Blow-Ups of Toric Surfaces and the Mori Dream Space Property

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NEW EXAMPLES AND NON-EXAMPLES OF MORI DREAM SPACES WHEN BLOWING UP TORIC SURFACES

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Abstract. We study the question of whether the blow-ups of toric surfaces of Picard number one at the identity point of the torus are Mori Dream Spaces. For some of these toric surfaces, the question whether the blow-up is a Mori Dream Space is equivalent to countably many planar interpolation problems. We state a conjecture which generalizes a theorem of González and Karu. We give new examples and non-examples of Mori Dream Spaces among these blow-ups.

1. Introduction

Mori Dream Spaces (MDS) were introduced by Hu and Keel in [HK00] as normal, $\mathbb{Q}$-factorial projective varieties such that

(1) $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$.
(2) $\mathrm{Nef}(X)$ is generated by finitely many semiample divisors.
(3) There exist finitely many small $\mathbb{Q}$-factorial modifications (SQMs) $g_i : X \dashrightarrow X_i$, $i = 1, \cdots r$, such that every X_i satisfies (1) and (2), and the cone of movable divisors $\mathrm{Mov}(X)$ is the union $\bigcup_{i=1}^{r} g_i^*(\mathrm{Nef}(X_i))$.

Mori Dream Spaces are important examples of varieties where Mori's program can be run for every divisor [HK00, Prop. 1.11].

Mori Dream Spaces are related to Cox rings. Assume X is a projective variety over the complex numbers $\mathbb{C}$, with finitely generated Picard group. Choose line bundles $L_1, \cdots, L_s$ which span $\mathrm{Pic}(X)_{\mathbb{Q}}$. A Cox ring of X is defined as the direct sum

$$\mathrm{Cox}(X) := \bigoplus_{(m_1,\cdots,m_s)\in\mathbb{Z}^s} H^0(X, m_1L_1 + \cdots + m_sL_s).$$

The finite generation of $\mathrm{Cox}(X)$ does not depend on the choice of $L_1, \cdots, L_s$ [HK00]. Furthermore, it is shown in [HK00, Prop. 2.9] that a $\mathbb{Q}$-factorial projective variety X with $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$ is a MDS if and only if $\mathrm{Cox}(X)$ is a finitely generated $\mathbb{C}$-algebra.

One basic source of examples of Mori Dream Spaces is from toric varieties. However, the blow-ups of MDS can fail to be MDS (as we recall below). For instance, blow-ups of toric varieties, in particular weighted projective planes, can fail to be MDS. A weighted projective plane $\mathbb{P}(a,b,c)$ is the quotient of $\mathbb{C}^3 - \{0\}$ by the following $\mathbb{C}^*$-action:

$$\mathbb{C}^* \times (\mathbb{C}^3 - \{0\}) \to \mathbb{C}^3 - \{0\}$$
$$(t,(x,y,z)) \mapsto (t^a x, t^b y, t^c z),$$

where a, b and c are positive integers. Note that $\mathbb{P}(a,b,c)$ is a toric projective surface of Picard number one.

We denote $X = \mathbb{P}(a, b, c)$. Let $X' = \mathrm{Bl}_e \mathbb{P}(a, b, c)$ be the blow-up of X at the identity point e of the open torus.

Question: For which a, b, c is the blow-up $X' = \mathrm{Bl}_e \mathbb{P}(a, b, c)$ a MDS?

Historically, Cutkosky gave many sufficient conditions for $\mathrm{Cox}(X')$ to be finitely generated, and equivalently, X' to be a MDS. For example, if $-K_{X'}$ is big, then X' is a MDS [Cut91]. In particular, if $a+b+c > \sqrt{abc}$ (for example, when one of a, b, c is at most 4), then $-K_{X'}$ is big, and X' is a MDS. Based on the work of Cutkosky, Srinivasan [Sri91] attained several numerical conditions for X' to be a MDS, including that if one of a, b, c is 6, then X' is a MDS. Recently, Hausen, Keicher and Laface gave an algorithm which provides more examples of MDS [HKL16].

On the other hand, by 2013 the only known examples of blow-ups at a general point of toric varieties failing to be a MDS were given by Goto, Nishida and Watanabe [GNW94] (1994). For example, they showed that $X' = \mathrm{Bl}_e \mathbb{P}(a, b, c)$ is not MDS when $(a, b, c) = (7N-3, 8N-3, (5N-2)N)$ for $N \geq 4$, and $3 \nmid N$. In 2013, Castravet and Tevelev [CT15] proved, using the [GNW94] results, that blow-ups of Losev-Manin moduli spaces at a general point are not MDS in sufficient large dimensions. Using this, they proved that the moduli space of curves $\overline{M}_{0,n}$ is not a Mori Dream Space for $n > 133$. In 2014, González and Karu [GK16] provided more examples of triples (a, b, c) such that X' is not a MDS, and lowered the bound to $n > 12$, which was further improved to $n > 9$ by Hausen, Keicher and Laface in [HKL16].

In this paper, we further develop and generalize González and Karu's idea, and show that for some toric surfaces X of Picard number one, the blow-up X' at the identity point of the torus is not a MDS if and only if a family of countable many planar interpolation problems in $\mathbb{P}^2$ all have solutions. (Proposition 2.1, Corollaries 4.3 and 6.1). As an application, we provide new examples (Theorem 2.6) and new non-examples (Theorems 2.13, 2.14) of MDS, from blow-ups of toric surfaces of Picard number one, in particular blow-ups of weighted projective planes (Example 5.6). Our method is different from Hausen, Keicher and Laface's. The results above can be combined into a numerical criterion (Corollary 2.16). We make a conjecture (Conjecture 2.10) generalizing the non-examples of [GK16].

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Contents

2. Main Results

Notations and Settings We work over $\mathbb{C}$. Let $N = \mathbb{Z}^2$ be the plane lattice, and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^2$.

A triangle $\Delta \subseteq N_\mathbb{R}$ with rational slopes s_1, s_2 and s_3 determines a polarized toric variety (X_Δ, H), in the way that the fan of $X = X_\Delta$ is the normal fan of Δ, and H is the $\mathbb{Q}$-Cartier torus-invariant divisor determined by Δ. Notice that two triangles Δ and Δ' determine the same X if they have the same slopes s_1, s_2, s_3. Hence, without specific mention, we assume every triangle Δ is at the following position: one vertex is at $(0, 0)$, and the opposite side of $(0, 0)$ passes through $(0, y_0)$ for $y_0 > 0$.

Define Δ_0 as the triangle with $y_0 = 1$.

A triangle in $N_\mathbb{R}$ is a *lattice triangle* if all its vertices are in N. We say a lattice triangle Δ is *good* if in addition the y-intercept y_0 is an integer. A good lattice triangle Δ is an integer multiple of the Δ_0 with the same slopes, and there exists a smallest good lattice triangle with the given slopes, which we denote as Δ_1. Let m be the integer such that $\Delta_1 = m\Delta_0$.

A *column* of a lattice triangle Δ is the set $\Delta \cap \{(x, y) \in \mathbb{Z}^2 \mid x = x_0\}$ for some integer x_0, such that x_0 does not equal the x-coordinates of the leftmost and rightmost vertices of Δ.

Given $s_1 < s_2 < s_3 \in \mathbb{Q}$, the *width* of s_1, s_2, s_3 is

$$w = \frac{1}{s_2 - s_1} + \frac{1}{s_3 - s_2},$$

which equals the width of Δ_0. Let $W := kmw$, which is the width of $k\Delta_1$.

Assumption: $w < 1$.

Finally, $X' = \mathrm{Bl}_e X_\Delta$ is the blow-up at the torus identity point e of X.

Proposition 2.1. (*see* [Cas16, Prop. 8.6]) *Given $s_1 < s_2 < s_3$ rational numbers with $w < 1$, Let X the toric variety defined by s_1, s_2, s_3. Let X' be the blow-up of X at the torus identity point e. Then the blow-up X' is not a MDS if and only if for every sufficiently divisible integer $k > 0$, there exists a curve Y in $\mathbb{P}^2$, of degree up to $W - 1 = kmw - 1$, and a vertex $p \neq (0, 0)$ of Δ_1, such that Y passes through all the points $(i, j) \in k\Delta_1 \cap \mathbb{Z}^2$ but does not pass through kp.*

Proof. See Section 3. □

We consider the interpolation problem proposed in Proposition 2.1. For any $\Delta = k\Delta_1$, the column at $\{x = 0\}$ contains $km + 1$ points. Suppose such a curve Y in degree $\leq kmw - 1 = W - 1 < km + 1$ exists, then Y passes through all those $km + 1$ points. By Bézout's Theorem, Y must contain the line $\{x = 0\}$ as a component. Hence the existence of Y is equivalent to the existence of Y_1 of degree $\leq kmw-2$, passing through all integer points except the column at $\{x=0\}$, and kq, but not passing through kp.

Indeed this argument by Bézout's Theorem can be run for the rest of columns in $k\Delta_1$ until step n, where n is the integer such that every remaining columns in $k\Delta_1$ after step n contain no more than $kmw-1-n = W-1-n$ points. Notice that when we stop, the existence of Y is equivalent to the existence of a curve Y_n of degree $\leq W-1-n$, passing through all columns remaining, and kq, but not passing through kp.

Definition 2.2. The *reduced degree* of a good lattice triangle Δ of width W equals $W-1-n$, where n is the maximal number of steps we can run as above. Equivalently, d equals the number of remaining columns (the left and right vertices excluded by definition) in Δ after we deleted all columns through the above reduction process.

We now introduce the reduced degree d and minimal degree d' of a triple of slopes s_1, s_2, s_3.

Definition 2.3. Given rational numbers $s_1 < s_2 < s_3$, with $w < 1$, let Δ_1 be the smallest good lattice triangle with slopes $\{s_i\}$. The *reduced degree* of s_1, s_2, s_3 is the largest nonnegative integer d, such that there are exactly d columns in Δ_1 containing $\leq d$ points.

Definition 2.4. Given rational numbers $s_1 < s_2 < s_3$, with $w < 1$, let Δ_1 be the smallest good lattice triangle with slopes $\{s_i\}$. The *minimal degree* of s_1, s_2, s_3 is the smallest positive integer d', such that there are exactly d' columns in Δ_1 containing $\leq d'$ points. When no such $d' > 0$ exists, we define the minimal degree to be zero.

Theorem 2.5. *Consider any rational numbers $s_1 < s_2 < s_3$ such that the width $w < 1$.*

(1) The reduced degrees of the good lattice triangles Δ with slopes s_1, s_2, s_3 all equal the reduced degree of s_1, s_2, s_3 (Definition 2.3). In particular, the reduced degree of s_1, s_2, s_3 equals the reduced degree of Δ_1.

(2) The reduced degree d satisfies the following inequality:

$$d < \frac{w}{1-w}.$$

For example, the triangle $\Delta_1 = 7\Delta_0$ with slopes $(-3/4, 1, 9/2)$ is shown in the Figure 1. There are 5 columns in Δ_1, at $x = -3, -2, -1, 0, 1$. The number of points in each column is $2, 4, 6, 8, 4$. The numbers of columns with $\leq i$ points are given below for $0 \leq i \leq 8$. As a result, the reduced degree $d = 0$. Further, for any integer multiple $k\Delta_1$, the reduced degrees will not increase, and all equal to 0. Finally, the minimal degree d' equals 0.

i	0	1	2	3	4	5	6	7	8
number of columns with $\leq i$ points	0	0	1	1	3	3	4	4	5

More interesting examples are the triangles in Figure 2 which have minimal degrees $d' = 3, 5, 7$ respectively. (Recall that we do not include the leftmost and rightmost vertices as columns)

Theorem 2.6. *Consider rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Let d be the reduced degree and d' be the minimal degree. Let X the toric variety defined by s_1, s_2, s_3. Let X' be the blow-up of X at the torus identity point e.*

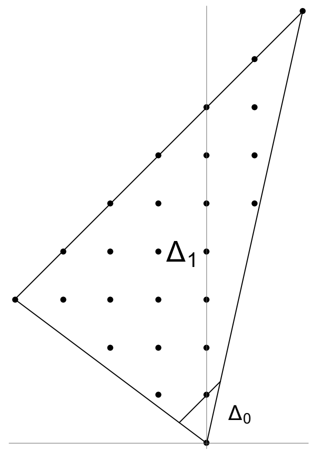

FIGURE 1. The triangle Δ_1 with slopes $(-3/4, 1, 9/2)$ has reduced degree 0

(1) If $d = 0$, then the blow-up X' is a Mori Dream Space (MDS).
(2) If $d' = 1$, then $s_2 \notin \mathbb{Z}$, and the blow-up X' is not a MDS.
(3) In particular, if $d = 1$, then X' is not a MDS.

For the proofs of Theorems 2.5 and 2.6, see Section 4.

Since X' is always a MDS when $d = 0$, it is worthwhile to develop a criterion for having $d = 0$. In Section 8 we prove the following combinatorial criterion:

Proposition 2.7. *Consider rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Let X, X' be defined as in Theorem 2.6. Let*

$$l_k = \lfloor ks_2 \rfloor - \lceil ks_1 \rceil + 1, \quad r_k = \lfloor ks_3 \rfloor - \lceil ks_2 \rceil + 1, \quad \text{for } k = 1, 2, \cdots$$

Let γ be the smallest positive integer such that $\gamma s_2^2, \gamma s_3$ and $\gamma s_2 s_3$ are all integers. Then the reduced degree d equals 0 if all the following three conditions hold:

(1) $l_1 \neq 1$ and $r_1 \neq 1$;
(2) If $\gamma > 1$, then

$$s_2 - s_1 \geq \max_{1 \leq t \leq \gamma - 1} \frac{r_t + \{(r_t - t)s_2\}}{r_t - t};$$

(3)

$$s_2 - s_1 \geq \frac{\gamma(s_3 - s_2) + 1 + \{s_2\}}{\gamma(s_3 - s_2 - 1) + 1}$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. In particular, X' is a MDS when all the conditions above hold.

Conversely, if in addition we assume

(4) For some t where $\dfrac{r_t + \{(r_t - t)s_2\}}{r_t - t}$ achieves its maximum among $1 \leq t \leq \gamma - 1$, t satisfies

$$\gamma(s_3 - s_2 - 1)(r_t - 1 + \{(r_t - t)s_2\}) - (r_t - t)(\gamma(s_3 - s_2) + 1) \leq 0,$$

Then $d = 0$ implies (1), (2), (3) all hold.

Corollary 2.8. *Consider rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. If $s_2 \in \mathbb{Z}$, then the reduced degree d equals zero, and X' is a MDS.*

Proof. It suffices to assume $s_2 = 0$ (See Remark 2.17). Then $l_k = -\lceil ks_1 \rceil + 1$ and $r_k = \lfloor ks_3 \rfloor + 1$. Since $w = 1/s_3 - 1/s_1 < 1$, we have $s_3 > 1$ and $s_1 \leq -1$. So $l_1 \geq 2$ and $r_1 \geq 2$. This shows (1) of Proposition 2.7.

For (2), we find $\{(r_t - t)s_2\} = 0$. We claim that for every $t \geq 1$, $r_t/(r_t - t) \leq s_3/(s_3 - 1)$. Indeed this simplifies to $r_t \geq s_3 t$, which holds since $r_t = \lfloor s_3 t \rfloor + 1 \geq s_3 t$. Now $w < 1$ implies that $-s_1 \geq s_3/(s_3 - 1)$. Therefore (2) holds.

Finally (3) follows from the inequality

$$-s_1 \geq \frac{s_3}{s_3 - 1} > \frac{\gamma s_3 + 1}{\gamma(s_3 - 1) + 1}.$$

Now Proposition 2.7 shows that $d = 0$, so X' is a MDS. ∎

Corollary 2.9. *For any $s_2, s_3 \in \mathbb{Q}$, with $\lfloor s_3 \rfloor - \lceil s_2 \rceil \geq 1$, there exists a rational number s_0, depending on s_2 and s_3 only, such that if $s_1 < s_0$, then the blow-up X' is a MDS.*

Proof. Firstly, $\lfloor s_3 \rfloor - \lceil s_2 \rceil \geq 1$ is equivalent to $r_1 \geq 2$. If in addition $w < 1$ and (1)-(3) of Proposition 2.7 all hold, then X' is a MDS. The condition $w < 1$ gives $s_1 < (s_2 s_3 - s_2^2 - s_3)/(s_3 - s_2 - 1)$. Assuming $w < 1$, l_1 cannot be zero. Then $l_1 \geq 2$ is equivalent to $s_1 \leq \lfloor s_2 \rfloor - 1$. Now s_0 can be taken as the minimum of the 4 upper bounds above from $w < 1$, $l_1 \geq 2$, and (2)(3) of Proposition 2.7. ∎

For example, consider $s_2 = 1/2$, and $s_3 = 3$ (see Example 5.5). Here $w < 1$ if and only if $s_1 < -7/6$. $l_1 \geq 2$ if and only if $s_1 \leq -1$. In Proposition 2.7, we have $\gamma = 4$. Then (2) says $s_1 \leq -6/5$, and (3) says $s_1 \leq -8/7$. Therefore we can take $s_0 = -6/5$, so that when $s_1 < -6/5$, $s_2 = 1/2$ and $s_3 = 3$, the blow-up X' is a MDS.

For the case when the minimal degree d' satisfies $d' \geq 2$ (hence, $d \geq 2$), we have the following conjecture:

Conjecture 2.10. *Consider rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Let d' be the minimal degree. Let X the toric variety defined by s_1, s_2, s_3, and X' be the blow-up of X at the torus identity point e. If $d' \geq 2$, and $d' \cdot s_2 \notin \mathbb{Z}$, then the blow-up X' is not a MDS.*

Remark 2.11. This conjecture, together with Theorem 2.6 (2), generalizes González and Karu's non-examples. Recall that in [GK16] González and Karu showed that if the lattice triangle Δ satisfy that

(1) The first column from left has n points;
(2) The i-th column from the right have $i + 1$ points, for $i = 1, 2, \cdots, n - 1$.

Then the blow-up X' is not a MDS if $ns_2 \notin \mathbb{Z}$. Indeed, here the triangle Δ has minimal degree $d' = n$. Also note that Theorem 2.6 (2) is exactly the case of $n = 1$.

Remark 2.12. The main observation is that we can classify all possible triangles Δ_1 of a given minimal degree d', by the numbers of lattice points on the columns with $< d'$ points.

Suppose $d' \geq 2$. Since $d' \neq 1$, Lemma 8.1 implies that the numbers of points on each columns are strictly increasing. Now there are exactly d' columns in Δ_1 with $\leq d'$ points, so those d' columns must have $2, 3, \cdots, d' - 1, d', d'$ points respectively. Hence Δ_1 determines a partition $\{2, \cdots, d' - 1\} = S \sqcup T$, such that the number of lattice point on the columns starting from the left (right) vertex are given by S (T respectively). In particular, González and Karu's non-examples are given by $S = \emptyset$ and $T = \{2, \cdots, d'-1\}$.

As a result, it is helpful to classify all possible triangles Δ_1 of a given minimal degree d', by the numbers of lattice points on the columns with $< d'$ points, which are given by a partition $\{2, \cdots, d'-1\} = S \sqcup T$, up to a horizontal reflection about the y-axis.

With the assistance of computer programs (in *Mathematica 10* [Wol16]), we have:

Theorem 2.13. *Conjecture 2.10 holds for $d' \leq 9$. That is, If $2 \leq d' \leq 9$, and $d' \cdot s_2 \notin \mathbb{Z}$, then the blow-up X' is not a MDS.*

Theorem 2.14. *Consider rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Let d' be the minimal degree. If $2 \leq d' \leq 9$ and $d' \cdot s_2 \notin \mathbb{Z}$, then either we are in the case of González and Karu's non-examples* [GK16] *(Remark 2.11), or $d' = 5, 7$ or 9, and up to adding a same integer t to all the three slopes, and a reflection about the y-axis, their slopes satisfy one of the following:*

$$\begin{cases} d' = 5; \\ -2 - \frac{1}{2} < s_1 \leq -2; \\ \frac{1}{3} < s_2 < \frac{1}{2}; \\ 2 \leq s_3 < 2 + \frac{1}{3}. \end{cases}, \begin{cases} d' = 7; \\ -2 - \frac{1}{3} < s_1 \leq -2; \\ \frac{1}{4} < s_2 < \frac{1}{3}; \\ 2 \leq s_3 < 2 + \frac{1}{4}. \end{cases}, \begin{cases} d' = 9; \\ -2 - \frac{1}{4} < s_1 \leq -2; \\ \frac{1}{5} < s_2 < \frac{1}{4}; \\ 2 \leq s_3 < 2 + \frac{1}{5}. \end{cases}$$

respectively. Conversely, any combination of slopes which satisfies one of the system of inequalities above as well as $w < 1$ and $d' \cdot s_2 \notin \mathbb{Z}$ (equivalently, $d' \cdot s_2 \neq 2$) determines a blow-up X' which is not a MDS. Figure 2 shows the relative positions of the lattice points on the columns with at most d' lattice points in each lattice triangle.

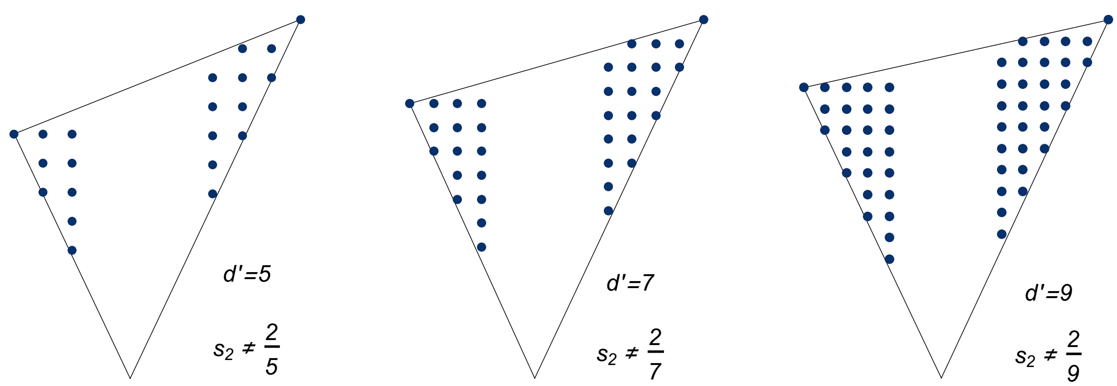

Figure 2. New non-examples of MDS given by the position of lattice points at corners

We will prove Theorem 2.13 and 2.14 in Section 6 and Section 9.

The inequalities of Theorem 2.14 come from the following general result.

Lemma 2.15. *Let Δ_1 be the smallest good lattice triangle with slopes $s_1 < s_2 < s_3$. Let $n \in \mathbb{Z}_{>0}$. Suppose in Δ_1 the first n columns from the left have $3, 5, \cdots, 2n-1, 2n+1$ points,* *and the first $n+1$ columns from the right have $2, 4, \cdots, 2n, 2n+1$ lattice points. Then up to adding a same integer t to all the three slopes, and a reflection about the y-axis, the slopes satisfy the inequalities*

$$\begin{cases} -2 - \dfrac{1}{n} < s_1 \leq -2; \\ \dfrac{1}{n+1} < s_2 < \dfrac{1}{n}; \\ 2 \leq s_3 < 2 + \dfrac{1}{n+1}. \end{cases}$$

Proof. See Section 9. □

Combining Theorem 2.5 (2), 2.6, 2.13 and 2.14, we have the following corollary.

Corollary 2.16. *Give rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Let d, d', X, X' be defined as in Theorem 2.6. Then*

(1) If $w \leq 1/2$, then $d = 0$, and X' is a MDS.
(2) If $w \leq 10/11$ and $d' \cdot s_2 \notin \mathbb{Z}$, then $1 \leq d' \leq d \leq 9$, and X' is not a MDS.

Remark 2.17. Indeed, adding the same integer l to all slopes is equivalent to a shear transformation $(x, y) \mapsto (x, y + lx)$ of the triangles, which induces one-to-one correspondences between the solutions of the corresponding interpolation problems, and further induces isomorphisms between the corresponding toric varieties.

Application 2.18. We apply Theorem 2.6 and Proposition 2.7 to weighted projective planes $\mathbb{P}(a, b, c)$ in Section 5.

By [Cut91, Lem. 10], for any triple (a, b, c), there exist another triple (a', b', c') of positive integers, such that a', b', c' are pairwise coprime, and $\mathrm{Bl}_e \mathbb{P}(a, b, c)$ is isomorphic with $\mathrm{Bl}_e \mathbb{P}(a', b', c')$. Therefore we can assume that a, b, c are pairwise coprime. For every pairwise coprime triple (a, b, c), there exists at most one triple of integers $(e, f, -g)$ such that $e, f, g > 0$, $ae + bf = cg$, and $\gcd(e, f, g) = 1$, even after permuting a, b and c (See [GK16]). We say such $(e, f, -g)$ a *relation* of the triple (a, b, c), and always rearrange the triple (a, b, c) in the order such that $ae + bf = cg$.

In Example 5.6 we provide two tables of the complete lists of triples (a, b, c) with $a \leq 15$ and $w < 1$ such that the minimal degree $d' = 1$ or $d' \geq 2$. In other words, any pairwise coprime triples with $a \leq 15$ that do not appear in any of the tables have $d = d' = 0$, and give blow-ups which are MDS.

Remark 2.19. Compared with the examples given in [HKL16], we find that when $a, b, c \leq 30$, our examples of MDS form a proper subset of their examples. Specifically, when $a, b, c \leq 30$ and are pairwise coprime, there are four different cases:

(1) There exists no relation such that $w < 1$, even after permuting a, b and c.

(2) There exists a relation with $w < 1$. The reduced degree $d \geq 1$ and it belongs to González and Karu's non-examples.

(3) There exists a relation with $w < 1$. The reduced degree $d \geq 1$, and it does not belong to González and Karu's non-examples. There are only four of them $(8, 13, 15)$, $(8, 13, 25)$, $(15, 19, 29)$ and $(15, 26, 29)$. For those four triples, $d = d'$ and $d' \cdot s_2 \in \mathbb{Z}$ (See the table below).

(4) There exists a relation with $w < 1$ and $d = 0$, hence giving MDS blow-ups.

(a,b,c)	$(8,13,15)$	$(8,13,25)$	$(15,19,29)$	$(15,26,29)$
minimal degree	3	2	3	4
reduced degree	3	2	3	4
s_2	2/3	1/2	2/3	3/4

It is worth pointing out that the examples of MDS by [HKL16] not only include all triples in cases (3) and (4), but also some in (1), for example, $\mathbb{P}(7,18,19)$.

On the other side, Theorem 2.14 provides the following new non-examples, all of minimal degree $d'=5$, including:

(1) $a=11, f=1, g=7$, such that $49/11 < b/c < 41/9$. The first examples are $(11,58,13),(11,140,31),(11,157,35),\cdots$.
(2) $a=13, f=1, g=8$, such that $64/13 < b/c < 46/9$. The first examples are $(13,84,17),(13,149,30),(13,157,31),\cdots$.
(3) $a=14, f=1, g=9$, such that $81/14 < b/c < 53/9$. The first examples are $(14,181,31),(14,309,53),(14,331,57),\cdots$.
(4) $\cdots$

3. The Interpolation Problem

We recall the following set-up from [GK16]. Fix rational numbers $s_1 < s_2 < s_3$. Consider the unique triangle Δ_0 given by slopes (s_1,s_2,s_3), such that one vertex of Δ_0 at $(0,0)$ and the opposite side of $(0,0)$ passes through $(0,1)$. Let (X,H) be the polarized toric surface determined by Δ_0. That is, the normal fan of Δ_0 is the fan of X, and H is the $\mathbb{Q}$-Cartier divisor corresponding to the polytope Δ_0. Then it follows that $H^2 = w$, which equals to twice the area of Δ_0.

Now $\mathrm{Pic}(X)\otimes\mathbb{Q}$ is generated by H. Let $\pi : X' \to X$ be the blow-up of X at the torus identity point e. Let E be the exceptional divisor of X'. Then $\mathrm{Pic}(X')\otimes\mathbb{Q}$ is two-dimensional, generated by the classes of $H' = \pi^*H$ and E. :

The section $1-y$ in $\mathcal{O}_X(H)$ defined by the lattice points $(0,0)$ and $(0,1)$ (both on sides of Δ_0) gives us an effective divisor in the class $[H]$ of X. Define C as the proper transform of this section. Then C is irreducible. We have $C \in [H'-E]$ and $C^2 = w-1 < 0$. By [KM08, Lem. 1.22], $\overline{\mathrm{NE}}(X')$ has extremal rays $\mathbb{R}_{\geq 0}[C]$ and $\mathbb{R}_{\geq 0}[E]$. $\mathrm{Nef}(X')$ is dual to $\overline{\mathrm{NE}}(X')$ under the pairing given by the intersection product, hence, spanned by the extremal rays $\mathbb{R}_{\geq 0}[H'-wE]$ and $\mathbb{R}_{\geq 0}[H']$.

By the Theorem of Zariski-Fujita [Laz04, Rem. 2.1.32], a movable divisor on a normal projective surface is semiample. Hence, X' is a MDS if and only if the ray $\mathbb{R}_{\geq 0}[H'-wE]$ contains a semiample divisor, which is further equivalent to the following: for some $k>0$, some effective Cartier divisor $F \in [kH'-kwE]$ does not contain C as a component. Note that for any integer $r>0$, such F exists if and only if some $G \in [krH'-krwE]$ exists which does not contain C as a component. Hence we can replace the triangle Δ_0 by $r\Delta_0$. In particular, recall $\Delta_1 = m\Delta_0$ is the smallest good lattice triangle of the given slopes. We will then replace Δ_0 by Δ_1.

An effective divisor F in the class $[km(H'-wE)]$ is defined by the Laurent polynomial

$$f(x,y)=\sum_{(i,j)\in k\Delta_1\cap\mathbb{Z}^2} b_{ij}x^iy^j,$$

whose partial derivatives up to order $kmw-1$ all vanish at $e=(1,1)$. Notice that the curve C passes through the two torus invariant points, which corresponds to the two vertices kp and kq of $k\Delta_1$ different from $(0,0)$. As pointed out in [GK16], f vanishes on C if and only if f vanishes at kp or kq, which is equivalent to that the coefficient b_{kp} or b_{kq} is zero. Combining these results, one has the following result:

Lemma 3.1 ([GK16]). *The blow-up X' is a MDS if and only if there exists an integer $k>0$, and some effective divisor F in the class $[km(H'-wE)]$ given by $f(x,y)=\sum_{(i,j)\in k\Delta_1\cap\mathbb{Z}^2} b_{ij}x^iy^j$, such that the coefficients b_{kp} and b_{kq} are nonzero.*

Further González and Karu proved that if there exists a derivative D with order up to $kmw-1$, which vanishes at every monomial x^iy^j indexed by $(i,j)\in k\Delta\cap\mathbb{Z}^2$ when evaluated at $(x,y)=(1,1)$, but does not vanish at $(i,j)=P$ or Q, then all $f(x,y)$ whose partial derivatives up to order $kmw-1$ will vanish at $e=(1,1)$ must have coefficients b_{kp} or b_{kq} zero. In fact, this can be translated into the following statements.

Lemma 3.2 (See also [Cas16]). *Fix an integer $k>0$ and a vertex $p\neq(0,0)$ of Δ_1. Then the following two conditions are equivalent:*

(1) For every

$$f(x,y)=\sum_{(i,j)\in k\Delta_1\cap\mathbb{Z}^2} b_{ij}x^iy^j,$$

such that the partial derivatives up to order $kmw-1$ all vanish at $e=(1,1)$, $f(x,y)$ has zero coefficient b_{kp};

(2) There exists a derivative D of order up to $kmw-1$,

$$D=\sum_{0\leq u,v\leq kmw-1}\alpha_{u,v}\partial_x^u\partial_y^v,$$

such that $D(x^iy^j)\mid_{(1,1)}=0$ for all $(i,j)\in k\Delta_1\cap\mathbb{Z}^2$ except kp, and $D(x^iy^j)\mid_{(1,1)}\neq 0$ at kp.

Proof. Consider the $\mathbb{C}$-vector space U of all $f(x,y)=\sum_{(i,j)\in k\Delta_1\cap\mathbb{Z}^2} b_{ij}x^iy^j$, and the $\mathbb{C}$-vector space V of all derivatives $D=\sum_{0\leq u,v\leq kmw-1}\alpha_{u,v}\partial_x^u\partial_y^v$. There is a natural pairing $\langle\cdot,\cdot\rangle: U\times V\to\mathbb{C}$ by $\langle f,D\rangle=D(f)\mid_{(x,y)=(1,1)}$, which induces natural morphisms $\phi: V\to U^*$ and $\tau: U\to V^*$. Let $W\subset U$ be the codimension-one subspace of U spanned by x^iy^j with $(i,j)\in k\Delta_1\cap\mathbb{Z}^2$ except kp. Define $W^0:=\{D\in V\mid \langle f,D\rangle=0,\ \text{for all } f\in W\}$. Then condition (1) in Proposition 3.2 is equivalent to $\ker\tau\subseteq W$, and condition (2) is equivalent to that $\ker\phi\subsetneqq W^0$. So we need only show $\ker\tau\subseteq W$ if and only if $\ker\phi\subsetneqq W^0$.

Indeed, if there exists a $D_0\in W^0$ but $D_0\notin\ker\phi$. Since W is codimension-one in U, $U=W\oplus\mathbb{C}x$ for some $x\in U-W$, then $\langle x,D_0\rangle\neq 0$. Now for every $f\in\ker\tau$, $f=zx+g$ for some $z\in\mathbb{C}, g\in W$. Then $\langle zx,D_0\rangle=\langle f,D_0\rangle-\langle g,D_0\rangle=0$, hence $z=0$, so $f\in W$. Therefore $\ker\tau\subseteq W$. Conversely, suppose $\ker\tau\subseteq W$, then $\tau(W)\cong W/\ker\tau$ is a subspace in V^*, which is a proper subspace of $\tau(U)\cong U/\ker\tau$. Let $\langle\cdot,\cdot\rangle_v$ be the dual paring between V and V^*. Then $\langle\cdot,\cdot\rangle_v$ is perfect. Let $\tau(U)^0=\{D\in V\mid\langle D,d\rangle_v=0,\ \text{for all } d\in\tau(U)\}$, and

$\tau(W)^0 = \{D \in V \mid \langle D, d\rangle_v = 0, \text{ for all } d \in \tau(W)\}$. Then we find $\tau(U)^0 \subsetneqq \tau(W)^0 = W^0$. Since $\ker\phi \subseteq \tau(U)^0$, we conclude that $\ker\phi \subsetneqq \tau(W)^0$. □

Proposition 3.3. *The blow-up X' is not a MDS if and only if for every integer $k > 0$, there exists a nonzero vertex p of Δ_1 and a derivative D of order up to $kmw - 1$:*

$$D = \sum_{0\le u,v\le kmw-1} \alpha_{u,v}\partial_x^u\partial_y^v,$$

such that $D(x^iy^j)\mid_{(1,1)}= 0$ for all $(i, j) \in k\Delta_1 \cap \mathbb{Z}^2$ except kp, and $D(x^iy^j)\mid_{(1,1)}\neq 0$ at kp.

Proof. The sufficiency is clear from Lemma 3.1 and Proposition 3.2. For the necessity, suppose X' is not a MDS, then there exists $k > 0$, such that every such $f(x, y)$ has zero coefficients b_{kp} or b_{kq}. Because all such $f(x, y)$ form a vector space, it must be the case that either $b_{kp} = 0$ for all $f(x, y)$ or $b_{kq} = 0$ for all $f(x, y)$. Without loss of generality, we assume $b_{kp} = 0$ for all $f(x, y)$. Then Lemma 3.2 implies the existence of such derivative D, hence the necessity follows. □

Finally, let M be the matrix associated to the paring $\langle\cdot,\cdot\rangle$ in the proof of Proposition 3.2. If we adopt the notation that $(a)_b := a(a-1)\cdots(a-b+1)$, then $M_{(i,j),(u,v)} = (i)_u(j)_v$. (See Section 6)

Lemma 3.4. *For every $n > 0$, the vector space spanned by polynomials $\{(x)_u \cdot (y)_v \mid 0 \le u + v \le n\}$ has a basis $\{x^uy^v \mid 0 \le u + v \le n\}$.*

Proof. We need only show every x^uy^v with $u + v \le n$ is generated by $\{(x)_u(y)_v \mid 0 \le u+v \le n\}$. We make inductions on $k = u+v$. For $k = 0$ the lemma is already true. Suppose this claim holds for $1, \cdots, k$. For $s + t = k + 1$, $x^sy^t - (x)_s(y)_t$ is a polynomial of degree at most k, and hence is spanned by $\{(x)_u(y)_v \mid 0 \le u + v \le k\}$ by the induction hypothesis. Hence x^sy^t is spanned by $\{(x)_u(y)_v \mid 0 \le u + v \le k + 1\}$. □

Proof of Proposition 2.1. By Lemma 3.4, the existence of such derivative D in Proposition 3.2 is equivalent to the existence of a polynomial

$$f = \sum_{u+v\le kmw-1} \alpha_{u,v}x^uy^v,$$

which vanishes at every $(i, j) \in k\Delta_1 \cap \mathbb{Z}^2$ except the vertex kp. This polynomial f defines a curve Y in $\mathbb{P}^2$ with degree up to $kmw - 1$, such that Y passes through the complement of kp in $k\Delta_1 \cap \mathbb{Z}^2$ but does not pass kp. □

4. The Reduced Degree

We prove Theorem 2.5 and 2.6 in this section. For rational slopes $s_1 < s_2 < s_3$, recall the *width* of s_1, s_2, s_3 equals $w(s_1, s_2, s_3) = 1/(s_2 - s_1) + 1/(s_3 - s_2)$, and we assume $w < 1$. As defined in Section 2, Δ_0 is the smallest triangle with the given slopes, such that if one vertex is placed at $(0, 0)$, then the opposite side passes through $(0, 1)$. Next, Δ_1 is the smallest good lattice triangle with the given slopes. Finally, let m be the integer such that $\Delta_1 = m\Delta_0$.

Let the leftmost and rightmost vertices of Δ_1 be $p = (x_1, y_1)$ and $q = (x_2, y_2)$ respectively. Then it is easy to show that

$$x_1 = \frac{1}{s_1 - s_2}, \quad x_2 = \frac{1}{s_3 - s_2}, \quad y_1 = \frac{s_1}{s_1 - s_2}, \quad y_2 = \frac{s_3}{s_3 - s_2}.$$

Recall that the reduced degree of s_1, s_2, s_3 equals to the largest nonnegative integer d, such that the number of columns in Δ_1 containing at most d points equals to d (Definition 2.2)

We define $l_k = \lfloor ks_2 \rfloor - \lceil ks_1 \rceil + 1$, and $r_k = \lfloor ks_3 \rfloor - \lceil ks_2 \rceil + 1$, for $k = 1, 2, \cdots$.

Definition 4.1. $\pi(n)$ is the total number of entries in the two sequences $\{l_k\}_{k\geq 1}$ and $\{r_k\}_{k\geq 1}$ which do not exceed n.

From a geometrical view, l_k and r_k are the number of lattice points on the k-th column from the left and right of a lattice triangle $k'\Delta_1$ for all k' such that $k'\Delta_1$ has as least k columns on left and right of the line $\{x = 0\}$. It follows that $\pi(n)$ is the number of columns with no more than n points in a sufficiently large lattice triangle $k'\Delta_1$. Finally, it is clear that $\pi(0) = 0$.

Proposition 4.2. *Given any rational numbers $s_1 < s_2 < s_3$ such that the width $w < 1$. Assume $\Delta_1 = m\Delta_0$. We have*

(1) $\pi(n) \leq n$ for all $n = 0, 1, 2, \cdots$.
(2) If $n \geq mw$, then $\pi(n) < n$.

Proof. See Section 7.

Proof of Theorem 2.5. For any $k > 0$ fixed, we run the reduction for the triangle $k\Delta_1$ by Bézout's Theorem as described in Definition 2.2. By Proposition 4.2, $\Delta_1 = m\Delta_0$, so the width of $k\Delta_1$ equals to $W := kmw$. Let the reduced degree of $k\Delta_1$ be d_k.

By Definition 2.3, we denote the reduced degree of s_1, s_2, s_3 as d. We need only show that all $d_k = d$ for $k = 1, 2, 3, \cdots$.

By sorting the numbers of lattice points in each column of $k\Delta_1$ increasingly, we obtain a sequence $U = \{u_i\}_{i=1}^{W-1}$ of length $W - 1 = kmw - 1$ (because we exclude the two vertices kp and kq) and the last term of U equals to $km + 1$. Hence at the i-th step, we are comparing u_{W-i} with $W - i$. So d_k equals to the largest integer $d_k < km + 1$ such that $u_{d_k} \leq d_k$, or equivalently, the largest integer d_k such that there exist at least d_k columns of $\leq d_k$ integer points in $k\Delta_1$. Notice the number of columns in $k\Delta_1$ of at most s integer points is at most $\pi(s)$, which is further bounded by s by Proposition 4.2. Hence there can exist at most d_k columns of $\leq d_k$ integer points in $k\Delta_1$. Hence d_k is the largest integer d_k such that there exist exactly d_k columns of $\leq d_k$ integer points in Δ_1, which by Definition 2.3 is the reduced degree d of s_1, s_2, s_3.

Recall the definition of $\pi(n)$ again, we find that $d = \pi(d)$. By Proposition 7.4, we have $d = \pi(d) < w(d + 1)$. Since $w < 1$, it follows that $d < w/(1 - w)$. □

Corollary 4.3. *Given rational slopes $s_1 < s_2 < s_3$ with width $w < 1$. Denote the two nonzero vertices of the smallest lattice triangle Δ_1 as p and q. Let d be the reduced degree of s_1, s_2, s_3. Let X be the toric surface determined by Δ_1. Then the blow-up $X' = \mathrm{Bl}_e X$ at the torus identity point e is not a MDS if and only if for every integer $k > 0$ there exists a curve Y in $\mathbb{P}^2$, of degree up to d, and a nonzero vertex p of Δ_1, such that*

(1) Y passes through all the points in those columns of $k\Delta_1$ of at most d points, and the vertex kq, but
(2) Y does not pass through kp.

Proof. Suppose the curve Y of degree $\leq W-1 = kmw-1$ in Theorem 2.1 exists. Then we can run the reduction via Bézout's Theorem and conclude that there exists a curve Y_d of degree $\leq d$ satisfying the conditions in the corollary, when d is the reduced degree.

Conversely, given a curve Y_d of degree $\leq d$, passing through all points in columns with at most d points, and the vertex kq, but not kp. Then there are exactly $W-d-1$ columns left in $k\Delta_1$. As a result, the union of Y_d with all the lines lying under every such column is a curve of degree $\leq W-1$, passing through all lattice points in $k\Delta_1$ but not kp. Therefore the theorem holds. □

Proof of Theorem 2.6. When $d=0$, such curve Y in Corollary 4.3 does not exist, hence X' is a MDS.

For (2), suppose $d'=1$. We claim $s_2 \notin \mathbb{Z}$. Indeed, if $s_2 \in \mathbb{Z}$, without loss of generality we can assume $s_2 = 0$. Since $w<1$, we find $s_1 < -1$ and $s_3 > 1$. Therefore $l_1 \geq 2$ and $r_1 \geq 2$. By Lemma 7.3, $\{l_k\}$ and $\{r_k\}$ are both increasing. So $\pi(1)=0$, which contradicts to $d'=1$.

Now $d'=1$ implies $l_1=1$ or $r_1=1$. By symmetry we need only prove for the case $l_1=1$. Now In $k\Delta_1$, the first column from the left contains only one point t. Since $s_2 \notin \mathbb{Z}$, t is not on the side between kq and kp. That is, kp, kq and t are not collinear. Then there exists a curve Y whose irreducible components are

- The line L through kq and t;
- All vertical lines lying under the rest columns of at most d points, except the first column from the left,

where d is the reduced degree. Then this curve Y has degree $1+(d-1)\cdot 1 = d$. Further Y does not passes through kp since L does not passes through kp and none of the other irreducible components pass through kp. Hence X' is not a MDS by Corollary 4.3.

Finally if $d=1$, then $d' \leq d = 1$, and $d' \neq 0$. So $d'=1$. So it follows from (2). □

5. Examples from Weighted Projective Planes

In this section we apply Theorems 2.6 and Proposition 2.7 to blow-ups of weighted projective planes.

Throughout this section, we assume that a,b,c are pairwise coprime, such that there exists a relation $(e,f,-g)$ of (a,b,c) (See Application 2.18). That is, $ae+bf-cg=0$, with e,f,g positive integers and $\gcd(e,f,g)=1$.

Proposition 5.1. *For every pairwise coprime triple (a,b,c) with a relation $(e,f,-g)$, there exists a unique integer r such that $1 \leq r \leq g$, $g \mid er-b$ and $g \mid fr+a$. Let Δ be the triangle with slopes*

$$s_1 = \frac{er-b}{eg}, \quad s_2 = \frac{r}{g}, \quad s_3 = \frac{fr+a}{fg}.$$

Then $\mathbb{P}(a,b,c)$ is isomorphic to the toric variety X_Δ.

Proof. The integer r is the solution of the system of congruence equations:

$$\begin{cases} ex \equiv b & \pmod g \\ fx \equiv -a & \pmod g. \end{cases} \tag{1}$$

Let $\alpha = \gcd(e,g)$, $\beta = \gcd(f,g)$, with $e = \alpha e_0$, $f = \beta f_0$. Since $\gcd(e,f,g)=1$, we find $\gcd(\alpha,\beta) = \gcd(\alpha,f) = \gcd(\beta,e) = 1$. Hence $\alpha\beta \mid g$. Further $\alpha \mid cg-ae = bf$. Since

$\gcd(\alpha, f) = 1$, $\alpha \mid b$. Similarly $\beta \mid a$. Now let $g = \alpha\beta g_0$, $b = \alpha b_0$, and $a = \beta a_0$. It follows that (1) is equivalent to

$$\begin{cases} e_0 x \equiv b_0 & \pmod{\beta g_0} \\ f_0 x \equiv -a_0 & \pmod{\alpha g_0}. \end{cases} \tag{2}$$

By Chinese reminder theorem, the system (2) has a solution if and only if $e_0^{-1} b_0 \equiv -f_0^{-1} a_0 \pmod{\gcd(\beta g_0, \alpha g_0)}$, where the inverse of g_0(or f_0) is taken in the multiplicative group $(\mathbb{Z}/\beta g_0 \mathbb{Z})^\times$ (respectively, in $(\mathbb{Z}/\alpha g_0 \mathbb{Z})^\times$). Notice that $\gcd(\beta g_0, \alpha g_0) = g_0$. Then $(e_0^{-1} b_0 + f_0^{-1} a_0) e_0 f_0 = f_0 b_0 + e_0 a_0 = c g_0$. Hence $(e_0^{-1} b_0 + f_0^{-1} a_0) e_0 f_0 \equiv 0 \pmod{g_0}$. Since $\gcd(e_0, g_0) = 1$, and $\gcd(f_0, g_0) = 1$, we find $(e_0^{-1} b_0 + f_0^{-1} a_0) \equiv 0 \pmod{g_0}$. Finally, this solution is unique modulo $\mathrm{lcm}(\alpha g_0, \beta g_0)$, which equals to g.

By the definition of s_i in the Proposition, we have

$$\vec{n}_1 = (\frac{er - b}{g}, -e), \quad \vec{n}_2 = (-r, g), \quad \vec{n}_3 = (\frac{fr + a}{g}, -f)$$

are normal vectors of the sides of Δ. They satisfy the relation that $a\vec{n}_1 + c\vec{n}_2 + b\vec{n}_3 = \vec{0}$. It remains to show that $\vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ span the lattice $N = \mathbb{Z}^2$, and are all primitive vectors.

In order to generate the lattice $\mathbb{Z}^2$, it suffices to show that $\vec{e}_1$ and $\vec{e}_2$ are linear combinations of $\vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ with integer coefficients. Since a, b, c are assumed as pairwise coprime and $ae \equiv -bf \pmod{c}$, we have $a^{-1} f \equiv -b^{-1} e \pmod{c}$ (inverses taken in $(\mathbb{Z}/c\mathbb{Z})^\times$), so the following system of equation of y:

$$\begin{cases} ay \equiv f & \pmod{c} \\ by \equiv -e & \pmod{c}. \end{cases} \tag{3}$$

has a unique solution (which we still call y) mod c. Therefore, there exist integers x, z such that $xc = ay - f$ and $zc = by + e$. Then direct calculation shows $x\vec{n}_1 + y\vec{n}_2 + z\vec{n}_3 = (1, 0) = \vec{e}_1$, using that $ae + bf = cg$.

On the other hand, the following system of equation of y':

$$\begin{cases} ay' \equiv (fr + a)/g & \pmod{c} \\ by' \equiv -(er - b)/g & \pmod{c}. \end{cases} \tag{4}$$

has a unique solution y' mod c. This follows from that $a(er - b)/g + b(fr + a)/g = c \equiv 0 \pmod{c}$. There exist integers x', z' such that $x'c = ay' - (fr + a)/g$ and $z'c = by' + (er - b)/g$. It can be calculated then that $x'\vec{n}_1 + y'\vec{n}_2 + z'\vec{n}_3 = (0, 1) = \vec{e}_2$.

It remains to show that $\vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ are primitive vectors in $\mathbb{Z}^2$. Indeed, suppose $d > 0$ with $d \mid r$ and $d \mid g$, then $d \mid er - b$ and $d \mid fr + a$. So $d \mid b$ and $d \mid a$, therefore $d = 1$. This shows $\gcd(r, g) = 1$, so $\vec{n}_2$ is primitive. For $\vec{n}_1$, suppose $t > 0$ with $t \mid (er - b)/g$ and $t \mid e$. Then $t \mid er - b$, so $t \mid b$. On the other hand, since $f \cdot (er - b)/g - e \cdot (fr + a)/g = -c$ and $(fr + a)/g$ is an integer, we have $t \mid c$. By assumption, b and c are coprime, so $t = 1$, and $\vec{n}_2$ is primitive. The result for $\vec{n}_3$ follows from symmetry. ∎

Now if the slopes are given by Proposition 5.1, then we have

$$w = \frac{1}{s_2 - s_1} + \frac{1}{s_3 - s_2} = \frac{eg}{b} + \frac{fg}{a} = \frac{cg^2}{ab}.$$

Therefore, all our results and definitions apply for weighted projective surfaces $\mathbb{P}(a,b,c)$ such that $w = cg^2/ab < 1$.

Corollary 5.2. *For every pairwise coprime triple (a,b,c) with a relation $(e,f,-g)$ such that $g = 1$ and $c < ab$, the reduced degree of the corresponding slopes is zero, and the blow-up X' is a MDS.*

Proof. If $g = 1$ and $c < ab$, then the width $w = c/ab < 1$. By Proposition 5.1, $r = 1$, so $s_2 = 1$. By Corollary 2.8, the reduced degree equals zero, and X' is a MDS. □

Remark 5.3. The MDS claim of Corollary 5.2 also follows from Cutkosky's results. It is shown in [Cut91] that if $-K_{X'}$ is a big divisor, then X' is a MDS. We claim if the width $w < 1$, then $-K_{X'}$ is big if and only if $cg < a+b+c$. Therefore if $w < 1$ and $cg < a+b+c$, then X' is a MDS. In particular, when $g = 1$, $-K_{X'}$ is big, and X' is a MDS.

Indeed, let $A = \mathcal{O}_X(1)$. Let H be defined as in Section 3. Then $H = \alpha A$ for some $\alpha \in \mathbb{Q}$. Since $A^2 = 1/abc$ [Cut91, Lem. 9] and $H^2 = w = cg^2/ab$, we find $r = cg$. Therefore $H = cgA$. The canonical divisor of X is $K_X \equiv \mathcal{O}_X(-a-b-c) = -(a+b+c)A = -\frac{a+b+c}{cg}H$. Hence $-K_{X'} \equiv \frac{a+b+c}{cg}H' - E$.

In Section 3, we showed that when the width $w < 1$, there exists a negative curve C on X' in the class $H' - E$. Therefore C and E span the two extremal rays of $\overline{\mathrm{NE}}(X')$. Now $-K_{X'}$ is big if and only if $-K_{X'}$ is in the interior of $\overline{\mathrm{NE}}(X')$. That is, $cg < a+b+c$. □

Definition 5.4. Let a, f, g, r be positive integers such that $g \mid fr + a$. Define $\Phi(a,f,g,r)$ as the set of pairwise coprime triples (a,b,c) such that there exists a relation (e,f,g) with $w < 1$.

We further use $\Phi(a,f,g,r)_0$, $\Phi(a,f,g,r)_1$ and $\Phi(a,f,g,r)_{\geq 2}$ to denote the subsets of $\Phi(a,f,g,r)$, consisting of triples of minimal degree 0, 1, and at least 2 respectively.

It is shown in [Cut91] and [Sri91] that when one of a,b,c is ≤ 4 or equal to 6, $-K_{X'}$ is big, and $X' = \mathrm{Bl}_e\,\mathbb{P}(a,b,c)$ is MDS. So the smallest unknown case is when one of a,b,c is 5. We apply Proposition 2.7 to classify all $\mathbb{P}(5,b,c)$ with $w < 1$, by whether the reduced degree d is zero, one, or at least 2.

Example 5.5. Assume $a = 5$ (equivalently $b = 5$ by symmetry).

An important observation emerges that $fg/a < w = fg/a + eg/b < 1$, so that $fg < a = 5$. Therefore, there are only finite many choices of f and g. The case $g = 2, f = 2$ contradicts to the assumption that $\gcd(e,f,g) = 1$. On the other hand, the case $g = 1$ gives reduced degree zero by Corollary 5.2.

So the remaining cases are: 1). $f = 1, g = 2$; 2). $f = 1, g = 3$; 3). $f = 1, g = 4$.

Case I. $f = 1, g = 2$. Here $r = 1$, so $s_2 = 1/2$, and $s_3 = 3$. In addition, $\gamma = 4$.

We can check that (4) of Proposition 2.7 is satisfied. So Proposition 2.7 shows that the reduced degree is zero if and only if $s_1 \leq -6/5$. Notice when s_2 and s_3 are fixed, the following are equivalent:

(1) $s_1 \leq -6/5$;
(2) the width $w \leq 84/85$;
(3) $b/c \geq 17/21$.

Further $\Phi(5,1,2,1)_1 = \emptyset$ since $l_1 \geq 2$ and $r_1 \geq 2$. As a result, a triple in this case has $d' \geq 2$ if and only if $84/85 < w < 1$, or equivalently $4/5 < b/c < 17/21$. Hence the above argument shows

$$\begin{aligned}
\Phi(5,1,2,1)_0 =& \{(5,b,c) \mid b/c \geq 17/21 \text{ and } 5 \mid 2c-b\}.\\
\Phi(5,1,2,1)_{\geq 2} =& \{(5,b,c) \mid 4/5 < b/c < 17/21 \text{ and } 5 \mid 2c-b\}.\\
\Phi(5,1,2,1)_1 =& \emptyset.
\end{aligned}$$

It is easy to prove by slopes that all the triple in $\Phi(5,1,2,1)_{\geq 2}$ have minimal degree 8.

Calculation shows the triples with smallest b and c in $\Phi(5,1,2,1)_{\geq 2}$ are:

$$(5,37,46), (5,54,67), (5,57,71), (5,71,88), \ldots.$$

Case II. $f = 1, g = 3$. We have $r = 1$, $s_2 = 1/3$, $s_3 = 2$, and $\gamma = 9$.

(4) of Proposition 2.7 is satisfied. The reduced degree is zero if and only if $s_1 \leq -11/5$, or equivalently

(1) the width $w \leq 189/190$;
(2) $b/c \geq 38/21$.

We have

$$\begin{aligned}
\Phi(5,1,3,1)_0 =& \{(5,b,c) \mid b/c \geq 38/21 \text{ and } 5 \mid 3c-b\}.\\
\Phi(5,1,3,1)_{\geq 2} =& \{(5,b,c) \mid 9/5 < b/c < 38/21 \text{ and } 5 \mid 3c-b\}.\\
\Phi(5,1,3,1)_1 =& \emptyset.
\end{aligned}$$

Similarly, it is easy to show that all the triples in $\Phi(5,1,3,1)_{\geq 2}$ have minimal degree 12. The smallest examples in $\Phi(5,1,3,1)_{\geq 2}$ are

$$(5,83,46), (5,121,67), (5,128,71), \cdots.$$

Case III. $f = 1, g = 4$. We have $r = 3$, $s_2 = 3/4$, $s_3 = 2$, and $\gamma = 16$.

In this case, Proposition 2.7 (1) shows that all triples have reduced degree zero. So $\Phi(5,1,4,3) = \Phi(5,1,4,3)_0$.

Conclusion. When $5, b, c$ are coprime and $w = cg^2/(5b) < 1$, X' is a MDS unless in the following two cases.

(i) $4/5 < b/c < 17/2$ and $5 \mid 2c - b$;
(ii) $9/5 < b/c < 38/21$ and $5 \mid 3c - b$.

Example 5.6. Suppose (4) of Proposition 2.7 holds. It follows from Proposition 2.7 that every nonempty $\Phi(a,f,g,r)_\alpha$ has the form

$$\Phi(a,f,g,r)_\alpha = \{(a,b,c) \mid b/c \in I \text{ and } a \mid cg - bf\},$$

where $\alpha \in \{1, \geq 2\}$, and I is an interval in $(g^2/a, \infty)$.

Therefore it suffices to determine the range I of b/c for each set $\Phi(a,f,g,r)_\alpha$.

For all the possible combinations of (a,f,g,r) such that $a \leq 15$, we find by a computer program that (4) of Proposition 2.7 holds. Therefore, we can apply Proposition 2.7 when $a \leq 15$. We use a computer program to obtain the following tables (Tables 1, 2) of all nonempty $\Phi(a,f,g,r)_1$ and $\Phi(a,f,g,r)_{\geq 2}$, for $a \leq 15$.

In other words, any pairwise coprime triple (a, b, c) such that $a \leq 15$, and $w < 1$, which appear in neither of the two tables, gives a blow-up which is MDS.

$(a; f, g; r)$	range of b/c	$(a; f, g; r)$	range of b/c
(7; 1, 2; 1)	$\frac{4}{7} < \frac{b}{c} < \frac{3}{5}$	(13; 1, 3; 2)	$\frac{9}{13} < \frac{b}{c} < \frac{5}{6}$
(7; 3, 2; 1)	All	(13; 4, 3; 2)	All
(7; 2, 3; 1)	All	(13; 1, 4; 3)	$\frac{16}{13} < \frac{b}{c} < \frac{7}{5}$
(9; 1, 2; 1)	$\frac{4}{9} < \frac{b}{c} < \frac{1}{2}$	(13; 3, 4; 1)	All
(10; 4, 2; 1)	All	(13; 2, 5; 1)	All
(10; 1, 3; 2)	$\frac{9}{10} < \frac{b}{c} < 1$	(14; 2, 2; 1)	$\frac{2}{7} < \frac{b}{c} < \frac{3}{10}$
(10; 2, 4; 1)	All	(14; 6, 2; 1)	All
(11; 1, 2; 1)	$\frac{4}{11} < \frac{b}{c} < \frac{3}{7}$	(14; 1, 3; 1)	$\frac{9}{14} < \frac{b}{c} < \frac{2}{3}$
(11; 5, 2; 1)	All	(14; 4, 3; 1)	All
(11; 2, 5; 2)	All	(15; 1, 2; 1)	$\frac{4}{15} < \frac{b}{c} < \frac{1}{3}$
(12; 3, 3; 1)	All	(15; 7, 2; 1)	All
(13; 1, 2; 1)	$\frac{4}{13} < \frac{b}{c} < \frac{3}{8}$	(15; 2, 7; 3)	All
(13; 5, 2; 1)	All		

TABLE 1. All nonempty $\Phi(a, f, g, r)_1$ such that $a \leq 15$. Every pairwise coprime triple (a, b, c) with a relation $(e, f, -g)$ such that $w < 1$, which appear in this table, has minimal degree 1, and gives a blow-up which is not a MDS.

$(a;f,g;r)$	range of b/c	$(a;f,g;r)$	range of b/c	$(a;f,g;r)$	range of b/c
(5; 1, 2; 1)	$\frac{4}{5}<\frac{b}{c}<\frac{17}{21}$	(11; 1, 4; 1)	$\frac{16}{11}<\frac{b}{c}<\frac{14}{9}$	(14; 4, 2; 1)	$\frac{2}{7}<\frac{b}{c}<\frac{5}{17}$
(5; 1, 3; 1)	$\frac{9}{5}<\frac{b}{c}<\frac{38}{21}$	(11; 1, 5; 4)	$\frac{25}{11}<\frac{b}{c}<\frac{104}{45}$	(14; 2, 3; 2)	$\frac{9}{14}<\frac{b}{c}<\frac{19}{29}$
(7; 1, 3; 2)	$\frac{9}{7}<\frac{b}{c}<\frac{38}{29}$	(11; 1, 6; 1)	$\frac{36}{11}<\frac{b}{c}<\frac{149}{45}$	(14; 2, 4; 1)	$\frac{8}{7}<\frac{b}{c}<\frac{67}{58}$
(7; 1, 4; 1)	$\frac{16}{7}<\frac{b}{c}<\frac{67}{29}$	(11; 1, 7; 3)	$\frac{49}{11}<\frac{b}{c}<\frac{41}{9}$	(14; 2, 4; 3)	$\frac{8}{7}<\frac{b}{c}<\frac{11}{9}$
(7; 1, 5; 3)	$\frac{25}{7}<\frac{b}{c}<\frac{18}{5}$	(11; 1, 8; 5)	$\frac{64}{11}<\frac{b}{c}<\frac{327}{56}$	(14; 1, 5; 1)	$\frac{25}{14}<\frac{b}{c}<\frac{17}{9}$
(8; 2, 2; 1)	$\frac{1}{2}<\frac{b}{c}<\frac{5}{9}$	(11; 1, 9; 7)	$\frac{81}{11}<\frac{b}{c}<\frac{52}{7}$	(14; 2, 5; 3)	$\frac{25}{14}<\frac{b}{c}<\frac{9}{5}$
(8; 1, 3; 1)	$\frac{9}{8}<\frac{b}{c}<\frac{11}{9}$	(12; 1, 5; 3)	$\frac{25}{12}<\frac{b}{c}<\frac{29}{13}$	(14; 1, 9; 4)	$\frac{81}{14}<\frac{b}{c}<\frac{53}{9}$
(8; 1, 5; 2)	$\frac{25}{8}<\frac{b}{c}<\frac{29}{9}$	(12; 1, 7; 2)	$\frac{49}{12}<\frac{b}{c}<\frac{55}{13}$	(14; 1, 11; 8)	$\frac{121}{14}<\frac{b}{c}<\frac{26}{3}$
(9; 1, 4; 3)	$\frac{16}{9}<\frac{b}{c}<\frac{67}{37}$	(13; 3, 2; 1)	$\frac{4}{1}<\frac{b}{c}<\frac{5}{14}$	(15; 3, 2; 1)	$\frac{4}{15}<\frac{b}{c}<\frac{17}{63}$
(9; 1, 5; 1)	$\frac{25}{9}<\frac{b}{c}<\frac{104}{37}$	(13; 2, 3; 1)	$\frac{9}{13}<\frac{b}{c}<\frac{7}{9}$	(15; 3, 3; 1)	$\frac{3}{5}<\frac{b}{c}<\frac{38}{63}$
(9; 1, 7; 5)	$\frac{49}{9}<\frac{b}{c}<\frac{11}{2}$	(13; 1, 5; 2)	$\frac{25}{13}<\frac{b}{c}<\frac{19}{9}$	(15; 3, 3; 2)	$\frac{3}{5}<\frac{b}{c}<\frac{11}{16}$
(10; 2, 2; 1)	$\frac{2}{5}<\frac{b}{c}<\frac{17}{42}$	(13; 1, 6; 5)	$\frac{36}{13}<\frac{b}{c}<\frac{149}{53}$	(15; 1, 4; 1)	$\frac{16}{15}<\frac{b}{c}<\frac{19}{16}$
(10; 2, 3; 1)	$\frac{9}{10}<\frac{b}{c}<\frac{19}{21}$	(13; 1, 7; 1)	$\frac{49}{13}<\frac{b}{c}<\frac{202}{53}$	(15; 1, 7; 6)	$\frac{49}{15}<\frac{b}{c}<\frac{202}{61}$
(10; 1, 7; 4)	$\frac{49}{10}<\frac{b}{c}<5$	(13; 1, 8; 3)	$\frac{64}{13}<\frac{b}{c}<\frac{46}{9}$	(15; 1, 8; 1)	$\frac{64}{15}<\frac{b}{c}<\frac{263}{61}$
(11; 3, 2; 1)	$\frac{4}{11}<\frac{b}{c}<\frac{5}{13}$	(13; 1, 9; 5)	$\frac{81}{13}<\frac{b}{c}<\frac{32}{5}$	(15; 1, 11; 7)	$\frac{121}{15}<\frac{b}{c}<\frac{131}{16}$
(11; 1, 3; 1)	$\frac{9}{11}<\frac{b}{c}<\frac{47}{56}$	(13; 1, 10; 7)	$\frac{100}{13}<\frac{b}{c}<\frac{47}{6}$	(15; 1, 13; 11)	$\frac{169}{15}<\frac{b}{c}<\frac{34}{3}$
(11; 2, 3; 2)	$\frac{9}{11}<\frac{b}{c}<\frac{8}{9}$	(13; 1, 11; 9)	$\frac{121}{13}<\frac{b}{c}<\frac{75}{8}$		

TABLE 2. All nonempty $\Phi(a,f,g,r)_{\geq 2}$ such that $a \leq 15$. In other word, every pairwise coprime triple (a,b,c) with a relation $(e,f,-g)$ such that $w<1$ and $a \leq 15$, which do not appear in this table and Table 1, has reduced degree zero, and gives a blow-up which is a MDS.

6. Non-examples when the Minimal Degree $d' \geq 2$

6.1. The interpolation problem of polynomials of degree d'. When the minimal degree $d' \geq 2$, the reduced degree $d \geq 2$ too. Corollary 4.3 motivates us to search for non-examples of MDS by solving the reduced interpolation problems for every $k>0$. Applying the proof of Theorem 2.6 and Theorem 2.1, we have a sufficient condition for the blow-up X' not to be a MDS:

Corollary 6.1. *Give rational numbers $s_1 < s_2 < s_3$ with width $w<1$. Let X the toric variety defined by s_1, s_2, s_3. Let X' be the blow-up of X at the torus identity point.*

Assume the minimal degree $d' \geq 2$. If for every integer $k>0$, there exists a nonzero vertex p of Δ_1 and a curve Y in $\mathbb{P}^2$, of degree up to d', such that

(1) Y passes through all the points in columns of $k\Delta_1$ containing $\leq d'$ lattice points, and the vertex kq,
(2) Y does not pass kp.

then the blow-up X' is not a MDS.

As a result, we try to solve the interpolation problems in Proposition 6.1 in search of non-examples of MDS. However, in general the reduced degree do not equals to the minimal degree. So the reverse direction of (6.1) can fail.

Recall Proposition 3.3. Suppose a curve Y in $\mathbb{C}^2$ is given by a bivariate polynomial $f(x,y) = \sum_{u+v\le n} a_{u,v}x^u y^v = 0$, of total degree n. Let $N = \binom{n+1}{2}$. Let I be a set of N distinct points in $\mathbb{C}^2$. Then we can define an $N\times N$ matrix M, whose rows are parametrized by $(i,j)\in I$, and columns parametrized by $J = \{(u,v) \mid 0 \le u+v \le n\}$, via

$$M_{(i,j),(u,v)} = i^u \cdot j^v.$$

We say in the following that $M = M_{I,J}$ is the matrix parametrized by I and J, where I, J are sets of tuples of the same size. It follows from linear algebra that Y passes through all points in I if and only $M\xi^T = 0$, where $\xi = (a_{u,v})_J$. Further if $\det M \neq 0$, then no curve of degree $\le n$ passing through all the l points.

In our case, for every $k > 0$, let I_k be the set of points in the columns of $\le d'$ points in $k\Delta_1$, together with the two vertices kp and kq. Let $J = \{(u,v) \mid 0 \le u+v \le d'\}$. We obtain a matrix M'_k parametrized by I_k and J. Now if $\det M'_k \neq 0$ for all $k \ge 0$, then there is a unique curve D of degree $\le d$ passing through $I - \{kp\}$. However, there is no curve of degree $\le d$ passing through all points in I. Hence D does not pass kq. By Corollary 6.1, X' is not a MDS. In summary, we have proved

Corollary 6.2. *If $\det M'_k \neq 0$ for all $k \ge 0$, then the blow-up X' is not a MDS.*

Remark 6.3. Corollary 6.2 motivates us to calculate $\det M'_k$. Indeed the minimal degree $d' = s + t$, where s is the number of columns of $\le d'$ points on the left, and t the number on the right. Recall that a shear translation $(x,y) \mapsto (x, y+lx)$ for $l \in \mathbb{Z}$ on I keeps the property that $M \neq 0$. Hence we shift the triangle $k\Delta_1$ so that

(1) the right-most vertex kq is at $(t, 0)$;
(2) $0 \le s_3 < 1/t$, so that the first t columns from the right are all in the first quadrant $\{(x,y) \mid x, y \ge 0\}$.

Notice it may be impossible to satisfy (2), but in all the following examples, (2) will be satisfied.

Now we introduce Dumnicki's notation from [Dum06].

Definition 6.4. Let $a_1, \cdots, a_n, u_1, \cdots, u_n \in \mathbb{Z}$. We define

$$(a_1^{\uparrow u_1}, \cdots, a_n^{\uparrow u_n}) := \bigcup_{i=1}^{n} (\{i-1\} \times \{u_i, u_i+1, \cdots, u_i + a_i - 1\}) \subset \mathbb{Z}^2.$$

For example, the set $(2^{\uparrow 2}, 1^{\uparrow 1}, 1^{\uparrow 0})$ is shown in Figure 3. The set $(3^{\uparrow 0}, 2^{\uparrow 0}, 1^{\uparrow 0})$ is shown in Figure 4.

Here we consider the case when all $u_i = 0$. In this case, $(a_1^{\uparrow 0}, \cdots, a_n^{\uparrow 0})$ have two boundaries lying on the two coordinate axis.

Recall that we say a finite set I of planar points *imposes independent conditions* on forms of degree n if the linear conditions of I on the coefficients of a polynomial of degree n are independent. Equivalently, for an interpolation problem of a polynomials of degree n

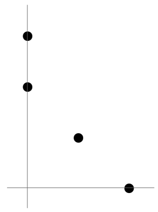

FIGURE 3. The set $(2^{\uparrow 2}, 1^{\uparrow 1}, 1^{\uparrow 0})$

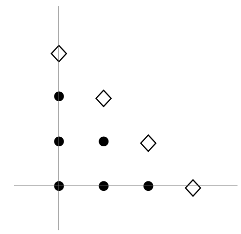

FIGURE 4. Solid points are $(3^{\uparrow 0}, 2^{\uparrow 0}, 1^{\uparrow 0})$. Diamond points index the basis $\mathcal{A}$

vanishing on I, each point in I makes the dimension of the solution space drop by exactly one.

Fixing an integer $d > 0$, we choose $\sigma = \{a_1, a_2, \cdots, a_n\}$ a subset of $\{1, 2, \cdots, d\}$ containing d, and suppose $a_1 = d > a_2 > \cdots a_n$. We claim that

Lemma 6.5. *Consider the vector space U of bivariate polynomials $f(x, y)$ of degree up to d, vanishing on the lattice set $(a_1^{\uparrow 0}, \cdots, a_n^{\uparrow 0})$ defined by σ.*

Then U have the following basis:

$$\mathcal{A} = \left\{ \binom{x}{u}\binom{y}{v} \mid (u, v) \in ((d+1)^{\uparrow 0}, \cdots, 1^{\uparrow 0}) - (a_1^{\uparrow 0}, \cdots, a_n^{\uparrow 0})) \right\}.$$

In particular, the lattice set $(a_1^{\uparrow 0}, \cdots, a_n^{\uparrow 0}))$ imposes independent conditions.

Remark 6.6. If we let $(x)_u = x(x-1)\cdots(x-u+1)$, the basis can also be chosen as

$$\mathcal{A}' = \left\{ (x)_u (y)_v \mid (u, v) \in ((d+1)^{\uparrow 0}, \cdots, 1^{\uparrow 0}) - (a_1^{\uparrow 0}, \cdots, a_n^{\uparrow 0})) \right\}.$$

Proof. Clearly every function in $\mathcal{A}$ vanishes on every point of $(a_1^{\uparrow 0}, \cdots, a_n^{\uparrow 0})$. It is also clear that they are linearly independent. So it remains to prove that they span the vector space U. The lattice set $S = ((d+1)^{\uparrow 0}, \cdots, 1^{\uparrow 0}))$ consists of exactly $\binom{d+1}{2}$ points. By Lemma 4.20 in [Dum06], Y imposes independent conditions, so the corresponding interpolation matrix is nonsingular. In particular the subset $(a_1^{\uparrow 0}, \cdots, a_n^{\uparrow 0})$ imposes independent conditions, hence the dimension of U equals to $|\mathcal{A}|$. This shows that $\mathcal{A}$ span the vector space U. □

Here we divide the set I_k into union of I_k^- and I_k^+, consisting of the points in the left and right corners of $k\Delta_1$ respectively. Now I_k^+ satisfies the conditions of Lemma 6.5. Hence I_k^+ determines a basis $\mathcal{A}_k$. Now we can define a matrix M_k which is parametrized by I_k^- and $\mathcal{A}_k$.

Corollary 6.7. *Suppose we can shift the triangles $k\Delta_1$ so that (1) and (2) are satisfied. Let M_k be defined as above. If $\det M_k \neq 0$ for all $k \geq 0$, then the blow-up X' is not a MDS.*

Proof. Indeed $\mathcal{A}_k$ span the vector space of bivariate polynomials of degree $\leq d'$ vanishing on I_k^+. Hence any polynomial of degree $\leq d'$ vanishing on all I_k is spanned by $\mathcal{A}_k$. As a result, $\det M_k' = 0$ if and only if $\det M_k = 0$. □

6.2. **González and Karu's example.** We define the generalized falling factorial $(x)_n$ by letting $(x)_n := \binom{x}{n}/n!$, for $n \in \mathbb{Z}$. For $n > 0$ this coincides with the usual definition $(x)_n = x(x-1)\cdots(x-n)$. For $n = 0$, $(x)_0 = 1$. For $n < 0$, we have $(x)_n = 1/\left((x+1)\cdots(x+(-n))\right)$. So when $n < 0$, $(x)_n$ is defined for any $x \notin \{-1, \cdots, -n\}$.

In [GK16] the non-examples of MDS are given by triangles with minimal degrees $d' = n$, $s = 1$ and $t = n - 1$. The lattice set I_k is given by

$$I^- = (1^{\uparrow 0}, n^{\uparrow -n-1}) + (A, B), \quad I^+ = (n^{\uparrow 0}, \cdots, 1^{\uparrow 0}).$$

where $A = n - 1 - kmw$, $B = -s_2 kmw$, and $I_k = I^- \cup I^+$. Further, I^+ defines the basis

$$\mathcal{A} = \{(x)_n, \cdots, (x)_{n-i}(y)_i, \cdots, (y)_n\}_{i=0}^n.$$

As a result, to prove these examples gives non-MDS blow-ups, we need only show $\det M_k \neq 0$. Instead of working for an individual k, we prefer leaving A and B as indeterminate, so that the matrix M parametrized by I^- and $\mathcal{A}$ is a matrix polynomial in variables A and B. Then the determinant of M is a polynomial of A and B.

Now the key idea is to partition I^- into $I^- = K_0 \cup K_1$. We let $K_0 = \{(A, B)\}$, so $K_1 = I^- \backslash K_0 = \{(A+1, B-2-j)\}_{j=0}^{n-1}$. Let M_α be the submatrix of M parametrized by K_α, for $\alpha = 0, 1$.

Proposition 6.8. *(1) Define a $(n+1) \times 1$ vector*

$$\xi = \left((-1)^i \binom{n}{i} (A + 1 - n + i)_i (B - 2 - i)_{n-i}\right)_{i=0}^n.$$

Then $M_1 \xi = 0$. Furthermore if $A \neq -1, 0, 1, \cdots, n-2$, ξ is the unique solution of $M_1 \xi = 0$ up to a scalar.

(2) $\det M = 0$ if and only if $(n+1)(A - n + 1) + nB = 0$ or $A = -1, 0, \cdots, n-2$.

Remark 6.9. Let $A = n - 1 - kmw$, $B = -s_2 kmw$. Then $(n+1)(A - n + 1) - nB) = -kmw(ns_2 + n + 1)$. Since there are at least n columns in Δ_1, $mw \geq n + 1$. Hence $A \leq -2$. Therefore $\det M_k = 0$ if and only if $ns_2 \in \mathbb{Z}$, which reproves the main theorem in [GK16].

Proof. The product of the j-th row of M_1 and ξ is

$$\begin{aligned}
&\sum_{i=0}^n (A+1)_{n-i}(B-2-j)_i(-1)^i \binom{n}{i} (A+1-n+i)_i (B-2-i)_{n-i} \\
=&\frac{(A+1)!}{(A+1-n)!}\frac{(B-2-j)!}{(B-2-n)!}\sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(B-2-i)!}{(B-2-j-i)!}.
\end{aligned}$$

Notice that $(B-2-i)!/(B-2-j-i)! = (B-2-i)_j$ is a polynomial of i with degree $j \leq n-1$. Hence the sum above is zero by Lemma 6.11, which proves that $M_1 \xi = 0$.

When $A \neq -1, 0, \cdots, n-2$, we can divide the i-th column of M_1 by $(A+1)_{n-i}$ to obtain a matrix M_2 of a single indeterminate B. That is, $(M_2)_{j,i} = (B-2-j)_i$. By Lemma 6.10, M_2 has rank n, so M_1 has rank n, too. Therefore ξ span the space of solutions of $M_1 x = 0$.

For (2), when $A \in \{-1, 0, 1, \cdots, n-2\}$, $(A)_n = (A+1)_n = 0$, so the first column of A is zero. When $A = -1$, $(A+1)_i = 0$ for all $0 \leq i \leq n$, hence $M_1 = 0$. In both cases, $\det M = 0$. When $A \notin \{-1, 0, \cdots, n-2\}$, by (1), $\det M = 0$ if and only if $M\xi = 0$, which is equivalent to $M_0\xi = 0$. Here

$$\begin{aligned}M_0\xi =&\sum_{i=0}^{n}(A)_{n-i}(B)_i(-1)^i\binom{n}{i}(A+1-n+i)_i(B-2-i)_{n-i}\\ =&\frac{A!}{(A+1-n)!}\frac{B!}{(B-2-n)!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(A+1-n+i)}{(B-i-1)(B-i)}\\ =&(A)_{n-1}(B)_{n+2}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\Big(-\frac{B-A+n-2}{B-1-i}+\frac{B-A+n-1}{B-i}\Big).\end{aligned}$$

By Lemma 6.12,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{B-1-i}=-n!(-B)_{-n-1},\quad \sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{B-i}=-n!(-B-1)_{-n-1},$$

Hence

$$\begin{aligned}M_0\xi =&n!(A)_{n-1}(B)_{n+2}\Big((B-A+n-2)(-B)_{-n-1}-(B-A+n-1)(-B-1)_{-n-1}\Big)\\ =&(-1)^{n+1}n!(A)_{n-1}\Big((B-A+n-2)(B)-(B-A+n-1)(B-n-1))\Big)\\ =&(-1)^{n+1}n!(A)_{n-1}((n+1)(A-n+1)+nB).\end{aligned}$$

□

Lemma 6.10. *Given n distinct number $a_1,\cdots,a_n\in\mathbb{R}$, the matrix $U=(u_{i,j})$ where $u_{i,j}=(a_i)_{j-1}$, $1\le i,j\le n$ is nonsingular.*

Proof. Recall the identity

$$\sum_{i=0}^{n}\left\{{n \atop i}\right\}(x)_i=x^n$$

for $n>0$, $(x)_i$ the i-th falling factorial, and $\left\{{n \atop i}\right\}$ the Stirling numbers of the second kind, which is the number of partitions of a set of n elements into i nonempty subsets. Hence the Stirling numbers $\left\{{j \atop i}\right\}$, $0\le i,j\le n-1$ give a matrix P which transforms U into a Vandermonde Matrix $V=(v_{i,j})$ where $v_{i,j}=(a_i)^{j-1}$. That is, $UP=V$. The matrix C is nonsingular because all a_is are distinct. The matrix P is unit triangular because $\left\{{j \atop i}\right\}=0$ when $i>j$ and $\left\{{i \atop i}\right\}=1$, therefore nonsingular. Hence U is nonsingular. □

Lemma 6.11. *(See [GK16, Lem. 2.3]) Let $n>0$ be an integer and $p(x)$ be a polynomial of degree $<n$. Then*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}p(i)=0.$$

Lemma 6.12. *Let $n\in\mathbb{Z}_{\ge0}$. Then*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{x+i}=n!(x-1)_{-n-1}=\frac{n!}{x(x+1)\cdots(x+n)}.$$

Proof. We make induction on n. When $n = 0$ this obviously holds. Suppose the identity holds for $n-1$. Then for n, we use

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}.$$

So

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{x+i} &= \sum_{i=0}^{n}(-1)^i\binom{n-1}{i}\frac{1}{x+i} + \sum_{i=0}^{n}(-1)^i\binom{n-1}{i-1}\frac{1}{x+i}\\ &= \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{1}{x+i} + \sum_{i=0}^{n-1}(-1)^{i+1}\binom{n-1}{i}\frac{1}{x+1+i}\\ &= (n-1)!(x-1)_{-n} - (n-1)!(x)_{-n}\\ &= n!(x-1)_{-n-1}.\end{aligned}$$

□

6.3. **The Case $d' = 5$ (Proof of Theorem 2.14, First case).** Here we prove that the first case in Theorem 2.14 of $d' = 5$ gives non-MDS blow-ups if $5s_2 \notin \mathbb{Z}$. For the rest two cases $d' = 7$ and 9, we managed to calculate $\det M$ via a computer program. The whole proof of Theorem 2.13 and 2.14 is given in Section 9.

We add -2 to all the slopes and translate the triangle so that the assumptions (1) and (2) in Remark 6.3 are satisfied. Define

$$I^- = (1^{\uparrow 0}, 3^{\uparrow -4}, 5^{\uparrow -8}) + (A, B), I^+ = (5^{\uparrow 0}, 4^{\uparrow 0}, 2^{\uparrow 0}, 1^{\uparrow 0}).$$

where $-A = kmw - 3$ and $B = -s_2kmw$. Then in the triangle $k\Delta_1$, the lattice sets $I_k = I^- \cup I^+$, for every k. Further I^+ define the basis

$$\mathcal{A} = \{(x)_5, \cdots, (x)_{5-i}(y)_i, \cdots, (y)_5, (x)_4, (x)_3 y, (x)_2(y)_2\}_{i=0}^5.$$

See Figure 5. In the following we treat A and B as indeterminate.

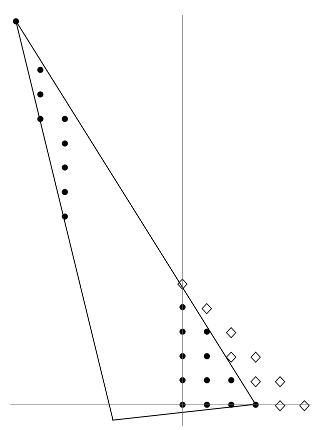

FIGURE 5. The triangle obtained by adding -2 to all the slopes of $k\Delta_1$ and then applying a shear translation. Solid points are the set I_k. Diamond points index the basis $\mathcal{A}$

The matrix M is defined as the matrix parametrized by I^- and $\mathcal{A}$. For convenience, we choose the order on I_k as follows: the vertex (A, B) is the first. Then the first column next to (A, B) from top to bottom. Finally the second column from top to bottom.

We divide the matrix M into blocks:

$$M = \left(\begin{array}{c|c} E & P \\ \hline F & Q \\ \hline G & R \end{array}\right)$$

so that E is of size 1×6, F of 3×6, and G of 5×6. That is, E, F, G are indexed by the points with x-coordinates $A, A+1, A+2$, respectively. We claim:

Lemma 6.13. *Define a* 6×1 *vector*

$$\xi = \left((-1)^i \binom{5}{i} (A+2-5+i)_i (B-4-i)_{n-i}\right)_{i=0}^{5}$$

Then $\det M = 0$ *if and only if* $A = -2, -1, 0, 1, 2$ *or* $(E - 2PQ^{-1}F)\xi = 0.$

Proof. The same proof of Proposition 6.8 shows that ξ is the unique solution of $G\xi = 0$ up to a scalar if $(A+2)_5 \neq 0$. Solve $(A+2)_5 = 0$ and we have $A = -2, -1, 0, 1, 2$.

When $A = 0, 1, 2$, the first column of M is zero. When $A = -1$, M has a 8×7 submatrix which is zero. When $A = -2$, M has a 5×8 submatrix which is zero. Hence in all these cases $\det M = 0$.

So we assume $(A+2)_5 \neq 0$. By Lemma 6.10, Q is nonsingular. Let $Q^{-1}F\xi = (a_0, a_1, a_2)^T$. Define the vector η such that

$$\eta^T := (\xi^T, 0, 0, 0) + (a_0, a_1, a_2, 0, 0, 0, -(A-2)a_0, -(A-1)a_1, -Aa_2).$$

We claim that $(F\ Q)\eta = (G\ R)\eta = 0$. Let $A_{(i)}$ denote the i-th row of a matrix A. Indeed, $(G\ R)_{(i)}\eta = (G\xi)_{(i)} + a_0(A+2)_5 + a_1(A+2)_4(B-4-i) + a_2(A+2)_3(B-4-i)_2 - a_0(A-2)(A+2)_4 - a_1(A-1)(A+2)_3(B-4-i) - a_2(A)(A+2)_2(B-4-i)_2 = 0$.

Similarly $(F\ Q)_{(i)}\eta = (F\xi)_{(i)} + a_0(A+1)_5 + a_1(A+1)_4(B-2-i) + a_2(A+1)_3(B-2-i)_2 - a_0(A-2)(A+1)_4 - a_1(A-1)(A+1)_3(B-2-i) - a_2(A)(A+1)_2(B-2-i)_2$, which equals to $(F\xi)_{(i)} - a_0(A+1)_4 - a_1(A+1)_3(B-2-i) - a_2(A+1)_2(B-2-i)_2$. In matrix form, this says $(F\ Q)\eta = F\xi - Q\eta = F\xi - Q(Q^{-1}F\xi) = 0$.

We now claim that the rank of the submatrix $N := \begin{pmatrix} F & Q \\ G & R \end{pmatrix}$ equals to 8, so η spans the solution space of N. As a result, $\det M = 0$ if and only if $(E\ P)\eta = 0$. Now similar calculation shows that

$$\begin{aligned} \begin{pmatrix} E & P \end{pmatrix} \eta =& E\xi - 2a_0(A)_4 - 2a_1(A)_3(B) - 2a_2(A)_2(B)_2 \\ =& E\xi - 2P(Q^{-1}F\xi). \end{aligned}$$

So we prove that $\operatorname{rank} N = 8$, assuming $(A+2)_5 \neq 0$. We divide the first three columns of N by $-(A-2), -(A-1), -A$ respectively, and add to the 7, 8, 9-th column. This reduces R to zero. To find what Q is transformed to, we have

$$\frac{(A+1)_{5-k}(B-s)_k}{-(A-2+k)} + (A+1)_{4-k}(B-s)_k = (A+1)_{3-k}(B-s)_k,$$

for $s = 2, 3, 4$. Therefore $N \sim N_1 = \begin{pmatrix} F & Q_1 \\ G & 0 \end{pmatrix}$ where

$$Q_1 = \begin{pmatrix} (A+1)_3(B-2)_0 & (A+1)_2(B-2)_1 & (A+1)_1(B-2)_2 \\ (A+1)_3(B-3)_0 & (A+1)_2(B-3)_1 & (A+1)_1(B-3)_2 \\ (A+1)_3(B-4)_0 & (A+1)_2(B-4)_1 & (A+1)_1(B-4)_2 \end{pmatrix}.$$

Since $(A+2)_5 \neq 0$, we can divide on each column of N by the corresponding factors of A appearing in G and R_1. This reduced G and Q_1 to Vandermonde-like matrices as in Lemma 6.10. Therefore, Lemma 6.10 shows that Q and the first five columns of G_1 are nonsingular. Hence we can reduce the matrix N_1 further to the row echelon form, which shows that N_1 has full row rank, which equals 8. Therefore $\operatorname{rank} N = \operatorname{rank} N_1 = 8$. □

Proof of the first case of Theorem 2.14. By Corollary 6.7, we need to show $\det M \neq 0$ when $5s_2 \notin \mathbb{Z}$. By Lemma 6.13, it suffices to calculate $E\xi - 2P(Q^{-1}F\xi)$. We will find PQ^{-1}, $F\xi$ and $E\xi$ respectively.

Suppose $PQ^{-1} = \zeta$, then $P^T = Q^T\zeta^T$, we divide the j-th column of Q by $(A+1)_{4-j}$ to obtain a matrix Q_1 where the (i,j)-th entry is $(B-2-i)_j$, for $0 \leq i, j \leq 2$. We divide the i-th entry of P by $(A+1)_{4-j}$ to get P_1. Now $Q_1^T\zeta^T = P_1^T$, and Q_1^T satisfies the assumption of Lemma 6.10. Observe that the j-th entry of P_1 is

$$\frac{(A)_{4-j}}{(A+1)_{4-j}}(B)_j = \frac{A-3+j}{A+1}(B)_j = \frac{1}{A+1}((A-3)(B)_j + B(B)_j - B(B-1)_j).$$

Hence by linearity and Lemma 6.13, we find

$$\begin{aligned}(PQ^{-1})^T = \zeta^T =& \left(\frac{(-1)^j}{2!(A+1)}\binom{2}{j}\left(\frac{(A-3+B)(4)_3}{2+j} - \frac{B(3)_3}{1+j}\right)\right)_{j=0}^{2} \\ =& \frac{1}{A+1}\Big(6A+3B-18, -8A-5B+24, 3A+2B-9\Big).\end{aligned}$$

Next the j-th entry of $F\xi$ equals to

$$\begin{aligned}&\sum_{i=0}^{5}(A+1)_{5-i}(B-2-j)_i(-1)^i\binom{5}{i}(A-3+i)_i(B-4-i)_{5-i} \\ =&\frac{(A+1)!}{(A-3)!}\frac{(B-2-j)!}{(B-9)!}\sum_{i=0}^{5}(-1)^i\binom{5}{i}\frac{(A-3+i)(B-4-i)!}{(B-2-j-i)!} \\ =&(A+1)_4(B-j)_{7-j}\sum_{i=0}^{5}(-1)^i\binom{5}{i}(A-3+i)(B-4-i)_{j-2}.\end{aligned}$$

For $j = 0, 1, 2$. Using Lemma 6.17, the above equals to

$$\begin{aligned}&(A+1)_4(B-j)_{7-j}(-1)(B-9)_{j-7}\Big((A-3)(6-j)_5 + (5-j)_5 5(B-2-j)\Big). \\ =&-(A+1)_4(5-j)_4\Big((6-j)(A-3) + 5(B-2-j)\Big).\end{aligned}$$

In other words, $F\xi = -5!(A+1)_4\Big(6A+5B-28, A+B-6, 0\Big)$. For $E\xi$:

$$\begin{aligned} E\xi =& \sum_{i=0}^{5} (A)_{5-i}(B)_i(-1)^i \binom{5}{i}(A-3+i)_i(B-4-i)_{5-i} \\ =& (A)_3(B)_9 \sum_{i=0}^{5}(-1)^i\binom{5}{i}(A-3+i)(A-4-i)(B-4-i)_{-4}. \end{aligned}$$

Use Lemma 6.17 again,

$$\begin{aligned} E\xi =& (A)_3(B)_9\Bigg((A-3)(A-4)(-1)^5(B-9)_{-9}(8)_5 + \\ & (2A-7)(-1)^5(B-9)_{-9}(7)_4(5B) + (-1)^5(B-9)_{-9}(6)_3(5B)(4B+3)\Bigg) \\ =& -(A)_3\Bigg((A-3)(A-4)(8)_5 + (7)_4(5B) + (6)_3(5B)(4B+3)\Bigg). \end{aligned}$$

Combining the results above, we find

$$E\xi - 2(PQ^{-1})(F\xi) = -5!(A)_3(8(A-3)+5B).$$

Hence $\det M \neq 0$ if $8(A-3)+5B \neq 0$. Let $-A = kmw - 3$ and $B = -s_2 kmw$, then this is equivalent to $5s_2 + 8 \neq 0$. Since $-2 + 1/3 < s_2 < -2 + 1/2$, this is equivalent to $5s_2 \notin \mathbb{Z}$. □

Remark 6.14. The determinant $\det M$ can be calculated by a computer program. We used *Mathematica 10* [Wol16] to obtain that

$$\det M = -2^{10}\cdot 3^3 \cdot 5(-2+A)(-1+A)^3A^5(1+A)^6(2+A)^4(-24+8A+5B),$$

which supports our calculation.

Lemma 6.15. *Let $x \in \mathbb{R}$, $m, n \in \mathbb{Z}_{>0}$. Define a $(n+1)\times(n+1)$ matrix U by*

$$U_{i,j} := (x-m-j)_i$$

where $0 \leq i, j \leq n$. Let

$$\sigma = \left((-1)^j\binom{n}{j}\frac{(n+m)_{n+1}}{n!(m+j)}\right)_{j=0}^{n}.$$

Then $U\sigma = (1, (x)_1, \cdots, (x)_n)^T$.

Proof. The product of the i-th row of U with σ is

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(x-m-j)_i\frac{(n+m)_{n+1}}{n!(m+j)} = \frac{(n+m)_{n+1}}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(x-m-j)_i}{m+j}.$$

Consider as polynomials of j, then $(x-m-j)_i = f(j)(m+j)+g(j)$. The quotient $f(j)$ is of degree $i-1$, and $g(j) = (x-m-(-m))_j = (x)_j$ by the reminder theorem. Hence

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(x-m-j)_i}{m+j} =& \sum_{j=0}^{n}(-1)^j\binom{n}{j}f(j)+\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(x)_j}{m+i}\\ =&0+(x)_i\cdot n!(m-1)_{-n-1}\\ =&\frac{n!(x)_j}{(n+m)_{n+1}},\end{aligned}$$

where we applied Lemma 6.11 and 6.12. Now the product equals to $(x)_i$. □

For the following lemmas, we let $\Delta f(x) := f(x+1) - f(x)$ be the finite difference of $f(x)$ with respect of x.

Lemma 6.16. *For $n \neq 0$, $\Delta(x)_n = n(x)_{n-1}$.*

Proof. $\Delta(x)_n = (x+1)_n - (x)_n = ((x+1)-(x-n+1))(x)_{n-1} = n(x)_{n-1}$. □

Lemma 6.17. *Consider the sum*

$$g(s) := \sum_{i=0}^{n}(-1)^i\binom{n}{i}i^s(x-i)_{-m}$$

for $n, m > 0$, $x \in \mathbb{R}$, and $s \geq 0$. Then

(1) $g(0) = (-1)^n(x-n)_{-n-m}(n+m-1)_n$*;*
(2) $g(1) = (-1)^n(x-n)_{-n-m}(n+m-2)_{n-1}n(x+m)$*;*
(3) $g(2) = (-1)^n(x-n)_{-n-m}(n+m-3)_{n-2}n(x+m)((n-1)x+mn-1)$*.*

Proof. First for $g(0)$ this is a generalization of Lemma 6.12. Indeed, apply Lemma 6.16 repeatedly. We have

$$(x)_{-m} = \frac{\Delta(x)_{-m+1}}{-m+1} = \cdots = \frac{\Delta^{m-1}(x)_{-1}}{(-1)^{m-1}(m-1)!}.$$

Notice that replacing x by $x-i$ for an integer i does not change the finite difference in Lemma 6.12. Hence we have

$$\begin{aligned}g(0) &= \frac{1}{(-1)^{m-1}(m-1)!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\Delta^{m-1}\frac{1}{x-i+1}\\ &= \frac{1}{(-1)^{m-1}(m-1)!}\Delta^{m-1}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{x-i+1}\\ &= \frac{1}{(-1)^{m-1}(m-1)!}\Delta^{m-1}(n!(-1)(-x-2)_{-n-1})\\ &= \frac{n!}{(-1)^m(m-1)!}\Delta^{m-1}((-1)^{n+1}(x-n)_{-n-1})\\ &= \frac{n!}{(-1)^m(m-1)!}((-1)^{n+1}(x-n)_{-n-m})(-1)^{m-1}(n+m-1)_{m-1})\\ &= (-1)^n(n+m-1)_n(x-n)_{-n-m}.\end{aligned}$$

For $s = 1$, notice $(x-i+1)_{-m+1} = (x-i+1)(x-i)_{-m}$. So $i(x-i)_{-m} = (x+1)(x-i)_{-m} - (x-i+1)_{-m+1}$ for $m \geq 1$. Hence

$$\begin{aligned} g(1) =&(x+1)\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)_{-m} - \sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i+1)_{-m+1} \\ =&(x+1)(-1)^n(n+m-1)_n(x-n)_{-n-m} - (-1)^n(n+m-2)_n(x+1-n)_{-n-m+1} \\ =&(-1)^n(x-n)_{-n-m}(n+m-2)_{n-1}n(x+m). \end{aligned}$$

Similarly for $s = 2$, we have

$$\begin{aligned} i^2(x-i)_{-m} =&i(x+1)(x-i)_{-m} - i(x-i+1)_{-m+1} \\ =&(x+1)^2(x-i)_{-m} - (2x-3)(x+1-i)_{-m+1} + (x+2-i)_{-m+2}. \end{aligned}$$

Hence

$$\begin{aligned} g(2) =&(x+1)^2(-1)^n(n+m-1)_n(x-n)_{-n-m} \\ &- (2x-3)(-1)^n(n+m-2)_n(x+1-n)_{-n-m+1} \\ &+ (-1)^n(n+m-3)_n(x+2-n)_{-n-m+2} \end{aligned}$$

which simplifies to the required result. □

6.4. **The Cases $d' = 7$ or 9.** We calculated the determinant of the interpolation matrix M by computer programs in the cases when $d' = 7$ and 9, and the slopes are given by the inequalities in Theorem 2.14. The codes for the three cases ($d' = 5, 7, 9$) are available online at `https://hezhuangblog.wordpress.com/research`.

In both cases, the construction of the shifted triangle $k\Delta_1$, the index sets I^-, I^+, $\mathcal{A}$, and the matrix M are all similar with the case $d' = 5$. We used a program in *Mathematica 10* to calculate $\det M$. As in the case $d' = 5$, we let the coordinate of the vertex in I^- be (A, B). Then $\det M = P(A, B)$ is a polynomial in A and B. Let $b_1, \cdots, b_r$ be distinct integers for r sufficiently large. For each b_i, evaluating $P(A, b_i)$ at sufficiently many values of A gives a polynomial $Q_{b_i}(A)$ such that $P(A, b_i) = Q_{b_i}(A)$ is the Lagrange polynomial in A. Then $P(A, B)$ equals the Lagrange polynomial of the data set $\{b_i, Q_{b_i}(A)\}$. Specifically:

(1) for the case $d' = 7$ (within the given constraints on slopes),

$$\begin{aligned} \det M = &- 2^{26} \cdot 3^{10} \cdot 5^3 \cdot 7(-3+A)(-2+A)^3(-1+A)^6 \cdot \\ &A^9(1+A)^{11}(2+A)^{10}(3+A)^6(12A - 48 + 7B). \end{aligned}$$

(2) For the case $d' = 9$ (within the given constraints on slopes),

$$\begin{aligned} \det M = &- 2^{59} \cdot 3^{24} \cdot 5^7 \cdot 7^3((-4+A)(-3+A)^3(-2+A)^6(-1+A)^{10} \cdot \\ &A^{14}(1+A)^{17}(2+A)^{17}(3+A)^{14}(4+A)^8(16A - 80 + 9B). \end{aligned}$$

Therefore $\det M = 0$ for all $k > 0$ is equivalent to $(12A - 48 + 7B) = 0$ or $(16A - 80 + 9B) = 0$ respectively. In either case, it is equivalent to $d' \cdot s_2 \in \mathbb{Z}$ by direct calculation.

7. Proof of Proposition 4.2

For every real number x, $\{x\} := x - \lfloor x \rfloor$ is the fraction part of x.

Lemma 7.1. *For any two real numbers x, y, integer n, we have:*

(1) $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$*;*
(2) $\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$*;*
(3) $n \leq x \iff n \leq \lfloor x \rfloor$*;*
(4) $n > x \iff n \geq \lfloor x \rfloor + 1$*.*

Proof. Obvious from definition. □

Lemma 7.2. *Let x, y be real numbers. Define $\phi(x,y) := \lfloor x + y \rfloor - \lceil x \rceil + 1$, then $\lfloor y \rfloor \leq \phi(x,y) \leq \lfloor y \rfloor + 1$.*

Proof. When $x \in \mathbb{Z}$, by Lemma 7.1, $\phi(x,y) = x + \lfloor y \rfloor - x + 1 = \lfloor y \rfloor + 1$. When $x \notin \mathbb{Z}$, Lemma 7.1 shows that $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$. Now $\lceil x \rceil - \lfloor x \rfloor = 1$, hence $\phi(x,y) = \lfloor y \rfloor$ or $\lfloor y \rfloor + 1$. □

Lemma 7.3. *Consider rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Let $l_k = \lfloor ks_2 \rfloor - \lceil ks_1 \rceil + 1$, and $r_k = \lfloor ks_3 \rfloor - \lceil ks_2 \rceil + 1$. Then for every integer $k \geq 1$ we have:*

$$\lfloor k(s_2 - s_1) \rfloor \leq l_k \leq \lfloor k(s_2 - s_1) \rfloor + 1,$$
$$\lfloor k(s_3 - s_2) \rfloor \leq r_k \leq \lfloor k(s_3 - s_2) \rfloor + 1.$$

Furthermore, both sequences $\{l_k\}_k$ and $\{r_k\}_k$ are increasing and positive.

Proof. Let ϕ be as defined in Lemma 7.2. Then $l_k = \phi(ks_1, k(s_2 - s_1))$, and $r_k = \phi(ks_2, k(s_3 - s_2))$. Hence the inequalities follows.

Next, since $w < 1$, we have $s_2 - s_1 > 1$. Therefore $l_{k+1} \geq \lfloor (k+1)(s_2 - s_1) \rfloor \geq \lfloor k(s_2 - s_1) \rfloor + \lfloor s_2 - s_1 \rfloor \geq \lfloor k(s_2 - s_1) \rfloor + 1 \geq l_k$. So $\{l_k\}_k$ is increasing. Similarly, since $s_3 - s_2 > 1$, the sequence $\{r_k\}_k$ is increasing. In particular, let $k = 1$. Then $l_1 \geq \lfloor s_2 - s_1 \rfloor \geq 1$ and $r_1 \geq \lfloor s_3 - s_2 \rfloor \geq 1$. □

Proposition 7.4. *Consider rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Then for all $n \geq 0$, we have $\pi(n) < w(n + 1)$.*

Proof. We define two auxiliary functions $\pi^-(n)$ (and $\pi^+(n)$) as the number of positive integer k such that $l_k \leq n$ ($r_k \leq n$ respectively). Then by definition, $\pi^-(n) + \pi^+(n) = \pi(n)$.

Next we find bounds for $\pi^-(n)$. Since $\{l_k\}_k$ is increasing by Lemma 7.3, $\pi^-(n)$ equal to the unique integer k such that $l_k \leq n < l_{k+1}$, or equivalently, $l_k \leq n \leq l_{k+1} - 1$. By Lemma 7.3, we have $\lfloor k(s_2 - s_1) \rfloor \leq n \leq (\lfloor (k+1)(s_2 - s_1) \rfloor + 1) - 1$, i.e., $\lfloor k(s_2 - s_1) \rfloor \leq n \leq \lfloor (k+1)(s_2 - s_1) \rfloor$. Applying Lemma 7.1 we find

$$n \geq \lfloor k(s_2 - s_1) \rfloor \iff n + 1 > k(s_2 - s_1),$$
$$n \leq \lfloor (k+1)(s_2 - s_1) \rfloor \iff n \leq (k+1)(s_2 - s_1).$$

Hence $k(s_2 - s_1) - 1 < n \leq (k+1)(s_2 - s_1)$. Now we can solve for $k = \pi^-(n)$:

$$\frac{n}{s_2 - s_1} - 1 \leq \pi^-(n) < \frac{n+1}{s_2 - s_1}. \tag{5}$$

Similar argument for $\{r_k\}$ and $\pi^+(n)$ shows that

$$\frac{n}{s_3 - s_2} - 1 \leq \pi^+(n) < \frac{n+1}{s_3 - s_2}. \tag{6}$$

Adding (5) and (6), and noticing that $w = (s_2 - s_1)^{-1} + (s_3 - s_2)^{-1}$, we find

$$wn - 2 \leq \pi(n) < w(n + 1).$$

□

Proof of Proposition 4.2. Part (1) follows from Proposition 7.4. Indeed $\pi(n) \le w(n+1) < n+1$. To prove (2), it suffices to show

$$\pi(n+m) = \pi(n) + mw. \tag{7}$$

for every $n \geq 1$.

Indeed, assume this holds, then by $w < 1$, and part one, we must have $\pi(n+m) < \pi(n)+m \le n+m$. Therefore, if $n > m$, then $\pi(n) < n$. It remains to prove for $mw \le n \le m$. First let $n = 1$ in (7), then $\pi(m+1) = \pi(1)+mw$. By the first part of Theorem 4.2, $\pi(1) = 0$ or 1. Suppose there exists some n' such that $mw \le n' \le m$ and $\pi(n') = n'$. By the definition of $\pi(n)$, π in non-decreasing, so

$$\pi(n') \le \pi(m+1).$$

Hence there are two possible cases:

1). If $\pi(1) = 0$, then $n' = \pi(n') \le \pi(m+1) = mw$. Because $mw \le n'$, we find $n' = mw$. Hence $\pi(mw) = \pi(m+1) = mw$. However, the column of Δ_1 on $x = 0$ contains exactly $m+1$ lattice points. Therefore, there exists $i, j \ge 1$ such that $l_i = r_j = m+1$. Hence for any $z < m+1$, $\pi(z) \le \pi(m+1) - 2$. In particular, $\pi(mw) \le \pi(m+1) - 2 = mw - 2$, a contradiction.

2). If $\pi(1) = 1$, then $n' = \pi(n') \le \pi(m+1) = mw+1$. Because $mw \le n'$, we find $n' = mw$ or $mw+1$. If $n' = mw$, then $\pi(mw) = mw$. If $n' = mw$, then $\pi(mw+1) = mw+1$. Now the same argument in 1) shows $\pi(mw) \le \pi(m+1) - 2$ and $\pi(mw+1) \le \pi(m+1)-2$, where $\pi(m+1) - 2 = mw - 1$. Hence we reach a contradiction.

Therefore, no such n' exists. By part (1) of the Theorem again, for every $mw \le n \le m$, $\pi(n) < n$.

Now we prove (7) in the following. By definition, m is the smallest integer such that $m\Delta_0$ is a good lattice triangle. Recall that the coordinates $p = (x_1, y_1)$, $q = (x_2, y_2)$, and the y-intercept is $y_0 = 1$. Hence m equals to the lowest common denominator of $\{x_1, x_2, y_1, y_2\}$. Let $s_k = u_k/v_k$ be in the lowest terms of s_k for $i = 1, 2, 3$. Let $\alpha = \operatorname{lcm}(v_1, v_2)$, and $\beta = \operatorname{lcm}(v_2, v_3)$. Then αs_1 and αs_2 are both integers, and $\gcd(\alpha s_1, \alpha s_2, \alpha) = 1$ by the choice of α. We let m_1 be the lowest common denominator of x_1 and y_1, then $m_1 \mid \alpha s_2 - \alpha s_1$, so there exists an integer t_1 such that $m_1 t_1 = \alpha s_2 - \alpha s_1$. We claim $t_1 \mid \alpha$ and $t_1 \mid \alpha s_1$. Indeed, let $g = \gcd(\alpha s_2 - \alpha s_1, \alpha)$, then $(\alpha s_2 - \alpha s_1)/g$ equals to the denominator of the lowest term of x_1, hence it divides m_1. Since $(\alpha s_2 - \alpha s_1)/t_1 = m_1$, we have $t_1 \mid g$, and hence $t_1 \mid \alpha$. Similarly, $t_1 \mid \alpha s_1$. Note that $t_1 \mid \alpha s_2 - \alpha s_1$, hence $t_1 \mid \alpha s_2$. However, $\gcd(\alpha s_1, \alpha s_2, \alpha) = 1$, so $t = 1$, and $m_1 = \alpha s_2 - \alpha s_1$.

Similarly, the lowest common denominator m_2 of x_2 and y_2 equals to $\beta s_2 - \beta s_3$. Hence $m = \operatorname{lcm}(m_1, m_2) = \operatorname{lcm}(\alpha(s_2 - s_1), \beta(s_3 - s_2))$.

Now for every $k > 0$, $l_{k+\alpha} = \lfloor (k+\alpha)s_2 \rfloor - \lceil (k+\alpha)s_1 \rceil + 1 = \lfloor ks_2 \rfloor - \lceil ks_1 \rceil + 1 + \alpha s_2 - \alpha s_1 = l_k + \alpha(s_2 - s_1)$. Because the sequence $\{l_k\}$ is non-decreasing, for any given positive integer $n \ge 1$, there exists a unique k such that $l_k \le n < l_{k+1}$. By the definition of $\pi^+(n)$ (see proof of Proposition 7.4), $\pi^+(n) = k$. Adding $\alpha(s_2 - s_1)$ to the inequality, we find

$$l_{k+\alpha} = l_k + \alpha(s_2 - s_1) \le n + \alpha(s_2 - s_1) < l_{k+1} + \alpha(s_2 - s_1) = l_{k+1+\alpha}.$$

This implies that $\pi^+(n + \alpha(s_2 - s_1)) = k + \alpha = \pi^+(n) + \alpha$ for all n.

Similarly, $r_{k+\beta} = r_k + \beta(s_3 - s_2)$ for every $k > 0$, and $\pi^-(n + \beta(s_3 - s_2)) = \pi^-(n) + \beta$ for all $n > 0$.

Finally, recall $m = \mathrm{lcm}(\alpha(s_2 - s_1), \beta(s_3 - s_2))$. If we let $t = \gcd(\alpha(s_2 - s_1), \beta(s_3 - s_2))$, then $mt = \alpha(s_2 - s_1)\beta(s_3 - s_2)$. By iteration, we find

$$\pi^+(n + m) = \pi^+(n + \alpha(s_2 - s_1) \cdot \frac{\beta(s_3 - s_2)}{t}) = \pi^+(n) + \frac{\alpha\beta(s_3 - s_2)}{t}.$$

and

$$\pi^-(n + m) = \pi^-(n + \beta(s_3 - s_2) \cdot \frac{\alpha(s_2 - s_1)}{t})) = \pi^-(n) + \frac{\alpha\beta(s_2 - s_1)}{t}.$$

Adding them up, we conclude

$$\pi(n + m) = \pi(n) + \frac{\alpha\beta(s_3 - s_1)}{t} = \pi(n) + mw.$$

□

8. Proof of Proposition 2.7

We first point out that for rational numbers $s_1 < s_2 < s_3$ with width $w < 1$, the reduced degree $d = 0$ if and only if the minimal degree $d' = 0$. Let $l_k = \lfloor ks_2 \rfloor - \lceil ks_1 \rceil + 1$, and $r_k = \lfloor ks_3 \rfloor - \lceil ks_2 \rceil + 1$ as in Theorem 2.13. Let $\pi(n)$ be defined as in 4.1. Then the following are all equivalent.

(1) $d' = 1$;
(2) $\pi(1) = 1$;
(3) Exactly one of l_1 and r_1 equals one.

Clearly, (3) $\implies$ (1) $\iff$ (2) by Proposition 4.2. By Lemma 7.3, $l_1 \geq 1$ and $r_1 \geq 1$. Therefore if $\pi(1) = 1$, then one of l_1 and r_1 is one but not both. So (2) $\implies$ (3).

Lemma 8.1. *Given rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. For all positive integers k, define $l_k = \lfloor ks_2 \rfloor - \lceil ks_1 \rceil + 1$, and $r_k = \lfloor ks_3 \rfloor - \lceil ks_2 \rceil + 1$. Then:*

(1) If $l_1 \neq 1$, then the sequence $\{l_k\}$ is strictly increasing.
(2) If $r_1 \neq 1$, then the sequence $\{r_k\}$ is strictly increasing.

Proof. We need only prove this for $\{r_k\}$. Since $w < 1$, $r_{k+1} \geq r_k$ for any k. Suppose there exists $k \geq 1$ such that $r_k = r_{k+1} = d + 1$. Then by definition we find $\lfloor ks_3 \rfloor - \lceil ks_2 \rceil = d$, and $\lfloor (k+1)s_3 \rfloor - \lceil (k+1)s_2 \rceil = d$. By adding the same integer to s_i, for $i = 1, 2, 3$, we can assume $0 < s_2 \leq 1$, without changing the values of $\{r_k\}$. Hence $\lceil s_2 \rceil = 1$. Further, $r_1 \neq 1$, and $w < 1$ implies that $r_1 \geq 1$, so $r_1 \geq 2$. This implies $s_3 \geq 2$. By Lemma 7.3 we have $\lceil (k+1)s_2 \rceil \leq \lceil ks_2 \rceil + \lceil s_2 \rceil = \lceil ks_2 \rceil + 1$. Hence

$$\lfloor (k+1)s_3 \rfloor = \lfloor (k+1)s_2 \rfloor + d \leq \lceil ks_2 \rceil + 1 + d = \lfloor ks_3 \rfloor + 1.$$

On the other hand, $\lfloor (k+1)s_3 \rfloor \geq \lfloor ks_3 \rfloor + \lfloor s_3 \rfloor \geq \lfloor ks_3 \rfloor + 2$. So we reached a contradiction. Hence $\{r_k\}$ is strictly increasing. □

Let $\pi(n)$ be as defined in Section 2. Let $\delta(n) := \pi(n) - n$. By Proposition 4.2, $\delta(n) \leq 0$ for all $n \geq 1$.

Lemma 8.2. *Given rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Suppose $\pi(1) = 0$ (so that $\delta(1) = -1$). Then for any $n > 1$, the following two are equivalent:*

(1) $\pi(n) = n$ (so that $\delta(n) = 0$).
(2) There exists a positive integer $n_0 \leq n$ such that $\delta(n_0) = \delta(n_0 + 1) = \cdots = \delta(n) = 0$, and $n_0 = r_v = l_u$ for some $u, v \geq 1$.

Proof. $\pi(n) = n$ if and only if $\delta(n) = 0$. Since $\pi(1) = 0$, $l_1 \neq 1$ and $r_1 \neq 1$. By Proposition 8.1, both $\{r_j\}$ and $\{l_i\}$ is strictly increasing. Hence for any given integer m, there are at most one r_j and at most one l_i equaling to m.

The sufficiency in the corollary is clear from definition. So we prove the necessity. Suppose for a given $n > 1$, $\delta(n) = 0$, then there are four cases:

- There exists $v, u \geq 1$ such that $n = r_v = l_u$. Then $\pi(n-1) = \pi(n) - 2 = n-2$. Hence $\delta(n-1) = \pi(n-1) - (n-1) = -1$. This shows $n_0 = n$.
- There exists $v \geq 1$ such that $n = r_v$, and no $l_u = n$. Then $\pi(n-1) = \pi(n) - 1 = n-1$. Hence $\delta(n-1) = 0$. This shows we can reduce the argument from n to $n-1$.
- There exists $l \geq 1$ such that $n = l_u$, and no $r_v = n$. Then $\pi(n-1) = \pi(n) - 1 = n-1$. Hence $\delta(n-1) = 0$. This shows we can reduce the argument from n to $n-1$.
- n does not equals to any elements in $\{l_v\}$ and $\{r_v\}$. In this case $\pi(n-1) = \pi(n) = n$, which contradicts to $\pi(n-1) \leq n-1$, hence impossible.

As a result, when $n_0 \neq n$, we can run the argument again for $n-1$. We claim that eventually it will terminate at some $n_0 \geq 2$. Indeed, if it stops at 1, then there exist $v, u \geq 1$ such that $r_v = l_u = 1$, which contradicts to the assumption that $\pi(1) = 0$. If it reduces to 0, then there exist $v \geq 1$ or $u \geq 1$ such that $r_v = 0$ or $l_u = 0$, which contradicts to $w < 1$. As a result, $n_0 = l_{u'} = r_{v'}$ for some positive integers u', v' and $n_0 \geq 2$. This finishes the proof of necessity. □

Corollary 8.3. *Given rational numbers $s_1 < s_2 < s_3$ with width $w < 1$. Assume $\pi(1) = 0$. Then the minimal degree $d' \geq 2$ if and only if there exists a positive integer v such that*

$$r_v = l_{r_v - v}.$$

Proof. Since $\pi(1) = 0$, by Lemma 8.1 and Lemma 7.3, both $\{l_k\}$ and $\{r_k\}$ are strictly increasing and at least 2. If $d' \geq 2$, then there exists $n \geq 2$ such that $\pi(n) = n$. Then by Lemma 8.2, there exists $n_0 \leq n$ and $u, v \geq 1$, such that $n_0 = r_v = l_u$ and $\pi(n_0) = n_0$. Since both $\{l_k\}$ and $\{r_k\}$ are strictly increasing, so $\pi(n_0) = u + v$ by definition of π. Now $u + v = \pi(n_0) = n_0 = r_v$, so $u = r_v - v$.

Conversely, suppose $r_v = l_{r_v - v}$ for some $v \geq 1$. Let $n_0 = r_v$. Then $n_0 \geq r_1 \geq 2$. Since both $\{l_k\}$ and $\{r_k\}$ are strictly increasing, $\pi(n_0) = v + r_v - v = r_v = n_0$. Therefore $d' \geq 2$ by definition of the minimal degree. □

Proof of Proposition 2.7. Suppose $d \geq 1$, we prove that one of (1), (2), (3) is false. Since $w < 1$, both l_1 and r_1 are positive. Since $d \geq 1$, $d' \geq 1$ too. If $d' = 1$, then $l_1 = 1$ or $r_1 = 1$, so (1) fails. Otherwise, assume $d' \geq 2$. Now $\pi(1) = 0$. By Corollary 8.3, we conclude that there exists a positive integer v such that $r_v = l_{r_v - v}$. Let $u(v) = r_v - v$.

Recall γ is the smallest positive integer such that $\gamma s_2^2 \in \mathbb{Z}$, $\gamma s_3 \in \mathbb{Z}$, and $\gamma s_2 s_3 \in \mathbb{Z}$. Therefore, $\gamma s_2 \in \mathbb{Z}$ also holds. Let q be the quotient of v by γ, and t be the remainder. That is, $v = q\gamma + t$ with $0 \leq t < \gamma$. Then q and t are unique. We define $r_0 = 1$. Now $r_v = \lfloor v s_3 \rfloor - \lceil v s_2 \rceil + 1 = \lfloor (q\gamma + t) s_3 \rfloor - \lceil (q\gamma + t) s_2 \rceil + 1$, which equals to $\lfloor t s_3 \rfloor - \lceil t s_2 \rceil + 1 + q\gamma(s_3 - s_2) = r_t + q\gamma(s_3 - s_2)$ as $\gamma s_2 \in \mathbb{Z}$ and $\gamma s_3 \in \mathbb{Z}$. Hence

$$r_v - v = r_t - t + q\gamma(s_3 - s_2 - 1).$$

Let $A = \gamma(s_3 - s_2 - 1)$, so that $r_v - v = r_t - t + qA$. Then $A > 0$ since $w < 1$. Next we calculate $l_u = l_{r_v - v}$. We have

$$\begin{aligned} l_u &= \lfloor us_2 \rfloor - \lceil us_1 \rceil + 1 \\ &= \lfloor (r_t - t + qA)s_2 \rfloor - \lceil (r_t - t + qA)s_1 \rceil + 1 \\ &= \lfloor (r_t - t)s_2 \rfloor + qAs_2 - \lceil (r_t - t + qA)(s_1 - s_2) + (r_t - t + qA)s_2 \rceil + 1 \\ &= \lfloor (r_t - t)s_2 \rfloor + qAs_2 - \lceil (r_t - t + qA)(s_1 - s_2) + (r_t - t)s_2 \rceil - qAs_2 + 1 \\ &= \lfloor (r_t - t)s_2 \rfloor - \lceil (r_t - t + qA)(s_1 - s_2) + (r_t - t)s_2 \rceil + 1. \end{aligned}$$

where we applied the fact that $qAs_2 = q\gamma(s_3 s_2 - s_2^2 - s_2)$ is an integer.

Replacing $l_u = r_v$ by the expressions of them above, we find

$$\begin{aligned} &\lfloor (r_t - t)s_2 \rfloor - \lceil (r_t - t + qA)(s_1 - s_2) + (r_t - t)s_2 \rceil + 1 = r_t + q\gamma(s_3 - s_2) \\ &\qquad \lfloor (r_t - t + qA)(s_2 - s_1) - (r_t - t)s_2 \rfloor = r_t - 1 + q\gamma(s_3 - s_2) - \lfloor (r_t - t)s_2 \rfloor. \end{aligned}$$

Hence by definition of the floor function,

$$\begin{aligned} r_t - 1 + q\gamma(s_3 - s_2) - \lfloor (r_t - t)s_2 \rfloor &\le (r_t - t + qA)(s_2 - s_1) - (r_t - t)s_2 \\ &< r_t + q\gamma(s_3 - s_2) - \lfloor (r_t - t)s_2 \rfloor. \end{aligned}$$

Adding $(r_t - t)s_2$ to all sides, we obtain that

$$r_t - 1 + q\gamma(s_3 - s_2) + \{(r_t - t)s_2\} \le (r_t - t + qA)(s_2 - s_1) < r_t + q\gamma(s_3 - s_2) + \{(r_t - t)s_2\}.$$

Recall $r_t - t + qA = r_v - v$. Since $\{r_k\}$ is strictly increasing and $r_1 \ge 2$, we have $r_k > k$ for all $k \ge 1$. Hence $r_v > v$, so we can divide by $r_t - t + qA$:

$$\frac{r_t - 1 + q\gamma(s_3 - s_2) + \{(r_t - t)s_2\}}{r_t - t + qA} \le s_2 - s_1 < \frac{r_t + q\gamma(s_3 - s_2) + \{(r_t - t)s_2\}}{r_t - t + qA}.$$

Let $B = r_t + \{(r_t - t)s_2\}$ and $C = r_t - t$. Then the above is equivalent to

$$\frac{q(A + \gamma) + B - 1}{qA + C} \le s_2 - s_1 < \frac{q(A + \gamma) + B}{qA + C}. \tag{8}$$

Notice that all steps above are equivalent. Consequently, when $d' \ne 1$, for any $v \ge 1$, $l_{r_v - v} = r_v$ if and only if $v = q\gamma + t$ and γ, t satisfy (8).

Now suppose $d' \ge 2$ and $l_{r_v - v} = r_v$ for some $v \ge 1$. Then $v = q\gamma + t$ and γ, t satisfy (8). Since $w < 1$, we have $s_2 - s_1 > \dfrac{s_3 - s_2}{s_3 - s_2 - 1} = (A + \gamma)/A$. As a result,

$$\frac{A + \gamma}{A} < \frac{q(A + \gamma) + B}{qA + C}.$$

Since $A, B, C > 0$, we find $(A + \gamma)C < AB$. Therefore, if $t \ne 0$, then $q \ge 0$, so

$$s_2 - s_1 < \frac{q(A + \gamma) + B}{qA + C} \le \frac{B}{C}.$$

which shows that (2) of Proposition 2.7 is false. If $t = 0$, then $q > 0$ because $v = q\gamma + 0 > 0$, so

$$s_2 - s_1 < \frac{q(A + \gamma) + B}{qA + C} \le \frac{(A + \gamma) + B}{A + C}.$$

Here $t = 0$, so $r_0 = 1$, $B = 1 + \{s_2\}$ and $C = 1$. Therefore

$$s_2 - s_1 < \frac{\gamma(s_3 - s_2) + 1 + \{s_2\}}{\gamma(s_3 - s_2 - 1) + 1}.$$

which says (3) of Proposition 2.7 is false. This proves the first part of the proposition.

Conversely, assume (4) of Proposition 2.7 holds. If (1) fails then $d' = 1$, so $d \neq 0$. Otherwise, suppose (1) holds (so $d' \neq 1$) and one of (2) and (3) is false, we will prove that $d' \geq 2$, so $d \neq 0$.

Case I. Assume (1) holds and (2) is false. Let t be the integer such that $1 \leq t \leq \gamma - 1$ and

$$\frac{r_t + \{(r_t - t)s_2\}}{r_t - t} = \max_{1 \leq i \leq \gamma - 1} \frac{r_i + \{(r_i - i)s_2\}}{r_i - i},$$

Then by $w < 1$ we have

$$\frac{s_3 - s_2}{s_3 - s_2 - 1} < s_2 - s_1 < \frac{r_t + \{(r_t - t)s_2\}}{r_t - t},$$

which is

$$\frac{A + \gamma}{A} < s_2 - s_1 < \frac{B}{C}.$$

So $(A + \gamma)C < AB$. Define two sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_n = \frac{n(A + \gamma) + B - 1}{nA + C}, \quad y_n = \frac{n(A + \gamma) + B}{nA + C}.$$

Then $x_n < y_n$ for all $n \geq 0$. Further

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \frac{s_3 - s_2}{s_3 - s_2 - 1} = \frac{A + \gamma}{A}.$$

Since $y_{n+1} - y_n = (nA + C)^{-1}((n + 1)A + C)^{-1}((A + \gamma)C - AB)$, and $(A + \gamma)C < AB$, $\{y_n\}$ is strictly decreasing, We now prove the following interval inclusion:

$$(9) \qquad (\frac{A + \gamma}{A}, \frac{B}{C}) \subset \bigcup_{n \geq 0} [x_n, y_n).$$

Then there exists a non-negative integer q such that $x_q \leq s_2 - s_1 < y_q$, which is exactly the equation (8). Let $v = q\gamma + t$, then $v \geq 1$ and $l_{r_v - v} = r_v$. Hence $d' \geq 2$ by Corollary 8.3.

So we prove the inclusion (9) above. Indeed, $\{y_n\}$ is strictly decreasing, so we need only to show that $x_n \leq y_{n+1}$ for all $n \geq 0$. We have

$$x_q - y_{q+1} = \frac{n(A + \gamma) + B - 1}{nA + C} - \frac{(n + 1)(A + \gamma) + B}{(n + 1)A + C}.$$

Since $w < 1$, $A, C > 0$. Hence $x_n - y_{n+1} \leq 0$ if and only if

$$(n(A + \gamma) + B - 1)((n + 1)A + C) - ((n + 1)(A + \gamma) + B)(nA + C) \leq 0.$$

which simplifies to

$$A(B - n - 1) \leq (A + \gamma + 1)C.$$

Now (4) of Proposition 2.7 says $A(B - 1) \leq (A + \gamma + 1)C$. Since $A > 0$, $n \geq 0$, $A(B - n - 1) \leq (A + \gamma + 1)C$ follows.

Case II. Suppose (1) holds and (3) is false. Then by $w < 1$ we have

$$\frac{s_3 - s_2}{s_3 - s_2 - 1} < s_2 - s_1 < \frac{\gamma(s_3 - s_2) + 1 + \{s_2\}}{\gamma(s_3 - s_2 - 1) + 1}.$$

which is

$$\frac{A+\gamma}{A} < s_2 - s_1 < \frac{A + \gamma + 1 + \{s_2\}}{A+1}. \tag{10}$$

So $(A+\gamma) < A(1 + \{s_2\})$, i.e., $\gamma < A\{s_2\}$.

Consider two sequences z_n and w_n defined by

$$z_n = \frac{n(A+\gamma) + \{s_2\}}{nA+1}, \quad w_n = \frac{n(A+\gamma) + 1 + \{s_2\}}{nA+1}.$$

Since $\{s_2\} < 1 < (s_3 - s_2)/(s_3 - s_2 - 1)$, we have $z_n < (A+\gamma)/A < w_n$ for all $n \geq 0$. Furthermore, $w_{n+1} - w_n = (nA+1)^{-1}((n+1)A+1)^{-1}(\gamma - A\{s_2\})$. So $\gamma < A\{s_2\}$ implies that $\{w_n\}$ is strictly decreasing. Finally, $\lim_{n\to\infty} w_n = (A+\gamma)/A$. Now (10) says $\lim_{n\to\infty} w_n < s_2 - s_1 < w_1$. Hence, there exists a positive integer q such that $z_q < (A+\gamma)/A < s_2 - s_1 < w_q$.

Note that for $t = 0$, $r_t = 1$, so $B = 1 + \{s_2\}$ and $C = 1$. Hence the equation $z_q < s_2 - s_1 < w_q$ is the equation (8) with $t = 0$. Therefore $v = q\gamma + 0$ is a positive integer and $l_{r_v - v} = r_v$. Hence $d' \geq 2$. □

9. Proof of Theorems 2.13 and 2.14

We have shown that the first case of $d' = 5$ in Theorem 2.13 gives a blow-up which is not a MDS if $5s_2 \notin \mathbb{Z}$. The result for $d' = 7, 9$ are given by using a computer program. Now to finish the proof of Theorem 2.14, we need only to show that if $2 \leq d' \leq 9$ and $d' \cdot s_2 \notin \mathbb{Z}$, then

(1) Either Δ_1 belongs to González and Karu's nonexamples (see Remark 2.11), or
(2) $d' = 5, 7$ or 9, and the slopes satisfy the inequalities in Theorem 2.13.

Then Theorem 2.13 and Theorem 2.14 will hold.

Consider the triangle Δ_1 of given slopes (s_1, s_2, s_3). We assume $d' \geq 2$. Since $d' \neq 1$, Lemma 8.1 shows that the numbers of points on each columns are strictly increasing from the leftmost and rightmost vertices to the center column. Notice that there are exactly d' columns in Δ_1 with $\leq d'$ points. Hence these columns must have $2, 3, \cdots, d'-1, d', d'$ points respectively.

Now it is clear that every such triangle Δ_1 determines a subset $S = \{u_1, \cdots, u_{s-1}\} \subset \{2, \cdots, d'-1\}$, whose complement is $\{v_1, \cdots, v_{t-1}\}$, such that leftmost columns of Δ_1 have $u_1, \cdots, u_{s-1}, d'$ points, and the rightmost columns have $v_1, \cdots, v_{t-1}, d'$ points. In particular, González and Karu's nonexamples correspond to $S = \emptyset$ and $T = \{2, \cdots, d'-1\}$.

Conversely, every such subset S determines a system of inequalities on the three slopes, by requiring that the leftmost s columns and rightmost t columns have the given number of points, from S, T, and d'. Therefore, to find all possible triangles with the given d', it suffices to solve the systems of inequalities for all subsets of $\{2, \cdots, d'-1\}$ and collect those who have solutions. This job is best done by a computer program. Here we will show an alternative proof for $2 \leq d' \leq 6$.

Lemma 9.1. *Let Δ be a lattice triangle given by slopes $s_1 < s_2 < s_3$. Let l_k be the number of lattice points on the k-th column from the left. Let the left vertex be p. Let $n \in \mathbb{Z}_{>0}$. Suppose there are at least nk columns with x-coordinate < 0. Then*

(1) $l_k - l_{k-1} \geq l_1 - 1$;
(2) $l_k \geq kl_1 - (k-1)$.
(3) $l_{nk} \geq kl_n - (k-1)$.

Similarly, let r_k be the number of lattice points on the k-th column from the right. Suppose there are at least nk columns with x-coordinate > 0. Then the same inequalities hold for r_k.

Proof. By the hypothesis, l_k has the following expression:

$$l_k = \lfloor ks_2 \rfloor - \lceil ks_1 \rceil + 1.$$

Hence

$$l_k - l_{k-1} = \lfloor ks_2 \rfloor - \lfloor (k-1)s_2 \rfloor - (\lceil ks_1 \rceil - \lceil (k-1)s_1 \rceil).$$

By Lemma 7.1,

$$\lfloor ks_2 \rfloor - \lfloor (k-1)s_2 \rfloor \geq \lfloor s_2 \rfloor, \quad \lceil ks_1 \rceil - \lceil (k-1)s_1 \rceil \leq \lceil s_1 \rceil.$$

Therefore $l_k - l_{k-1} \geq \lfloor s_2 \rfloor - \lceil s_1 \rceil = l_1 - 1$. Now adding the inequalities in (1) gives (2). Finally, notice that the proof of (1) works when replacing the sequence $\{l_k\}_k$ by $\{l_{nk}\}_k$. Hence, (3) holds. The proof for $\{r_k\}$ is identical. □

We can now exclude many subsets S and T for which the conditions in Lemma 9.1 cannot be satisfied. As in the lemma, we let l_k (r_k) be the number of lattice points on the k-th column from the left (right).

Now we prove the classification part of Theorem 2.14 for $d' \leq 6$. Firstly by symmetry we can always assume $|S| \leq \lfloor d'/2 - 1 \rfloor$. For $d' = 2, 3$, the only possible case is $S = \emptyset$. For $d' = 4$, $S = \emptyset$ or $\{2\}$ or $\{3\}$. If $S = \{2\}$, then $T = \{3\}$, so that the $r_1 = 3$, $r_2 = 4$. Now $r_2 < 2r_1 - 1$ contradicts to Lemma 9.1. If $S = \{3\}$, then $T = \{2\}$, so by symmetry this case does not exist.

For $d' = 5$, if $S = \{2\}$ then $T = \{3, 4\}$. Then $(r_1 = 3, r_2 = 4, r_3 = 5)$. If $S = \{4\}$ then $l_1 = 4, l_2 = 5$. In both case, $r_2 < 2r_1 - 1$, contradicts to Lemma 9.1. So $S = \emptyset$ or $S = \{3\}$.

For $d' = 6$, Either $2 \in S$ or $2 \in T$. Assume $2 \in S$. Then $l_1 = 2$, and $r_1 \geq 3$. If $r_1 = 4$ or 5, then by Lemma 9.1, $r_2 \geq 7$, contradiction. If $r_1 = 6$, then $T = \emptyset$, contradicts to $|S| \leq 2$. If $r_1 = 3$, then either $r_2 = 5, r_3 = 6$, or $r_2 = 6$. The first gives a contradiction since $r_3 - r_2 \leq 2$. The second gives $S = \{2, 4, 5\}$, contradicts to $|S| \leq 2$. Therefore $2 \notin S$.

Now we have $2 \in T$. Then $3 \leq l_1 \leq 6$. If $l_1 = 6$ then $S = \emptyset$. Otherwise $l_1 = 3$ or 4 or 5. In all cases, $l_3 \geq 3l_1 - 2 \geq 7$, so $l_2 = 6$. But this shows $S = \{3\}$, and $T = \{2, 4, 5\}$. Then $r_2 = 4$ and $r_4 = 6$ contradicts to (3) of Lemma 9.1.

In conclusion, when $2 \leq d' \leq 6$, the only possible triangles are González and Karu's nonexamples, and the one where $S = \{3\}$, $T = \{2, 4\}$. We can solve the corresponding system of inequalities given by $l_1 = 3, l_2 = 5$, $r_1 = 2$, $r_2 = 4$, and $r_3 = 5$. Assuming that $0 \leq s_2 < 1$, the solution follows form Lemma 2.15:

$$\begin{cases} -2 - \frac{1}{2} < s_1 \leq -2; \\ \frac{1}{3} < s_2 < \frac{1}{2}; \\ 2 \leq s_3 < 2 + \frac{1}{3}. \end{cases}$$

The rest cases for $d' = 7, 8, 9$ are calculated by computer programs. Note that when $d' \leq 6$ we did not invoke the condition of $w < 1$. However we need the condition $w < 1$ when d' is at least 7. Assume $7 \leq d' \leq 9$, and $|S| \leq \lfloor d'/2 - 1 \rfloor$. The only possible triangles other than the $S = \emptyset$ case are given by

(a) $S = \{4\}$, $T = \{2, 3, 5, 6\}$.
(b) $S = \{3, 5\}$, $T = \{2, 4, 6\}$.
(c) $S = \{3, 6\}$, $T = \{2, 4, 5, 7\}$.
(d) $S = \{5\}$, $T = \{2, 3, 4, 6, 7, 8\}$.
(e) $S = \{3, 5, 7\}$, $T = \{2, 4, 6, 8\}$.

We show that (a), (c) and (d) do not satisfy the assumptions of Theorem 2.13. By a computer program, the slopes in case (a) satisfies the following inequalities up to adding a same integers to all the slopes:

$$\begin{cases} -\frac{5}{2} < s_1 \leq -2; \\ s_2 = 1; \\ \frac{7}{3} \leq s_3 < \frac{12}{5}. \end{cases}$$

Therefore the width w is at least the width of triple $(-5/2, 1, 12/5)$, which equals to 1. This contradicts to our assumption that $w < 1$.

Similarly, the slopes in case (d) satisfies

$$\begin{cases} -\frac{7}{2} < s_1 \leq -3; \\ s_2 = 1; \\ \frac{9}{4} \leq s_3 < \frac{16}{7}. \end{cases}$$

so that the width is at least 1, a contradiction.

The slopes in case (c) satisfies

$$\begin{cases} -\frac{7}{3} < s_1 \leq -2; \\ s_2 = \frac{1}{2}; \\ 2 \leq s_3 < \frac{11}{5}. \end{cases} \quad \text{or} \quad \begin{cases} -\frac{5}{3} < s_1 \leq -\frac{3}{2}; \\ s_2 = 1; \\ \frac{5}{2} \leq s_3 < \frac{13}{5}. \end{cases}$$

The second system of inequalities gives $w \geq 1$, hence a contradiction. The first system of inequalities gives $d' \cdot s_2 = 4 \in \mathbb{Z}$, so it does not satisfy the hypothesis of Conjecture 2.10. In conclusion, when assuming $d' \cdot s_2 \notin \mathbb{Z}$, the only non-examples of MDS are given by

- $S = \{3\}$, $T = \{2, 4\}$, $d' = 5$;
- $S = \{3, 5\}$, $T = \{2, 4, 6\}$, $d' = 7$;
- $S = \{3, 5, 7\}$, $T = \{2, 4, 6, 8\}$, $d' = 9$.

which proves Theorem 2.13 and Theorem 2.14.

Proof of Lemma 2.15. Using the notations l_k and r_k (Lemma 9.1), the hypothesis says $l_k = 2k + 1$ for $1 \leq k \leq n$. On the right side, $r_k = 2k$ for $1 \leq k \leq n$ and $r_{n+1} = 2n + 1$.

We can assume $0 \leq s_2 < 1$. First $l_1 = \lfloor s_2 \rfloor - \lceil s_1 \rceil + 1 = 3$, and $\lfloor s_2 \rfloor = 0$. Hence $\lceil s_1 \rceil = -2$. Therefore $\lceil ns_1 \rceil \leq -2n$. Now $l_n = \lfloor ns_2 \rfloor - \lceil ns_1 \rceil + 1 = 2n + 1$. So we have $\lfloor ns_2 \rfloor \leq 0$. Since $s_2 \geq 0$, the only possibility is $\lfloor ns_2 \rfloor = 0$ and $\lceil ns_1 \rceil = -2n$. As a result, $ns_2 < 1$, and $-2n - 1 < ns_1 \leq -2n$. So $s_2 < 1/n$, and $-2 - 1/n < s_1 \leq -2$.

Next we examine the columns on the right. Since $r_1 = \lfloor s_3 \rfloor - \lceil s_2 \rceil + 1 = 2$, and $\lceil s_2 \rceil = 1$, we have $\lfloor s_3 \rfloor = 2$, so that $\lfloor (n+1)s_3 \rfloor \geq 2(n+1)$. From $r_{n+1} = \lfloor (n+1)s_3 \rfloor - \lceil (n+1)s_2 \rceil + 1 =$

$2n+1$, we find $\lceil (n+1)s_2 \rceil \geq 2$. Since $s_2 < 1/n$, the only possibility is $\lceil (n+1)s_2 \rceil = 2$ and $\lfloor (n+1)s_1 \rfloor = 2n+2$. Therefore $(n+1)s_2 > 1$ and $2n+2 \leq (n+1)s_1 < 2n+3$, so $s_2 > 1/(n+1)$ and $2 \leq s_1 < 2 + 1/(n+1)$. □

Appendix A. Figures Classifying triples $(7, b, c)$ where $b, c \leq 70$

We classify triple $(7, b, c)$ where $7 \leq b, c \leq 70$ by various conditions.

In Figure 6, we classify the triples by Cutkosky, González and Karu's results. We first exclude the triples such that $b > c$. Then we paint the blocks in the following order:

(1) Those such that $(a+b+c)^2 > abc$, so $-K_{X'}$ is big, and hence the blow-up X' is a MDS [Cut91, Cor. 1].
(2) Then we exclude triples where a, b, c are not pairwise coprime.
(3) Those in the remaining such that no relation $(e, f, -g)$ with width $w = cg^2/ab < 1$ exists, even after permuting $7, b$ and c.
(4) Those in the remaining such that $cg < a+b+c$, so $-K_{X'}$ is big (See Remark 5.3), and X' is a MDS.
(5) Those in the remaining which belong to González and Karu's non-examples [GK16].
(6) The rest.

In Figure 7, we classify the triples by their reduced degrees. Firstly we exclude the triples such that $b > c$, or not pairwise coprime. For every triple remaining, if there exists a relation $(e, f, -g)$ such that the width $w = cg^2/ab < 1$, then the reduced degree can be defined by Proposition 5.1. Then we divide the triples into three types: $d = 0$, $d = 1$, or $d \geq 2$. Indeed $d = 0$ implies the blow-up is a MDS, and $d = 1$ implies the blow-up is not a MDS. For some triples of $d \geq 2$ with small minimal degrees, whether the blow-up X' is a MDS is known by Theorems 2.13, 2.14, or by González and Karu's non-example criterion [GK16].

FIGURE 6. Classification of $(7, b, c)$ where $b, c \leq 70$ by Cutkosky, González and Karu's results.

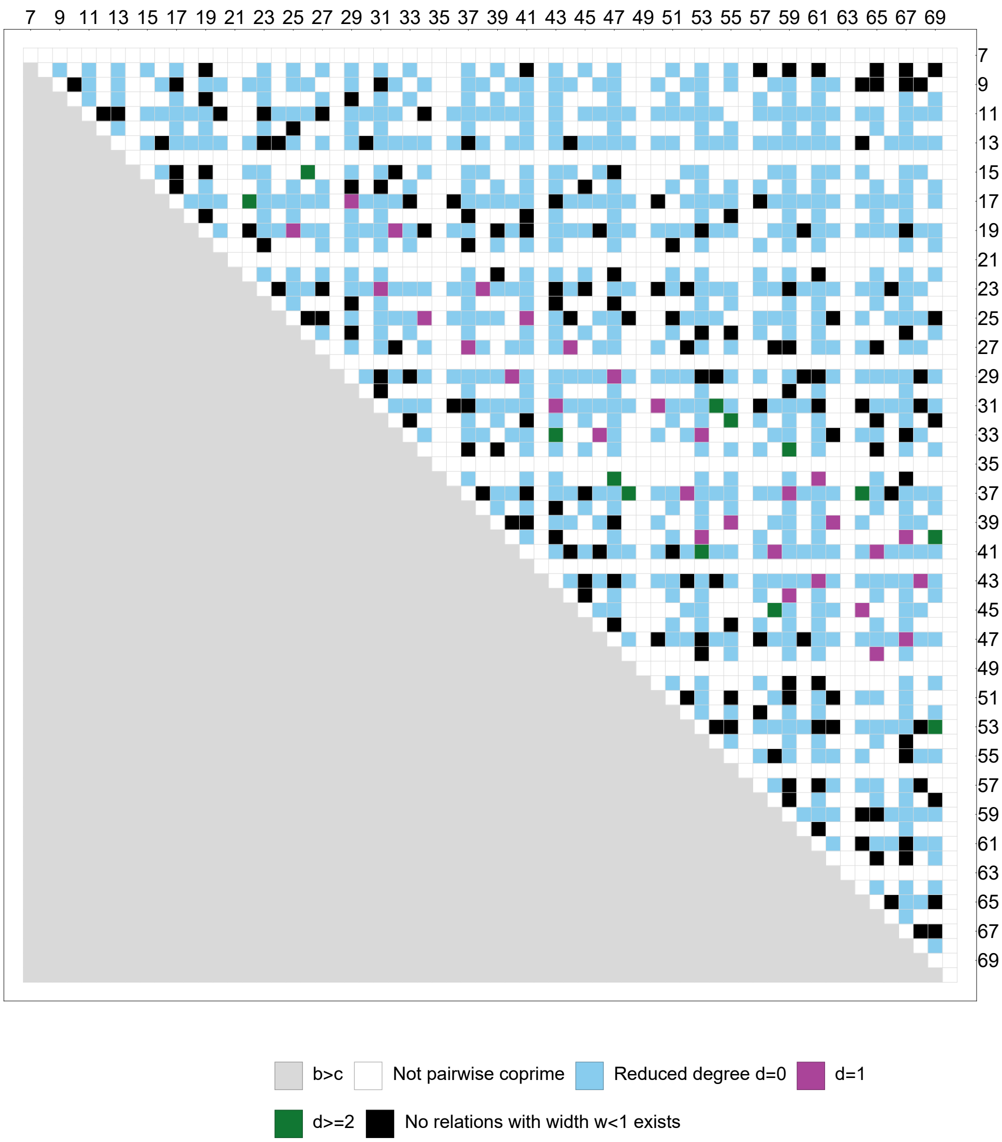

FIGURE 7. Classification of $(7, b, c)$ where $b, c \leq 70$ by reduced degrees

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