] and [25] the same author showed respectively that surfaces S with $p_g(S) = 0$ and $q = 1$ are Ulrich-wild, and the existence of special Ulrich bundles on regular surfaces with non-negative Kodaira dimension. For what concerns del Pezzo surfaces E. Coskun, Kulkarni and Mustopa in [36] characterized the divisors class on del Pezzo surfaces which represent the first Chern class of a rank r Ulrich bundle. In [23] Casnati completely classified rank two Ulrich bundles on anticanonically embedded surfaces and in [81] Miró Roig and Pons-Llopis dealt with the minimal resolution conjecture on del Pezzo surfaces and its connection with Ulrich bundles. In [82] the same authors showed that all smooth rational aCM surfaces in $\mathbb{P}^4$ are Ulrich-wild and that a general determinantal variety of codimension one or two supports indecomposable Ulrich sheaves of rank one and two. In [77] proved the existence of Ulrich bundles on some surfaces of maximal Albanese dimension. In [37] I. Coskun, Costa, Huizenga, Miró-Roig and Woolf studied equivariant vector bundles on partial flag varieties arising from Schur functors and they classify Ulrich bundles of this form on some two-steps flags. In [40] Costa and Miró-Roig classified all homogeneous Ulrich bundles on the Grassmannian of k-planes in the projective n-space. Finally in [**?**] the authors dealt with the existence of Ulrich line bundles and stable rank two Ulrich bundles on geometrically ruled surfaces and in [83] the authors proved the existence of rank two special and simple Ulrich bundles on Weierstrass fibrations.

In this context the contribution of the present thesis can be found in chapter 3 (see the introduction of the chapter for the statements of the main results), where we study Ulrich bundles over Hirzebruch surfaces, i.e. geometrically rationally ruled surfaces. Aprodu, Costa and Miró-Roig proved in [6], the existence of stable, rank two special Ulrich bundles on ruled surfaces over a curve of genus $g \geq 1$. Faenzi and Malaspina in [51] and Miró-Roig in [79] considered the case of Hirzebruch surfaces embedded as rational normal scrolls. In chapter 3, using derived categories techniques, we prove that any Ulrich bundle admits a two-terms resolution on the Hirzebruch surface X. Conversely, given an injective map between totally decomposed vector bundles on X, we give a necessary and sufficient condition such that its cokernel is an Ulrich bundle. In this way we are able to prove and classify, with different techniques with respect to [6], the existence of rank two special (i.e. $\det(E) = 3h+K$) Ulrich bundles on the Hirzebruch surfaces X. Furthermore, using a result of I. Coskun and Huizenga [38], we are able to prove the existence of Ulrich bundles of any admissible rank and first Chern class on X with respect to all very ample divisors h.

Moreover this characterization of Ulrich bundles on Hirzebruch surfaces leads to a description of the moduli space of such vector bundles (for previous results see [42], [43] and the references therein). The study of moduli spaces of sheaves has been an important and central topic in algebraic geometry in the recent years. In particular we show that the moduli space of stable Ulrich bundles (whenever non-empty) can be realized as the quotient of an open subset of some space of matrices. Furthermore we are able to describe, inside this space of matrices, the locus corresponding to non-Ulrich bundles.

Finally we give criterions of admissibility for the first Chern class and the rank of an Ulrich bundle and we construct several examples of indecomposable rank two and three Ulrich bundles both via the Hartshorne-Serre correspondence and using the computer software *Macaulay2*.

In the second part of the thesis we focus on instanton bundles. Instanton bundles on $\mathbb{P}^3$ were first defined in [12] by Atiyah, Drinfel'd, Hitchin and Manin. Their importance arises from quantum physics; in fact these particular bundles correspond (through the Penrose-Ward transform) to self-dual solutions of the Yang-Mills equation over the real sphere S^4. We recall that a mathematical instanton bundle E with charge (or quantum number k) on $\mathbb{P}^3$ is a stable rank two vector bundle E such that $c_1(E) = 0$, $c_2(E) = k$ and with the property (called instantonic condition) that

$$H^1(E(-2)) = 0.$$

Every instanton of charge k on $\mathbb{P}^3$ can be represented as the cohomology of a monad (a three-term self dual complex). Starting from $\mathbb{P}^3$, instanton bundles were generalized to other projective spaces and the study of their moduli spaces became an important topic in algebraic geometry (see for example [63], [65], [66], [64] and [85]).

Coming back to the connections with quantum physics, in [59] Hitchin showed that the only twistor spaces of four dimensional (real) differential varieties which are Kähler (and a posteriori, projective) are $\mathbb{P}^3$ and the flag variety $F(0,1,2)$, which is the twistor space of $\mathbb{P}^2$.

On $F(0,1,2)$ instanton bundles have been studied in [20], [45] and more recently in [78]. $F(0,1,2)$ is a Fano threefold with Picard number two. Let us call h_1 and h_2 the two generators of $\mathrm{Pic}(F(0,1,2))$. In [78] Malaspina, Marchesi and Pons-Llopis gave the following definition:

a rank two vector bundle E on the Fano threefold $F(0,1,2)$ is an instanton bundle of charge k if the following properties hold:

- $c_1(E) = 0$, $c_2(E) = kh_1h_2$;
- $h^0(E) = h^1(E(-1,-1)) = 0$ and E is μ-semistable.

Apart from the connection to physics, another line of work is to generalize the notion of instanton bundles on other threefolds, also as part of the larger goal of understanding vector bundles on threefolds. Furthermore there have been generalizations of instanton bundles on higher dimensional varieties (see for example [2], [39], [41], [85]).

In [49] (see also [73] in the case $i_X = 2$ and [87] for details in the case of the del Pezzo threefold of degree 5), the author generalizes the notion of instanton bundle on $\mathbb{P}^3$ to any Fano threefold X with Picard number one. He defines instanton bundles starting from a stable vector bundle G with

- $G \cong G^\vee \otimes \omega_X$;
- $H^1(X,G) = 0$.

Let E be the twist $G(t)$ of G such that $c_1(E)$ is either -1 or 0, and let $c_2(E) = k$. Then E is a k-instanton on X.

In [49] Faenzi considered Fano threefolds X with Picard number one. In this case the index i_X of the threefold is a number $i_X \in \{1,2,3,4\}$. $i_X = 4$ implies $X = \mathbb{P}^3$, while $i_X = 3$ implies that X is a smooth quadric hypersurface in $\mathbb{P}^4$. In case $i_X = 2$, the threefold X is called a del Pezzo threefold, while for $i_X = 1$ the variety X is called a *prime* Fano threefold. The author showed the non-emptiness of the moduli space of instanton bundles of charge k for all the Fano threefolds with $i_X > 1$, and when $i_X = 1$ under the assumption that the anticanonical bundle is very ample and that X contains a line with normal bundle $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$. Then the author considers varieties X such that the intermediate Jacobian of X is trivial. In this case he obtains a monadic description of instanton bundles on a Fano threefold of Picard number one.

In [73] Kuznetsov focused on Fano threefolds of Picard number 1, index 2 and degree 4 and 5, both via a monadic description and by studying the behaviour of such vector bundles when restricted to lines. In the case of degree 5 there is only one such threefold Y_5 which can be constructed as a linear section of codimension 3 of the Grassmannian $\mathrm{Gr}(2,5)$ embedded into the Plücker space $\mathbb{P}(\Lambda^2 k^5)$. On Y_5 any instanton is described, similarly to case of $\mathbb{P}^3$, as the cohomology of a self-dual monad. Furthermore the Hilbert scheme of lines on Y_5 can be identified with the projective plane $\mathbb{P}^2$ and the locus of jumping lines is a curve inside $\mathbb{P}^2$. In degree

4 each Fano threefold Y_4 of index two is an intersection of two quadrics in $\mathbb{P}^5$. In the pencil of quadrics passing through Y_4 there are 6 degenerate quadrics. Consider the double covering C of $\mathbb{P}^1$ parameterizing quadrics in the pencil. Let τ be the hyperelliptic involution of C. Then the acyclic extension $\widetilde{E}$ of an instanton bundle E of charge n corresponds to a semistable rank n vector bundle F on C such that $\pi^* F \cong F^\vee$. Moreover the scheme of lines on Y_4 is isomorphic to the abelian surface $\mathrm{Pic}^0(C)$ and the curve of jumping lines coincides with the theta-divisor on $\mathrm{Pic}^0 C$ associated with the bundle F. In [73] the author offers also some remarks on the case of degree two and three.

Comparing the definition of instanton bundle on Fano threefolds with different Picard number, it is worth to notice that when the Picard number is one, the condition $H^0(E) = 0$ implies the μ-stability. When the Picard number is larger than one, however, this is not true and it is natural to consider also μ-semistable bundles (see [45] and [78] Remark 2.2).

In this line also the definition on $F(0,1,2)$ may be generalized to any Fano threefold with Picard number larger than one (see [27] for the case of the blow up of the projective 3-space at a point).

A really interesting subject concerning instanton bundles is the study of their moduli space. The subspace of stable instanton bundles with a given c_2 can be identified with the open subspace of the Maruyama moduli space of stable rank two bundles with those fixed Chern classes satisfying the cohomological vanishing condition. For a large family of Fano threefolds with Picard number one, Faenzi in [49, Theorem A] proves that the moduli spaces of instanton bundles has a generically smooth irreducible component. An analogous result has been obtained in [78] for the flag threefold. In the case of $\mathbb{P}^3$, it is known that the moduli space of instantons of arbitrary charge is affine (see [44]), irreducible (see [89], [90]) and smooth (see [32], [71] for charge smaller than 5 and [68] for arbitrary charge). The rationality is still an open problem in general, being settled only for charges 1, 2, 3, and 5 (see [15], [58], [48] and [70]).

In chapter 4 of this thesis (see the introduction of the chapter for the statements of the main results) we contribute to the study of instanton bundles on Fano threefolds, dealing with $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which has the same index and degree of $F(0,1,2)$ but its Picard number is three. We also slightly generalize some results on the flag variety $F(0,1,2)$. Let us call h_1, h_2 and h_3 the three generators of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. The only difference with respect to the definition of instanton bundle on $F(0,1,2)$ is that on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ we allow any possible second Chern class $c_2(E)=k_1h_2h_3+k_2h_1h_3+k_3h_1h_2$ instead of restricting to $c_2(E)=kh_1h_2$, and we define the charge to be $k=k_1+k_2+k_3$. By using a Beilinson type spectral sequence with suitable full exceptional collections we construct two different monads which are the analog of the monads for instanton bundles on $\mathbb{P}^3$ and on $F(0,1,2)$.

This part of the the thesis is also related to the first one concerning Ulrich bundles. Indeed there is a connection between Ulrich and instanton bundles. Notice that when the charge is minimal, namely $k=2$, any instanton bundle is, up to twist, an Ulrich bundle. In the case of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, in [28] Casnati, Faenzi and Malaspina showed that there exist rank two Ulrich bundles with either $c_1(E)=2h$ or $c_1=h_1+2h_2+3h_3$. It is straightforward to see that any rank two Ulrich bundle with $c_1=2h$ is an instanton bundle twisted by h. However each rank two Ulrich bundle with $c_1=h_1+2h_2+3h_3$ cannot be the twist of an instanton bundle because of numerical conditions on $c_1(E)$. On the two Fano threefolds with index two and Picard number two every instanton bundle with minimal degree of $c_2(E)$ is a twist of an Ulrich bundle (see [78] and [27]). If E is an instanton bundle on a Fano threefold X with index two and Picard number one we have $h^1(E)=c_2(E)-2$ and $h^0(E)=h^2(E)=h^3(E)=0$ (see section 3 of [49]). So when the charge is minimal, namely $c_2(E)=2$, by Serre duality we have $h^i(E)=h^i(E(-1))=h^i(E(-2))=0$ for any i, so E is Ulrich up to a twist. On the three dimensional quadric (the only case of index three) each instanton bundle of minimal charge is the spinor bundle which is Ulrich. The monad for instanton bundles on $\mathbb{P}^3$ is

$$0\to\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k}\xrightarrow{\alpha}\mathcal{O}_{\mathbb{P}^3}^{\oplus 2k+2}\xrightarrow{\beta}\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus k}\to 0,$$

so when $k=0$ we get $E=\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$. This is not an instanton bundle because is not simple. However, from a monadic point of view, we may say that the trivial bundle $E=\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$, which is Ulrich, is a limit case. The case of index one is much more complicated (see section 4 of [49]).

In chapter 4, starting from Ulrich bundles, we are able to show, via an induction process, the existence instanton bundles on $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ for every charge and every second Chern class. In this process we are able to construct, similarly to [49] and [78], a nice component of the moduli space of instanton bundles, i.e. generically smooth and irreducible. Furthermore the vector bundles living in this component are generically trivial when restricted to lines. Since minimal charge instanton bundles are Ulrich, we show that the generic Ulrich bundle is trivial when restricted to the generic line. Apart from being the base case of induction, this could lead to a better understanding of rank two Ulrich bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The construction proceeds as follows: take a rank two Ulrich bundle which is trivial when restricted to the generic line. Then apply an elementary transformation along a generic line to obtain a torsion free sheaf with increased c_2 and deform it to a locally free sheaf. As a consequence, the generic instanton bundle on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ will be trivial when restricted on the generic line of each family. Thus it is natural to study the locus inside the Hilbert scheme of lines consisting of lines for which the restriction of the generic instanton is not trivial, i.e. the locus of *jumping lines*. We show that the locus of jumping lines form a divisor inside the Hilbert scheme of lines and we describe it.

Another approach to better study and understand the moduli space of instanton bundles has been given by the *'t Hooft* bundles. They are defined to be instanton bundles on $\mathbb{P}^3$ having sections at the first twist (i.e. $H^0(E(1)) \neq 0$) and with $c_1 = 0$ and $c_2 \geq 2$. These vector bundles correspond to smooth points in the moduli space of instanton bundles. In particular the zero locus of a section of $E(1)$ is given by a disjoint union of lines. Starting from this notion, we generalized the concept of 't Hooft bundles to the Fano varieties $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $F(0,1,2)$. The main difference with the $\mathbb{P}^3$ case is that it is no longer possible to consider only lines, because through the Hartshorne-Serre correspondence, such curves are in correspondence with bundles with sections. Thus in our case a 't Hooft bundle E is an instanton bundle such that $H^0(E(h_i)) \neq 0$ for some i, and the generic section of $E(h_i)$ vanishes along a one-dimensional subscheme which is the disjoint union of lines and conics. These vector bundles are special points in the moduli space of instanton bundles, indeed one expects that the generic instanton start to have section when twisted by a large number. Nevertheless they allow us to construct a nice component of the moduli space of instanton bundles.

Chapter 2

Preliminary notions

2.1 Vector bundles

We will work over an algebraically closed field k of characteristic zero.

For any coherent sheaf E on X we will denote the twisted sheaf $E \otimes \mathcal{O}_X(l)$ by $E(l)$. $E^\vee$ will indicate the dual sheaf $\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ and $\mathcal{E}nd(E) = \mathcal{H}om_{\mathcal{O}_X}(E, E)$ stands for the sheaf of endomorphisms of E while $\mathrm{End}(E) = \mathrm{Hom}(E, E)$ will denote the group. $H^i(X, E)$ will denote, as usual, the cohomology groups of the sheaf E and $h^i(X, E)$ will denote their dimensions. We will also write $\mathrm{ext}^i(E, F) = \dim_k \mathrm{Ext}^i(E, F)$ for the dimension of the extension groups. We will use the notation $H^i_*(X, E)$ for the graded module $\bigoplus_{l \in \mathbb{Z}} H^i(X, E(l))$ and ω_X will denote the dualizing sheaf. We will write $c_i(E)$ for the i-th Chern class of a coherent sheaf E and we will denote the Picard group of X, i.e. the group isomorphism classes of line bundles, by $\mathrm{Pic}(X)$.

In this work by a vector bundle on X we will mean a locally free sheaf of constant rank over a projective variety X.

2.1.1 Construction of vector bundles

In this section we present the main tools to construct vector bundles that we will use in the next chapters. Let us start with the Hartshorne-Serre correspondence.

Let F be a rank two vector bundle on X and let $s \in H^0(X,F)$ be a section. In general its zero-locus $(s)_0 \subseteq X$ is either empty or its codimension is at most 2. We can always write $(s)_0 = S \cup Z$ where Z has codimension 2 (or it is empty) and S has pure codimension 1 (or it is empty). In particular $F(-S)$ has a section vanishing on Z, thus we can consider its Koszul complex

$$0 \to \mathcal{O}_X(S) \to F \to \mathcal{I}_{Z|X}(-S) \otimes \det(F) \to 0. \tag{2.1.1}$$

Sequence (2.1.1) tensorized by $\mathcal{O}_Z$ yields $\mathcal{I}_{Z|X}/\mathcal{I}^2_{Z|X} \cong F^\vee(S) \otimes \mathcal{O}_Z$, thus

$$\mathcal{N}_{Z|X} \cong F(-S) \otimes \mathcal{O}_Z. \tag{2.1.2}$$

If $S = 0$, then Z is locally complete intersection inside X, because $\mathrm{rk}(F) = 2$. In particular, it has no embedded components. The above construction can be reversed as follows.

Theorem 2.1. *Let $Z \subseteq X$ be a local complete intersection subscheme of codimension 2. If $\det(\mathcal{N}_{Z|X}) \cong \mathcal{O}_Z \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Pic}(X)$ such that $h^2(X,\mathcal{L}^\vee) = 0$, then there exists a vector bundle F of rank two on X such that:*

(i) $\det(F) \cong \mathcal{L}$;

(ii) F has a section s such that Z coincides with the zero locus $(s)_0$ of s.

Moreover, if $H^1(X,\mathcal{L}^\vee) = 0$, the above two conditions determine F up to isomorphism.

Proof. See [9]. □

Let us focus on the surface case. Let X be a smooth, projective irreducible surface. As in Theorem 2.1 we will relate rank two vector bundles on a surface X with subscheme of codimension two.

Theorem 2.2. *[61, Theorem 5.1.1] Let $Z \subset X$ be a local complete intersection of codimension two, and let L and M be line bundles on X. Then there exists an extension*

$$0 \to L \to E \to M \otimes \mathcal{I}_{Z|X} \to 0$$

such that E is locally free if and only if the pair $(L^\vee \otimes M \otimes K_X, Z)$ has the Cayley-Bacharach property:
(CB) If $Z' \subset Z$ is a subscheme with $l(Z') = l(Z) - 1$ and $s \in H^0(X, L^\vee \otimes M \otimes K_X)$ with $s_{|Z'} = 0$, then $s_{|Z} = 0$.

Observe that the Cayley-Bacharach property clearly holds for all Z if $H^0(X, L^\vee \otimes M \otimes K_X) = 0$.

We will conclude this section by describing how to construct vector bundles on surfaces via elementary transformation.

Definition 2.3. Let C be an effective divisor on the surface X. If F and G are vector bundles on X and C, respectively, then a vector bundle E is obtained by an elementary transformation of F along G if there exists an exact sequence

$$0 \to E \to F \to i_*G \to 0$$

where i denotes the embedding $C \subset X$.

Proposition 2.4. *If F and G are locally free on X and C, respectively, then the kernel E of any surjection $\phi : F \to i_*G$ is locally free. Moreover, if ρ denotes the rank of G, one has* $\det(E) \cong \det(F) \otimes \mathcal{O}_X(-\rho C)$ *and* $c_2(E) = c_2(F) - \rho C c_1(F) + \frac{1}{2}\rho C(\rho C + K_X) + \chi(G)$.

Observe that a trivial example of an elementary transformation is given by $\mathcal{O}_X(-C)$ which fits into the short exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

i.e. is the elementary transformation of $\mathcal{O}_X$ along $\mathcal{O}_C$.

2.1.2 Moduli spaces

In this section we will state the main definitions and results about moduli spaces of sheaves. For the proofs of the statements and more details see [61, Chapter 4].

Let $X \subset \mathbb{P}^N$ be a smooth n-dimensional projective variety and let H be a very ample divisor on X. Given a coherent sheaf E on X, the *Euler characteristic* of E is defined as

$$\chi(E) = \sum_{i=0}^{n} (-1)^i h^i(E).$$

The Hilbert polynomial $P_E(m)$ is given by

$$m \to \chi\big(E \otimes \mathcal{O}_X(m)\big) / \operatorname{rk}(E).$$

Definition 2.5. Let X be a smooth irreducible projective variety of dimension n and let H be an ample line bundle on X.

(i) For a torsion free sheaf E on X let

$$\mu_H(E) := \frac{c_1(E)H^{n-1}}{\operatorname{rk}(E)}$$

be the slope of E with respect to H. The sheaf E is μ-*semistable* with respect to the polarization H if and only if

$$\mu_H(F) \leq \mu_H(E)$$

for all non-zero subsheaves $F \subset E$ with $\operatorname{rk}(F) < \operatorname{rk}(E)$. If the strictly inequality holds then E is μ-*stable* with respect to H.

(ii) For a torsion free sheaf E on X let

$$P_E(m) := \frac{\chi\big(E \otimes \mathcal{O}_X(m)\big)}{\operatorname{rk}(E)}$$

be its Hilbert polynomial. The sheaf E is *Gieseker semistable* with respect to the polarization H if and only if

$$P_F(m) \leq P_E(m) \quad \text{for} \quad m >> 0$$

for all non-zero subsheaves $F \subset E$ with $\operatorname{rk}(F) < \operatorname{rk}(E)$. If the strictly inequality holds then E is *Gieseker stable* with respect to H.

Remark 2.6. We will simply say μ-(semi)stable or stable when there is no confusion on H. Furthermore the two notions of stability are related by the following implications

$$\mu-\text{stable} \Rightarrow \text{Gieseker stable} \Rightarrow \text{Gieseker semistable} \Rightarrow \mu-\text{semistable}.$$

Also it is worth to notice that both notion of stability strongly depends on the choice of the ample line bundle H.

Theorem 2.7. *Let X be a smooth projective variety of dimension $n \geq 2$ and let H be an ample line bundle on X. Then for any rank r torsion-free, μ-semistable with respect to H, sheaf E on X we have*

$$\Delta(E) := (2rc_2(E) - (r-1)c_1^2(E))H^{n-2} \geq 0.$$

The class $\Delta(E)$ is called the discriminant of E.

Let $\mathcal{C}$ be a category and let $\mathcal{M} : \mathcal{C} \to (Sets)$ be a contravariant moduli functor.

Definition 2.8. We say that a moduli functor $\mathcal{M} : \mathcal{C} \to (Sets)$ is represented by an object $M \in Ob(\mathcal{C})$ if it is isomorphic to the functor of points of M defined by $h_M(S) = Hom_{\mathcal{C}}(S, M)$. The object M is called a *fine moduli space for the moduli functor $\mathcal{M}$.*

If a fine moduli space exists, it is unique up to isomorphism. Since it is quite rare that a fine moduli space exists it is necessary to find some other weaker conditions. So we have the following

Definition 2.9. We say that a moduli functor $\mathcal{M} : \mathcal{C} \to (Sets)$ is corepresented by an object $M \in Ob(\mathcal{C})$ if there is a natural transformation $\alpha : \mathcal{M} \to h_M$ such that $\alpha(\{pt\})$ is bijective and for any object $N \in Ob(\mathcal{C})$ and any natural transformation $\beta : \mathcal{M} \to h_N$ there exists a unique morphism $\phi : M \to N$ such that $\beta = h_\phi \alpha$. The object M is called a *coarse moduli space* for the contravariant moduli functor $\mathcal{M}$. Furthermore if such a space exists, it is unique up to isomorphism.

In this work we will deal with the following contravariant moduli functor. Let X be a smooth, irreducible projective variety over an algebraically closed field k of characteristic zero. For a fixed polynomial $P \in \mathbb{Q}[t]$, we consider the contravariant moduli functor

$$\mathcal{M}_X^P(-) : (Sch/k) \to (Sets)$$
$$S \mapsto \mathcal{M}_X^P(S),$$

where $\mathcal{M}_X^P(S) = \{$S-flat families $F \to X \times S$ of vector bundles on X all whose fibers have Hilbert polynomial P$\}/\sim$, with $F \sim F'$ if and only if $F \cong F' \otimes p^*L$ for some $L \in \mathrm{Pic}(S)$ where $p : S \times X \to S$ is the natural projection. If $f : S \to S'$ is a morphism of schemes, $\mathcal{M}_X^P(f)(-)$ is the map obtained by pulling-back sheaves via $f_X = f \times id_X$.

If $\mathcal{M}_X^P$ exists is unique up to isomorphism. Nevertheless, in general the contravariant moduli functor $\mathcal{M}_X^P(-)$ is not representable. To get a coarse moduli space we must restrict to stable vector bundles.

Definition 2.10. Let X be a smooth, irreducible, projective variety of dimension n over an algebraically closed field k of characteristic 0 and let H be an ample divisor on X. For a fixed polynomial $P \in Q[t]$, we consider the contravariant moduli subfunctor $\mathcal{M}_X^{s,H,P}(-)$ of the contravariant moduli functor $\mathcal{M}_X^P(-)$, where $\mathcal{M}_X^{s,H,P}(S) = \{$S-flat families $F \to X \times S$ of vector bundles on X all whose fibers are μ-stable with respect to H and have Hilbert polynomial P$\}/\sim$.

Theorem 2.11. *The contravariant moduli functor $\mathcal{M}_X^{s,H,P}$ has a coarse moduli scheme $M_X^{s,H,P}$ which is separated and of finite type over k, i.e.:*

(i) There exists a natural transformation

$$\Phi : \mathcal{M}_X^{s,H,P}(-) \to \mathrm{Hom}(-, M_X^{s,H,P})$$

which is bijective on any reduced point x.

(ii) For every scheme N and any natural transformation

$$\Psi : \mathcal{M}_X^{s,H,P}(-) \to \mathrm{Hom}(-, N),$$

there exists a unique morphism $\alpha : M_X^{s,H,P} \to N$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}_X^{s,H,P}(-) & \xrightarrow{\quad\Phi\quad} & \mathrm{Hom}(-, M_X^{s,H,P}) \\ & \searrow^{\Psi} \quad \swarrow_{\alpha_*} & \\ & \mathrm{Hom}(-,N) & \end{array}$$

Moreover, $M_X^{s,H,P}$ *decompose as a disjoint union* $M^s_{X,H}(r;c_1,\dots,c_{\min(r,n)})$ *of moduli spaces of rank r* μ*-stable vector bundles with Chern classes* $(c_1,\dots,c_{\min(r,n)})$ *up to numerical equivalence.*

One of the main problems in algebraic geometry is to determine the local and global structure of $M^s_{X,H}(r;c_1,\dots,c_{\min(r,n)})$. We have the following bounds for the local dimension:

Proposition 2.12. *Let X be a n-dimensional smooth irreducible projective variety, let H be an ample divisor and let E be a* μ*-stable vector bundle of rank r with Chern classes* $c_i \in H^{2i}(X,\mathbb{Z})$*, represented in* $M^s_{X,H}(r;c_1,\dots,c_{\min(r,n)})$ *by a point* $[E]$*. Then the Zarinski tangent space of* $M^s_{X,H}(r;c_1,\dots,c_{\min(r,n)})$ *at* $[E]$ *is canonically isomorphic to*

$$T_{[E]}M^s_{X,H}(r;c_1,\dots,c_{\min(r,n)}) \cong \mathrm{Ext}^1(E,E).$$

Moreover we have the following bounds

$$\mathrm{ext}^1(E,E) \geq \dim_{[E]} M^s_{X,H}(r;c_1,\dots,c_{\min(r,n)}) \geq \mathrm{ext}^1(E,E) - \mathrm{ext}^2(E,E).$$

In particular, if $\mathrm{Ext}^2(E,E)=0$ *then* $M^s_{X,H}(r;c_1,\dots,c_{\min(r,n)})$ *is smooth at* $[E]$ *and*

$$\dim_{[E]} M^s_{X,H}(r;c_1,\dots,c_{\min(r,n)}) = \mathrm{ext}^1(E,E).$$

We conclude this section by recalling the definition of simple and indecomposable vector bundles.

Definition 2.13. Let X be a projective variety and let E be a vector bundle on X. E is called *simple* if the only endomorphisms are the homoteties, i.e. $\mathrm{End}(E) = k$. E is called *indecomposable* if it cannot be written as $E \cong F \oplus G$ where F and G are non-zero vector bundles.

These notions are related as follows:

Lemma 2.14. *Let $X \subseteq \mathbb{P}^N$ be a projective variety and let E be a vector bundle on X. Then we have the following implications:*

$$E \text{ is } \mu\text{-stable} \Rightarrow E \text{ is simple} \Rightarrow E \text{ is indecomposable}.$$

2.2 Derived categories

In this section we will state some preliminary facts on derived categories. For more details on the topic see for example [54], [56].

Definition 2.15. Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ and $B^\bullet$ be two complexes over $\mathcal{A}$ and let $k = (k^i) : A^i \to B^{i-1}$ be a collection of morphisms of sheaves. Then the maps

$$h = kd + dk : A^\bullet \to B^\bullet,$$

i.e.

$$h^i = k^{i+1} d_A^i + d_B^{i-1} k^i : A^i \to B^i$$

form a morphism of complexes.

The morphism $h = A^\bullet \to B^\bullet$ is said to be homotopic to 0, and we will write $h \sim 0$. Morphisms $f, g : A^\bullet \to B^\bullet$ are said to be homotopic if $f - g = kd + dk \sim 0$ and k is the corresponding homotopy. If $f \sim g$, then $H^\bullet(f) = H^\bullet(g)$ where $H^\bullet(\bullet)$ is the induced map in cohomology.

Definition 2.16. A morphism $f : A^\bullet \to B^\bullet$ of complexes in an abelian category $\mathcal{A}$ is said to be a *quasi-isomorphism* if the corresponding cohomology morphisms $H^n(f) : H^n(A^\bullet) \to H^n(B^\bullet)$ are isomorphisms for any n.

Definition 2.17. Let $\mathcal{A}$ be and abelian category and $\mathrm{Kom}(\mathcal{A})$ the category of complexes over $\mathcal{A}$. There exists a category $D(\mathcal{A})$ and a functor $Q : \mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$ with the following properties:

a) $Q(f)$ is an isomorphism for any quasi-isomorphism f.

b) Any functor $F : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ transforming quasi-isomorphisms into isomorphisms can be uniquely factorized through $D(\mathcal{A})$ i.e. there exists a unique functor $G : D(\mathcal{A}) \to \mathcal{D}$ with $F = G \circ Q$.

The category $D(\mathcal{A})$ is called the derived category of the abelian category $\mathcal{A}$

In what follows we will work with a full subcategory of $D(\mathcal{A})$. Let

$$\mathrm{Kom}^{+}(\mathcal{A}) : K^{i} = 0 \text{ for } i \leq i_0(K^{\bullet}),$$
$$\mathrm{Kom}^{-}(\mathcal{A}) : K^{i} = 0 \text{ for } i \geq i_0(K^{\bullet})$$

$$\mathrm{Kom}^{b}(\mathcal{A}) = \mathrm{Kom}^{+}(\mathcal{A}) \cap \mathrm{Kom}^{-}(\mathcal{A}).$$

These are full subcategories in $\mathrm{Kom}(\mathcal{A})$ and if we localize by quasi-isomorphism it is possible to form the corresponding derived categories $D^{+}(\mathcal{A})$, $D^{-}(\mathcal{A})$ and $D^{b}(\mathcal{A})$.

2.2.1 Derived category of coherent sheaves

Given a smooth projective variety X, we will denote by $D^b(X)$ be the bounded derived category of coherent sheaves on X.

Definition 2.18. Let X be a smooth projective variety.

(i) An object $E \in D^b(X)$ is called *exceptional* if $\mathrm{Hom}^{\bullet}_{D^b(X)}(E,E)$ is a 1-dimensional algebra generated by the identity.

(ii) An ordered collection $(E_0, E_1, \ldots, E_m)$ of objects in $D^b(X)$ is called an *exceptional collection* if $\mathrm{Hom}^{\bullet}_{D^b(X)}(E_i, E_j) = 0$ for $j < i$.

(iii) An exceptional collection $(E_0, E_1, \ldots, E_m)$ of objects in $D^b(X)$ is called a *strongly exceptional collection* if $\mathrm{Ext}^{k}_{D^b(X)}(E_i, E_j) = 0$ for $j \geq i$ and $k \neq 0$.

(iv) An ordered collection $(E_0, E_1, \ldots, E_m)$ of objects in $D^b(X)$ is a *full (strongly) exceptional collection* if it is a (strongly) exceptional collection and $E_0, E_1, \ldots, E_m$ generate the bounded derived category $D^b(X)$.

Remark 2.19. The existence of a full strongly exceptional collection $(E_0, \ldots, E_m)$ of coherent sheaves on a smooth projective variety X imposes that the Grothendieck group $K_0(X) = K_0(\mathcal{O}_X - mod)$ is isomorphic to $\mathbb{Z}^{m+1}$.

Definition 2.20. Let X be a smooth projective variety and let (A, B) be an exceptional pair of objects of $D^b(X)$. Let us consider the following distinguished triangles on the category $D^b(X)$

$$L_A B \to \mathrm{Hom}^\bullet(A,B) \otimes A \to B \to L_A B[1] \tag{2.2.1}$$

$$R_B A[-1] \to A \to \mathrm{Hom}^{\vee\bullet}(A,B) \otimes B \to R_B A. \tag{2.2.2}$$

A *left mutation* of an exceptional pair $\sigma = (A,B)$ is the pair

$$L_A \sigma = (L_A B, A) = (LB, A)$$

and a *right mutation* of an exceptional pair $\sigma = (A,B)$ is the pair

$$R_B \sigma = (B, R_B A) = (B, RA).$$

Lower indices will be omitted whenever this does not cause confusion.

Definition 2.21. Let X be a smooth projective variety and let $\sigma = (E_0, \dots, E_m)$ be an exceptional collection of objects of $D^b(X)$. A *left mutation* (resp. *right mutation*) of σ is defined as a mutation of a pair of adjacent objects in this collection, i.e. for any $i \le m$ a left mutation L_i replaces the i-th pair of consequent elements (E_{i-1}, E_i) by its left mutation $(L_{E_{i-1}} E_i, E_{i-1})$ and a right mutation R_i replaces the same pair of consequent elements (E_{i-1}, E_i) by its right mutation $(E_i, R_{E_i} E_{i-1})$:

$$L_i \sigma = L_{E_{i-1}} \sigma = (E_0, \dots, L_{E_{i-1}} E_i, E_{i-1}, \dots, E_m),$$
$$R_i \sigma = R_{E_i} \sigma = (E_0, \dots, E_i, R_{E_i} E_{i-1}, \dots, E_m).$$

Remark 2.22. (i) If X is a smooth projective variety and $\sigma = (F_0, \dots, F_m)$ is an exceptional collection of objects of $\mathcal{D}$, then any mutation of σ is an exceptional collection. Moreover, if σ generates the category $D^b(X)$, then the mutated collection also generates $D^b(X)$.

(ii) In general the mutation of strongly exceptional collection is not a strongly exceptional collection. In fact, take $X = \mathbb{P}^1 \times \mathbb{P}^1$ and consider the full strongly exceptional collection $\sigma = (\mathcal{O}_X, \mathcal{O}_X(1,0), \mathcal{O}_X(0,1), \mathcal{O}_X(1,1))$ of line bundles on X. It is not difficult to check that the mutated collection

$$(\mathcal{O}_X, \mathcal{O}_X(1,0), L_{\mathcal{O}_X(0,1)} \mathcal{O}_X(1,1), \mathcal{O}_X(0,1))$$

is no more a strongly exceptional collection of line bundles on X.

In the next sections we will use the Beilinson's spectral sequence in order to obtain a complex whose cohomology coincides with certain vector bundles (for this version of the theorem see [8, Theorem 2.5]). We start by stating that for every coherent sheaves there exists a spectral sequence which degenerates to the sheaf itself.

Theorem 2.23 (Beilinson-type spectral sequence)**.** *Let X be a smooth projective variety with a full exceptional collection $(E_0,\dots,E_n)$ where $E_i = \mathcal{E}_i^*[-k_i]$, with each $\mathcal{E}_i$ a vector bundle, and $(k_0,\dots,k_n)\in\mathbb{Z}^{n+1}$ such that there exists a sequence $(\mathcal{F}_0 = F_0,\dots,\mathcal{F}_n = F_n)$ of vector bundles satisfying*

$$\mathrm{Ext}^k(E_i,F_j) = H^{k+k_i}(\mathcal{E}_i\otimes\mathcal{F}_j) = \begin{cases}\mathbb{C} & \text{if } i=j=k,\\ 0 & \text{otherwise.}\end{cases} \tag{2.2.3}$$

$(F_n,\dots,F_0)$ is the right dual collection of $(E_0,\dots,E_n)$. Then for every coherent sheaf A on X there is a spectral sequence in the square $-n\le p\le 0,\ 0\le q\le n$ with E_1-term

$$E_1^{p,q} = H^{q+k_{-p}}(\mathcal{E}_{-p}\otimes A)\otimes\mathcal{F}_{-p}$$

which is functorial in A and converges to

$$E_\infty^{p,q} = \begin{cases} A & \text{if } p+q=0,\\ 0 & \text{otherwise.}\end{cases}$$

It is possible to state a stronger version of the Beilinson's theorem. Namely, given additional conditions on a exceptional collection E of the variety, every coherent sheaf is quasi isomorphic to a complex exact everywhere except in degree 0. Every term of this complex is a direct sum of objects of the dual collection of E.

Theorem 2.24. *[18][1] Let A_i^j, B_i^j, with $i=1,\dots,n$ and $j=1,\dots,k_i$ be bundles on a smooth projective variety X and denote by p and q the two projections of $X\times_X X$ on X. Suppose that we have*

(i) a resolution of the diagonal $\Delta_X\subset X\times X$ given by

$$\cdots\to\bigoplus_{j=1}^{k_2} p^*A_2^j\otimes q^*B_2^j\to\bigoplus_{j=1}^{k_1} p^*A_1^j\otimes q^*B_1^j\to\mathcal{O}_{X\times X}\to\mathcal{O}_\Delta\to 0.$$

(ii) $\mathrm{Ext}^p_X(B^j_i, B^s_t) = 0$ *for* $p > 0$ *and for all* i, j, t, s.

Then each complex $F^\bullet$ *on* X *is obtained as the cohomology of a complex* $\mathcal{C}^\bullet_F$ *with*

$$\mathcal{C}^p_F = \bigoplus_{s-i=p} \bigoplus_{j=1}^{k_i} H^s(X, F^\bullet \otimes A^j_I) \otimes B^j_i$$

so that B^j_i *are building block for the sheaves on* X

The idea is to take the resolution of the diagonal and apply the Fourier-Mukai functor

$$Rq_*(p^*(-) \otimes \mathcal{C}_\Delta)$$

where $\mathcal{C}$ is the complex resolving the diagonal and p and q are the two natural projections from $X \times X$ to X. In this way we obtain a complex quasi isomorphic to $F^\bullet$, but with morphism defined in derived category. Condition (ii) ensures that these morphisms actually arise from true morphism of sheaves.

Kapranov showed that if $D^b(X)$ admits a strong full exceptional collection of locally free sheaves $(E_0, \dots, E_m)$ then there exists a resolution of diagonal in terms of these sheaves and their derived category duals, and every complex on X is quasi-isomorphic to a complex whose terms are direct sums of the $E_i's$ [69, 2.14-2.17].

Let us rephrase these results.

Proposition 2.25. *Let X be a smooth projective variety and let A be a coherent sheaf on X. Let* $(E_0, \dots, E_n)$ *be a full exceptional collection of locally free objects and* $(F_n, \dots, F_0)$ *its right dual collection as in* (2.2.3) *and such that* $F_i's$ *are locally free. If* $\mathrm{Ext}^k(F_i, F_j) = 0$ *for* $k > 0$ *and all i, j, i.e.* $(F_n, \dots, F_0)$ *is strong, then there exists a complex of vector bundles* $L^\bullet$ *such that*

1. $H^k(L^\bullet) = \begin{cases} A & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$
2. $L^k = \bigoplus_{k=p+q} H^{q+k-p}(A \otimes \mathcal{E}_{-p}) \otimes \mathcal{F}_{-p}$ *with* $0 \le q \le n$ *and* $-n \le p \le 0$.

Example 2.26. The projective space $\mathbb{P}^n$ admits a full exceptional collection of the form $(\mathcal{O}_{\mathbb{P}^n}(-n), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n})$ whose dual is $(\mathcal{O}_{\mathbb{P}^n}(-1), \dots, \Omega^1_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n})$. Both collections are strong, thus both the $\Omega^j_{\mathbb{P}^n}(j)$ and $\mathcal{O}_{\mathbb{P}^n}(-j)$ with $j = 0, \dots, n$, can be considered as the building blocks of any coherent sheaf on the projective space (see [18] for the original result).

In the rest of this section we will recall to construct full exceptional collections for projective bundles. Indeed this will cover the construction of full exceptional collections for the varieties we considered in this thesis.

Let E be a rank r vector bundle over a smooth projective variety X. Then there exists a projective bundle $\mathbb{P}(E)$ with projection $p : \mathbb{P}(E) \to X$. We will denote by $D^b(X)$ and $D^b(E)$ the bounded derived category of coherent sheaves on X and $\mathbb{P}(E)$ respectively.

Proposition 2.27. *[86, Corollary 2.7] If there exists a full exceptional collection in the derived category $D^b(X)$ then the derived category $D^b(E)$ also posses an exceptional collection. Indeed, let $(E_0,\dots,E_n)$ be a full exceptional collection in $D^b(X)$. Then the collection*

$$(p^*E_0 \otimes \mathcal{O}_E(-r+1),\dots, p^*E_n \otimes \mathcal{O}_E(-r+1),\dots, p^*E_0,\dots, p^*E_n)$$

is a full exceptional collection in $D^b(E)$.

Let us consider the Hirzebruch surface $X_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \to \mathbb{P}^1$. Using Proposition 2.27 we have that the collection (corresponding to $\mathcal{E}_i$ in Theorem 2.23)

$$(\mathcal{O}_{X_e}(-1,-1), \mathcal{O}_{X_e}(-1,0), \mathcal{O}_{X_e}(0,-1), \mathcal{O}_{X_e}) \tag{2.2.4}$$

is a full exceptional collection for X_e, whose dual collection ($\mathcal{F}_i$ in Theorem 2.23)

$$(\mathcal{O}_{X_e}(-1,-e-1)[-1], \mathcal{O}_{X_e}(-1,-e)[-1], \mathcal{O}_{X_e}(0,-1), \mathcal{O}_{X_e}) \tag{2.2.5}$$

is obtained using the duality condition expressed in (2.2.3).

In an analogous way it is possible to compute a full exceptional collection for the Segre threefold $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Indeed we have that the collection

$$\begin{gathered}(\mathcal{O}_X(-h), \mathcal{O}_X(-h_2-h_3), \mathcal{O}_X(-h_1-h_3),\\ \mathcal{O}_X(-h_1-h_2), \mathcal{O}_X(-h_3), \mathcal{O}_X(-h_2), \mathcal{O}_X(-h_1), \mathcal{O}_X)\end{gathered} \tag{2.2.6}$$

which is self dual up to shift, i.e. the dual collection using (2.2.3) is given by

$$(\mathcal{O}_X(-h)[-4], \mathcal{O}_X(-h_2-h_3)[-4], \mathcal{O}_X(-h_1-h_3)[-3],$$
$$\mathcal{O}_X(-h_1-h_2)[-2], \mathcal{O}_X(-h_3)[-2], \mathcal{O}_X(-h_2)[-1], \mathcal{O}_X(-h_1), \mathcal{O}_X).$$

We conclude this section with some remarks on the resolution of the diagonal and mutations.

Example 2.28. Let us consider the quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. In [69] the author shows that the diagonal $\Delta \subset Q \times Q$ admits the following resolution

$$0 \to \begin{array}{c} \mathcal{O}_Q(-1,-2) \boxtimes \mathcal{O}_Q(-1,0) \\ \oplus \\ \mathcal{O}_Q(-2,-1) \boxtimes \mathcal{O}_Q(0,-1) \end{array} \to \mathcal{O}_Q(-1,-1) \boxtimes \Psi \to \mathcal{O}_{Q\times Q} \to \mathcal{O}_\Delta \to 0. \quad (2.2.7)$$

where Ψ is the restriction of $\Omega_{\mathbb{P}^3}$ to Q.

However it is also possible to consider the following resolution of the diagonal

$$0 \to \mathcal{O}_Q(-1,-1) \boxtimes \mathcal{O}_Q(-1,-1) \to \begin{array}{c} \mathcal{O}_Q(-1,0) \boxtimes \mathcal{O}_Q(-1,0) \\ \oplus \\ \mathcal{O}_Q(0,-1) \boxtimes \mathcal{O}_Q(0,-1) \end{array} \to \mathcal{O}_{Q\times Q} \to \mathcal{O}_\Delta \to 0. \quad (2.2.8)$$

In this example we show how to obtain (2.2.7) from (2.2.8) through consecutive mutations. Let us consider the full exceptional collection

$$E = (\mathcal{O}_Q(-1,-1), \mathcal{O}_Q(-1,0), \mathcal{O}_Q(0,-1), \mathcal{O}_Q),$$

whose dual is

$$F = (\mathcal{O}_Q(-1,-1)[-1], \mathcal{O}_Q(-1,0)[-1], \mathcal{O}_Q(0,-1), \mathcal{O}_Q).$$

Let us apply a left mutation to the pair $(\mathcal{O}_Q(-1,-1), \mathcal{O}_Q(-1,0))$ of objects of E. The mutated pair, using the Euler sequence of $\mathbb{P}^1$, is given by $(\mathcal{O}_Q(-1,-2), \mathcal{O}_Q(-1,-1))$. Since F is the dual collection of E, to a left mutation of objects of E correspond a right mutation of objects of F. In particular to a left mutation of the pair $(\mathcal{O}_Q(-1,-1),\mathcal{O}_Q(-1,0))$, corresponds a right mutation of the pair

$$(\mathcal{O}_Q(-1,-1)[-1],\mathcal{O}_Q(-1,0)[-1])$$

of objects of F. Thus the mutated pair will be $(\mathcal{O}_Q[-1](-1,0),\mathcal{O}_Q(-1,1)[-1])$. So the two mutated collections are

$$E' = (\mathcal{O}_Q(-1,-2),\mathcal{O}_Q(-1,-1),\mathcal{O}_Q(0,-1),\mathcal{O}_Q)$$

$$F' = (\mathcal{O}_Q(-1,0)[-1],\mathcal{O}_Q(-1,1)[-1],\mathcal{O}_Q(0,-1),\mathcal{O}_Q)$$

and (2.2.8) becomes

$$0 \to \mathcal{O}_Q(-1,-2)\boxtimes\mathcal{O}_Q(-1,0) \to \begin{matrix}\mathcal{O}_Q(-1,-1)\boxtimes\mathcal{O}_Q(-1,1)\\ \oplus \\ \mathcal{O}_Q(0,-1)\boxtimes\mathcal{O}_Q(0,-1)\end{matrix} \to \mathcal{O}_{Q\times Q} \to \mathcal{O}_\Delta \to 0.$$

Now let us consider the pair $(\mathcal{O}_Q(-1,-1),\mathcal{O}_Q(0,-1))$ in E'. Its left mutation is given by the pair $(\mathcal{O}_Q(-2,-1),\mathcal{O}_Q(-1,-1))$. The correspondent right mutation in F' is given by $R_{\mathcal{O}_Q(0,-1)}(\mathcal{O}_Q(-1,1)[-1],\mathcal{O}_Q(0,-1)) = (\mathcal{O}_Q(0,-1),R\mathcal{O}_Q(-1,1)[-1])$. Now we compute $R\mathcal{O}_Q(-1,1)[-1]$ using the distinguished triangle (2.2.2). In derived category $\mathrm{Hom}_D^\vee(\mathcal{O}_Q(-1,1)[-1],\mathcal{O}_Q(0,-1)) = \mathrm{Ext}^1_Q(\mathcal{O}_Q(-1,1),\mathcal{O}_Q(0,-1))^\vee$. Since $\mathrm{ext}^1_Q(\mathcal{O}_Q(-1,1),\mathcal{O}_Q(0,-1)) = 2$, we have $R\mathcal{O}_Q(-1,1)[-1] = \Psi$ where Ψ is an extension of type

$$0 \to \mathcal{O}_Q(0,-1)^2 \to \Psi \to \mathcal{O}_Q(-1,1) \to 0$$

i.e. Ψ is the restriction of $\Omega_{\mathbb{P}^3}$ to Q. Now the mutated collections are

$$E'' = (\mathcal{O}_Q(-1,-2),\mathcal{O}_Q(-2,-1),\mathcal{O}_Q(-1,-1),\mathcal{O}_Q)$$

$$F'' = (\mathcal{O}_Q(-1,0)[-1],\mathcal{O}_Q(0,-1),\Psi,\mathcal{O}_Q)$$

and we obtain the resolution of the diagonal as in (2.2.7)

$$0 \to \begin{matrix}\mathcal{O}_Q(-1,-2)\boxtimes\mathcal{O}_Q(-1,0)\\ \oplus \\ \mathcal{O}_Q(-2,-1)\boxtimes\mathcal{O}_Q(0,-1)\end{matrix} \to \mathcal{O}_Q(-1,-1)\boxtimes\Psi \to \mathcal{O}_{Q\times Q} \to \mathcal{O}_\Delta \to 0.$$

Chapter 3

Ulrich bundles on Hirzebruch surfaces

In this chapter we will characterize Ulrich bundles over Hirzebruch surfaces. In [6] Aprodu, Costa and Miró-Roig discussed the existence of Ulrich line bundles and special rank two Ulrich bundles, i.e. rank two Ulrich bundles E such that $\det(E) = 3h + K_X$, over ruled surfaces. Their existence implies that the associated Cayley-Chow form is represented as a linear pfaffian [47]. The authors proved that Ulrich line bundles over ruled surfaces exist only for a particular choice of the polarization $\mathcal{O}_X(h)$, and they proved the existence of special Ulrich bundles under a mild assumption on the polarization.

In [51] and [79] the authors considered the case of Hirzebruch surfaces embedded as rational normal scrolls.

In this chapter we prove the following theorem:

Theorem 3.1. *Let $(X_e, \mathcal{O}_{X_e}(a,b))$ be a polarized Hirzebruch surface and E a rank r Ulrich bundle with $c_1(E) = \alpha C_0 + \beta f$.*

1. *Then E fits into a short exact sequence of the form*

$$0 \to \mathcal{O}_{X_e}^{\gamma}(a-1, b-e-1) \to \mathcal{O}_{X_e}^{\delta}(a-1, b-e) \oplus \mathcal{O}_{X_e}^{\tau}(a, b-1) \to E \to 0$$

where $\gamma = \alpha + \beta - r(a+b-1) - e(\alpha - ar)$, $\delta = \beta - r(b-1) - e(\alpha - ar)$ and $\tau = \alpha - r(a-1)$.

2. *Then E fits into a short a exact sequence of the form*

$$0 \to E \to \mathcal{O}_{X_e}^{\lambda}(2a-1,2b-2) \oplus \mathcal{O}_{X_e}^{\mu}(2a-2,2b-1-e) \longrightarrow \mathcal{O}_{X_e}^{\nu}(2a-1,2b-1) \to 0$$

where $\lambda = r(2b-1-e) - \beta - e(r(2a-2) - \alpha)$, $\mu = r(2a-1) - \alpha$ *and* $\nu = r(2a+2b-3-e) - \alpha - \beta - e(r(2a-2) - \alpha)$.

Thus we are able to express E as the cokernel (resp. kernel) of a certain injective (resp. surjective) map, using derived categories techniques. A result of this type was obtained in [33] for the Veronese surface, and in [76] for the projective space $\mathbb{P}^N$ embedded with a very ample line bundle $\mathcal{O}_{\mathbb{P}^N}(d)$.

In [38] I. Coskun and Huizenga found a similar resolution with totally different methods. They used it to classify Chern characters such that the correspondent general stable bundle on X_e is globally generated.

Then we discuss the inverse problem: given an injective map ϕ as above, is $E = \mathrm{Coker}(\phi)$ an Ulrich bundle? Our answer is given by

Theorem 3.2. *Let* $(X_e, \mathcal{O}_{X_e}(h))$ *with* $h = aC_0 + bf$ *be a polarized Hirzebruch surface.*

1. *Let* ϕ *be an injective map*

$$\mathcal{O}_{X_e}^{\gamma}(a-1,b-e-1) \xrightarrow{\phi} \mathcal{O}_{X_e}^{\delta}(a-1,b-e) \oplus \mathcal{O}_{X_e}^{\tau}(a,b-1)$$

with δ, τ *non-negative,* γ *positive and* $\delta + \tau > \gamma$. *Let us call* $r = \delta + \tau - \gamma$ *and denote with E the cokernel of* ϕ. *In particular* $c_1(E) = \alpha C_0 + \beta f$ *with* $\alpha = \tau + r(a-1)$ *and* $\beta = r(b-1) + \delta - e(r - \tau)$. *If* $c_1(E)h = \frac{r}{2}(3h^2 + hK_{X_e})$ *then E is an Ulrich bundle if and only if* $H^2(E(-2h)) = 0$.

2. *If* ψ *is a surjective map*

$$\mathcal{O}_{X_e}^{\lambda}(2a-1,2b-2) \oplus \mathcal{O}_{X_e}^{\mu}(2a-2,2b-1-e) \xrightarrow{\psi} \mathcal{O}_{X_e}^{\nu}(2a-1,2b-1)$$

with λ, μ *non-negative,* ν *positive and* $\lambda + \mu > \nu$. *Let us call* $r = \lambda + \mu - \nu$ *and denote by E the kernel of* ψ. *In particular* $c_1(E) = \alpha C_0 + \beta f$ *with* $\alpha = r(2a-1) - \mu$ *and* $\beta = r(2b-1-e) - \lambda - e(\mu - r)$. *If* $c_1(E)h = \frac{r}{2}(3h^2 + hK_{X_e})$ *then E is an Ulrich bundle if and only if* $H^0(E(-h)) = 0$.

For the first case, by computing the long exact sequence in cohomology we see that $h^0(E(-h)) = h^1(E(-h)) = 0$ and that $h^1(E(-2h)) = h^2(E(-2h))$. Thus as soon as one of them vanishes, also the other does, and this is equivalent to the injectivity of the induced map $H^2(\phi(-2h))$. The second case is treated in an analogous way.

In the particular case of special Ulrich bundles, the induced map $H^2(\phi(-2h))$ is always an isomorphism. So we obtain an alternative (with respect to [6, Theorem 3.4]) proof of the existence of special Ulrich bundles, characterizing them as the cokernel of a map between very well understood vector bundles. For the other cases we offer a family of counterexamples which shows that the cohomological condition $h^1(E(-2h)) = 0$ is necessary in general, and we describe the locus where $\mathrm{Coker}(\phi)$ fails to be Ulrich. Moreover we are able to prove the following existence theorem:

Theorem 3.3. *Let us consider $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$. If the map ϕ as in Theorem 3.2 is general, then $E = \mathrm{Coker}(\phi)$ is Ulrich. In particular on $(X_e, \mathcal{O}_{X_e}(a,b))$ there exist Ulrich bundles for any admissible rank and first Chern class.*

The strategy is to use [38, Theorem 1.1] to obtain that the bundle E, realized as the cokernel of a general map ϕ, has the property that $E(-2h)$ has natural cohomology. Since $h^1(E(-2h)) = h^2(E(-2h))$ they cannot be both different from zero and thus E is Ulrich. This shows that every Hirzebruch surface admits Ulrich bundles of any admissible first Chern class, any admissible rank and with respect to every very ample polarization.

Starting from Theorem 3.3 we also discussed the existence of stable Ulrich bundles of rank greater than two with respect to polarizations that do not factor through a Veronese embedding.

Furthermore, using the computer software *Macaulay2*, Theorem 3.2 gives us a useful tool to construct examples of Ulrich bundles of higher rank with a fixed first Chern class.

Part of what follows can be found in [4].

3.1 General facts on Hirzebruch surfaces and Ulrich bundles

Let us begin with some definitions and results on Hirzebruch surfaces. For more details see [57, V.2].

Definition 3.4. A *geometrically ruled surface*, or simply *ruled surface*, is a surface X, together with a surjective morphism $\pi : X \to C$ to a (nonsingular) curve C, such that the fibre X_y is isomorphic to $\mathbb{P}^1$ for every point $y \in C$.

Note that as a consequence of this definition, π admits a section (i.e. a morphism $\sigma : C \to X$ such that $\pi \circ \sigma = \mathrm{id}_C$).

Proposition 3.5. *If $\pi : X \to C$ is a ruled surface, then there exists a locally free sheaf E of rank 2 on C such that $X \cong \mathbb{P}(E)$ over C. Conversely every such $\mathbb{P}(E)$ is a ruled surface over C. If E and E' are two locally free sheaves of rank 2 on C, then $\mathbb{P}(E)$ is a ruled surface over C, then $\mathbb{P}(E)$ and $\mathbb{P}(E')$ are isomorphic as ruled surfaces over C if and only if there exists an invertible sheaf L on C such that $E' \cong E \otimes L$.*

Now we continue with a description of the Picard group.

Proposition 3.6. *Let $\pi : X \to C$ be a ruled surface, let $C_0 \subseteq X$ be a section, and let f be a fibre. Then*

$$\operatorname{Pic} X \cong \mathbb{Z} \oplus \pi^* \operatorname{Pic} C,$$

where $\mathbb{Z}$ is generated by C_0. Also

$$\operatorname{Num} X \cong \mathbb{Z} \oplus \mathbb{Z},$$

generated by C_0, f, and satisfying $C_0 f = 1$, $f^2 = 0$.

Proposition 3.7. *If $\pi : X \to C$ is a ruled surface, it is possible to write $X \cong \mathbb{P}(E)$ where E is a vector bundle on C such that E has section, but for all invertible sheaves L on C with negative degree we have $H^0(E(L)) = 0$. In this case the integer $e = -\deg E$ is an invariant of X. Furthermore there is a section $\sigma : C \to X$ with image C_0, such that $L(C_0) \cong \mathcal{O}_X(1)$.*

Now we describe the canonical divisor on X.

Lemma 3.8. *The canonical divisor K_X on X is given by*

$$K_X = -2C_0 + (2g-2-e)f$$

where g is the genus of the curve C.

Definition 3.9. An *Hirzebruch surface* is a geometrically ruled surface over $\mathbb{P}^1$. In this case $e \geq 0$ and for each $e \geq 0$ there is exactly one Hirzebruch surface with invariant e, given by $\mathbb{P}(E) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ over $\mathbb{P}^1$. We will denote the Hirzebruch surface with invariant e by X_e.

Now let us consider a divisor D. It will be of the form $D = aC_0 + bf$. We have the following

Proposition 3.10. *Let D as above be a divisor on X_e. Then:*

$$D \text{ is very ample} \Leftrightarrow D \text{ is ample} \Leftrightarrow a > 0 \text{ and } b > ae.$$

Given a divisor $D = aC_0 + bf$ we will write the associated line bundle as $\mathcal{O}_{X_e}(a,b)$ or $\mathcal{O}_{X_e}(aC_0 + bf)$.

Furthermore on X_e there are two natural short exact sequences. The first one is

$$0 \to \mathcal{O}_{X_e}(0,-1) \to \mathcal{O}_{X_e}^2 \to \mathcal{O}_{X_e}(0,1) \to 0, \tag{3.1.1}$$

which is the pullback on X_e of the Euler sequence over $\mathbb{P}^1$. Other than that we also have a second natural exact sequence

$$0 \to \Omega_{X_e/\mathbb{P}^1} \to \pi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{X_e}(-1,0) \to \mathcal{O}_{X_e} \to 0$$

which, in this case, will take the form

$$0 \to \mathcal{O}_{X_e}(-1,-e) \to \mathcal{O}_{X_e} \oplus \mathcal{O}_{X_e}(0,-e) \to \mathcal{O}_{X_e}(1,0) \to 0. \tag{3.1.2}$$

Now we state the following lemma which shows how to compute the cohomology of the line bundles over X_e.

Lemma 3.11. *[57, Exercise III.8.1, III.8.4] Given $\mathcal{O}_{X_e}(tC_0+sf)$ a line bundle on $\pi : X_e \to \mathbb{P}^1$ then*

$$H^i(X_e, \mathcal{O}_{X_e}(tC_0+sf)) = \begin{cases} 0 \text{ if } t=-1 \\ H^i(\mathbb{P}^1, \mathrm{Sym}^t(\mathcal{E})\otimes \mathcal{O}_{\mathbb{P}^1}(s)) \text{ if } t \geq 0 \\ H^{2-i}(\mathbb{P}^1, \mathrm{Sym}^{-2-t}(\mathcal{E})\otimes \mathcal{O}_{\mathbb{P}^1}(-e-s-2)) \text{ if } t \leq -2, \end{cases}$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$.

Now we recall the main definitions and properties of Ulrich bundles and we present the main results on Hirzebruch surfaces. Let $X \subset \mathbb{P}^N$ be a smooth irreducible closed subscheme, let F be a vector bundle on X and let $\mathcal{O}_X(h)$ be the induced polarization. We say that:

- F is *initialized* if $h^0(X,F(-h)) = 0 \neq h^0(X,F)$.
- F is *aCM* if $h^i(X,F(th)) = 0$ for each $t \in \mathbb{Z}$ and $i = 1,\dots,\dim(X)-1$.
- F is *Ulrich* if $h^i(X,F(-ih)) = h^j(X,F(-(j+1)h)) = 0$ for each $i > 0$ and $j < \dim(X)$.

Proposition 3.12. *[24, Proposition 2.1] Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h)$. If E a vector bundle on S, then the following are equivalent:*

1. *E is an Ulrich bundle;*
2. *$E^\vee(3h+K_S)$ is an Ulrich bundle;*
3. *E is an aCM bundle and*

$$\begin{aligned} c_1(E)h &= \frac{\mathrm{rk}(E)}{2}(3h^2+hK_S), \\ c_2(E) &= \frac{1}{2}(c_1(E)^2 - c_1(E)K_S) - \mathrm{rk}(E)(h^2-\chi(\mathcal{O}_S)); \end{aligned} \tag{3.1.3}$$

4. *$h^0(S,E(-h)) = h^0(S,E^\vee(2h+K_S)) = 0$ and equalities (3.1.3) hold.*

Moreover, the Riemann-Roch theorem on a surface S is

$$\chi(F) = \frac{c_1^2(F)}{2} - \frac{c_1(F)K_S}{2} - c_2(F) + rk(F)\chi(\mathcal{O}_S) \tag{3.1.4}$$

for each locally free sheaf F on S.

We continue with some properties of Ulrich bundles.

Theorem 3.13. *[22, Theorem 2.9] Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, closed variety. If E is an Ulrich bundle on X then the following assertions hold.*

a) E is semistable and μ-semistable.

b) E is stable if and only if it is μ-stable.

c) if

$$0 \to L \to E \to M \to 0$$

is an exact sequence of coherent sheaves with M torsion free and $\mu(L) = \mu(E)$, then both L and M are Ulrich bundles.

Finally we state the main result about Ulrich bundles on Hirzebruch surfaces.

Proposition 3.14. *[6, Theorem 2.1] Let us consider $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ and $e > 0$. Then there exists Ulrich line bundles with respect to h if and only if $a = 1$, and they are*

$$\mathcal{O}_{X_e}(0, 2b-1-e) \text{ and } \mathcal{O}_{X_e}(1, b-1).$$

Now we consider rank two Ulrich bundles. Although the following result was proved in a more general context, we state it in the case of Hirzebruch surfaces.

Proposition 3.15. *[6, Proposition 3.1, 3.3, Theorem 3.4] Let us consider $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ and $e > 0$. Then*

1. *if $a = 1$ there exists a family of dimension $2b - e - 3$ of indecomposable, rank-two, simple, strictly semistable, special Ulrich bundles on X_e.*

2. *if $a \geq 2$ there exists special Ulrich bundles with respect to h given by extensions*

$$0 \to \mathcal{O}_{X_e}(a, b-1) \to E \to \mathcal{I}_Z(2a-2, 2b-1-e) \to 0,$$

where Z is a general zero-dimensional subscheme of X_e with $l(Z) = (a-1)(b - \frac{ea}{2})$.

In what follows we will show that every Ulrich bundle of any rank fits into a short exact sequence of totally decomposed vector bundles on X_e. Furthermore, given an injective map between these vector bundles we give necessary and sufficient conditions such that the cokernel of the map is Ulrich. In this way we are able to obtain the existence of Ulrich bundles of any admissible rank and first Chern class on X_e.

3.2 From the Ulrich bundle to the resolution

We start this section by describing the cohomology of an Ulrich bundle E on X_e.

Lemma 3.16. *Let E be a rank r Ulrich bundle on $(X_e, \mathcal{O}_{X_e}(a,b))$. Then*

1. $h^0(X_e, E(t,s)) = h^2(X_e, E(t,s)) = 0$ *for all* $-2a \leq t \leq -a$ *and* $-2b \leq s \leq -b$
2. $h^1(X_e, E(-a,s)) = h^2(X_e, E(-a,s)) = 0$ *for all* $s \geq -b$.
3. $h^1(X_e, E(t,s)) = h^2(X_e, E(t,s)) = 0$ *for all* $t \geq -a$ *and* $s \geq -b+e$.
4. $h^0(X_e, E(t,s)) = h^1(X_e, E(t,s)) = 0$ *for all* $t \leq -2a$ *and* $s \leq -2b-e$.

In particular $E(t,s)$ has natural cohomology (i.e. there exists at most one k such that $h^k(E(t,s)) \neq 0$) for such t and s.

Proof. For the first part of the lemma, since E is Ulrich, then $E(-a,-b)$ has no cohomology. For any effective divisor D we have the following short exact sequence

$$0 \to \mathcal{O}_{X_e}(-D) \to \mathcal{O}_{X_e} \to \mathcal{O}_D \to 0. \tag{3.2.1}$$

Tensoring (3.2.1) by $E(-a,-b)$ and considering the long exact sequence in cohomology, we obtain $h^0(X_e, E(t,s)) = 0$ for all $t \leq -a$ and $s \leq -b$. Using Serre's duality and the fact that $E^\vee(3h + K_{X_e})$ is Ulrich, we obtain $h^2(X_e, E(t,s)) = 0$ for all $t \geq -2a$ and $s \geq -2b$.

For the second part, we proceed by induction on s. We have $h^1(X_e, E(-a,-b)) = 0$ because E is Ulrich. Suppose $h^1(X_e, E(-a,k)) = 0$ for all $-b \leq k \leq s$, tensor (3.1.1) by $E(-a,s)$ and consider the long exact sequence in cohomology. Since

$h^1(X_e, E(-a,s)) = 0$ by inductive hypothesis, we have $h^1(X_e, E(-a,s+1)) = 0$, which proves (2).

For the third part we want to show that $h^1(X_e, E(t,s)) = 0$ for all $t \geq -a$ and $s \geq -b+e$, so we proceed by a double induction on t and s. Suppose $s \geq -b$, by (2) we have that $h^1(X_e, E(-a,s)) = 0$. Suppose that $h^1(X_e, E(k,s)) = 0$ for all $-a \leq k \leq t$ and $s \geq -b$, and tensor (3.1.2) by $E(t,s)$. Considering the long exact sequence induced in cohomology we have that if $s \geq -b+e$, then $h^1(X_e, E(t,s)) = h^1(X_e, E(t,s-e)) = 0$ by inductive hypothesis, and $h^2(X_e, E(t-1,s-e)) = 0$ because E is Ulrich, so we can conclude that $h^1(X_e, E(t+1,s)) = 0$ which proves (3).

For the last part recall that since E is Ulrich then the same holds for $E^\vee(3h+K_{X_e})$, so we obtain (4) using (3) and Serre's duality.

□

Now we will prove one of the main theorems of this work.

Theorem 3.17. *Let E be an Ulrich bundle of rank r on $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ and with first Chern class $c_1(E) = \alpha C_0 + \beta f$.*

1. *Then E fits into a short exact sequence of the form*

$$0 \to \mathcal{O}_{X_e}^{\gamma}(a-1, b-e-1) \to \mathcal{O}_{X_e}^{\delta}(a-1, b-e) \oplus \mathcal{O}_{X_e}^{\tau}(a, b-1) \to E \to 0 \tag{3.2.2}$$

   where $\gamma = \alpha + \beta - r(a+b-1) - e(\alpha - ar)$, $\delta = \beta - r(b-1) - e(\alpha - ar)$ and $\tau = \alpha - r(a-1)$.

2. *Then E fits into a short a exact sequence of the form*

$$0 \to E \to \mathcal{O}_{X_e}^{\lambda}(2a-1, 2b-2) \oplus \mathcal{O}_{X_e}^{\mu}(2a-2, 2b-1-e) \xrightarrow{\psi} \mathcal{O}_{X_e}^{\nu}(2a-1, 2b-1) \to 0 \tag{3.2.3}$$

   where $\lambda = r(2b-1-e) - \beta - e(r(2a-2) - \alpha)$, $\mu = r(2a-1) - \alpha$ and $\nu = r(2a+2b-3-e) - \alpha - \beta - e(r(2a-2) - \alpha)$.

Proof. We apply the Beilinson's Theorem to $E(-a,-b)$ as in Proposition 2.25. We start by computing the cohomology table of $E(-a,-b)$.

$\mathcal{O}_{X_e}(-1,-e-1)$	$\mathcal{O}_{X_e}(-1,-e)$	$\mathcal{O}_{X_e}(0,-1)$	$\mathcal{O}_{X_e}$	
0	0	0	0	h^3
γ	δ	0	0	h^2
0	0	τ	0	h^1
0	0	0	0	h^0
$E(-a-1,-b-1)[-1]$	$E(-a-1,-b)[-1]$	$E(-a,-b-1)$	$E(-a,-b)$	

Every column represents the dimension of the cohomology groups of the vector bundle at the bottom. The therms on the top of the table are the vector bundles that will appear in the Beilinson resolution as in Proposition 2.25. The shifts in the last two columns represent the k_i's in Theorem 2.23 and Proposition 2.25.

Since X_e is a surface, it follows that:

$$h^3(X_e, E(-a,-b)) = h^3(X_e, E(-a,-b-1)) = 0$$

and trivially

$$h^0(X_e, E(-a-1,-b-1))[-1] = h^0(X_e, E(-a-1,-b))[-1] = 0.$$

All the zeroes in the table are obtained using Lemma 3.16. Since all the vector bundles in the table have natural cohomology, we will use the Riemann-Roch theorem to compute the only non-zero cohomology groups. In general given a divisor D on X_e we have

$$c_1(E(D)) = c_1(E) + rD$$
$$c_2(E(D)) = c_2(E) + (r-1)c_1(E)D + \frac{r(r-1)}{2}D^2.$$

So by Riemann-Roch and using Proposition 3.12 we have

$$\chi(E(D)) = rh^2 + c_1(E)D + \frac{r}{2}D(D - K_{X_e}). \tag{3.2.4}$$

- $E(-a-1,-b-1)$.

  In this case $D = -h - C_0 - f$ so using Proposition 3.12 and equality (3.2.4) we have
  $$\chi(E(D)) = -\alpha - \beta + r(a+b-1) + e(\alpha - ra).$$

- $E(-a-1,-b)$.

  In this case $D = -h - C_0$ so as above
  $$\chi(E(D)) = -\beta + r(b-1) + e(\alpha - ar).$$

- $E(-a,-b-1)$.

  In this case $D = -h + f$ and
  $$\chi(E(D)) = -\alpha + r(a-1).$$

By Proposition 2.25 we have a short exact sequence

$$0 \to \mathcal{O}_{X_e}^{\gamma}(-1,-1-e) \to \mathcal{O}_{X_e}^{\delta}(-1,-e) \oplus \mathcal{O}_{X_e}^{\tau}(0,-1) \to E(-a,-b) \to 0 \tag{3.2.5}$$

with $\gamma = \alpha + \beta - r(a+b-1) - e(\alpha - ar)$, $\delta = \beta - r(b-1) - e(\alpha - ar)$ and $\tau = \alpha - r(a-1)$. Tensoring the sequence by $\mathcal{O}_{X_e}(h)$ we obtain part (1) of the theorem.

For part (2) recall that if E is Ulrich, then the same is true for $E^\vee(3h + K_{X_e})$. Applying Beilinson's theorem to $E^\vee(3h + K_{X_e})$, and dualizing the sequence, we obtain (2). □

Observe that Theorem 3.17 imposes some numerical necessary conditions that a vector bundle must satisfy in order to be Ulrich.

Corollary 3.18. Let E be a rank r vector bundle on $(X_e, \mathcal{O}_{X_e}(a,b))$ with first Chern class $c_1(E) = \alpha C_0 + \beta f$. If E is Ulrich then, using the notation of Theorem 3.17, δ, τ and γ (resp. λ, μ and ν) are non-negative, with δ and τ (resp. λ and μ) not both zero. Moreover if $a > 1$ and $e > 0$ then γ and τ (resp. μ and ν) are positive.

Proof. The non-negativity follows directly from the fact that the exponents of the resolutions (3.2.2) and (3.2.3) must be non-negative since they represent the dimension of a cohomology group. Since $\delta+\tau-\gamma=r>0$, δ and τ cannot be both zero. Furthermore, suppose $\gamma=0$. Then we will have $E\cong\mathcal{O}_{X_e}^{\delta}(a-1,b-e)\oplus\mathcal{O}_{X_e}^{\tau}(a,b-1)$ but by Propositions 3.14 and 3.23, if $e=0$ then both $\mathcal{O}_{X_e}^{\delta}(a-1,b-e)$ and $\mathcal{O}_{X_e}^{\tau}(a,b-1)$ are Ulrich. If $e>1$ then $\mathcal{O}_{X_e}^{\delta}(a-1,b-e)$ is not Ulrich and $\mathcal{O}_{X_e}^{\tau}(a,b-1)$ is Ulrich only when $a=1$. So γ can be zero only if $e=0$ or $a=1$ (and in this case also $\delta=0$). The part regarding ν is completely analogous. Now we prove the last part of the statement. Suppose $e>0$ and $a>1$. If $\tau=0$ then E would be the pull-back of a vector bundle of $\mathbb{P}^1$. By Grothendieck's theorem every vector bundle on $\mathbb{P}^1$ is the direct sum of line bundles, i.e $E=\pi^*(\bigoplus_i L_i)$. Since E is Ulrich, each $\pi^*(L_i)$ is Ulrich, but this is not possible since, by Proposition 3.14, X_e does not admit Ulrich line bundles if $e>0$ and $a>1$. □

Using similar techniques it is also possible to retrieve each Ulrich bundle on X_e as the cohomology of a monad.

Proposition 3.19. *Let E be a rank r Ulrich bundle on $(X_e,\mathcal{O}_{X_e}(a,b))$ with $a>1$ and with first Chern class $c_1(E)=\alpha C_0+\beta f$. Then E is the cohomology of a monad of the form*

$$0\to\mathcal{O}_{X_e}^{\varepsilon}(a-1,b-e)\to\mathcal{O}_{X_e}^{\zeta}(a-1,b+1-e)\oplus\mathcal{O}_{X_e}^{\eta}(a,b)\to\mathcal{O}_{X_e}^{\vartheta}(a,b+1)\to 0 \tag{3.2.6}$$

where $\varepsilon=2\alpha+\beta-r(2a+b-1)-e(\alpha-ar)$, $\zeta=2\alpha-2r(a-1)$, $\eta=\alpha+\beta-r(a+b-1)-e(\alpha-ar)$ and $\vartheta=\alpha-r(a-1)$.

Proof. We apply Beilinson's theorem to retrieve the monad. In order to do so, we compute the cohomology table of $E(-a,-b-1)$.

$\mathcal{O}_{X_e}(-1,-e-1)$	$\mathcal{O}_{X_e}(-1,-e)$	$\mathcal{O}_{X_e}(0,-1)$	$\mathcal{O}_{X_e}$	
0	0	0	0	h^3
ε	ζ	0	0	h^2
0	0	η	ϑ	h^1
0	0	0	0	h^0
$E(-a-1,-b-2)[-1]$	$E(-a,-b-2)[-1]$	$E(-a-1,-b-1)$	$E(-a,-b-1)$	

Since X_e is not embedded as a scroll, we obtain all the vanishings in the table with Lemma 3.16. To compute the dimension of the only non-zero cohomology groups we use Riemann-Roch. So we have

- $\varepsilon = -\chi(E(-a-1,-b-2)) = 2\alpha+\beta-r(2a+b-1)-e(\alpha-ar)$.
- $\zeta = -\chi(E(-a,-b-2)) = 2\alpha-2r(a-1)$.
- $\eta = -\chi(E(-a-1,-b-1)) = \alpha+\beta-r(a+b-1)-e(\alpha-ar)$.
- $\vartheta = -\chi(E(-a,-b-1)) = \alpha-r(a-1)$.

By Proposition 2.25 we get that $E(-a,-b-1)$ is the cohomology of the monad

$$0 \to \mathcal{O}_{X_e}^{\varepsilon}(-1,-e-1) \to \mathcal{O}_{X_e}^{\zeta}(-1,-e) \oplus \mathcal{O}_{X_e}^{\eta}(0,-1) \to \mathcal{O}_{X_e}^{\vartheta} \to 0$$

and tensoring it by $\mathcal{O}_{X_e}(a,b+1)$ we obtain the desired result. □

3.3 From the resolution to the Ulrich bundle

Now we study the inverse problem: given a coherent sheaf E which is the cokernel (resp. kernel) of a map as in (3.2.2) (resp in (3.2.3)), is it an Ulrich sheaf on X_e?

Theorem 3.20. *Let $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ be a polarized Hirzebruch surface.*

1. *Let ϕ be an injective map*

$$\mathcal{O}_{X_e}^{\gamma}(a-1, b-e-1) \xrightarrow{\phi} \mathcal{O}_{X_e}^{\delta}(a-1, b-e) \oplus \mathcal{O}_{X_e}^{\tau}(a, b-1) \tag{3.3.1}$$

with δ, τ non-negative, γ positive and $\delta + \tau > \gamma$. Let us call $r = \delta + \tau - \gamma$ and denote with E the cokernel of ϕ. In particular $c_1(E) = \alpha C_0 + \beta f$ with $\alpha = \tau + r(a-1)$ and $\beta = r(b-1) + \delta - e(r-\tau)$. If $c_1(E)h = \frac{r}{2}(3h^2 + hK_{X_e})$ then E is an Ulrich bundle if and only if $H^2(X_e, E(-2h)) = 0$.

2. *If ψ is a surjective map*

$$\mathcal{O}_{X_e}^{\lambda}(2a-1, 2b-2) \oplus \mathcal{O}_{X_e}^{\mu}(2a-2, 2b-1-e) \xrightarrow{\psi} \mathcal{O}_{X_e}^{\nu}(2a-1, 2b-1) \tag{3.3.2}$$

with λ, μ non-negative, ν positive and $\lambda + \mu > \nu$. Let us call $r = \lambda + \mu - \nu$ and denote by E the kernel of ψ. In particular $c_1(E) = \alpha C_0 + \beta f$ with $\alpha = r(2a-1) - \mu$ and $\beta = r(2b-1-e) - \lambda - e(\mu - r)$. If $c_1(E)h = \frac{r}{2}(3h^2 + hK_{X_e})$ then E is an Ulrich bundle if and only if $H^0(X_e, E(-h)) = 0$.

Proof. We only prove (1), since the proof of (2) is completely analogous. First of all observe that the existence of an injective map ϕ is guaranteed by the fact that $\mathcal{O}_{X_e}^{\gamma\delta}(0,1) \oplus \mathcal{O}_{X_e}^{\gamma\tau}(1,e)$ is globally generated [14, §4.1].

Let E be the cokernel of ϕ, thus E fits into an exact sequence like (3.2.2). So as soon as we check that

$$H^0(X_e, E(-h)) = H^1(X_e, E(-h)) = 0 \qquad \text{and} \qquad H^1(X_e, E(-2h)) = H^2(X_e, E(-2h)) = 0,$$

then E is an Ulrich vector bundle. Let us consider $E(-h) = E(-a, -b)$. Now tensor (3.2.2) by $\mathcal{O}_{X_e}(-a, -b)$. Since

$$H^i(X_e, \mathcal{O}_{X_e}^{\gamma}(-1, -e-1)) = H^i(X_e, \mathcal{O}_{X_e}^{\delta}(-1, -e) \oplus \mathcal{O}_{X_e}^{\tau}(0, -1)) = 0 \text{ for all } i,$$

we get

$$h^0(X_e, E(-a, -b)) = h^1(X_e, E(-a, -b)) = 0.$$

Now we focus on $E(-2h) = E(-2a,-2b)$. We tensor (3.2.2) by $\mathcal{O}_{X_e}(-2a,-2b)$. Setting $A = \mathcal{O}_{X_e}^{\gamma}(-a-1,-b-e-1)$ and $B = \mathcal{O}_{X_e}^{\delta}(-a-1,-b-e) \oplus \mathcal{O}_{X_e}^{\tau}(-a,-b-1)$ the induced long exact sequence in cohomology takes the form

$$0 \to H^1(X_e, E(-2h)) \to H^2(X_e, A) \xrightarrow{H^2(\phi(-2h))} H^2(X_e, B) \to H^2(X_e, E(-2h)) \to 0.$$

We show that $h^2(X_e, A) = h^2(X_e, B)$. This will imply $H^1(X_e, E(-2h)) = H^2(X_e, E(-2h))$. By Riemann-Roch we obtain that if $c_1(E)h = \frac{r}{2}(3h^2 + hK_{X_e})$, then $\chi(A) = \chi(B)$. In particular $\chi(E(-2h)) = 0$ and since $H^0(X_e, E(-2h)) = 0$ we have that

$$H^1(X_e, E(-2h)) = H^2(X_e, E(-2h)).$$

Thus $E = \text{Coker}(\phi)$ is Ulrich if and only if $H^2(X_e, E(-2h)) = 0$ (which is equivalent to the injectivity of the map $H^2(\phi(-2h))$).

□

In [6] the authors proved the existence of special rank two Ulrich bundles on ruled surfaces. Thanks to Theorem 3.20 we obtain the existence of special Ulrich bundles on Hirzebruch surfaces for any very ample polarization in a different way. Furthermore we characterize them as the cokernel (resp. kernel) of an injective (resp. surjective) map between certain totally decomposed vector bundles.

Corollary 3.21. Let E be a rank two vector bundle on $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ and $c_1(E) = \alpha C_0 + \beta f = 3h + K_{X_e}$. Then E is Ulrich if and only if it is the cokernel of an injective map ϕ

$$\mathcal{O}_{X_e}^{\gamma}(a-1, b-e-1) \xrightarrow{\phi} \mathcal{O}_{X_e}^{\delta}(a-1, b-e) \oplus \mathcal{O}_{X_e}^{\tau}(a, b-1)$$

with $\gamma = \alpha + \beta - 2(a+b-1) - e(\alpha - 2a)$, $\delta = \beta - 2(b-1) - e(\alpha - 2a)$ and $\tau = \alpha - 2(a-1)$.

Proof. We showed in Theorem 3.20 that as soon as a vector bundle has a resolution of the form (3.2.2) then $h^i(X_e, E(-h)) = 0$ for all i. Furthermore, E is a rank two vector bundle, so $E^\vee \cong E(-c_1)$. From the resolution we are able to conclude that $h^1(X_e, E(-2h)) = h^2(X_e, E(-2h))$ and using Serre's duality we get

$$h^2(X_e, E(-2h)) = h^0(X_e, E^\vee(2h + K_{X_e})) = h^0(X_e, E(-h)) = 0,$$

thus E is Ulrich. Notice that in the case of special Ulrich bundles the resolutions (3.2.2) and (3.2.3) are dual to each other (up to twist). □

Remark 3.22. In the case of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ one can find a resolution of an Ulrich bundle using similar techniques (see [33], [76]). In that case, every rank r vector bundle admitting a resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}{}^{\frac{r}{2}(d-1)}(d-2) \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^2}{}^{\frac{r}{2}(d+1)}(d-1) \to E \to 0 \qquad (3.3.3)$$

is Ulrich if and only if $H^2(\mathbb{P}^2, E(-2d)) = 0$. When we consider rank two vector bundles sitting in the previous short exact sequence, they are automatically Ulrich, i.e. the cohomological condition is trivially satisfied (using the fact that $E^\vee \cong E(-c_1)$). However we will see that the situation on rationally ruled surfaces is quite different.

In order to show that the vanishing of $H^1(X_e, E(-2h))$ is needed in general, we focus our attention on rank two Ulrich bundles on $(X_0, \mathcal{O}_{X_0}(d,d))$. We will see in Section 5 that, thanks to Proposition 3.29, we have $d+1$ admissible first Chern classes $\alpha C_0 + \beta f$ (up to an exchange of α and β) and we have that for $(\alpha, \beta) = (2d-2, 4d-2)$ the Ulrich bundle splits since $\mathrm{Ext}^1(\mathcal{O}_{X_0}(d-1, 2d-1), \mathcal{O}_{X_0}(d-1, 2d-1)) = 0$.

We start recalling which are the Ulrich line bundles on $(X_0, \mathcal{O}_{X_0}(a,b))$.

Proposition 3.23. *[24, Example 2.1] Let L be a line bundle on the polarized surface $(X_0, \mathcal{O}_{X_0}(a,b))$. Then L is Ulrich if and only if*

$$L = \mathcal{O}_{X_0}(2a-1, b-1) \quad \text{or} \quad L = \mathcal{O}_{X_0}(a-1, 2b-1).$$

Now we are able to construct a counterexample of a vector bundle realized as the cokernel of a map as in Theorem 3.20 which is not Ulrich (in particular with $H^1(X_e, E(-2h)) \cong H^2(X_e, E(-2h)) \neq 0$).

Example 3.24. Let us consider the polarized Hirzebruch surface $(X_0, \mathcal{O}_{X_0}(d,d))$ and let u be an integer such that $1 \le u \le d-1$. We construct a rank 2 vector bundle sitting in a resolution of the form (3.2.2), with first Chern class $c_1 = (2d-2+u, 4d-2-u)$ that is not Ulrich. Let us consider the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}^{u-1}(d-1) \to \mathcal{O}_{\mathbb{P}^1}^{u}(d) \to \mathcal{O}_{\mathbb{P}^1}(u+d-1) \to 0$$

and let us pull it back on X obtaining

$$0 \to \mathcal{O}_{X_0}^{u-1}(d-1,d-1) \to \mathcal{O}_{X_0}^{u}(d,d-1) \to \mathcal{O}_{X_0}(u+d-1,d-1) \to 0.$$

With the same argument we can find a second short exact sequence

$$0 \to \mathcal{O}_{X_0}^{2d-u-1}(d-1,d-1) \to \mathcal{O}_{X_0}^{2d-u}(d-1,d) \to \mathcal{O}_{X_0}(d-1,3d-u-1) \to 0.$$

If we set $E = \mathcal{O}_{X_0}(u+d-1,d-1) \oplus \mathcal{O}_{X_0}(d-1,3d-u-1)$ and we combine the two sequences we obtain a resolution of the form (3.2.2):

$$0 \to \mathcal{O}_{X_0}^{2d-2}(d-1,d-1) \to \mathcal{O}_{X_0}^{2d-u}(d-1,d) \oplus \mathcal{O}_{X_0}^{u}(d,d-1) \to E \to 0.$$

Every direct summand of an Ulrich bundle is also Ulrich. By Proposition 3.23, we know that both $\mathcal{O}_{X_0}(u+d-1,d-1)$ and $\mathcal{O}_{X_0}(d-1,3d-1-u)$ are Ulrich only when $u = d$, so the bundle E constructed in this way is not Ulrich.

We conclude this section with the following remark.

Remark 3.25. In the same hypothesis of Theorem 3.20, the locus of maps ϕ which do not give rise to Ulrich bundles is a divisor in the open space of maximal rank matrices which represent morphisms ϕ. In fact it is the locus where the induced map in cohomology

$$H^2(X_e, \mathcal{O}_{X_e}^{\gamma}(-a-1,-b-e-1)) \xrightarrow{H^2(\phi(-2h))} H^2(X_e, \mathcal{O}_{X_e}^{\delta}(-a-1,-b-e) \oplus \mathcal{O}_{X_e}^{\tau}(-a,-b-1))$$

is not an isomorphism. Since this two vector spaces have the same dimension, the locus where $E = \mathrm{Coker}(\phi)$ is not Ulrich is given by $\det(H^2(\phi(-2h))) = 0$. Now we produce an example where we explicitly describe this locus.

Consider X_0 embedded with $\mathcal{O}_{X_0}(2,2)$. By [23, Theorem 6.7] there exists a rank two Ulrich bundle F on $(X_0, \mathcal{O}_{X_0}(2,2))$ with $c_1(F) = 3C_0 + 5f$. Consider a rank two vector bundle E with $c_1(E) = 3C_0 + 5f$ realized as the cokernel of an injective map

$$\mathcal{O}_{X_0}^{2}(1,1) \xrightarrow{\phi} \mathcal{O}_{X_0}^{3}(1,2) \oplus \mathcal{O}_{X_0}(2,1).$$

Now we describe the locus where $E = \mathrm{Coker}(\phi)$ fails to be Ulrich. Recall by Theorem 3.20 that E is Ulrich if and only if the induced map in cohomology $H^2(\phi(-2h))$ is injective. By Serre's duality this is equivalent to the surjectivity of a map $\psi : \mathrm{Hom}(\mathcal{O}_{X_0}^3(-3,-2) \oplus \mathcal{O}_{X_0}(-2,-3), \mathcal{O}_{X_0}(-2,-2)) \to \mathrm{Hom}(\mathcal{O}_{X_0}^2(-3-3), \mathcal{O}_{X_0}(-2,-2))$ where $\psi(f) = f \circ \phi$.

Now take

$$f \in \mathrm{Hom}(\mathcal{O}_{X_0}^3(-3,-2) \oplus \mathcal{O}_{X_0}(-2,-3), \mathcal{O}_{X_0}(-2,-2)).$$

Let us denote $[Y_0 : Y_1]$ the coordinates of the first factor $\mathbb{P}^1$ and $[Y_2 : Y_3]$ the coordinates of the second factor. Then the matrices of f and ϕ are

$$f = \begin{pmatrix} \alpha_1^0 Y_0 + \alpha_1^1 Y_1 & \alpha_2^0 Y_0 + \alpha_2^1 Y_1 & \alpha_3^0 Y_0 + \alpha_3^1 Y_1 & \alpha_4^2 Y_2 + \alpha_4^3 Y_3 \end{pmatrix}$$

and

$$\phi = \begin{pmatrix} \beta_{1,1}^2 Y_2 + \beta_{1,1}^3 Y_3 & \beta_{1,2}^2 Y_2 + \beta_{1,2}^3 Y_3 \\ \beta_{2,1}^2 Y_2 + \beta_{2,1}^3 Y_3 & \beta_{2,2}^2 Y_2 + \beta_{2,2}^3 Y_3 \\ \beta_{3,1}^2 Y_2 + \beta_{3,1}^3 Y_3 & \beta_{3,2}^2 Y_2 + \beta_{3,2}^3 Y_3 \\ \beta_{4,1}^0 Y_0 + \beta_{4,1}^1 Y_1 & \beta_{4,2}^0 Y_0 + \beta_{4,2}^1 Y_1 \end{pmatrix}.$$

Imposing $\psi(f) = f\phi = (0)$ we obtain a system 8 equations given by

$$\sum_{l=1}^{3} \alpha_l^i \beta_{l,k}^j + \alpha_4^j \beta_{4,k}^i = 0 \tag{3.3.4}$$

where $i = 0,1$, $j = 2,3$ and $k = 1,2$. Now if we consider α_b^a as variables, the matrix of the system (3.3.4) is given by

$$\mathcal{B} = \begin{pmatrix} \beta_{1,1}^2 & \beta_{2,1}^2 & \beta_{3,1}^2 & 0 & 0 & 0 & \beta_{4,1}^0 & 0 \\ \beta_{1,2}^2 & \beta_{2,2}^2 & \beta_{3,2}^2 & 0 & 0 & 0 & \beta_{4,2}^0 & 0 \\ \beta_{1,1}^3 & \beta_{2,1}^3 & \beta_{2,2}^3 & 0 & 0 & 0 & 0 & \beta_{4,1}^0 \\ \beta_{1,2}^3 & \beta_{2,2}^3 & \beta_{3,2}^3 & 0 & 0 & 0 & 0 & \beta_{4,2}^0 \\ 0 & 0 & 0 & \beta_{1,1}^2 & \beta_{2,1}^2 & \beta_{3,1}^2 & \beta_{4,1}^1 & 0 \\ 0 & 0 & 0 & \beta_{1,2}^2 & \beta_{2,2}^2 & \beta_{3,2}^2 & \beta_{4,2}^1 & 0 \\ 0 & 0 & 0 & \beta_{1,1}^3 & \beta_{2,1}^3 & \beta_{2,2}^3 & 0 & \beta_{4,1}^1 \\ 0 & 0 & 0 & \beta_{1,2}^3 & \beta_{2,2}^3 & \beta_{3,2}^3 & 0 & \beta_{4,2}^1 \end{pmatrix} \tag{3.3.5}$$

and the locus where E is not Ulrich is described by $\det(\mathcal{B}) = 0$.

3.4 Admissible ranks and Chern classes for Ulrich bundles

In this section we deal with the admissible first Chern classes and admissible ranks of Ulrich bundles on $(X_e, \mathcal{O}_{X_e}(a,b))$. We start with the following definition:

Definition 3.26. Let $D = \alpha C_0 + \beta f$ be a divisor on $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ and let r be a positive integer. We say that the pair (r, D) is an admissible Ulrich pair with respect to h if and only if the following conditions hold

- $Dh = \frac{r}{2}(3h^2 + hK_{X_e})$;
- α and β satisfy the numerical conditions

$$r(a-1) \leq \alpha \leq r(2a-1), \tag{3.4.1}$$

$$r(b-1) - e(\alpha - r(2a-2)) \leq \beta \leq r(2b-1) - e(r(2a-1) - \alpha), \tag{3.4.2}$$

  with strict equalities in (3.4.1) if $e > 0$ and $a > 1$.

We will omit h in the notation when no confusion arises. It follows trivially from the definition that if (r, D) is not an admissible Ulrich pair, then there cannot exist a rank r Ulrich bundles on $(X_e, \mathcal{O}_{X_e}(h))$ with $c_1(E) = D$. Furthermore, once the rank is fixed, we will sometimes consider admissible first Chern classes instead of admissible Ulrich pairs.

Remark 3.27. The bounds in Definition 3.26 are obtained using Corollary 3.3. In particular the bound on β would be $r(b-1) - e(\alpha - ar) \leq \beta \leq r(2b-1) - e(r(2a-1) - \alpha)$ but it is possible to improve it. Recall by Proposition 3.12 that E is Ulrich if and only if $E^\vee(3h + K_{X_e})$ is Ulrich, so the bounds on β should be centred in $\frac{r}{2}(3b - 2 - e)$. This gives us the actual bounds on β in Definition 3.26.

In the next proposition we characterize admissible Ulrich pairs.

Proposition 3.28. *Let $D=\alpha C_0+\beta f$ be a divisor on $(X_e,\mathcal{O}_{X_e}(h))$ with $h=aC_0+bf$. If (r,D) is an admissible Ulrich pair then it satisfies*

$$T=\frac{er}{2}(3a-1)+\frac{b}{a}(\alpha+r)\in\mathbb{Z}$$
$$\text{and} \tag{3.4.3}$$
$$r(a-1)+\frac{era}{2b}(a-1)\leq\alpha\leq r(2a-1)-\frac{era}{2b}(a-1).$$

Conversely any pair (r,D) satisfying (3.4.3) *and $Dh=\frac{r}{2}(3h^2+hK_{X_e})$ is an admissible Ulrich pair.*

Proof. Since (r,D) is an admissible Ulrich pair then $Dh=\alpha(b-ea)+\beta a=\frac{r}{2}(3h^2+hK_{X_e})$. Now we express β as a function of α, thus

$$\beta=r(3b-1)+\alpha e-\frac{er}{2}(3a-1)-\frac{b}{a}(r+\alpha).$$

However α and β represent the coefficients of the divisor D, thus they must be integers. In particular we obtain that

$$\frac{er}{2}(3a-1)+\frac{b}{a}(\alpha+r)\in\mathbb{Z}.$$

Moreover observe that since we expressed β as a function of α, the numerical conditions (3.4.2) on β give us the bound on α. In fact by imposing $\beta\leq r(2b-1)-e(r(2a-1)-\alpha)$ one obtains

$$r(a-1)+\frac{era}{2b}(a-1)\leq\alpha.$$

The upper bound is obtained as in Remark 3.27 by noticing that the interval giving the bounds for α should be centered in $\frac{r}{2}(3a-2)$. For the second part of the statement it is enough to show that the bound (3.4.3) on α gives us the bound on β, but this follows from the fact that we obtained (3.4.3) by imposing the inequalities (3.4.2) on β. □

Now we focus on rank two Ulrich bundles. Notice that in this case $\frac{er}{2}(3a-1)$ is always even, thus the admissibility of the first Chern class depends on whether $\frac{b}{a}(\alpha+r)$ is an integer or not.

Proposition 3.29. *Let E be a rank two Ulrich bundle on $(X_0, \mathcal{O}_{X_0}(a,b))$. If* $\mathrm{GCD}(a,b) = s$ *then we have* $2s+1$ *possible first Chern classes for E given by*

$$(2a-2+kq)C_0 + (4b-2-kp)f$$

with $0 \le k \le 2s$, $p = \frac{b}{s}$ *and* $q = \frac{a}{s}$.

Proof. Suppose that $c_1(E)$ is given by $c_1(E) = \alpha C_0 + \beta f$. Then by Proposition 3.12 we have:

$$c_1(E)h = a\beta + b\alpha = 6ab - 2a - 2b \tag{3.4.4}$$

which is an integer. Now solving with respect to β we obtain

$$\beta = \frac{1}{a}(6ab - 2a - 2b - b\alpha).$$

In order for $c_1(E)$ to be an admissible first Chern class of an Ulrich bundle, by Proposition 3.28 we have the following bounds for α and β

$$2a-2 \le \alpha \le 4a-2 \qquad \text{and} \qquad 2b-2 \le \beta \le 4b-2.$$

Now define

$$\begin{cases} \alpha_i = 2a-2+i \\ \beta_i = \frac{1}{a}\big(6ab-2a-2b-b(2a-2)-b\alpha_i\big) = 4b-2-\frac{bi}{a}, \end{cases}$$

By Proposition 3.28, $(2, \alpha_i C_0 + \beta_i f)$ is an admissible Ulrich pair if and only if $\frac{bi}{a}$ is an integer. Let us define $s = \mathrm{GCD}(a,b)$. Then the only possibilities for $\frac{bi}{a}$ to be integer are when $i = k\frac{a}{s}$, with $0 \le k \le 2s$ and $k \in \mathbb{Z}$. So we have $2s+1$ admissible first Chern classes for E. □

Remark 3.30. In the case of $(X_0, \mathcal{O}_{X_0}(a,b))$, to satisfy the Bogomolov's inequality for semistable rank two vector bundles is equivalent to satisfy the numerical conditions in Corollary 3.18. In fact for a semistable rank two vector bundle E with first Chern class $c_1(E) = \alpha C_0 + \beta f$ the Bogomolov's inequality gives us

$$\Delta = 4c_2(E) - c_1^2(E) \ge 0,$$

while the numerical conditions for the Beilinson's resolution are

$$\begin{cases} 4a-2 \geq \alpha \geq 2a-2 \\ 4b-2 \geq \beta \geq 2b-2. \end{cases}$$

Using Proposition 3.12 we have

$$c_2(E) = \frac{c_1(E)^2}{2} - \frac{c_1(E)K_{X_0}}{2} - 2(h^2-1),$$

so that

$$\Delta = c_1(E)^2 - 2c_1(E)K_{X_0} - 2(h^2-1).$$

Recall that for a rank two Ulrich bundle we have

$$c_1(E)h = \alpha b + \beta a = 3h + K_{X_0},$$

so expressing Δ in terms of α we have $\Delta(\alpha) \geq 0$ for all $2a-2 \leq \alpha \leq 4a-2$. We conclude that any rank two vector bundle on X_0 fitting into the resolution (3.2.2) satisfies Bogomolov's inequality and, conversely, any vector bundle satisfying the Bogomolov's inequality also satisfies the conditions in Corollary 3.18

For $(X_e, \mathcal{O}_{X_e}(a,b))$ with $e > 0$ the situation is slightly different from X_0, because when the invariant e is positive is not always guaranteed the existence of Ulrich line bundles.

Remark 3.31. Polarizations $h = C_0 + bf$ with $b > e$ are the only ones such that X_e admits Ulrich line bundles. In this case it is immediate to see that we have three admissible first Chern classes for a rank two Ulrich bundle, which are

$$(4b-2-2e)f \qquad C_0 + (3b-2-e)f \qquad 2C_0 + (2b-2)f.$$

For all the other cases we have the following proposition.

Proposition 3.32. *Let E be a rank two Ulrich bundle on $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ such that $a > 1$. If* $\mathrm{GCD}(a,b) = s$ *then the admissible first Chern classes for E are given by*

$$(2a-2+kq)C_0 + (4b-2-e-kp+kqe)f$$

with $k \in \mathbb{Z}$ *such that* $\frac{es}{b}(a-1) \leq k \leq 2s - \frac{es}{b}(a-1)$, $p = \frac{b}{s}$ *and* $q = \frac{a}{s}$.

Proof. The strategy is completely analogous to the one of Proposition 3.29 but the computations are a bit more tedious. Since the degree of $c_1(E)$ is fixed for any rank two Ulrich bundle, this will give us an equation in the coefficients of $c_1(E)$, namely α and β. By solving for β and imposing that it is an integer number, we obtain all the admissible first Chern classes. □

Remark 3.33. Suppose X_e is embedded with a very ample divisor $h = aC_0 + bf$ such that $\mathrm{GCD}(a,b) = 1$, then the only possibility for the first Chern class of a stable, rank two Ulrich bundle E on X_e is $c_1(E) = 3h + K_{X_e}$ (i.e. E is special).

In what follows we focus on the admissibility of ranks and first Chern classes of Ulrich bundles in some particular cases. In light of Remark 3.33, one expects that for polarizations $\mathcal{O}_{X_e}(a,b)$ with $\mathrm{GCD}(a,b) = 1$ we have the least possible number of admissible first Chern classes for E.

Let E be a rank r Ulrich bundle on X_e with respect to $\mathcal{O}_{X_e}(a,b)$ with $\mathrm{GCD}(a,b) = 1$ and let $c_1(E) = \alpha C_0 + \beta f$ be its first Chern class. Suppose $e > 0$ and $a > 1$. If a is odd or e is even, then by Proposition 3.28 the pair $(r, c_1(E))$ is an admissible Ulrich pair if and only if

$$\alpha = ka - r \tag{3.4.5}$$

with

$$k \in \mathbb{Z} \qquad \text{and} \qquad r + \frac{er}{2b}(a-1) \leq k \leq 2r - \frac{er}{2b}(a-1).$$

Now we describe explicitly all the admissible Ulrich pairs for such polarized Hirzebruch surfaces.

Proposition 3.34. *Let* E *be a rank* r *Ulrich bundle on* $(X_e, \mathcal{O}_{X_e}(h))$ *with* $h = aC_0 + bf$ *such that* $\mathrm{GCD}(a,b) = 1$. *Suppose* $e > 0$ *and* $a > 1$. *If* a *is odd or* e *is even then the admissible first Chern classes for* E *are given by*

$$(ka - r)C_0 + \Big(r\big(3b - 1 - \frac{e}{2}(3a+1)\big) + k(ae - b)\Big)f \tag{3.4.6}$$

with $k \in \mathbb{Z}$ *such that* $r + \frac{er}{2b}(a-1) \leq k \leq 2r - \frac{er}{2b}(a-1)$.

Proof. In the hypothesis of this Proposition $c_1(E) = \alpha C_0 + \beta f$ is admissible if and only if $\alpha = ka - r$ with $r + \frac{er}{2b}(a-1) \leq k \leq 2r - \frac{er}{2b}(a-1)$. For each of such α use the relation

$$\alpha(b-ae)+\beta a=\frac{r}{2}(3h^2+hK_{X_e})$$

to compute β. □

3.5 Existence and moduli spaces

In this section we will discuss some results on the existence of Ulrich bundles and their moduli spaces. Let us fix the notation. Let X be a smooth algebraic surface, we will denote by $M_h(r;c_1,c_2)$ the moduli space of rank two locally free sheaves E on X stable with respect to a polarization h and with $\det(E)=c_1\in \mathrm{Pic}(X)$ and $c_2(E)=c_2\in\mathbb{Z}$. We start by recalling the following proposition concerning the rank two case.

Proposition 3.35. *[42, Theorem 4.7] Let X be a smooth, irreducible, projective, minimal, rational surface, $c_1\in \mathrm{Pic}(X)$ and $c_2\in\mathbb{Z}$. Then, for any polarization h on X, the moduli space $M_h(2;c_1,c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2-c_1^2-3$, whenever non-empty.*

The moduli space of stable rank r Ulrich bundles E with $\det(E)=c_1$ and $c_2(E)=c_2$ is an open subset in $M_h(r;c_1,c_2)$ and we will denote it by $M_h^U(r;c_1,c_2)$. In what follows we show how we can use Theorem 3.17 to study the moduli spaces of Ulrich bundles of rank greater than two. We start by giving an existence theorem.

We showed in the previous section that given an injective map ϕ as in Theorem 3.20, in general the vanishing $H^1(X_e,E(-2h))=0$ is necessary to obtain that $\mathrm{Coker}(\phi)$ is Ulrich. However we are able to prove the following existence result.

Theorem 3.36. *$(X_e,\mathcal{O}_{X_e}(h))$ supports Ulrich bundle of any admissible Ulrich pair (r,c_1).*

Proof. First of all recall that by Proposition 3.14 and 3.23, $(X_e,\mathcal{O}_{X_e}(h))$ admits Ulrich line bundles if and only if $e=0$ or $h=C_0+bf$ and $e>0$. Now let us consider a map ϕ, as in (3.3.1), general. Then ϕ would be injective and let us denote by E its cokernel. Corollary 3.18 implies that the Chern character of $E(-2h)$ satisfies the hypothesis of [38, Theorem 1.1]. In particular we obtain that $E(-2h)$ has natural cohomology and, since $\chi(E(-2h))=0$, we have $h^1(X_e,E(-2h))=h^2(X_e,E(-2h))=0$ and E is Ulrich. □

Once the existence is settled we focus on moduli spaces of Ulrich bundles.

Lemma 3.37. *If E is a rank r Ulrich bundle on $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ and $a, b > 1$, then* $\operatorname{Ext}^2(E,E) = 0$.

Proof. Consider the short exact sequence (3.2.2) and tensor it by $E^\vee$. The long exact sequence in cohomology gives us

$$\delta h^2(X_e, E^\vee(a-1, b-e)) + \tau h^2(X_e, E^\vee(a, b-1)) \geq h^2(X_e, E \otimes E^\vee).$$

Since E is Ulrich, the same is true for $E^\vee(3h + K_{X_e})$. Now using Lemma 3.16 and the fact that $a, b > 1$, we obtain $h^2(X_e, E^\vee(a-1, b-e)) = h^2(X_e, E^\vee(a, b-1)) = 0$, thus $H^2(X_e, E \otimes E^\vee) \cong \operatorname{Ext}^2(E,E) = 0$. □

In light of this we can state the following proposition.

Proposition 3.38. *Let us consider $(X_e, \mathcal{O}_{X_e}(h))$ with $h = aC_0 + bf$ and let (r, c_1) be an admissible Ulrich pair. Then the moduli space $M_h^U(r; c_1, c_2)$ is a smooth, irreducible, unirational, quasi-projective variety of dimension $c_1^2 - rc_1K_{X_e} - r^2(2h^2 - 1) + 1$, for $r = 2, 3$, $e > 0$ and $a > 1$.*

Proof. Smoothness comes from Lemma 3.37. To any element

$$\phi \in \operatorname{Hom}(\mathcal{O}_{X_e}^{\gamma}(a-1, b-e-1), \mathcal{O}_{X_e}^{\delta}(a-1, b-e) \oplus \mathcal{O}_{X_e}^{\tau}(a, b-1))$$

as in Theorem 3.20, we can associate its cokernel, forming a flat family. Thanks to Theorem 3.36 the generic element in this family is an Ulrich bundle. Theorem 3.13 tells us that an Ulrich bundle can only be destabilized over an Ulrich bundle and the cokernel of the inclusion map is also Ulrich. In particular, if we consider the rank r to be two or three, the existence of a strictly semistable Ulrich bundle would imply the existence of Ulrich line bundles. Using Proposition 3.14, we see that $(X_e, \mathcal{O}_{X_e}(h))$ does not admit Ulrich line bundles for $e > 0$ and $a > 1$. We conclude that in these cases all the Ulrich bundles are stable, so $M_h^U(r; c_1, c_2)$ is non-empty. For irreducibility and unirationality observe that $M_h^U(r; c_1, c_2)$ is dominated by an open subset of a space of matrices, which is irreducible and unirational. Finally, for the dimension, recall that for stable bundles we have $h^0(E \otimes E^\vee) = 1$. Since $\dim M_h^U(r; c_1, c_2) = \dim \operatorname{Ext}^1(E,E)$, and $\operatorname{Ext}^2(E,E) = 0$ by Lemma 3.37, we obtain

$\dim M_h^U(r;c_1,c_2) = 1-\chi(E\otimes E^\vee)$. Using [22, Proposition 2.12] we have the desired result. □

Remark 3.39. When $(X_e, \mathcal{O}_{X_e}(a,b))$ admits Ulrich line bundles the situation is different. In [51] the authors proved that when $a=1$ we have exactly two Ulrich line bundles and all the Ulrich bundles of rank greater than two are strictly semistable, i.e $M_h^U(r;c_1,c_2)$ is empty for $r\geq 2$. If we consider $(X_0, \mathcal{O}_{X_0}(a,b))$ with $a,b>1$, by Proposition 3.23 it always admits two Ulrich line bundles. However we will see in Section 7 that there exists a stable rank two Ulrich bundle for every admissible first Chern class. Thus we can describe the moduli space $M^U_{X_0,h}(2;c_1,c_2)$ using the same argument of Proposition 3.38.

We continue this section dealing with higher ranks Ulrich bundles. Suppose X_e is embedded with a very ample line bundle $\mathcal{O}_{X_e}(h)$ with $h=aC_0+bf$ and such that $\mathrm{GCD}(a,b)=1$, i.e. the embedding $X_e \hookrightarrow \mathbb{P}^N$ does not factor through a Veronese embedding, and with $a>1$ and odd. Once we fix the rank r the admissible Ulrich pairs are given by (3.4.6).

Remark 3.40. It is worth to notice that in Proposition 3.34 we have $r+\frac{er}{2b}(a-1) < 2r-\frac{er}{2b}(a-1)$. However k is an integer and it could happen that there is no integer between $r+\frac{er}{2b}(a-1)$ and $2r-\frac{er}{2b}(a-1)$, i.e. in that cases there are no admissible Ulrich pairs (r,D) with respect to $h=aC_0+\beta f$.

Proposition 3.41. *Let us consider $(X_e, \mathcal{O}(a,b))$ with* $\mathrm{GCD}(a,b)=1$*, $e>0$ and $a>1$. If a is odd or e is even then for any even rank r the pair $(r, \frac{r}{2}(3h+K_{X_e}))$ is an admissible Ulrich pair.*

Proof. Let us set $D=\alpha C_0+\beta f = \frac{r}{2}(3h+K_{X_e})$. In particular $\alpha = \frac{3}{2}ra - r$ so by Proposition 3.34 it is enough to show that $r+\frac{er}{2b}(a-1)\leq \frac{3}{2}r \leq 2r-\frac{er}{2b}(a-1)$. This is equivalent to $r(b-ea+e)\geq 0$ which is always true by the very ampleness of h. □

In the cases when r is odd then the divisor $\frac{r}{2}(3h+K_{X_e})$ does not have integer coefficients, thus the “nearest" admissible Ulrich pair would be $\frac{r}{2}(3h+K_{X_e})-\frac{1}{2}D$ where $D=aC_0+(ae-b)f$. Let us denote by Δ_t the positive number

$$\Delta_t = \frac{tb}{b-ae+e}.$$

Observe that if r is odd and $r < \Delta_1$, then there are no integers between $r + \frac{er}{2b}(a-1)$ and $2r - \frac{er}{2b}(a-1)$, thus we have the following proposition.

Proposition 3.42. *Let us consider $(X_e, \mathcal{O}(a,b))$ with $\mathrm{GCD}(a,b) = 1$, $e > 0$ and $a > 1$. If a is odd or e is even then there are no admissible Ulrich pair (r,D) with r odd and $r < \Delta_1$, i.e. there cannot exist odd rank Ulrich bundles of rank $r < \Delta_1$.*

Remark 3.43. In the same setting of Proposition 3.42, let us consider $\bar{r}$ to be the first odd integer such that $\bar{r} \geq \Delta_1$. Then there exists two admissible Ulrich pairs $\left(\bar{r}, \frac{\bar{r}}{2}(3h + K_{X_e}) - \frac{1}{2}D\right)$ and $\left(\bar{r}, \frac{r}{2}(3h + K_{X_e}) + \frac{1}{2}D\right)$ with $D = aC_0 + (ae-b)f$. By Proposition 3.36 there exist an Ulrich bundle corresponding to these admissible Ulrich pairs. Observe that such a bundle is stable. In fact if it were semistable, then would be an extension of an odd and an even Ulrich bundle with rank smaller than $\bar{r}$, but this is not possible since there are no Ulrich bundles of odd rank smaller than $\bar{r}$.

Now we prove a Lemma which will be useful in the next propositions

Lemma 3.44. *Let E_i be Ulrich bundles on $(X_e, \mathcal{O}_{X_e}(h))$ such that $\mathrm{rk}(E_i) = r_i$ are even. Then the admissible first Chern classes for E_i are*

$$c_1(E_i) = \frac{r_i}{2}(3h + K_{X_e}) + k_i D \quad \textit{with} \quad -\left(\frac{r_i}{2} - \frac{er}{2b}(a-1)\right) < k_i < \frac{r_i}{2} - \frac{er}{2b}(a-1) \tag{3.5.1}$$

with $k_i \in \mathbb{Z}$ and $D = aC_0 + (ae-b)f$. In particular

$$\chi(E_i \otimes E_j^\vee) = -\frac{r_i r_j}{4}(h^2 - 4) + K_{X_e} D \left(\frac{r_i}{2}k_j - \frac{r_j}{2}k_i\right) + k_i k_j h^2.$$

Proof. The first part of the Lemma is a direct consequence of Proposition 3.34. For the computation of the Euler characteristics see [22, Proposition 2.12] using the relations $D^2 = -h^2$ and $Dh = 0$. □

Proposition 3.45. *Let us consider $(X_e, \mathcal{O}(a,b))$ with $\mathrm{GCD}(a,b) = 1$, $e > 0$ and $a > 1$. Suppose a is odd or e is even. Then for any even rank $r < \Delta_1$ there exists a stable rank r Ulrich bundle E with $c_1(E) = \frac{r}{2}(3h + K_{X_e})$.*

Proof. First of all observe that if $\Delta_1 \leq 2$ then there are no r satisfying the hypothesis, thus let us suppose $\Delta_1 \geq 3$. We will use the same idea of a method that M. Casanellas and R. Hartshorne used in [22, Theorem 4.3]. To show the existence of rank $2t < \Delta_1$ simple Ulrich bundles with $c_1(E) = t(3h + K_{X_e})$, we proceed by induction on half the rank t. The existence of stable special rank two Ulrich bundles is given by Proposition

3.38 and 3.41. Now suppose that for any $s < t$ there exists a rank $2s$ stable Ulrich bundle with first Chern class equal to $s(3h+K_{X_e})$. By inductive hypothesis there exist stable Ulrich bundles F and G of ranks 2 and $2t-2$ respectively, such that $c_1(F)+c_1(G)=t(3h+K_{X_e})$. Now consider a non-split extension

$$0 \to F \to E \to G \to 0.$$

The bundle E is a simple Ulrich bundle of rank $2t$ (see [22, Lemma 4.2]). Notice that this is possible since $\dim \mathrm{Ext}^1(G,F) = h^1(X_e, F \otimes G^\vee) > 0$ by Lemma 3.44. Now consider the modular family of simple bundles E. We can compute its dimension using Lemma 3.44 as $h^1(X_e, E \otimes E^\vee) = t^2(h^2-4)+1$. Now we show that the dimension of each family of strictly semistable Ulrich bundles of rank $2t$ which are obtained as an extension of two stable Ulrich bundles is strictly smaller than $h^1(X_e, E \otimes E^\vee)$. Consider two stable Ulrich bundles F_1 and F_2 of rank $2t_1$ and $2t_2$ respectively, such that $t_1+t_2=t$. We have $c_1(F_j)=t_j(3h+K_{X_e})$. Now we show that

$$\dim\{F_1\} + \dim\{F_2\} + \dim(\mathrm{Ext}^1(F_2,F_1)) - 1 < h^1(X_e, E \otimes E^\vee), \qquad (3.5.2)$$

i.e. we want to show that

$$(t_1^2+t_2^2+t_1t_2)(h^2-4) < (t_1+t_2)^2(h^2-4)$$

which is equivalent to

$$t_1t_2(h^2-4) > 0.$$

Since we supposed $e > 0$ and $a > 1$, we have that $h^2 \geq 12$, thus $t_1t_2(h^2-4) > 0$. In particular, we have that the general element in the modular family of simple Ulrich bundles of rank $2(t_1+t_2) = 2t$ and $c_1 = t(3h+K_{X_e})$ is stable. □

We continue with some remarks.

Remark 3.46. In the proof of Proposition 3.45 it is enough to consider strictly semistable Ulrich bundles E which are extensions of stable Ulrich bundles. Indeed suppose E is a strictly semistable Ulrich bundle. Then each term of his Jordan-Hölder filtration is a stable Ulrich bundle [34, Lemma 2.15]. Let F be one of them and consider the quotient $F_1 = E/F$. Observe that F_1 is Ulrich by Proposition 3.13. In this way we can always assume that a strictly semistable Ulrich bundle is an extension of a stable bundle F and a semistable Ulrich bundle F_1. Now, if F_1 is extension of two stable Ulrich bundles, then using the same dimensional count as in the proof of Proposition 3.45, the family parametrizing F_1 has dimension strictly smaller than the family of simple Ulrich bundles with the same invariants as F_1. If F_1 is an extension of a stable Ulrich bundle and a strictly semistable Ulrich bundle then we iterate this process until we obtain an Ulrich bundle F_l which is extension of two stable Ulrich bundles. However at each step of this process we have an inequality as in (3.5.2), thus in the end $h^1(X_e, E \otimes E^\vee)$ is strictly greater than the dimension of any family parametrizing strictly semistable Ulrich bundles with the same invariants as E.

Remark 3.47. The hypothesis $r < \Delta_1$ in Proposition 3.45 is necessary to exclude the existence of rank odd Ulrich bundles. The inequality $\Delta_{j-1} \leq r < \Delta_j$ gives us information about the admissible Ulrich pairs (r, D). We saw that if $r < \Delta_1$ there are no admissible Ulrich pairs (r, D) with r odd. Using Proposition 3.41 it is possible to see that if $\Delta_1 \leq r < \Delta_2$ then if r is even the only admissible Ulrich pair is given by $\left(r, \frac{r}{2}(3h + K_{X_e})\right)$ and if r is odd then we have exactly two admissible Ulrich pairs given by $\left(r, \frac{r}{2}(3h + K_{X_e}) - \frac{1}{2}D\right)$ and $\left(r, \frac{r}{2}(3h + K_{X_e}) + \frac{1}{2}D\right)$, with $D = aC_0 + (ae - b)f$.

In general given $t \in \mathbb{Z}_{\geq 0}$, if $\Delta_{t-1} \leq r < \Delta_t$ then

- there exist $n = 2\left\lfloor \frac{t}{2} \right\rfloor$ admissible Ulrich pairs (r, D) with r odd;
- there exist $m = 2\left\lceil \frac{t}{2} \right\rceil - 1$ admissible Ulrich pairs (r, D) with r even.

Observe that $r < \Delta_r$ for each r, thus the maximum number of admissible Ulrich pairs (r, D) is $r - 1$. Moreover it is worth to notice that, without the hypothesis $r < \Delta_1$, it is considerably more difficult to use the same strategy of Proposition 3.45 to prove the existence of stable Ulrich bundles. In fact, in these cases, an Ulrich bundle can be realized as an extension of two Ulrich bundles in several different ways.

In the remaining part of this section we compare the results we obtained with the existing literature.

Remark 3.48. In [92] it has been proved that for any birationally ruled surface S endowed with an ample divisor h, the moduli spaces $\overline{M}_h(r; c_1, c_2)$ of semistable vector bundles, whenever non-empty, are irreducible and normal. In addition, the open subspace of stable vector bundles $M_h(r; c_1, c_2)$ is smooth. Theorem 3.36, Proposition 3.38 and 3.45 extend these results in the case of Hirzebruch surfaces showing the unirationality of an open subset and the non-emptiness for some admissible Ulrich pairs (r, c_1) and polarizations.

In [75] the authors proved the unirationality, smoothness, irreducibility and non-emptiness of the moduli spaces $M_h(3; c_1, c_2)$ of rank three stable vector bundles on polarized Hirzebruch surfaces for some Chern classes. Thanks to Proposition 3.38 we partially extend this result for $r = 3$, $e > 0$, $a > 1$ and all the admissible Chern classes of an Ulrich bundle.

3.6 Indecomposable rank two Ulrich bundles

In this section we will construct rank two stable Ulrich bundles on X_e with respect to a very ample polarization $aC_0 + bf$. Using Serre's correspondence on surfaces, we will construct stable Ulrich bundles on X_0 for two of the admissible first Chern classes. Then we will show how to use *Macaulay2* to produce examples of Ulrich bundle on X_e for several different polarization, Chern classes and ranks.

Proposition 3.49. *Let us consider* $(X_0, \mathcal{O}_{X_0}(h))$ *with* $h = aC_0 + bf$ *and* $\mathrm{GCD}(a, b) = s > 1$. *Then there exists non-special rank two Ulrich bundles with* $c_1(E) = (3a - 2 - \frac{a}{s})C_0 + (3b - 2 + \frac{b}{s})f$ *given by*

$$0 \to \mathcal{O}_{X_0}(a-1, b+\frac{b}{s}-1) \to E \to \mathcal{I}_Z(2a-1-\frac{a}{s}, 2b-1) \to 0, \qquad (3.6.1)$$

with Z *a general zero dimensional subscheme of* X_0 *with* $l(Z) = ab(\frac{s-1}{s})$.

Proof. First we prove that there exists vector bundles realized as an extension (3.6.1). In order to do so we need to verify that the pair $((a-2-\frac{a}{s})C_0 + (b-\frac{b}{s}-2)f, Z)$ has the Cayley-Bacharach property. We have

$$h^0(X_0, \mathcal{O}_{X_0}(a-2-\frac{a}{s}, b-\frac{b}{s}-2)) = ab(1-\frac{1}{s})^2 - (a+b)(1-\frac{1}{s}) + 1.$$

An easy computation shows that

$$h^0(X_0, \mathcal{O}_{X_0}(a-2-\frac{a}{s}, b-\frac{b}{s}-2)) \leq l(Z) - 1.$$

It follows that for a general Z, the pair $\big((a-2-\frac{a}{s})C_0+(b-\frac{b}{s}-2)f, Z\big)$ verifies the Cayley-Bacharach property, so in any extension of type (3.6.1) there are rank two vector bundles.

By Proposition 3.12, in order for E to be Ulrich we need to verify the equalities on the Chern classes and the vanishings in cohomology. Every vector bundle in the extension (3.6.1) has first Chern class $c_1(E) = 3h + K_{X_0} + D$ where $D = (-\frac{a}{s})C_0 + (\frac{b}{s})f$, so we have $c_1(E)h = 3h^2 + K_{X_0}h$. Furthermore a direct computation shows that

$$\begin{aligned} c_2(E) &= l(Z) + \big((a-\frac{a}{s})C_0+(b-\frac{b}{s})f\big)\big((a-1)C_0+(b+\frac{b}{s}-1)f\big)+ \\ &+ \big((a-1)C_0+(b+\frac{b}{s}-1)f\big)^2 \\ &= \frac{1}{2}(c_1(E)^2 - c_1(E)K_{X_0}) + r(h^2-1). \end{aligned}$$

So it remains to check that $h^0(X_0, E(-h)) = h^0(X_0, E^\vee(2h+K_{X_0})) = 0$. Twisting (3.6.1) by $\mathcal{O}_{X_0}(-h)$ and considering the long exact sequence in cohomology we obtain:

$$h^0(X_0, E(-h)) = h^0(X_0, \mathcal{I}_Z(a-1-\frac{a}{s}, b-1)).$$

Furthermore

$$h^0(X_0, \mathcal{O}_{X_0}(a-1-\frac{a}{s}, b-1)) = l(Z)$$

thus for a general Z we have $h^0(X_0, \mathcal{I}_Z(a-1-\frac{a}{s}, b-1)) = 0$ and $h^0(X_0, E(-h)) = 0$. For the second vanishing recall that $E^\vee(2h+K_{X_0}) \cong E(-h-D) = E(-a+\frac{a}{s}, -b-\frac{b}{s})$. Now tensoring (3.6.1) by $\mathcal{O}_{X_0}(-a+\frac{a}{s}, -b-\frac{b}{s})$ and considering the long exact sequence in cohomology we get

$$h^0(X_0, E(-h-D)) = h^0(X_0, \mathcal{I}_Z(a-1, b-\frac{b}{s}-1)).$$

Furthermore

$$h^0(X_0, \mathcal{O}_{X_0}(a-1, b-\frac{b}{s}-1)) = l(Z),$$

thus for a general Z we have $h^0(X_0, \mathcal{I}_Z(a-1, b-\frac{b}{s}-1)) = 0$ and $h^0(X_0, E(-h-D)) = 0$, so by Proposition 3.12, E is a rank two Ulrich bundle on X_0. □

Remark 3.50. Recall by Proposition 3.23 that the only two Ulrich line bundles on $(X_0, \mathcal{O}_{X_0}(a,b))$ are $L = \mathcal{O}_{X_0}(2a-1, b-1)$ and $M = \mathcal{O}_{X_0}(a-1, 2b-1)$. We conclude that a non-special rank two Ulrich bundle, apart from the trivial ones $E = L^2$ and $E = M^2$, is always stable. In fact we can check the (semi)stability of a sheaf E by considering the subsheaves F such that the quotient E/F is torsion free (see [**?** , Theorem 1.2.2]). A rank two Ulrich bundle can only destabilize on an Ulrich line bundle, but in this way the quotient would also be an Ulrich line bundle thanks to Theorem 3.13. However, this is not possible because of the numerical conditions imposed by the first Chern classes.

Remark 3.51. In a completely similar way it is possible to construct rank two special Ulrich bundles for any very ample polarization $h = aC_0 + bf$ as extensions

$$0 \to \mathcal{O}_{X_0}(a-1, b+\frac{b}{s}-1) \to E \to \mathcal{I}_Z(2a-1, 2b-\frac{b}{s}-1) \to 0, \tag{3.6.2}$$

with Z a general zero dimensional subscheme of X_0 with $l(Z) = ab(\frac{s-1}{s})$. Although examples of special rank two Ulrich bundles on X_0 had already been given by extensions of the two Ulrich line bundles $L = \mathcal{O}_{X_0}(2a-1, b-1)$ and $M = \mathcal{O}_{X_0}(a-1, 2b-1)$, the Ulrich bundles constructed as in (3.6.2) are stable. In fact if we tensor (3.6.2) by $L^\vee$ and consider the long exact sequence in cohomology we get

$$0 \to H^0(X_0, E \otimes L^\vee) \to H^0(X_0, \mathcal{I}_Z(0, b-\frac{b}{s})),$$

but $h^0(X_0, \mathcal{O}_{X_0}(0, b-\frac{b}{s})) = b - \frac{b}{s} + 1 \leq l(Z)$ so it is not possible to have non-zero maps between L and E. Similarly, tensoring (3.6.2) by $M^\vee$ and taking the induced sequence in cohomology we have

$$0 \to H^0(X_0, E \otimes M^\vee) \to H^0(X_0, \mathcal{I}_Z(a, -\frac{b}{s})).$$

but $H^0(X_0, \mathcal{I}_Z(a, -\frac{b}{s})) = 0$, so does not exist a non-zero map between M and E. Since a rank two Ulrich bundle can only destabilize on an Ulrich line bundle, E is stable, thus indecomposable.

Now we produce an alternative description of a rank two non-special Ulrich bundle on $(X_0, \mathcal{O}_{X_0}(d,d))$ as a non-trivial extension of two non-Ulrich line bundles.

Proposition 3.52. *Let E be a rank two Ulrich bundle on $\big(X_0, \mathcal{O}_{X_0}(d,d)\big)$ with $c_1(E) = (4d-3)C_0 + (2d-1)f$, then E can be represented by an element $\xi \in$*

$\mathrm{Ext}^1(\mathcal{O}_{X_0}(2d-2,2d-1),\mathcal{O}_{X_0}(2d-1,0))$ *i.e. there exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}_{X_0}(2d-1,0) \longrightarrow E \longrightarrow \mathcal{O}_{X_0}(2d-2,2d-1) \longrightarrow 0. \tag{3.6.3}$$

Conversely, if E is a rank two vector bundle corresponding to $\xi \in \mathrm{Ext}^1(\mathcal{O}_{X_0}(2d-2,2d-1),\mathcal{O}_{X_0}(2d-1,0))$ then E is Ulrich if and only if it is initialized.

Proof. Let us build the Beilinson's table of $E(-2d+1,-2d+1)$.

$\mathcal{O}_{X_0}(-1,-1)$	$\mathcal{O}_{X_0}(-1,0)$	$\mathcal{O}_{X_0}(0,-1)$	$\mathcal{O}_{X_0}$	
0	0	0	0	h^3
0	1	0	0	h^2
0	0	$2d-1$	2d-2	h^1
0	0	0	0	h^0
$E(-2d,-2d)[-1]$	$E(-2d,-2d+1)[-1]$	$E(-2d+1,-2d)$	$E(-2d+1,-2d+1)$	

Observe that the zeroes in the table are obtained using Lemma 3.16. So in order to compute the numbers in the cohomology table we use the Riemann-Roch theorem. Thus we have

- $\chi(E(-2d+1,-2d+1)) = -h^1(X_0,E(-2d+1,-2d+1)) = -2(d-1)$.
- $\chi(E(-2d+1,-2d)) = -h^1(X_0,E(-2d+1,-2d)) = 1-2d$.
- $\chi(E(-2d,-2d+1)) = -h^1(X_0,E(-2d,-2d+1)) = -1$.

The first page of the Beilinson's spectral sequence will give us

$$0 \to \operatorname{Ker}\phi \to \mathcal{O}_{X_0}(0,-1)^{2d-1} \xrightarrow{\phi} \mathcal{O}_{X_0}^{2d-2} \to \operatorname{Coker}\phi \to 0, \tag{3.6.4}$$

and looking at the second (and infinity) page we have

$$0 \to \operatorname{Ker}\psi \to \mathcal{O}_{X_0}(-1,0) \xrightarrow{\psi} \operatorname{Coker}\phi \to 0. \tag{3.6.5}$$

So in the end we obtain $E(-2d+1,-2d+1)$ as an extension

$$0 \to \operatorname{Ker}\phi \to E(-2d+1,-2d+1) \to \operatorname{Ker}\psi \to 0. \tag{3.6.6}$$

Observe that $\operatorname{Ker}\phi$ is locally free since ϕ is the pull-back of a map on $\mathbb{P}^1$ and the kernel of a map on a smooth curve between locally free sheaves is locally free. Furthermore, by (3.6.6) $\operatorname{Ker}\phi$ can have rank at most 2. We say that $\operatorname{Ker}\phi$ has rank 1. In fact the rank cannot be zero because ϕ in (3.6.4) cannot be injective and the rank cannot be 2 because in that case $\operatorname{Ker}\psi$ would be a torsion sheaf which is in contradiction with (3.6.5).

So $\operatorname{Ker}\phi = \mathcal{O}_{X_0}(0,x)$. Consider (3.6.5). Since E is Ulrich we have that $E^\vee(3d-2,3d-2)$ is also Ulrich. In particular $h^i(X_0,E^\vee(2d-2,2d-2)) = 0$ for all i. But E is a rank two vector bundle, so $E \cong E^\vee(c_1)$ and we have the following short exact sequence

$$0 \to \operatorname{Ker}\phi \otimes \mathcal{O}_{X_0}(0,2d-2) \to E^\vee(2d-2,2d-2) \to \operatorname{Ker}\psi \otimes \mathcal{O}_{X_0}(0,2d-2) \to 0. \tag{3.6.7}$$

Now if we tensor (3.6.5) by $\mathcal{O}_{X_0}(0,2d-2)$ we get $h^0(X_0,\operatorname{Ker}\psi \otimes \mathcal{O}_{X_0}(0,2d-2)) = 0$ so, considering the long exact sequence in cohomology induced by (3.6.7) we have

$$h^0(X_0,\operatorname{Ker}\phi \otimes \mathcal{O}_{X_0}(0,2d-2)) = h^1(X_0,\operatorname{Ker}\phi \otimes \mathcal{O}_{X_0}(0,2d-2)) = 0$$

and the only possibility is to have $x+2d-2=-1$, i.e. $\operatorname{Ker}\phi = \mathcal{O}_{X_0}(0,1-2d)$.

Now we deal with $\operatorname{Coker}\phi$. Consider (3.6.5), then the only two possibilities for $\operatorname{Ker}\psi$ are

i) $\operatorname{Ker}\psi = \mathcal{I}_Z(0,-1)$ with Z a non-empty zero dimensional subscheme of X_0.

ii) $\operatorname{Ker}\psi = \mathcal{O}_{X_0}(-D)$ with D an effective divisor on X_0.

If $i)$ holds then we have $c_2(\operatorname{Coker}\phi) = -l(Z)$. But using (3.6.4) we observe that $c_2(\operatorname{Coker}\phi) = 0$ which is in contradiction with Z being non-empty. So it must be $\operatorname{Ker}\psi = \mathcal{O}_{X_0}(-D)$. Since $c_1(\operatorname{Coker}\phi) = 0$ then the only possibility is to have $\mathcal{O}(-D) = \mathcal{O}_{X_0}(0,-1)$ and $\operatorname{Coker}\phi = 0$. In this way we obtain

$$0 \to \mathcal{O}_{X_0}(0,1-2d) \to E(-2d+1,-2d+1) \to \mathcal{O}_{X_0}(-1,0) \to 0.$$

and tensoring it by $\mathcal{O}_{X_0}(2d-1,2d-1)$ we obtain the desired result.

Conversely take an extension

$$0 \to \mathcal{O}_{X_0}(2d-1,0) \to E \to \mathcal{O}_{X_0}(2d-2,2d-1) \to 0. \qquad (3.6.8)$$

Twisting it by $\mathcal{O}_{X_0}(-2d,-2d)$, we have

$$h^1(X_0,E(-2d,-2d)) = h^2(X_0,E(-2d,-2d)) = 0.$$

Now twist (3.6.8) by $\mathcal{O}_{X_0}(-d,-d)$ and consider the long exact sequence in cohomology. Since

$$h^1(X_0,\mathcal{O}_{X_0}(d-1,-d)) = h^0(X_0,\mathcal{O}_{X_0}(d-2,d-1)) = d(d-1),$$

we have $h^0(X_0,E(-d,-d)) = h^1(X_0,E(-d,-d))$. So as soon as one of the cohomology groups vanishes, also the other does. □

We end this paper with an example of a code which allows us to construct Ulrich bundles on $(X_e,\mathcal{O}_{X_e}(h))$ for an admissible Ulrich pair. We will use the resolution (3.2.3).

Example 3.53. In this example we will construct non-special rank two Ulrich bundles on X_1. In *Macaulay2*, given a divisor $D = tC_0 + sf$ on the Hirzebruch surface X_e the notation for line bundles is $\mathcal{O}_{X_e}(D) = \mathcal{O}_{X_e}(DC_0, Df) = \mathcal{O}_{X_e}(s-et,t)$, i.e. $\mathcal{O}_{X_e}(D) = \mathcal{O}_{X_e}(a,b)$ where a and b are respectively the intersection between D and the generators C_0 and f of $\mathrm{Pic}(X_e)$.

```
i1 : loadPackage "NormalToricVarieties";
```

Choose the self intersection invariant e

```
i2 : e=1;

i3 : FFe=hirzebruchSurface(e, CoefficientRing => ZZ/32003, Variable => y);

i4 : S = ring FFe;

i5 : loadPackage "BoijSoederberg";

i6 : loadPackage "BGG";

i7 : cohomologyTable(ZZ,CoherentSheaf,List,List):=(k,F,lo,hi)->(new Cohom
...
```

Fix a polarization $h = aC_0 + bf$, the rank r of our bundle and the first Chern class $uC_0 + vf$.

```
i8 : a=3;

i9 : b=6;

i10 : r=2;

i11 : u=6;

i12 : v=16;

i13 : exp1=r*(2*b-1-e)-v-e*(r*(2*a-2)-u);

i14 : exp2=r*(2*a-1)-u;

i15 : exp3=r*(2*a+2*b-3-e)-u-v-e*(r*(2*a-2)-u);
```

we construct two random matrices to obtain, as in Theorem 3.20(2), two maps

$$\mathcal{O}_{X_e}^{exp1}(0,-1) \to \mathcal{O}_{X_e}^{exp3}$$
$$\mathcal{O}_{X_e}^{exp2}(-1,-e) \to \mathcal{O}_{X_e}^{exp3}$$

```
i16 : M=random(S^exp3,S^{exp1:{-1,0}});

             4       2
o16 : Matrix S  <--- S

i17 : P=random(S^exp3,S^{exp2:{0,-1}});

             4       4
o17 : Matrix S  <--- S

i18 : Mtot=M|P;

             4       6
o18 : Matrix S  <--- S

i19 : F=minimalPresentation ker Mtot;

i20 : ShF= (sheaf(FFe,F))(2*b-1-e*(2*a-1),2*a-1);
```

Finally we check that the sheaf constructed in this way satisfies the vanishing of $H^0(X_e, F(-h))$ (or equivalently of $H^1(X_e, F(-h))$) required in Theorem 3.20(2).

```
i21 : HH^0(FFe,ShF(e*a-b,-a))
```

```
o21 = 0

        ZZ
o21 : ------module
      32003

i22 : exit
```

Chapter 4

Instanton bundles on Fano threefolds

In this chapter we study instanton bundles over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and over the flag variety $F(0,1,2)$, which are Fano threefolds of degree 6. We divide this chapter in two sections. In the first one we consider $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We describe each instanton as the cohomology of a monad and we prove the existence of instanton bundles of every charge and every possible c_2 via both a deformation argument and using the Hartshorne-Serre's correspondence. Finally we describe the locus of jumping lines. In the second section we consider the flag variety $F(0,1,2)$. We recall the known results which can be found in [78]. Then we generalize them by showing the existence of instanton bundles of every possible charge and c_2.

4.1 Instanton bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

In this section we consider $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which is a Fano threefold of Picard number three. Let us call h_1, h_2 and h_3 the three generators of the Picard group. By using a Beilinson type spectral sequence with suitable full exceptional collections we construct two different monads which are the analog of the monads for instanton bundles on $\mathbb{P}^3$ and on $F(0,1,2)$. We will prove the following theorem:

Theorem 4.1. *Let E be a charge k instanton bundle on X with $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$, then E is the cohomology of a monad of the form*

(i)

$$\begin{array}{ccccc} & \mathcal{O}_X^{k_3}(-h_1-h_2) & & \mathcal{O}_X^{k_2+k_3}(-h_1) & \\ & \oplus & & \oplus & \\ 0 \to & \mathcal{O}_X^{k_2}(-h_1-h_3) & \to & \mathcal{O}_X^{k_1+k_3}(-h_2) & \to \mathcal{O}_X^{k-2} \to 0. \\ & \oplus & & \oplus & \\ & \mathcal{O}_X^{k_1}(-h_2-h_3) & & \mathcal{O}_X^{k_1+k_2}(-h_3) & \end{array}$$

Conversely any μ-semistable bundle defined as the cohomology of such a monad is a charge k instanton bundle.

(ii)

$$\begin{array}{ccccc} & \mathcal{O}_X^{k_3}(-h_1-h_2) & & \mathcal{O}_X^{k_2+k_3}(h_1) & \\ & \oplus & & \oplus & \\ 0 \to & \mathcal{O}_X^{k_2}(-h_1-h_3) & \to \mathcal{O}_X^{3k+2} \to & \mathcal{O}_X^{k_1+k_3}(h_2) & \to 0 \\ & \oplus & & \oplus & \\ & \mathcal{O}_X^{k_1}(-h_2-h_3) & & \mathcal{O}_X^{k_1+k_2}(h_3) & \end{array}$$

Conversely any μ-semistable bundle with $H^0(E) = 0$ defined as the cohomology of such a monad is a charge k instanton bundle.

Furthermore we show that the Gieseker strictly semistable instanton bundles are extensions of line bundles and can be obtained as pullbacks from $\mathbb{P}^1 \times \mathbb{P}^1$. The cases where the degree of $c_2(E)$ is minimal, namely $k = k_1 + k_2 + k_2 = 2$, has been studied in [28]. In fact we get, up to twist, Ulrich bundles.

Here we show that Ulrich bundles is generically trivial on the lines. So we use this case as a starting step in order to prove by induction the existence of μ-stable instanton bundles generically trivial on the lines for any possible $c_2(E)$. In particular we prove the following

Theorem 4.2. *For each non-negative $k_1, k_2, k_3 \in \mathbb{Z}$ with $k = k_1 + k_2 + k_3 \geq 2$ there exists a μ-stable instanton bundle E with $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$ on X such that*

$$\mathrm{Ext}^1_X(E,E) = 4k-3, \qquad \mathrm{Ext}^2_X(E,E) = \mathrm{Ext}^3_X(E,E) = 0$$

and such that E is generically trivial on lines.

In particular there exists, inside the moduli space $MI(k_1e_1 + k_2e_2 + k_3e_3)$ of instanton bundles with $c_2 = k_1e_1 + k_2e_2 + k_3e_3$, a generically smooth irreducible component of dimension $4k-3$.

Finally we also study the locus of jumping lines obtaining the following result:

Proposition 4.3. *Let E be a generic instanton on X with $c_2 = k_1e_1 + k_2e_2 + k_3e_3$. Then the locus of jumping lines in the family $|e_1|$, denoted by $\mathcal{D}^1_E$, is a divisor given by $\mathcal{D}^1_E = k_3l + k_2m$ equipped with a sheaf G fitting into*

$$0 \to \mathcal{O}^{k_3}_{\mathcal{H}}(-1,0)\oplus\mathcal{O}^{k_2}_{\mathcal{H}}(0,-1) \to \mathcal{O}^{k_2+k_3}_{\mathcal{H}} \to i_*G \to 0.$$

Permuting indices we are also able to describe the locus of jumping lines in the other two rulings of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

Part of what follows can be found in the paper [5] by Antonelli and Malaspina.

4.1.1 First properties and monads

Let V_1,V_2,V_3 be three 2-dimensional vector spaces with the coordinates $[x_{1i}],[x_{2j}],[x_{3k}]$ respectively with $i,j,k \in \{1,2\}$. Let $X \cong \mathbb{P}(V_1)\times\mathbb{P}(V_2)\times\mathbb{P}(V_3)$ and then it is embedded into $\mathbb{P}^7 \cong \mathbb{P}(V)$ by the Segre map where $V = V_1\otimes V_2\otimes V_3$.

The intersection ring $A(X)$ is isomorphic to $A(\mathbb{P}^1)\otimes A(\mathbb{P}^1)\otimes A(\mathbb{P}^1)$ and so we have

$$A(X) \cong \mathbb{Z}[h_1,h_2,h_3]/(h_1^2,h_2^2,h_3^2).$$

We may identify $A^1(X) \cong \mathbb{Z}^{\oplus 3}$ by $a_1h_1 + a_2h_2 + a_3h_3 \mapsto (a_1,a_2,a_3)$. Similarly we have $A^2(X) \cong \mathbb{Z}^{\oplus 3}$ by $k_1e_1 + k_2e_2 + k_3e_3 \mapsto (k_1,k_2,k_3)$ where $e_1 = h_2h_3, e_2 = h_1h_3, e_3 = h_1h_2$ and $A^3(X) \cong \mathbb{Z}$ by $ch_1h_2h_3 \mapsto c$. Then X is embedded into $\mathbb{P}^7$ by the complete linear system $h = h_1 + h_2 + h_3$ as a subvariety of degree 6 since $h^3 = 6$.

If E is a rank two bundle with the Chern classes $c_1 = (a_1,a_2,a_3)$, $c_2 = (k_1,k_2,k_3)$ we have:

$$\begin{aligned} c_1(E(s_1,s_2,s_3)) &= (a_1+2s_1, a_2+2s_2, a_3+2s_3) \\ c_2(E(s_1,s_2,s_3)) &= c_2 + c_1\cdot(s_1,s_2,s_3) + (s_1,s_2,s_3)^2 \end{aligned} \tag{4.1.1}$$

for $(s_1,s_2,s_3) \in \mathbb{Z}^{\oplus 3}$.

Let us recall the Riemann-Roch formula:

$$\chi(E) = (a_1+1)(a_2+1)(a_3+1)+1-\frac{1}{2}((a_1,a_2,a_3)\cdot(k_1,k_2,k_3)+2(k_1+k_2+k_3)) \tag{4.1.2}$$

Let us also recall the description of the Hilbert scheme of lines on X. A line on X is in the class of $|e_i|$ for some i.

Proposition 4.4. *[29, Proposition 4.1][62, Proposition 3.5.6] The Hilbert scheme $Hilb_{t+1}(X)$ has exactly three disjoint components. Each of them is the locus of points representing one and the same class inside $A^2(X)$ and it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

Now we define instanton bundles on X.

Definition 4.5. A μ-semistable vector bundle E on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is called an instanton bundle of charge k if and only if $c_1(E) = 0$,

$$H^0(E) = H^1(E(-h)) = 0$$

and $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$ with $k_1 + k_2 + k_3 = k$.

Remark 4.6. It is worthwhile to point out that, exactly as in the case of $F(0,1,2)$ (see [78] Remark 2.2), the condition $H^0(E) = 0$ does not follow from the other conditions defining an instanton bundle. Indeed we may consider the rank two aCM bundles with $c_1(E) = 0$ and $H^0(E) \neq 0$ given in [28] Theorem B.

Now we recall the Hoppe's criterion for semistable vector bundles over polycyclic varieties, i.e. varieties X such that $\mathrm{Pic}(X) = \mathbb{Z}^l$.

Proposition 4.7. *[67, Theorem 3] Let E be a rank two holomorphic vector bundle over a polycyclic variety X and let L be a polarization on X. E is μ-(semi)stable if and only if*

$$H^0(X, E \otimes \mathcal{O}_X(B)) = 0$$

for all $B \in Pic(X)$ such that $\delta_L(B) \underset{(<)}{\leq} -\mu_L(E)$, where $\delta_L(B) = \deg_L(\mathcal{O}_X(B))$.

In order to get a monadic description of instanton bundles, we need to apply Proposition 2.25. We start by constructing the full exceptional collections that we will use in the next theorems.

Let us consider on the three copies of $\mathbb{P}^1$ the full exceptional collection

$$\{\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}\}.$$

We may obtain the full exceptional collection $\langle E_7, \dots, E_0\rangle$ (see Proposition 2.27):

$$\begin{gathered}\{\mathcal{O}_X(-h)[-4], \mathcal{O}_X(-h_2-h_3)[-4], \mathcal{O}_X(-h_1-h_3)[-3],\\ \mathcal{O}_X(-h_1-h_2)[-2], \mathcal{O}_X(-h_3)[-2], \mathcal{O}_X(-h_2)[-1], \mathcal{O}_X(-h_1), \mathcal{O}_X\}.\end{gathered} \tag{4.1.3}$$

The associated full exceptional collection $\langle F_7 = \mathcal{F}_7, \dots, F_0 = \mathcal{F}_0\rangle$ of Theorem 2.23 is

$$\begin{gathered}\{\mathcal{O}_X(-h), \mathcal{O}_X(-h_2-h_3), \mathcal{O}_X(-h_1-h_3), \mathcal{O}_X(-h_1-h_2),\\ \mathcal{O}_X(-h_3), \mathcal{O}_X(-h_2), \mathcal{O}_X(-h_1), \mathcal{O}_X\}.\end{gathered} \tag{4.1.4}$$

From (4.1.3) with a left mutation of the pair $\{\mathcal{O}_X(-h_1), \mathcal{O}_X\}$ we obtain:

$$\begin{gathered}\{\mathcal{O}_X(-h)[-4], \mathcal{O}_X(-h_2-h_3)[-4], \mathcal{O}_X(-h_1-h_3)[-3],\\ \mathcal{O}_X(-h_1-h_2)[-2], \mathcal{O}_X(-h_3)[-2], \mathcal{O}_X(-h_2)[-1], \mathcal{O}_X(-2h_1), \mathcal{O}_X(-h_1)\}.\end{gathered} \tag{4.1.5}$$

From the above collection with a left mutation of the pair $\{\mathcal{O}_X(-h_2)[-1], \mathcal{O}_X(-2h_1)\}$ we obtain:

$$\begin{gathered}\{\mathcal{O}_X(-h)[-4], \mathcal{O}_X(-h_2-h_3)[-4], \mathcal{O}_X(-h_1-h_3)[-3],\\ \mathcal{O}_X(-h_1-h_2)[-2], \mathcal{O}_X(-h_3)[-2], A[-1], \mathcal{O}_X(-h_2)[-1], \mathcal{O}_X(-h_1)\}\end{gathered} \tag{4.1.6}$$

where A is given by the extension

$$0 \to \mathcal{O}_X(-2h_1) \to A \to \mathcal{O}_X(-h_2)^{\oplus 2} \to 0. \tag{4.1.7}$$

From the above collection with a left mutation of the pair $\{\mathcal{O}_X(-h_3), A\}$ we obtain:

$$\begin{aligned}&\{\mathcal{O}_X(-h)[-4], \mathcal{O}_X(-h_2-h_3)[-4], \mathcal{O}_X(-h_1-h_3)[-3],\\ &\mathcal{O}_X(-h_1-h_2)[-2], B[-2], \mathcal{O}_X(-h_3)[-2], \mathcal{O}_X(-h_2)[-1], \mathcal{O}_X(-h_1)\}\end{aligned} \tag{4.1.8}$$

where B is given by the extension

$$0 \to A \to B \to \mathcal{O}_X(-h_3)^{\oplus 2} \to 0. \tag{4.1.9}$$

Making the respective right mutation of (4.1.4) we obtain the full exceptional collection $\langle F_7 = \mathcal{F}_n, \dots, F_0 = \mathcal{F}_0 \rangle$ of Theorem 2.23:

$$\{\mathcal{O}_X(-h), \mathcal{O}_X(-h_2-h_3), \mathcal{O}_X(-h_1-h_3), \mathcal{O}_X(-h_1-h_2), \mathcal{O}_X, \mathcal{O}_X(h_3), \mathcal{O}_X(h_2), \mathcal{O}_X(h_1)\}. \tag{4.1.10}$$

It is easy to check that the conditions (2.2.3) are satisfied. Observe that both collections (4.1.3) and (4.1.10) are strong.

Theorem 4.8. *Let E be a charge k instanton bundle on X with $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$, then E is the cohomology of a monad of the form*

$$0 \to \begin{array}{c}\mathcal{O}_X^{k_3}(-h_1-h_2)\\ \oplus\\ \mathcal{O}_X^{k_2}(-h_1-h_3)\\ \oplus\\ \mathcal{O}_X^{k_1}(-h_2-h_3)\end{array} \to \begin{array}{c}\mathcal{O}_X^{k_2+k_3}(-h_1)\\ \oplus\\ \mathcal{O}_X^{k_1+k_3}(-h_2)\\ \oplus\\ \mathcal{O}_X^{k_1+k_2}(-h_3)\end{array} \to \mathcal{O}_X^{k-2} \to 0. \tag{4.1.11}$$

Conversely any μ-semistable bundle defined as the cohomology of such a monad is a charge k instanton bundle.

Proof. We consider the Beilinson type spectral sequence associated to an instanton bundle E and identify the members of the graded sheaf associated to the induced filtration as the sheaves mentioned in the statement of Theorem 2.23 and Proposition

2.25. We consider the full exceptional collection $\langle E_7, \dots, E_0\rangle$ given in (4.1.3) and the full exceptional collection $\langle F_7, \dots, F_0\rangle$ given in (4.1.4).

First of all, let us observe that since $H^0(E) = 0$ we have $H^0(E(-D)) = 0$ for every effective divisor D. Furthermore by Serre's duality we have also $H^2(E(K+D)) = 0$ for all effective divisors D. Since $c_1(E) = 0$ using Serre's duality and $H^1(E(-h)) = 0$ we obtain

$$H^i(E(-h)) = H^{3-i}(E(-h)) = 0 \text{ for all } i.$$

We want to show that for each twist in the table, there's only one non vanishing cohomology group, so that we can use the Riemann-Roch formula to compute the dimension of the remaining cohomology group. Let us consider the pull-back of the Euler sequence from one of the $\mathbb{P}^1$ factors

$$0 \to \mathcal{O}_X(-h_a) \to \mathcal{O}_X^2 \to \mathcal{O}_X(h_a) \to 0 \tag{4.1.12}$$

and tensor it by $E(-h)$. We have

$$0 \to E(-2h_a - h_b - h_c) \to E^2(-h) \to E(-h_b - h_c) \to 0$$

with $a, b, c \in \{1, 2, 3\}$ and they are all different from each other. Since $H^i(E(-h)) = 0$ for all i and $H^0(E(-2h_a - h_b - h_c)) = H^3(E(-2h_a - h_b - h_c)) = 0$, considering the long exact sequence induced in cohomology we have $H^2(E(-h_b - h_c)) = 0$. Now we want to show that $H^2(E(-h_a)) = 0$ for all $a \in \{1, 2, 3\}$. Tensor (4.1.12) by $E(-h_b)$ with $b \neq a$ and we have:

$$0 \to E(-2h_a - h_b) \to E^2(-h_a - h_b) \to E(-h_b) \to 0.$$

Considering the long exact sequence induced in cohomology we have that $H^2(E(-h_b)) = 0$ since $H^2(E(-h_a - h_b)) = H^3(E(-2h_a - h_b)) = 0$. Finally if we tensor (4.1.12) by $E(-h_a)$ and we consider the long exact sequence in cohomology, we obtain $H^2(E) = 0$.

Now let us compute the Euler characteristic of E tensored by a line bundle $\mathcal{O}_X(D)$ so that we are able to compute all the numbers in the Beilinson's table. Combining (4.1.1) and (4.1.2) we have

$$\chi(E(D)) = \frac{1}{6}(2D^3 - 6c_2(E)D) + h(D^2 - c_2(E)) + Dh^2 + 2. \tag{4.1.13}$$

By (4.1.13) we have

- $h^1(E) = -\chi(E) = 2 - k_1 - k_2 - k_3 = 2 - k.$
- $h^1(E(-h_i)) = -\chi(E(-h_i)) = k_i - k.$
- $h^1(E(-h_i - h_j)) = -\chi(E(-h_i - h_j)) = k_i + k_j - k.$

So we get the following table:

$\mathcal{O}_X(-h)$	$\mathcal{O}_X(-h_2-h_3)$	$\mathcal{O}_X(-h_1-h_3)$	$\mathcal{O}_X(-h_1-h_2)$	$\mathcal{O}_X(-h_3)$	$\mathcal{O}_X(-h_2)$	$\mathcal{O}_X(-h_1)$	$\mathcal{O}_X$	
0	0	0	0	0	0	0	0	h^7
0	0	0	0	0	0	0	0	h^6
0	k_1	0	0	0	0	0	0	h^5
0	0	k_2	0	0	0	0	0	h^4
0	0	0	k_3	k_1+k_2	0	0	0	h^3
0	0	0	0	0	k_1+k_3	0	0	h^2
0	0	0	0	0	0	k_2+k_3	$k-2$	h^1
0	0	0	0	0	0	0	0	h^0
$E(-h)[-4]$	$E(-h_2-h_3)[-4]$	$E(-h_1-h_3)[-3]$	$E(-h_1-h_2)[-2]$	$E(-h_3)[-2]$	$E(-h_2)[-1]$	$E(-h_1)$	E	

Using Beilinson's theorem in the strong form (as in Proposition 2.25) we retrieve the monad (4.1.11).

Conversely let E be a μ-semistable bundle defined as the cohomology of a monad (4.1.11). We may consider the two short exact sequences:

$$0 \to G \to \begin{matrix} \mathcal{O}_X^{k_2+k_3}(-h_1) \\ \oplus \\ \mathcal{O}_X^{k_1+k_3}(-h_2) \\ \oplus \\ \mathcal{O}_X^{k_1+k_2}(-h_3) \end{matrix} \to \mathcal{O}_X^{k-2} \to 0 \tag{4.1.14}$$

and

$$0 \to \begin{matrix} \mathcal{O}_X^{k_3}(-h_1-h_2) \\ \oplus \\ \mathcal{O}_X^{k_2}(-h_1-h_3) \\ \oplus \\ \mathcal{O}_X^{k_1}(-h_2-h_3) \end{matrix} \to G \to E \to 0. \tag{4.1.15}$$

We deduce that $H^0(G) = H^0(E) = 0$. By (4.1.14) and (4.1.15) tensored by $\mathcal{O}_X(-h)$ we obtain $H^1(G(-h)) = H^1(E(-h)) = 0$ so E is an instanton. □

Proposition 4.9. *Let E be an instanton bundle on X, then $h^1(E(-h-D)) = 0$ for every effective divisor D.*

Proof. Let us consider the two short exact sequences (4.1.14) and (4.1.15) tensored by $\mathcal{O}_X(-h+D)$. By Künneth formula we have that $h^i(\mathcal{O}_X(-h-D)) = 0$ for all i, and thus taking the cohomology of (4.1.14) we get $h^i(G(-h-D)) = 0$ for $i \neq 3$. Combining this with the induced sequence in cohomology of (4.1.15) we obtain $h^0(E(-h-D)) = h^1(E(-h-D)) = 0$. □

In the next theorem we obtain a description of instanton bundles as the cohomology of a different monad.

Theorem 4.10. *Let E be a charge k instanton bundle on X with $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$, then E is the cohomology of a monad of the form*

$$0 \to \begin{matrix} \mathcal{O}_X^{k_3}(-h_1-h_2) \\ \oplus \\ \mathcal{O}_X^{k_2}(-h_1-h_3) \\ \oplus \\ \mathcal{O}_X^{k_1}(-h_2-h_3) \end{matrix} \to \mathcal{O}_X^{3k+2} \to \begin{matrix} \mathcal{O}_X^{k_2+k_3}(h_1) \\ \oplus \\ \mathcal{O}_X^{k_1+k_3}(h_2) \\ \oplus \\ \mathcal{O}_X^{k_1+k_2}(h_3) \end{matrix} \to 0 \qquad (4.1.16)$$

Conversely any μ-semistable bundle with $H^0(E) = 0$ defined as the cohomology of such a monad is a charge k instanton bundle.

Proof. We consider the Beilinson type spectral sequence associated to an instanton bundle E and identify the members of the graded sheaf associated to the induced filtration as the sheaves mentioned in the statement of Theorem 2.23. We consider the full exceptional collection $\langle E_7, \dots, E_0 \rangle$ given in (4.1.8) and the full exceptional collection $\langle F_7, \dots, F_0 \rangle$ given in (4.1.10).

First of all, let us observe that since since E is μ-semistable, by Hoppe's criterion we have $H^0(E(-D)) = 0$ for every effective divisor D. Furthermore we have all the vanishing computed in Theorem 4.8. Moreover by (4.1.7) and (4.1.9) tensored by E

we get

$$\begin{aligned}\chi(E\otimes B) &= \chi(E\otimes A)+2\chi(E(-h_3))\\ &= \chi(E(-2h_1))+2\chi(E(-h_3))+2\chi(E(-h_2))\\ &= -2+k_1-k_2-k_3-2(k_1+k_2)-2(k_1+k_3)\\ &= -2-3k.\end{aligned}$$

So we get the following table:

$\mathcal{O}_X(-h)$	$\mathcal{O}_X(-h_2-h_3)$	$\mathcal{O}_X(-h_1-h_3)$	$\mathcal{O}_X(-h_1-h_2)$	$\mathcal{O}_X(-h_3)$	$\mathcal{O}_X(-h_2)$	$\mathcal{O}_X(-h_1)$	$\mathcal{O}_X$	
0	0	0	0	0	0	0	0	h^7
0	0	0	0	0	0	0	0	h^6
0	k_1	0	0	0	0	0	0	h^5
0	0	k_2	0	a	0	0	0	h^4
0	0	0	k_3	b	k_1+k_2	0	0	h^3
0	0	0	0	0	0	k_1+k_3	0	h^2
0	0	0	0	0	0	0	k_2+k_3	h^1
0	0	0	0	0	0	0	0	h^0
$E(-h)[-4]$	$E(-h_2-h_3)[-4]$	$E(-h_1-h_3)[-3]$	$E(-h_1-h_2)[-2]$	$E\otimes B[-2]$	$E(-h_3)[-2]$	$E(-h_2)[-1]$	$E(-h_1)$	

where $a-b=-2-3k$. Since the spectral sequence converges to an object in degree 0 and there no maps involving a we deduce that $a=0$ and $b=3k+2$. So we get the following table:

$\mathcal{O}_X(-h)$	$\mathcal{O}_X(-h_2-h_3)$	$\mathcal{O}_X(-h_1-h_3)$	$\mathcal{O}_X(-h_1-h_2)$	$\mathcal{O}_X(-h_3)$	$\mathcal{O}_X(-h_2)$	$\mathcal{O}_X(-h_1)$	$\mathcal{O}_X$	
0	0	0	0	0	0	0	0	h^7
0	0	0	0	0	0	0	0	h^6
0	k_1	0	0	0	0	0	0	h^5
0	0	k_2	0	0	0	0	0	h^4
0	0	0	k_3	3k+2	k_1+k_2	0	0	h^3
0	0	0	0	0	0	k_1+k_3	0	h^2
0	0	0	0	0	0	0	k_2+k_3	h^1
0	0	0	0	0	0	0	0	h^0
$E(-h)[-4]$	$E(-h_2-h_3)[-4]$	$E(-h_1-h_3)[-3]$	$E(-h_1-h_2)[-2]$	$E\otimes B[-2]$	$E(-h_3)[-2]$	$E(-h_2)[-1]$	$E(-h_1)$	

Using Beilinson's theorem as in Proposition 2.25 we retrieve the monad (4.1.16).

Conversely let E be a μ-semistable bundle with no global sections defined as the cohomology of a monad (4.1.11). We may consider the two short exact sequences:

$$0 \to G \to \mathcal{O}_X^{3k+2} \to \begin{array}{c} \mathcal{O}_X^{k_2+k_3}(h_1) \\ \oplus \\ \mathcal{O}_X^{k_1+k_3}(h_2) \\ \oplus \\ \mathcal{O}_X^{k_1+k_2}(h_3) \end{array} \to 0 \tag{4.1.17}$$

and

$$0 \to \begin{array}{c} \mathcal{O}_X^{k_3}(-h_1-h_2) \\ \oplus \\ \mathcal{O}_X^{k_2}(-h_1-h_3) \\ \oplus \\ \mathcal{O}_X^{k_1}(-h_2-h_3) \end{array} \to G \to E \to 0. \tag{4.1.18}$$

By (4.1.17) and (4.1.18) tensored by $\mathcal{O}_X(-h)$ we obtain $H^1(G(-h)) = H^1(E(-h)) = 0$ so E is an instanton. □

Remark 4.11. It is possible to construct vector bundles which are realized as the cohomology of a monad as in Theorem 4.8 and 4.10 but that are not μ-semistable. Let us consider a generic line l in the ruling e_1. It has the following resolution on X

$$0 \to \mathcal{O}_X(-h_2-h_3) \to \mathcal{O}_X(-h_2) \oplus \mathcal{O}_X(-h_3) \to \mathcal{O}_X \to \mathcal{O}_l \to 0. \tag{4.1.19}$$

By adjunction formula we have $\mathcal{N}^\vee_{l/X} \cong \mathcal{I}_{l|X} \otimes \mathcal{O}_l$, and using (4.1.19) we obtain $\mathcal{N}^\vee_{l/X} \cong \mathcal{O}_l^2$ and in particular $\det \mathcal{N}_{l/X} \otimes \mathcal{O}_l \cong \mathcal{O}_X(D) \otimes \mathcal{O}_l$ where D is a divisor of the form $D = ah_2 + bh_3$. Choosing $D = 2h_2 - 4h_3$, since $h^2(\mathcal{O}_X(-D)) = 0$, it is possible to construct a vector bundle E with $c_1(E) = 0$ and $c_2(E) = 5e_1$ through the Hartshorne-Serre correspondence (Theorem 2.1) which fits into

$$0 \to \mathcal{O}_X(-h_2+2h_3) \to E \to \mathcal{I}_{l|X}(h_2-2h_3) \to 0. \tag{4.1.20}$$

The vector bundle constructed in this way has no sections, i.e. $H^0(E) = 0$ and if we tensor (4.1.20) by $\mathcal{O}_X(-h)$ and we take the cohomology, we obtain

$$H^i(E(-h)) \cong H^i(\mathcal{I}_{l|X}(-h_1-3h_3)).$$

Now consider the sequence

$$0 \to \mathcal{I}_{l|X} \to \mathcal{O}_X \to \mathcal{O}_l \to 0. \tag{4.1.21}$$

Tensoring (4.1.21) by $\mathcal{O}_X(-h_1-3h_3)$, we get $h^1(\mathcal{I}_{l|X}(-h_1-3h_3)) = h^0(\mathcal{O}_l(-h_1-3h_3)) = h^0(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. Thus we obtain $H^1(E(-h)) = 0$. In this way we constructed a vector bundle E with $c_1(E) = 0$ satisfying all the instantonic conditions but the μ-semistability. In fact by Proposition 4.7 E is not μ-semistable since $H^0(E(h_2-2h_3)) \neq 0$. Furthermore E has the same cohomology table of an instanton bundle, thus it is realized as the cohomology of the monads

$$0 \to \mathcal{O}_X^5(-h_2-h_3) \to \mathcal{O}_X^5(-h_2) \oplus \mathcal{O}_X^5(-h_3) \to \mathcal{O}_X^3 \to 0$$

and

$$0 \to \mathcal{O}_X^5(-h_2-h_3) \to \mathcal{O}_X^{17} \to \mathcal{O}_X^5(h_2) \oplus \mathcal{O}_X^5(h_3) \to 0.$$

Remark 4.12. Let us remark that the monad (4.1.16) is the analog of the monad for instanton bundles on $\mathbb{P}^3$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2k+2} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus k} \to 0,$$

and the monad (4.1.11) is the analog of the second monad for instanton bundles on $\mathbb{P}^3$ (see for instance [3] display (1.1))

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k} \xrightarrow{\alpha} \Omega_{\mathbb{P}^3}(1)^{\oplus k} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2k-2} \to 0.$$

A very similar behaviour was shown for the two monads for instanton bundles on the flag threefold in [78].

As in the case of instanton bundles on the projective space and flag varieties, the two monads (4.1.16) and (4.1.11) are closely related. Indeed, sequence (4.1.17) fits in the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & \mathcal{O}^{\oplus 3k+2} & \xrightarrow{\beta} & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_{i+1}+k_{i+2}}(h_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_{i+1}+k_{i+2}}(-h_i) & \longrightarrow & \mathcal{O}^{\oplus 4k} & \xrightarrow{\beta'} & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_{i+1}+k_{i+2}}(h_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{\oplus k-2} & \longrightarrow & \mathcal{O}^{\oplus k-2} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

So we get sequence (4.1.14) as the first column. Moreover sequence (4.1.18) fits in the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_i}(-h_{i+1}-h_{i+2}) & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_i}(-h_{i+1}-h_{i+2}) & \longrightarrow & \oplus_{i\in\mathbb{Z}_3}\mathcal{O}_X^{k_{i+1}+k_{i+2}}(-h_i) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{O}^{\oplus k-2} & \longrightarrow & \mathcal{O}^{\oplus k-2} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

which is the display of monad (4.1.11).

Finally, for the monad (4.1.16) is not necessary the assumption $H^0(E) = 0$. Exactly the same behavior was shown for the analog monad on $F(0,1,2)$ (see [78] Theorem 4.2).

We end this section by characterizing the strictly Gieseker semistable instanton bundles on X

Proposition 4.13. *Let E be an instanton bundle of charge k. If E is not μ-stable then $k = 2l^2$ for some $l \in \mathbb{Z}$, $l \neq 0$. Moreover $c_2(E) = 2l^2 e_i$, $i = 1,2,3$ and E can be constructed as an extension*

$$0 \to \mathcal{O}_X(-lh_i + lh_j) \to E \to \mathcal{O}_X(lh_i - lh_j) \to 0 \tag{4.1.22}$$

with $i \neq j$.

Proof. Suppose $H^0(X, E(ah_1 + bh_2 - (a+b)h_3)) \neq 0$ for some $a, b \in \mathbb{Z}$. So E fits into an exact sequence

$$0 \to \mathcal{O}_X \to E(ah_1 + bh_2 - (a+b)h_3) \to \mathcal{I}_Z(2ah_1 + 2bh_2 - 2(a+b)h_3) \to 0$$

where $Z \subset X$ is a subscheme of X. Since $H^0(E(ah_1+bh_2-(a+b)h_3) \otimes \mathcal{O}_X(-h_j)) = 0$ for all $j = 1,2,3$ by Proposition 4.7, we have that $Z \subset X$ is either empty or purely 2-codimensional. Suppose we are dealing with the latter case, since E in Gieseker semistable we have that

$$P_{\mathcal{O}_X}(t) \leq P_{E(ah_1+bh_2-(a+b)h_3)}(t) \leq P_{\mathcal{I}_Z(2ah_1+2bh_2-2(a+b)h_3)}(t)$$

and

$$P_{\mathcal{I}_Z(2ah_1+2bh_2-2(a+b)h_3)}(t) = P_{\mathcal{O}_X(2ah_1+2bh_2-2(a+b)h_3)}(t) - P_{\mathcal{O}_Z(2ah_1+2bh_2-2(a+b)h_3)}(t)$$

where $P(t)$ is the Hilbert polynomial. So we have

$$\begin{aligned} P_{\mathcal{O}_Z(2ah_1+2bh_2-2(a+b)h_3)}(t) &\leq P_{\mathcal{O}_X(2ah_1+2bh_2-2(a+b)h_3)}(t) - P_{\mathcal{O}_X}(t) \\ &= (2a+t+1)(2b+t+1)(t+1-2a-2b) - (t+1)^3 \\ &= -4(t+1)(a^2+b^2+ab) < 0 \text{ for } t >> 0. \end{aligned}$$

contradicting Serre's vanishing theorem. Se we can conclude that Y is empty and E fits into

$$0 \to \mathcal{O}_X(-ah_1 - bh_2 + (a+b)h_3) \to E \to \mathcal{O}_X(ah_1 + bh_2 - (a+b)h_3) \to 0.$$

Now computing $c_2(E)$ we obtain

$$\begin{aligned} c_2(E) &= (-ah_1 - bh_2 + (a+b)h_3) \cdot (ah_1 + bh_2 - (a+b)h_3) \\ &= 2b(a+b)e_1 + 2a(a+b)e_2 - 2abe_3. \end{aligned}$$

Since E is an instanton bundle on X, all the summands of $c_2(E)$ must be nonnegative. In fact as we saw in Proposition 4.8 they represent the dimension of a cohomology group. So either a or b is 0 (but not both since the charge k must be greater than two) or $a = -b$. In all three cases we obtain the desired result. □

4.1.2 Splitting behaviour of Ulrich bundles

In the next sections we will construct, through an induction process, stable k-instanton bundles on X for each charge k and all second Chern classes.

Let us consider the base case of induction, which consists of charge 2 instantons on X, i.e. rank two Ulrich bundles (up to twisting by $\mathcal{O}_X(-h)$). For further details about Ulrich bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ see [29]. We have two possible alternatives for the second Chern class of an Ulrich bundle:

(a) $c_2(E) = 2e_i$ for some $i \in \{1,2,3\}$.

(b) $c_2(E) = e_i + e_j$ with $i \neq j$.

We show that in both in cases the generic Ulrich bundle has trivial restriction with respect to a generic line of each family. In both cases we have $\mathrm{Ext}^2(E,E) = \mathrm{Ext}^3(E,E) = 0$ by [29, Lemma 2.3].

Proposition 4.14. *The generic instanton bundle of minimal charge $k = 2$ has trivial restriction with respect to the generic line of each family $|e_1|$, $|e_2|$ and $|e_3|$.*

Proof. We will separate the proof treating both cases (a) and (b).

Case (a)

Let us begin with the first case. By Theorem 4.8 we see that every rank two Ulrich bundle with this second Chern class is the pullback of a vector bundle on a quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$. In this case, by Proposition 4.13, there exist strictly semistable

Ulrich bundle realized as extensions

$$0 \to \mathcal{O}_X(h_j - h_k) \to E \to \mathcal{O}_X(h_k - h_j) \to 0 \qquad (4.1.23)$$

with $j \neq k \neq i \neq j$. For these vector bundles, by restricting (4.1.23) to a line in each family, we observe that in the family h_jh_k there are not jumping lines, i.e. $E_l = \mathcal{O}_l^2$ for each $l \in |h_jh_k|$. On the other hand, E_l is never trivial when $l \in |h_ih_k|$ or $l \in |h_jh_i|$. However the generic bundle will be stable, so let us focus on stable Ulrich bundles. They are pull back via the projection on the quadric, of stable bundles on Q. By [88, Lemma 2.5] every such bundle can be deformed to a stable bundle which is trivial when restricted to the generic line of each family.

Case (b)

Now let us consider the second case. The details of what follows can be found in [28]. Up to a permutation of the indices we can assume $c_2(E) = e_2 + e_3$. Let us denote by H a general hyperplane section in $\mathbb{P}^7$ and let S be $S = X \cap H$. S is a del Pezzo surface of degree 6, given as the blow up of $\mathbb{P}^2$ in 3 points. Let us denote by F the pullback to S of the class of a line in $\mathbb{P}^2$ and by E_i the exceptional divisors. Take a general curve C of class $3F - E_1$, so that C is a smooth, irreducible, elliptic curve of degree 8. Moreover we have $h^0(C, \mathcal{N}_{C|X}) = 16$ and $h^1(C, \mathcal{N}_{C|X}) = 0$, so the Hilbert scheme $\mathscr{H} = \mathscr{H}^{8t}$ of degree 8 elliptic curves is smooth of dimension 16 [28, Proposition 6.3] and the general deformation of C in $\mathscr{H}$ is non-degenerate [28, Proposition 6.6]. Let $\mathcal{C} \subset X \times B \to B$ a flat family of curves in $\mathscr{H}$ with special fibre $\mathcal{C}_{b_0} \cong C$ over b_0. To each curve in the family $\mathcal{C}$ we can associate a rank two vector bundle via the Serre's correspondence:

$$0 \to \mathcal{O}_X(-h) \to E_b \to I_{C_b|X}(h) \to 0 \qquad (4.1.24)$$

where C_b is the curve in $\mathcal{C}$ over $b \in B$. The general fiber C_b correspond via (4.1.24) to rank two Ulrich bundle of the desired c_2.

Now choose a line L in S, such that $L \cap C$ is a single point x. In order to do so, we deal with the classes of F and E_i in $A^2(X)$. One obtain that the classes of F, E_1, E_2 and E_3 are $e_1 + e_2 + e_3$, e_1, e_2 and e_3 respectively. In particular, there exists a line L in the system $|E_1|$ (corresponding to $|e_1|$ in $A^2(X)$) which intersects the curve C in the class $3F - E_1$ in one point. It follows that $I_{C|X}(1) \otimes \mathcal{O}_L \cong \mathcal{O}_x \oplus \mathcal{O}_L$. Tensoring

(4.1.24) by $\mathcal{O}_L$ we obtain a surjection

$$E_{b_{0|L}} \to \mathcal{O}_x \oplus \mathcal{O}_L \to 0.$$

In particular $E_{b_{0|L}}$ cannot be $\mathcal{O}_L(-t) \oplus \mathcal{O}_L(t)$ for any $t > 0$, thus $E_{b_{0|L}}$ is trivial, which is equivalent to $h^0(L, E_{b_{0|L}}(-1)) = 0$. By semicontinuity we have that $h^0(L, E_{b_{|L}}(-1)) = 0$ for all b in an open neighborhood of $b_0 \in B$, thus the vector bundle corresponding to the general fiber C_b is trivial over the line L. Since this is an open condition on the variety of lines contained in X, it takes place for the general line in $|e_1|$.

To deal with the other families of lines let us consider a general quadric Q in $|h_1|$. Let C be a smooth, irreducible, non-degenerate elliptic curve in the class $2e_1 + 3e_2 + 3e_3$. $\mathrm{Pic}(Q) \cong \mathbb{Z}^2$ is generated by two lines $< l, m >$ which correspond respectively to e_3 and e_2. Since Q is general then $Z = C \cap Q$ consist of two points. Following the previous strategy, we say that E restricted to a generic line of the family e_2 (resp. e_3) is trivial if Z is not contained in a line of the ruling m (resp. l). As in the previous case, let us consider the del Pezzo surface $S = X \cap H$ with H a general hyperplane section. The intersection between Q and S is a curve in the class $e_2 + e_3 \in A^2(X)$. Let us denote by Y the curve $Y = Q \cap S$. We compute the class of Y in S. We have the following short exact sequences

$$0 \to \mathcal{O}_X(-2h_1 - h_2 - h_3) \to \mathcal{O}_X(-h_1) \oplus \mathcal{O}_X(-h) \to \mathcal{O}_X \to I_Y \to 0$$
$$0 \to I_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

and computing the cohomology we find that $h^1(Y, \mathcal{O}_Y) = g = 0$. In particular we have that Y is a degree two curve of genus 0 on S, thus it must be in the class of $F - E_1$. Furthermore, observe that every line of each ruling of Q intersect S in only one point. Now let us take a general curve C in the class $3F - E_1$, so that C is a smooth, irreducible, elliptic curve of degree 8. Computing the intersection product between C and Y, we see that $C \cap Y$ consists of two points. Those two points cannot lie on a line in Q, because each line in Q intersect S only in one point. As before, let $\mathcal{C} \subset X \times B \to B$ a flat family of curves in $\mathscr{H}$ with special fibre $\mathcal{C}_{b_0} \cong C$ over b_0. To each curve in the family $\mathcal{C}$ we can associate a rank two vector bundle via the sequence (4.1.24). Let $Z_b = C_b \cap Q$ and denote by l and m the two rulings of Q. We observed that Z_{b_0} is not contained in a line either of l or m, i.e C_{b_0} intersects the generic line of both $|l|$ and $|m|$ in one point. But the rulings of Q correspond to the rulings e_2 and e_3 of X, thus we can repeat the same argument used for the generic line in e_1. In this way we conclude that the vector bundle corresponding to the general fiber C_b is trivial over the generic line of each of the families $|e_1|$, $|e_2|$ and $|e_3|$. □

4.1.3 Construction of instanton bundles of higher charge

In this section we will construct instanton bundles of every charge generically trivial on lines, through an induction process starting from Ulrich bundles. By doing so, we will also construct a nice component of the moduli space $MI(c_2)$ of μ-stable instanton bundles on X with fixed c_2.

Theorem 4.15. *For each non-negative $k_1, k_2, k_3 \in \mathbb{Z}$ with $k = k_1 + k_2 + k_3 \geq 2$ there exists a μ-stable instanton bundle E with $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$ on X such that*

$$\mathrm{Ext}^1_X(E,E) = 4k-3, \qquad \mathrm{Ext}^2_X(E,E) = \mathrm{Ext}^3_X(E,E) = 0$$

and such that E is generically trivial on lines.

In particular, there exists inside $MI(k_1e_1 + k_2e_2 + k_3e_3)$ a generically smooth irreducible component of dimension $4k-3$.

Proof. We will divide the proof in two steps. In the first one we will construct a torsion free sheaf with increasing c_2. In the second step we deform it to a locally free sheaf.

Step 1: Defining a sheaf G with increased c_2.

Let us consider a charge k instanton bundle E on X with $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$. Suppose $E_{|l_i} = \mathcal{O}^2_{l_i}$, with l_i is a generic line of each family e_i and $\mathrm{Ext}^2(E,E) = \mathrm{Ext}^3(E,E) = 0$.

Let us consider the short exact sequence

$$0 \to G \to E \to \mathcal{O}_l \to 0. \tag{4.1.25}$$

G is a torsion free sheaf which is not locally free. Using the resolution of $\mathcal{O}_l$:

$$0 \to \mathcal{O}_X(-h_2-h_3) \to \mathcal{O}_X(-h_2) \oplus \mathcal{O}_X(-h_3) \to \mathcal{O}_X \to \mathcal{O}_l \to 0 \tag{4.1.26}$$

we obtain $c_1(\mathcal{O}_l) = 0$ and $c_2(\mathcal{O}_l) = -e_1$ so using the sequence (4.1.25) we have that $c_1(G) = 0$, $c_2(G) = (k_1+1)e_1 + k_2e_2 + k_3e_3$ and $c_3(G) = 0$.

Now, applying the functor $\mathrm{Hom}(E,-)$ to (4.1.25) we obtain $\mathrm{Ext}^2(E,G) = 0$. In fact we have $\mathrm{Ext}^2(E,E) = 0$ by hypothesis and $\mathrm{Ext}^1(E,\mathcal{O}_l) = 0$ by Serre's duality since $E_{|_l} = \mathcal{O}_l^2$. Now apply the contravariant functor $\mathrm{Hom}(-,G)$ to (4.1.25). We have the following sequence

$$\mathrm{Ext}^2(E,G) \to \mathrm{Ext}^2(G,G) \to \mathrm{Ext}^3(\mathcal{O}_l,G).$$

Now we show that $\mathrm{Ext}^3(\mathcal{O}_l,G) = 0$ in order to obtain $\mathrm{Ext}^2(G,G) = 0$. By Serre's duality we have $\mathrm{Ext}^3(\mathcal{O}_l,G) = \mathrm{Hom}(G,\mathcal{O}_l(-2h))$. Consider the spectral sequence

$$E_2^{p,q} = H^p(X,\mathcal{E}xt^q(A,B)) \Rightarrow \mathrm{Ext}^{p+q}(A,B)$$

with $A,B \in \mathrm{Coh}(X)$. Setting $A = G$ and $B = \mathcal{O}_l(-2h)$ we obtain

$$\mathrm{Hom}(G,\mathcal{O}_l(-2h)) = H^0(\mathcal{H}om(G,\mathcal{O}_l(-2h))).$$

Now applying the functor $\mathcal{H}om(-,\mathcal{O}_l(-2h))$ to the sequence (4.1.25), we obtain

$$\begin{aligned} 0 &\to \mathcal{H}om(\mathcal{O}_l,\mathcal{O}_l(-2h)) \to \mathcal{H}om(E,\mathcal{O}_l(-2h)) \to \\ &\to \mathcal{H}om(G,\mathcal{O}_l(-2h)) \to \mathcal{E}xt^1(\mathcal{O}_l,\mathcal{O}_l(-2h)) \to 0. \end{aligned} \tag{4.1.27}$$

Now $\mathcal{H}om(\mathcal{O}_l,\mathcal{O}_l(-2h)) \cong \mathcal{O}_l(-2h)$, $\mathcal{H}om(E,\mathcal{O}_l(-2h)) \cong \mathcal{H}om(\mathcal{O}_X,\mathcal{O}_l) \otimes E_{|_l}^\vee(-2h) \cong \mathcal{O}_l^2(-2h)$ and $\mathcal{E}xt^1(\mathcal{O}_l,\mathcal{O}_l(-2h)) \cong N_l(-2h) = \mathcal{O}_l^2(-2h)$. If we split (4.1.27) in two short exact sequences we obtain

$$0 \to \mathcal{O}_l(-2h) \to \mathcal{H}om(G,\mathcal{O}_l(-2h)) \to \mathcal{O}_l^2(-2h) \to 0.$$

We deduce $\mathcal{H}om(G,\mathcal{O}_l(-2h)) \cong \mathcal{O}_l^3(-2h)$, thus

$$H^0(\mathcal{H}om(G,\mathcal{O}_l(-2h))) \cong H^0(\mathcal{O}_l^3(-2h)) = 0.$$

Finally we obtain $\mathrm{Ext}^3(\mathcal{O}_l,G) \cong \mathrm{Hom}(G,\mathcal{O}_l(-2h)) = 0$ from which it follows $\mathrm{Ext}^2(G,G) = 0$. This implies that $M_X(2,0,c_2(G))$ is smooth in the point correspondent to G. Now we show that $\mathrm{Ext}^3(G,G) = 0$. Applying the contravariant functor $\mathrm{Hom}(-,G)$ to (4.1.25) we get a surjection

$$\mathrm{Ext}^3(E,G) \to \mathrm{Ext}^3(G,G) \to 0.$$

If we apply $\mathrm{Hom}(E,-)$ to (4.1.25) we obtain $\mathrm{Ext}^3(E,G) = \mathrm{Ext}^2(E,\mathcal{O}_l)$ which vanishes since $E_{|_l} = \mathcal{O}_l^2$. Thus $\mathrm{Ext}^3(G,G) = 0$ and in particular we have that the dimension of the component of $M_X(2,0,c_2(G))$ containing G has dimension equal to $\dim \mathrm{Ext}^1(G,G) = 1-\chi(G,G)$. Applying $\mathrm{Hom}(G,-)$, $\mathrm{Hom}(-,E)$ and $\mathrm{Hom}(-,\mathcal{O}_l)$ to (4.1.25) we obtain

$$\chi(G,G) = \chi(E,E) - \chi(E,\mathcal{O}_l) - \chi(\mathcal{O}_l,E) + \chi(\mathcal{O}_l,\mathcal{O}_l).$$

By inductive hypothesis $\chi(E,E) = 4-4k$. We compute the remaining terms in the equation. Applying $\mathrm{Hom}(-,E)$, $\mathrm{Hom}(E,-)$ and $\mathrm{Hom}(\mathcal{O}_l,-)$ to (4.1.26), a Riemann-Roch computation yields $\chi(E,\mathcal{O}_l) = \chi(\mathcal{O}_l,E) = 2$ and $\chi(\mathcal{O}_l,\mathcal{O}_l) = 0$, thus

$$\dim \mathrm{Ext}^1(G,G) = 1-\chi(G,G) = 4k+1.$$

Furthermore tensor (4.1.25) by $\mathcal{O}_{m_i}(-h)$ where m_i is a generic line from the family e_i. Since m_i and l are disjoint for each i, tensoring by $\mathcal{O}_{m_i}(-h)$ leaves the sequence exact. Using the fact that $E_{|_{m_i}} = \mathcal{O}_{m_i}^{\oplus 2}$, we obtain $G_{|_{m_i}} = \mathcal{O}_{m_i}^{\oplus 2}$ and in particular $H^0(G \otimes \mathcal{O}_{m_i}(-h)) = 0$ for each i.

Step 2: Deforming G to a locally free sheaf F.

Now we take a deformation of G in $M_X(2,0,c_2(G))$ and let us call it F. For semicontinuity F satisfies

$$H^0(X,F\otimes \mathcal{O}_l(-h)) = 0 \quad \text{and} \quad H^1(X,F(-h)) = 0$$

Our goal is to show that F is locally free. Let us take E' and l' two deformations in a neighborhood of E and l respectively. The strategy is to show that if F is not locally free, then he would fit into a sequence

$$0 \to F \to E' \to \mathcal{O}_{l'} \to 0.$$

But such F's are parameterized by a family of dimension $4k$: indeed we have a $(4k-3)$-dimensional family for the choice of E', 2 for the choice of a line in the first family and we have 1 for $\mathbb{P}^1 = \mathbb{P}(H^0(l', E_{|_{l'}}))$, since $E_{l'} \cong \mathcal{O}_{l'}^2$. But we showed that G, and hence F, moves over a $(4k+1)$-dimensional component in $M_X(2,0,c_2(G))$, so F must be locally free.

Given such F let us consider the natural short exact sequence

$$0 \to F \to F^{\vee\vee} \to T \to 0. \tag{4.1.28}$$

Let us denote by Y the support of T. Since we supposed F not locally free, we have that $Y \neq \emptyset$. Furthermore T is supported in codimension at least two. We say that Y has pure dimension one.

In fact twisting (4.1.28) by $\mathcal{O}_X(-h)$ we observe that if $H^0(X, F^{\vee\vee}(-h)) \neq 0$ then a nonzero global section of $F^{\vee\vee}$ will induce via pull-back a subsheaf K of F with $c_1(K) = h$, which is not possible since F is stable. So we have $H^0(X, F^{\vee\vee}(-h)) \cong H^1(X, F(-h)) = 0$ which implies $H^0(X, T(-h)) = 0$. In particular Y has no embedded points, i.e. is pure of dimension one. We want to show that Y is actually a line.

Let H be a general hyperplane section which does not intersect the points where $F^{\vee\vee}$ is not locally free. Tensor (4.1.25) by $\mathcal{O}_H$. Since H is general the sequence remains exact and $\mathcal{O}_{l\cap H}$ is supported at one point, which represent the point where G_H fails to be reflexive (in this case also locally free). F is a deformation of G and because of the choice of H, restricting (4.1.28) to H does not affect the exactness of the short exact sequence. Moreover T_H is supported on points where F_H is not reflexive. Since being reflexive is an open condition, by semicontinuity T_H is supported at most at one point. But Y cannot be empty and is purely one dimensional, thus $Y \cap H$ consists of one point and Y must be a line L. Furthermore by semicontinuity T is of generic rank one and we have $c_2(T)h = -1$ (see [53, Example 15.3.1]).

Now we prove that $F^{\vee\vee}$ is locally free. Twist (4.1.28) by $\mathcal{O}_X(th)$ with $t << 0$. Considering the long exact sequence induced in cohomology we have $h^1(X, T(t)) \leq h^2(X, F(t))$ because $h^1(X, F^{\vee\vee}) = 0$ by Serre's vanishing. Observe that $c = c_3(F^{\vee\vee})$ and $c_2(T)$ are invariant for twists.

By computing the Chern classes using (4.1.28) we have $c_3(T)=c$ and $c_3(T(th))=c-2thc_2(T)$. For $t<<0$ we have

$$h^1(T(th))=-\chi(T(th))=(t+1)hc_2(T)-\frac{c}{2}.$$

By semicontinuity we have $h^2(F(th))\leq h^2(G(th))$, but using (4.1.25) and Hirzebruch-Riemann-Roch formula we obtain $h^2(G(th))=h^1(\mathcal{O}_{l_1}(t))=-(t+1)$ for $t<<0$. Now we have

$$(t+1)hc_2(T)-\frac{c}{2}=h^1(T(th))\leq h^2(X,F(t))\leq -(t+1)$$

so that

$$hc_2(T)\geq -1+\frac{c}{2(t+1)}, \tag{4.1.29}$$

which holds for all $t<<0$. Now using (4.1.29) and substituting $hc_2(T)=-1$ we get $c\leq 0$. Since $F^{\vee\vee}$ is reflexive, $c\geq 0$ so we obtain $c_3(T)=c=0$.

Now it remains to show that $F^{\vee\vee}$ is a deformation of E. The first step is to show that L is a deformation of the line l. In order to do so we compute the class of L in $A^2(X)$, which is represented by $c_2(T)=a_1e_1+a_2e_2+a_3e_3$. Consider a divisor $D=\beta_1h_1+\beta_2h_2+\beta_3h_3$, by (4.1.2) and $c=0$ we have

$$h^1(L,T(D))=(D+2)c_2(T).$$

Suppose $\beta_i<<0$ for all i. Then

$$a_1(\beta_1+1)+a_2(\beta_2+1)+a_3(\beta_3+1)=h^1(L,T(D))=h^2(X,F(D))\leq h^2(X,G(D)) \tag{4.1.30}$$

where the last inequality is by semicontinuity. Furthermore $\beta_i<<0$ implies that $h^1(X,E(D))=h^2(X,E(D))=0$ and thus

$$h^2(X,G(D))=h^1(l,\mathcal{O}_l(D))=-1-\beta_1. \tag{4.1.31}$$

We showed that $a_1+a_2+a_3=c_2(T)h=-1$ and combining this with (4.1.30) and (4.1.31) we obtain

$$a_2(\beta_2-\beta_1)+a_3(\beta_3-\beta_1)\leq 0$$

for all $\beta_i << 0$, thus we must have $a_2 = a_3 = 0$ and $a_1 = -1$, i.e. L lives in a neighborhood of l. Since $c = 0$ we have that $F^{\vee\vee}$ is locally free and we computed $c_2(T) = -e_1$, so we get $c_2(F^{\vee\vee}) = k_1e_1 + k_2e_2 + k_3e_3$, which implies that $F^{\vee\vee}$ has the same Chern classes as E. Therefore, $F^{\vee\vee}$ is a flat deformation of E and also semistable, so $F^{\vee\vee}$ lies in a neighborhood of E in $M_X(2,0,c_2(E))$. Observe that, by semicontinuity, F has trivial splitting type on the generic line of each family. To summarize, we showed that if F is not locally free it fits into a sequence

$$0 \to F \to E' \to \mathcal{O}_{l'} \to 0$$

with E' and l' flat deformation of E and l. But we observed that this is not possible, thus F must be locally free.

□

4.1.4 Existence via Serre's correspondence

In this section we will construct instantons of each possible $c_2(E)$ using an alternative strategy. Suppose $c_2(E) = k_1e_1 + k_2e_2 + k_3e_3$ with $k_1 \geq k_2 \geq k_3$, otherwise just apply a permutation on the $e_i's$ in the following argument.

We start by describing the Hilbert scheme of conics on X. Recall that if C is a conic, then it is either in the class $|e_1 + e_2|$, $|e_2 + e_3|$ or $|e_1 + e_3|$.

Proposition 4.16. *The Hilbert scheme $Hilb_{2t+1}(X)$ has exactly three disjoint components. Each of them is the locus of points representing one and the same class inside $A^2(X)$, is smooth, unirational and has dimension 4.*

Proof. We will follow the same idea as in [29, Proposition 4.2]. We want to prove that the locus $\mathcal{H}_{c_2} \subseteq \mathrm{Hilb}_{2t+1}(X)$ of points representing curves whose class in $A^2(X)$ is c_2, is irreducible.

Let us consider $c_2 = e_1 + e_2$ the other cases being similar. To give a morphism $\alpha : \mathbb{P}^1 \to X$ such that the class $\deg(\alpha)\mathrm{im}(\alpha)$ in $A^2(X)$ is c_2 is the same as to give two pairs of linearly independent sections in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)), H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, thus a general element of

$$F = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus^2} \times H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus^2}.$$

For the general choice of the general element the map α is an isomorphism onto its image. Let $F_0 \subseteq F$ be the open and non-empty locus of points satisfying such a condition. We have a natural family $\mathcal{F}_0 \subseteq F_0 \times X$ whose fibres are smooth conics, hence the family is flat. The universal property of the Hilbert scheme yields the existence of a unique morphism $F_0 \to \mathrm{Hilb}_{2t+1}(X)$ whose image is $\bar{\mathcal{H}}_{c_2}$ which is the locus of non-necessarily skew curves whose class is c_2. Thus $\bar{\mathcal{H}}_{c_2}$ is irreducible and since F_0 is a rational variety, it follows that $\bar{\mathcal{H}}_{c_2}$ is also unirational. In particular $\mathcal{H}_{c_2}$ is open inside $\bar{\mathcal{H}}_{c_2}$ because it trivially coincides with $\bar{\mathcal{H}}_{c_2} \cap \mathrm{Hilb}_{2t+1}(X)$.

Now we prove that $\mathcal{H}_{c_2}$ is smooth of dimension 4. Let us consider a point in $\mathcal{H}_{c_2}$ corresponding to a smooth, connected conic C and we compute $h^0(X, \mathcal{N}_{C/X})$ and $h^1(X, \mathcal{N}_{C/X})$. Since C is rational we know that $\mathcal{N}_{C/X} = \mathcal{O}_{\mathbb{P}^1}(a) + \mathcal{O}_{\mathbb{P}^1}(b)$ for some integer a and b.

By adjunction we have $\det(\mathcal{N}_{C/X}) \cong \mathcal{O}_{\mathbb{P}^1}(2)$, thus $a+b=2$. Recall that there is a surjection $\Omega_X^\vee \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{N}_{C/X}$. Since $\Omega_X \cong \bigoplus_{i=1}^3 \mathcal{O}_X(-2h_i)$, it follows that $\mathcal{N}_{C/X}$ is globally generated, thus $a, b \geq 0$. We conclude that $h^0(X, \mathcal{N}_{C/X}) = 4$ and $h^1(X, \mathcal{N}_{C/X}) = 0$ so that $\mathcal{H}_{c_2}$ is globally smooth of dimension 4 and we also conclude that the components of $\mathrm{Hilb}_{2t+1}(X)$ are necessarily disjoint. □

Now let us consider $L_1, L_2, \dots, L_\alpha$ disjoint lines from the family e_1, $M_1, M_2, \dots, M_\beta$ disjoint lines from the family e_3 and $C_1, C_2, \dots, C_\gamma$ disjoint conics in the class $e_2 + e_3$. First of all observe that we can choose such curves so that they are all pairwise disjoint. Let us denote by Y the one-dimensional scheme

$$Y = \bigcup_{i=1}^{\alpha} L_i \cup \bigcup_{j=1}^{\beta} M_j \cup \bigcup_{k=1}^{\gamma} C_k \tag{4.1.32}$$

We claim that that $\det \mathcal{N}_{Y/X} \cong \mathcal{O}_X(2h_2) \otimes \mathcal{O}_Y$. We can verify such an isomorphism component by component. Let us consider the sequence defining a line in e_1.

$$0 \to \mathcal{O}_X(-h_2 - h_3) \to \mathcal{O}_X(-h_2) \oplus \mathcal{O}_X(-h_3) \to \mathcal{O}_X \to \mathcal{O}_{L_i} \to 0$$

and split it into two short exact sequences

$$0 \to \mathcal{O}_X(-h_2 - h_3) \to \mathcal{O}_X(-h_2) \oplus \mathcal{O}_X(-h_3) \to \mathcal{I}_{L_i} \to 0 \tag{4.1.33}$$
$$0 \to \mathcal{I}_{L_i} \to \mathcal{O}_X \to \mathcal{O}_{L_i} \to 0.$$

Now recall that by adjunction formula we have $\mathcal{N}^\vee_{L_i/X} \cong \mathcal{I}_{L_i} \otimes \mathcal{O}_{L_i}$. Now using (4.1.35) we obtain $\mathcal{N}^\vee_{L_i/X} \cong \mathcal{O}^2_{L_i}$. In particular

$$\det \mathcal{N}_{L_i/X} \otimes \mathcal{O}_{L_i} \cong \mathcal{O}_X(D) \otimes \mathcal{O}_{L_1}.$$

with $D = ah_2 + bh_3$ a divisor on X. Repeating the same reasoning on every component of Y we obtain that $\det \mathcal{N}_{Y/X} \cong \mathcal{O}_X(2h_2) \otimes \mathcal{O}_Y$, i.e. the determinant of the normal bundle of Y is extendable on X. Since $h^2(X, \mathcal{O}_X(-2h_2)) = 0$, it follows that there exists a vector bundle F on X with a section s vanishing exactly along Y and with $c_1(F) = 2h_2$ and $c_2(F) = Y$. Thus $E = F(-h_2)$ has $c_1(E) = 0$ and $c_2(E) = c_2(F) = Y$ and it fits into

$$0 \to \mathcal{O}_X(-h_2) \to E \to \mathcal{I}_{Y|X}(h_2) \to 0. \qquad (4.1.34)$$

Now we prove the following

Proposition 4.17. *Let E be a vector bundle with $c_1(E) = 0$, $c_2(E) = Y$ with Y as in* (4.1.32) *and $\alpha + \beta + 2\gamma \geq 2$. Suppose $E(h_2)$ has a section vanishing along Y, i.e. E fits into*

$$0 \to \mathcal{O}_X(-h_2) \to E \to \mathcal{I}_{Y|X}(h_2) \to 0$$

then E is a μ-stable instanton bundle with charge $k = \alpha + \beta + 2\gamma$ such that

$$\dim \operatorname{Ext}^1_X(E,E) = 4k - 3, \qquad \operatorname{Ext}^2_X(E,E) = \operatorname{Ext}^3_X(E,E) = 0.$$

Proof. By construction $c_1(E) = 0$ and $c_2(E) = \alpha e_1 + \gamma e_2 + (\beta + \gamma) e_3$.

Since $k = \alpha + \beta + 2\gamma \geq 2$, then Y contains at least two disjoint components, we have $h^0(\mathcal{I}_{Y|X}(h_2)) = 0$. Taking the cohomology of (4.1.34) we obtain that $h^0(E) = h^0(\mathcal{I}_{Y|X}(h_2)) = 0$. Tensoring (4.1.34) by $\mathcal{O}_X(-h)$ we have $h^1(E(-h)) = h^1(\mathcal{I}_{Y|X}(-h_1 - h_3))$. Now consider

$$0 \to \mathcal{I}_{Y|X} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0. \qquad (4.1.35)$$

Taking the cohomology of the above sequence tensored by $\mathcal{O}_X(-h_1 - h_3)$, we have $h^1(\mathcal{I}_{Y|X}(-h_1 - h_3)) = h^0(\mathcal{O}_Y(-h_1 - h_3)) = 0$ because each connected component Z of Y is isomorphic to $\mathbb{P}^1$ and $(-h_1 - h_3)Z = -1$.

Now we prove the μ-stability of E. In order to do so we will use the Hoppe's criterion. Let us take a divisor D such that $Dh \geq 0$, thus D must be of the form $D = d_1h_1 + d_2h_2 + d_3h_3$, with $\sum_{i=1}^{3} d_i \geq 0$. Now let us consider the short exact sequence

$$0 \to \mathcal{O}_X(-D-h_2) \to E(-D) \to \mathcal{I}_{Y|X}(-D+h_2) \to 0.$$

By Proposition 4.7, E is μ-stable if and only if $h^0(E(-D)) = 0$. Using (4.1.35) it is clear that $h^0(E(-D)) = 0$ when $d_1 > 0$, $d_3 > 0$ or $d_2 > 1$. So we only have four cases left:

- $D = 0$.
- $D = h_2$.
- $D = -h_1 + h_2$.
- $D = h_2 - h_3$.

In all the cases we have $h^0(E(-D)) = h^0(\mathcal{I}_{Y|X}(-D+h_2)) = 0$ because Y contains at least two disjoint connected components, thus E is μ-stable.

Now we focus on the Ext groups. Since E is μ-stable, it is simple. Hence we have $\mathrm{Ext}^3(E,E) = 0$. Now we show $\mathrm{Ext}^2(E,E) = 0$. Take the short exact sequence (4.1.34) and tensor it by $E^\vee \cong E$. Now taking cohomology we have

$$H^2(X, E(-h_2)) \to \mathrm{Ext}^2_X(E,E) \to H^2(X, E \otimes \mathcal{I}_{Y|X}(h_2)).$$

We show that both $H^2(X,E(-h_2))$ and $H^2(X,\mathcal{I}_{Y|X}(h_2))$ are zeros. Take the cohomology of the short exact sequence (4.1.34) tensorized by $\mathcal{O}_X(-h_2)$. We obtain $H^2(X,E(-h_2)) \cong H^2(X,\mathcal{I}_{Y|X}) \cong H^1(Y,\mathcal{O}_Y) \cong 0$ because Y is the disjoint union of smooth rational curves. It remains to show that $H^2(X,E\otimes\mathcal{I}_{Y|X}(h_2)) \cong 0$. In order to do so let us take the short exact sequence (4.1.34) and tensorize it by $\mathcal{O}_X(h_2)$. Taking cohomology we obtain $h^2(X,E(h_2)) = h^2(X,\mathcal{I}_{Y|X}(2h_2))$. Now if we tensorize (4.1.35) by $\mathcal{O}_X(2h_2)$ and we take cohomology we have $h^2(X,\mathcal{I}_{Y|X}(2h_2)) = h^1(Y,\mathcal{O}_X(2h_2)\otimes\mathcal{O}_Y) = 0$ since $\mathcal{O}_X(2h_2)$ restricts to each component of Y to a degree two line bundle. Thus we have $h^2(X,E(h_2)) = 0$. Now if we take the cohomology of (4.1.35) tensorized by $E(h_2)$ we have

$$h^2(X,E(h_2)\otimes\mathcal{I}_{Y|X}) \leq h^1(Y,E(h_2)\otimes\mathcal{O}_Y).$$

But now using the fact that $E \otimes \mathcal{O}_Y \cong \mathcal{N}_{Y/X}^\vee$ we have $h^1(Y, E(h_2) \otimes \mathcal{O}_Y) = 0$ and thus $h^2(X, E(h_2) \otimes \mathcal{I}) = 0$. Finally we obtain $\mathrm{Ext}^2_X(E,E) = 0$ and the assertion on the dimension of $\mathrm{Ext}^1_X(E,E)$ follows from Riemann-Roch, since E is simple and $\chi(E \otimes E^\vee) = 4 - c_2(E)h$. □

Proposition 4.18. *The vector bundles E constructed in Proposition 4.17 are generically trivial on lines.*

Proof. Following the same reasoning as in Section 4.3, we have that E is generically trivial on lines if the generic line from each family intersect the zero locus Y of a section $s \in H^0(X, E(h_2))$ in only one point. By our choice of Y it is clear that we can choose a line in each family which intersect Y in only one point, but this is an open condition, so it will also hold for the generic line.

□

4.1.5 Jumping lines

In this section we describe the locus of jumping lines inside the Hilbert scheme of lines in X. Let us recall the definition of a jumping line:

Definition 4.19. Let E be a rank two vector bundle on X with $c_1(E) = 0$. A jumping line for E is a line L such that $H^0(E_L(-r)) = 0$ for some $r > 0$. The largest such integer is called the order of the jumping line L.

Let us consider a line in the first family $e_1 = h_2h_3$. Then we have the following resolution

$$0 \to \mathcal{O}_X(-h_2 - h_3) \to \mathcal{O}_X(-h_2) \oplus \mathcal{O}_X(-h_3) \to \mathcal{O}_X \to \mathcal{O}_L \to 0. \tag{4.1.36}$$

Let $\mathcal{H}$ be the Hilbert scheme of lines of the family h_2h_3. In particular we have $\mathcal{H} = \mathbb{P}^1 \times \mathbb{P}^1$, and we will denote by l and m the generators of $\mathrm{Pic}(\mathcal{H})$. Writing the sequence (4.1.36) with respect to global sections of $\mathcal{O}_X(-h_2) \oplus \mathcal{O}_X(-h_3)$ we get the description of the universal line $\mathcal{L} \subset X \times \mathcal{H}$

$$0 \to \mathcal{O}_X(-h_2-h_3) \boxtimes \mathcal{O}_\mathcal{H}(-1,-1) \to \begin{array}{c} \mathcal{O}_X(-h_2) \boxtimes \mathcal{O}_\mathcal{H}(-1,0) \\ \oplus \\ \mathcal{O}_X(-h_3) \boxtimes \mathcal{O}_\mathcal{H}(0,-1) \end{array} \to \mathcal{O}_{X \times \mathcal{H}} \to \mathcal{O}_\mathcal{L} \to 0. \tag{4.1.37}$$

Let us denote by $\mathcal{D}^1_E$ the locus of jumping lines (from the first family) of an instanton bundle E, and by i its embedding in $\mathcal{H}$. Let us consider the following diagram

$$\begin{array}{ccc} & \mathcal{L} \subset X \times \mathcal{H} & \\ {}^{q}\swarrow & & \searrow^{p} \\ X & & \mathcal{H} \end{array} \tag{4.1.38}$$

where q and p are the projection to the first and second factor respectively.

Lemma 4.20. *$\mathcal{D}^1_E$ is the support of the sheaf $R^1p_*(q^*(E(-h_1)) \boxtimes \mathcal{O}_{\mathcal{L}})$.*

Proof. See [84, p. 108] for a proof for $\mathbb{P}^n$. Since the argument is local, it can be generalized to our case. □

We recall two classical result that we need in order to describe the locus of jumping lines.

Theorem 4.21 (Grauert). *[57, Corollary 12.9] Let $f : X \to Y$ be a projective morphism of noetherian schemes with Y integral, and let F be a coherent sheaf on X, flat over Y. If for some i the function $h^i(Y,F)$ is constant on Y, then $R^if_*(F)$ is locally free on Y, and for every y the natural map*

$$R^if_*(F) \otimes k(y) \to H^i(X_y, F_y) \tag{4.1.39}$$

is an isomorphism.

Theorem 4.22. *[57, Theorem 5.3, Appendix A] Let $f : X \to Y$ be a smooth projective morphism of nonsingular quasi projective varieties. Then for any $x \in K(X)$ we have*

$$ch(f_!(x)) = f_*(ch(x).td(T_f)) \tag{4.1.40}$$

in $A(Y) \otimes \mathbb{Q}$, where T_f is the relative tangent sheaf of f.

Now we are ready to state the following

Proposition 4.23. *Let E be a generic instanton on X with $c_2 = k_1e_1 + k_2e_2 + k_3e_3$. Then $\mathcal{D}^1_E$ is a divisor given by $\mathcal{D}^1_E = k_3l + k_2m$ equipped with a sheaf G fitting into*

$$0 \to \mathcal{O}_{\mathcal{H}}^{k_3}(-1,0) \oplus \mathcal{O}_{\mathcal{H}}^{k_2}(0,-1) \to \mathcal{O}_{\mathcal{H}}^{k_2+k_3} \to i_*G \to 0. \tag{4.1.41}$$

Proof. By Lemma 4.20 a line L is jumping for E if and only if the point of $\mathcal{H}$ corresponding to L lies in the support of $R^1 p_*(q^*(E(-h_1)) \boxtimes \mathcal{O}_{\mathcal{L}})$.

Let us consider the Fourier-Mukai functor

$$\Phi_{\mathcal{L}} : D^b(X) \to D^b(\mathcal{H})$$

with kernel the structure sheaf of $\mathcal{L}$. We need to compute the transform of the bundles appearing in the monad (4.1.11) tensorized by $\mathcal{O}_X(-h_1)$.

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-2h_1 - h_2))$.

  By (4.1.37) tensored by $\mathcal{O}_X(-2h_1 - h_2) \boxtimes \mathcal{O}_{\mathcal{H}}$, since the only non zero cohomology on X is $h^2(\mathcal{O}_X(-2h_1 - 2h_2)) = 1$ we get $R^i p_*(q^*(\mathcal{O}_X(-2h_1 - h_2)) \boxtimes \mathcal{O}_{\mathcal{L}}) = 0$ for $i \neq 1$. Using the projection formula we obtain

  $$R^1 p_*(q^*(\mathcal{O}_X(-2h_1 - h_2)) \boxtimes \mathcal{O}_{\mathcal{L}}) \cong R^2 p_*(q^*(\mathcal{O}_X(-2h_1 - 2h_2))) \boxtimes \mathcal{O}_{\mathcal{H}}(-1,0).$$

  Observe that by Theorem 4.21 we have that $R^2 p_*(q^*(\mathcal{O}_X(-2h_1 - 2h_2)))$ is a rank one vector bundle on $\mathcal{H}$. Using (4.1.40) it follows trivially that

  $$c_1(R^2 p_* q^*(\mathcal{O}_X(-2h_1 - 2h_2))) = 0.$$

  In fact consider the diagram (4.1.38). Since X is a threefold and $\mathcal{H}$ is a surface, we have that after being pulled-back on $X \times \mathcal{H}$ and push-forwarded to $\mathcal{H}$ all the cycles on X became either zero or points. So we obtain

  $$R^1 p_*(q^*(\mathcal{O}_X(-2h_1 - h_2)) \boxtimes \mathcal{O}_{\mathcal{L}}) \cong \mathcal{O}_{\mathcal{H}}(-1,0).$$

  We continue with the other terms of the monad (4.1.11). The computations are completely analogous.

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-2h_1 - h_3))$.

  By (4.1.37) tensored by $\mathcal{O}_X(-2h_1 - h_3) \boxtimes \mathcal{O}_{\mathcal{H}}$, since the only non zero cohomology on X is $h^2(\mathcal{O}_X(-2h_1 - 2h_3)) = 1$ we get $R^i p_*(q^*(\mathcal{O}_X(-2h_1 - h_3)) \boxtimes \mathcal{O}_{\mathcal{L}}) = 0$ for $i \neq 1$ and

  $$R^1 p_*(q^*(\mathcal{O}_X(-2h_1 - h_3)) \boxtimes \mathcal{O}_{\mathcal{L}}) \cong \mathcal{O}_{\mathcal{H}}(0,-1).$$

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-h_1-h_2-h_3))$.

  By (4.1.37) tensored by $\mathcal{O}_X(-h_1-h_2-h_3)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the cohomology on X is all zero we get $R^i p_*(q^*(\mathcal{O}_X(-h_1-h_2-h_3))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for all i.

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-h_1-h_2))$.

  By (4.1.37) tensored by $\mathcal{O}_X(-h_1-h_2)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the cohomology on X is all zero we get $R^i p_*(q^*(\mathcal{O}_X(-h_1-h_2))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for all i.

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-h_1-h_3))$.

  By (4.1.37) tensored by $\mathcal{O}_X(-h_1-h_3)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the cohomology on X is all zero we get $R^i p_*(q^*(\mathcal{O}_X(-h_1-h_3))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for all i.

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-2h_1))$.

  By (4.1.37) tensored by $\mathcal{O}_X(-2h_1)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the only non zero cohomology on X is $h^2(\mathcal{O}_X(-2h_1))=1$ we get $R^i p_*(q^*(\mathcal{O}_X(-2h_1))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for $i\neq 1$ and

  $$R^1 p_*(q^*(\mathcal{O}_X(-2h_1))\boxtimes\mathcal{O}_{\mathcal{L}})\cong\mathcal{O}_{\mathcal{H}}.$$

- $\Phi_{\mathcal{L}}(\mathcal{O}_X(-h_1))$.

  By (4.1.37) tensored by $\mathcal{O}_X(-h_1)\boxtimes\mathcal{O}_{\mathcal{H}}$, since the cohomology on X is all zero we get $R^i p_*(q^*(\mathcal{O}_X(-h_1))\boxtimes\mathcal{O}_{\mathcal{L}})=0$ for all i.

Now we apply the $\Phi_{\mathcal{L}}$ to the monad (4.1.11). First we apply $\Phi_{\mathcal{L}}$ to the sequence

$$\begin{array}{c} \mathcal{O}_X^{k_2+k_3}(-h_1) \\ \oplus \\ 0\to K\to \mathcal{O}_X^{k_1+k_3}(-h_2) \to \mathcal{O}_X^{k-2}\to 0 \\ \oplus \\ \mathcal{O}_X^{k_1+k_2}(-h_3) \end{array}$$

we get $R^i p_* q^*(K\otimes\mathcal{O}_X(-h_1))=0$ for $i\neq 1$ and

$$R^1 p_*(q^*(K\otimes\mathcal{O}_X(-h_1))\boxtimes\mathcal{O}_{\mathcal{L}})\cong\mathcal{O}_{\mathcal{H}}^{k_2+k_3}.$$

From

$$\begin{array}{c} \mathcal{O}_X^{k_3}(-h_1-h_2) \\ \oplus \\ 0 \to \mathcal{O}_X^{k_2}(-h_1-h_3) \to K \to E \to 0 \\ \oplus \\ \mathcal{O}_X^{k_1}(-h_2-h_3) \end{array}$$

we get

$$0 \to R^0 p_*(q^*(E \otimes \mathcal{O}_X(-h_1)) \boxtimes \mathcal{O}_{\mathcal{L}}) \to \mathcal{O}_{\mathcal{H}}^{k_3}(-1,0) \oplus \mathcal{O}_{\mathcal{H}}^{k_2}(0,-1) \xrightarrow{\gamma}$$
$$\mathcal{O}_{\mathcal{H}}^{k_2+k_3} \to R^1 p_*(q^*(E \otimes \mathcal{O}_X(-h_1)) \to 0.$$

so γ is a $(k_2+k_3)\times(k_2+k_3)$ matrix made by two blocks. The first one is a $(k_2+k_3)\times(k_3)$ linear matrix in the first variables of $\mathcal{H}$ and the second one a $(k_2+k_3)\times(k_2)$ linear matrix in the second variables of $\mathcal{H}$. We observe that Ker(γ) is zero since is a torsion free sheaf which is zero outside $\mathcal{D}_E^1$, and Coker$(\gamma) \cong R^1 p_*(q^*(E \otimes \mathcal{O}_X(-h_1)) \boxtimes \mathcal{O}_{\mathcal{L}})$ is an extension to $\mathcal{H}$ of a rank 1 sheaf on $\mathcal{D}_E^1$ denoted by G. That is a divisor $k_3 l + k_2 m$ given by the vanishing of the determinant of γ.

□

Remark 4.24. The Hilbert space of lines on X is made of three disjoint connected component, each of which is isomorphic to the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ (see [29, Proposition 4.1]). So we can repeat this exact same reasoning to the lines of the family e_2 and e_3, i.e. permuting the indices $(1,2,3)$, we can describe the locus $\mathcal{D}_E^i$ as a divisor of type $k_j l + k_h m$ with $i \neq j \neq h \neq i$.

In a completely analogous way, it is possible to use the monad (4.1.16) to study the locus of jumping lines, obtaining the same result.

4.2 Instanton bundles on the Flag variety $F(0,1,2)$

In this section we deal with the Flag variety $F(0,1,2)$, which is the other Fano threefold of degree 6. Let us call h_1 and h_2 the two generators of the Picard group. Recall that the Chow group in codimension two is generated by h_1^2 and h_2^2 with the relation $h_1^2 + h_2^2 = h_1 h_2$. In the first part of the section we summarize the known results about instanton bundles on $F(0,1,2)$ and we slightly generalize them taking into account the differences between our definition and the definition given in [78]. In particular in this work we will not assume any restriction on the second Chern class of the bundle. In this case it is possible to obtain a monadic description of any instanton bundle, similarly to the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 4.25. *Let E be an instanton bundle with charge k on F, i.e. $c_2(E) = k_1 h_1^2 + k_2 h_2^2$ with $k = k_1 + k_2$.*

- *Then, up to permutation, E is the cohomology of a monad*

$$0 \to \begin{matrix} \mathcal{O}_F(-1,0)^{\oplus k_1} \\ \oplus \\ \mathcal{O}_F(0,-1)^{\oplus k_2} \end{matrix} \xrightarrow{\alpha} \begin{matrix} G_1(-1,0)^{\oplus k_1} \\ \oplus \\ G_2(0,-1)^{\oplus k_2} \end{matrix} \xrightarrow{\beta} \mathcal{O}_F^{\oplus k-2} \to 0, \tag{4.2.1}$$

  where G_i is the pull-back of the twisted cotangent bundle $\Omega_{\mathbb{P}^2}(2)$ from the two natural projections $p_i : F \subset \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$.

  Reciprocally, the cohomology of such a monad defines a k-instanton.

- *Then, up to permutation, E is the cohomology of a monad*

$$0 \to \begin{matrix} \mathcal{O}_F(-1,0)^{\oplus k_1} \\ \oplus \\ \mathcal{O}_F(0,-1)^{\oplus k_2} \end{matrix} \xrightarrow{\alpha} \mathcal{O}_F^{\oplus 2k+2} \xrightarrow{\beta} \begin{matrix} \mathcal{O}_F(1,0)^{\oplus k_1} \\ \oplus \\ \mathcal{O}_F(0,1)^{\oplus k_2} \end{matrix} \to 0. \tag{4.2.2}$$

  Moreover, the monad obtained is self-dual, i.e. it is possible to find a non degenerate symplectic form $q : W \to W^*$, with W a $(2k+2)$-dimensional vector space describing the copies of the trivial bundle in the monad, such that $\beta = \alpha^\vee \circ (q \otimes id_{\mathcal{O}_F})$.

In [78] it has been proved the existence of stable instanton bundles of every second Chern class of the form kh_1h_2. In this thesis we generalize this result to any second Chern class $k_1h_1^2 + k_2h_2^2$. Let us consider $C_1, C_2, \dots, C_{k_1}$ disjoint conics which are represented by h_1h_2 in $A^2(F)$, and $L_1, L_2, \dots L_{k_2+1}$ lines represented by h_2^2 in $A^2(F)$. Let Y be the one dimensional subscheme of F given by

$$Y = \bigcup_{i=1}^{k_1} C_i \cup \bigcup_{j=1}^{k_2+1} L_j. \tag{4.2.3}$$

Proposition 4.26. *Let E be a vector bundle with $c_1(E) = 0$, $c_2(E) = Y + h_2^2$ with and $k_1 + k_2 \geq 2$. Suppose $E(0,1)$ has a section vanishing along Y, i.e. E fits into*

$$0 \to \mathcal{O}_F(0,-1) \to E \to \mathcal{I}_{Y|F}(0,1) \to 0$$

then E is a $\mu-stable$ instanton bundle with charge $k = k_1 + k_2$ such that

$$\dim \mathrm{Ext}^1_F(E,E) = 4k-3, \qquad \mathrm{Ext}^2_F(E,E) = \mathrm{Ext}^3_F(E,E) = 0.$$

As a consequence, we construct a nice component of the moduli space of instanton bundles.

Corollary 4.27. For each non-negative k_1 and k_2 such that $k_1 + k_2 \geq 2$ there exists an irreducible component

$$MI_F^0(k_1h_1^2 + k_2h_2^2) \subseteq MI_F(k_1h_1^2 + k_2h_2^2)$$

which is generically smooth of dimension $4(k_1 + k_2) - 3$.

4.2.1 The Flag variety $F(0,1,2)$

Let $F \subseteq \mathbb{P}^7$ be the del Pezzo threefold of degree 6. We can construct F as the general hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$. The projections π_i induce maps $p_i \colon F \to \mathbb{P}^2$ by restriction, $i = 1,2$ and such maps are isomorphic to the canonical map $\mathbb{P}(\Omega^1_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$. Thinking of the second copy of $\mathbb{P}^2$ as the dual of the first one, then F can also be viewed naturally as the flag variety of pairs point–line in $\mathbb{P}^2$. We denote by $A(F)$ the Chow ring of F. Let h_i, $i = 1,2$, be the respective classes of $p_i^*\mathcal{O}_{\mathbb{P}^2}(1)$ in $A^1(F)$. The class of the hyperplane divisor on F is $h = h_1 + h_2$.

The above discussion proves the isomorphisms

$$A(F) \cong A(\mathbb{P}^2)[h_1]/(h_1^2 - h_1h_2 + h_2^2) \cong \mathbb{Z}[h_1,h_2]/(h_1^2 - h_1h_2 + h_2^2, h_1^3, h_2^3).$$

In particular, $Pic(F) \cong \mathbb{Z}^{\oplus 2}$ with generators h_1 and h_2. We will denote the Chern polynomial of a given coherent sheaf E by

$$c_E(t) := 1 + (\alpha_1h_1 + \alpha_2h_2)t + (\beta_1h_1^2 + \beta_1h_2^2)t^2 + \gamma h_1^2h_2t^3,$$

where the coefficient of degree i is $c_i(E)$.
Recall that F contains two families of lines Λ_1,Λ_2, each isomorphic to $\mathbb{P}^2$. Their representatives in the Chow ring $A(F)$ are h_1^2, h_2^2. Notice that if we look at F as the projective bundle $\mathbb{P}(\Omega^1_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$, these families correspond to the fibers over points of $\mathbb{P}^2$. We have a geometrical description (using the notion of flag variety): given $p \in \mathbb{P}^2$, $\lambda_p := \{L \in \mathbb{P}^{2\vee} \mid p \in L\} \in \Lambda_1$. Analogously, given a line $L \subset \mathbb{P}^2$, $\lambda_L := \{x \in \mathbb{P}^2 \mid x \in L\} \in \Lambda_2$. Notice $\lambda_x \cap \lambda_y = \emptyset$ if $x \neq y$ (clear from cohomological product $h_1^2 h_2^2 = 0$) and $\lambda_x \cap \lambda_L = \emptyset$ (resp. $\{x, L\}$) if $x \in l$ (resp. $x \notin L$). If L_1 (resp. L_2) is a line from the family Λ_1 (resp. Λ_2) it holds that

$$\mathcal{O}_F(\alpha,\beta) \otimes \mathcal{O}_{L_1} \cong \mathcal{O}_{\mathbb{P}^1}(\beta) \quad (\text{resp.} \mathcal{O}_F(\alpha,\beta) \otimes \mathcal{O}_{L_2} \cong \mathcal{O}_{\mathbb{P}^1}(\alpha))$$

since $h_1^2(\alpha h_1 + \beta h_2) = \beta h_1^2 h_2$. The $\mathcal{O}_F$-resolutions of a line L_1 is:

$$0 \longrightarrow \mathcal{O}_F(-2,0) \longrightarrow \mathcal{O}_F(-1,0)^2 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_{L_1} \longrightarrow 0; \tag{4.2.4}$$

(or the analogous one for the second family of lines L_2).

In order to compute the $\mathcal{O}_F$-resolution of a point $p \in F$, we can consider its $\mathcal{O}_L$-resolution

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{O}_L \longrightarrow \mathcal{O}_p \longrightarrow 0$$

and use the mapping cone construction to conclude that

$$0 \longrightarrow \mathcal{O}_F(-3,-1) \longrightarrow \begin{array}{c} \mathcal{O}_F(-2,0) \\ \oplus \\ \mathcal{O}_F(-2,-1)^2 \end{array} \longrightarrow \begin{array}{c} \mathcal{O}_F(-1,0)^2 \\ \oplus \\ \mathcal{O}_F(-1,-1) \end{array} \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_p \longrightarrow 0. \tag{4.2.5}$$

The flag variety F also contains a family of conics C (see the next subsection) whose O_F-resolution is:

$$0 \longrightarrow \mathcal{O}_F(-1,-1) \longrightarrow \begin{array}{c}\mathcal{O}_F(-1,0)\\ \oplus \\ \mathcal{O}_F(0,-1)\end{array} \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_C \longrightarrow 0. \qquad (4.2.6)$$

We have to distinguish two different cases (see Remark 4.30). In the case of a smooth conic $C \cong \mathbb{P}^1$, it holds:

$$\mathcal{O}_F(\alpha,\beta) \otimes \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1}(\alpha+\beta)$$

which will be denoted either by $\mathcal{O}_C(\alpha,\beta)$ or $\mathcal{O}_C(\alpha+\beta)$, to remember we are restricting at a conic.
In the case of a reducible conic $C = L_1 \cup L_2$, we will always use the notation $\mathcal{O}_C(\alpha,\beta)$ to keep track of the degree on each one of the lines.

We will now recall how to compute the cohomology of the line bundles on F (see [30] Proposition 2.5):

Proposition 4.28. *For each $\alpha_1, \alpha_2 \in \mathbb{Z}$ with $\alpha_1 \leq \alpha_2$, we have*

$$h^i\big(F, \mathcal{O}_F(\alpha_1,\alpha_2)\big) \neq 0$$

if and only if

- $i = 0$ *and* $\alpha_1 \geq 0$*;*
- $i = 1$ *and* $\alpha_1 \leq -2$*,* $\alpha_1 + \alpha_2 + 1 \geq 0$*;*
- $i = 2$ *and* $\alpha_2 \geq 0$*,* $\alpha_1 + \alpha_2 + 3 \leq 0$*;*
- $i = 3$ *and* $\alpha_2 \leq -2$*.*

In all these cases

$$h^i\big(F, \mathcal{O}_F(\alpha_1,\alpha_2)\big) = (-1)^i \frac{(\alpha_1+1)(\alpha_2+1)(\alpha_1+\alpha_2+2)}{2}.$$

It could be thought that the study of the geometry of lines of the F will be enough to define and understand instanton bundles, as it turned out to be in the case of instanton bundles on $\mathbb{P}^3$. Nevertheless, in the case of the flag variety, the main kind of rational curve we are interested in is the conic. In fact, through the Ward correspondence, instanton bundles on F have trivial splitting on "real" conics (this is explained in [20] and [45] without explicitly mentioning the degree) and therefore, by semicontinuity, on the general element of $\mathscr{C} := Hilb^{2t+1}(F)$. Therefore, we devote this subsection to study the main properties of the conics on F.

Lemma 4.29. *[78, Lemma 1.5] The Hilbert scheme of rational curves of degree two $\mathscr{C} := Hilb^{2t+1}(F)$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. The open set $\mathbb{P}^2 \times \mathbb{P}^2 \backslash F$ corresponds to smooth conics. Moreover, the canonical map $p : \mathcal{C} \to F$ from the universal conic $\mathcal{C}$ to F endows $\mathcal{C}$ with the structure of a quadric bundle of relative dimension 2 over F.*

Remark 4.30. Indeed, it is known (see for instance [74, Lemma 2.1.1]) that any subscheme of F with Hilbert polynomial $2t+1$ will be a smooth conic, a pair of distinct lines intersecting on a point, or a line with a double structure. In order to see that there is no such non-reduced subscheme on F we should observe that for any line L on F, we have $\mathcal{N}_{L|F} \cong \mathcal{O}_L^2$. Therefore there is no surjective map $\mathcal{N}_{L|F}^\vee \to \mathcal{O}_L(-1)$ and we conclude again by [74, Lemma 2.1.1].

4.2.2 Definition of instantons, properties and monads

Similarly to the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we will give the following definition

Definition 4.31. For any integer $k \geq 1$ we will call an *instanton bundle* with charge k (or, for short, a k-instanton) a rank two μ-semistable bundle E on F with $H^0(E) = 0$, $c_1(E) = (0,0)$, $c_2(E) = k_1h_1^2 + k_2h_2^2$ with $k_1 + k_2 = k$ and $H^1(E(-1,-1)) = 0$.

Observe that in this case the definition is slightly different from the one found in [78] where it is required that the second Chern class $c_2(E)$ is concentrated in the term h_1h_2 (i.e. $k_1 = k_2$). Using Beilinson's spectral sequences techniques, in [78] obtained a monadic description of instanton bundles over $F(0,1,2)$. Adapting their results to our definition it is possible to state the following theorems:

Theorem 4.32. *[78, Theorem 4.1] Let E be an instanton bundle with charge k on F. Then, up to permutation, E is the cohomology of a monad*

$$0 \to \mathcal{O}_F(-1,0)^{\oplus k_1} \oplus \mathcal{O}_F(0,-1)^{\oplus k_2} \xrightarrow{\alpha} G_1(-1,0)^{\oplus k_1} \oplus G_2(0,-1)^{\oplus k_2} \xrightarrow{\beta} \mathcal{O}_F^{\oplus k-2} \to 0, \tag{4.2.7}$$

where G_i is the pull-back of the twisted cotangent bundle $\Omega_{\mathbb{P}^2}(2)$ from the two natural projections $p_i : F \subset \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$.

Reciprocally, the cohomology of such a monad defines a k-instanton.

Theorem 4.33. *[78, Theorem 4.2] Let E be an instanton bundle with charge k on F. Then, up to permutation, E is the cohomology of a monad*

$$0 \to \mathcal{O}_F(-1,0)^{\oplus k_1} \oplus \mathcal{O}_F(0,-1)^{\oplus k_2} \xrightarrow{\alpha} \mathcal{O}_F^{\oplus 2k+2} \xrightarrow{\beta} \mathcal{O}_F(1,0)^{\oplus k_1} \oplus \mathcal{O}_F(0,1)^{\oplus k_2} \to 0. \tag{4.2.8}$$

Moreover, the monad obtained is self-dual, i.e. it is possible to find a non degenerate symplectic form $q : W \to W^*$, with W a $(2k+2)$-dimensional vector space describing the copies of the trivial bundle in the monad, such that $\beta = \alpha^\vee \circ (q \otimes id_{\mathcal{O}_F})$.

Reciprocally, any vector bundle with no global sections defined as the cohomology of such a monad is a k-instanton bundle.

Moreover in [78] the authors characterized strictly semistable instanton bundles with the second Chern class concentrated in the term h_1h_2.

Proposition 4.34. *[78, Proposition 2.5] Let E be an instanton bundle on F with $c_2(E) = kh_1h_2$. Then, it is also Gieseker semistable. Moreover, if E is not μ-stable, then $k = l^2$ for some $l \in \mathbb{Z}, l \neq 0$ and it can be constructed as an extension Λ_l of the form*

$$0 \to \mathcal{O}_F(l,-l) \to E \to \mathcal{O}_F(-l,l) \to 0. \tag{4.2.9}$$

The only common element of the two families of extensions Λ_l and Λ_{-l} is the decomposable bundle $\mathcal{O}_F(l,-l) \oplus \mathcal{O}_F(-l,l)$.

Through an induction process, they also constructed stable instanton bundles with $c_2(E) = kh_1h_2$ on the flag variety for each charge k. More concretely, they proved the following

Theorem 4.35. *[78, Theorem 5.1] Let $F \subset \mathbb{P}^7$ be the flag variety. The moduli space $MI_F^s(k)$ of stable instanton bundles with $c_2(E) = kh_1h_2$ is non empty and has a generically smooth irreducible component of dimension $8k-3$.*

We conclude this section by stating the behaviour of instanton bundles when restricted to conics. In [78] the authors defined the notion of jumping conic.

Definition 4.36. Let E be an instanton bundle on the flag variety F. A conic $C \subset F$ (irreducible or not) is a jumping conic of type (a,b) if it satisfies $H^1(E_{|C}(-1,0)) = a$ and $H^1(E_{|C}(0,-1)) = b$. C is said to have trivial splitting type when it has type $(0,0)$.

Suppose first that $C \subset F$ is an irreducible conic, $C \cong \mathbb{P}^1$. In that case, $\mathcal{O}_F(-1,0)_{|C} = \mathcal{O}_F(0,-1)_{|C} = \mathcal{O}_C(-1)$ and for an instanton bundle E we have $E_{|C} \cong \mathcal{O}_C(-a) \oplus \mathcal{O}_C(a)$ if and only if $H^1(E_{|C}(-1,0)) = H^1(E_{|C}(0,-1)) = a$ if and only if it is a jumping conic of type (a,a).

On the other hand, for a reducible conic $C = L_1 \cup L_2$ for lines L_i intersecting transversely on a single point. In this case, it is well-known that $Pic(C) \cong \mathbb{Z}^2$, where the isomorphism is given by $\mathcal{L} \to (deg_{L_1}(\mathcal{L}), deg_{L_2}(\mathcal{L}))$. Therefore, for an instanton E on F the restriction to C is of the form $E_C \cong \mathcal{O}_C(a,b) \oplus \mathcal{O}_C(-a,-b)$ if and only if it is a jumping conic of type (a,b).

Let us denote by $\mathcal{D}_E \subset \mathscr{C}$ the locus of jumping conics of an instanton E, and by i its embedding in H.

Proposition 4.37. *[78, Proposition 6.2] Let E be an instanton bundle with $c_2(E) = kh_1h_2$ on F. Then $\mathcal{D}_E$ is a divisor of type (k,k) equipped with a sheaf G fitting into*

$$0 \to \mathcal{O}_H(-1,-1)^{\oplus k} \oplus \mathcal{O}_H(-1,0)^{\oplus k} \to \mathcal{O}_H^{\oplus k} \oplus \mathcal{O}_H(-1,0)^{\oplus k} \to i_*G \to 0. \quad (4.2.10)$$

In the next section we will slightly generalize Theorem 4.35 by allowing every possible second Chern class.

4.2.3 Construction of instantons via Serre's correspondence

In this section we will construct instantons for each possible $c_2(E)$ and we will show that they are smooth points of an irreducible component of the moduli space of instanton bundles. Suppose $c_2(E) = k_1h_1^2 + k_2h_2^2$ with $k_1 \leq k_2$.

Let us consider $C_1, C_2, \ldots, C_{k_1}$ disjoint conics which are represented by h_1h_2 in $A^2(F)$, and $L_1, L_2, \ldots L_{k_2+1}$ lines represented by h_2^2 in $A^2(F)$. Let Y be the one dimensional subscheme of F given by

$$Y = \bigcup_{i=1}^{k_1} C_i \cup \bigcup_{j=1}^{k_2+1} L_j. \tag{4.2.11}$$

We claim that $\det \mathcal{N}_{Y/F} \cong \mathcal{O}_F(0,2) \otimes \mathcal{O}_Y$. Let us work component by component, similarly to the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. By adjunction formula $\mathcal{N}^{\vee}_{C_i/F} \cong \mathcal{I}_{C_i} \otimes \mathcal{O}_{C_i}$, so splitting (4.2.4) in two short exact sequences and tensorizing by $\mathcal{O}_{C_i}$ we get $\mathcal{N}_{C_i/F} \cong \mathcal{O}^2_{C_i}(-1)$. In particular we have $\det \mathcal{N}_{C_i/F} \cong \mathcal{O}_F(D) \otimes \mathcal{O}_{C_i}$ where D is a divisor of the form $ah_1 + bh_2$ with $a+b=2$. Doing the same for lines L_j we obtain $\mathcal{N}_{L_j/F} \cong \mathcal{O}^2_{L_j}$ and thus $\det \mathcal{N}_{L_j/F} \cong \mathcal{O}_F(D) \otimes \mathcal{O}_{L_j}$ where D is a divisor of the form ch_2. Combining these two results we obtain that $\det \mathcal{N}_{Y/F} \cong \mathcal{O}_F(0,2) \otimes \mathcal{O}_Y$, i.e the determinant of the normal bundle of Y is extendable on F. Since $h^2(F, \mathcal{O}_F(-2h_2)) = 0$, there exists a vector bundle G on F with a section s vanishing along Y with $c_1(G) = 2h2$ and $c_2(G) = Y$. Thus $E = G(-h_2)$ has $c_1(E) = 0$, $c_2(E) = c_2(F) + h_2^2$ and it fits into

$$0 \to \mathcal{O}_F(0,-1) \to E \to \mathcal{I}_{Y|F}(0,1) \to 0. \tag{4.2.12}$$

So we have the following Proposition

Proposition 4.38. *Let E be a vector bundle with $c_1(E) = 0$, $c_2(E) = Y + h_2^2$ with Y as in* (4.2.11) *and $k_1 + k_2 \geq 2$. Suppose $E(0,1)$ has a section vanishing along Y, i.e. E fits into*

$$0 \to \mathcal{O}_F(0,-1) \to E \to \mathcal{I}_{Y|F}(0,1) \to 0$$

then E is a $\mu - stable$ instanton bundle with charge $k = k_1 + k_2$ such that

$$\dim \operatorname{Ext}^1_F(E,E) = 4k - 3, \qquad \operatorname{Ext}^2_F(E,E) = \operatorname{Ext}^3_F(E,E) = 0.$$

Proof. The proof is similar to the proof of Proposition 4.17. By construction $c_1(E) = 0$ and $c_2(E) = k_1 h_1^2 + k_2 h_2^2$.

Taking the cohomology of (4.2.12), we obtain $h^0(E) = h^0(\mathcal{I}_{Y|F}(0,1)) = 0$ because Y contains at least three disjoint components. Tensoring (4.2.12) by $\mathcal{O}_F(-1,-1)$ we have $h^1(E(-1,-1)) = h^1(\mathcal{I}_{Y|F}(-1,0))$. Now considering the defining sequence of the ideal $\mathcal{I}_{Y|F}$ tensorized by $\mathcal{O}_F(-1,0)$ we obtain $h^1(\mathcal{I}_{Y|F}(-1,0)) = h^0(\mathcal{O}_Y(-1,0)) = 0$ because each connected component Z of Y is isomorphic to $\mathbb{P}^1$ and $-h_1 Z = -1$.

Now we prove the μ-stability of E. By Proposition 4.7 E is μ-stable if and only if $h^0(E(-D)) = 0$ for each divisor D such that $Dh^2 \geq 0$. Let us take such a divisor $D = d_1h_1 + d_2h_2$ with $d_1 + d_2 \geq 0$ and consider the short exact sequence

$$0 \to \mathcal{O}_F(-d_1, -d_2 - 1) \to E(-d_1, -d_2) \to \mathcal{I}_{Y|F}(-d_1, -d_2 + 1) \to 0.$$

Now

$$h^0(\mathcal{I}_{Y|F}(-d_1, -d_2 + 1)) \leq h^0(\mathcal{O}_F(-d_1, -d_2 + 1)).$$

So it is clear that $h^0(\mathcal{I}_{Y|F}(-d_1, -d_2 + 1)) = 0$ whenever $d_1 > 0$ or $d_2 > 1$. In these cases we have $h^0(E(-D)) = 0$. It remains to study the cases $D = h_2$ and $D = -h_1 + h_2$. In both cases $h^0(E(-d_1, -d_2)) = h^0(\mathcal{I}_{Y|F}(-d_1, -d_2 + 1)) = 0$ because Y contains at least three disjoint components, thus E is μ-stable.

Now we prove the part of the statement regarding the Ext groups. Since E is μ-stable, it is simple. Thus we have $\mathrm{Ext}^3_F(E, E) = 0$. Now we show that $\mathrm{Ext}^2(E, E) = 0$ and the assertion on the dimension of $\mathrm{Ext}^1_F(E, E)$ will follow from Riemann-Roch. Consider the short exact sequence (4.2.12) and tensor it by $E \cong E^\vee$. Taking cohomology we have

$$H^2(F, E(0, -1)) \to \mathrm{Ext}^2_F(E, E) \to H^2(F, E \otimes \mathcal{I}_{Y|F}(0, 1)).$$

Using the short exact sequence (4.2.12) we obtain $H^2(F, E(0, -1)) \cong H^2(F, \mathcal{I}_{Y|F}) \cong H^1(Y, \mathcal{O}_Y) \cong 0$ because Y is the disjoint union of smooth rational curves. So $\mathrm{Ext}^2_F(E, E) = 0$ as soon as $H^2(F, E \otimes \mathcal{I}_{Y|F}(0, 1))$ vanishes. In order to show this vanishing let us take the short exact sequence (4.2.12) and tensorize it by $\mathcal{O}_F(0, 1)$. Taking cohomology we obtain $h^2(F, E(0, 1)) = h^2(X, \mathcal{I}_{Y|X}(0, 2))$. Now if we tensorize

$$0 \to \mathcal{I}_{Y|F} \to \mathcal{O}_F \to \mathcal{O}_Y \to 0$$

by $\mathcal{O}_F(0, 2)$ and we take cohomology we have $h^2(F, \mathcal{I}_{Y|F}(0, 2)) = h^1(Y, \mathcal{O}_F(0, 2) \otimes \mathcal{O}_Y) = 0$ since $\mathcal{O}_F(0, 2)$ restricts to each component of Y to a degree two line bundle. Thus we have $h^2(F, E(h_2)) = 0$. Now if we take the cohomology of the defining sequence of $\mathcal{I}_{Y|F}$ tensorized by $E(h_2)$ we have

$$h^2(F, E(0, 1) \otimes \mathcal{I}_{Y|F}) \leq h^1(Y, E(0, 1) \otimes \mathcal{O}_Y).$$

But now using the fact that $E \otimes \mathcal{O}_Y \cong \mathcal{N}^\vee_{Y/F}$ we have $h^1(Y, E(0,1) \otimes \mathcal{O}_Y) = 0$ and thus $h^2(F, E(0,1)) \otimes \mathcal{I}_{Y|F}) = 0$. Finally we obtain $\mathrm{Ext}^2_X(E,E) = 0$. To compute the dimension of $\mathrm{Ext}^1_F(E,E)$ we use Riemann-Roch. Since $c_1(E \otimes E^\vee) = c_3(E \otimes E^\vee) = 0$ and $c_2(E \otimes E^\vee) = 4c_2(E)$ we have

$$\dim \mathrm{Ext}^1_F(E,E) = h^0(E \otimes E^\vee) + h^2(E \otimes E^\vee) - \chi(E \otimes E^\vee) = 4c_2(E)(h_1 + h_2) - 3,$$

thus $\dim \mathrm{Ext}^1_F(E,E) = 4k - 3$. □

As a consequence we are able to describe the component of the moduli space of instanton bundles containing the vector bundles constructed in Proposition 4.38.

Corollary 4.39. For each non-negative k_1 and k_2 such that $k_1 + k_2 \geq 2$ there exists an irreducible component

$$MI^0_F(k_1h_1^2 + k_2h_2^2) \subseteq MI_F(k_1h_1^2 + k_2h_2^2)$$

which is generically smooth of dimension $4(k_1 + k_2) - 3$ and containing all points corresponding to the bundles obtained in Proposition 4.38.

Proof. The schemes as in (4.2.11) represent points in a non-empty open subset $\mathcal{U} \subset \mathscr{C} \times \Lambda_2$. Since the latter product is a product of irreducible varieties, it is irreducible. It follows that $\mathcal{U}$ is irreducible as well.

Since the vector bundle E in Sequence (4.2.12) is uniquely determined by the scheme Y, we obtain in this way a flat family of bundles containing all the bundles obtained via Proposition 4.38 and parameterized by $\mathcal{U}$. Thus there exists a morphism $u : \mathcal{U} \to MI_F(k_1h_1^2 + k_2h_2^2)$. Every point in the image of u is smooth because $\mathrm{Ext}^2_F(E,E) = 0$ by Proposition 4.38, thus there exists a unique component $MI^0_F(k_1h_1^2 + k_2h_2^2)$ containing $u(\mathcal{U})$. Then by Proposition 4.38 we have

$$\dim MI^0_F(k_1h_1^2 + k_2h_2^2) = \dim \mathrm{Ext}^1_F(E,E) = 4k - 3,$$

which completes the proof. □

Acknowledgements

At the end of this journey I would like to spend a few words to thank the people who I shared the last years with.

To my advisor Professor Gianfranco Casnati, for your guidance through my PhD. It has been a pleasure to work together and to have the opportunity to learn so many things. Thank you because you always had the right words to motivate me. To my co-advisor Professor Francesco Malaspina, for your optimism that helped me in every challenging moment of the last three years. Thank you because every time I needed help I knew your door was always open. To my referees and committee members Professor Enrique Arrondo, Professor Cinzia Casagrande and Professor Rosa Maria Miró-Roig, I am very grateful for your suggestions and remarks on the thesis which considerably helped to improve the whole work. To my parents, for their unconditional support for my choices and for always being there when I need them. To my colleagues, because it's very rare to find good friends in the workplace, but it's definitely unique to find a family. To every member of the department, because it was never a burden to come in the office. To all the people I met during my PhD, every single one of you contributed to this journey. To Cumpi, because every time just a glance is enough to understand and encourage each other.

To Rosi, no words can even slightly describe what you do every single day for me. Thank you cause none of this would have been possible without you.

References

[1] V. Ancona and G. Ottaviani. Canonical resolutions of sheaves on Schubert and Brieskorn varieties. In Klas Diederich, editor, *Complex Analysis*, pages 14–19, 1991.

[2] V. Ancona and G. Ottaviani. Stability of special instanton bundles on $\mathbb{P}^{2n+1}$. *Trans. Amer. Math. Soc.*, 341(2):677–693, 1994.

[3] V. Ancona and G. Ottaviani. On singularities of $M_{\mathbb{P}^3}(c_1, c_2)$. *International Journal of Mathematics*, 9:407–419, 1998.

[4] V. Antonelli. Characterization of Ulrich bundles on Hirzebruch surfaces. arXiv:1806.10380 [math.AG], to appear on Rev. Math. Compl., 2018.

[5] V. Antonelli and F. Malaspina. Instanton bundles on the Segre threefold with Picard number three. to appear on Math. Nach., 2019.

[6] M. Aprodu, L. Costa, and R. M. Miró-Roig. Ulrich bundles on ruled surfaces. *Journal of Pure and Applied Algebra*, 222(1):131 – 138, 2018.

[7] M. Aprodu, G. Farkas, and A. Ortega. Minimal resolutions, Chow forms and Ulrich bundles on K3 surfaces. *Journal für die reine und angewandte Mathematik*, 2017:225, 2017.

[8] M. Aprodu, S. Huh, F. Malaspina, and J. Pons-Llopis. Ulrich bundles on smooth projective varieties of minimal degree. *arXiv:1705.07790 [math.AG]*, 2017. Preprint.

[9] E. Arrondo. A Home-Made Hartshorne-Serre correspondence. *Rev. Mat. Complut.*, 20(2):423 – 443, 2007.

[10] E. Arrondo and L. Costa. Vector bundles on Fano 3-folds without intermediate cohomology. *Communications in Algebra*, 28(8):3899–3911, 2000.

[11] E. Arrondo and B. Graña. Vector bundles on $G(1,4)$ without intermediate cohomology. *Journal of Algebra*, 214:128–142, 04 1999.

[12] M. F. Atiyah, V.G. Drinfeld, N. J. Hitchin, and Y.I. Manin. Construction of instantons. *Phys. Lett. A*, 65:185–187, 1978.

[13] J. Backelin and J. Herzog. On Ulrich-modules over hypersurface rings. In M. Hochster, C. Huneke, and J. D. Sally, editors, *Commutative Algebra*, pages 63–68, New York, NY, 1989. Springer New York.

[14] C. Bănică. Smooth reflexive sheaves. In *Proceedings of the Col- loquium on Complex Analysis and the Sixth Romanian-Finnish Seminar*, volume 36, pages 571–593, 1991.

[15] W. Barth. Moduli of vector bundles on the projective plane. *Inventiones mathematicae*, 42(1):63–91, Dec 1977.

[16] A. Beauville. Determinal hypersurface. *Michigan Math. J.*, 48:39–64, 2000.

[17] A. Beauville. An introduction to Ulrich bundles. *arXiv:1610.02771 [math.AG]*, 2016.

[18] A. A. Beilinson. Coherent sheaves on Pn and problems of linear algebra. *Functional Analysis and Its Applications*, 12(3):214–216, Jul 1978.

[19] J. P. Brennan, J. Herzog, and B. Ulrich. Maximally generated Cohen-Macaulay modules. *Math. Scand.*, 61(2):181–203, 1987.

[20] N. P. Buchdahl. Instantons on $\mathbb{CP}^2$. *J. Diff. Geometry*, 24:19–52, 1986.

[21] R. O. Buchweitz, G. M. Greuel, and F. O. Schreyer. Cohen-macaulay modules on hypersurface singularities ii. *Inventiones mathematicae*, 88(1):165–182, Feb 1987.

[22] M. Casanellas, R. Hartshorne, F. Geiss, and F. O. Schreyer. Stable Ulrich bundles. *International Journal of Mathematics*, 23(08):1250083, 2012.

[23] G. Casnati. Rank two stable Ulrich bundles on anticanonically embedded surfaces. *Bulletin of the Australian Mathematical Society*, 95(1):22–37, 2017.

[24] G. Casnati. Special Ulrich bundles on non-special surfaces with $p_g = q = 0$. *Int. J. Math.*, 28(8), 2017.

[25] G. Casnati. Special Ulrich bundles on regular surfaces with non-negative Kodaira dimension. arXiv:1809.08565 [math.AG], Preprint, 2019.

[26] G. Casnati. Ulrich bundles on non-special surfaces with $p_g = 0$ and $q = 1$. *Revista Matemática Complutense*, 32(2):559–574, May 2019.

[27] G. Casnati, E. Coskun, Ö. Genç, and F. Malaspina. Instanton bundles on the blow-up of the projective 3-space at a point. Preprint, 2019.

[28] G. Casnati, D. Faenzi, and F. Malaspina. Rank two aCM bundles on the del Pezzo threefold with Picard number 3. *Journal of Algebra*, 429:413 – 446, 2015.

[29] G. Casnati, D. Faenzi, and F. Malaspina. Moduli spaces of rank two aCM bundles on the Segre product of three projective lines. *Journal of Pure and Applied Algebra*, 220(4):1554 – 1575, 2016.

[30] G. Casnati, D. Faenzi, and F. Malaspina. Rank two aCM bundles on the del Pezzo fourfold of degree 6 and its general hyperplane section. *Journal of Pure and Applied Algebra*, 222(3):585 – 609, 2018.

[31] G. Casnati and F. Galluzzi. Stability of rank 2 vector bundle on K3 surfaces. *Math. Scand.*, 122(2):239–256, apr 2018.

[32] I. Coandă, A. S. Tikhomirov, and G. Trautmann. Irreducibility and smoothness of the moduli space of mathematical 5-instantons over $\mathbb{P}_3$. *International Journal of Mathematics*, 14(01):1–45, 2003.

[33] E. Coskun and Ö. Genç. Ulrich bundles on Veronese surfaces. *Proceedings of AMS*, 145(11):4687–4701, 2017.

[34] E. Coskun, R. S. Kulkarni, and Y. Mustopa. On representations of clifford algebras of ternary cubic forms. *Contemp. Math.*, 562:91–99, 2011.

[35] E. Coskun, R. S. Kulkarni, and Y. Mustopa. Pfaffian quartic surfaces and representations of Clifford algebras. *Doc. Math.*, 17:1003–1028, 2012.

[36] E. Coskun, R. S. Kulkarni, and Y. Mustopa. The geometry of Ulrich bundles on del Pezzo surfaces. *Journal of Algebra*, 375:280 – 301, 2013.

[37] I. Coskun, L. Costa, J. Huizenga, R. M. Miró-Roig, and M. Woolf. Ulrich Schur bundles on flag varieties. *Journal of Algebra*, 474:49 – 96, 2017.

[38] I. Coskun and J. Huizenga. Brill-Noether theorems and globally generated vector bundles on Hirzebruch surfaces. *Nagoya Mathematical Journal*, 2018.

[39] L. Costa and R. M. Miró-Roig. Monads and instanton bundles on smooth hyperquadrics. *Mathematische Nachrichten*, 282(2):169–179, 2009.

[40] L. Costa and R. M. Miró-Roig. $GL(V)$-invariant Ulrich bundles on Grassmannians. *Mathematische Annalen*, 361(1):443–457, Feb 2015.

[41] L. Costa, R. M. Miró-Roig, and J. Pons-Llopis. An account of instanton bundles on hyperquadrics. In Cham, editor, *From classical to modern algebraic geometry*, Trends Hist. Sci., pages 409–428. Birkhäuser/Springer, 2016.

[42] L. Costa and R.M. Miró-Roig. Rationality of moduli spaces of vector bundles on rational surfaces. *Nagoya Math. J.*, 165:43–69, 2002.

[43] L. Costa and R.M. Miró-RoigR. Rationality of moduli spaces of vector bundles on Hirzebruch surfaces. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, (509):151–166, 1999.

[44] L. Costa and G. Ottaviani. Nondegenerate multidimensional matrices and instanton bundles. *Trans. Amer. Math. Soc.*, 355(1):49–55, 2003.

[45] S. K. Donaldson. Vector bundles on flag manifolds and the Ward correspondence. In *Geometry today (Rome)*, volume 60 of *Progr. Math.*, pages 109–119. Birkhäuser Boston, Boston, MA, 1985.

[46] D. Eisenbud and J. Herzog. The classification of homogeneous Cohen-Macaulay rings of finite representation type. *Mathematische Annalen*, 280(2):347–352, Mar 1988.

[47] D. Eisenbud, F. O. Schreyer, and J. Weyman. Resultants and Chow forms via exterior syzigies. *J. Amer. Math. Soc.*, 16:537–579, 2003.

[48] G. Ellingsrud and S. A. Strømme. Stable rank-2 vector bundles on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 3$. *Mathematische Annalen*, 255(1):123–135, Mar 1981.

[49] D. Faenzi. Even and odd instanton bundles on Fano threefolds of Picard number one. *Manuscripta Mathematica*, 144(1):199–239, May 2014.

[50] D. Faenzi. Ulrich bundles on K3 surfaces. Preprint, arXiv:1807.07826 [math.AG], 2018.

[51] D. Faenzi and F. Malaspina. Surfaces of minimal degree of tame representation type and mutations of Cohen–Macaulay modules. *Advances in Mathematics*, 310:663 – 695, 2017.

[52] D. Faenzi and J. Pons-Llopis. The Cohen-Macaulay representation type of arithmetically Cohen-Macaulay varieties. arXiv:1504.03819 [math.AG], Preprint., 2015.

[53] W. Fulton. *Intersection Theory*. Springer, 1984.

[54] S. I. Gelfand and Y. I. Manin. *Methods of homological algebra*. Springer Science & Business Media, 2013.

[55] D. E. Haile. On the Clifford algebra of a binary cubic form. *Amer. J. Math.*, 106(6):1269–1280, 1984.

[56] R. Hartshorne. *Residues and duality*, volume 20. Springer, 1966.

[57] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Text in Mathematics*. Springer-Verlag New York, 1977.

[58] R. Hartshorne. Stable vector bundles of rank 2 on $\mathbb{P}^3$. *Mathematische Annalen*, 238:229–280, 1978.

[59] N. J. Hitchin. Kählerian twistor spaces. *Proc. Lond. Math. Soc.*, 43(1):133–150, 1981.

[60] G. Horrocks. Vector bundles on the punctured spectrum of a local ring. *Proceedings of the London Mathematical Society*, s3-14(4):689–713, 1964.

[61] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Cambridge University Press, second edition, 2010.

[62] V. A. Iskovskikh and Û. V. Prokhorov. *Algebraic geometry V: Fano varieties*, volume 5. Boom Koninklijke Uitgevers, 1999.

[63] M. Jardim. Instanton sheaves on complex projective spaces. *Collect. Math*, 57(1):69–91, 2006.

[64] M. Jardim, M. Maican, and A. S. Tikhomirov. Moduli spaces of rank 2 instanton sheaves on the projective space. *Pacific J. Math.*, 291(2):399–424, 2017.

[65] M. Jardim, S. Marchesi, and A. Wissdorf. Moduli of autodual instanton bundles. *Bull. Braz. Math. Soc.*, 47(3):823–843, 2016.

[66] M. Jardim, D. Markushevich, and A. S. Tikhomirov. New divisors in the boundary of the instanton moduli space. *Mosc.Math. J.*, 18(1):117–148, 2018.

[67] M. Jardim, G. Menet, D. Prata, and H. Sá Earp. Holomorphic bundles for higher dimensional gauge theory. *Bull. London Math. Soc.*, 49:117–132, 2017.

[68] M. Jardim and M. Verbitsky. Trihyperkähler reduction and instanton bundles on $\mathbb{CP}^3$. *Compositio Mathematica*, 150(11):1836–1868, 2014.

[69] M. M. Kapranov. On the derived categories of coherent sheaves on some homogeneous spaces. *Inventiones mathematicae*, 92(3):479–508, Oct 1988.

[70] P. I. Katsylo. Rationality of the module variety of mathematical instantons with $c_2 = 5$. In *Lie groups, their discrete subgroups, and invariant theory*, volume 8 of *Adv. Math. Sci.*, pages 105–111. Amer. Math. Soc., 1992.

[71] P. I. Katsylo and G. Ottaviani. Regularity of the moduli space of instanton bundles $M_{\mathbb{P}^3}(5)$. *Transformation Groups*, 8(2):147–158, Jun 2003.

[72] H. Knörrer. Cohen-Macaulay modules on hypersurface singularities. *Inventiones mathematicae*, 88:153–164, 1987.

[73] A. Kuznetsov. Instanton bundles on Fano threefolds. *Central European Journal of Mathematics*, 4:1198–1231, 2012.

[74] A. Kuznetsov, Y. Prokhorov, and C. Shramov. Hilbert scheme of lines and conics and automorphism groups of Fano threefolds. *Jpn. J. Math.*, 13(1):109–185, 2018.

[75] W.-P. Li and Z. Qin. Rank-3 stable bundles on rational ruled surfaces. *Mathematische Zeitschrift*, 222(2):279–303, 1996.

[76] Z. Lin. Ulrich bundles on projective spaces. *arXiv:1703.06424 [math.AG]*, 2017. Preprint.

[77] A. F. Lopez. On the existence of ulrich vector bundles on some surfaces of maximal albanese dimension. *Eur. J. Math.*, 5(3):958–963, 2019.

[78] F. Malaspina, S. Marchesi, and Joan Pons-Llopis. Istanton bundles on the flag variety $F(0,1,2)$. arXiv:1706.06353 [math.AG], to appear on Ann Scuola Norm-Sci, 2018.

[79] R. M. Miró-Roig. The representation type of rational normal scrolls. *Rend. Circ. Mat. Palermo*, 62(153), 2013.

[80] R. M. Miró-Roig. Lectures on the representation type of a projective variety. In Springer, editor, *Commutative algebra and its interactions to algebraic geometry*, volume 2210 of *Lecture Notes in Math.*, pages 165–216. 2018.

[81] R. M. Miró-Roig and J. Pons-Llopis. The minimal resolution conjecture for points on del pezzo surfaces. *Algebra Number Theory*, 6(1):27–46, 2012.

[82] R. M. Miró-Roig and J. Pons-Llopis. Representation type of rational aCM surfaces $X \subseteq \mathbb{P}^4$. *Algebras and Representation Theory*, 16(4):1135–1157, 2013.

[83] R. M. Miró-Roig and J. Pons-Llopis. Special Ulrich bundles on regular Weierstrass fibrations. *Mathematische Zeitschrift*, 2018.

[84] C. Okonek, M. Schneider, and H. Spindler. *Vector bundles on complex projective spaces*, volume 3 of *Progress in Mathematics*. Springer-Verlag New York, 1980.

[85] C. Okonek and H. Spindler. Mathematical instanton bundles on $\mathbb{P}^{2n+1}$. *Journal für die reine und angewandte Mathematik*, 364, 1986.

[86] D. O. Orlov. Projective bundles, monoidal transformations and derived categories of coherent sheaves. *Russian Acad. Sci. Ivz. Math.*, 41(1):133–141, 1993.

[87] G. Sanna. Small charge instantons and jumping line on quintic del Pezzo. *Int. Math. Res. Notices*, 21:6523–6583, 2017.

[88] S. Soberon-Chavez. Rank 2 vector bundles over a complex quadric surface. *The Quarterly Journal of Mathematics*, 36(2):159–172, 06 1985.

[89] A. S. Tikhomirov. Moduli of mathematical instanton vector bundles with odd $c2$ on projective space. *Izv. Math.*, 76(5), 2012.

[90] A. S. Tikhomirov. Moduli of mathematical instanton vector bundles with even $c2$ on projective space. *Izv. Math.*, 77(6), 2013.

[91] B. Ulrich. Gorenstein rings and modules with high number of generators. *Math. Z.*, 188(1):23–32, 1984.

[92] C. Walter. Irreducibility of moduli spaces of vector bundles on birationally ruled surfaces. *Algebraic Geometry (Catania, 1993/Barcelona, 1994), Lecture Notes in Pure and Appl. Math.*, 200:201–211, 1998.