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Normalized solutions of L^2-supercritical NLS equations on compact metric graphs

Xiaojun Chang *[1], Louis Jeanjean†[2], and Nicola Soave‡[3]

[1]*School of Mathematics and Statistics & Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024, Jilin, PR China*
[2]*Laboratoire de Mathématiques (CNRS UMR 6623), Université de Franche-Comté, Besancon 25030, France*
[3]*Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133, Milano, Italy*

Abstract

This paper is devoted to the existence of non-trivial bound states of prescribed mass for the mass-supercritical nonlinear Schrödinger equation on compact metric graphs. The investigation is based upon a variational principle which combines the monotonicity trick and a min-max theorem with second order information for constrained functionals, and upon the blow-up analysis of bound states with prescribed mass and bounded Morse index.

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1 Introduction and main results

In this paper we investigate the existence of non-constant critical points for the *mass supercritical* NLS energy functional $E(\cdot,\mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$ defined by

$$E(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,dx, \qquad p>6 \tag{1.1}$$

under the mass constraint

$$\int_{\mathcal{G}} |u|^2\,dx = \mu > 0, \tag{1.2}$$

where $\mathcal{G}$ is a *compact metric graph*. Critical points, also called *bound states*, solve the stationary nonlinear Schrödinger equation (NLS) on $\mathcal{G}$

$$-u'' + \lambda u = |u|^{p-2}u \tag{1.3}$$

for some Lagrange multiplier λ, coupled with Kirchhoff condition at the vertexes (see (1.4) below). In turn, solutions to (1.4) give standing waves of the time-dependent NLS on $\mathcal{G}$.

*changxj100@nenu.edu.cn
†louis.jeanjean@univ-fcomte.fr
‡nicola.soave@polimi.it

There are several physical motivations to consider Schrödinger equations on metric graphs. We refer the interested reader to the recent paper [24], to [6, 9, 28], and to the references therein. In addition, the problem on metric graphs presents interesting new mathematical features with respect to the Euclidean case. For these reasons, the problem of existence of bound states on metric graphs attracted a lot of attention in the past decade, mainly in the *subcritical* or *critical regimes*, which correspond to $p \in (2,6)$ or $p = 6$, respectively. In such frameworks, a particularly relevant issue concerns the existence of *ground states*, that is, global minimizers of the energy under the mass constraint, see [2, 3, 4, 5] for non-compact $\mathcal{G}$, and [12, 13] for the compact case. We also refer the interested reader to [10, 15, 29, 31, 32, 34] and references therein for strictly related issues (problems with localized nonlinearities, combined nonlinearities, existence of critical points in absence of ground states), always in subcritical and critical regimes.

In striking contrast, as far as we know, the supercritical regime on general graphs is essentially untouched. In this case the energy is always unbounded from below, and ground states never exist. However, it is natural to discuss the existence of bound states, and in this paper we address this problem on any *compact* graph $\mathcal{G}$. An interesting feature of this setting is that there always exists a constrained constant (trivial) critical point of $E(\cdot,\mathcal{G})$, obtained by taking the constant function $\kappa_\mu := (\mu/\ell)^{1/2}$, where ℓ denotes the total length of $\mathcal{G}$. Thus, in order to obtain a non-trivial result, one has to focus on existence of non-constant bound states.

Basic notations and main result

We recall that a metric graph $\mathcal{G} = (\mathcal{E},\mathcal{V})$ is a connected metric space obtained by glueing together a number of closed line intervals, the edges in $\mathcal{E}$, by identifying some of their endpoints, the vertexes in $\mathcal{V}$. The peculiar way in which these identifications are performed defines the topology of $\mathcal{G}$. Any bounded edge e is identified with a closed bounded interval I_{e}, typically $[0,\ell_{\mathrm{e}}]$ (where ℓ_{e} is the length of e), while unbounded edges are identified with (a copy of) the closed half-line $[0,+\infty)$. A metric graph is compact if and only if it has a finite number of edges, and none of them is unbounded.

A function u on $\mathcal{G}$ is a map $u : \mathcal{G} \to \mathbb{R}$, which is identified with a vector of functions $\{u_{\mathrm{e}}\}$, where each u_{e} is defined on the corresponding interval I_{e}. Endowing each edge with Lebesgue measure, one can define L^p spaces over $\mathcal{G}$, denoted by $L^p(\mathcal{G})$, in a natural way, with norm

$$\|u\|^p_{L^p(\mathcal{G})} = \sum_{\mathrm{e}} \|u_{\mathrm{e}}\|^p_{L^p(\mathrm{e})}.$$

The Sobolev space $H^1(\mathcal{G})$ is defined as the set of functions $u : \mathcal{G} \to \mathbb{R}$ such that $u_{\mathrm{e}} \in H^1([0,\ell_{\mathrm{e}}])$ for every bounded edge e, $u_{\mathrm{e}} \in H^1([0,+\infty))$ for every unbounded edge e, and u is continuous on $\mathcal{G}$ (in particular, if a vertex v belongs to two or more edges e_i, the corresponding functions u_{e_i} take the same value on v); the norm in $H^1(\mathcal{G})$ is naturally defined as

$$\|u\|^2_{H^1(\mathcal{G})} = \sum_{\mathrm{e}} \|u'_{\mathrm{e}}\|^2_{L^2(\mathrm{e})} + \|u_{\mathrm{e}}\|^2_{L^2(\mathrm{e})}.$$

We aim at proving the existence of non-constant critical points of the energy $E(\cdot,\mathcal{G})$, defined in (1.1), constrained on the L^2-sphere

$$H^1_\mu(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) : \int_{\mathcal{G}} |u|^2\,dx = \mu \right\}.$$

If $u \in H^1_\mu(\mathcal{G})$ is such a critical point, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that u satisfies the following problem:

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{for every edge } \mathrm{e} \in \mathcal{E}, \\ \sum_{\mathrm{e} \succ \mathrm{v}} u'_{\mathrm{e}}(\mathrm{v}) = 0 & \text{at every vertex } \mathrm{v} \in \mathcal{V}, \end{cases} \tag{1.4}$$

where e $\succ$ v means that the edge e is incident at v, and the derivative $u_{\mathrm{e}}'(v)$ is always an outer derivative. The second equation is the so called *Kirchhoff condition*. Notice that the positive constant function $\kappa_\mu = (\mu/\ell)^{1/2}$ trivially satisfies (1.4), for $\lambda = (\mu/\ell)^{(p-2)/2}$.

Our main existence result is as follows.

Theorem 1.1. *Let $\mathcal{G}$ be any compact metric graph, and $p > 6$. There exists $\mu_1 > 0$ depending on $\mathcal{G}$ and on p such that, for any $0 < \mu < \mu_1$, problem (1.4) with the mass constraint (1.2) has a positive non-constant solution which corresponds to a mountain pass critical point of $E(\cdot, \mathcal{G})$ on $H^1_\mu(\mathcal{G})$, at a strictly larger energy level than κ_μ.*

Remark 1.2. Note that the Lagrange multiplier associated with any positive solution u to (1.4) is positive. Indeed, by standard arguments, we know that $u \in C^2(\mathrm{e})$ on every edge. Then, integrating the first equation in (1.4) on every edge, summing over the edges and making use of the Kirchhoff condition, we obtain

$$\lambda \|u\|_{L^1(\mathcal{G})} = \|u\|^{p-1}_{L^{p-1}(\mathcal{G})},$$

whence we deduce that $\lambda > 0$.

Remark 1.3. The theorem is not a perturbation result, in the sense that the value μ_1 will not be obtained by any limit process, and can be explicitly estimated. We refer to Proposition 2.1 and Remark 2.2 for more details.

On the other hand, one may wonder whether or not the restriction $\mu < \mu_1$ can be removed. This is an open problem, our min-max approach fails for large masses. Observing that our solutions will have Morse index at most 2 as critical points of the associated action functional (see Section 3), another related issue could be to investigate if it is possible to find solutions of (1.4), possibly non-positive, with any mass $\mu > 0$ and Morse index bounded by 2. For the NLS equations with Dirichlet conditions in bounded Euclidean domains, this question has a negative answer, see [33, Theorem 1.2]. Even if the two problems are not equivalent, this result suggests that a bound of type $\mu < \mu_1$ may be necessary.

The proof of Theorem 1.1 is divided into some intermediate steps. At first, in Section 2, we observe the local minimality of the constant solution for $\mu < \mu_1$, following [12].

Since in addition $E(\cdot, \mathcal{G})$ is unbounded from below, as $p > 6$, this naturally suggests the possible existence of a second critical point, of mountain pass type. However, we have to face some severe compactness issues, and, in particular, the existence of a *bounded* Palais-Smale sequence at the mountain pass level is not straightforward. We point out that the techniques based on scalings, usually employed in the Euclidean setting and related to the validity of a Pohozaev identity (see [22] or [7, 21]), do not work, since $\mathcal{G}$ is not scale invariant. To overcome this obstruction, a first natural attempt is to adapt the monotonicity trick of [23]: we first introduce a family of functionals $E_\rho(\cdot, \mathcal{G}) : H^1_\mu(\mathcal{G}) \to \mathbb{R}$ defined by

$$E_\rho(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{\rho}{p}\int_{\mathcal{G}} |u|^p\,dx, \quad \rho \in \left[\frac{1}{2}, 1\right].$$

Exploiting the monotonicity of $E_\rho(u, \mathcal{G})$ with respect to ρ, we can easily show that $E_\rho(\cdot, \mathcal{G})|_{H^1_\mu(\mathcal{G})}$ has a bounded Palais-Smale sequence of mountain pass type, for almost every $\rho \in [1/2, 1]$. Since $\mathcal{G}$ is compact, this ensures the existence of a critical point u_ρ of $E_\rho(u, \mathcal{G})$, for almost every $\rho \in [1/2, 1]$. Now the idea is to take the limit of $\{u_{\rho_n}\}$ along a sequence $\rho_n \to 1^-$, in order to obtain a critical point of the original functional. However, once again the boundedness of $\{u_{\rho_n}\}$ is an issue. In order to gain compactness, we use a general principle which combines the monotonicity trick, as presented in [23], and the mountain pass theorem with second order information for constrained functionals, Theorem 3.10 below (this is [11, Theorem 1]). A similar result was recently proved in [8, 26] in the unconstrained setting.

Mountain pass or min-max theorems with second order information have been introduced in [18, 19]. The second order information turns out to be extremely useful in proving the compactness of Palais-Smale sequences when the problem is not scale-invariant (and hence a Pohozaev identity is not available).

With Theorem 3.10, we prove the existence of a sequence $\{u_{\rho_n}\}$ critical points for $E_\rho(\cdot,\mathcal{G})|_{H^1_\mu(\mathcal{G})}$ (which in particular are solutions of approximating problems) *with uniformly bounded Morse index*. In Section 4, we perform a detailed blow-up analysis for this type of sequences, in the spirit of [17] (see also [33]). We think that this analysis is of independent interest and, for the sake of generality, we perform it on graphs which are not necessarily compact. In Theorem 4.2, we characterize the blow-up behavior of solutions close to local maximum points, both when they accumulate in the interior of one edge, or when they accumulate on a vertex; in the latter case, the limit problem is an NLS equation posed on a *star-graph*, which is a new phenomenon with respect to the Euclidean case. In Theorem 4.6, we establish a relation between the upper bound on the Morse index and the number of maximum points of the solutions, and describe the behavior far away from them.

Afterwards, Theorems 4.2 and 4.6 are used in Section 5 to finally deduce, via a contradiction argument, that also the sequence $\{u_{\rho_n}\}$ is bounded, and converges to the non-constant mountain pass solution of Theorem 1.1.

2 Local minimality of the constant solution

Let $\kappa_\mu := (\mu/\ell)^{1/2}$ with $\ell := |\mathcal{G}|$ being the total length of the graph $\mathcal{G}$. Clearly, the constant function κ_μ is always a solution to (1.4) in $H^1_\mu(\mathcal{G})$ for some $\lambda \in \mathbb{R}$, and hence a constrained critical point of $E(\cdot,\mathcal{G})$ on $H^1_\mu(\mathcal{G})$. Furthermore, following [12], we can give a variational characterization of κ_μ.

Proposition 2.1. *Assume that $\mathcal{G}$ is a compact graph and $p > 6$. Then there exists $\mu_1 > 0$ depending on $\mathcal{G}$ and on p such that*

(i) *if $0 < \mu < \mu_1$, then κ_μ is a strict local minimizer of $E(u,\mathcal{G})$ in $H^1_\mu(\mathcal{G})$;*

(ii) *if $\mu > \mu_1$, then κ_μ is not a local minimizer of $E(u,\mathcal{G})$ in $H^1_\mu(\mathcal{G})$.*

Proof. To characterize the variational properties of κ_μ, we shall evaluate the sign of the quadratic form $\varphi \in T_{\kappa_\mu} H^1_\mu(\mathcal{G}) \mapsto d^2|_{H^1_\mu(\mathcal{G})} E(\kappa_\mu,\mathcal{G})[\varphi,\varphi]$, where $d^2|_{H^1_\mu(\mathcal{G})} E(u,\mathcal{G})$ denotes the constrained Hessian of $E(\cdot\,\mathcal{G})$ on $H^1_\mu(\mathcal{G})$ and $T_{\kappa_\mu} H^1_\mu(\mathcal{G})$ is the tangent space of $H^1_\mu(\mathcal{G})$ at κ_μ, defined as follows:

$$T_{\kappa_\mu} H^1_\mu(\mathcal{G}) := \left\{ \phi \in H^1(\mathcal{G}) : \int_{\mathcal{G}} \phi\, dx = 0 \right\}.$$

From [12, Proposition 4.1], which remains valid with the same proof for $p > 6$ we obtain

$$d^2|_{H^1_\mu(\mathcal{G})} E(\kappa_\mu,\mathcal{G})[\phi,\phi] = \int_{\mathcal{G}} |\phi'|^2\, dx - (p-2)\kappa_\mu^{p-2} \int_{\mathcal{G}} |\phi|^2\, dx, \qquad \forall \phi \in T_{\kappa_\mu} H^1_\mu(\mathcal{G}). \tag{2.1}$$

Denote now by $\lambda_2(\mathcal{G})$ the smallest positive eigenvalue of the Kirchhoff Laplacian on $\mathcal{G}$ (that is $-(\cdot)''$ on $\mathcal{G}$, coupled with the Kirchhoff condition at the vertexes), namely

$$\lambda_2(\mathcal{G}) = \inf_{\phi \in H^1(\mathcal{G}), \int_{\mathcal{G}} \phi\, dx = 0} \frac{\int_{\mathcal{G}} |\phi'|^2 dx}{\int_{\mathcal{G}} |\phi|^2 dx}.$$

Let us suppose that $0 < \mu < \mu_1$, where

$$\mu_1 := \ell \left(\frac{\lambda_2(\mathcal{G})}{p-2} \right)^{\frac{2}{p-2}}, \tag{2.2}$$

and let $\beta \in (0,1)$ be such that

$$\beta \lambda_2(\mathcal{G}) - (p-2) \left(\frac{\mu}{\ell} \right)^{\frac{p-2}{2}} > 0. \tag{2.3}$$

In view of (2.1), it follows that

$$d^2|_{H^1_\mu(\mathcal{G})}E(\kappa_\mu,\mathcal{G})[\phi,\phi] \geq (1-\beta)\int_{\mathcal{G}}|\phi'|^2 + [\beta\lambda_2(\mathcal{G})-(p-2)\kappa_\mu^{p-2}]\int_{\mathcal{G}}|\phi|^2dx$$

for every $\phi \in T_{\kappa_\mu}H^1_\mu(\mathcal{G})$, which implies that $d^2|_{H^1_\mu(\mathcal{G})}E(\kappa_\mu,\mathcal{G})$ is positive definite whenever $0<\mu<\mu_1$. Hence, for any such μ, the constant κ_μ is a strict local minimizer of $E(\cdot,\mathcal{G})$ on $H^1_\mu(\mathcal{G})$.

If instead $\mu>\mu_1$, taking an eigenfunction ϕ_2 corresponding to $\lambda_2(\mathcal{G})$, we obtain

$$d^2|_{H^1_\mu(\mathcal{G})}E(\kappa_\mu,\mathcal{G})[\phi_2,\phi_2] = [\lambda_2(\mathcal{G})-(p-2)\kappa_\mu^{p-2}]\int_{\mathcal{G}}|\phi_2|^2dx<0,$$

which implies that κ_μ is not a local minimizer of $E(u,\mathcal{G})$ in $H^1_\mu(\mathcal{G})$. □

Remark 2.2. By [20, Theorem 1], we have $\lambda_2(\mathcal{G})\geq \pi^2/\ell^2$. Then by (2.2) it follows that

$$\mu_1 \geq \ell^{\frac{p-6}{p-2}}\left(\frac{\pi^2}{p-2}\right)^{\frac{2}{p-2}}.$$

In particular, $\mu_1\to+\infty$ as $\ell\to+\infty$.

3 Mountain pass solutions for approximating problems

When κ_μ is a local minimizer of the energy, and since the energy is unbounded from below on $H^1_\mu(\mathcal{G})$ in the supercritical regime, one may consider the question of finding a non-constant solution of mountain pass (MP) type. The existence of a MP solution will be the content of this and the next two sections. Before proceeding, it is convenient to recall a preliminary result and a definition.

Lemma 3.1 (Proposition 3.1 in [13])**.** *Assume that $\mathcal{G}$ is a compact graph and $\{u_n\}\subset H^1_\mu(\mathcal{G})$ is a bounded Palais-Smale sequence of $E(\cdot,\mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$. Then there exists $u\in H^1(\mathcal{G})$ such that, up to a subsequence, $u_n\to u$ strongly in $H^1_\mu(\mathcal{G})$.*

Definition 3.2. For any graph $\mathcal{F}$ (not necessarily compact) and any solution $U\in C(\mathcal{F})\cap H^1_{\mathrm{loc}}(\mathcal{F})$, not necessarily in $H^1(\mathcal{F})$, of

$$\begin{cases} -U''+\lambda U = \rho|U|^{p-2}U & \text{in } \mathcal{F}, \\ \sum_{\mathrm{e}\succ\mathrm{v}} U'(\mathrm{v})=0 & \text{for any vertex v of } \mathcal{F}, \end{cases} \tag{3.1}$$

with $\lambda,\rho\in\mathbb{R}$, we consider

$$Q(\varphi;U,\mathcal{F}) := \int_{\mathcal{F}}\left(|\varphi'|^2+(\lambda-(p-1)\rho|U|^{p-2})\varphi^2\right)dx, \quad \forall\varphi\in H^1(\mathcal{F})\cap C_c(\mathcal{F}). \tag{3.2}$$

The *Morse index of* U, denoted by $m(U)$, is the maximal dimension of a subspace $W\subset H^1(\mathcal{F})\cap C_c(\mathcal{F})$ such that $Q(\varphi;U,\mathcal{F})<0$ for all $\varphi\in W\setminus\{0\}$.

Note that this is the definition of Morse index as solution to (3.1), and not as critical point of the energy functional under the L^2 constraint (see Definition 3.9 below).

Lemma 3.1 is a useful result which exploits the compactness of the reference graph $\mathcal{G}$. However, as already anticipated in the introduction, in the present setting even the existence of a *bounded* Palais-Smale sequence at the mountain pass level is not straightforward. To overcome this issue, we introduce the family of functionals

$$E_\rho(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{\rho}{p}\int_{\mathcal{G}} |u|^p\,dx,$$

depending on the parameter $\rho \in [1/2,1]$. The idea is to adapt the monotonicity trick [23] on this family.

The main result of this section is the following:

Proposition 3.3. *Let $\mu \in (0,\mu_1)$. For almost every $\rho \in [1/2,1]$, there exists a critical point u_ρ of $E_\rho(\cdot,\mathcal{G})$ on $H^1_\mu(\mathcal{G})$, at level $c_\rho > E_\rho(\kappa_\mu,\mathcal{G})$, which solves*

$$\begin{cases} -u_\rho'' + \lambda_\rho u_\rho = \rho u_\rho^{p-1}, \quad u_\rho > 0 & \text{in } \mathcal{G}, \\ \sum_{\mathrm{e}\succ \mathrm{v}} u_\rho'(\mathrm{v}) = 0 & \text{for any vertex } \mathrm{v}, \end{cases} \tag{3.3}$$

for some $\lambda_\rho > 0$. Moreover, its Morse index satisfies $m(u_\rho) \le 2$.

In the proof of the proposition, the value of $\mu \in (0,\mu_1)$ is fixed and will not change. As a first step, we show that the family of functionals $E_\rho(\cdot,\mathcal{G})$ has a mountain pass geometry on $H^1_\mu(\mathcal{G})$ around the constant local minimizer κ_μ, uniformly with respect to ρ.

Lemma 3.4. *There exists $w \in H^1_\mu(\mathcal{G})$ such that, setting*

$$\Gamma := \left\{\gamma \in C([0,1], H^1_\mu(\mathcal{G})) :\ \gamma(0) = \kappa_\mu,\ \gamma(1) = w\right\},$$

we have that

$$c_\rho := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} E_\rho(\gamma(t),\mathcal{G}) > E_\rho(\kappa_\mu,\mathcal{G}) = \max\{E_\rho(\gamma(0),\mathcal{G}), E_\rho(\gamma(1),\mathcal{G})\}, \quad \forall \rho \in \left[\frac{1}{2},1\right].$$

Remark 3.5. Note that the functions κ_μ and w, and hence also Γ, are independent of ρ.

Proof. Since $\rho \le 1$, and taking advantage of the monotonicity, we see from the proof of Proposition 2.1 that κ_μ remains a strict local minimizer of $E_\rho(\cdot,\mathcal{G})$ in $H^1_\mu(\mathcal{G})$ for all $\rho \in [1/2,1]$.
More precisely, for any $\rho \in [1/2,1]$ there exists a ball $B(\kappa_\mu, r_\rho)$ of center κ_μ in $H^1_\mu(\mathcal{G})$ and radius $r_\rho > 0$ such that κ_μ strictly minimizes $E_\rho(\cdot\,\mathcal{G})$ in $\overline{B(\kappa_\mu, r_\rho)}$, and

$$\inf_{u\in\partial B(\kappa_\mu,r_\rho)} E_\rho(u,\mathcal{G}) > E_\rho(\kappa_\mu,\mathcal{G}) > E_1(\kappa_\mu,\mathcal{G}). \tag{3.4}$$

Let e be any edge of $\mathcal{G}$; we identify e with the interval $[-\ell_{\mathrm{e}}/2, \ell_{\mathrm{e}}/2]$. Then any compactly supported H^1 function v on such interval, with mass μ, can be seen as a function in $H^1_\mu(\mathcal{G})$. Denoting by $v_t(x) := t^{1/2}v(tx)$, with $t > 1$, it is not difficult to check that $v_t \in H^1_\mu(\mathcal{G})$ (notice in particular that the support of v_t is shrinking as t becomes larger), and that

$$E_\rho(v_t,\mathcal{G}) = \frac{t^2}{2}\int_{\mathrm{e}} |v'|^2\,dx - \frac{\rho t^{\frac{p-2}{2}}}{p}\int_{\mathrm{e}} |v|^p\,dx \le \frac{t^2}{2}\left(\int_{\mathrm{e}} |v'|^2\,dx - \frac{t^{\frac{p-6}{2}}}{p}\int_{\mathrm{e}} |v|^p\,dx\right),$$

for every $\rho \in [1/2,1]$. Since $p > 6$,

$$E_\rho(v_t,\mathcal{G}) < E_1(\kappa_\mu,\mathcal{G}) < E_\rho(\kappa_\mu,\mathcal{G})$$

for t sufficiently large (independent of ρ). Taking now $w = v_t$ with any such choice of t in the definition of Γ, the above estimate and the minimality of κ_μ in $\overline{B(\kappa_\mu, r_\rho)}$ imply that $w \notin B(\kappa_\mu, r_\rho)$. Therefore, by continuity, for any $\gamma \in \Gamma$ there exist $t_\gamma \in [0,1]$ such that $\gamma(t_\gamma) \in \partial B(\kappa_\mu, r_\rho)$; and hence, by (3.4),

$$\max_{t\in[0,1]} E_\rho(\gamma(t),\mathcal{G}) \ge E_\rho(\gamma(t_\gamma),\mathcal{G}) > \inf_{u\in\partial B(\kappa_\mu,r_\rho)} E_\rho(u,\mathcal{G}) > E_\rho(\kappa_\mu,\mathcal{G}) = \max\{E_\rho(\kappa_\mu,\mathcal{G}), E_\rho(w,\mathcal{G})\},$$

which completes the proof. ☐

At this point we wish to use the monotonicity trick on the family of functionals $E_\rho(\cdot\,\mathcal{G})$, in order to obtain a bounded Palais-Smale sequence at level c_ρ for almost every $\rho \in [1/2,1]$. In fact, we need a stronger result carrying also a "approximate Morse-index" information, Theorem 3.10 below, proved in [11].

We recall the general setting in which the theorem is stated. Let $(E,\langle\cdot,\cdot\rangle)$ and $(H,(\cdot,\cdot))$ be two *infinite-dimensional* Hilbert spaces and assume that:

$$E \hookrightarrow H \hookrightarrow E',$$

with continuous injections. For simplicity, we assume that the continuous injection $E \hookrightarrow H$ has norm at most 1 and identify E with its image in H. We also introduce:

$$\begin{cases} \|u\|^2 = \langle u,u\rangle, \\ |u|^2 = (u,u), \end{cases} \qquad u \in E,$$

and, for $\mu \in (0,+\infty)$, we define

$$S_\mu = \{u \in E, |u|^2 = \mu\}.$$

For our application, it is plain that $E = H^1(\mathcal{G})$ and $H = L^2(\mathcal{G})$.

Definition 3.6. Let $\phi : E \to \mathbb{R}$ be a C^2-functional on E and $\alpha \in (0,1]$. We say that ϕ' and ϕ'' are α-Hölder continuous on bounded sets if for any $R > 0$ one can find $M = M(R) > 0$ such that for any $u_1,u_2 \in B(0,R)$:

$$||\phi'(u_1) - \phi'(u_2)|| \le M||u_2 - u_1||^\alpha, \quad ||\phi''(u_1) - \phi''(u_2)|| \le M||u_1 - u_2||^\alpha. \tag{3.5}$$

Definition 3.7. Let ϕ be a C^2-functional on E, for any $u \in E$ define the continuous bilinear map:

$$D^2\phi(u) = \phi''(u) - \frac{\phi'(u)\cdot u}{|u|^2}(\cdot,\cdot).$$

Remark 3.8. If u is a critical point of the functional $\phi|_{S_\mu}$ then the restriction of $D^2\phi(u)$ to T_uS_μ coincides with the constrained *Hessian* of $\phi|_{S_\mu}$ at u (as introduced in Proposition 2.1.)

Definition 3.9. Let ϕ be a C^2-functional on E, for any $u \in S_\mu$ and $\theta > 0$, we define the *approximate Morse index* by

$$\tilde{m}_\theta(u) = \sup\left\{\dim L \,\middle|\, L \text{ is a subspace of } T_uS_\mu \text{ such that: } D^2|_{S_\mu}\phi(u)(\varphi,\varphi) < -\theta\|\varphi\|^2, \quad \forall \varphi \in L \right\}.$$

If u is a critical point for the constrained functional $\phi|_{S_\mu}$ and $\theta = 0$, we say that this is the *Morse index of u as constrained critical point*.

Theorem 3.10 (Theorem 1 in [11]). *Let $I \subset (0,+\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \to \mathbb{R}$ of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u), \qquad \rho \in I,$$

where $B(u) \ge 0$ for every $u \in E$, and

$$\text{either } A(u) \to +\infty \text{ or } B(u) \to +\infty \text{ as } u \in E \text{ and } \|u\| \to +\infty. \tag{3.6}$$

Suppose moreover that Φ'_ρ and Φ''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0,1]$. Finally, suppose that there exist $w_1,w_2 \in S_\mu$ (independent of ρ) such that, setting

$$\Gamma = \left\{\gamma \in C([0,1],S_\mu) : \gamma(0) = w_1, \quad \gamma(1) = w_2\right\},$$

we have

$$c_\rho := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\rho(\gamma(t)) > \max\{\Phi_\rho(w_1),\Phi_\rho(w_2)\}, \quad \rho \in I. \tag{3.7}$$

Then, for almost every $\rho \in I$, there exist sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \to 0^+$ such that, as $n \to +\infty$, *(i)* $\Phi_\rho(u_n) \to c_\rho$;

(ii) $||\Phi'_\rho|_{S_\mu}(u_n)|| \to 0$;

(iii) $\{u_n\}$ *is bounded in* E;

(iv) $\tilde{m}_{\zeta_n}(u_n) \le 1$.

We are ready to give the proof of Proposition 3.3.

Proof of Proposition 3.3. We apply Theorem 3.10 to the family of functionals $E_\rho(\cdot, \mathcal{G})$, with $E = H^1(\mathcal{G})$, $H = L^2(\mathcal{G})$, $S_\mu = H^1_\mu(\mathcal{G})$, and Γ defined in Lemma 3.4. Setting

$$A(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx \quad \text{and} \quad B(u) = \frac{\rho}{p}\int_{\mathcal{G}} |u|^p.$$

assumption (3.6) holds, since we have that

$$u \in H^1_\mu(\mathcal{G}),\ \|u\| \to +\infty \quad \Longrightarrow \quad A(u) \to +\infty.$$

Moreover, assumption (3.5) holds since the unconstrained first and second derivatives of E_ρ are of class C^1, and hence locally Hölder continuous, on $H^1_\mu(\mathcal{G})$.

In this way, for almost every $\rho \in [1/2, 1]$ there exist a bounded Palais-Smale sequence $\{u_n\}$ for the constrained functional $E_\rho(\cdot, \mathcal{G})|_{H^1_\mu(\mathcal{G})}$ at level c_ρ, and $\zeta_n \to 0^+$, such that $\tilde{m}_{\zeta_n}(u_n) \le 1$. Moreover, as explained in [11, Remark 1.4], since $u \in S_\mu \;\mapsto\; |u| \in S_\mu$, $w_1, w_2 \ge 0$, the map $u \mapsto |u|$ is continuous, and $E_\rho(u, \mathcal{G}) = E_\rho(|u|, \mathcal{G})$, it is possible to choose $\{u_n\}$ with the property that $u_n \ge 0$ on $\mathcal{G}$. By Lemma 3.1, we have that $u_n \to u_\rho$ strongly in $H^1(\mathcal{G})$, and $u_\rho \ge 0$ is a constrained critical point, thus a non-negative solution to (3.3), for $\lambda_\rho = \lambda(u_\rho)$ (Lemma 3.1 is stated for the particular value $\rho = 1$; however, it is immediate to check that this choice does not play any role in the proof). The case when u_ρ vanishes in one (or more) vertexes can be easily ruled out by the Kirchhoff condition, the uniqueness theorem for ODEs, and the fact that $u_\rho \ge 0$. Thus, u_ρ is strictly positive on each vertex, whence $u_\rho > 0$ in $\mathcal{G}$ by the strong maximum principle.

It remains to show that the Morse index $m(u_\rho)$, defined in Definition 3.2 with $\lambda = \lambda(u_\rho)$ is at most 2. This result can be directly deduce from [11, Theorem 3] but we prove it here in our setting for completeness. We omit the dependence of the functionals $E_\rho(\cdot, \mathcal{G})$ on $\mathcal{G}$, to simplify the notation. Defining

$$\overline{\lambda}_\rho := -\frac{1}{\mu} E'_\rho(u_\rho) \cdot u_\rho = -\lim_{n\to\infty} \frac{1}{\mu} E'_\rho(u_n) \cdot u_n,$$

we conclude from Theorem 3.10 (ii) that $\overline{\lambda}_\rho = \lambda_\rho$, we refer to [11, Remark 1.2] for more detail.

To show that $u_\rho \in S_\mu$ has Morse index at most 1 as constrained critical point, see Definition 3.9, we assume by contradiction that there exists a $W_0 \subset T_u S_\mu$ with $\dim W_0 = 2$ such that

$$D^2 E_\rho(u_\rho)(w, w) < 0, \quad \text{for all } w \in W_0 \backslash \{0\}.$$

Since W_0 is of finite dimension, by compactness and homogeneity, there exists a $\beta > 0$ such that

$$D^2 E_\rho(u_\rho)(w, w) < -\beta ||w||^2, \quad \text{for all } w \in W_0.$$

Now, from [11, Corollary 1] or using directly that E'_ρ and E''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0, 1]$, we deduce that there exists a $\delta_1 > 0$ such that, for any $v \in S_\mu$ such that $||v - u|| \le \delta_1$,

$$D^2 E_\rho(v)(w, w) < -\frac{\beta}{2} ||w||^2 \quad \text{for all } w \in W_0. \tag{3.8}$$

Since $\{u_n\} \subset S_\mu$ converges to u we have that $||u_n - u|| \leq \delta_1$ for $n \in \mathbb{N}$ large enough. Then since $\dim W_0 > 1$, (3.8) provides a contradiction with Theorem 3.10 (iv) where we recall that $\zeta_n \to 0^+$. Finally, recalling that S_μ is of codimension 1 in $H^1(\mathcal{G})$ and observing that, for any $w \in H^1(\mathcal{G})$,

$$D^2 E_\rho(u_\rho)(w,w) := E''_\rho(u)(w,w) + \lambda_\rho(w,w) = \int_{\mathcal{G}} \left[|w'|^2 + \left(\lambda_\rho - (p-1)|u_\rho|^{p-2}\right) w^2\right] dx,$$

we obtain that $m(u_\rho) \leq 2$. □

4 Blow-up Phenomena

Proposition 3.3 does not ensure the existence of a mountain pass solution for the original problem obtained when $\rho = 1$. However, it gives the existence of a sequence $\rho_n \to 1^-$, with a corresponding sequence of mountain pass critical points $u_{\rho_n} \in H^1_\mu(\mathcal{G})$ of $E_{\rho_n}(\cdot, \mathcal{G})$, constrained on $H^1_\mu(\mathcal{G})$. We aim to show that $\{u_{\rho_n}\}$ converges to a constrained critical point of $E_1(\cdot, \mathcal{G})$. To this purpose, it is sufficient to prove that $\{u_{\rho_n}\}$ is bounded in $H^1(\mathcal{G})$, thanks to Lemma 3.1. The advantage of working with $\{u_{\rho_n}\}$ is that this is a sequence of *solutions of approximating problems with uniformly bounded Morse index*. In this section we perform a blow-up analysis for this type of sequences, in the spirit of [17]. This analysis, of independent interest, will be used in the next section to gain the desired boundedness of $\{u_{\rho_n}\}$.

A somehow related study, regarding least action solutions, was previously performed in [14].

General setting for the blow-up analysis.

For the sake of generality, in what follows we consider a general metric graph satisfying the following assumption:

$\mathcal{G}$ has a finite number of vertexes and edges (but is not necessarily compact).

Let $\{u_n\} \in H^1(\mathcal{G})$ be a sequence of positive solutions of the NLS equation, coupled with Kirchhoff condition at the vertexes:

$$\begin{cases} -u''_n + \lambda_n u_n = \rho_n u_n^{p-1} & \text{on } \mathcal{G}, \\ u_n > 0 & \text{on } \mathcal{G}, \\ \sum_{\mathrm{e} \succ \mathrm{v}} u'_{e,n}(\mathrm{v}) = 0 & \forall \mathrm{v} \in \mathcal{V}, \end{cases} \tag{4.1}$$

where $\rho_n \to 1$ (in fact, it would be sufficient to ask that $\rho_n \to \rho > 0$, regardless of the value of ρ), and $\lambda_n \in \mathbb{R}$.

We denote by $B_r(x_0) = \{x \in \mathcal{G} : \operatorname{dist}(x, x_0) < r\}$. Moreover, we denote by $\mathcal{G}_m$ the star-graph with $m \geq 1$ half-lines glued together at their common origin 0 (note that $\mathcal{G}_1 = \mathbb{R}^+$, and $\mathcal{G}_2$ is isometric to $\mathbb{R}$).

It is also convenient to recall the definition of $Q(\phi; u, \mathcal{G})$, see (3.2).

At first, we note that if $\lambda_n \to +\infty$, then u_n blows-up along any sequence of local maximum points.

Lemma 4.1. *Let $x_n \in \mathcal{G}$ be a local maximum point for u_n. Then*

$$u_n(x_n) \geq \lambda_n^{\frac{1}{p-2}}.$$

Proof. Let e be an edge of $\mathcal{G}$ such that $x_n \in \mathrm{e} \simeq [0, \ell_\mathrm{e}]$; it is plain that $u_n|_e \in C^2([0, \ell_e])$, by regularity. If x_n is in the interior of e, then $u''_n(x_n) \leq 0$; if instead x_n is a vertex of e, then, by the Kirchhoff condition, $u'_n(x_n)$ must vanish, and hence $u''_n(x_n) \leq 0$ again. In both cases, the equation of u_n (which holds on the whole closed interval $[0, \ell_e]$) yields

$$\lambda_n u_n(x_n) - \rho_n u_n^{p-1}(x_n) = u''_n(x_n) \leq 0,$$

whence the thesis follows. □

The next theorem provides a precise behavior, close to a local maximum point, of the sequence $\{u_n\}$, as $\lambda_n \to +\infty$ while $m(u_n)$ remains bounded. In the statement and in the proof, we will systematically identify an edge e with the interval $[0, \ell_{\mathrm{e}}]$, where ℓ_{e} denotes the length of e. Since in this section we allow $\mathcal{G}$ to be non-compact, it is admissible that $\ell_{\mathrm{e}} = +\infty$ (clearly, in such case e $\simeq [0, +\infty)$; unless it is necessary, we will not distinguish these cases).

Theorem 4.2. *Suppose that*

$$\lambda_n \to +\infty \quad \textit{and} \quad m(u_n) \le \bar{k} \quad \textit{for some } \bar{k} \ge 1.$$

Let $x_n \in \mathcal{G}$ be such that, for some $R_n \to \infty$,

$$u_n(x_n) = \max_{B_{R_n \tilde{\varepsilon}_n}(x_n)} u_n \quad \textit{where } \tilde{\varepsilon}_n = (u_n(x_n))^{-\frac{p-2}{2}} \to 0. \tag{4.2}$$

Suppose moreover that

$$\limsup_{n\to\infty} \frac{\mathrm{dist}(x_n, \mathcal{V})}{\tilde{\varepsilon}_n} = +\infty. \tag{4.3}$$

Then, up to a subsequence, the following holds:

(i) all the x_n lie in the interior of the same edge e $\simeq [0, \ell_e]$.

(ii) Setting $\varepsilon_n = \lambda_n^{-\frac{1}{2}}$, we have that

$$\begin{gathered} \frac{\tilde{\varepsilon}_n}{\varepsilon_n} \to (0,1], \\ \frac{\mathrm{dist}(x_n, \mathcal{V})}{\varepsilon_n} \to +\infty \quad \textit{as } n \to \infty, \end{gathered} \tag{4.4}$$

and the scaled sequence

$$v_n(y) := \varepsilon_n^{\frac{2}{p-2}} u_n(x_n + \varepsilon_n y) \quad \textit{for } y \in \frac{[0, \ell_e] - x_n}{\varepsilon_n} \tag{4.5}$$

converges to V in $C^2_{\mathrm{loc}}(\mathbb{R})$ as $n \to \infty$, where $V \in H^1(\mathbb{R})$ is the (unique) positive finite energy solution to

$$\begin{cases} -V'' + V = V^{p-1}, \quad U_0 > 0 & \textit{in } \mathbb{R}, \\ V(0) = \max_{\mathbb{R}} V, \\ V(x) \to 0 & \textit{as } |x| \to +\infty. \end{cases}$$

(iii) There exists $\phi_n \in C_c^\infty(\mathcal{G})$, with $\mathrm{supp}\,\phi_n \subset B_{\bar{R}\varepsilon_n}(x_n)$ for some $\bar{R} > 0$, such that

$$Q(\phi_n; u_n, \mathcal{G}) < 0.$$

(iv) For all $R > 0$ and $q \ge 1$, we have that

$$\lim_{n\to\infty} \lambda_n^{\frac{1}{2} - \frac{q}{p-2}} \int_{B_{R\varepsilon_n}(x_n)} u_n^q \, dx = \lim_{n\to\infty} \int_{B_R(0)} v_n^q \, dy = \int_{B_R(0)} V^q \, dy.$$

If, instead of (4.3), we suppose that

$$\limsup_{n\to\infty} \frac{\mathrm{dist}(x_n, \mathcal{V})}{\tilde{\varepsilon}_n} < +\infty, \tag{4.6}$$

then, up to a subsequence,

(i′) $x_n \to \mathrm{v} \in \mathcal{V}$*, and all the* x_n *lie on the same edge* $\mathrm{e}_1 \simeq [0,\ell_1]$*, where the vertex* v *is identified by the coordinate* 0 *on* e_1.

(ii′) *Let* $\mathrm{e}_2 \simeq [0,\ell_2]$, $\dots$, $\mathrm{e}_m \simeq [0,\ell_m]$ *be the other edges of* $\mathcal{G}$ *having* v *as a vertex (if any), where* v *is identified by the coordinate* 0 *on each* e_i*. Setting* $\varepsilon_n = \lambda_n^{-\frac{1}{2}}$*, we have that*

$$\begin{aligned} &\frac{\tilde{\varepsilon}_n}{\varepsilon_n} \to (0,1], \\ &\limsup_{n\to\infty} \frac{\mathrm{dist}(x_n,\mathcal{V})}{\varepsilon_n} < +\infty, \end{aligned} \tag{4.7}$$

and the scaled sequence defined by

$$v_n(y) := \varepsilon_n^{\frac{2}{p-2}} u_n(\varepsilon_n y) \quad \textit{for } y \in \frac{\mathrm{e}_i}{\varepsilon_n}, \textit{ for } i = 1,\dots,m,$$

converges to a limit V *in* $C^0_{\mathrm{loc}}(\mathcal{G}_m)$ *as* $n\to\infty$*. Denoting by* V_i *the restriction of* V *to the i-th half-line* ℓ_i *of* $\mathcal{G}_m$*, and by* $v_{i,n}$ *the restriction of* v_n *to* $\mathrm{e}_i/\varepsilon_n$*, we have moreover that* $v_{i,n} \to V_i$ *in* $C^2_{\mathrm{loc}}([0,+\infty))$*. Finally,* $V \in H^1(\mathcal{G}_m)$ *is a positive finite energy solution to the NLS equation on the star-graph*

$$\begin{cases} -V'' + V = V^{p-1}, \quad V > 0 & \textit{in } \mathcal{G}_m, \\ \sum_{i=1}^m V_i'(0^+) = 0, & \\ V(x) \to 0 & \textit{as } \mathrm{dist}(x,0) \to \infty \end{cases}$$

with a global maximum point $\bar{x}$ *located on* ℓ_1*, whose coordinate is*

$$\bar{x} = \lim_{n\to\infty} \bar{x}_n \in [0,+\infty), \quad \textit{where} \quad \bar{x}_n := \frac{\mathrm{dist}(x_n,\mathcal{V})}{\varepsilon_n}.$$

(iii′) *There exists* $\phi_n \in C_c^\infty(\mathcal{G})$*, with* $\mathrm{supp}\,\phi_n \subset B_{\bar{R}\varepsilon_n}(x_n)$ *for some* $\bar{R} > 0$*, such that*

$$Q(\phi_n; u_n, \mathcal{G}) < 0.$$

(iv′) *For all* $R > 0$ *and* $q \geq 1$*, we have that*

$$\lim_{n\to\infty} \lambda_n^{\frac{1}{2}-\frac{q}{p-2}} \int_{B_{R\varepsilon_n}(x_n)} u_n^q\,dx = \lim_{n\to\infty} \int_{B_R(\bar{x}_n)} v_n^q\,dy = \int_{[0,\bar{x}+R]} V_1^q\,dy + \sum_{i=2}^m \int_{[0,R-\bar{x}]} V_i^q\,dy = \int_{B_R(\bar{x})} V^q\,dy$$

(where $B_R(\bar{x}_n)$ *and* $B_R(\bar{x})$ *denote the balls in the scaled and in the limit graphs, respectively).*

The proof of the theorem is divided into several intermediate steps. We start with some preliminary results.

Lemma 4.3. *Let* $U \in H^1_{\mathrm{loc}}(\mathcal{G}_m)$ *be a solution to*

$$\begin{cases} -U'' + \lambda U = \rho U^{p-1} & \textit{in } \mathcal{G}_m, \\ U > 0 & \textit{in } \mathcal{G}_m, \\ \sum_{i=1}^m U_i'(0) = 0, & \end{cases} \tag{4.8}$$

for some $p > 2$*,* $\rho, \lambda > 0$*, where* U_i *denotes the restriction of* U *on the i-th half-line of* $\mathcal{G}_m$*. Suppose that* U *is stable outside a compact set* K*, in the sense that* $Q(\varphi; U, \mathcal{G}_m) \geq 0$ *for all* $\varphi \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m \setminus K)$*. Then* $U(x) \to 0$ *as* $\mathrm{dist}(x,0) \to +\infty$*, and* $U \in H^1(\mathcal{G})$*.*

The proof is analogue to the one of [17, Theorem 2.3], and hence we omit it.

Remark 4.4. Clearly, by the density of $H^1([0,+\infty))\cap C_c([0,+\infty))$ in $H^1([0,+\infty))$, any solution with finite Morse index is stable outside a compact set.

Lemma 4.5. *Let $U\in H^1(\mathcal{G}_m)$ be any non-trivial solution of (4.8). Then its Morse index $m(U)$ is strictly positive.*

Proof. Thanks to the Kirchhoff condition, it is not difficult to check that

$$\int_{\mathcal{G}_m}\left(|U'|^2+\lambda U^2\right)dx=\int_{\mathcal{G}_m}\rho|U|^p\,dx.$$

Therefore

$$Q(U;U,\mathcal{G}_m)=(2-p)\int_{\mathcal{G}_m}|U|^p\,dx<0,$$

and the thesis follows by density of $H^1(\mathcal{G}_m)\cap C_c(\mathcal{G}_m)$ in $H^1(\mathcal{G}_m)$. □

Proof of Theorem 4.2 under assumption (4.3). This case is simpler than the one when (4.6) holds, since, roughly speaking, after rescaling we do not see the vertexes of $\mathcal{G}$, and we obtain a limit problem on the line. We present in any case the proof for the sake of completeness. Since $\mathcal{G}$ has a finite number of edges, up to a subsequence all the points x_n belong to same edge e, and (i) holds. Let $\tilde{u}_n$ be defined by

$$\tilde{u}_n(y):=\tilde{\varepsilon}_n^{\frac{2}{p-2}}u_n(x_n+\tilde{\varepsilon}_n y)\quad\text{for } y\in\tilde{\mathrm{e}}_n:=\frac{\mathrm{e}-x_n}{\tilde{\varepsilon}_n}.$$

Notice that any interval $[-a,a]$, with $a>0$, is contained in $\tilde{\mathrm{e}}_n$ for sufficiently large n. Indeed, $(\mathrm{e}-x_n)/\tilde{\varepsilon}_n$ contains the set

$$\{y\in\mathbb{R}:\ |\tilde{\varepsilon}_n y|<\operatorname{dist}(x_n,\mathcal{V})\}=\left\{y\in\mathbb{R}:\ |y|<\frac{\operatorname{dist}(x_n,\mathcal{V})}{\tilde{\varepsilon}_n}\right\},$$

which exhausts the whole line $\mathbb{R}$ as $n\to\infty$, by (4.3).

Now, on every compact $[-a,a]$ we have that $\tilde{u}_n(0)=1=\max_{[-a,a]}\tilde{u}_n$ for n large (since $u_n(x_n)=\max_{B_{R_n\tilde{\varepsilon}_n}(x_n)}u_n$ for some $R_n\to+\infty$), and

$$-\tilde{u}_n''+\tilde{\varepsilon}_n^2\lambda_n\tilde{u}_n=\rho_n\tilde{u}_n^{p-1},\quad \tilde{u}_n>0\quad\text{in }\tilde{\mathrm{e}}_n.$$

Furthermore, by Lemma 4.1

$$\tilde{\varepsilon}_n^2\lambda_n\in(0,1],\quad\forall n.$$

Thus, by elliptic estimates, we have that $\tilde{u}_n\to\tilde{u}$ in $C^2_{\mathrm{loc}}(\mathbb{R})$, and the limit $\tilde{u}$ solves

$$-\tilde{u}''+\tilde{\lambda}\tilde{u}=\tilde{u}^{p-1},\quad \tilde{u}\ge 0\quad\text{in }\mathbb{R}\tag{4.9}$$

for some $\tilde{\lambda}\in[0,1]$. By local uniform convergence, $\tilde{u}(0)=1$, and hence $\tilde{u}>0$ in $\mathbb{R}$ by the strong maximum principle. We claim that

$$\text{the Morse index of }\tilde{u}\text{ is bounded by }\bar{k}.\tag{4.10}$$

If by contradiction this is false, then there exists $k>\bar{k}$ functions $\phi_1,\dots,\phi_k\in H^1(\mathbb{R})\cap C_c(\mathbb{R})$, linearly independent in $H^1(\mathbb{R})$, such that $Q(\phi_i;\tilde{u},\mathbb{R})<0$ for every $i\in\{1,\cdots,k\}$. Let then

$$\phi_{i,n}(x)=\tilde{\varepsilon}_n^{\frac{1}{2}}\phi_i\left(\frac{x-x_n}{\tilde{\varepsilon}_n}\right)$$

Since ϕ_i has compact support, the functions $\phi_{i,n}$ can be regarded as functions in $H^1(\mathrm{e})$, and hence in $H^1(\mathcal{G})$, for every n large, thanks to (4.3). Indeed, if $\operatorname{supp}\phi_i \subset [-M,M]$, then

$$\left\{x \in \mathbb{R} : \frac{x-x_n}{\tilde{\varepsilon}_n} \subset [-M,M]\right\} = [x_n - \tilde{\varepsilon}M, x_n + \tilde{\varepsilon}_n M] \\ \subset \left[x_n - \tilde{\varepsilon}_n \frac{\operatorname{dist}(x_n,\mathcal{V})}{\tilde{\varepsilon}_n}, x_n + \tilde{\varepsilon}_n \frac{\operatorname{dist}(x_n,\mathcal{V})}{\tilde{\varepsilon}_n}\right] \subset \mathrm{e}.$$

Moreover, $\phi_{1,n},\dots,\phi_{k,n}$ are linearly independent in $H^1(\mathcal{G})$, and, by scaling,

$$Q(\phi_{i,n};u_n,\mathcal{G}) = Q(\phi_{i,n};u_n,\mathrm{e}) = Q(\phi_i;\tilde{u}_n,\tilde{\mathrm{e}}_n) \to Q(\phi_i;\tilde{u},\mathbb{R}) < 0.$$

This implies that $m(u_n) \geq k > \bar{k}$ for sufficiently large n, a contradiction. Therefore, claim (4.10) is proved. To sum up, $\tilde{u}$ is a finite Morse index non-trivial solution to (4.9), for some $\tilde{\lambda} \in [0,1]$. Having $\tilde{\lambda} = 0$ is however not possible, since by phase plane analysis the equation $\tilde{u}'' + \tilde{u}^{p-1} = 0$ in $\mathbb{R}$ has only periodic sign-changing solution, but the trivial one. Now, by Lemma 4.3, $\tilde{u} \to 0$ as $|x| \to +\infty$, and $\tilde{u} \in H^1(\mathbb{R})$. Therefore,

$$0 < \liminf_{n\to\infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \leq \limsup_{n\to\infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \leq 1, \tag{4.11}$$

which proves the first estimate in (4.4). At this point it is equivalent, but more convenient, to work with v_n defined by (4.5) rather than with $\tilde{u}_n$. By (4.3) and (4.11),

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n,\mathcal{V})}{\varepsilon_n} = +\infty.$$

Thus, similarly as done before, one can show that v_n converges to a limit function v in $C^2_{\mathrm{loc}}(\mathbb{R})$, such that

$$-v'' + v = v^{p-1} \quad v \geq 0 \quad \text{in } \mathbb{R};$$

moreover, v has a positive global maximum $v(0) \geq 1$ (thus $v > 0$ in $\mathbb{R}$), has finite Morse index $m(v) \leq \bar{k}$, and hence, by Lemma 4.3, $v \to 0$ as $|x| \to \infty$, and $v \in H^1(\mathbb{R})$. It is well known that there exists only one such solution, denoted by V. Thus, (ii) is proved. Point (iv) follows directly by local uniform convergence. Finally, point (iii) is a consequence of the fact that the Morse index of V is positive (see Lemma 4.5; in fact, it is well known that in fact $m(V)$ is precisely equal to 1). This implies that there exists $\phi \in C^1_c(\mathbb{R})$ such that $Q(\phi;V,\mathbb{R}) < 0$; thus, defining

$$\phi_{i,n}(x) = \varepsilon_n^{\frac{1}{2}} \phi_i\left(\frac{x-x_n}{\varepsilon_n}\right),$$

we deduce that for sufficiently large n we have $Q(\phi_{i,n};u_n,\mathcal{G}) < 0$, and $\operatorname{supp}\phi_{i,n} \subset B_{\bar{R}\varepsilon_n}(x_n)$ for some $\bar{R} > 0$. □

Proof of Theorem 4.2 under assumption (4.6). Since $\tilde{\varepsilon}_n \to 0$ and $\mathcal{G}$ has a finite number of vertexes and edges, up to a subsequence the maximum points x_n converge to a vertex v, and belong to same edge $\mathrm{e}_1 \simeq [0,\ell_1]$; thus, (i') holds, and we can suppose that

$$\frac{d_n}{\tilde{\varepsilon}_n} \to \eta \in [0,+\infty), \quad d_n := \operatorname{dist}(x_n,\mathcal{V}) = x_n.$$

Let

$$\tilde{u}_n(y) := \tilde{\varepsilon}_n^{\frac{2}{p-2}} u_n(\tilde{\varepsilon}_n y) \quad \text{for } y \in \tilde{\mathrm{e}}_{i,n} := \frac{\mathrm{e}_i}{\tilde{\varepsilon}_n}, \text{ for } i = 1,\dots,m.$$

Note that $\tilde{u}_n$ is defined on a graph $\mathcal{G}_{m,n}$ consisting in m expanding edges, glued together at their common origin, which is identified with the coordinate 0 on each edge $\tilde{e}_{i,n}$. In the limit $n \to \infty$, this graph converges to the star-graph $\mathcal{G}_m$. Plainly, for every $a > \eta + 1$ and large n

$$\tilde{u}_n\left(\frac{x_n}{\tilde{\varepsilon}_n}\right) = 1 = \max_{B_a(0)} \tilde{u}_n$$

(since $u_n(x_n) = \max_{B_{R_n\tilde{\varepsilon}_n}(x_n)} u_n$ for some $R_n \to +\infty$),

$$-\tilde{u}_n'' + \tilde{\varepsilon}_n^2 \lambda_n \tilde{u}_n = \rho_n \tilde{u}_n^{p-1}, \quad \tilde{u}_n > 0$$

on any edge of $\mathcal{G}_{m,n}$, and the Kirchhoff condition at the origin holds. Also, by Lemma 4.1,

$$\tilde{\varepsilon}_n^2 \lambda_n \in (0,1] \quad \forall n.$$

Thus, by elliptic estimates, we have that $\tilde{u}_n|_{\tilde{e}_{i,n}} =: \tilde{u}_{i,n} \to \tilde{u}_i$ in $C^2_{\text{loc}}([0,+\infty))$ for every i, and the limit $\tilde{u}_i$ solves

$$-\tilde{u}_i'' + \tilde{\lambda}\tilde{u}_i = \tilde{u}_i^{p-1}, \quad \tilde{u}_i \geq 0 \quad \text{in } (0,+\infty) \tag{4.12}$$

for some $\tilde{\lambda} \in [0,1]$. Moreover, since $\tilde{u}_n$ is continuous on $\mathcal{G}_{m,n}$ and by uniform convergence, $\tilde{u}_i(0) = \tilde{u}_j(0)$ for every $i \neq j$, so that $\tilde{u} \simeq (\tilde{u}_1, \ldots, \tilde{u}_m)$ can be regarded as a function defined on $\mathcal{G}_m$. Since the convergence $\tilde{u}_{i,n} \to \tilde{u}_i$ takes place in C^2 up to the origin, also the Kirchhoff condition passes to the limit. Now we exclude the case that $\tilde{u} \equiv 0$ on some half-line of $\mathcal{G}_m$. By local uniform convergence, we have that

$$\tilde{u}_1(\rho) = \lim_{n\to\infty} \tilde{u}_{1,n}\left(\frac{d_n}{\tilde{\varepsilon}_n}\right) = 1.$$

This implies that $\tilde{u}_1 > 0$ in $(0,+\infty)$, by the strong maximum principle. In turn, the Kirchhoff condition, the uniqueness theorem for ODEs, and the strong maximum principle again, ensure that $\tilde{u}_i > 0$ on $(0,+\infty)$ for every i. Finally, we claim that

$$\text{the Morse index of } \tilde{u} \text{ is bounded by } \bar{k}. \tag{4.13}$$

The proof of this claim is completely analogue to the one of (4.10). If by contradiction this is false, then there exists $k > \bar{k}$ functions $\phi_1, \ldots, \phi_k \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m)$, linearly independent in $H^1(\mathcal{G}_m)$, such that $Q(\phi_i; \tilde{u}, \mathcal{G}_m) < 0$ for every $i \in \{1, \cdots, k\}$. Let then

$$\phi_{i,n}(x) = \tilde{\varepsilon}_n^{\frac{1}{2}} \phi_i\left(\frac{x}{\tilde{\varepsilon}_n}\right).$$

Since ϕ_i has compact support, the functions $\phi_{i,n}$ can be regarded as functions in $H^1(\mathcal{G}) \cap C_c(\mathcal{G})$ for every n large; precisely, $\text{supp}(\phi_{i,n}) \subset B_{R\tilde{\varepsilon}_n}(x_n)$ for some $R > 2\rho$. Moreover, $\phi_{1,n}, \ldots, \phi_{k,n}$ are linearly independent in $H^1(\mathcal{G}_m)$ and, by scaling,

$$Q(\phi_{i,n}; u_n, \mathcal{G}) = Q(\phi_{i,n}; u_n, \mathrm{e}) = Q(\phi_i; \tilde{u}_n, \tilde{\mathrm{e}}_n) \to Q(\phi_i; \tilde{u}, \mathbb{R}) < 0.$$

This implies that $m(u_n) \geq k > \bar{k}$ for sufficiently large n, a contradiction. Therefore, claim (4.13) is proved.

To sum up, $\tilde{u}$ is a finite Morse index non-trivial solution to (4.12), for some $\tilde{\lambda} \in [0,1]$. As before, the case $\tilde{\lambda} = 0$ can be ruled out by phase-plane analysis, and hence, by Lemma 4.3, $\tilde{u} \to 0$ as $|x| \to +\infty$, and $\tilde{u} \in H^1(\mathcal{G}_m)$. Therefore,

$$0 < \liminf_{n\to\infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \leq \limsup_{n\to\infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \leq 1, \tag{4.14}$$

which proves the first estimate in (4.7). At this point it is equivalent, but more convenient, to work with v_n defined in point (ii') of the theorem, rather than with $\tilde{u}_n$. By (4.6) and (4.14),

$$\limsup_{n\to\infty} \frac{\mathrm{dist}(x_n, \mathcal{V})}{\varepsilon_n} < +\infty.$$

Thus, similarly as done before, one can show that v_n converges, in $C^0_{\mathrm{loc}}(\mathcal{G}_m)$ and in $C^2_{\mathrm{loc}}([0,+\infty))$ on every half-line, to a limit function $V \simeq (V_1, \dots, V_m)$, which solves

$$\begin{cases} -V'' + V = V^{p-1}, \quad V \geq 0 \quad \text{in } \mathcal{G}_m, \\ \sum_{i=1}^m V_i'(0^+) = 0; \end{cases} \tag{4.15}$$

furthermore, V has a positive global maximum on the half-line ℓ_1, $V_1(\bar{x}) \geq 1$ (thus $V > 0$ in $\mathcal{G}_m$), and has finite Morse index $m(V) \leq \bar{k}$. Moreover, by Lemma 4.3, $V \to 0$ as $|x| \to \infty$. Thus, (ii') is proved. Point (iv') follows directly by local uniform convergence. Finally, point (iii') is a consequence of Lemma 4.5. This implies that there exists $\phi \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m)$ such that $Q(\phi; V, \mathcal{G}_m) < 0$; thus, defining

$$\phi_{i,n}(x) = \varepsilon_n^{\frac{1}{2}} \phi_i\left(\frac{x - x_n}{\varepsilon_n}\right),$$

it is not difficult to deduce that for sufficiently large n we have $Q(\phi_{i,n}; u_n, \mathcal{G}) < 0$, and $\mathrm{supp}\,\phi_{i,n} \subset B_{\bar{R}\varepsilon_n}(x_n)$ for some positive $\bar{R}$. □

Theorem 4.2 allows to describe the pointwise blow-up behavior close to local maximum points. In what follows, we focus on the global behavior, and, in particular, on what happens far away from local maxima.

Theorem 4.6. *Let $\{u_n\} \subset H^1(\mathcal{G})$ be a sequence of solutions to (4.1) such that $\lambda_n \to +\infty$ and $m(u_n) \leq \bar{k}$ for some $\bar{k} \geq 1$. There exist $k \in \{1, \dots, \bar{k}\}$, and sequences of points $\{P_n^1\}$, ..., $\{P_n^k\}$, such that*

$$\lambda_n \mathrm{dist}(P_n^i, P_n^j) \to +\infty, \quad \forall i \neq j, \tag{4.16}$$

$$u_n(P_n^i) = \max_{B_{R_n \lambda_n^{-1/2}}(P_n^i)} u_n \quad \text{for some } R_n \to +\infty, \text{ for every } i, \tag{4.17}$$

and constants $C_1, C_2 > 0$ such that

$$u_n(x) \leq C_1 \lambda_n^{\frac{1}{p-2}} \sum_{i=1}^k e^{-C_2 \lambda_n^{\frac{1}{2}} \mathrm{dist}(x, P_n^i)} + C_1 \lambda_n^{\frac{1}{p-2}} \sum_{j=1}^h e^{-C_2 \lambda_n^{\frac{1}{2}} \mathrm{dist}(x, \mathrm{v}_j)}, \quad \forall x \in \mathcal{G} \setminus \bigcup_{i=1}^k B_{R\lambda_n^{-1/2}}(P_n^i), \tag{4.18}$$

where $\mathrm{v}_1, \dots, \mathrm{v}_h$ are all the vertexes of $\mathcal{G}$.

Proof. The proof follows closely the one of [17, Theorem 3.2], and is divided into two steps.

Step 1) There exist $k \in \{1, \dots, \bar{k}\}$, and sequences of points $\{P_n^1\}$, ..., $\{P_n^k\}$, such that (4.16) *and* (4.17) *hold, and moreover*

$$\lim_{R\to+\infty} \left(\limsup_{n\to\infty} \lambda_n^{-\frac{1}{p-2}} \max_{d_n(x) \geq R\lambda_n^{-1/2}} u_n(x) \right) = 0. \tag{4.19}$$

where $d_n(x) = \min\{\mathrm{dist}(x, P_n^i) : \ i = 1, \dots, k\}$ is the distance function from $\{P_1^n, \dots, P_k^n\}$.

Thanks to Theorem 4.2, we can adapt the proof of [17, Theorem 3.2] with minor changes (some details are actually simpler in the present setting, since here we deal with a constant potential, differently to [17]). In adapting Theorem 3.2 from [17], it is important to point out that any limit of u_n, given by Theorem 4.2, tends to 0 at infinity. This fact is crucial in the proof of (4.19).

Moreover, if the reference graph is unbounded, it is important to observe that $u_n(x) \to 0$ as $|x| \to +\infty$ on each half-line, since $u_n \in H^1(\mathcal{G})$ by assumption. This implies that, if $\{P_n^1\}, \dots, \{P_n^h\}$ are local maximum points of u_n, then there exists a maximum point on $\mathcal{G} \setminus \bigcup_{i=1}^h B_{R\lambda_n^{-1/2}}(P_n^i)$.

Step 2) Conclusion of the proof. By (4.19), for every $\varepsilon \in (0,1)$ small, to be chosen later, there exist $R > 0$ and $n_R \in \mathbb{N}$ large such that

$$\max_{d_n(x) > R\lambda_n^{-1/2}} u_n(x) \le \lambda_n^{\frac{1}{p-2}} \varepsilon, \quad \forall n \ge n_R. \tag{4.20}$$

Thus, in the set $A_n := \{d_n(x) > R\lambda_n^{-1/2}\}$, in addition to (4.20) we also have that

$$u_n'' = (\lambda_n - u_n^{p-2})u_n \quad \Longrightarrow \quad -u_n'' + \frac{\lambda_n}{2} u_n \le 0 \tag{4.21}$$

provided that $\varepsilon > 0$ is small enough.

We want to exploit (4.20) and (4.21) in a comparison argument, as in [17] (or [16, Theorem 3.1]). However, the presence of the vertexes makes the argument a little bit more involved in our setting.

Let us denote by $\{\mathrm{v}_j\}_{j=1}^{h_1}$ the set of vertexes which are not included in one of the balls $B_{R\lambda_n^{-1/2}}(P_n^i)$ for large n. On any such vertex, by (4.20),

$$u_n(\mathrm{v}_j) \le \lambda_n^{\frac{1}{p-2}} \varepsilon. \tag{4.22}$$

For any edge e, we consider the restriction of u_n on $\mathrm{e} \cap A_n$. Since k is independent of n, $\mathrm{e} \cap A_n$ consists in finitely many relatively open intervals (which may be unbounded, if $\mathcal{G}$ is non-compact).

Let I_n be any such *bounded* interval; then the following alternative holds: $\partial I_n \cap \{\mathrm{v}_j\}_{j=1}^{h_1}$ can either be empty (case 1), or be a single vertex , say v_1 (case 2), or be a pair of vertexes, say v_1 and v_2 (case 3).

Assume at first that case 1 holds. Then there exist two indexes $i, j \in \{1, \dots, k\}$ such that ∂I_n consists in one point at distance $R\lambda_n^{-1/2}$ from P_n^i, and one point at distance $R\lambda_n^{-1/2}$ from P_n^j. Consider the function

$$\phi_n(x) = e^{-\gamma \lambda_n^{\frac{1}{2}} |x - P_n^i|} + e^{-\gamma \lambda_n^{\frac{1}{2}} |x - P_n^j|},$$

which solves $\phi_n'' = \gamma^2 \lambda_n \phi_n$ in I_n. By taking $\gamma < 1/4$, we have that

$$-\phi_n'' + \frac{\lambda_n}{2} \phi_n \ge 0 \quad \text{in } I_n.$$

Moreover,

$$\left(e^{\gamma R} \lambda_n^{\frac{1}{p-2}} \phi_n - u_n \right) \Big|_{\partial I_n} \ge \lambda_n^{\frac{1}{p-2}} (1 - \varepsilon) > 0,$$

and hence, by the comparison principle, we have that

$$u(x) \le e^{\gamma R} \lambda_n^{\frac{1}{p-2}} \phi_n(x), \qquad \forall x \in I_n,$$

which clearly implies the validity of the thesis on I_n in this case.

If case 2 holds, then there exists an index $i \in \{1, \dots, k\}$ such that ∂I_n consists in a point at distance $R\lambda_n^{-1/2}$ from P_n^i, plus the vertex v_1. Arguing as before, it is not difficult to check that

$$u(x) \le e^{\gamma R} \lambda_n^{\frac{1}{p-2}} e^{-\gamma \lambda_n^{\frac{1}{2}} |x - P_n^i|} + \lambda_n^{\frac{1}{p-2}} e^{-\gamma \lambda_n^{\frac{1}{2}} |x - \mathrm{v}_1|}, \quad \forall x \in I_n,$$

which gives the thesis in case 2.

In case 3, an analogue argument ensures that

$$u(x) \leq \lambda_n^{\frac{1}{p-2}} e^{-\gamma\lambda_n^{\frac{1}{2}}|x-\mathrm{v}_1|} + \lambda_n^{\frac{1}{p-2}} e^{-\gamma\lambda_n^{\frac{1}{2}}|x-\mathrm{v}_2|}, \quad \forall x \in I_n,$$

whence the thesis follows once again.

Finally, let us consider the case when I_n is an *unbounded* interval of $\mathrm{e} \cap A_n$. Then we only have two possibilities: either ∂I_n consists in a point at distance $R\lambda_n^{-1/2}$ from P_n^i, or ∂I_n consists in a vertex, say v_1.

In the former case, we argue as before with the comparison function

$$\psi_n(x) = e^{-\gamma R}\lambda_n^{\frac{1}{p-2}} e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^i|},$$

where $\gamma < 1/4$. In the latter one, we can use

$$\psi_n(x) = \lambda_n^{\frac{1}{p-2}} e^{-\gamma\lambda_n^{\frac{1}{2}}|x-\mathrm{v}_1|}.$$

To sum up, slightly modifying the choice of the comparison functions, according to the structure of ∂I_n, it is possible to prove the validity of (4.18) in all the possible cases. ☐

5 Mountain pass solution for the original problem

In this section we complete the proof of the main existence result, Theorem 1.1. Let $\mu \in (0, \mu_1)$. As already anticipated in Section 4, Proposition 3.3 gives a sequence of mountain pass critical points $u_{\rho_n} \in H^1_\mu(\mathcal{G})$ of $E_{\rho_n}(\cdot\,, \mathcal{G})$ on $H^1_\mu(\mathcal{G})$ with $\rho_n \to 1^-$ and $m(u_{\rho_n}) \leq 2$. Moreover, the energy level c_{ρ_n} is bounded, since

$$E_1(\kappa_\mu, \mathcal{G}) \leq E_\rho(\kappa_\mu, \mathcal{G}) \leq c_\rho \leq c_{1/2}, \qquad \forall \rho \in \left[\frac{1}{2}, 1\right]$$

(the first and the second inequalities are proved in Lemma 3.4; the third one follows directly from the monotonicity of c_ρ). Thus, Theorem 1.1 is a direct corollary of the next statement.

Proposition 5.1. *Let $\mathcal{G}$ be a metric graph, $\{u_n\} \subset H^1(\mathcal{G})$ a sequence of solutions to (4.1) for some $\lambda_n \in \mathbb{R}$ and $\rho_n \to 1$. Suppose that*

$$\int_{\mathcal{G}} |u_n|^2\,dx = \mu, \quad m(u_n) \leq \bar{k}, \qquad \forall n,$$

for some $\mu > 0$ and $\bar{k} \in \mathbb{N}$, and that

the sequence of the energy levels $\{c_n := E_{\rho_n}(u_n, \mathcal{G})\}$ is bounded.

Then the sequences $\{\lambda_n\} \subset \mathbb{R}$ and $\{u_n\} \subset H^1(\mathcal{G})$ must be bounded. In addition, $\{u_n\}$ is a (bounded) Palais-Smale sequence for $E_1(\cdot\,, \mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$.

Proof of Theorem 1.1. It is sufficient to apply Proposition 5.1 on the sequence $\{u_{\rho_n}\}$ which, as observed, fulfills the assumptions. Indeed, applying Lemma 3.1 we then deduce that $u_n \to \bar{u}$ strongly in $H^1(\mathcal{G})$. ☐

Proof of Proposition 5.1. Since

$$\int_{\mathcal{G}} \left(|u_n'|^2 + \lambda_n u_n^2\right) dx = \rho_n \int_{\mathcal{G}} |u_n|^p\,dx,$$

it follows that

$$c_n = E_{\rho_n}(u_n, \mathcal{G}) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} |u_n'|^2\,dx - \frac{\lambda_n \mu}{p};$$

therefore

$$\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathcal{G}}|u_n'|^2\,dx = c_n + \frac{\lambda_n\mu}{p}. \tag{5.1}$$

This estimate gives the boundedness of $\{u_n\}$ in $H^1(\mathcal{G})$, provided that $\{\lambda_n\}$ is bounded (recall that $\{c_n\}$ is bounded as well). Once the boundedness of $\{u_n\}$ in $H^1(\mathcal{G})$ is proved, and since $\rho_n \to 1$, the fact that it is a Palais-Smale sequence for $E_1(\cdot,\mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$ is straightforward.

Therefore, we only have to show that $\{\lambda_n\}$ is bounded. By contradiction, we suppose that this is not the case. By (5.1), we have that $\lambda_n \to +\infty$, up to a subsequence. Thus, Theorems 4.2 and 4.6 hold for $u_n := u_{\rho_n}$. For $\{P_n^1\},\dots,\{P_n^k\}$ given by Theorem 4.6, Theorem 4.2 ensures the existence of blow-up limits, which can be either defined on $\mathbb{R}$, or on a star graph $\mathcal{G}_m$. In the rest of the proof:

- $\{v_n^i\}$ denotes the scaled sequence around P_n^i;
- V^i denotes the limit of $\{v_n^i\}$;
- $\bar{x}_n^i$ denotes the global maximum point of v_n^i;
- $\bar{x}^i$ denotes the global maximum point of V^i.

Then, for $R>0$, on one hand we have that

$$\left|\lambda_n^{\frac{1}{2}-\frac{2}{p-2}}\int_{\mathcal{G}}u_n^2\,dx - \sum_{i=1}^k\int_{B_R(\bar{x}_n^i)}(v_n^i)^2\,dx\right| \to +\infty \tag{5.2}$$

(in the second integral, the ball $B_R(\bar{x}_n^i)$ is the ball in the scaled graph). Indeed, the first term inside the absolute value satisfies

$$\lambda_n^{\frac{1}{2}-\frac{2}{p-2}}\int_{\mathcal{G}}u_n^2\,dx = \lambda_n^{\frac{1}{2}-\frac{2}{p-2}}\mu \to +\infty,$$

since $p>6$. While the second term is bounded, since by Theorem 4.2

$$\sum_{i=1}^k\int_{B_R(\bar{x}_n^i)}(v_n^i)^2\,dx \to \int_{B_R(\bar{x}^i)}(V^i)^2\,dx,$$

and it is the sum of a finite number of bounded integrals, being $V^i \in H^1(\mathcal{G}_m)$.

On the other hand, by Theorem 4.6, for some positive constant C which changes from one line to another,

$$\begin{aligned}
\left|\lambda_n^{\frac{1}{2}-\frac{2}{p-2}}\int_{\mathcal{G}}u_n^2\,dx - \sum_{i=1}^k\int_{B_R(x_n^i)}(v_n^i)^2\,dx\right| &= \lambda_n^{\frac{1}{2}-\frac{2}{p-2}}\left|\int_{\mathcal{G}}u_n^2\,dx - \sum_{i=1}^k\int_{B_{R\lambda_n^{-1/2}}(P_n^i)}u_n^2\,dx\right| \\
&= \lambda_n^{\frac{1}{2}-\frac{2}{p-2}}\int_{\mathcal{G}\setminus\bigcup_i B_{R\lambda_n^{-1/2}}(P_n^i)}u_n^2\,dx \\
&\le C_1\lambda_n^{\frac{1}{2}}\sum_{i=1}^k\int_{\mathcal{G}\setminus\bigcup_i B_{R\lambda_n^{-1/2}}(P_n^i)}e^{-C_2\lambda_n^{\frac{1}{2}}\operatorname{dist}(x,P_n^i)}\,dx + C_1\lambda_n^{\frac{1}{2}}\sum_{j=1}^h\int_{\mathcal{G}}e^{-C_2\lambda_n^{\frac{1}{2}}\operatorname{dist}(x,\mathrm{v}_j)}\,dx \\
&\le C\lambda_n^{\frac{1}{2}}\sum_{i=1}^k\int_{\mathcal{G}\setminus B_{R\lambda_n^{-1/2}}(P_n^i)}e^{-C\lambda_n^{\frac{1}{2}}\operatorname{dist}(x,P_n^i)}\,dx + C\lambda_n^{\frac{1}{2}}\sum_{j=1}^h\int_{\mathcal{G}}e^{-C\lambda_n^{\frac{1}{2}}\operatorname{dist}(x,\mathrm{v}_j)}\,dx \\
&\le C\lambda_n^{\frac{1}{2}}\int_{R\lambda_n^{-1/2}}^{+\infty}e^{-C\lambda_n^{\frac{1}{2}}y}\,dy + C\lambda_n^{\frac{1}{2}}\int_0^{+\infty}e^{-C\lambda_n^{\frac{1}{2}}y}\,dy \\
&\le C\int_R^{+\infty}e^{-Cz}\,dz + C\int_0^{\infty}e^{-Cz}\,dz \\
&\le Ce^{-CR}+C,
\end{aligned}$$

By taking the limit as $n \to \infty$, we deduce that

$$\limsup_{n\to\infty} \left| \lambda_n^{\frac{1}{2}-\frac{2}{p-2}} \int_{\mathcal{G}} u_n^2\,dx - \sum_{i=1}^{k} \int_{B_r(\bar{x}_i)} (V^i)^2\,dx \right| \leq Ce^{-CR} + C,$$

in contradiction with (5.2). □

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