ON FUNCTIONAL SUCCESSIVE MINIMA.

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`This note is dedicated to Andrzej Schinzel in fond memory. One of his favourite english authors was Hilaire Belloc, who wrote (with a single word changed):`

When I am dead, I hope it may be said:
'His sins were scarlet, but his papers were read.'

Abstract. *In the classical Geometry of Numbers the calculation of successive minima may be quite difficult, even in* $\mathbf{R}^2$ *using the norm coming from a distance function associated to a set. In the literature there seem to be hardly any analogues when* $\mathbf{R}$ *is replaced by the algebraic closure of a function field in one variable and one uses a norm arising from the absolute height. Here we calculate a one-parameter family of examples that naturally arose in our recent paper on bounded heights. We also comment on whether the minima are attained.*

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1. introduction

Siegel's Lemma was originally constructed for use in diophantine approximation and transcendence theory, but has since found applications elsewhere, for example to complexity theory (see [9] p.98), to integer-valued entire functions (see [9] chapter 10), or to counting rational points (see [9] chapter 18). In its simplest form it says that a system of $M \geq 1$ homogeneous linear equations (assumed for convenience linearly independent) in $N \geq 1$ unknowns over $\mathbf{Z}$ has a small non-trivial solution in $\mathbf{Z}^N$ provided $M < N$. More general versions show that there are $L = N - M$ linearly independent solutions which are usually not much bigger.

It is convenient to take advantage of the homogeneity by working with $\mathbf{Q}$ and $\mathbf{Q}^N$.

More precisely, we define the projective height of non-zero $\mathbf{q} = (q_1, \ldots, q_K)$ in $\mathbf{Q}^K$ as

$$H(\mathbf{q}) = \max\{|qq_1|, \ldots, |qq_K|\} \tag{1.1}$$

where q is anything in $\mathbf{Q}$ such that $qq_1, \ldots, qq_K$ are in $\mathbf{Z}$ and coprime. The linear equations define a variety V of dimension L in $\mathbf{Q}^N$ whose Grassmannian coordinates form a non-zero vector $\mathbf{v}$ in $\mathbf{Q}^K$ for $K = \binom{N}{L}$, and we may define $H(V) = H(\mathbf{v})$. Then there are independent $\mathbf{x}_1, \ldots, \mathbf{x}_L$ in V with

$$H(\mathbf{x}_1) \cdots H(\mathbf{x}_L) \leq cH(V)$$

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where c depends only on N.

There are extensions with $\mathbf{Q}$ replaced by any number field, but now c depends on the field (and indeed must). For all this see [4] (pp. 72-80).

Roy and Thunder [10] in 1996 succeeded in replacing $\mathbf{Q}$ by its algebraic closure $\overline{\mathbf{Q}}$; and now c again depends only on N. In fact they proved that for any $\epsilon > 0$ there are independent $\mathbf{x}_1, \ldots, \mathbf{x}_L$ in V with

$$H(\mathbf{x}_1) \cdots H(\mathbf{x}_L) \leq (2^{L(L-1)/2} + \epsilon) H(V)$$

(see also David and Philippon [7] for an explicit deduction of a better bound on $H(\mathbf{x}_1)$ from a 1995 work [13] of Zhang - they remark that a similar deduction can be made for the product of heights).

In the same [10], they proved results when $\overline{\mathbf{Q}}$ is replaced by the function field $\overline{k(t)}$, where k is any field and t is transcendental over k. Now one gets (see Theorem 2.2 p.6 with $\delta = 0$)

$$H(\mathbf{x}_1) \cdots H(\mathbf{x}_L) \leq (1 + \epsilon) H(V), \tag{1.2}$$

which is sharp because they note (also p.6) that for any independent $\mathbf{x}_1, \ldots, \mathbf{x}_L$ in V one has

$$H(\mathbf{x}_1) \cdots H(\mathbf{x}_L) \geq H(V). \tag{1.3}$$

They reformulate these results in terms of successive minima already familiar from the classical Geometry of Numbers.

But from now on we will go additive with $h = \log H$.

Thus for $i = 1, \ldots, L$ define $\mu_i(V)$ as the infimum of all real μ for which there exist i linearly independent elements $\mathbf{x}$ of V with $h(\mathbf{x}) < \mu$. Thus

$$\mu_1(V) \leq \cdots \leq \mu_L(V).$$

Then (1.2) and (1.3) are equivalent to the single statement

$$\mu_1(V) + \cdots + \mu_L(V) = h(V). \tag{1.4}$$

At last we give the definition of h in $\overline{k(t)}^N$. As in (1.1) but with degrees we define $h(\mathbf{q})$ for $\mathbf{q}$ in $k(t)^K$ by

$$h(\mathbf{q}) = \max\{\deg(qq_1), \ldots, \deg(qq_K)\} \tag{1.5}$$

where q is anything in $k(t)$ such that $qq_1, \ldots, qq_K$ are in $k[t]$ and coprime. This is the same as

$$\sum_v \log \max\{|q_1|_v, \ldots, |q_K|_v\} \tag{1.6}$$

where v runs over all valuations of $k(t)$ that are trivial on k. Standard height theory [4] then extends this to $\overline{k(t)}^K$, where an analogue of (1.6) holds with extra rational coefficients. Then $h(V)$ is defined as above with the Grassmannian.

Now in the classical Geometry of Numbers the calculation of successive minima even in rather simple situations can be difficult. For example let S be the (long thin) set of (x, y) in $\mathbf{R}^2$ with $|\pi x - y| \leq 1/113, |x| \leq 113$. Then by Minkowski the infimum $\mu_1(113)$ of all $\mu \geq 0$ such that μS contains a non-zero point of $\mathbf{Z}^2$ satisfies $\mu_1(113) \leq 1$. Here it is not so hard to show that in fact $\mu_1(113) = 113(106\pi - 33) > .99$. But if we replace 113 by 1936 then

$$\mu_1(1936) = \frac{113}{1936} = .058367\ldots < \frac{1}{\sqrt{292}}. \tag{1.7}$$

However $\mu_1(33102) > .99$ again. The behaviour with parameters lies very deep, as one can show that $\liminf_{\lambda\to\infty} \mu_1(\lambda) = 0$ if and only if π has unbounded partial quotients in its continued fraction (as might be guessed from (1.7) above). Of course the transcendence of π plays no role here; for example the real α with $\alpha^3 - 8\alpha - 10 = 0$ has a reasonably early partial quotient 16467250 (see [6] Churchhouse and Muir). For related problems see also Cassels [5] Theorem VIIA (p.92) on binary cubic forms and section XI.4 (p.329) in connexion with Minkowski's Conjecture.

An example more in our context is the plane V in $\mathbf{Q}^3$ defined by $a_1x_1 + a_2x_2 + a_3x_3 = 0$ with integers a_1, a_2, a_3. On it the smallest value of $h(\mathbf{x})$ with non-zero $\mathbf{x}$ in $\mathbf{Q}^3$ can be found in principle for any specific integers, but there is probably no simple closed formula; and *a fortiori* for two independent solutions. Further if we replace $\mathbf{Q}$ by $\overline{\mathbf{Q}}$ the problems are unlikely to get easier. See however Sombra [11] for the explicit calculation of certain related successive minima for toric varieties.

Things may look easier in the function field case (where for example all valuations are ultrametric); thus for V_0 defined by $x_1 + x_2 - x_3 = 0$ with $h(V_0) = 0$, corresponding to the affine line $x + y = 1$, we have $(x, y) = (1, 0), (0, 1)$ so that $\mu_1(V) = 0, \mu_2(V) = 0$. Or for the line $tx + (1-t)y = 1$ with $h(V_1) = 1$ we have $(1, 1), (t, 1+t)$ so that $\mu_1(V) = 0, \mu_2(V) = 1$. But just for $t^{1/2}x + (1-t)^{1/2}y = 1$ the values are not so clear (see section 4). Indeed we know of no other calculations of this sort in the literature.

In a recent paper [2] we proved (among other things) that algebraic numbers τ satisfying $\tau^\lambda + (1-\tau)^\lambda = 1$ usually have height bounded above independently of the positive rational $\lambda \geq 0$ (see also [1] for integral λ). The proof involved the V_λ corresponding to

$$t^\lambda x + (1-t)^\lambda y = 1$$

with $h(V_\lambda) = \lambda$. There it sufficed to know when $\mu_1(V_\lambda)$ could be near zero. Here we push the techniques further to calculate explicitly $\mu_1(V_\lambda), \mu_2(V_\lambda)$ for all $\lambda \geq 0$, at least in characteristic zero. Thus from now on we will assume that k has characteristic zero, and (for convenience) that k is algebraically closed. By (1.4) it will suffice to treat the first minimum.

Theorem 1.1. *Define the integer part* $l = [3\lambda/2] \geq 0$*, so that*

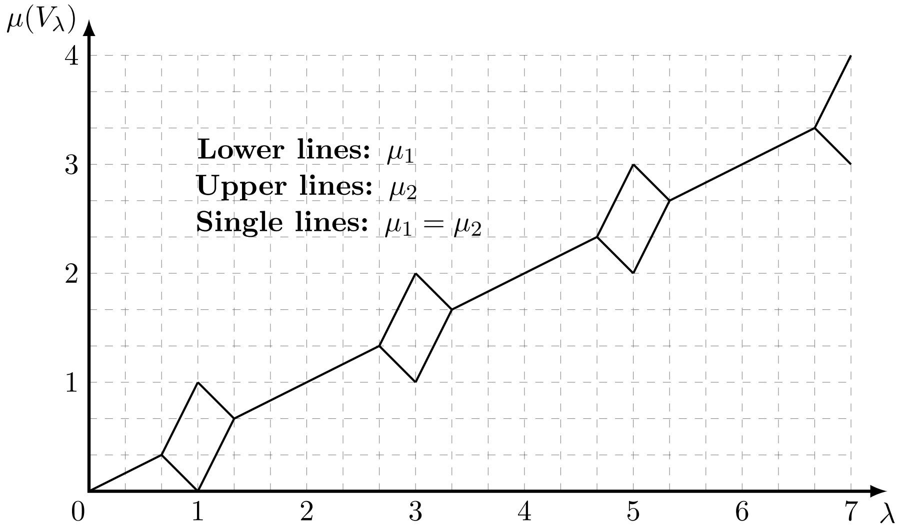
$$\frac{2l}{3} \leq \lambda < \frac{2l+2}{3}.$$

Then $\mu_1(V_\lambda) = \lambda/2$ *unless* $l \equiv 1 \bmod 3$*, in which case*

$$\mu_1(V_\lambda) = l - \lambda \qquad \left(\frac{2l}{3} \leq \lambda < \frac{2l+1}{3}\right),$$

$$\mu_1(V_\lambda) = 2\lambda - l - 1 \qquad \left(\frac{2l+1}{3} \leq \lambda < \frac{2l+2}{3}\right).$$

Thus by plotting also the second minimum $\mu_2(V_\lambda) = \lambda - \mu_1(V_\lambda)$ we see an infinite sequence of "lozenges", as in the picture.



2. PRELIMINARIES

The first is very well-known.

Lemma 2.1. *For non-zero coprime A, B, C in $k[t]$, not all in k, with $A + B + C = 0$ we have*

$$\max\{\deg A, \deg B, \deg C\} \leq -1 + \sum_{\tau \in k, ABC(\tau)=0} 1.$$

Proof. The account in [8] (p.194) is simple but unfortunately does not explicitly exclude A, B, C being all in k; however if one adds the assumption $n_0(abc) \geq 1$ there, then all becomes fine. Alternatively see the third author [12] (p.121) or the second author [9] (p.153). □

The second has a similar flavour but seems to be new. For a multiplicative abelian group G we define G^{div} as the set of g for which there exists a positive integer d with g^d in G. We shall need this only for $G = k(t)^*$.

Lemma 2.2. *For α, β, γ in $k(t)^{*\text{div}}$ with $\alpha + \beta + \gamma = 0$ there exists δ in $k(t)^{*\text{div}}$ such that $\delta\alpha, \delta\beta, \delta\gamma$ are in $k(t)^*$.*

Proof. Indeed this mimics the first step in the proof of Lemma 2.1. Write $u = -\alpha/\gamma, v = -\beta/\gamma$ so that $u + v = 1$. With the extension of $\mathrm{d}/\mathrm{d}t$ to $\overline{k(t)}$, we have $u' + v' = 0$ or better $(u'/u)u + (v'/v)v = 0$; and, if $u'/u \neq v'/v$, then solving the two linear equations for u, v yields

$$u = -\frac{v'/v}{u'/u - v'/v}, \quad v = \frac{u'/u}{u'/u - v'/v}.$$

There is a positive integer d with $w = u^d$ in $k(t)$, and so $u'/u = (w'/w)/d$ lies also in $k(t)$. Similarly for v'/v, and so u, v lie in $k(t)$. Thus $\delta = 1/\gamma$ will do. And if $u'/u = v'/v$ then $w = u/v$ lies in k so $k(t)$, so also $v = 1/(1 + w)$ and u too. □

3. Proof of Theorem

We may assume $\lambda > 0$. By (1.4) we have $\mu_1 = \mu_1(V_\lambda) \leq \lambda/2$. Take any ξ, η in $\overline{k(t)}$ with $h(\xi, \eta, 1) < \lambda$ and

$$\xi t^\lambda + \eta(1 - t)^\lambda = 1. \tag{3.1}$$

Then ξ, η are both non-zero.

Let σ be any element of $\mathrm{Gal}(\overline{k(t)}/k(t))$. Applying it to (3.1), we get

$$\xi^\sigma \theta_\sigma t^\lambda + \eta^\sigma \phi_\sigma (1 - t)^\lambda = 1.$$

with roots of unity $\theta_\sigma, \phi_\sigma$. Eliminating $(1 - t)^\lambda$ from this and (3.1) gives $\Delta t^\lambda = \delta$ with

$$\Delta = \Delta_\sigma = \xi \eta^\sigma \phi_\sigma - \xi^\sigma \eta \theta_\sigma, \qquad \delta = \delta_\sigma = \eta^\sigma \phi_\sigma - \eta.$$

.

If $\Delta \neq 0$ for some σ then

$$\lambda = h(t^\lambda, 1) = h(\delta, \Delta).$$

Here δ and Δ are bihomogeneous in (ξ, ξ^σ),(η, η^σ), of bidegrees $(0, 1)$ and $(1, 1)$ respectively. Now

$$\max\{y, y', xy', x'y\} \leq \max\{x, y, 1\} \max\{x', y', 1\}$$

for $x, y, x', y' \geq 0$. In the analogue of (1.6) for $h(\delta, \Delta)$ a typical term is

$$\log \max\{|\delta|_v, |\Delta|_v\} \leq \log \max\{|\eta|_v, |\eta^\sigma|_v, |\xi\eta^\sigma|_v, |\xi^\sigma\eta|_v\}$$

which is therefore at most

$$\log \max\{|\xi|_v, |\eta|_v, 1\} + \log \max\{|\xi^\sigma|_v, |\eta^\sigma|_v, 1\}.$$

Summing over v, we end up with

$$h(\delta, \Delta) \leq h(\xi, \eta, 1) + h(\xi^\sigma, \eta^\sigma, 1) = 2h(\xi, \eta, 1).$$

So we get

$$h(\xi, \eta, 1) \geq \frac{\lambda}{2} \tag{3.2}$$

which would imply $\mu_1(V_\lambda) \geq \lambda/2$. If that were true then we have equality by (1.4). This means that we are on the "single lines" in the picture or possibly their continuations through the lozenges.

But what if $\Delta = 0$ above for all σ?

Then $\delta = 0$ as well. And eliminating t^λ instead shows that

$$\xi^\sigma \theta_\sigma = \xi, \qquad \eta^\sigma \phi_\sigma = \eta. \tag{3.3}$$

By raising to suitable powers we see that ξ, η lie in $k(t)^{*\mathrm{div}}$.

Now Lemma 2.2 on (3.1) shows that $\xi t^\lambda, \eta(1-t)^\lambda$ are in $k(t)^*$.

So $\rho = \xi t^\lambda,\ 1-\rho = \eta(1-t)^\lambda$ for some $\rho \neq 0, 1$ in $k(t)$, and with $\lambda = p/q$ and $\rho = Y/Z$ with coprime Y, Z in $k[t]$ we contradict (3.2) if

$$h^* = qh(\xi, \eta, 1) = h((1-t)^p Y^q, t^p(Z-Y)^q, t^p(1-t)^p Z^q) < p/2 \tag{3.4}$$

now over $k[t]$. If the three expressions were coprime then h^* would be $\max\{p + nq, 2p + mq\}$, where

$$n = \max\{\deg Y, \deg Z\}, \qquad m = \deg Z. \tag{3.5}$$

Now the only possible common prime factors are $t, 1-t$. Write

$$r = \mathrm{ord}_{t=0} Y, \quad s = \mathrm{ord}_{t=1}(Z - Y). \tag{3.6}$$

If t occurs as a common factor then $r \geq 1$ so $Z(0) \neq 0$, and the largest power of t occurring is $\min\{rq, p\}$ (and this holds even if t does not occur, because then $r = 0$).

Similarly the largest power of $1-t$ occurring is $\min\{sq, p\}$.

Thus

$$h^* = \max\{p + nq, 2p + mq\} - \min\{rq, p\} - \min\{sq, p\}, \tag{3.7}$$

and we have to figure out how much smaller than $p/2$ this can be. Now $h^* < p/2$ is equivalent to eight inequalities according to the choices in max and the two min. Dehomogenizing these gives

$$n - r - s < -\frac{\lambda}{2}, \quad n - r < \frac{\lambda}{2}, \quad n - s < \frac{\lambda}{2}, \quad n < \frac{3\lambda}{2}, \tag{3.8}$$

$$m - r - s < -\frac{3\lambda}{2}, \quad m - r < -\frac{\lambda}{2}, \quad m - s < -\frac{\lambda}{2}, \quad m < \frac{\lambda}{2}. \tag{3.9}$$

Adding the first of (3.8) and the fourth of (3.9) gives $r + s > n + m$. We apply Lemma 2.1 to $Y + (Z - Y) - Z$. The terms are not all constant, else ρ would be, and then $h(\xi, \eta, 1) = \lambda$. Counting the zeroes $\tau = 0, 1$ separately gives

$$n \leq -1 + 2 + (n - r) + (n - s) + m,$$

that is, $r + s \leq n + m + 1$. Therefore

$$r + s = n + m + 1. \tag{3.10}$$

And the fourth of (3.8) and the first of (3.9) now give $(3\lambda/2) - 1 < n < 3\lambda/2$. Thus

$$n = [3\lambda/2] = l \tag{3.11}$$

as in the theorem.

Similarly the first of (3.8) and the fourth of (3.9) give

$$\frac{l}{3} - 1 \leq \frac{\lambda}{2} - 1 < m < \frac{\lambda}{2} < \frac{l+1}{3}$$

so $m \le l/3$, and then

$$m = \left[\frac{l}{3}\right]. \tag{3.12}$$

Now the second and third of (3.8) give

$$\min\{r,s\} > n - \frac{\lambda}{2} = l - \frac{\lambda}{2} > l - \frac{l+1}{3} = \frac{2l-1}{3}. \tag{3.13}$$

Next we examine the cases $l \equiv 2, 0, 1 \bmod 3$ in turn.

When $l \equiv 2 \bmod 3$ we have by (3.12) $m = (l-2)/3$; and (3.13) gives $\min\{r,s\} \ge (2l+2)/3$. But then

$$r + s \ge \frac{4l+4}{3} > \frac{4l+1}{3} = n + m + 1$$

a contradiction to (3.10).

When $l \equiv 0 \bmod 3$ we have $m = l/3$. Now the second and third of (3.9) give

$$\min\{r,s\} > m + \frac{\lambda}{2} = \frac{l}{3} + \frac{\lambda}{2} \ge \frac{2l}{3}$$

so $\min\{r,s\} \ge (2l+3)/3$. Thus now

$$r + s \ge \frac{4l+6}{3} > \frac{4l+3}{3} = n + m + 1$$

a similar contradiction. Thus in both cases $l \equiv 2, 0 \bmod 3$ we cannot have $\Delta = 0$ and (3.2) holds.

Finally when $l \equiv 1 \bmod 3$ then $m = (l-1)/3$ and (3.13) implies $\min\{r,s\} \ge (2l+1)/3$. But then

$$r + s \ge \frac{4l+2}{3} = n + m + 1$$

now no contradiction to (3.10). Instead it forces

$$r = s = \frac{2l+1}{3}.$$

Next (3.7) gives

$$h(\xi,\eta,1) = \frac{h^*}{q} = \max\{\lambda + n, 2\lambda + m\} - \min\{r,\lambda\} - \min\{s,\lambda\}.$$

If $\lambda \le (2l+1)/3$ this is

$$(\lambda + l) - \lambda - \lambda = l - \lambda \le \frac{\lambda}{2}$$

so $h(\xi,\eta,1)$ decreases from $l/3$ to $(l-1)/3$ as λ increases from $2l/3$.

And if $\lambda \ge (2l+1)/3$ it is

$$(2\lambda + m) - r - s = 2\lambda - l - 1 \le \frac{\lambda}{2}$$

so $h(\xi,\eta,1)$ increases from $(l-1)/3$ to $(l+1)/3$ as λ increases from $(2l+1)/3$ to $(2l+2)/3$.

These describe precisely how $h(\xi,\eta,1)$ falls below $\lambda/2$, which is in accordance with the assertions of the theorem, and we are now on the "lower lines" of the picture.

The above arguments show that if we are not on the single lines then $l \equiv 1 \bmod 3$ as well as $n = l,\ m = (l-1)/3,\ r = s = (2l+1)/3$ and we are on the lower lines. We complete the proof by showing that in this case coprime Y, Z actually exist satisfying (3.5), (3.6) and $0 \neq Y \neq Z \neq 0$, thus producing a point on the lower lines.

In fact linear algebra gives Y, Z, not both zero, of degrees at most n, m respectively and $\mathrm{ord}_{t=0} Y$ at least r and $\mathrm{ord}_{t=1}(Z - Y)$ at least s. The rest is relatively routine, but we give some details.

For example $Y = 0$ implies $Z \neq 0$ and so $s \leq m$ a contradiction; similarly for $Y = Z$ and $Z = 0$. If we assume for the moment that Y, Z are coprime, then for example $\deg Y < n$ leads to a contradiction using Lemma 2.1; similarly for $\deg Z < m$ and $\mathrm{ord}_{t=0} Y > r$, $\mathrm{ord}_{t=1}(Z - Y) > s$. And finally if Y, Z have a common factor $D = t^a(1-t)^b C$ with $C(0) \neq 0, C(1) \neq 0$ and $\deg D = d \geq 1$, then we check that $Y/D = t^{r-a} Y'$ with $\deg Y' \leq n-d-r+a$, $(Z-Y)/D = (1-t)^{s-b} X$ with $\deg X \leq n-d-s+b$, and of course $\deg(Z/D) \leq m-d$. Now Lemma 2.1 gives yet another contradiction (using $a+b-2d = (a+b-d)-d < 0$).

Here are some examples.

For $l = 1$ we have $n = 1, m = 0, r = s = 1$ and we can take

$$Y = t, \quad Z = 1, \quad Z - Y = 1 - t$$

corresponding to $x = t$ so (3.1) holds for

$$\xi = t^{1-\lambda}, \quad \eta = (1-t)^{1-\lambda};$$

here $2/3 < \lambda < 4/3$.

For $l = 4$ we have $n = 4, m = 1, r = s = 3$ and we can take

$$Y = -t^3(t-2), \quad Z = 2t - 1, \quad Z - Y = -(1-t)^3(t+1)$$

so (3.1) holds for

$$\xi = -t^{3-\lambda}\frac{t-2}{2t-1}, \quad \eta = -(1-t)^{3-\lambda}\frac{t+1}{2t-1}; \tag{3.14}$$

here $8/3 < \lambda < 10/3$.

And finally for $l = 7$ we have $n = 7, m = 2, r = s = 5$ and we can take

$$Y = t^5(2t^2 - 7t + 7), \quad Z = 7t^2 - 7t + 2, \quad Z - Y = (1-t)^5(2t^2 + 3t + 2)$$

and (3.1) for

$$\xi = t^{5-\lambda}\frac{2t^2 - 7t + 7}{7t^2 - 7t + 2}, \quad \eta = (1-t)^{5-\lambda}\frac{2t^2 + 3t + 2}{7t^2 - 7t + 2}$$

and $14/3 < \lambda < 16/3$.

4. Additional remarks

The above discussion shows that the infimum $\mu_1(V_\lambda)$ is attained when $l \equiv 1 \bmod 3$. We do not know if this is generally the case for other l.

An interesting value with $l = 0$ is $\lambda = 1/2$. Here one can check that (ξ_1, η_1) and (ξ_2, η_2) are independent solutions, where

$$\xi_1 = \frac{1 + (1-t)^{1/2}}{t^{1/2}}, \quad \eta_1 = -1$$

$$\xi_2 = \frac{-20t + 20 + 9t^{1/2} + 12(1-t)^{1/2} + 15t^{1/2}(1-t)^{1/2}}{25t - 16},$$

$$\eta_2 = \frac{-15t - 12t^{1/2} - 16(1-t)^{1/2} - 20t^{1/2}(1-t)^{1/2}}{25t - 16}.$$

We found these by noting that $k(t, t^{1/2}, (1-t)^{1/2})$ has genus zero and is in fact $k(T)$ for

$$T = \frac{-10t + 4 - 3t^{1/2} - 4(1-t)^{1/2} - 5t^{1/2}(1-t)^{1/2}}{25t - 16}$$

with

$$t^{1/2} = \frac{4T^2 + 2T}{5T^2 + 4T + 1}, \quad (1-t)^{1/2} = \frac{3T^2 + 4T + 1}{5T^2 + 4T + 1}.$$

Now the usual linear algebra in $k[T]$ leads to the two solutions above. One can verify that

$$h(\xi_1, \eta_1, 1) = h(\xi_2, \eta_2, 1) = \frac{1}{4}$$

(only the valuations above $t = 0$ and $t = 16/25$ are needed). This corresponds to a point on the first single line of the graph. Thus $\mu_1(V_\lambda) = \mu_2(V_\lambda) = 1/4$ are both attained.

When $l = 1$ and $\lambda = 3/4$ it is easy to see that $\mu_1(V_\lambda) = 1/4$ is attained, even in the form just after (3.3) with ξt^λ in $k(t)$. It is almost as easy to show that $\mu_2(V_\lambda) = 1/2$ is not attained in this form, but we do not know if it is attained in some other form. Perhaps the effective Theorem 5.1 (p.15) of [10] may help with such problems.

From now on let us suppose that λ is in $\mathbf{Z}$. Then we have $\mu_1(V_\lambda) = [\lambda/2]$, $\mu_2(V_\lambda) = \lambda - [\lambda/2]$ and we will show that both are attained in this form, so with ξ, η now in $k(t)$.

To start the proof one sees again by linear algebra that the space of $\mathcal{S} = (P, Q, R)$ in $k[t]^3$ with $Pt^\lambda + Q(1-t)^\lambda + R = 0$ and the degrees of P, Q, R at most d has k-dimension $e \geq 2d - \lambda + 2$. We note that if $d < \lambda$ then any $\mathcal{S}$ must have $R \neq 0$.

For even $\lambda \geq 2$ (we already treated $\lambda = 0$) and $d = \lambda/2$ we get $e \geq 2$. Any solution must have P, Q, R coprime, else there would be a solution with height strictly less than $\lambda/2$, contradicting the fact that this is $\mu_1(V_\lambda)$. Now any two solutions that are k-independent must be $k(t)$-independent (and so trivially $\overline{k(t)}$-independent). They thus correspond to $\mu_2(V_\lambda) = \lambda/2$.

For example with $\lambda = 2$ and $d = 1$ we get for (3.1) two reasonable solutions

$$(\xi, \eta) = (-2t + 3, 2t + 1), \quad \left(-\frac{t-2}{t}, 1\right)$$

of which the first is unique in $k[t]^2$.

For odd $\lambda \geq 3$ (we already treated $\lambda = 1$) and $d = (\lambda - 1)/2$ we get $e \geq 1$ and so any solution $\mathcal{S}_1 = (P_1, Q_1, R_1)$ must have P_1, Q_1, R_1 coprime and correspond to $\mu_1(V_\lambda) = (\lambda - 1)/2$. For $d = (\lambda + 1)/2$ we get $e \geq 3$. We already have two k-independent solutions $\mathcal{S}_1$ and $t\mathcal{S}_1$, so there must be another solution $\mathcal{S}_2 = (P_2, Q_2, R_2)$ which is k-independent of these. If P_2, Q_2, R_2 were not coprime then any common factor D must have degree 1 because of $\mu_1(V_\lambda) = (\lambda - 1)/2$. Then $\mathcal{S}_2$ would be k-proportional to $D\mathcal{S}_1$ contradicting the way it was chosen. Thus P_2, Q_2, R_2 must be coprime. As above now the k-independence of $\mathcal{S}_1, \mathcal{S}_2$ implies their $k(t)$-independence, and so $\mathcal{S}_2$ corresponds to $\mu_2(V_\lambda) = (\lambda + 1)/2$.

For example with $\lambda = 3$ and $d = 1$ we get the unique solution (3.14) and for $d = 2$ we get a solution

$$(\xi, \eta) = (6t^2 - 15t + 10, 6t^2 + 3t + 1)$$

which is unique in $k[t]^2$.

The case of odd λ and $d = (\lambda - 1)/2$ corresponds exactly to the Padé approximations originally considered by Beukers and Tijdeman [3] - see also [4] Theorem 5.2.10 (p.130) with $L = M = N = d$.

The other cases, of odd λ and $d = (\lambda + 1)/2$, or even λ and $d = \lambda/2$, do not strictly speaking correspond to Padé because they are not unique (up to constants). They do however come from linear combinations of Padé approximations (see for example the proof of Lemma 6 of [3] p.199). A similar phenomenon occurs for $\lambda = 1/2$ above.

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