Pseudodifferential Operators with Completely Periodic Symbols and Gabor Frames

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Chapter 1
Pseudodifferential operators with completely periodic symbols and Gabor frames

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Abstract We investigate continuity and invertibility of a family of pseudodifferential operators with symbol $p(x, \omega)$ periodic on both variables (x, ω). Some applications to Gabor frames are considered.

1.1 Introduction

Let us consider the continuous *Time-Frequency* (TF) representation of a signal $f = f(t) \in L^2(\mathbb{R}^d)$, given by

$$f = \frac{1}{(g,\gamma)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} V_g f(x,\omega) M_\omega T_x \gamma \, dx \, d\omega, \quad g, \gamma \in L^2(\mathbb{R}^d), \ (g,\gamma) \neq 0, \quad (1.1)$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\mathbb{R}^d)$, $V_g f(x,\omega) := (f, M_\omega T_x g)$ is the *short-time Fourier transform* (STFT) of f with respect to the *analysis window* $g = g(t) \in L^2(\mathbb{R}^d)$ and $M_\omega T_x \gamma(t) := e^{2\pi i \omega \cdot t} \gamma(t - x)$ is the *time-frequency shift* of the *reconstruction window* $\gamma = \gamma(t) \in L^2(\mathbb{R}^d)$.

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Due expecially to numerical purposes, one of the main task of Time Frequency Analysis is replacing the *continuous* representation in (1.1) by an analogous *discrete* representation in terms of countably many shifts $M_{\beta k}T_{\alpha h}\gamma$, for $(h,k) \in \mathbb{Z}^d \times \mathbb{Z}^d$, where the STFT $V_g f$ is sampled at the *discrete lattice* made by points $(\alpha h, \beta k)$, with the *width parameters* $\alpha, \beta > 0$, that is

$$f = \sum_{h,k \in \mathbb{Z}^d} (f, M_{\beta k}T_{\alpha h}g)M_{\beta k}T_{\alpha h}\gamma. \tag{1.2}$$

Differently from the continuous case where g and γ do not need to satisfy any condition unless $(g, \gamma) \neq 0$, to get the discrete representation (1.2) g and γ cannot be chosen arbitrarily. This naturally leads to the study of *Gabor frames*. For $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ and positive numbers α, β, the system

$$\mathscr{G}(g, \alpha, \beta) := \{M_{\beta k}T_{\alpha h}g, \, (h,k) \in \mathbb{Z}^d \times \mathbb{Z}^d\} \tag{1.3}$$

is said to be a *Gabor frame* provided that for some constants $A, B > 0$:

$$A\|f\|_2^2 \leq \sum_{h,k \in \mathbb{Z}^d} |(f, M_{\beta k}T_{\alpha h}g)|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d). \tag{1.4}$$

A classical problem is to find conditions on (g, α, β) under which (1.3) is a Gabor frame. The available literature devoted to such a problem is wide; without being exhaustive, we quote among the others [4], [9], [12], [10], [3].

For every system $\mathscr{G}(g, \alpha, \beta)$ one can formally define the *Gabor operator* as

$$S_g f := \sum_{h,k \in \mathbb{Z}^d} (f, M_{\beta k}T_{\alpha h}g)M_{\beta k}T_{\alpha h}g, \quad f \in L^2(\mathbb{R}^d). \tag{1.5}$$

It is well-known that $\mathscr{G}(g, \alpha, \beta)$ is a Gabor frame if and only S_g is invertible as a linear bounded operator on $L^2(\mathbb{R}^d)$, see [13], [2]. The above considerations motivate the interest to study the L^2 boundedness and invertibiliy of the Gabor operator (1.5). Let us notice that in [2] the Gabor operator (1.5) is expressed in terms of a pseudodifferential operator with Kohn-Nirenberg quantization, that is

$$S_g f(x) = \int e^{2\pi i x \cdot \omega} p(x, \omega)\widehat{f}(\omega)d\omega, \tag{1.6}$$

where the symbol $p(x, \omega)$ reads as

$$p(x, \omega) := \sum_{h,k \in \mathbb{Z}^d} q(x + \alpha h, \omega + \beta k) \quad \text{with} \quad q(x, \omega) := e^{-2\pi i x \cdot \omega}g(x)\overline{\widehat{g}}(\omega). \tag{1.7}$$

Notice that the symbol $p(x.\omega)$ above is periodic in both x and ω. This assures that it is bounded, but prevents any decay behavior at infinity. Applying to (1.6) *Calderón-Vaillancourt Theorem* of L^2-boundedness, in [2] the authors find sufficient conditions on g and α, β such that S_g is continuous and invertible on $L^2(\mathbb{R}^d)$.

The aim of this paper is to give results of continuity and invertibility for pseudodifferential operators with a general *periodic* symbol and provide applications for finding conditions on the the window g and the lattice widths α, β for a Gabor system $\mathscr{G}(g, \alpha, \beta)$ to be a Gabor frame, in view of (1.6), (1.7). The matter is studied in the framework of generalized $L-periodic$ symbols, for any invertible matrix L.

1.2 Notation and Tools

In this section some notation and tools used throughout the presentation are collected. Later on, we set for shortness $\mathbb{Z}_0^n := \mathbb{Z}^n \setminus \{0\}$. For $x, \omega \in \mathbb{R}^n$, $x \cdot \omega = \langle x, \omega \rangle = \sum_{j=1}^n x_j \omega_j$ is the inner product of x and ω and $|x|$ the Euclidean norm of x. We also set $\langle x \rangle := \sqrt{1 + |x|^2}$. We write $\mathscr{F}f(\omega) = \hat{f}(\omega) = \int f(x)e^{-2\pi i x \cdot \omega} dx$ for the Fourier transform of $f \in \mathscr{S}(\mathbb{R}^n)$, with the well known extension to $u \in \mathscr{S}'(\mathbb{R}^n)$.
The *polynomial weight function v* is defined for some $s \geq 0$ by

$$v(z) = (1 + |z|^2)^{s/2}, \quad \forall z \in \mathbb{R}^n. \tag{1.8}$$

A non negative measurable function $m = m(z)$ on $\mathbb{R}^n$ is said to be a *polynomially moderate* if there exists a positive constant C such that

$$m(z_1 + z_2) \leq Cv(z_1)m(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{R}^n. \tag{1.9}$$

For other details about weight functions see [9, §11.1].
Let $GL(n)$ be the space of all invertible real matrices of size $n \times n$. We define the *symplectic matrix* in $GL(2d)$, see [9, §9.4], by

$$\mathscr{I} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad \text{with } I = I_d \text{ the identity matrix of size } d. \tag{1.10}$$

We recall now the definition of the main building block operators in Time Frequency Analysis. For all $z = (x, \omega) \in \mathbb{R}^{2d}$ and $f \in \mathscr{S}(\mathbb{R}^d)$ we set:

$$T_x f(t) = f(t - x) \text{ (translation)}; \quad M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \text{ (modulation)};$$
$$\pi_z f = M_\omega T_x f \text{ (time frequency shift)}, \tag{1.11}$$

with suitable extension to distributions $f \in \mathscr{D}'(\mathbb{R}^d)$.

Througout the rest of the paper, we consider $m = m(z)$ to be a given polynomially moderate weight function on $\mathbb{R}^{2d}$, with respect to some polynomial weight (1.8), in the sense of (1.9)

Definition 1 For $p, q \in [1, +\infty]$, the $m-$weighted space $L_m^{p,q}(\mathbb{R}^{2d})$ consists of all (Lebesgue) measurable functions F on $\mathbb{R}^{2d}$, such that the norm

$$\|F\|_{L_m^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x,\omega)|^p m(x,\omega)^p dx \right)^{q/p} d\omega \right)^{1/q}$$

is finite (with the expected modification in the case when at least one among p or q equals $+\infty$).

Definition 2 For a fixed $g \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$ and $p,q \in [1,+\infty]$, the $m-$weighted modulation space $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M_m^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p dx \right)^{q/p} d\omega \right)^{1/q} \tag{1.12}$$

is finite.

The definition of the space $M_m^{p,q}$ is independent of the choice of the window g in the STFT, and $M_m^{p,q}$ turns out to be a Banach space with respect to a norm (1.12) corresponding to any nonzero window g (norms (1.12) related to different windows are shown to be equivalent to each other). We definitely address the reader to [9] and the references therein for a thorough study of weighted modulation spaces and their properties. Following [9], in the case of $p = q$ we denote $M_m^p := M_m^{p,p}$ and in the *unweighted case* of $m(x,\omega) \equiv 1$ we write $M^{p,q}$ and M^p instead of $M^{p,q}$.

Remark 1 In order to stay in the classical setup of rapidly decreasing functions and tempered distributions, when dealing with modulation spaces, we reduce our previous study to the case when $v(z)$, $z = (x,\omega)$, is a polynomial weight (1.8). However, weighted modulation spaces can be defined even for more general types of non polynomial weight functions, that are only sub-multiplicative, namely satisfying

$$v(z_1 + z_2) \le v(z_1)v(z_2), \quad \forall z_1, z_2 \in \mathbb{R}^{2d}.$$

It is likely expected that our main results in Theorem 1 and Theorem 2 could be extended to the case of non polynomial weight functions, by working in the more general setting of "tempered"ultradistristributions introduced in [5], instead of standard tempered distributions in $\mathscr{S}'(\mathbb{R}^d)$, see also [1].

1.3 Pseudodifferential operators with completely periodic symbols

Definition 3 For $L \in GL(n)$, we say that a distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ is $L-$periodic if

$$T_{L\kappa}u = u \quad \text{for any } \kappa \in \mathbb{Z}^n. \tag{1.13}$$

From Hörmander [14, §7.2], see also [6] for additional details, there is a canonical way of identifying $L-$periodic distributions with *distributions on the $L-$torus*

$\mathbb{T}_L^n := \mathbb{R}^n / L\mathbb{Z}^n$, namely linear continuous forms on $C^\infty(\mathbb{T}_L^n)$. Here after $\mathscr{D}'(\mathbb{T}_L^n)$ denotes the space of distributions on $\mathbb{T}_L^n$. Up to the identification above, one has $\mathscr{D}'(\mathbb{T}_L^n) \subset \mathscr{S}'(\mathbb{R}^n)$. In particular, this means there is a natural way of testing $L-$periodic distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ against $L-$periodic smooth functions $\phi \in C^\infty(\mathbb{T}_L^n)$. This provides a way to define the *Fourier coefficients* of a general $L-$periodic distribution u by setting

$$c_{\kappa,L}(u) = \frac{1}{|\det L|}\langle u, e^{-2\pi i\langle L^{-T}\kappa,\cdot\rangle}\rangle_{\mathbb{T}_L^n}, \tag{1.14}$$

where by $\langle\cdot,\cdot\rangle_{\mathbb{T}_L^n}$ we mean the duality pair betweeen $C^\infty(\mathbb{T}_L^n)$ and its dual space $\mathscr{D}'(\mathbb{T}_L^n)$ and we have set $L^{-T} := (L^{-1})^T$. On can also proves that the distribution u as above enjoys the following *Fourier expansion*

$$u = \sum_{\kappa \in \mathbb{Z}^n} c_{\kappa,L}(u)e^{2\pi i\langle L^{-T}k,\cdot\rangle}, \tag{1.15}$$

with convergence in $\mathscr{S}'(\mathbb{R}^n)$, see again [6] for more details.
For short in the following we set $c_\kappa(u) = c_{\kappa,L}(u)$.

We recall that the τ *pseudodifferential operator*, $0 \le \tau \le 1$, with symbol $p(z) = p(x,\omega) \in \mathscr{S}'(\mathbb{R}^{2d})$, $z = (x,\omega)$, is the operator acting from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ defined by

$$\mathrm{Op}_\tau(p)u(x) := \iint e^{2\pi i(x-y)\cdot\omega} p((1-\tau)x + \tau y, \omega)u(y)\,dy\,d\omega, \quad u \in \mathscr{S}(\mathbb{R}^d). \tag{1.16}$$

The formal integration must be understood in distribution sense. For the definition and development of pseudodifferential operators see the basic texts [17], [15].
Consider now the case when the symbol $p(z)$ in (1.16) is $L-$periodic for a given invertible matrix $L \in GL(2d)$, in the sense of Definition 3. The following result was proved in [6].

Proposition 1 *Consider* $p \in \mathscr{D}'(\mathbb{T}_L^{2d})$, $L \in GL(2d)$. *Then for any* $0 \le \tau \le 1$ *and* $u \in \mathscr{S}(\mathbb{R}^d)$ *we can write*

$$\mathrm{Op}_\tau(p)u = \sum_{\kappa \in \mathbb{Z}^{2d}} c_\kappa(p)e^{2\pi i\tau\langle I_2 L^{-T}\kappa, I_1 L^{-T}\kappa\rangle}\pi_{\mathscr{J}L^{-T}\kappa}u, \tag{1.17}$$

with convergence in $\mathscr{S}'(\mathbb{R}^{2d})$ *of the series above, where* $c_\kappa(p)$ *are the Fourier coefficients of p defined in* (1.14), $\mathscr{J}$ *the matrix introduced in* (1.10) *and* I_1, I_2 *are the* $d \times 2d$ *matrices*

$$I_1 = (I,0), \quad I_2 = (0,I). \tag{1.18}$$

Formula (1.17) provides a simple representation of the $\tau-$pseudodifferential operator (1.16) with $L-$periodic symbol $p(z)$ as a discrete superposition of time-frequency shift operators. This naturally suggests to study the action of such operators on a suitable class of function spaces invariant under time-frequency shifts.

1.4 Continuity and invertibility on time-frequency invariant spaces

Definition 4 We say that a Banach space $\mathscr{S}(\mathbb{R}^d) \hookrightarrow X \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$, with $\mathscr{S}(\mathbb{R}^d)$ dense in X, is *time frequency shifts invariant* (tfs invariant from now on) if for some polynomial weight function v and $C > 0$

$$\|\pi_z u\|_X \leq C v(z) \|u\|_X, \quad u \in X, \quad z \in \mathbb{R}^{2d}. \tag{1.19}$$

Some examples

Consider $p, q \in [1, +\infty)$ and a polynomially moderated weight function $m(z)$ on $\mathbb{R}^{2d}$.

- The $m-$weighted modulation spaces $M_m^{p,q}(\mathbb{R}^d)$, are tfs invariant, see [9, Theorem 11.3.5].
- The $m-$weighted Lebesgue space $L_m^p(\mathbb{R}^d)$ and $m-$weighted Fourier-Lebesgue space $\mathscr{F}L_m^p(\mathbb{R}^d)$, defined as the sets of measurable functions and tempered distributions in $\mathbb{R}^d$, making finite the norms $\|f\|_{L_m^p} := \|m(\cdot, \omega_0)f\|_{L^p}$ and $\|f\|_{\mathscr{F}L_m^p} = \|m(x_0, \cdot)\hat{f}\|_{L^p}$, whatever are $(x_0, \omega_0) \in \mathbb{R}^{2d}$, are tfs invariant.

In both the examples the positive constants C are directly obtained from (1.9) and depend only on the weights m.

The results on pseudodifferential operators presented here after are proven in [6].
The time-frequency invariance condition (1.19) means that time-frequency shift operators are continuous on any tfs invariant space X. Then a sufficient condition for a $\tau-$pseudodifferential operator $\mathrm{Op}_\tau(p)$ with $L-$periodic symbol $p(z)$ to be continuous on any such X directly comes out from the discrete decomposition formula (1.17)

Theorem 1 *Let X be a tfs invariant space, $L \in GL(2d)$, $p \in \mathscr{D}'(\mathbb{T}_L^{2d})$. Assume that the Fourier coefficients $c_\kappa(p)$ defined in* (1.14) *satisfy,*

$$\|c_\kappa(p)\|_{\ell_{L,v}^1} := \sum_{\kappa \in \mathbb{Z}^{2d}} v\left(\mathscr{J}L^{-T}\kappa\right) |c_\kappa(p)| < +\infty. \tag{1.20}$$

Then for any $\tau \in [0,1]$ the operator $\mathrm{Op}_\tau(p)$ is bounded on X and

$$\|\mathrm{Op}_\tau(p)\|_{\mathscr{L}(X)} \leq C \|c_\kappa(p)\|_{\ell_{L,v}^1}, \tag{1.21}$$

Where C is the constant in (1.19).

Cncerning the invertibility of pseudodifferential operators with $L-$periodic symbols we have the following result.

Theorem 2 *Let X be a tfs invariant space, $L \in GL(2d)$, $p \in \mathscr{D}'(\mathbb{T}_L^{2d})$. Assume that the Fourier coefficients $c_\kappa(p)$, $\kappa \in \mathbb{Z}^{2d}$, satisfy*

$$c_0(p) \neq 0 \quad and \quad \sum_{\kappa \in \mathbb{Z}_0^{2d}} |c_\kappa(p)| v \left(\pi_{\mathscr{I}L^{-T}\kappa} \right) < \frac{|c_0(p)|}{C}, \qquad (1.22)$$

where C is the constant in (1.19). *Then for any* $0 \leq \tau \leq 1$

i) *the operator* $\mathrm{Op}_\tau(p)$ *is invertible in* $\mathscr{L}(X)$;
ii) *the norm in* $\mathscr{L}(X)$ *of the inverse operator satisfies the following estimate*

$$\|(\mathrm{Op}_\tau(p))^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{(1+C)\,|c_0(p)| - C\|c_k(p)\|_{\ell^1_{L,m}}}. \qquad (1.23)$$

1.5 Application to Gabor frames

For the proof of the results presented in this section, the reader is addressed to [7].
We say *(generalized) Gabor system* a sequence $\mathscr{G}(g,L) := \{\pi_{L\kappa}g\}_{\kappa \in \mathbb{Z}^{2d}}$, where $L \in GL(2d)$ and g is a generic measurable function on $\mathbb{R}^d$.
We can associate to a Gabor system the operator

$$S_{g,\gamma}^L u := \sum_{\kappa \in \mathbb{Z}^{2d}} (u, \pi_{-L\kappa}g)\, \pi_{-L\kappa}\gamma, \qquad (1.24)$$

said *Gabor operator* with windows g, γ. Here $\gamma = \gamma(t)$, $g = g(t)$ are regular enough to guarantee the convergence of the series in the right-hand side of (1.24) at least in $\mathscr{S}'(\mathbb{R}^d)$, whenever $u \in \mathscr{S}(\mathbb{R}^d)$. By reducing L to a suitable diagonal matrix and setting $\gamma = g$, we obtain the Gabor system (1.3) and operator (1.5) in classical terms, see e.g. [9], [11], [13], [2] and the reference therein.
We take now $\gamma \in L^1(\mathbb{R}^d)$ and $\hat{g} \in L^1(\mathbb{R}^d)$, and let $q(x,\omega)$ be the symbol defined by

$$q(x,\omega) := e^{-2\pi i x \cdot \omega}(\gamma \otimes \bar{\hat{g}})(x,\omega) = e^{-2\pi i x \cdot \omega}\gamma(x)\bar{\hat{g}}(\omega), \quad (x,\omega) \in \mathbb{R}^{2d}, \qquad (1.25)$$

so that $q \in L^1(\mathbb{R}^{2d})$.
For a given matrix $L \in GL(2d)$, we define the $L-$periodic symbol

$$q_L(x,\omega) = \sum_{\kappa \in \mathbb{Z}^{2d}} q(x + I_1 L\kappa, \omega + I_2 L\kappa), \quad (x,\omega) \in \mathbb{R}^{2d}, \qquad (1.26)$$

where I_1 and I_2 are the $d \times 2d$ matrices defined in (1.18)
Following Boggiatto-Garello [2], we consider the pseudodifferential operator $q_L(\cdot,D) = \mathrm{Op}_0(q_L)$ with Kohn-Nirenberg quantization (i.e. $\tau = 0$) and symbol $q_L(x,\omega)$.
In [6], we proved the following.

Proposition 2 *Consider* g, γ *measurable functions such that* $\gamma \in L^1(\mathbb{R}^d)$, $\hat{g} \in L^1(\mathbb{R}^d)$ *and* $L \in GL(2d)$. *Then*

$$S_{g,\gamma}^L u = q_L(\cdot,D)u \quad \forall u \in \mathscr{S}(\mathbb{R}^d), \qquad (1.27)$$

where the symbol $q_L(x,\omega)$ is defined in (1.26).

The above proposition shows that the generalized Gabor operator coincides with the pseudodifferential operator with periodic symbol (1.26). Therefore, applying to such operator the continuity and the invertibility results given by Theorem 1 and Thorem 2 we obtain a sufficient condition of continuity or invertibility for the Gabor operator $S^L_{g,\gamma}$. To do so, we need to compute the Fourier coefficients of the periodic symbol $q_L(x,\omega)$. It can be shown, see [6], that

$$c_\kappa(q_L) = \frac{1}{|\det L|} V_g\gamma(\mathscr{J}L^{-T}\kappa), \quad \kappa \in \mathbb{Z}^{2d}. \tag{1.28}$$

As an application of Theorem 1 to the symbol $q_L(x,\omega)$ defined as in (1.25) we get the following

Proposition 3 *Let the measurable functions g,γ satisfy $\gamma\in L^1(\mathbb{R}^d)$, $\hat{g}\in L^1(\mathbb{R}^d)$ and let $L\in GL(2d)$ be such that*

$$\sigma_{L,g,\gamma} := \sum_{\kappa\in\mathbb{Z}^{2d}} v\left(\mathscr{J}L^{-T}\kappa\right)|V_g\gamma(\mathscr{J}L^{-T}\kappa)| < +\infty. \tag{1.29}$$

Then the Gabor operator $S^L_{g,\gamma}$ extends to a linear bounded operator in $\mathscr{L}(X)$ and the operator norm of $S^L_{g,\gamma}$ in $\mathscr{L}(X)$ enjoys the estimate

$$\|S^L_{g,\gamma}\|_{\mathscr{L}(X)} \le \frac{C}{|\det L|}\sigma_{L,g,\gamma}, \tag{1.30}$$

being $C>0$ the constant in (1.19).

As a consequence of Proposition 3 together with some known properties of Modulation spaces, see [9, Proposition 12.1.4, Proposition 12.1.11] we get the following.

Corollary 1 *Assume that $\gamma,g\in M^1_v$. Then for every $L\in GL(2d)$ the Gabor operator $S^L_{g,\gamma}$ extends to a linear bounded operator in $\mathscr{L}(X)$; as such, the operator norm of $S^L_{g,\gamma}$ satisfies the following estimate*

$$\|S^L_{g,\gamma}\|_{\mathscr{L}(X)} \le C\|g\|_{M^1_v}\|\gamma\|_{M^1_v}, \tag{1.31}$$

with $C>0$ depending only on the weight v.

As for the invertibility of $S^L_{g,\gamma}$, the following result comes from applying Theorem 2 to the symbol $q_L(x,\omega)$.

Proposition 4 *Let g,γ satisfy $\gamma\in L^1(\mathbb{R}^d)$, $\hat{g}\in L^1(\mathbb{R}^d)$, $(\gamma,g)_{L^2}\neq 0$ and let $L\in GL(2d)$ such that*

$$\sum_{\kappa\in\mathbb{Z}^{2d}_0} v(\mathscr{J}L^{-T}\kappa)|V_g\gamma(\mathscr{J}L^{-T}\kappa)| < \frac{1}{C}|(\gamma,g)_{L^2}|, \tag{1.32}$$

where C is the constant in (1.19). Then $S_{g,\gamma}^L$ is invertible as an element of $\mathscr{L}(X)$. The norm of the inverse operator satisfies

$$\|(S_{g,\gamma}^L)^{-1}\|_{\mathscr{L}(X)} \leq \frac{|\det L|}{(1+C)\,|(\gamma,g)_{L^2}| - C\sigma_{L,g,\gamma}}, \tag{1.33}$$

where $\sigma_{L,g,\gamma}$ is defined in (1.29) and C is again the constant in (1.19).

Remark 2 Notice that condition (1.29), ensuring that $S_{g,\gamma}^L$ extends to an element of $\mathscr{L}(X)$, trivially follows from (1.32).

Corollary 2 *Consider $L \in GL(2d)$ and $g \in L^1 \setminus \{0\}$, such that $\hat{g} \in L^1$, which satisfy*

$$\sum_{\kappa \in \mathbb{Z}_0^{2d}} v(\mathscr{J}L^{-T}\kappa)\left|V_g g\left(\mathscr{J}L^{-T}\kappa\right)\right| < \|g\|_{L^2}^2\,, \tag{1.34}$$

then the Gabor system $\mathscr{G}_L := \{\pi_{L\kappa}g\}_{\kappa \in \mathbb{Z}^{2d}}$ is a frame in L^2, with possible frame bounds

$$A = \frac{2\|g\|_{L^2}^2 - \sigma_{L,g}}{|\det L|}; \tag{1.35}$$

$$B = \frac{\sigma_{L,g}}{|\det L|} < \frac{2}{|\det L|}\|g\|_{L^2}^2,$$

where $\sigma_{L,g} := \sigma_{L,g,g}$.

With the aim of making more explicit than in (1.32) the conditions on g, γ and L for the invertibility of the operator $S_{g,\gamma}^L$, we consider now the case of a diagonal matrix

$$L = diag(\alpha_1,\ldots,\alpha_d,\beta_1,\ldots,\beta_d)\,, \quad \alpha_j,\,\beta_j > 0\,, \quad \text{for } j = 1,\ldots,d\,; \tag{1.36}$$

in this case we write $S_{g,\gamma}^{\alpha,\beta} := S_{g,\gamma}^L$.

Corollary 3 *Let $\gamma, g \in M_v^1$ satisfy $(\gamma,g)_{L^2} \neq 0$ and $\theta := \max_{j=1,\ldots,d}\{\alpha_j,\beta_j\}$. Then there exists $0 < \theta_0 < 1$ depending only on the weight v and $(\gamma,g)_{L^2}$ such that if $\theta \leq \theta_0$ then the Gabor operator $S_{g,\gamma}^{\alpha,\beta}$ is invertible as an element of $\mathscr{L}(X)$.*

Unfortunately it seems that no explicit upper bound of θ is available without imposing additional decay estimates on the growth at infinity of γ and g. Looking for such an explicit upper bound will be postponed to future research.

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