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Model theory and partition theorems for monoid actions on semigroups

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Contents

Chapter 1

Introduction

1.1 Description of the content

This thesis consists of three chapters, which correspond to three articles. The first appears in [12] and the second is under review [1]. We include them here without modifications, except for some footnotes. The third chapter is part of a work in progress.

In the first chapter we introduce a new method. This method uses basic concepts from model theory to prove old and new Ramsey theorems. It is versatile and easy to learn. A concise summary of this method is given in Section 3.5.

In Chapter 4 we improve upon the results of Chapter 3, but at the cost of introducing more technical definitions.

The rest of this introduction is devoted to motivating this work and highlighting the main results.

1.2 Results

1.2.1 Motivation

We study monoid actions on (partial) semigroups by endomorphisms, a very broad subject that is of interest to many mathematicians. To motivate our theorems, we first review the existing literature.

One of the most celebrated theorems in Ramsey theory is due to Hindman [27]: it states that for every semigroup $(S, \cdot)$, for every finite coloring of S there is an infinite sequence $\bar{s} = (s_i)_{i\in\omega}$ such that the following set is monochromatic [1]

$$\mathrm{fp}(\bar{s}) = \{s_{i_0} \cdot \ldots \cdot s_{i_n} : n \in \omega, i_0 < \cdots < i_n\}.$$

The original theorem is stated for $(\mathbb{N}, +)$, but it is easy to see that if the statement holds for $(\mathbb{N}, +)$ then it holds for any semigroup. [2]

The set $\mathrm{fp}(\bar{s})$ has the following closure property: if two elements $s = s_{i_0} \cdot \ldots \cdot s_{i_m}$ and $t = s_{i_{m+1}} \cdot \ldots \cdot s_{i_n}$ belong to $\mathrm{fp}(\bar{s})$, where $i_0 < \cdots < i_n$, then $s \cdot t$ belongs to $\mathrm{fp}(\bar{s})$. When there is an action on a semigroup, we would like a monochromatic set which is also closed under the action.

In this thesis, we study the case where a monoid M acts by endomorphisms on a partial semigroup[3] S. Ideally, for every finite coloring of S we would like to find a sequence $(s_n)_{n\in\omega}$ of elements of S such that the following set is monochromatic

$$* = \left\{m_0\, s_{i_0} \cdot \ldots \cdot m_n\, s_{i_n} : n \in \omega, i_0 < \cdots < i_n, m_i \in M\right\}.$$

This is not always possible (see the discussion right after corollary 3.6.2). Therefore, we require that suitable subsets of $*$ are monochromatic.

To sum up, let S be a partial semigroup and let M act on S by endomorphisms. Our problem is to find for every coloring of S large monochromatic subsets of a set of the form $*$. The definition of these subsets is postponed for some paragraphs.

This problem is far from new. Many theorems in Ramsey theory can be interpreted as an answer to this problem. Each theorem answers the problem for a particular choice of M, S, and a subset of $*$. These are e.g. Bergelson-Blass-Hindman theorem [6], Carlson's theorem [9], Furstenberg and Katznelson's Ramsey theorem [19], Gowers' FIN_k theorem[24], Hales-Jewett theorem, ([26]), Lupini's theorem [33], which is an infinitary version of Bartosova and Kwiatkowska's theorem [4], and Solecki's theorem [43]. Hindman's theorem can also be interpreted in this setting, taking M as the trivial monoid and S as $(\mathbb{N}, +)$.

The results above play an important role in many fields of mathematics. Carlson's theorem is the motivating example to introduce the notion of topological Ramsey space [9]. In the same direction, it is the motivating example for the notion of Ramsey space, which is deeply studied in the monograph of Todorcevic [47]. Gowers proved his FIN_k theorem (and its symmetric version) as a tool to solve an old problem in Banach spaces [24]. Hales-Jewett theorem has among its corollaries Van der

[1]A finite coloring of S is a map from S to a finite set $\{0, \ldots, k-1\}$, and a monochromatic set is a subset of S which is contained in the preimage of some $i < k$.

[2]Given a semigroup S and an element $s \in S$ one can consider the map $n \mapsto s^n$ to define a coloring on $\mathbb{N}$ and then derive the theorem for the semigroup S.

[3]A partial semigroup is a set S with an operation $(S \cup \{\perp\})^2 \to S \cup \{\perp\}$ -where $\perp$ stands for "not defined"- such that $\perp x = \perp = x \perp$ and such that if $(xy)z \neq \perp$ or $x(yz) \neq \perp$ then $(xy)z = x(yz)$. An endomorphism is a map $x \mapsto mx$ such that if $xy \neq \perp$ then $m(xy) = (mx)y \neq \perp$.

Waerden's theorem [48]. Its density version, proved by Furstenberg and Katznelson with techniques from ergodic theory [20], is one of the highest peaks of the area, being stronger than Szemeredi's theorem on arithmetic progressions [44]. Another important generalization of Hales-Jewett theorem is its polynomial extension, found by Bergelson and Leibman [7]. Bartosova and Kwiatkowska applied their results to prove that a certain group of homeomorphisms of the Lelek fan is extremely amenable. Lupini continued his work in a slightly different direction. He studied actions of trees on semigroups, and applied his results to amenable groups [34].

1.2.2 Ramsey monoids

Let us introduce a partial semigroup, together with a monoid action, which is universal in a sense explained in the proposition below.

For a monoid M, let FIN_M be the set of all partial functions with finite domain from $\mathbb{N}$ to M. Elements of FIN_M are called *located words*. There is a partial associative operation on FIN_M: given f and g such that all elements of the domain of f are smaller than all elements of the domain of g, define the product to be the union of the two functions. We consider FIN_M together with the coordinate-wise action of M on FIN_M, i.e. the action $f \mapsto af$ such that $(af)(n) = af(n)$ for every $f \in FIN_M$ and $n \in \mathbb{N}$. A *variable located word* is an element of FIN_M that has at least a 1 in its range.

We say that a sequence $\bar{t}$ in a partial semigroup S with an action of M on S is *basic* if $m_0 t_{i_0} \cdot \ldots \cdot m_n t_{i_n}$ is defined for every $i_0 < \ldots < i_n$ and $m_0, \ldots, m_n \in M$.

Let us introduce one of the most important subsets of $*$: it is closely related to the notion of block subspace in Banach space theory, which is pervasive in the work of Gowers (see for example [22], [23]).

Definition *Let M be a monoid acting by endomorphisms on a partial semigroup S, and let $\bar{s}$ be a basic sequence of elements of S. The (combinatorial) M*-span *of $\bar{s}$ is the set*

$$\langle \bar{s} \rangle_M = \left\{ m_0\, s_{i_0} \cdot \ldots \cdot m_n\, s_{i_n} : n \in \omega, i_0 < \cdots < i_n, m_i \in M, \text{ at least one } m_i \text{ is } 1_M \right\}.$$ [4]

The next definition introduces a quasi-order which is central in the definition of topological Ramsey space by Carlson [9].

Definition *Given a partial semigroup S and two infinite sequences $\bar{s}$, $\bar{t}$ of elements of S, we say that $\bar{s}$ is* extracted *from $\bar{t}$, or $\bar{s} \leq_M \bar{t}$, if there is an increasing sequence $(i_n)_{n\in\omega}$ of natural numbers such that $s_n \in \langle t_{i_n}, \ldots, t_{(i_{n+1})-1} \rangle_M$.*

As mentioned above, FIN_M is universal: the existence of monochromatic M-spans

[4] Notice that if $\bar{s} = s_0 \ldots s_{k-1}$ is a sequence of variable located words then every element of $\langle \bar{s} \rangle_M$ can be written as $m_0 s_{i_0} \cdot \ldots \cdot m_n s_{i_n}$ in an unique way. This implies that $\langle \bar{s} \rangle_M$ is as large as possible.

in FIN_M implies the existence of monochromatic M-spans in every other semigroup on which M acts.

Proposition (Proposition 3.6.1) *For a monoid M the following are equivalent.*

1. *For every finite coloring of FIN_M there is an infinite basic sequence of variable located words with monochromatic M-span.*
2. *For every partial semigroup S on which M acts by endomorphisms, for every basic sequence $\bar{t} \in S^\omega$, for every finite coloring of S there is a sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s} \rangle_M$ is monochromatic.*

Finally, one of the main definitions of this work:

Definition *We say that a monoid is* Ramsey *if one of the equivalent statements in the Proposition above holds.*[5]

If a monoid is Ramsey then apparently stronger statements hold. For example, one can always find a sequence $\bar{s}$ such that the set $*$ is partitioned in at most $|\mathbb{X}(M)|$ many monochromatic subsets, where $\mathbb{X}(M) = \{aM : a \in M\}$ (see Corollary 3.6.2).

Carlson's theorem and Gowers' FIN_k theorem correspond to the statement that certain monoids are Ramsey. It can be shown that Bergelson-Blass-Hindman theorem and Hales-Jewett theorem are a consequence of Carlson's theorem.

After Carlson's and Gowers' theorems, the main result about Ramsey monoids was found by Solecki. A monoid is said *almost $\mathcal{R}$-trivial* if $aM = bM$ and $a \neq b$ imply $Ma = a$.

Theorem (Solecki, [43]) *If a monoid is finite, almost $\mathcal{R}$-trivial, and $\mathbb{X}(M)$ is linearly ordered by inclusion, then M is Ramsey. If M is Ramsey then $\mathbb{X}(M)$ is linearly ordered by inclusion.*

Our theorem generalizes Solecki's theorem and gives an algebraic characterization of Ramsey monoids. A monoid M is *aperiodic* if for every $a \in M$ there is n such that $a^n = a^{n+1}$: one of our main results is the following theorem.

Theorem (Corollary 4.3.8) *A monoid is Ramsey if and only it is finite, aperiodic and $\mathbb{X}(M)$ is linearly ordered by inclusion.*

1.2.3 $\mathbb{Y}$-controllable monoids

Being Ramsey is a very strong condition. Some theorems have a weaker statement, such as Furstenberg and Katznelson's Ramsey theorem [19]. Inspired by this theo-

[5]The original definition, given by Solecki, has some advantages but is slightly longer: we postpone it to Chapter 3. It is equivalent, as shown in Proposition 3.6.1.

rem, Solecki isolated a condition for monoids, weaker than being Ramsey. We call the monoids satisfying this condition $\mathbb{Y}$*-controllable*. Furstenberg and Katznelson's theorem is a consequence of the fact that some monoids are $\mathbb{Y}$-controllable.

Roughly speaking, a monoid M is $\mathbb{Y}$-controllable if in a big portion of a set of the form $*$, the colors are easily controlled by a partial order, called $(\mathbb{Y}(M), \leq_{\mathbb{Y}})$. This partial order captures the behaviour of the linear parts of $(\mathbb{X}(M), \subseteq)$.

We will see that a monoid is Ramsey if and only if it is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linearly ordered by inclusion. If a monoid M is $\mathbb{Y}$-controllable we have a Ramsey theorem even if $\mathbb{X}(M)$ is not linearly ordered by inclusion.

Definition *Given a monoid M, $\mathbb{Y}(M) \subseteq \mathscr{P}(\mathbb{X}(M))$ consists of the non-empty subsets of $\mathbb{X}(M)$ which are linearly ordered by inclusion. Given $x, y \in \mathbb{Y}(M)$, define $x \leq_{\mathbb{Y}(M)} y$ if $x \subseteq y$ and all elements of $y \setminus x$ are larger with respect to $\subseteq$ than all elements of x.*

Let $\langle \mathbb{Y}(M) \rangle$, with operation $\vee$, be the semigroup freely generated by $\mathbb{Y}(M)$ modulo the relations

$$p \vee q = q = q \vee p \text{ for } p \leq_{\mathbb{Y}(M)} q.$$

Definition *A monoid M is* $\mathbb{Y}$-controllable *if for every finite $F \subseteq \langle \mathbb{Y}(M) \rangle$, for every $\boldsymbol{y}$ maximal element in $\mathbb{Y}(M)$, and for every finite coloring of FIN_M there is a sequence $\bar{s} \in (\mathrm{FIN}_M)^\omega$ of variable words such that for every $f \in F$ the following set is monochromatic*

$$*_f = \left\{ m_0\, s_{i_0} \cdot \ldots \cdot m_n\, s_{i_n} : n \in \omega, i_0 < \cdots < i_n, m_i \in M, m_0 \boldsymbol{y} \vee \cdots \vee m_n \boldsymbol{y} = f \right\}.$$

Solecki proved the following.

Theorem (Solecki, [43]) *If a monoid is finite and almost $\mathcal{R}$-trivial then it is $\mathbb{Y}$-controllable.*

In a monoid M, define the equivalence relation $\mathcal{R}$ as $a\mathcal{R}b$ if and only if $aM = bM$. Let $\mathbb{X}_{\mathcal{R}}(M) = \{aM : [a]_{\mathcal{R}}$ has more than one element$\}$.

Our main results on $\mathbb{Y}$-controllable monoids are summarized in the following theorem. Our theorem gives examples of $\mathbb{Y}$-controllable monoids of any cardinality.

Theorem (Theorem 4.5.4) *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linearly ordered by inclusion. Assume that there are no infinite chains in $(\mathbb{X}(M), \subseteq)$ and there are no infinite $\mathcal{R}$-classes. Then, M is $\mathbb{Y}$-controllable.*
In the other direction, let M be a $\mathbb{Y}$-controllable monoid. Then, M is aperiodic. Also, for every element $a \in M$ and maximal $\boldsymbol{y} \in \langle \mathbb{Y}(M) \rangle$ the set $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} a\boldsymbol{y}\}$ is finite.

In the following proposition (Proposition 3.7.1) we find $\mathbb{Y}$-controllable monoids where $\mathbb{X}_{\mathcal{R}}(M)$ is not linear.

Proposition *Let M be a finite, aperiodic monoid, such that for every distinct $a, b \in M$ with $a\, \mathcal{R}\, b$, we have $a^2 = a$ and $ax = bx$ for every $x \in M \setminus \{1\}$. Then, M is $\mathbb{Y}$-controllable.*

1.2.4 Locally Ramsey, locally $\mathbb{Y}$-controllable

One of the most immediate questions at this point is what can be said for infinite monoids. To obtain monochromatic M-spans, our techniques require finiteness at a certain point. Namely, we have to encode the monoid action with a first-order formula (see methods section 1.3).

The most natural approach for infinite monoids is to search for monochromatic subsets of $*$, where the action is limited to finite subsets of the monoid. Following what has been done for Carson's theorem, also called infinitary Hales-Jewett theorem ([47], Theorem 4.21), we give the following definitions.

Definition *Let M be a monoid acting on a partial semigroup S and let $\bar{s} \in S^\omega$ be a basic sequence. Let $(M_i)_{i\in\omega}$ be a sequence of finite subsets of M. The (M_i)-span of S is the set*

$$\langle \bar{s} \rangle_{(M_i)} = \{ m_0\, s_{i_0} \cdot \ldots \cdot m_n\, s_{i_n} : n \in \omega, i_0 < \cdots < i_n, m_i \in M_i, \text{ at least one } m_i \text{ is } 1_M \}.$$

Definition *A monoid M is* locally Ramsey *if for every finite coloring of FIN_M and for every sequence $(M_i)_{i\in\omega}$ of finite subsets of M there is an infinite sequence $\bar{s}$ of variable located words with monochromatic (M_i)-span.*

In Chapter 3, we obtain the following characterization.

Theorem (Theorem 4.5.5) *Let M be a monoid. Then, M is locally Ramsey if and only if M is aperiodic and $\mathbb{X}(M)$ is finite and linearly ordered by inclusion.*

The next definition is a natural generalization of $\mathbb{Y}$-controllable. As before, we require $m_i \in M_i$, where M_i are finite subsets of M.

Definition *A monoid M is* locally $\mathbb{Y}$-controllable *if for every finite $F \subseteq \langle \mathbb{Y}(M) \rangle$, for every $\boldsymbol{y}$ maximal element in $\mathbb{Y}(M)$, for every sequence $(M_i)_{i\in\omega}$ of finite subsets of M, and for every finite coloring of FIN_M there is a sequence $\bar{s} \in (\mathrm{FIN}_M)^\omega$ of variable words such that for every $f \in F$ the following set is monochromatic*

$$*_{f,(M_i)} = \{ m_0\, s_{i_0} \cdot \ldots \cdot m_n\, s_{i_n} : n \in \omega, i_0 < \cdots < i_n, m_i \in M_i, m_0\boldsymbol{y} \vee \cdots \vee m_n\boldsymbol{y} = f \}.$$

As in the case of $\mathbb{Y}$-controllable monoids, we do not have an algebraic characterization of locally $\mathbb{Y}$-controllable monoids. However, our results may suggest the possible final characterization (see open problems section).

Theorem (Theorem 4.5.3) *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linearly ordered by inclusion. Assume that there are no infinite chains in $(\mathbb{X}(M), \subseteq)$. Then, M is locally $\mathbb{Y}$-controllable. In the other direction, let M be a locally $\mathbb{Y}$-controllable monoid. Then, M is aperiodic. Also, for every $a \in M$ and maximal $\boldsymbol{y} \in \langle \mathbb{Y}(M) \rangle$ the set $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} a\boldsymbol{y}\}$ is finite.*

Proposition 3.7.1 gives examples of (locally) $\mathbb{Y}$-controllable monoids where $\mathbb{X}_{\mathcal{R}}(M)$

is not linear.

1.2.5 Other results

In Chapter 2, we introduce a relation $\prec$ in order to strengthen some classical results. We further require that the sequence $\bar{s} \in S^\omega$ with monochromatic span can be taken such that $s_n \prec s_{n+1}$ for every $n \in \omega$.

Taking Hindman's theorem as example, $\prec$ can be any relation satisfying the following two conditions:

1. for every finite $A \subseteq S$ there is a c such that $A \prec c$.

2. $a \prec b \prec c$ implies $a \prec b \cdot c$ for all $a, b, c \in S$.

In Lemma 2.8.1 a generalization of Gowers' FIN_k Theorem is found. There, we consider a semigroup together with a set of endomorphisms. This set of endomorphisms may be not closed under composition. Hence, Lemma 2.8.1 does not fit easily in the context of monoid actions.

1.3 Methods

The most celebrated proof of Hindman's theorem is due to Galvin and Glazer. It was first published in Comfort's survey [13]. Galvin and Glazer's proof uses idempotents in $\beta\mathbb{N}$, the space of ultrafilters over $\mathbb{N}$. The space $\beta\mathbb{N}$ is a compact right topological semigroup[6], with product defined by

$$\mathcal{U} * \mathcal{V} = \big\{A \subseteq \mathbb{N} : \{x \in \mathbb{N} : \{y \in \mathbb{N} : xy \in A\} \in \mathcal{V}\} \in \mathcal{U}\big\}.$$

Idempotents in compact right topological semigroups continue to have a central role in this area of Ramsey theory. Meanwhile, new techniques have emerged to avoid the use of ultrafilters and to bring new insights from other areas of mathematics. Most notably, two disciplines play an important role in Ramsey theory: ergodic theory [5], and methods from non-standard analysis [14].

In this thesis, we use model theory instead of ultrafilters, and the space of types instead of $\beta\mathbb{N}$.

We define a semigroup operation $\cdot_G$ on the space of types $S(G)$ over a semigroup G. It is convenient to use a saturated elementary extension $\mathcal{G}$ of the semigroup G. The extension $\mathcal{G}$ is often called a monster model.

We write $a \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.5ex\hbox{$|$}}_G b$, for $a, b \in \mathcal{G}$, if $\mathrm{tp}(a/Gb)$ is finitely satisfied in G. We think $\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.5ex\hbox{$|$}}_G$ as a form of independence. We assume it is 1-stationary, which is defined as follows.

[6] A compact right topological semigroup is a semigroup U with a compact topology such that the map $x \mapsto xu$ is continuous, for every $u \in U$.

Definition *If $a \equiv_G x \mathop{\smile\!\!\!\!\!\!\downarrow}_G b$ is a complete type (over G, b) for every $a \in \mathcal{G}$ and $b \in \mathcal{G}^{<\omega}$ we say that $\mathop{\smile\!\!\!\!\!\!\downarrow}_G$ is* 1-stationary.

Now we define the product of two types. It corresponds to the product of ultrafilters but it is arguably more intuitive. The product of two types is the type of two independent realizations of the types.

Definition *Assume $\mathop{\smile\!\!\!\!\!\!\downarrow}_G$ is 1-stationary. Let $a, b \in G$, then $tp(a/G) \cdot_G tp(b/G)$ is the type $tp(a' \cdot b'/G)$, for any couple $a' \equiv_G a$, $b' \equiv_G b$ such that $a' \mathop{\smile\!\!\!\!\!\!\downarrow}_G b'$.*

If $\mathop{\smile\!\!\!\!\!\!\downarrow}_G$ is 1-stationary, then the product $\cdot_G$ is well-defined. In this case, $(S(G), \cdot_G)$ is a compact right topological semigroup (Proposition 3.5.2).

If the theory of G is stable, then $\mathop{\smile\!\!\!\!\!\!\downarrow}_G$ is 1-stationary. A trick allows us to work with any semigroup G. If we add every subset of G to the language, then $\mathop{\smile\!\!\!\!\!\!\downarrow}_G$ is 1-stationary.

The key notion for this method is an old one: a coheir sequence.

Definition *We say that the tuple $\bar{c}$ is a* coheir sequence *of $p(x)$ over G if $c_n \models p(x)$ and $c_n \mathop{\smile\!\!\!\!\!\!\downarrow}_G \bar{c}_{\restriction n}$ and $c_{n+1} \equiv_{G, \bar{c}_{\restriction n}} c_n$ for every $n < \omega$.*

In particular, a coheir sequence $\bar{c}$ is indiscernible over G, i.e. $\bar{c}_{\restriction I_0} \equiv_G \bar{c}_{\restriction I_1}$ for every $I_0, I_1 \subseteq \omega$ of equal finite cardinality.

Here, we outline the strategy we use to prove that a monoid M is Ramsey.

- Find an idempotent type $u \in S(\mathrm{FIN}_M)$ such that $mu \cdot_G u = u = u \cdot_G mu$ for every $m \in M$. We require that elements satisfying u are variable words. [7]
- Consider a coheir sequence of u, with reverse order. Prove that this sequence has a monochromatic M-span.
- Use the coheir sequence to obtain a sequence of words in FIN_M with the same properties, i.e. a sequence of variable words with monochromatic M-span.

A coheir sequence has two key features for the last two steps of this strategy to work.

- $c_i \mathop{\smile\!\!\!\!\!\!\downarrow}_G c_{i-1} \cdot \ldots \cdot c_0$ for every $i \in \omega$: this implies that $\mathrm{tp}(\prod_{i \le n} c_i/G) = \prod_{i \le n} \mathrm{tp}(c_i/G)$. It also implies that $\mathrm{tp}(\prod_{i \le n} m_i c_i/G) = \prod_{i \le n} \mathrm{tp}(m_i c_i/G)$ for any $m_i \in M$, $i \le n$.
- $\mathrm{tp}(c_n/Gc_{n-1} \ldots c_0)$ is finitely satisfied in G: this can be used to find a sequence in the model with the same first-order properties of the coheir sequence. Here we need that the colors and the monoid action are definable.

[7] We prove a more general statement. In fact, there is an idempotent with these properties for every action of M by continuous endomorphisms on any compact right topological semigroup. The variable word condition becomes an ideal condition.

This strategy was first developed to prove Ramsey's theorem, Hindman's theorem, and Carlson's theorem. This strategy is versatile. For example, we use it to prove that some monoids are $\mathbb{Y}$-controllable, even in the elaborate context of sequences of M-pointed sets.

Our method has connections with the non-standard method. There, the product is defined through the star map $b \mapsto^* b$. Given two elements a and b in a semigroup $(S, \cdot)$, they consider the product $a \cdot^* b$ (see [15]). This corresponds to considering the product of two independent elements, in the sense of $\perp_G$. The non-standard method proved to be a powerful tool in Ramsey theory. This work wants to show that model theory may have a similar role.

1.4 Open problems

The following open problems are a natural continuation of our work.

Conjecture *A monoid M is $\mathbb{Y}$-controllable if and only if it is aperiodic, there are no infinite chains in $(\mathbb{X}(M), \subseteq)$, and there are no infinite $\mathcal{R}$-classes.*

From this conjecture would follow that a finite monoid is aperiodic if and only it is $\mathbb{Y}$-controllable. This could uncover an interesting connection with automata theory, since the class of finite aperiodic monoids corresponds to the class of star-free languages, through the celebrated Schützenberger's theorem [41]. A partial connection with Schützenberger's theorem is already given by the characterization of Ramsey monoids.

Conjecture *A monoid M is locally $\mathbb{Y}$-controllable if and only if it is aperiodic and there are no infinite chains in $(\mathbb{X}(M), \subseteq)$.*

In this thesis, we characterize Ramsey monoids. This does not characterize all the monoid actions on a semigroup S such that for every finite coloring of S there is a monochromatic span. Hence, the following is an interesting research line.

Open Problem *Characterize the couples (S, M), where S is a partial semigroup, and M is a monoid acting on S by endomorphisms, for which for every finite coloring of S, and for every ideal $I \subseteq S$ there is a sequence $\bar{s}$ of elements of I such that $\langle \bar{s} \rangle_M$ is monochromatic*[8].

This problem is particularly appealing in the case of $S = \mathrm{FIN}_M$ for some M. Lupini's theorem exemplifies an action of a monoid M' on FIN_M such that M' is not Ramsey, and such that there is always a monochromatic M'-span [33].

[8] An ideal I of a semigroup S is a set $I \subseteq S$ such that $IS \subseteq I$ and $SI \subseteq I$.

Chapter 2

Ramsey's coheirs

Abstract

We use the model theoretic notion of coheir to give short proofs of old and new theorems in Ramsey Theory. As an illustration we start from Ramsey's theorem itself. Then we prove Hindman's theorem and the Hales-Jewett theorem. Finally, we prove two Ramsey theoretic principles that have among their consequences partition theorems due to Carlson and to Gowers.

2.1 Introduction

Ramsey theory has substantial and diverse applications to many parts of mathematics. In particular, Ramsey's theorem has foundational applications to model theory through the Ehrenfeucth-Mostowski construction of indiscernibles and generalizations thereof. In this paper we explore the converse direction, that is, we use model theory to obtain new proofs of classical results in Ramsey Theory.

The Stone-Čech compactification, obtained via ultrafilters, is a widely employed method for proving Ramsey theoretic results. One of its first major applications is the celebrated Galvin-Glazer proof of Hindman's theorem, see e.g. [8]. Our methods are related, but alternative, to the ultrafilter approach. We replace βG (the Stone-Čech compactification of a semigroup G) with a large saturated elementary extension of G, i.e. a monster model of $\mathrm{Th}(G/G)$. One immediate advantage is that we work with elements of a natural semigroup with a natural operation. In contrast, elements of βG are ultrafilters, that is, sets of sets, and the semigroup operation among ultrafilters is far from straightforward.

This idea is not completely new: in his seminal work on the applications of topological dynamics to model theory [37, 38], Newelski replaces the semigroup βG with the space of types over G with a suitably defined operation. Our approach is similar, except that, unlike Newelski, we do not pursue connections with topological dynamics, but rather offer an alternative realm of application. The investigation of alternative methods in the study of regularity phenomena has been called for by Di Nasso [14, Open problem #1]. This article contains a possible answer.

The model theoretic tools employed in this paper are relatively basic. Section 2.2 is meant to give an accessible overview of the necessary notions for readers whose expertise is not primarily in model theory. Our results do not require assumptions of model theoretic tameness such as stability, NIP, etc., much like those that use nonstandard analysis, for example in [15]. Investigating the effect of such assumptions remains as future work.

The second author is grateful to Pierre Simon for suggesting the comparison with nonstandard analysis. Both authors would like to thank Vassilis Kanellopoulos for helpful conversation. When this paper was essentially complete, we became aware of [2], which is worth mentioning since it employs similar methods in a related context.

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The paper is divided into two parts. In the first part we prove that the notion of coheir leads to short and elegant proofs of well-known results. Most proofs in this part may be considered folklore, though they have not appeared in the literature so far. They are included here to provide a self-contained, gentle introduction to the techniques that are used in the second part.

As a preliminary illustrative step, we present a proof of Ramsey's theorem (Theorem 2.3.1). Then we prove a generalization of Hindman's theorem (Theorem 2.5.1), which is required in the second part of the paper. We also show how to combine Ramsey's and Hindman's theorems in a single proposition – the Milliken–Taylor theorem (Theorem 2.5.3). Finally, we prove an abstract algebraic version of the Hales-Jewett theorem (Theorem 2.6.4) due to Sabine Koppelberg [31].

In the second part of the paper we prove two Ramsey-theoretic properties of semigroups (Lemmas 2.7.1 and 2.8.1). As an application, we derive a generalization of Carlson's theorem on colourings of variable words which we present in the style of Koppelberg (Theorem 2.7.2) and in its classical form (Corollary 2.7.3). Lemma 2.8.1 is a partition theorem that generalizes Gowers' FIN_k Theorem [24] in a different direction than [33].

The extent of the generalizations mentioned above is limited, and they could be obtained in other ways, but our motivation here is to show the use and relevance of model theoretic methods. Numerous papers in the literature strengthen or generalize the partition theorems considered here. The comparison of the results that appear in these papers is not always straightforward – a few are compared in Chapter 3[1]

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The proofs in this paper require a modicum of familiarity with model theory. However, the results can be stated in an elementary language, and in the rest of this introduction we introduce the necessary terminology.

Throughout the paper G is a semigroup and Σ a non-empty set of endomorphisms of G. For $\bar a \in G^{\leq\omega}$ we write

$$\mathrm{fp}^{\Sigma}\,\bar a = \Big\{\sigma_0\, a_{i_0} \cdot \dots \cdot \sigma_k\, a_{i_k} \;:\; i_0 < \dots < i_k < |\bar a|,\;\; \bar\sigma \in \big(\Sigma \cup \{\mathrm{id}_G\}\big)^{k+1},\; k < |\bar a|\Big\}$$

Overlined symbols, such as $\bar a$ or $\bar\sigma$, always denote a tuple, and a_i, σ_i denotes the i-th entry of that tuple.

When Σ is empty, we write $\mathrm{fp}\,\bar a$.

2.1.1 Example For future reference, we instantiate the definition above in the context of free semigroups. Let G be the set of words on a finite alphabet $A \cup \{x\}$, where x is a symbol not in A which we call *variable*. Let C be the set of words on the alphabet A. Words in C are called *constant words*, while those in $G \smallsetminus C$ are called *variable words*. When G is endowed with the operation of concatenation of words, C and $G \smallsetminus C$ are subsemigroups of G. For $t \in G$ and $a \in A$, let $t(a)$ be the word obtained by replacing all the occurrences of x in t by a. Note that the map $\sigma_a : t \mapsto t(a)$ is an endomorphism of G. In the literature, when G is as above and $\Sigma = \{\sigma_a : a \in A\}$, the elements of $\mathrm{fp}^{\Sigma}\,\bar s$ are called *extracted words*. We say that a tuple $\bar a \in \big(\mathrm{fp}^{\Sigma}\bar s\big)^{\omega}$ is an *extracted sequence* if $a_i \in \mathrm{fp}^{\Sigma} s_{\restriction [n_i, n_{i+1})}$ for some increasing sequence of positive integers $\langle n_i : i < \omega\rangle$. If, moreover, $a_i \notin C$ for all i, we say that $\bar a$ is an *extracted variable sequence* of $\bar s$. □

The following definition will be used to express our results in the general context of semigroups.

2.1.2 Definition *Let $\lessdot$ be a binary relation on G. We say that G is* $\lessdot$-covered *if for every finite $A \subseteq G$ there is a c such that $A \lessdot c$. If c can be found in some fixed $B \subseteq G$, we say $\lessdot$-covered* by *B. We say that G is* $\lessdot$-closed *if $a \lessdot b \lessdot c$ implies $a \lessdot b\cdot c$ for all $a, b, c \in G$. A* $\lessdot$-chain *in G is a tuple $\bar a \in G^{\leq\omega}$ such that $a_i \lessdot a_{i+1}$.*

The preorder relation given by the length of the words on a free semigroup G is a natural example that is both $\lessdot$-closed and $\lessdot$-covered. A less straightforward relation is used in the proof of Theorem 2.7.2.

Finally, we recall two standard notions. Let $C \subseteq G$ be a subsemigroup. We say that C is *nice* if $a \cdot b \in C$ implies $a, b \in C$[2]. A homomorphism $\sigma : G \to C$ such that $\sigma_{\restriction C} = \mathrm{id}_C$ is called *retraction* of G onto C. Note that the set of constant words in

[1]See Proposition 3.6.1 and the discussion after Open Problem 3.7.2.

[2]Equivalently, if $G \setminus C$ is a both-sided ideal.

Example 2.1.1 is a nice subsemigroup and that the maps σ_a are retractions.

We are now ready to state Lemma 2.7.1.

Lemma Let Σ be a finite set of retractions of G onto a nice subsemigroup C. Let $\lessdot$ be a relation on G that makes it $\lessdot$-closed and $\lessdot$-covered by $G \smallsetminus C$. Then, for every finite coloring of G, there is a $\lessdot$-chain $\bar{a} \in (G \smallsetminus C)^\omega$ such that $\mathrm{fp}^\Sigma \bar{a} \smallsetminus C$ is monochromatic. □

When C and Σ are empty and $\lessdot$ holds for all pairs, the lemma reduces to Hindman's theorem (Theorem 2.5.1).

The appropriate choice of G, C, Σ and $\lessdot$ yields Carlson's partition theorem (in particular no model theoretic argument is necessary, see Theorem 2.7.2 and its Corollary 2.7.3).

In the last section we prove Lemma 2.8.1 which is similar to the lemma above but deals with composition of homomorphisms. This is also stated in an elementary language and a general version of a partition theorem by Gowers is derived from it.

2.2 Coheirs, and coheir sequences

We assume that the reader is familiar with undergraduate model theory and in this section we only review the few prerequisites that go beyond that. Proofs are omitted. The reader may consult any standard model theory textbook e.g. [46] (the intrepid reader may consult [49], some lecture notes which use the same notation and quirks as this paper). The notation and terminology are standard with the possible exception of Definitions 2.2.3 and 2.2.5.

A *sequence* is a function whose domain is a linear order. A *tuple* is a sequence whose domain is an ordinal. The domain of the tuple c is denoted by $|c|$ and is called the *length* of c.

2.2.1 Notation Sometimes (i.e. not always) we may overline tuples as mnemonic. When a tuple $\bar{c}$ is introduced, c_i denotes the i-th element of $\bar{c}$. We write $c_{\restriction I}$, where $I \subseteq |\bar{c}|$, for the tuple which is naturally associated to the restriction of $\bar{c}$ to I. The bar is dropped for ease of notation. □

We denote the monster model by $\mathcal{U}$ or, when dealing with semigroups, by $\mathcal{G}$. We always work over a fixed set of parameters $A \subseteq \mathcal{U}$. When this set is a model, as it will often be, we denote it by M, or G in the case of semigroups.

We say that a type $p(x)$ is *finitely satisfied* in A if every conjunction of formulas in $p(x)$ has a solution in $A^{|x|}$. A global type that is finitely satisfiable in A is invariant over A.

If M is a model every consistent type $p(x) \subseteq L(M)$ is finitely satisfied in M. For this reason in a few points in this paper it is necessary to work over a model. For simplicity, we always assume this.

The following is an easy, well-known fact.

2.2.2 Proposition *Every type $q(x) \subseteq L(\mathcal{U})$ that is finitely satisfiable in M has an extension to a global type finitely satisfiable in M.* □

If $p(x)$ is finitely satisfied in M, the extensions of $p(x)$ that are also finitely satisfied in M are called *coheirs* of $p(x)$.

In many cases it is useful to focus on elements instead of their types. We introduce the following notation to express that $\mathrm{tp}(a/M,b)$ is finitely satisfied in M. (The notion is standard in model-theory, it has no standard notation though.)

2.2.3 Definition *For every $a \in \mathcal{U}^{|x|}$ and $b \in \mathcal{U}^{|z|}$ we define*

$$a \downfree_M b \Leftrightarrow \varphi(a\,;b) \text{ for all } \varphi(x\,;z) \in L(M) \text{ such that } M^{|x|} \subseteq \varphi(\mathcal{U}^{|x|}\,;b)$$

We call this the coheir-heir *relation. We define the type*

$$x \downfree_M b = \Big\{ \varphi(x;b) \ : \ \varphi(x;b) \in L(M) \text{ and } M^{|x|} \subseteq \varphi(\mathcal{U}^{|x|};b) \Big\}.$$

The tuples a realizing this type are those such that $a \downfree_M b$. We will use the symbol $a \equiv_M x \downfree_M b$ for the union of the types $x \downfree_M b$ and $\mathrm{tp}(a/M)$. □

We imagine $a \downfree_M b$ as saying that a is *independent* from b over M. This is a very strong form of independence. In general it is not symmetric, that is, $a \downfree_M b$ is not the same as $b \downfree_M a$ (symmetry is equivalent to stability).

We shall use, sometimes without reference, the following easy lemma.

2.2.4 Lemma *The following properties hold for all small M, a, b, and c*

1. $a \downfree_M b \ \Rightarrow \ fa \downfree_M fb$ *for every $f \in Aut(\mathcal{U}/M)$* — *invariance*
2. $a \downfree_M b \ \Leftrightarrow \ a_0 \downfree_M b_0$ *for all finite $a_0 \subseteq a$ and $b_0 \subseteq b$* — *finite character*
3. $a \downfree_M b, c$ *and* $b \downfree_M c \ \Rightarrow \ a, b \downfree_M c$ — *transitivity*
4. $a \downfree_M b \ \Rightarrow$ *there exists $a' \equiv_{M,b} a$ such that $a' \downfree_M b, c$* — *coheir extension* □

Note that $a \equiv_M x \downfree_M b$ is the intersection of all types in $S(M,b)$ that are coheirs of $\mathrm{tp}(a/M)$. As there may be more than one of such coheirs, $a \equiv_M x \downfree_M b$ need not be a complete over M, b. In fact, completeness is a rather strong property.

2.2.5 Definition *If $a \equiv_M x \downfree_M b$ is a complete type (over M, b) for every $a \in \mathcal{U}^{|x|}$, every $b \in \mathcal{U}^{<\omega}$, and every tuple of variables x, then we say that $\downfree_M$ is* stationary. *We say*

n-stationary *if the requirement above is restricted to* $|x| = n$. □

Stationarity is often ensured by the following property.

2.2.6 Proposition *Fix a tuple of variable x of length n. If for every $\varphi(x) \in L(\mathfrak{U})$ there is a formula $\psi(x) \in L(M)$ such that $\varphi(M^{|x|}) = \psi(M^{|x|})$ then $\downfree_M$ is n-stationary.* □

2.2.7 Remark Stationarity of $\downfree_M$ over every model M is equivalent to the stability of T. However, in unstable theories the assumption may hold for some particular model. For example, if every subset of M^n is the trace of a definable set, then $\downfree_M$ is n-stationary by the proposition above. This simple observation will be of help in the proof of Theorem 2.5.1. For natural example let $T = T_{\rm dlo}$ and let $M \subseteq \mathfrak{U}$ have the order-type of $\mathbb{R}$. By quantifier elimination every definable of $\mathfrak{U}$ is union of finitely many intervals. By Dedekind completeness, the trace on A of any interval of $\mathfrak{U}$ coincides with that of an M-definable interval. □

Let $p(x) \in S(\mathfrak{U})$ be a global type that is finitely satisfiable in M. We say that the tuple $\bar{c}$ is a *coheir sequence* of $p(x)$ over M if for every $i < |\bar{c}|$

$$c_i \vDash p_{\restriction M, c_{\restriction i}}(x).$$

The following is a convenient characterization of coheir sequences.

2.2.8 Lemma *For $\bar{c}$ a tuple of length ω, the following are equivalent*

1. *$\bar{c}$ is a coheir sequence over M;*
2. *$c_n \downfree_M c_{\restriction n}$ and $c_{n+1} \equiv_{M, c_{\restriction n}} c_n$ for every $n < \omega$.* □

Let $I, <_I$ be a linear order. We call a function $\bar{a} : I \to \mathfrak{U}^{|x|}$ an *I-sequence*, or simply a *sequence* when I is clear.

If $I_0 \subseteq I$ we call $a_{\restriction I_0}$, the restriction of $\bar{a}$ to I_0, a *subsequence* of $\bar{a}$. When I_0 is finite we identify $a_{\restriction I_0}$ with a tuple of length $|I_0|$.

2.2.9 Definition *Let $I, <_I$ be an infinite linear order and let $\bar{a}$ be an I-sequence. We say that a is a* sequence of indiscernibles *over A or, a sequence of* A-indiscernibles, *if $a_{\restriction I_0} \equiv_A a_{\restriction I_1}$ for every $I_0, I_1 \subseteq I$ of equal finite cardinality.* □

The following can be easily derived from the lemma above by induction.

2.2.10 Proposition *Every sequence of coheirs over M is M-indiscernible.* □

2.3 Ramsey's theorem from coheir sequences

We illustrate the relation between coheirs and Ramsey phenomena in the simplest possible case: Ramsey's theorem. The subsequent sections build on this proof for more sophisticated results.

In this chapter we deal with finite partitions. The partition of a set X into k subsets is often represented by a map $f : X \to [k]$. The elements of $[k] = \{1, \ldots, k\}$ are also called *colors*, and the partition a *coloring*, or *k-coloring*, of X. We say that $Y \subseteq X$ is *monochromatic* if f is constant on Y.

Let M be an arbitrary infinite set. Fix $n, k < \omega$ and fix a coloring f of the set of all *n-subsets* of M, aleas the *complete n-uniform hypergraph* with vertex set M,

$$f\colon\binom{M}{n} \quad\to\quad [k].$$

We say that $H \subseteq M$ is a *monochromatic subgraph* if the subgraph induced by H is monochromatic. In the literature monochromatic subgraphs are also called *homogeneous sets.*

The following is a very famous theorem which we prove here in an unusual way. The proof will serve as a blueprint for other constructions in this paper.

2.3.1 Theorem *Let M be an infinite set. Then for every positive integer n and every finite coloring of the complete n-uniform hypergraph with vertex set M there is an infinite monochromatic subgraph.*

Proof Let L be a language that contains k relation symbols $r_1, \ldots, r_k$ of arity n. Given a k-coloring f we define a structure with domain M. The interpretation of the relation symbols is

$$r_i^M = \Big\{a_1, \ldots, a_n \in M \;:\; f\big(\{a_1, \ldots, a_n\}\big) = i\Big\}.$$

We may assume that M is an elementary substructure of some large saturated model $\mathcal{U}$. Pick any type $p(x) \in S(\mathcal{U})$ finitely satisfied in M but not realized in M and let $\bar{c} = \langle c_i : i < \omega \rangle$ be a coheir sequence of $p(x)$.

There is a first-order sentence saying that the formulas $r_i(x_1, \ldots, x_n)$ are a coloring of $\binom{M}{n}$. Then by elementarity the same holds in $\mathcal{U}$. By indiscernibility, all tuples of n distinct elements of $\bar{c}$ have the same color, say 1. We now prove that there is a sequence $\bar{a} = \langle a_i : i < \omega \rangle$ in M with the same property.

We construct $a_{\restriction i}$ by induction on i as follows.

Assume as induction hypothesis that the subsequences of length n of $a_{\restriction i}, c_{\restriction n}$ all have color 1. Our goal is to find $a_i \in M$ such that the same property holds for $a_{\restriction i}, a_i, c_{\restriction n}$. By the indiscernibility of $\bar{c}$, the property holds for $a_{\restriction i}, c_{\restriction n}, c_n$. And this can be written by a formula $\varphi(a_{\restriction i}, c_{\restriction n}, c_n)$. As $\bar{c}$ is a coheir sequence, by Lemma 2.2.8 we can find $a_i \in M$ such that $\varphi(a_{\restriction i}, c_{\restriction n}, a_i)$. So, as the order is irrelevant, $a_{\restriction i}, a_i, c_{\restriction n}$ satisfies the induction hypothesis. □

2.4 Idempotent orbits in semigroups

In this and the following sections we fix a semigroup G which we identify with a first-order structure. The language contains, among others, the symbol $\cdot$ which is interpreted as a binary associative operation on G. We write $\mathcal{G}$ for a large saturated elementary extension of G.

For any two sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$ we define

$$\mathcal{A} \cdot_G \mathcal{B} \quad = \quad \Big\{ a \cdot b \ : \ a \in \mathcal{A},\ b \in \mathcal{B} \text{ and } \ a \mathop{\smile\!\!\!\!\!\!\bigcup}_G b \Big\}$$

In this and the next section we abbreviate $\mathcal{O}(a/G)$, the orbit of a under $\mathrm{Aut}(\mathcal{G}/G)$, with a_G. We write $a \cdot_G \mathcal{B}$ for $\mathcal{O}(a/G) \cdot_G \mathcal{B}$. Similarly for $\mathcal{A} \cdot_G b$ and $a \cdot_G b$.

2.4.1 Lemma *If $\mathcal{A}$ is type definable over G then so is $\mathcal{A} \cdot_G b$ for any b.*

Proof The set $\mathcal{A} \cdot_G b$ is the union of $\mathcal{A} \cdot_G \{c\}$ as c ranges in b_G. The set $\mathcal{A} \cdot_G \{c\}$ is type definable, say by the the type $\exists y\, p(x, y, c)$ where

$$p(x, y, c) \quad = \quad y \mathop{\smile\!\!\!\!\!\!\bigcup}_G c \ \wedge \ y \cdot c = x \ \wedge \ y \in \mathcal{A}$$

Note that, by the invariance of $\mathop{\smile\!\!\!\!\!\!\bigcup}_G$, if $f \in \mathrm{Aut}(\mathcal{G}/G)$, then $\exists y\, p(x, y, fc)$ defines $\mathcal{A} \cdot_G \{fc\}$. Therefore if $q(z) = \mathrm{tp}(b/G)$ then $\exists y, z\, \big[q(z) \cup p(x, y, z)\big]$ defines $\mathcal{A} \cdot_G b$. □

By the invariance of $\mathop{\smile\!\!\!\!\!\!\bigcup}_G$, for every $f \in \mathrm{Aut}(\mathcal{G}/G)$ we have $f[\mathcal{A} \cdot_G \mathcal{B}] = f[\mathcal{A}] \cdot_G f[\mathcal{B}]$. Therefore when $\mathcal{A}$ and $\mathcal{B}$ are invariant over G, also $\mathcal{A} \cdot_G \mathcal{B}$ is invariant over G. Below we mainly deal with invariant sets.

2.4.2 Proposition *For all G-invariant sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$*

$$\mathcal{A} \cdot_G \big(\mathcal{B} \cdot_G \mathcal{C}\big) \quad \subseteq \quad \big(\mathcal{A} \cdot_G \mathcal{B}\big) \cdot_G \mathcal{C}.$$

Proof Let $a \cdot b \cdot c$ be an arbitrary element of the l.h.s. where $a \mathop{\smile\!\!\!\!\!\!\bigcup}_G b \cdot c$ and $b \mathop{\smile\!\!\!\!\!\!\bigcup}_G c$. By extension (Lemma 2.2.4), there exists a' such that $a \equiv_{G, b\cdot c} a' \mathop{\smile\!\!\!\!\!\!\bigcup}_G b \cdot c,\ b,\ c$. By transitivity (again Lemma 2.2.4), $a' \cdot b \mathop{\smile\!\!\!\!\!\!\bigcup}_G c$. Therefore $a' \cdot b \cdot c$ belongs to the r.h.s. Finally, as $a' \equiv_{G, b\cdot c} a$, also $a \cdot b \cdot c$ belongs to the r.h.s. by invariance. □

Let $\mathcal{A}$ be a non-empty set. When $\mathcal{A} \cdot_G \mathcal{A} \subseteq \mathcal{A}$, we say that it is *idempotent* (over G).

2.4.3 Corollary *Assume $\mathcal{B} \subseteq \mathcal{A}$ are both G-invariant. Then if $\mathcal{A}$ is idempotent, also $\mathcal{A} \cdot_G \mathcal{B}$ is idempotent.*

Proof We check that if $\mathcal{A}$ is idempotent so is $\mathcal{A} \cdot_G \mathcal{B}$

$$\begin{aligned} \big(\mathcal{A} \cdot_G \mathcal{B}\big) \cdot_G \big(\mathcal{A} \cdot_G \mathcal{B}\big) \quad &\subseteq \quad \mathcal{A} \cdot_G \big(\mathcal{A} \cdot_G \mathcal{B}\big) && \text{because } \mathcal{A} \cdot_G \mathcal{B} \subseteq \mathcal{A} \\ &\subseteq \quad \big(\mathcal{A} \cdot_G \mathcal{A}\big) \cdot_G \mathcal{B} && \text{by the lemma above} \end{aligned}$$

$$\subseteq \quad \mathcal{A} \cdot_G \mathcal{B} \qquad \square$$

We show that, under the assumption of stationarity, the operation $\cdot_G$ is associative. The quotient map $\mathcal{G} \to \mathcal{G}/{\equiv_G}$ is almost a homomorphism.

2.4.4 Proposition *Assume $\mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G$ is 1-stationary, see Definition 2.2.5. Fix $a \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G b$ arbitrarily. Then $a'{\cdot}b' \equiv_G a{\cdot}b$ for every $a' \equiv_G a$ and $b' \equiv_G b$ such that $a' \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G b'$. Or, in other words,*

$$(a{\cdot}b)_G \quad = \quad a \cdot_G b.$$[3]

Proof We prove two inclusions, only the second one requires stationarity.

$\subseteq$ As $a \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G b$ holds by hypothesis, $a{\cdot}b \in a \cdot_G b$. The inclusion follows by invariance.

$\supseteq$ By invariance it suffices to show that the l.h.s. contains $a \cdot_G \{b\}$. Let $a' \in a_G$ such that $a' \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G b$. We claim that $a'{\cdot}b \in (a{\cdot}b)_G$. Both a and a' satisfy $a \equiv_G x \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G b$. By 1-stationarity, $a \equiv_{G,b} a'$. Hence $a{\cdot}b \equiv_G a'{\cdot}b$. □

2.4.5 Corollary (associativity) *Assume $\mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G$ is 1-stationary. Then for all G-invariant sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$*

$$\mathcal{A} \cdot_G (\mathcal{B} \cdot_G \mathcal{C}) \quad = \quad (\mathcal{A} \cdot_G \mathcal{B}) \cdot_G \mathcal{C}.$$

Proof We can assume that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are G-orbits. Say of a, b, and c respectively. We can assume that $a \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G b{\cdot}c$ and $b \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G c$. By Proposition 2.4.4 the set on the l.h.s. equals $(a{\cdot}b{\cdot}c)_G$. By a similar argument the set on the r.h.s. equals $(a'{\cdot}b'{\cdot}c')_G$ for some elements a', b', and c'. Proposition 2.4.2 proves that inclusion $\subseteq$ holds in general. But inclusion between orbits amounts to equality. □

The following lemma proves the existence of idempotent orbits. The proof is self-contained, i.e. it does not use Ellis's theorem on the existence of idempotents in compact left topological semigroups (however, the argument is very similar). As a comparison, finding a proof in the setting of nonstandard analysis is listed as an open problem in [14].

2.4.6 Lemma *Assume $\mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\raise-0.2ex\hbox{$\mid$}}_G$ is 1-stationary. If $\mathcal{A}$ is minimal among the idempotent sets that are type-definable over G, then $\mathcal{A} = b_G$ for some (any) $b \in \mathcal{A}$.*

Proof Fix arbitrarily some $b \in \mathcal{A}$. By Corollary 2.4.3, the set $\mathcal{A} \cdot_G b$ is contained in $\mathcal{A}$, idempotent and type-definable over G by Lemma 2.4.1. Therefore by minimality $\mathcal{A} \cdot_G b = \mathcal{A}$. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the set of those a such that $a \cdot_G b = b_G$. This set is non-empty because $b \in \mathcal{A} \cdot_G b$. It is easy to verify that $\mathcal{A}'$ is type-definable over G, b. As it is clearly invariant over G, it is type-definable over G. By associativity it is idempotent. Hence, by minimality, $\mathcal{A}' = \mathcal{A}$. Then $b \in \mathcal{A}'$, which implies $b \cdot_G b = b_G$. That is, b has idempotent orbit. Finally, by minimality, $\mathcal{A} = b_G$. □

[3] Notice that in this proposition we do not need $\cdot$ to be associative.

2.4.7 Corollary *Under the same assumptions of the lemma above, every idempotent set that is type-definable over G contains an element with an idempotent orbit.* □

2.5 Hindman's theorem

In this section we merge the theory of idempotents presented in Section 2.4 with the proof of Ramsey's theorem to obtain Hindman's theorem.

Let $\bar{a}$ be a tuple of elements of G of length $\leq \omega$. In Section 2.1 we defined $\operatorname{fp}\bar{a}$ and the notions of $\lessdot$-closed and $\lessdot$-covered. The relation $\lessdot$ is introduced mainly for future reference. The classical Hindman's theorem is obtained with the positive integers (as an additive semigroup) for G and $<$ for $\lessdot$.

2.5.1 Hindman Theorem *Let $\lessdot$ be a relation on G that makes it $\lessdot$-closed and $\lessdot$-covered. Then for every finite coloring of G there is a $\lessdot$-chain $\bar{a}$ such that $\operatorname{fp}\bar{a}$ is monochromatic. If there is no $g \in G$ such that $G \lessdot g$, we may futher assume that the elements of the $\lessdot$-chain are all distinct.* [4]

Proof We interpret G as a structure in a language that extends the language of semigroups with a symbol for $\lessdot$ and one for each subset of G. Let $\mathcal{G}$ be a saturated elementary superstucture of G. As observed in Remark 2.2.7, the language makes $\mathop{\smile\!\!\!\!\!|}_G$ trivially 1-stationary.

We write $\mathcal{G}'$ for the type-definable set $\{g : G \lessdot g\}$, which is non-empty because G is $\lessdot$-covered. We claim that $\mathcal{G}'$ is idempotent. In fact, if $a, b \in \mathcal{G}'$ then, as $G \lessdot a, b$ and $a \mathop{\smile\!\!\!\!\!|}_G b$, we must have that $a \lessdot b$. Therefore, from the $\lessdot$-closedness of G we infer $a \cdot b \in \mathcal{G}'$.

Let g_0 be an element of $\mathcal{G}'$ with idempotent orbit as given by Corollary 2.4.7. We can assume that $g_0 \notin G$ otherwise the sequence that is identically g_0 trivially proves the theorem. If we want the elements of the chain $\bar{a}$ to be distinct it suffices require that $g_0 \notin G$. By definition of g_0, this can be directly assumed when there is no $g \in G$ such that $G \lessdot g$. Let $p(x) \in S(\mathcal{G})$ be a global coheir of $\operatorname{tp}(g_0/G)$. Let $\bar{g}$ be a coheir sequence of $p(x)$, that is

$$g_i \quad \models \quad p_{\restriction G, g_{\restriction i}}(x).$$

We write $\bar{g}_{\restriction i}$ for the tuple $g_{i-1}, \ldots, g_0$. By the idempotency of $(g_0)_G$ and Proposition 2.4.4, $h \equiv_G g_0$ for all $h \in \operatorname{fp}\bar{g}_{\restriction i}$ and all i. It follows in particular that $\operatorname{fp}\bar{g}_{\restriction i}$ is monochromatic, say all its elements have color 1. Now, we use the sequence $\bar{g}$ to define $\bar{a} \in G^\omega$ such that all elements of $\operatorname{fp}\bar{a}$ have color 1.

Assume as induction hypothesis that $\operatorname{fp}(a_{\restriction i}, g_0)$ is monochromatic of color 1. Our goal is to find a_i such that the same property holds for $\operatorname{fp}(a_{\restriction i+1}, g_0)$.

First we claim that from the induction hypothesis it follows that, for all j, all el-

[4] In [21] it is discussed when such elements are all distinct.

ements of $\text{fp}(a_{\restriction i}, \overleftarrow{g}_{\restriction j})$ have color 1. In fact, the elements of $\text{fp}(a_{\restriction i}, \overleftarrow{g}_{\restriction j})$ have the form $b \cdot h$ for some $b \in \text{fp}(a_{\restriction i})$ and $h \in \text{fp}(\overleftarrow{g}_{\restriction j})$. As $h \equiv_G g_0$, we conclude that $b \cdot h \equiv_G b \cdot g_0$, which proves the claim.

Let $\varphi(a_{\restriction i}, g_{i+1}, g_{\restriction i+1})$ say that all elements of $\text{fp}(a_{\restriction i}, \overleftarrow{g}_{\restriction i+2})$ have color 1. As $\bar{g}$ is a coheir sequence we can find a_i such that $\varphi(a_{\restriction i}, a_i, g_{\restriction i+1})$. Hence all elements of $\text{fp}(a_{\restriction i+1}, \overleftarrow{g}_{\restriction i+1})$ have color 1. Therefore a_i is as required. □

Hindman's theorem generalizes to a proposition that subsumes Ramsey's theorem. It is usually referred to as the Milliken–Taylor theorem [36] and [45]. By the following observation, we may use virtually the same proof.

2.5.2 Proposition *Assume $\perp_G$ is 1-stationary. Let $\bar{g} \in \mathcal{G}^\omega$ be a coheir sequence of some global coheir of* $\text{tp}(g/G)$ *where g has idempotent orbit. Let $\bar{h} \in \mathcal{G}^\omega$ be such that $h_i \in \text{fp}(\overleftarrow{g}_{\restriction I_i})$ for some finite non-empty $I_i \subseteq \omega$ such that $I_i < I_{i+1}$. Then $\bar{h} \equiv_G \bar{g}$.*

Proof Write n_i for the minimum of I_i. It suffices to prove that $h_i \equiv_{G, g_{\restriction n_i}} g_{n_i}$. Note that the type $g \equiv_G x \perp_G g_{\restriction n_i}$ is satisfied both by h_i and g_{n_i}, hence the claim follows by stationarity. □

Write $\text{fp}(\bar{a})_n$ for the n-uniform hypergraph with vertex set $\text{fp}(\bar{a})$ and as edges those sets $\{h_1, \ldots, h_n\}$ such that $h_i \in \text{fp}(a_{\restriction I_i})$ for some finite sets $I_1 < \cdots < I_n$.

2.5.3 Milliken-Taylor Theorem *Let $\prec$ be a relation on G that makes it $\prec$-closed and $\prec$-covered. Then for every positive integer n and every finite coloring of the complete n-uniform hypergraph with vertex set G there is a $\prec$-chain $\bar{a}$ such that* $\text{fp}(\bar{a})_n$ *is monochromatic.* □

Proof Given a coheir sequence $\bar{g}$ as in the proof of Theorem 2.5.1 we want to define $\bar{a} \in G^\omega$ such that $\text{fp}(\bar{a})_n$ is monochromatic. By the proposition above, $\text{fp}(\overleftarrow{g}_{\restriction i})_n$ is monochromatic for every $i \geq n$. As in the proof of Theorem 2.5.1, we define by induction $\bar{a} \in G^\omega$ in such a way that $\text{fp}(a_{\restriction i}, \overleftarrow{g}_{\restriction n})_n$ is a finite monochromatic subgraph of G. □

2.6 The Hales-Jewett theorem

The Hales-Jewett theorem is a purely combinatorial statement that implies the van der Waerden theorem. The original proof by Alfred Hales and Robert Jewett is combinatorial [26]. An alternative proof, also combinatorial, is due by Saharon Shelah [42]. Our proof is similar to the proof by Andreas Blass in [8] (based on ideas from [6]), but we use saturated models where he uses Stone-Čech compactification. We present three versions of the main theorem.

First we prove an abstract algebraic version due to Sabine Koppelberg [31] which is easier to state and to prove (this version comes in two variants). The classical version follows easily from the algebraic one.

We work with the same notation as in Section 2.4. We say that an element c is *left-minimal* (w.r.t. $\mathcal{A}$) if $c \in \mathcal{A} \cdot_G g$ for every $g \in \mathcal{A} \cdot_G c$.

2.6.1 Proposition *Assume $\mathop{\downfree}_G$ is 1-stationary. Let $\mathcal{A}$ be idempotent and type-definable over G. Then $\mathcal{A}$ contains a left-minimal element c with idempotent orbit.*

Proof Construct by induction a chain of type-definable idempotent sets $\mathcal{B}_\alpha \subseteq \mathcal{A}$ and elements $b_\alpha \in \mathcal{B}_\alpha$ such that $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_{\alpha+1} = \mathcal{A} \cdot_G b_\alpha$. For α limit take the intersection. By idempotency of $\mathcal{A}$, it is straightforward to check that $\mathcal{B}_{\alpha+1} \subseteq \mathcal{B}_\alpha$. The sets $\mathcal{B}_\alpha$ are type-definable and idempotent by 2.4.1 and 2.4.3. For α limit $\mathcal{B}_\alpha$ is non-empty by compactness, as it is intersection of a chain of closed sets.

For some α we cannot properly extend this construction. For this α, for every $c, g \in \mathcal{B}_\alpha$ we have $\mathcal{A} \cdot_G c = \mathcal{B}_\alpha = \mathcal{A} \cdot_G g$. Hence every $c \in \mathcal{B}_\alpha$ is left-minimal. As $\mathcal{B}_\alpha$ is idempotent, by Corollary 2.4.7 there is some $c \in \mathcal{B}_\alpha$ with idempotent orbit. □

2.6.2 Proposition *Assume $\mathop{\downfree}_G$ is 1-stationary. Let $\mathcal{A}$ be idempotent and type-definable over G. Let c_G be idempotent and such that $c \cdot_G \mathcal{A},\ \mathcal{A} \cdot_G c \subseteq \mathcal{A}$. Then*

1. *$c \cdot_G \mathcal{A} \cdot_G c$ contains some g with idempotent orbit;*
2. *if moreover c is left-minimal, then $c \equiv_G g$ for every g as in 1.*

Note, parenthetically, that the set in 1 may not be type-definable, therefore Corollary 2.4.7 does not apply directly and we need an indirect argument.

Proof 1. From $c \cdot_G \mathcal{A} \subseteq \mathcal{A}$ we obtain that $\mathcal{A} \cdot_G c$ is idempotent. As it is also type-definable, $\mathcal{A} \cdot_G c$ contains a b with idempotent orbit by Corollary 2.4.7. There is an $a \in \mathcal{A}$ such that $b_G = a \cdot_G c$, then $b \cdot_G c = b_G$. From this we obtain that $c \cdot_G b$ is idempotent and contained in $c \cdot_G \mathcal{A} \cdot_G c$.

2. From $g \in c \cdot_G \mathcal{A} \cdot_G c$ and the idempotency of c_G we obtain $g_G = c \cdot_G g$. As $g \in \mathcal{A} \cdot_G c$, from the left-minimality of c_G we obtain $c \in \mathcal{A} \cdot_G g$. Hence $c_G = c \cdot_G g$, by the idempotency of g_G. Therefore $c_G = g_G$, which proves 2. □

The following is a technical lemma that is required in many proofs below.

2.6.3 Proposition *Assume $\mathop{\downfree}_G$ is 1-stationary. Let $\sigma : \mathcal{G} \to \mathcal{G}$ be a semigroup homomorphism definable over G. Then for every $a, b \in \mathcal{G}$*

1. *$\sigma\big[a_G\big] \ = \ (\sigma\, a)_G$*
2. *$\sigma\big[a \cdot_G b\big] \ = \ \sigma\, a \cdot_G \sigma\, b.$*

Proof 1. As $a \equiv_G a'$ implies $\sigma\, a \equiv_G \sigma\, a'$, inclusion $\subseteq$ is clear. For the converse, note that the type $\exists y \, \big[\sigma\, y = x \wedge y \equiv_G a\big]$ is trivially realized by $\sigma\, a$. Therefore it is realized by all elements of $(\sigma\, a)_G$. Hence all elements of $(\sigma\, a)_G$ are the image of some element in a_G.

2. Let $a \equiv_G a' \downfree_G b' \equiv_G b$. By Proposition 2.4.4 we have $\sigma\big[a \cdot_G b\big] = \sigma\big[(a' \cdot b')_G\big]$. Then it suffices to prove that $\sigma\big[(a' \cdot b')_G\big] \subseteq \sigma\, a \cdot_G \sigma\, b$, because by 1 and Proposition 2.4.4 both sides of the equality are orbits. As σ preserves $\downfree_G$ and orbits, we obtain that $\sigma(a' \cdot b')$ is in $\sigma\, a \cdot_G \sigma\, b$, as well as all other elements of $\sigma\big[(a' \cdot b')_G\big]$. □

2.6.4 Hales-Jewett Theorem (Koppelberg's version) *Let G be an infinite semigroup and let $C \subseteq G$ be a nice subsemigroup. Let Σ be a finite set of retractions of G onto C. Then, for every finite coloring of C, there is an $a \in G \smallsetminus C$ such that $\{\sigma\, a : \sigma \in \Sigma\}$ is monochromatic.*

Proof Let $G \preceq \mathcal{G}$. Here $\mathcal{G}$ is a monster model in a language that expands the natural one with a symbol for all subsets of G and for every retraction in Σ. As observed in Remark 2.2.7, this makes $\downfree_G$ trivially 1-stationary. Let $\mathcal{C}$ be the definable set such that $C = G \cap \mathcal{C}$. By elementarity, $\mathcal{C}$ is a nice subsemigroup of $\mathcal{G}$. The language contains also symbols for the retractions $\sigma : \mathcal{G} \to \mathcal{C}$.

By Proposition 2.6.1, there is a left-minimal $c \in \mathcal{C}$ with idempotent orbit.

By niceness, $\mathcal{G} \smallsetminus \mathcal{C}$ and c satisfy the assumptions of Proposition 2.6.2. Hence, by the first claim of that proposition, there is an idempotent $g \in c \cdot_G (\mathcal{G} \smallsetminus \mathcal{C}) \cdot_G c$. In particular, $g \in \mathcal{G} \smallsetminus \mathcal{C}$. Now apply the second claim of Proposition 2.6.3, with $\mathcal{C}$ for $\mathcal{A}$ to obtain $\sigma\, g \in c \cdot_G \mathcal{C} \cdot_G c$ for all $\sigma \in \Sigma$. As $\sigma\, g$ is also idempotent, we apply Proposition 2.6.2 to conclude that $\sigma\, g \equiv_G c$. In particular the set $\{\sigma\, g : \sigma \in \Sigma\}$ is monochromatic.

Though the element g above need not belong to $G \smallsetminus C$, by elementarity $G \smallsetminus C$ contains some a with the same property and this proves the theorem. □

Finally we show how the classical Hales-Jewett theorem follows from its abstract version.

If C and X are two semigroups we denote by $C * X$ their free product. That is, $C * X$ contains finite sequences of elements of $C \cup X$, below called *words*, that alternate elements in C with elements in X. The product of two words is obtained concatenating them and, when it applies, replacing two contiguous elements of the same semigroup by their product. Note that C and X are nice subsemigroups of $C * X$. When X is the free semigroup generated by a variable x, we denote $C * X$ by $C[x]$. If $w(x)$ is an element of $C[x]$ and $a \in C$ we denote by $w(a)$ the result of replacing x by a in $w(x)$.

2.6.5 Hales-Jewett Theorem (classical version) *Let C be a semigroup generated by some finite set A. Let x be a variable. Then for every finite coloring of $C[x]$ there is a $w(x) \in C[x] \smallsetminus C$ such that $\{w(a) : a \in A\}$ is monochromatic.*

Proof Let $G = C[x]$. For every $a \in A$ the homomorphism $\sigma_a : w(x) \mapsto w(a)$ is a retraction of G onto C. Hence we can apply the theorem above. □

We conclude with a variant of Theorem 2.6.4 that applies to a broader class of semigroup homomorphisms. This result is not required for the following.

For Σ a set of maps $\sigma : G \to C$ and $c \in C$ we define

$$\Sigma^{-1}[c] \quad = \quad \bigcap_{\sigma \in \Sigma} \sigma^{-1}[c]$$

Clearly, when the maps in Σ are retractions, $\Sigma^{-1}[c]$ is non-empty for all $c \in C$ because it contains at least c.

2.6.6 Hales-Jewett Theorem (yet another variant) *Let C be a semigroup and let Σ be a finite set of homomorphisms $\sigma : G \to C$ such that $\Sigma^{-1}[c]$ is non-empty for all $c \in C$. Then, for every finite coloring of C, there is a $g \in G$ such that the set $\{\sigma g \ : \ \sigma \in \Sigma\}$ is monochromatic.*

Proof Let $G * C$ be the free product of the two semigroups. Any homomorphism $\sigma : G \to C$ extends canonically to a retraction of $G * C$ onto C. The elements of G that occur in a word are replaced by their image under σ, finally the elements in the resulting sequence are multiplied. This extension is denoted by the same symbol σ.

Apply Theorem 2.6.4 to obtain some $w \in G * C$ such that $\{\sigma w \ : \ \sigma \in \Sigma\}$ is monochromatic. Suppose $w = c_0 \cdot g_0 \cdot \cdots \cdot c_n \cdot g_n$ for some $g_i \in G$ and $c_i \in C$, where one or both of c_0 or g_n could be absent. Pick some $h_i \in \Sigma^{-1}[c_i]$ and let $g = h_0 \cdot g_0 \cdot \cdots \cdot h_n \cdot g_n$. Then $\{\sigma g \ : \ \sigma \in \Sigma\}$ is monochromatic as required to complete the proof. □

2.7 Carlson's theorem

This section is devoted to the following lemma and some of its consequences.

2.7.1 Lemma *Let Σ be a finite set of retractions of G onto a nice subsemigroup C. Let $\lessdot$ be a relation on G that makes it $\lessdot$-closed and $\lessdot$-covered by $G \smallsetminus C$. Then, for every finite coloring of G, there is a $\lessdot$-chain $\bar{a} \in (G \smallsetminus C)^\omega$ such that $\mathrm{fp}^\Sigma \bar{a} \smallsetminus C$ is monochromatic.*

Proof The models $\mathcal{G}$ and $\mathcal{C}$ are as in the proof of Theorem 2.6.4. The language is the same with $\lessdot$ included. Let $\mathcal{B} = \{g \in \mathcal{G} \smallsetminus \mathcal{C} : G \lessdot g\}$. By Proposition 2.6.1 there is some left-minimal $c \in \mathcal{C}$ with idempotent orbit. As G is $\lessdot$-covered by $G \smallsetminus C$, the set $\mathcal{B}$ is non-empty. As G is $\lessdot$-closed and C is nice, $\mathcal{B}$ and c satisfy the assumptions of Proposition 2.6.2. Then, $c \cdot_G \mathcal{B} \cdot_G c$ contains some g_0 with idempotent orbit. By Proposition 2.6.3, we obtain that $\sigma g_0 \in c \cdot_G \mathcal{C} \cdot_G c$ for all $\sigma \in \Sigma$. As $(\sigma g_0)_G$ is also idempotent, we apply the second claim of Proposition 2.6.3, with $\mathcal{C}$ for $\mathcal{A}$ to conclude that $\sigma g_0 \equiv_G c$ for all $\sigma \in \Sigma$. Now, let $\bar{g}$ be a coheir sequence as in Theorem 2.5.1, and assume the notation thereof. As $g_0 \in c \cdot_G \mathcal{B} \cdot_G c$ then $c \cdot_G g_0 = g_0 \cdot_G c = (g_0)_G$. Hence $h \equiv_G g_0$ for all i and all $h \in \mathrm{fp}\, \bar{g}_{\restriction i} \smallsetminus \mathcal{C}$. In particular all these h have the same color, say color 1. Now, we can use the sequence $\bar{g}$ to define $\bar{a} \in (G \smallsetminus C)^\omega$ such that all elements of $\mathrm{fp}^\Sigma \bar{a} \smallsetminus C$ have color 1 by reasoning as in the proof of Theorem 2.5.1. □

Carlson's theorem is a result that combines the theorems of Hindman and Hales-Jewett and has a number of important consequences. We refer the reader to [17] for a discussion of some of these consequences. The definitions in Example 2.1.1 will help matching the notation.

We first present a Koppelberg-style version of the theorem. It is obtained from the lemma above applying a suitable coding.

2.7.2 Carlson Theorem (à la Koppelberg) *Let Σ be a finite set of retractions of G onto a nice subsemigroup C. Let $\bar{s} \in (G \smallsetminus C)^\omega$. Then for every finite coloring of G, there is an increasing sequence of positive integers $\langle n_i : i < \omega \rangle$ and some $a_i \in \mathrm{fp}^\Sigma s_{\restriction [n_i, n_{i+1})} \smallsetminus C$ such that $\mathrm{fp}^\Sigma \bar{a} \smallsetminus C$ is monochromatic.*

Proof Let G_* be the free semigroup generated by the alphabet

$$\{\langle \sigma, g \rangle \ : \ \sigma \in \Sigma \cup \{\mathrm{id}_G\},\ g \in G \smallsetminus C\}.$$

The semigroup C_* is defined as G_*, only σ is restricted to range over Σ. Clearly C_* is a nice subsemigroup of G_*. We associate to each $\sigma \in \Sigma$ the endomorphism of G_* that substitutes σ for every occurrence of id_G in a word. These maps, which we denote by σ_*, are retractions of G_* onto C_*.

If $g_* \in G_*$ has the form $\langle \sigma_1, g_1 \rangle \cdots \langle \sigma_n, g_n \rangle$ we call $\sigma_1 g_1 \cdots \sigma_n g_n \in G$ the *evaluation* of g_*. We denote the evaluation by $\mathrm{eval}(g_*)$. As $\tau \sigma = \sigma$ for every $\tau, \sigma \in \Sigma$, we have that $\mathrm{eval}(\sigma_* g_*) = \sigma \, \mathrm{eval}(g_*)$. The evaluation of $g_* \in C_*$ belongs to C and, as C is nice, the evaluation of $g_* \in G_* \smallsetminus C_*$ belongs to $G \smallsetminus C$.

We color each element of G_* with the color of its evaluation.

We define the relation $\lessdot$ on G_*. First, we need to define the *well-formed* elements of G_*. These are elements of the form $\langle \sigma_1, s_{i_1} \rangle \cdots \langle \sigma_n, s_{i_n} \rangle$ for some $i_1 < \cdots < i_n$. Now, for $h_*, g_* \in G_*$ we define $h_* \lessdot g_*$ if one of the following holds

1. h_* is not well-formed while g_* is;
2. the product (i.e., concatenation) $h_* g_*$ is well-formed.

It is immediate to verify that $\lessdot$ is G_* is $\lessdot$-closed and $\lessdot$-covered by $G_* \smallsetminus C_*$. Therefore by Lemma 2.7.1 there is a $\lessdot$-chain $\bar{a}_* \in (G_* \smallsetminus C_*)^\omega$ such that $\mathrm{fp}^\Sigma \bar{a}_* \smallsetminus C_*$ is monochromatic. We can assume that all elements of $\bar{a}_*$ are well-formed (only the first element might be ill-formed, but we can drop it). Then the sequence $\langle \mathrm{eval}(a_{i*}) : i \in \omega \rangle$ is as required by the lemma. □

From the algebraic version of Carlson's theorem we obtain the classical one in the same way as for the Hales-Jewett theorem (Theorem 2.6.5), which we refer to for the notation.

2.7.3 Corollary (Carlson's theorem, classical version) *Let C be a semigroup generated by some finite set A. Let x be a variable. Let $\bar{s} \in \big(C[x] \smallsetminus C\big)^\omega$. Let Σ contain, for every $a \in A$, the function $w(x) \mapsto w(a)$. Then, for every finite coloring of $C[x]$, there is an* *increasing sequence of positive integers $\langle n_i : i < \omega \rangle$ and some $a_i \in \text{fp}^{\Sigma} s_{\restriction [n_i, n_{i+1})} \smallsetminus C$ such that $\text{fp}^{\Sigma}\bar{a} \smallsetminus C$ is monochromatic (with the terminology of Example 2.1.1, $\bar{a}$ is an extracted variable sequence of $\bar{s}$).* □

2.8 Gowers's partition theorem

The following is similar to Lemma 2.7.1 but here Σ contains compositions of homomorphisms.

2.8.1 Lemma *For $0 < i < n$, let G_i be a nice subsemigroup of G_{i+1} and let $\sigma_i : G_{i+1} \to G_i$ be surjective homomorphisms. Let $\prec$ be a relation on G_n that makes it $\prec$-closed and $\prec$-covered by $G_n \smallsetminus G_{n-1}$. Finally, let $\Sigma = \{\sigma_i \circ \cdots \circ \sigma_{n-1} : 0 < i < n\}$. Then, for every finite coloring of G_n, there is a $\prec$-chain $\bar{a} \in (G_n \smallsetminus G_{n-1})^{\omega}$ such that $\text{fp}^{\Sigma}\bar{a} \smallsetminus G_{n-1}$ is monochromatic.*

Proof For convenience, we let i run from 0, hence we agree that $\sigma_0 : G_1 \to G_0 = G_1$ is the identity. Let $\mathcal{B}_n = \{b \in \mathcal{G}_n \smallsetminus \mathcal{G}_{n-1} : G_n \prec b\}$ and $\mathcal{B}_i = \sigma_i[\mathcal{B}_{i+1}]$. Note that the $\mathcal{B}_i$ are non-empty because G_n is $\prec$-covered by $G_n \smallsetminus G_{n-1}$. Also, as $\mathcal{G}_i$ is a nice subsemigroup of $\mathcal{G}_{i+1}$, we have that $\mathcal{B}_i \cdot_G \mathcal{B}_{i+1}$, $\mathcal{B}_{i+1} \cdot_G \mathcal{B}_i \subseteq \mathcal{B}_{i+1}$.

We claim there is some $b_n \in \mathcal{B}_n$ with idempotent orbit such that, if we define $b_i = \sigma_i\, b_{i+1}$ for $0 \leq i < n$, the following holds

$$b_n \cdot_G b_i \quad = \quad b_i \cdot_G b_n \quad = \quad (b_n)_G.$$

Note that these equalities may be replaced by

$$\sharp_i \qquad b_i \cdot_G b_{i+1} \quad = \quad b_{i+1} \cdot_G b_i \quad = \quad (b_{i+1})_G.$$

Let $b_0 = b_1$ be any element of $\mathcal{B}_0$ with idempotent orbit. We assume as induction hypothesis that we have $b_i \in \mathcal{B}_i$ for $i \leq k$, with idempotent orbits, such that $b_i = \sigma_i\, b_{i+1}$ and $\sharp_i$ hold for all $i < k$. We show how to find b_{k+1}.

We prove that b_k and the set $\mathcal{B}_{k+1} \cap \sigma_k^{-1}[b_k]$, which below we denote by $\mathcal{A}$ for short, satisfy the assumptions of Proposition 2.6.2.[5] The proof of the idempotency of $\mathcal{A}$ is left to the reader. We prove that $b_k \cdot_G \mathcal{A} \subseteq \mathcal{A}$, the proof of $\mathcal{A} \cdot_G b_k \subseteq \mathcal{A}$ is similar. As $b_k \cdot_G \mathcal{B}_{k+1} \subseteq \mathcal{B}_{k+1}$ by nicety, it suffices to prove that $b_k \cdot_G \sigma_k^{-1}[b_k]$ is contained in $\sigma_k^{-1}[b_k]$. This latter inclusion holds because, by the induction hypothesis,

$$\sigma_k\Big[b_k \cdot_G \sigma_k^{-1}[b_k]\Big] \quad = \quad \sigma_k[b_k] \cdot_G b_k \quad = \quad b_{k-1} \cdot_G b_k \quad = \quad (b_k)_G.$$

Now we apply Proposition 2.6.2 to find an idempotent $b_{k+1} \in b_k \cdot_G \mathcal{A} \cdot_G b_k$. Therefore $\sharp_k$ is satisfied. Moreover $\sigma_k\, b_{k+1} \in (b_k)_G$ by Proposition 2.6.3, hence we can assume $b_k = \sigma_k\, b_{k+1}$ as claimed above.

Finally, as in the proof of Theorem 2.5.1, the required chain $\bar{a}$ is obtained from a coheir sequence of a global coheir of $\text{tp}(b_n/G)$. □

[5] We need the hypothesis that homomorphisms are surjective to say that $\mathcal{A}$ is non empty.

2.8.2 Remark The lemma above continues to hold, with essentially the same proof, if for Σ we take a set of the form

$$\Sigma \quad = \quad \bigcup_{i=1}^{n-1} \Sigma_i \circ \cdots \circ \Sigma_{n-1}$$

where

$$\Sigma_i \circ \cdots \circ \Sigma_{n-1} \quad = \quad \Big\{\sigma_i \circ \cdots \circ \sigma_{n-1} \;:\; \sigma_i \in \Sigma_i, \ldots, \sigma_{n-1} \in \Sigma_{n-1}\Big\}$$

and where Σ_i are some finite sets of homomorphisms $G_{i+1} \to G_i$ such that for every $g \in G_i$ the set $\Sigma_i^{-1}[g]$ is non-empty. □

Let G_i be the set of functions $a : \omega \to \{0, \ldots, i\}$ with finite *support* that is, the set $\mathrm{supp}(a) = \{x \in \omega \;:\; a\,x \neq 0\}$ is finite. We introduce a semigroup operation on G_i by defining $(a{\cdot}b)\,x = \max\{ax, bx\}$. This makes G_i a nice subsemigroup of G_{i+1}.

2.8.3 Corollary (Gowers Partition Theorem) *With G_i as above, let $\sigma_i : G_{i+1} \to G_i$ be homomorphisms and let Σ be as in Lemma 2.8.1. Then for every finite coloring of G_n there is an $\bar{a} \in \big(G_n \smallsetminus G_{n-1}\big)^{\omega}$ such that $\mathrm{fp}^{\Sigma}\bar{a} \smallsetminus G_{n-1}$ is monochromatic and $\mathrm{supp}(a_i) < \mathrm{supp}(a_{i+1})$.*

The homomorphisms σ_i usually considered in the literature are so-called *tetris* operations i.e. $(\sigma_i\, a)x \;=\; \max\{a\,x - 1,\; 0\}$, or generalizations thereof. However the theorem is more general.

Proof Let $\prec$ be the relation $\mathrm{supp}(a) < \mathrm{supp}(b)$ and apply Theorem 2.8.1. □

Chapter 3

Ramsey monoids

Abstract

Recently, Solecki introduced the notion of Ramsey monoid to produce a common generalization to theorems such as Hindman's theorem, Carlson's theorem, and Gowers' FIN_k theorem. He proved that an entire class of finite monoids is Ramsey. Here we improve this result, enlarging this class and finding a simple algebraic characterization of finite Ramsey monoids. We extend in a similar way a result of Solecki regarding a second class of monoids connected to the Furstenberg-Katznelson Ramsey Theorem. The results obtained suggest a possible connection with Schützenberger's theorem and finite automata theory.

3.1 Introduction

One of the most celebrated theorems in Ramsey theory is due to Hindman [27]. It states that for every semigroup $(S, \cdot)$, for every finite coloring of S there is an infinite sequence $\bar{s} = (s_i)_{i \in \omega} \in S^\omega$ such that the following set is monochromatic

$$\mathrm{fp}(\bar{s}) = \{s_{i_0} \cdot \cdots \cdot s_{i_n} : n \in \omega, i_0 < \cdots < i_n\}.$$

A natural question is whether a theorem of this kind can be proved if we allow the elements of the semigroup to be moved by some action.

The first answers to this question were given by Carlson [9] and Gowers [24] who studied actions of specific monoids. They used analogues of the following notion:

3.1.1 Definition *Let M be a monoid acting by endomorphisms on a partial semigroup S, and let $\bar{s}$ be a sequence of elements of S. The (combinatorial)* M-span *of $\bar{s}$ is the set*

$$\langle \bar{s} \rangle_M = \left\{ m_0\, s_{i_0} \cdots m_n\, s_{i_n} : n \in \omega, i_0 < \cdots < i_n, m_i \in M, \text{ at least one } m_i \text{ is } 1_M \right\}.$$

Solecki realized that these theorems share the same underlying structure, from which he isolated the notion of Ramsey monoid. We report here the original definition as stated in [43]. We refer the reader to Proposition 3.6.1 for other equivalent definitions.

Let M be a monoid and X be a set. If M acts on X, we say that X is an M-set. Suppose $(X_n)_{n\in\omega}$ is a family of M-sets. We say that the action of M on $(X_n)_{n\in\omega}$ is *uniform* if for every $k, n \in \omega$, $m \in M$ and $x \in X_k \cap X_n$, if $m_k x$ and $m_n x$ are the results of the action of m on x respectively in X_k and in X_n, then $m_k x = m_n x$. Notice that the action of M on $(X_n)_{n\in\omega}$ is uniform if and only if it extends to $\bigcup_{n\in\omega} X_n$. Let $X = \bigcup_{n\in\omega} X_n$. Define $W_X = (X^{<\omega}, ^\frown)$ to be the free semigroup on the alphabet X, with $^\frown$ the concatenation of sequences. Define $\langle (X_n)_{n\in\omega}\rangle$ to be the partial subsemigroup of W_X consisting of all sequences $x_1{}^\frown ... {}^\frown x_n \in W_X$ for which there exists $i_1 < ... < i_n \in \omega$ such that $x_k \in X_{i_k}$. If the action of M is uniform on $(X_n)_{n\in\omega}$, it is possible to define the *coordinate-wise action* of M on $\langle (X_n)_{n\in\omega}\rangle$ by setting $m(x_1{}^\frown ... {}^\frown x_n) = m(x_1){}^\frown ... {}^\frown m(x_n)$. This is an action by endomorphisms. An infinite sequence $\bar{s} \in (\langle (X_n)_{n\in\omega}\rangle)^\omega$ is said *basic* if the product $s_{i_0}{}^\frown \dots {}^\frown s_{i_n}$ is in $\langle (X_n)_{n\in\omega}\rangle$ for every $i_0 < \cdots < i_n$. This implies that if M acts uniformly on $(X_n)_{n\in\omega}$ and $\bar{s}$ is basic, then every product in $\langle \bar{s}\rangle_M$ is defined. A *pointed M-set* is an M-set X together with a *distinguished point* $x \in X$ such that $Mx = \{mx : m \in M\} = X$.

3.1.2 Definition *A monoid M is said* Ramsey *if for all sequences of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly and for all finite colorings of $\langle (X_n)_{n\in\omega}\rangle$ there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega}\rangle)^\omega$ such that s_n has a distinguished point for every $n \in \omega$ and the span $\langle \bar{s}\rangle_M$ is monochromatic.*

The notion of Ramsey monoid provides a common framework for many theorems in combinatorics. For example, Hindman's theorem can be restated as "*The trivial monoid $\{1\}$ is Ramsey*". Similarly, Carlson's theorem and Gowers' FIN_k theorem can be seen just as two examples of Ramsey monoids (see also Proposition 3.6.1). Hence, every new Ramsey monoid gives a new generalization of Hindman's theorem as powerful as Carlson's and Gowers' theorems.

Furthermore, from Carlson's and Gowers' theorems one can get examples of Ramsey spaces, see [47, Section 4.4]. In the same way, one can show that any new example of Ramsey monoid gives new examples of Ramsey spaces.

Let us recall some notation from [43]. Given a monoid M, define $\mathbb{X}(M) = \{aM : a \in M\}$. We say that $\mathbb{X}(M)$ is linear if it is linearly ordered by inclusion. Let $\mathcal{R}$ be the equivalence relation on M defined by $a\,\mathcal{R}\,b$ if $aM = bM$. A monoid M is called *almost $\mathcal{R}$-trivial* if for every $\mathcal{R}$-class $[a]_\mathcal{R}$ with more than one element we have $Ma = \{a\}$.

To the best of our knowledge, the only known examples of Ramsey monoids before Solecki's paper were given by Carlson's and Gowers' theorems. Solecki in [43, Corollary 4.5] proved that the class of finite almost $\mathcal{R}$-trivial monoids with linear $\mathbb{X}(M)$, that includes Carlson's and Gowers' monoids, is Ramsey. Also, he proved that Ramsey monoids have a linear $\mathbb{X}(M)$.

We work here with a well-known class of monoids that extends the one of almost $\mathcal{R}$-trivial monoids. A monoid is said *aperiodic* if for every $a \in M$ there exists $n \in \omega$ such that $a^n = a^{n+1}$ (see [40]). The first main achievement of this paper is an improvement of Solecki's result: first, we extend [43, Corollary 4.5] to the wider class of finite aperiodic monoids with linear $\mathbb{X}(M)$, and secondly, we prove that aperiodicity is also a necessary condition for being Ramsey, giving thus a complete characterization of finite Ramsey monoids.

3.1.3 Theorem (main theorem 1) *A finite monoid M is Ramsey if and only if it is aperiodic and $\mathbb{X}(M)$ is linear.*

To introduce the other peak of this paper we need some more notions from [43].

Given a monoid M, $\mathbb{Y}(M) \subseteq \mathcal{P}(\mathbb{X}(M))$ consists of the non-empty subsets of $\mathbb{X}(M)$ which are linearly ordered by inclusion. Given $x, y \in \mathbb{Y}(M)$, define $x \leq_{\mathbb{Y}(M)} y$ if $x \subseteq y$ and all elements of $y \setminus x$ are larger with respect to $\subseteq$ than all elements of x.

Let $\langle \mathbb{Y}(M) \rangle$, with operation $\vee$, be the semigroup freely generated by $\mathbb{Y}(M)$ modulo the relations

$$p \vee q = q = q \vee p \text{ for } p \leq_{\mathbb{Y}(M)} q.$$

We say that M is $\mathbb{Y}$*-controllable* if for every finite $F \subseteq \langle \mathbb{Y}(M) \rangle$, for every $\mathbf{y}$ maximal element in $\mathbb{Y}(M)$, for every sequence of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly and for every finite coloring of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ such that s_n has a distinguished point for every $n \in \omega$ and such that for every $m, n \in \omega$ and for every $a_i, b_j \in M$ <u>if</u> $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} \in F$ and $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = b_0\mathbf{y} \vee \cdots \vee b_m\mathbf{y}$, <u>then</u> $a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_n}$ has the same color of $b_0 s_{j_0} \cdot \ldots \cdot b_m s_{j_m}$, for every $i_0 < \cdots < i_n$, $j_0 < \cdots < j_m$.

Another major result of Solecki [43, Corollary 4.3] is that almost $\mathcal{R}$-trivial monoids are $\mathbb{Y}$-controllable. This has amongst its consequences a theorem of Furstenberg and Katznelson [19]. We refer the reader to [43, Section 4] for a detailed discussion about this connection. We extend Solecki's result to a larger class of monoids and we prove that aperiodicity is a necessary condition for being $\mathbb{Y}$-controllable.

Given a monoid M, define $\mathbb{X}_{\mathcal{R}}(M) = \{aM : [a]_{\mathcal{R}}$ has more than one element$\}$. We say that $\mathbb{X}_{\mathcal{R}}(M)$ is linear if it is linearly ordered by inclusion.

3.1.4 Theorem (main theorem 2) *Let M be a finite monoid. If M is aperiodic and $\mathbb{X}_{\mathcal{R}}(M)$ is linear, then M is $\mathbb{Y}$-controllable. If M is $\mathbb{Y}$-controllable, then it is aperiodic.*

The proof of Theorem 3.1.4 is first presented as divided into two parts: a new result about monoid actions on compact topological right topological semigroups and a reformulation of known results by Solecki (the latter are then re-proved in Section 3.5 using model theory). Namely, the main technical novelty of this paper, which allows us to prove one direction of Theorem 3.1.4 and consequently one direction of Theorem 3.1.3, is the following result. Relevant notions are defined in Section 3.3.

3.1.5 Theorem *Let M be a finite aperiodic monoid. Let U be a compact right topological semigroup on which M acts by continuous endomorphisms. If $\mathbb{X}_{\mathcal{R}}(M)$ is linear, then there exists a minimal idempotent $u \in U$ such that $a(u) = b(u)$ for all couples $a, b \in M$ such that $a \,\mathcal{R}\, b$.*

The notion of aperiodic monoid plays a central role in both our main results. This class of monoids is also involved in one of the most important theorems in finite automata theory, also dealing with the semigroup of words, due to Schützenberger [41]. This suggests there might be a possible connection between automata theory and Ramsey theory.

From now on, we review the structure of the paper section by section.

Section 3.2 is introductory to the theory of monoids and Green's relations. First, we recall some basic properties of the class of aperiodic monoids and of the class of monoids with linear $\mathbb{X}(M)$. This is meant to provide alternative necessary or sufficient conditions for a monoid to be Ramsey, using Theorem 3.1.3. Then, we introduce the class of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$, and show that this class properly extends both the one of almost $\mathcal{R}$-trivial monoids introduced by Solecki in [43] and that of aperiodic monoids with linear $\mathbb{X}(M)$.

In Section 3.3, we study actions of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ on compact right topological semigroups, proving Theorem 3.1.5. This theorem and Corollary 3.3.6 seem to show that aperiodicity plays a relevant role in dynamic theory. We complete the proofs of Theorems 3.1.3 and 3.1.4 in two different ways in Section 3.4 and Section 3.5.

In Section 3.4 we guide the reader to a rephrasing of Solecki's work, to show how one implication of Theorem 3.1.3 and one of Theorem 3.1.4 follow from Theorem 3.1.5. This is done in two steps. First, we use a lemma of Solecki [43, Lemma 2.5]. Secondly, we guide the reader through the ultrafilter proof of Solecki. In Theorem 3.4.6 we prove the remaining implication of Theorem 3.1.4, i.e. that every $\mathbb{Y}$-controllable monoid is aperiodic. The same argument shows that Ramsey monoids are aperiodic.

In section 3.5, we give an alternative to the ultrafilter proof of Solecki, using model theory. Here we use the space of types where Solecki uses the space of ultrafilters. Model theory is limited to Section 3.5, Theorem 3.6.3, and Proposition 3.7.1 and just basic notions are assumed.

In Section 3.6, first we explain in detail the relation between Ramsey monoids and Carlson's and Gowers' theorems. We show that the definition of Ramsey monoid presented so far can be reformulated in many equivalent ways, and in particular one can choose semigroups other than $\langle (X_n)_{n\in\omega} \rangle$. Secondly, we point out that Ramsey monoids satisfy stronger properties than being Ramsey. In particular, in Theorem 3.6.3 we prove a common generalization of Milliken-Taylor theorem [36], [45] and Theorem 3.1.3, combining Ramsey's theorem and the definition of Ramsey monoid.

In Section 3.7, we provide further examples of $\mathbb{Y}$-controllable monoids, through Proposition 3.7.1. This shows that there are $\mathbb{Y}$-controllable monoids such that $\mathbb{X}_{\mathcal{R}}(M)$ is not linear. Finally, we collect some open questions in the area.

3.2 Aperiodic monoids, $\mathbb{X}(M)$ and $\mathbb{X}_{\mathcal{R}}(M)$

In this section, we introduce the basic notions and definitions about monoids we are going to use throughout the paper.

One of the best ways to describe monoids and semigroups is using Green's relations. They were first introduced by Green in his doctoral thesis and in [25]. The Green's relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ on a monoid M are the equivalence relations defined by, respectively, $a \mathcal{R} b$ if $aM = bM$, $a \mathcal{L} b$ if $Ma = Mb$ and $a \mathcal{J} b$ if $MaM = MbM$. The Green's relation $\mathcal{H}$ is the intersection of $\mathcal{R}$ and $\mathcal{L}$, while the Green's relation $\mathcal{D}$ is the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$. In every finite monoid, we have $\mathcal{D} = \mathcal{J}$. The Green's relations induce quasi-orders on the monoid. Given two element $a, b \in M$, define $a \leq_{\mathcal{R}} b$ if $aM \subseteq bM$, $a \leq_{\mathcal{L}} b$ if $Ma \subseteq Mb$, $a \leq_{\mathcal{J}} b$ if $MaM \subseteq MbM$, and finally $a \leq_{\mathcal{H}} b$ if both $a \leq_{\mathcal{R}} b$ and $a \leq_{\mathcal{L}} b$ hold. If $\mathcal{K}$ is an equivalence relation, we say that an equivalence class $[a]_{\mathcal{K}}$ is *trivial* if it contains exactly one element, and we say that a monoid M is $\mathcal{K}$-*trivial* if every $\mathcal{K}$-class is trivial. For more information about Green's relations, see e.g. [11].

A monoid is said *aperiodic* if for all $a \in M$ there exists $n \in \omega$ such that $a^n = a^{n+1}$. The class of finite aperiodic monoids has been widely studied, as it is involved in one of the most important theorems in finite automata theory, due to Schützenberger [41]. It states that star-free languages are exactly those languages whose syntactic monoid is finite and aperiodic. By a result of McNaughton and Papert, these also correspond to the languages definable in FO$[<]$, i.e. first-order logic with signature $<$ [35].

Among finite monoids, the class of aperiodic monoids can be characterized in many ways. We report here some of the most famous options used in literature. Among all possibilities, we isolate the notion of $\mathcal{R}$-rigid monoid as the operative definition we are going to use in the proofs of the next section.

3.2.1 Definition *A monoid is said* $\mathcal{R}$-rigid *if for every* $a, b \in M$, *if* $ab \,\mathcal{R}\, b$, *then* $ab = b$.

3.2.2 Proposition *Let* M *be a finite monoid. The following are equivalent:*

1. *M is aperiodic.*
2. *For every* $g, a, g' \in M$, *if* $gag' = a$, *then* $ga = ag' = a$.
3. *M is $\mathcal{R}$-rigid.*
4. *M is $\mathcal{H}$-trivial.*
5. *M contains no non-trivial subgroup.*

For the ease of the reader, we report also a short proof of the equivalence.

Proof First, assume 1, and let $g, a, g' \in M$ be such that $gag' = a$. By induction this implies $g^n a (g')^n = a$ for every $n \in \omega$. Choose n such that $g^{n+1} = g^n$ and $(g')^{n+1} = (g')^n$. Then, 2 holds since

$$ga = g(g^n a (g')^n) = g^{n+1} a (g')^n = g^n a (g')^n = a$$

and similarly,

$$ag' = (g^n a (g')^n) g' = g^n a (g')^{n+1} = g^n a (g')^n = a.$$

If $ab \,\mathcal{R}\, b$, then by definition of $\mathcal{R}$ there exists $a' \in M$ such that $b = aba'$, hence 2 implies 3.

Notice that if $a, b \in M$ are such that $a \,\mathcal{H}\, b$, then in particular $a \,\mathcal{R}\, b$ and there is $x \in M$ such that $xb = a$, hence 3 implies 4.

Now if $G \subseteq M$ is a subgroup of M, then for every $a, b \in G$ there are x, y such that $ax = ya = b$, and symmetrically there are x', y' such that $bx' = y'b = a$, hence $a \,\mathcal{H}\, b$ and G is contained inside one single $\mathcal{H}$-class. Therefore, 4 implies 5.

Finally, notice that if M is finite, then for every $a \in M$ there are minimal $n, k \in \omega$ such that $a^{n+k} = a^n$. Then, the set $\{a^{n+i} : i < k\}$ is a subgroup of M, and it is trivial if and only if $k = 1$, hence 5 implies 1. □

The class of aperiodic monoids is closed under most basic operations. For example, the following holds:

3.2.3 Proposition *Let* $(S_1, *_1), \dots, (S_n, *_n)$ *be aperiodic semigroups. Then, the following are aperiodic:*

1. *The product monoid* $S_1 \times \cdots \times S_n$ *with coordinate-wise operation.*
2. *The disjoint union* $S_1 \sqcup \cdots \sqcup S_n$ *with operation* $a * b = a *_i b$ *when* $a, b \in S_i$, *and* $a * b = b * a = a$ *if* $a \in S_i$ *and* $b \in S_j$ *with* $i < j$.

For more information about aperiodic monoids and their relations with languages and automata, see for example [32] or [40].

Let us move to the next class. Given a monoid M, define $\mathbb{X}(M) = \{aM : a \in M\}$. We say that $\mathbb{X}(M)$ is linear if it is linearly ordered by inclusion (equivalently, if $\leq_{\mathcal{R}}$ is a total quasi-order). The existence of a total quasi-order affects the behaviour of Green's relations, and having that $\leq_{\mathcal{R}}$ is total has even stronger consequences. The next proposition collects some well-known properties of monoids where $\leq_{\mathcal{R}}$ is total (see [28, Proposition 3.18-3.20]).

3.2.4 Proposition *Let M be a finite monoid with linear $\mathbb{X}(M)$. Then, the following hold:*

1. *For every $a \in M$, the principal right ideal aM is a both-sided ideal.*
2. *$\mathcal{J} = \mathcal{D} = \mathcal{R}$ and $\mathcal{L} = \mathcal{H}$, while $\leq_{\mathcal{R}} = \leq_{\mathcal{J}}$ and $\leq_{\mathcal{L}} = \leq_{\mathcal{H}}$.*
3. *$\mathcal{R}$ is a congruence relation.*
4. *$\leq_{\mathcal{R}}$ is translation-invariant on both sides.*

For more information about monoids with linear $\mathbb{X}(M)$, see for example [28].

Combining results about aperiodic monoids with results about monoids with linear $\mathbb{X}(M)$, one can obtain further properties and characterizations of the class of finite aperiodic monoid with linear $\mathbb{X}(M)$. For example, a finite monoid with linear $\mathbb{X}(M)$ is aperiodic if and only if it is $\mathcal{L}$-trivial. Notice that in light of Theorem 3.1.3, every property of this class of monoids will give a necessary condition for a monoid to be Ramsey.

Finally, let us introduce a seemingly new class of monoids, the class of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$. It is one of the key notions for the other main result of this paper, Theorem 3.1.4.

Let $\mathbb{X}_{\mathcal{R}}(M)$ be the subset of $\mathbb{X}(M)$ of those aM such that $[a]_{\mathcal{R}}$ is non-trivial. We say that $\mathbb{X}_{\mathcal{R}}(M)$ is linear if it is linearly ordered by inclusion. Recall also that M is called *almost $\mathcal{R}$-trivial* if for every non-trivial $\mathcal{R}$-class $[a]_{\mathcal{R}}$ we have $Ma = \{a\}$ (see [43] and [30]).

3.2.5 Proposition *Every finite almost $\mathcal{R}$-trivial monoid M is aperiodic and has linear $\mathbb{X}_{\mathcal{R}}(M)$.*

Proof Let M be a finite almost $\mathcal{R}$-trivial monoid. First, we want to show that $\mathbb{X}_{\mathcal{R}}(M)$ has at most one element that is the minimum of $\mathbb{X}(M)$ (and so $\mathbb{X}_{\mathcal{R}}(M)$ is in particular linearly ordered by inclusion). If $[a]_{\mathcal{R}}$ is a non-trivial $\mathcal{R}$-class then, for every $m \in M$ we have $ma = a$, that means $a \in mM$ and $aM \subseteq mM$. Hence, if $[a]_{\mathcal{R}}$ and $[b]_{\mathcal{R}}$ are non-trivial $\mathcal{R}$-classes, then we have $aM = bM$. Now let us prove that M is aperiodic. Since M almost $\mathcal{R}$-trivial, then $Mb = \{b\}$ holds for every non-trivial $\mathcal{R}$-class. This in particular implies that $(Mb) \cap [b]_{\mathcal{R}} = \{b\}$ holds for every $\mathcal{R}$-class, and this is just a rephrasing of the $\mathcal{R}$-rigid condition. Then the claim follows from Proposition 3.2.2. □

Notice that the converse does not hold, as it is easy to show that there are aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ that have more than one non-trivial $\mathcal{R}$-class (for example, by combining almost $\mathcal{R}$-trivial monoids with point 2 of Proposition 3.2.3; see also Example 3.2.6). Also, there are aperiodic monoids that have non-trivial $\mathcal{R}$-classes $[a]_{\mathcal{R}}$ such that a is not idempotent (a minimal example is given by the monoid in Table 3.1, see also Example 3.2.6). These conditions are impossible for almost $\mathcal{R}$-trivial monoids, as shown in the proof of Proposition 3.2.5. Finally, there are examples of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ that do not have linear $\mathbb{X}(M)$ (the easiest examples coming from $\mathcal{R}$-trivial monoids). Thus, the class of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ properly extends both the class of almost $\mathcal{R}$-trivial monoids and the class of aperiodic monoids with linear $\mathbb{X}(M)$.

3.2.6 Example Consider the Gowers' monoid $G_k = (\{0,\dots,k-1\}, \bar{+})$ with operation $i \bar{+} j = \min(i+j, k-1)$. Consider also the Carlson's semigroup $C_A = (A, *)$, i.e. a finite set A with operation $a * b = b$ for every $a, b \in A$. Let $C_A^1 = C_A \cup \{1_{C_A^1}\}$ be the corresponding monoid. Then, for every k and A the monoid $M = (G_k \times C_A^1)$ is aperiodic and has linear $\mathbb{X}_{\mathcal{R}}(M)$, while $\tilde{M} = (G_k \times C_A) \cup \{1_{\tilde{M}}\}$ is aperiodic, has linear $\mathbb{X}(\tilde{M})$ and all its $\mathcal{R}$-classes other than $[1_{\tilde{M}}]_{\mathcal{R}}$ are non-trivial. If $k \geq 2$, neither of these monoids is almost $\mathcal{R}$-trivial.

For those familiar with finite automata theory, Schützenberger Theorem provides a wonderful way to produce examples of aperiodic monoids. Starting from a star-free language S, or from a formula in FO[<], we always generate a finite aperiodic syntactic monoid. For example, the monoid from Table 3.1 is the syntactic monoid of the star-free language S in the alphabet $A = \{a, g, h\}$ defined as

$$S = \{g,h\}^* h \cup \{g,h\}^* a \{g,h\}^* g \cup A^* a A^* a A^*$$

or, equivalently, defined by the formula in FO[<] that says *"the word is non-empty, and if there are no letters a, then the word ends with h, and if there is exactly one letter a, then the word ends with g"*.

1	0	a	b	g	h
0	0	0	0	0	0
a	0	0	0	b	a
b	0	0	0	b	a
g	0	a	b	g	h
h	0	a	b	g	h

Table 3.1: Syntactic monoid of the language S.

3.3 Dynamic theory

In this section, we study actions of aperiodic monoids with linear $\mathbb{X}_{\mathcal{R}}(M)$ on compact right topological semigroups. The main objective is to prove Theorem 3.3.5, i.e. Theorem 3.1.5 from the introduction. This result reveals the relation between aperiodic monoids and dynamic theory and it will be the key point to prove one direction of Theorem 3.1.3 and one direction of Theorem 3.1.4. The advantage to work with compact right topological semigroups is that they are the common ground for many different techniques, either from logic or ergodic theory (see e.g. [6], [19], [34], [43], [47]).

Let us recall some notions. A semigroup $(U, \cdot)$ with a topology τ is a right topological semigroup if the map $x \mapsto x \cdot u$ is continuous from U to U for every $u \in U$. It is called compact if τ is compact. A element u in a semigroup $(U, \cdot)$ is called idempotent if $u \cdot u = u$. The set of idempotents of U is denoted by $E(U)$. We define a partial order $\leq_U$ in $E(U)$ by

$$u \leq_U v \iff uv = u = vu.$$

Finally, let $I(U)$ be the smallest compact both-sided ideal of U. It exists by compactness of U.

We report some facts about idempotents, corresponding to [47, Lemma 2.1, Lemma 2.3, Corollary 2.5].

3.3.1 Proposition *Let U be a compact right topological semigroup. Then,*

1. *$E(U)$ is non-empty.*
2. *For every idempotent v there is a $\leq_U$-minimal idempotent u such that $u \leq_U v$.*
3. *Any both-sided ideal of U contains all the minimal idempotents of U.*

3.3.2 Fact *Let M be a monoid, let U be any set, and fix a left action of M on U. Then, for every $a, b \in M$ such that $aM \subseteq bM$ we have $a(U) \subseteq b(U)$.*

Proof In fact, if $bm = a$ for some $m \in M$, then $a(U) = b(m(U)) \subseteq b(U)$. □

In particular, if $a \mathcal{R} b$, then $a(U) = b(U)$.

3.3.3 Lemma *Let M be a finite aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear. Then, for every distinct $a, b \in M$ with $a \mathcal{R} b$ there are two distinct $g, h \in M$ such that $ag = b$, $bh = a$ and $gh = h$, $hg = g$. This in particular implies $gM = hM$.*

Proof Fix a non-trivial $\mathcal{R}$-class $[c]_{\mathcal{R}}$ and let $a, b \in [c]_{\mathcal{R}}$ with $a \neq b$. For every $y, z \in M$, define

$$G_{y,z} = \{g_{y,z} \in M : yg_{y,z} = z\}.$$

Notice that if $y \mathcal{R} z$, then $G_{y,z}$ is non-empty. Let $g \in G_{a,b}$ be such that gM is minimal in $\{xM : x \in G_{a,b}\}$, and similarly let $h \in G_{b,a}$ be such that hM is minimal in $\{xM : x \in G_{b,a}\}$.

We claim that $g \mathcal{R} h$. Notice that $hgh \in G_{b,a}$ since $bhgh = agh = bh = a$. Since $hghM \subseteq hgM \subseteq hM$, by minimality of hM we have $hghM = hgM = hM$, so $h \mathcal{R} hg$ and $h \mathcal{R} hgh$. Notice that $hg \in G_{b,b}$ and that $G_{b,a} \cap G_{b,b} = \emptyset$, so $h \neq hg$ and the class $[h]_{\mathcal{R}}$ is non-trivial. Similarly, $g \mathcal{R} gh \mathcal{R} ghg$, $gh \in G_{a,a}$ and so the class $[g]_{\mathcal{R}}$ is non-trivial. By hypothesis this implies either $gM \subseteq hM$ or $hM \subseteq gM$. Suppose for example $gM \subseteq hM$. Then,

$$|h(gM)| \leq |gM| \leq |hM| = |hgM|$$

and so $|hM| = |gM|$ and $hM = gM$. This implies that h, hg, g, gh, are all in the same $\mathcal{R}$-class, hence $gh = h$ and $hg = g$, by definition of $\mathcal{R}$-rigid and Proposition 3.2.2. □

3.3.4 Lemma *Let M be a finite aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear. Then, for every $a \in M$, if there are $b, c \in [a]_{\mathcal{R}}$ such that $bc = c$, then for every $b, c \in [a]_{\mathcal{R}}$ we have $bc = c$.*

Proof First, notice that if $xy = y$ for some $x, y \in M$, then $xz = z$ for every $z \in [y]_{\mathcal{R}}$ since $xzM = xyM = yM = zM$ and since M is $\mathcal{R}$-rigid by Proposition 3.2.2.

Hence, we just need to prove that given a non-trivial $\mathcal{R}$-class $[a]_{\mathcal{R}}$ such that $ax = x$ for every $x \in [a]_{\mathcal{R}}$, and given an element $b \in [a]_{\mathcal{R}}$ with $b \neq a$, then we have $ba = a$.

Let g, h be such that $ag = b$ and $bh = a$. Notice that $ha \mathcal{R} hb$ since $haM = hbM$, and also $ha \neq hb$ since $bha = a \neq b = bhb$. Then $haM \in \mathbb{X}_{\mathcal{R}}(M)$ and so $haM \subseteq aM$ or $aM \subseteq haM$. We have $|aM| = |bhaM| \leq |haM| \leq |aM|$, hence $|haM| = |aM|$ and by linearity of $\mathbb{X}_{\mathcal{R}}(M)$ we must have $haM = aM$. Since M is R-rigid, $ha = a$ holds and we have $ba = bha = a$. □

With this, we are ready to prove Theorem 3.1.5.

3.3.5 Theorem *Let M be a finite aperiodic monoid. Let U be a compact right topological semigroup on which M acts by continuous endomorphisms. If $\mathbb{X}_{\mathcal{R}}(M)$ is linear, then there exists a minimal idempotent $u \in E(U) \cap I(U)$ such that $a(u) = b(u)$ for all couples $a, b \in M$ such that $a \mathcal{R} b$.*

Proof Let $a_0M \subsetneq ... \subsetneq a_nM$ be an increasing enumeration of $\mathbb{X}_{\mathcal{R}}(M)$ and define $a_{n+1} = 1$. Every $a_i(U)$ is a semigroup, since $a_i(u_1) \cdot a_i(u_2) = a_i(u_1 \cdot u_2)$, and it is compact because it is a continuous image of a compact space. Then, $a_i(U)$ is a compact subsemigroup of the compact semigroup $a_{i+1}(U)$. We want to find a chain of idempotents u_i such that $u_{i+1} \leq_U u_i$ and such that u_i is minimal in $E(a_i(U))$ with respect to $\leq_{a_i(U)}$, for every $i \leq n+1$.

First, by points 1 and 2 of Proposition 3.3.1, we can find $u_0 \in a_0(U)$ satisfying the requirement. Then, suppose we have $u_i \in a_i(U)$ idempotent. Since $a_i(U) \subseteq a_{i+1}(U)$

we may apply point 2 of Proposition 3.3.1 to find $u_{i+1} \in a_{i+1}(U)$ idempotent such that $u_{i+1} \leq_{a_{i+1}(U)} u_i$ and u_{i+1} is minimal in $E(a_{i+1}(U))$, and this concludes the construction. Since $a_{n+1} = 1$ and $E(a_{n+1}(U)) = E(U)$, by point 3 of Proposition 3.3.1 we also know that $u_{n+1} \in I(U)$.

We claim that $u = u_{n+1}$ satisfies the requirements of the thesis.

First, we want to show that for each $\mathcal{R}$-class $[a_i]_{\mathcal{R}}$ with $a_i a_i = a_i$ we have

$$b(u) = u_i \text{ for all } b \in [a_i]_{\mathcal{R}}.$$

By Lemma 3.3.4, for every $b \in [a_i]_{\mathcal{R}}$ we have $ba_i = a_i$, and this implies that for every $v \in a_i(U)$, say $v = a_i(u_v)$, we have $b(v) = b(a_i(u_v)) = a_i(u_v) = v$. In particular for $v = u_i$, we have $b(u_i) = u_i$. Notice that the action of M is order preserving on $(U, \leq_U)$, since it is by endomorphisms. Since $u \leq_U u_i$ we get

$$b(u) \leq_U b(u_i) = u_i.$$

Thus, $b(u) \leq_{a_i(U)} u_i$, and since u_i is minimal in $a_i(U)$, we get $b(u) = u_i$.

Now consider a non trivial $\mathcal{R}$-class $[a_i]_{\mathcal{R}}$ such that $a_i a_i \notin [a_i]_{\mathcal{R}}$, and let $a, b \in [a_i]_{\mathcal{R}}$. Let g, h be given as in Lemma 3.3.3 such that $ag = b$ and $bh = a$ and $hg = g$. Notice that this implies $bg = bhg = ag = b$ and also $h(u) = g(u)$, since $[g]_{\mathcal{R}}$ belongs to the previous case. Then, $a(u) = bh(u) = bg(u) = b(u)$. □

We take the opportunity to state a corollary of Lemma 3.3.3.

3.3.6 Corollary *Let M be a finite aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear, let U be a set and fix a left action of M on U. Then, for every $a, b \in M$ with $a \,\mathcal{R}\, b$ and for every $u \in a(U)$ we have $a(u) = b(u)$.*

Proof Let $a, b \in M$ be such that $a \,\mathcal{R}\, b$ and $a \neq b$, and let $g, h \in M$ be given by Lemma 3.3.3 such that $ag = b$ and $bh = a$, and $gh = h$ and $hg = g$. This in particular implies $gg = ghg = hg = g$, and $bg = bhg = ag = b$. Notice that by linearity of $\mathbb{X}_{\mathcal{R}}(M)$ either $a(M) \subseteq g(M)$ or $g(M) \subseteq a(M)$ holds, since both $[a]_{\mathcal{R}}$ and $[g]_{\mathcal{R}}$ are non-trivial. Then, we have $a(M) \subseteq g(M)$, since $|aM| = |bhM| \leq |hM| = |gM|$, and also $a(U) \subseteq g(U)$, by Fact 3.3.2. Fix $u \in a(U)$ and find $v \in U$ such that $u = g(v)$. We have

$$a(u) = a(g(v)) = a((gg)(v))) = ag(g(v))) = b(g(v)) = b(u).$$ □

3.4 Coloring theorems and aperiodic monoids

In this section, we discuss how from Theorem 3.3.5 one can prove Theorems 3.1.3 and 3.1.4 following ideas from Solecki's paper. The main novelty introduced here is the proof that $\mathbb{Y}$-controllable monoids and Ramsey monoids are aperiodic.

Let us recall some relevant notions for this section. Given a monoid M, the set $\mathbb{Y}(M) \subseteq \mathcal{P}(\mathbb{X}(M))$ consists of the non-empty subsets of $\mathbb{X}(M)$ which are linearly ordered by inclusion. Define $x \leq_{\mathbb{Y}(M)} y$, for $x, y \in \mathbb{Y}(M)$, if and only if $x \subseteq y$ and all elements of $y \setminus x$ are larger with respect to $\subseteq$ than all elements of x.

There is a natural left M-action on $\mathbb{Y}(M)$ defined as $x \mapsto mx = \{maM : aM \in x\}$. Given two left actions of M on U and U', a map $f : U \to U'$ is said M*-equivariant* if it preserves the action of M, i.e. $f(ma) = mf(a)$.

For ease of notation, we isolate the following class of monoids.

3.4.1 Definition *A monoid M is called* good *if for every left action of M by continuous endomorphisms on a compact right topological semigroup U there exists a function $g\colon \mathbb{Y}(M) \to E(U)$ such that*

(i) g is M-equivariant;

(ii) g is order reversing with respect to $\leq_{\mathbb{Y}(M)}$ and $\leq_U$;

(iii) g maps maximal elements of $\mathbb{Y}(M)$ to $I(U)$.

The notion of good monoids was first used by Solecki in [43]. We borrow here three results that are contained or essentially proved therein.

The following useful lemma has the same function as two other lemmas by Lupini [33, Lemma 2.2] and Barrett [3, Theorem 5.8], i.e. to get stronger conclusions from results like Theorem 3.1.5.

3.4.2 Lemma [43, Lemma 2.5] *Let M be a finite monoid. Assume that for every left action of M by continuous endomorphisms on a compact right topological semigroup U there is a M-equivariant f from $\mathbb{Y}(M)$ to U such that f maps maximal elements of $\mathbb{Y}(M)$ to $I(U)$. Then, M is good.*

We isolate the following lemma from the proof of [43, Theorem 2.4] since it gives a sufficient condition for a monoid to be good.

3.4.3 Lemma *Let M be a finite monoid and assume that for every action by continuous endomorphisms of M on a compact right topological semigroup U there exists a minimal idempotent $u \in E(U) \cap I(U)$ such that $a(u) = b(u)$ for all couples $a, b \in M$ such that $a \mathcal{R} b$. Then, M is good.*

Proof Let $\pi : \mathbb{Y}(M) \to \mathbb{X}(M)$ be the function that maps a set $y \subseteq \mathbb{Y}(M)$ to the maximal element in y with respect to $\subseteq$. Let $u \in E(U) \cap I(U)$ be given by hypothesis. The function $f : \mathbb{X}(M) \to E(U)$ that maps aM to $a(u)$ is well-defined, and maps $1M$ to $u \in E(U) \cap I(U)$. Also, notice that if y is a maximal element of $\mathbb{Y}(M)$, then $1M \in y$ and so $\pi \circ f(y) = u \in E(U) \cap I(U)$. Since both f and π are M-equivariant the map $f \circ \pi : \mathbb{Y}(M) \to E(U)$ satisfies the assumptions of Lemma 3.4.2, from which we get that M is good. □

Solecki in [43, Corollary 4.3] states that every finite almost $\mathcal{R}$-trivial monoid is $\mathbb{Y}$-controllable, but in the proof he shows something stronger. In fact, the hypothesis that M is almost $\mathcal{R}$-trivial is used only to apply [43, Theorem 2.4], which states that every finite almost $\mathcal{R}$-trivial monoid is good. The remaining part of the proof never uses this hypothesis again, and relies instead on the fact that M is good. In other words, from the proof of [43, Corollary 4.3] one can derive also the following result.

3.4.4 Theorem *Let M be a finite monoid. If M is good, then it is $\mathbb{Y}$-controllable.*

However, the reader can find a short model-theoretic proof of this result in Section 3.5.

Finally, the following is a restatement of part of the proof of [43, Corollary 4.5 (i)].

3.4.5 Fact *If M is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear, then M is Ramsey.*

Proof Notice that $\mathbb{X}(M)$ is linear if and only if $\mathbb{X}(M) \in \mathbb{Y}(M)$. We want to use the definition of $\mathbb{Y}$-controllable with $\mathbf{y} = \mathbb{X}(M)$ and $F = \{\mathbf{y}\}$. It is enough to notice that for every $a \in M$ we have

$$a\mathbb{X}(M) = \{amM : mM \in \mathbb{X}(M)\} = \{xM : xM \subseteq aM\}.$$

Hence, if $aM \subseteq bM$, then $a\mathbf{y} \vee b\mathbf{y} = b\mathbf{y} = b\mathbf{y} \vee a\mathbf{y}$, and so $a_1\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = \mathbf{y} \in F$ for every $a_1, .., a_n \in M$ with at least one i such that $a_i = 1$. □

3.4.6 Theorem *Let M be a finite monoid.*

1. *If M is aperiodic and has a linear $\mathbb{X}_{\mathcal{R}}(M)$, then it is $\mathbb{Y}$-controllable.*

2. *If M is $\mathbb{Y}$-controllable, then it is aperiodic.*

Proof First, let M be a finite aperiodic monoid with linear $\mathbb{X}_{\mathcal{R}}(M)$. By Theorem 3.3.5 and Lemma 3.4.3, we get that M is good. Hence, Theorem 3.4.4 implies that M is $\mathbb{Y}$-controllable, and statement 1 holds.

In order to prove 2, let $(M^{<\omega}, ^\frown)$ be the free semigroup over M, with coordinatewise action. Notice that $(M^{<\omega}, ^\frown)$ can be seen as $\langle (X_n)_{n<\omega} \rangle$ setting all $X_n = M$, with 1 as distinguished point, and a word w has a distinguished point if and only if $1 \in \operatorname{ran} w$ (in which case we call w a variable word).

Suppose M is not aperiodic, and let $a \in M$ be such that $a^{n+1} \neq a^n$ for every $n \in \omega$. Let $A = \{a^n : n \in \omega\}$, and let $C = \{m \in M : a^n m \in A$ for some $n \in \omega\}$. Then, we have $ac \neq c$ for every $c \in C$, and $ac \in C$ if and only if $c \in C$.

Let $\mathbf{y} = \{a^n M : n \in \omega\}$, where we set $a^0 = 1$, and let $F = \{\mathbf{y}\}$. Then, $\mathbf{y}$ is a maximal element of $\mathbb{Y}(M)$, and $\mathbf{y} \vee \mathbf{y} = \mathbf{y} = a\mathbf{y} \vee \mathbf{y}$.

Let $C \cup \{\perp\}$ be the set of colors. Given a word $w \in M^{<\omega}$, let m be the first letter of w in C, if any. If there is such m, color w by m. Otherwise, color w by $\perp$. Consider any sequence of variable words $\bar{y} \in (M^{<\omega})^\omega$, and consider the words $y_0{}^\frown y_1$ and $a(y_0){}^\frown y_1$ with colors c_1 and c_2 respectively. Then, $c_1 \in C$, since y_0 is a variable word and $1 \in \operatorname{ran}(y_0)$. Hence, by definition of C we have $c_2 = ac_1$. Therefore, $c_2 = ac_1 \neq c_1$, contradicting the fact that M is $\mathbb{Y}$-controllable. □

3.4.7 Theorem *Let M be a finite monoid. The following are equivalent:*

1. *M is Ramsey.*
2. *M is aperiodic and $\mathbb{X}(M)$ is linear.*

Proof Proof of point 2 of Theorem 3.4.6 also shows that if M is Ramsey then it is aperiodic. If M is Ramsey, then $\mathbb{X}(M)$ is linear by [43, Corollary 4.5 (ii)]. Theorem 3.4.6 and Fact 3.4.5 prove that 2 implies 1. □

3.4.8 Corollary *Let M be a finite monoid. Then, M is Ramsey if and only if it is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear.*

We conclude this chapter with a corollary concerning the definition of Ramsey monoid. It is not clear to the authors whether the following result can be proved with methods similar to those developed in Section 3.6, without passing through Theorem 3.4.7.

Recall that a variable words is a word w such that $1 \in \operatorname{ran}(w)$.

3.4.9 Corollary *Let M be a finite monoid. The following are equivalent:*

1. *M is Ramsey.*
2. *For all finite coloring of $M^{<\omega}$ there are two variable words y_0 and y_1 such that $My_0{}^\frown y_1$ is monochromatic.*

Proof The exact same proofs of [43, Corollary 4.5 (ii)] and point 2 of Theorem 3.4.6 show that if condition 2 hold, then M is aperiodic and $\mathbb{X}(M)$ is linear. The rest follows from Theorem 3.4.7 and by definition of Ramsey monoid. □

3.5 A model-theoretic approach

In this section, we give a short explicit proof of Theorem 3.4.4. We shall use Proposition 3.5.2, which says that the space of types $S(G)$ over a semigroup G is a compact right topological semigroup if we add some symbol to the signature. This approach is discussed in [12] and is further developed here. The one difference in exposition is that here we define a product between types, while in [12] we define a product between type-definable sets. We assume some basic knowledge of model theory. We refer the reader to Tent's and Ziegler's book [46].

In what follows, we consider a semigroup G such that M acts by endomorphisms on G, and a monster model $\mathcal{G}$.

We say that a type $p(x)$ is *finitely satisfied* in G if every finite conjunction of formulas in $p(x)$ has a solution in $G^{|x|}$. We write $a \mathrel{\unicode{x2AEB}}_G b$ if $\operatorname{tp}(a/Gb)$ is finitely satisfied in G. In literature this relation is also denoted by $a \downarrow^{u}_{G} b$.

We say that the tuple $\bar{c}$ is a *coheir sequence* of $p(x)$ over G if $c_n \vDash p(x)$ and $c_n \mathrel{\unicode{x2AEB}}_G \bar{c}_{\restriction n}$ and $c_{n+1} \equiv_{G,\bar{c}_{\restriction n}} c_n$ for every $n < \omega$. In particular, $\bar{c}$ is indiscernible over G, i.e. $\bar{c}_{\restriction I_0} \equiv_G \bar{c}_{\restriction I_1}$ for every $I_0, I_1 \subseteq \omega$ of equal finite cardinality.

The following is an easy well-known fact.

3.5.1 Fact *For every type $p(x) \in S(G)$ there is a coheir sequence of $p(x)$.*

In order to define a product between types, we need some stationarity. Here, we obtain it by adding sets to the signature.

Let G be a semigroup on which a monoid M acts by endomorphisms and let L be its signature. Let $L^+ = L \cup \{A : A \subseteq G^n, n \in \omega\} \cup \{m : m \in M\}$ be the expansion of L where the symbol A is interpreted in G as the set A, and m is interpreted as the unary function $a \mapsto ma$.

From now on we consider G with this augmented signature.

For $a, b \in S(G)$ define $a \cdot_G b$ as $\operatorname{tp}(a' \cdot b'/G)$, for any $a', b' \in \mathcal{G}$ such that $a' \vDash a$, $b' \vDash b$ and $a' \mathrel{\unicode{x2AEB}}_G b'$.

As usual, we will consider the compact topology on $S(G)$ generated by the basic open sets $\{t \in S(G) : \varphi(x) \in t\}$, for $\varphi(x) \in L^+(G)$.

3.5.2 Proposition *If G is a model as above, then $(S(G), \cdot_G)$ is a compact right topological semigroup and the action of M defined by $m\operatorname{tp}(a/G) = \operatorname{tp}(ma/G)$ is by continuous endomorphisms.*

Proof Proposition 4.4 and Remark 2.7 of [12] prove that if G is considered with signature L^+ then $a \cdot_G b$ is well-defined for every $a, b \in S(G)$ and $(S(G), \cdot_G)$ is a semigroup. In [12, Proposition 6.3] it is proved that the action of M on $(S(G), \cdot_G)$ defined by $m\operatorname{tp}(a/G) = \operatorname{tp}(ma/G)$ gives well-defined endomorphisms of $(S(G), \cdot_G)$. It is straightforward to check that the maps $m\operatorname{tp}(a/G)$ are continuous. The one missing proof is that $x \mapsto x \cdot_G r$ is continuous from $S(G)$ to $S(G)$, for any $r \in S(G)$.

Let $b \in \mathcal{G}$ and let $q(x, y)$ be the type in $S(G)$ such that $a \vDash q(x, b)$ if and only if $a \mathrel{\unicode{x2AEB}}_G b$, i.e.

$$q(x, y) = \{\varphi(x; y) : \varphi(x; y) \in L^+(G) \text{ and } G^{|x|} = \varphi(G^{|x|}; b)\}.$$

Notice that if $b' \equiv_G b$, then $a \vDash q(x, b') \iff a \mathrel{\unicode{x2AEB}}_G b'$.

Let $p(z)$ be a partial type over G and let $r(y) = \mathrm{tp}(b/G)$. The type

$$\exists y, z\; r(y) \wedge q(x, y) \wedge z = x \cdot y \wedge p(z)$$

is satisfied by those $a \in \mathcal{G}$ such that $\mathrm{tp}(a/G) \cdot_G r$ satisfies $p(z)$. Hence, the preimage of the closed set $p(S(G))$ is closed. □

We are ready to prove Theorem 3.4.4. Let us introduce the following auxiliary definition to simplify the notation of the next proof.

3.5.3 Definition *Let F be a finite subset of the semigroup $\langle \mathbb{Y}(M) \rangle$, let $\boldsymbol{y}$ be a maximal element in $\mathbb{Y}(M)$, and let c be a finite coloring of a semigroup S on which M acts. We say that a sequence $\bar{s} \in S^{\leq \omega}$ is $(F, \mathbf{y}, c)$-controllable if for every $m, n \leq |\bar{s}|$ and for every $a_i, b_j \in M$ <u>if</u> $a_0\boldsymbol{y} \vee \cdots \vee a_n\boldsymbol{y}$ belongs to F and $a_0\boldsymbol{y} \vee \cdots \vee a_n\boldsymbol{y} = b_0\boldsymbol{y} \vee \cdots \vee b_m\boldsymbol{y}$, <u>then</u> $a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_n}$ has the same color of $b_0 s_{j_0} \cdot \ldots \cdot b_m s_{j_m}$, for every $i_0 < \cdots < i_n$, $j_0 < \cdots < j_m$.*

Proof of Theorem 3.4.4 Let $(X_n)_{n\in\omega}$ be a sequence of pointed M-sets on which M acts uniformly, and let $\perp$ be not in $\bigcup_{n\in\omega} X_n$. Define $G = (\langle (X_n)_{n\in\omega} \rangle \cup \{\perp\}, ^\frown)$ to be the semigroup extending $(\langle (X_n)_{n\in\omega} \rangle, ^\frown)$ defining $x ^\frown y = \perp$ if $x ^\frown y$ is not defined in the partial semigroup $\langle (X_n)_{n\in\omega} \rangle$. In particular, $x ^\frown \perp = \perp ^\frown x = \perp$. We write $x \prec y$ if and only if $x ^\frown y \neq \perp$ or $x = \perp$.

It is enough to prove that for every finite subset F of $\langle \mathbb{Y}(M) \rangle$, for every maximal element $\mathbf{y}$ in $\mathbb{Y}(M)$, and for every c finite coloring of G there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ that is $(F, \mathbf{y}, c)$-controllable and such that s_n has a distinguished point for every $n \in \omega$.

Let $L = \{^\frown, \prec\}$ and consider G with augmented signature L^+. Let

$$U = \{p \in S(G) : G \prec p\}$$

where $G \prec p$ is a shorthand for $\{g \prec x : g \in G\} \subseteq p(x)$.

Notice that U is non-empty, since for every finite set $A \subseteq G$ there is a $b \in G$ such that $A \prec \mathrm{tp}(b/G)$. Also, $\perp \notin U$. We claim that U is a compact subsemigroup of $(S(G), \cdot_G)$. Let $\mathcal{G}$ be a monster model in the language L^+. Let $a, b \in \mathcal{G}$ such that $\mathrm{tp}(a/G) \in U$, $\mathrm{tp}(b/G) \in U$ and $a \mathop{\downfootline}_G b$. Then, we must have that $a \prec b$, since $G \prec \mathrm{tp}(b/G)$ and $a \mathop{\downfootline}_G b$. Now, let $g \in G$ such that $g \neq \perp$. Then, $g ^\frown a \mathop{\downfootline}_G b$ and hence $g ^\frown a \prec b$. Since $g \neq \perp$ and $g \prec a$ we also have $g ^\frown a \neq \perp$. Hence,

$$g ^\frown (a ^\frown b) = (g ^\frown a) ^\frown b \neq \perp.$$

Therefore, we have that $g \prec (a ^\frown b)$ from which we get $\mathrm{tp}(a/G) \cdot_G \mathrm{tp}(b/G) \in U$. Also, U is type-definable over G and hence compact. Finally, notice that $a \prec mb$ for every $m \in M$ and for every $a, b \in G$ such that $a \prec b$. Hence, U is closed under the action of M. By Proposition 3.5.2, it is a compact right topological semigroup such that M acts on U by continuous endomorphisms.

Let $u = g(\mathbf{y}) \in E(U) \cap I(U)$, where $g : \mathbb{Y}(M) \to E(U)$ is the function given by definition of good monoid. Let DP be the set of elements of $\langle (X_n)_{n\in\omega} \rangle$ that have at least one distinguished point, and let $J = \{p \in S(G) : DP \in p\}$. Since $J \cap U$ is a non-empty both-sided ideal of U, and $u \in I(U)$, we have that u is in J. Let $(u_n)_{n\in\omega}$ be a coheir sequence of u. We write $\bar{u}_{\restriction i}$ for the tuple $u_{i-1}, \dots, u_0$. Notice that since the map g is order-reversing and M-equivariant, for every $a_0, \dots, a_n, b_0, \dots, b_m \in M$ if $a_0\mathbf{y} \vee \dots \vee a_n\mathbf{y} = b_0\mathbf{y} \vee \dots \vee b_m\mathbf{y}$ then also $a_0u \cdot_G \dots \cdot_G a_nu = b_0u \cdot_G \dots \cdot_G b_mu$. Hence, it is straightforward to check that $\bar{u}_{\restriction i}$ is $(F, \mathbf{y}, c)$-controllable for every $i \in \omega$. Notice that $\bar{u}_{\restriction i}$ is a basic sequence since products stay in U and $\perp \notin U$. Finally, u_n are elements of DP since $u \in J$. Now, we use the sequence $\bar{u}$ to define $\bar{s} \in G^\omega$ with same properties as $\bar{u}$.

Let $k \in \omega$ be such that for every element $f \in F$ there are $k' < \kappa$ and $a_0, \dots, a_{k'} \in M$ such that $f = a_0\mathbf{y} \vee \dots \vee a_{k'}\mathbf{y}$. Notice that this implies that for every $f, f' \in \langle \mathbb{Y}(M) \rangle$ such that $f \vee f' \in F$ there are $c_0, \dots c_j \in M$ with $j < k$ such that $f' = c_0\mathbf{y} \vee \dots \vee c_j\mathbf{y}$. This follows from the property that the set of predecessors of any element of $\mathbb{Y}(M)$ is linearly ordered by $\leq_{\mathbb{Y}(M)}$.

Assume as induction hypothesis that the tuple obtained by concatenation $s_{\restriction i} ^\frown \bar{u}_{\restriction k}$ is $(F, \mathbf{y}, c)$-controllable and $s_{\restriction i}$ is a basic sequence of elements of DP. Our goal is to find $s_i \in G$ such that the same properties hold for $s_{\restriction i+1}$.

From the induction hypothesis it follows that $s_{\restriction i} ^\frown \bar{u}_{\restriction l}$ is $(F, \mathbf{y}, c)$-controllable for any $l \in \omega$. In fact, let $w = b_0 s_{i_0} {}^\frown \dots {}^\frown b_m s_{i_m} {}^\frown b_{m+1} u_{i_{m+1}} {}^\frown \dots {}^\frown b_n u_{i_n}$ be such that $b_0\mathbf{y} \vee \dots \vee b_m\mathbf{y} \vee b_{m+1}\mathbf{y} \vee \dots \vee b_n\mathbf{y} \in F$. Let $j < k$ and $c_0, \dots, c_j \in M$ be such that $b_{m+1}\mathbf{y} \vee \dots \vee b_n\mathbf{y} = c_0\mathbf{y} \vee \dots \vee c_j\mathbf{y}$. Since $\bar{u}$ is a coheir sequence, we have that $\bar{u}_{\restriction I_0} \equiv_G \bar{u}_{\restriction j+1}$ for any $I_0 \subseteq l$ of size $j+1$. Hence, the type over G of w is equal to the type over G of $b_0 s_{i_0} {}^\frown \dots {}^\frown b_m s_{i_m} {}^\frown c_0 u_j {}^\frown \dots {}^\frown c_j u_0$. Therefore, we may use the induction hypothesis to conclude that $s_{\restriction i} ^\frown \bar{u}_{\restriction l}$ is $(F, \mathbf{y}, c)$-controllable. Also, $s_{\restriction i} ^\frown \bar{u}_{\restriction l}$ is a basic sequence by induction hypothesis and idempotence of u.

Let $\varphi(s_{\restriction i}, u_{i+1}, u_{\restriction i+1})$ say that $s_{\restriction i} ^\frown \bar{u}_{\restriction i+2}$ is $(F, \mathbf{y}, c)$-controllable and that $s_{\restriction i} ^\frown \bar{u}_{\restriction i+2}$ is a basic sequence of elements of DP. As $\bar{u}$ is a coheir sequence we can find $s_i \in G$ such that $\varphi(s_{\restriction i+1}, u_{\restriction i+1})$. Hence, s_i has the desired properties. □

3.6 Equivalent definitions of Ramsey monoid

In this section, we briefly prove the equivalence between different notions of being Ramsey.

First, in Proposition 3.6.1 we show that in this context results on located words are not stronger than results on words, but rather equivalent, since one can derive results about located words from results about words. While the converse is well-known, this implication apparently has been overlooked. For example, Bergelson-

Blass-Hindman theorem on located words [6] can be derived from Carlson's theorem on variable words [9], since Carlson's Theorem implies condition 3 in Proposition 3.6.1. Conditions 3 and 4 of Proposition 3.6.1 also show that Carlson's Theorem and Gowers' Theorem are indeed equivalent to the statement that a certain monoid is Ramsey.

Secondly, in Corollary 3.6.2 and Theorem 3.6.3 we state some equivalent definitions of Ramsey monoid which may be useful for applications.

First, let us recall some basic definitions. $(S, \cdot)$ is a partial semigroup if $\cdot$ is a partial binary function $\cdot : S^2 \to S$ such that $(s_1 \cdot s_2) \cdot s_3 = s_1 \cdot (s_2 \cdot s_3)$ whenever $(s_1 \cdot s_2) \cdot s_3$ and $s_1 \cdot (s_2 \cdot s_3)$ are both defined. An endomorphism on a partial semigroup S is a function $m : S \to S$, denoted by $s \mapsto ms$, such that for all $s_1, s_2 \in S$ for which $s_1 \cdot s_2$ is defined, then $ms_1 \cdot ms_2$ is defined and $m(s_1 \cdot s_2) = (ms_1) \cdot (ms_2)$.

Given a partial semigroup S and two sequences $\bar{s}$ and $\bar{t}$ in S^ω, we say that $\bar{s}$ is *extracted* from $\bar{t}$, or $\bar{s} \leq_M \bar{t}$, if there is an increasing sequence $(i_n)_{n\in\omega}$ of natural numbers such that $s_n \in \langle t_{i_n}, \ldots, t_{(i_{n+1})-1} \rangle_M$. As for pointed M-sets, we say that $\bar{t}$ is *basic* if $m_0 t_{i_0} \cdot \ldots \cdot m_n t_{i_n}$ is defined for every $i_0 < \ldots < i_n$ and $m_0, \ldots, m_n \in M$.

An infinite sequence $\bar{t} \in M^\omega$ is *rapidly increasing* if $|t_n| > \Sigma_{i=0}^{n-1} |t_i|$ for all $n \in \omega$. Let FIN_M be the partial semigroup of located words, i.e. $\mathrm{FIN}_M = \langle (X_n)_{n\in\omega} \rangle$ where $X_n = \{n\} \times M$ with the usual action. As for words, a variable located word is a located word w such that $(n, 1_M) \in \operatorname{ran} w$ for some n.

3.6.1 Proposition *The following are equivalent for a monoid M:*

1. *M is Ramsey.*

2. *For every partial semigroup S on which M acts by endomorphisms, for every basic sequence $\bar{t} \in S^\omega$, for every finite coloring of S there is a sequence $\bar{s} \leq_M \bar{t}$ such that $\langle \bar{s} \rangle_M$ is monochromatic.*

3. *There is a rapidly increasing sequence of variable words $\bar{x} \in (M^{<\omega})^\omega$ such that for all finite colorings of $M^{<\omega}$ there is a sequence $\bar{s} \leq_M \bar{x}$ with $\langle \bar{s} \rangle_M$ monochromatic.*

4. *For all finite colorings of FIN_M there is a sequence of variable located words with monochromatic M-span.*

Proof It is easy to check that points 1 and 2 are equivalent. Indeed, $\langle (X_n)_{n\in\omega} \rangle$ is a partial semigroup for every $(X_n)_{n\in\omega}$, hence 2 implies 1. Conversely, given a partial semigroup S and a basic sequence $(t_n)_{n\in\omega} \in S^\omega$, we may obtain results about S from $\langle (X_n)_{n\in\omega} \rangle$ choosing the sets $X_n = Mt_n \subseteq S$ with distinguished point t_n.

It is also straightforward to check that 2 implies 3 and 4. Hence, it remains to prove that 3 implies 2 and 4 implies 2.

So let us show that 3 implies 2. Let $\bar{x}$ be the sequence given by 3, let S be a partial semigroup and let $\bar{t}$ be a basic sequence in S^{ω}. We write $\frown$ to denote the operation of $M^{<\omega}$ and $\bullet$ to denote the operation of S.

By definition of rapidly increasing sequence and since each x_n is a variable word, every element of $\langle \bar{x} \rangle_M$ can be written as $m_1 x_{i_1} \frown \ldots \frown m_n x_{i_n}$ in a unique way. Hence, there is a function $f : \langle \bar{x} \rangle_M \to \langle \bar{t} \rangle_M$ defined as the surjective homomorphism of partial semigroups such that for every $n \in \omega$, $m_1 \ldots m_n \in M$, and $i_1 < \cdots < i_n$,

$$f(m_1 x_{i_1} \frown \ldots \frown m_n x_{i_n}) = m_1 t_{i_1} \bullet \ldots \bullet m_n t_{i_n}.$$

Let $\{B_i : i < n\}$ be a coloring of S into finitely many pieces. Define $A_i = f^{-1}[B_i]$, then $\{A_i : i < n\} \cup \{M^{<\omega} \setminus \langle \bar{x} \rangle_M\}$ is a finite coloring of $M^{<\omega}$. By 3 we may find $\bar{y} \leq_M \bar{x}$ such that $\langle \bar{y} \rangle_M$ is monochromatic. Notice that $\langle \bar{y} \rangle_M \subseteq \langle \bar{x} \rangle_M$, so there is $k < n$ such that $\langle \bar{y} \rangle_M \subseteq A_k$.

Set $f(\bar{y}) = (f(y_i))_{i \in \omega}$. Then, $f(\bar{y}) \leq_M f(\bar{x}) = \bar{t}$. It is enough to prove that $\langle f(\bar{y}) \rangle_M = f[\langle \bar{y} \rangle_M]$ and then we are done, since $f[\langle \bar{y} \rangle_M] \subseteq f[A_k] = B_k$.

Notice that $mf(y_j) = f(my_j)$ for all $m \in M$. In fact, if $y_j = m_1 x_{i_1} \frown \ldots \frown m_n x_{i_n}$, then

$$mf(y_j) = m(m_1 t_{i_1}) \bullet \ldots \bullet m(m_k t_{i_n}) = f(my_j).$$

Let $g \in \langle f(\bar{y}) \rangle_M$, say

$$g = m_1 f(y_{i_1}) \bullet \ldots \bullet m_n f(y_{i_n}) = f(m_1 y_{i_1}) \bullet \ldots \bullet f(m_n y_{i_n}).$$

Then, $g = f(m_1 y_{i_1} \frown \ldots \frown m_n y_{i_n})$.

The proof of 4 implies 2 proceeds in a similar manner, starting from the sequence $\bar{x} = ((n, 1_M))_{n \in \omega}$ from which every sequence of variable located words can be extracted. □

Notice that in the proof of Proposition 3.6.1, with the same notation and assumptions, we showed that the color of $g = m_1 f(y_{i_1}) \bullet \ldots \bullet m_n f(y_{i_n})$ is controlled by the color of $m_1 y_{i_1} \cdot \ldots \cdot m_n y_{i_n}$. Proposition 3.6.1 can be extended to $\mathbb{Y}$-controllable monoids, providing three equivalent definitions for this notion as well.

In the definition of M-span, we ask that at least one element of the basic sequence is moved by 1. Here, we show how to relax this condition.

3.6.2 Corollary *Let M be a finite Ramsey monoid. Then, for any partial semigroup S, for any finite coloring of S and for every sequence $\bar{t} \in S^{\omega}$ there is $\bar{s} \leq_M \bar{t}$ such that for every $a \in M$ the set*

$$\{m_0 \, s_{i_0} \cdots m_n \, s_{i_n} : i_0 < \cdots < i_n, m_i \in aM, m_i \mathrel{\mathcal{R}} a \text{ for at least one } i\}$$

is monochromatic.

Proof By Corollary 3.4.8, M is $\mathbb{Y}$-controllable. Then, the thesis follows from the definition of $\mathbb{Y}$-controllable monoid applied to the maximal element $\mathbf{y} = \mathbb{X}(M)$ and to $F = \{a\mathbf{y} : a \in M\}$, and the arguments of Proposition 3.6.1. □

In previous corollary, the action of M can be controlled with $|\mathbb{X}(M)|$-many colors. This is optimal, as in general it is not possible to get less than $|\mathbb{X}(M)|$-many monochromatic sets. For example, choose $\mathbb{X}(M)$ as set of colors, and color each word $w \in M^{<\omega}$ by the minimum aM such that $\mathrm{ran}(w) \subseteq aM$: then, if $\bar{t}$ is a sequence of variable words, for any $\bar{s} \leq_M \bar{t}$ each set defined above has a different color.

When instead M is $\mathbb{Y}$-controllable but $\mathbb{X}(M)$ is not linear, it is not difficult to see that for any $k \in \omega$ there are $\mathbf{y}$ and $F \subseteq \mathbb{Y}(M)$ and colorings of, say, $M^{<\omega}$ such that for every sequence of variable words $\bar{s}$ there are more than k-many $f \in F$ such that the sets

$$\langle \bar{s} \rangle_f = \{a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_m} : a_i \in M, i_0 < \cdots < i_m, a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = f\}$$

have different colors.

The next theorem is a generalization of both Theorem 3.4.7 and Milliken-Taylor theorem [36], [45]. It is a combination of Ramsey's theorem and Theorem 3.4.7, in the same way as Milliken-Taylor theorem is a combination of Ramsey's theorem and Hindman's theorem. For a sequence $\bar{s} \in S^\omega$ let $\bar{s}^{(n)}$ be the collection of n-subsets of $\{a \in S : a = s_i \text{ for some } i \in \omega\}$. Notice that for $n = 1$ the following is the content of Theorem 3.4.7.

3.6.3 Theorem *Let M be a finite Ramsey monoid. Then, for any $n \geq 1$, for all sequences of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly, for any finite coloring of n-subsets of $\langle (X_n)_{n\in\omega} \rangle$ there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ such that s_n has a distinguished point for every $n \in \omega$ and such that $\bigcup_{\bar{r} \leq_M \bar{s}} \bar{r}^{(n)}$ is monochromatic.*

Proof The proof goes as in Theorem 3.4.4, in section 3.5. Let G and $u = g(\mathbf{y})$ be defined as in Theorem 3.4.4, with $\mathbf{y} = \mathbb{X}(M)$, and let $(u_n)_{n\in\omega}$ be a coheir sequence of u. It is straightforward to check that all elements of the span of $\bar{u}_{\restriction i}$ satisfy the type u for every $i \in \omega$. Also, notice that with signature L^+, for every $a, a', b \in \mathcal{G}^{<\omega}$ if $a \equiv_G a'$, $a' \downarrow_G b$, and $a \downarrow_G b$, then $a \equiv_{Mb} a'$. Then, for any $\overleftarrow{h} \leq_M \overleftarrow{u}$ we have $\overleftarrow{h} \equiv_G \overleftarrow{u}$, by the remark above and the definition of coheir sequence. All the n-subsets of an indiscernible sequence have the same color, for any $n \in \omega$. The rest of the proof is the same as in Theorem 3.4.4. □

The same arguments of Proposition 3.6.1 allow to extend this result to any partial semigroup.

It can be easily seen that if a monoid satisfies the conclusions of Corollary 3.6.2 or the conclusions of Theorem 3.6.3, then it is Ramsey. Conversely, Corollary 3.6.2 and Theorem 3.6.3 hold for all finite Ramsey monoids. Hence, their conclusions hold for a finite monoid if and only if it is Ramsey.

3.7 Final remarks and open problems

We conclude with some open questions and remarks concerning the work done so far.

Our main theorems suggest a possible connection between Ramsey theory and automata theory, passing through Schützenberger's Theorem. Any result in that direction would be of the highest interest.

Limiting ourselves to Ramsey theory, there are still several challenging open questions in the context of monoid actions on semigroups.

One of the most immediate questions is what can be proven for infinite monoids. In a fore-coming paper, the authors prove that there are not infinite Ramsey monoids, and thus a monoid is Ramsey if and only if it is finite, aperiodic and has linear $\mathbb{X}(M)$.

Theorem 3.4.6 provides a sufficient condition for a monoid to be $\mathbb{Y}$-controllable. This condition is not necessary, as there are $\mathbb{Y}$-controllable monoids for which $\mathbb{X}_{\mathcal{R}}(M)$ is not linear.

3.7.1 Proposition *Let M be a finite aperiodic monoid such that for every distinct $a, b \in M$ with $a \mathcal{R} b$, we have $a^2 = a$ and $ax = bx$ for every $x \in M \setminus \{1\}$. Then, M is $\mathbb{Y}$-controllable.*

Proof To show that M is $\mathbb{Y}$-controllable is enough to work with $M^{<\omega}$, by the arguments of Proposition 3.6.1.

Consider the monoid $\tilde{M} = (M, *)$ where $x * y = y$ for all $x, y \neq 1$. It acts coordinatewise on $M^{<\omega}$, considered as $\tilde{M}^{<\omega}$.

Let G be $M^{<\omega}$ with the signature L^+ used in the proof of Theorem 3.4.4, plus an unary function $\tilde{a}$ for any $a \in M$, which is interpreted in G as the action of $\tilde{M}$. Since $\tilde{M}$ is Ramsey and since every element in $\tilde{M}$ different from 1 is in the same $\mathcal{R}$-class, one can find an idempotent u in the space of types $S(G)$ such that $\tilde{a}u = \tilde{b}u$ for every $a, b \neq 1$. Let v be an element of the monster model satisfying u. Then, if $a \mathcal{R} b$, we have

$$av = a\tilde{a}v \equiv_G a\tilde{b}v = bv,$$

where we use the fact that for every $x \in M^{<\omega}$, and hence for every x in the monster model, we have $a\tilde{a}x = ax$ and $a\tilde{b}x = bx$, by hypothesis. Hence, we can conclude that M is $\mathbb{Y}$-controllable, by the arguments of Theorem 3.4.4. □

An example of a monoid satisfying the hypothesis of Proposition 3.7.1 for which $\mathbb{X}_{\mathcal{R}}(M)$ is not linear is given by the following Cayley table.

On the other hand, it seems possible that the necessary condition of Theorem 3.4.6 is also sufficient, and that a finite monoid is aperiodic if and only if it is Y-controllable. If true, this would suggest an even stronger connection between Ramsey theory and

1	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Table 3.2: Example of (aperiodic) $\mathbb{Y}$-controllable monoid M such that $\mathbb{X}_{\mathcal{R}}(M)$ is not linear.

Schützenberger's Theorem.

3.7.2 Open Problem *Find an algebraic characterization of $\mathbb{Y}$-controllable monoids.*

If M is a Ramsey monoid, then *for every action of M on every partial semigroup* you have a monochromatic set as described in the definition. Lupini's in [33] gave examples of non-Ramsey monoids where the same statement holds for *certain actions on certain partial semigroups* (actually, he proved a stronger statement that can be seen as the analogue of Corollary 3.6.2).

Define I_k to be the set of functions f from k to k such that $f(0) = 0$ and such that if $f(i) = j$ then either $f(i+1) = j$ or $f(i+1) = j+1$. Then, I_k is a monoid with composition of functions as operation, and k is an I_k-set with distinguished point $k-1$, where the action is defined by $fi = f(i)$. This action induces a coordinate-wise action on $\mathrm{FIN}_k = \langle (\{n\} \times k)_{n\in\omega} \rangle$ (i.e. the set of all partial functions with finite domain from $\mathbb{N}$ to k). Lupini in [33] showed that for every $k \in \omega$ and for every finite coloring of FIN_k there is an infinite sequence of words in FIN_k each containing $k-1$ such that its I_k-span is monochromatic. Notice that this result implies that every $\mathcal{R}$-trivial monoid is Ramsey. In fact, let N be a $\mathcal{R}$-trivial monoid with linear $\mathbb{X}(N)$. Without loss of generality, we may assume that $N = \{0, \dots, k-1\}$ and that $0N \subseteq \cdots \subseteq (k-1)N$ is an increasing enumeration of $\mathbb{X}(N)$. Then, the coordinate-wise action of N on FIN_k coincides with the action of a submonoid of I_k, by Proposition 3.2.4, and Lupini's theorem implies point 4 of Proposition 3.6.1.

All monoids I_k are $\mathcal{R}$-trivial, but $\mathbb{X}(I_k)$ is linear if and only if $k \leq 3$ (see [43, Section 4 .4]). In particular, if $k > 3$ these monoids are not Ramsey, and Lupini's result does not follow from the theory of Ramsey monoids. It would be interesting to see if a similar statement holds for other (non-Ramsey) monoids.

3.7.3 Open Problem *Classify the couples (M, k) such that $k \in \omega$ is a pointed M-set and for every finite coloring of FIN_k there is a basic sequence $\bar{s}$ in FIN_k such that s_n has a distinguished point for every $n \in \omega$ and such that the M-span of $\bar{s}$ is monochromatic.*

In the same direction, the following seems a challenging problem.

3.7.4 Open Problem *Characterize the class of triples $(S, M, \bar{t})$, where S is a partial semigroup,*

M is a monoid acting on S by endomorphisms and $\bar{t}$ is a basic sequence in S, for which for every finite coloring of S there is a sequence $\bar{s} \leq_M \bar{t}$ in S such that its M-span is monochromatic.

One can check that every finite Ramsey monoid generates examples of Ramsey spaces. However, an even nicer property might be true: there are topological Ramsey spaces that induce a collection of projected spaces such that every metrically Baire set has the Ramsey property. A sufficient condition for the latter has been found by Dobrinen and Mijares in [16]. An example of a space of this form is Carlson-Simpson space, see [10] and [47, section 5.6]). See also [16, section 4] for generalizations of the latter.

3.7.5 Open Problem *Which topological Ramsey spaces given by finite Ramsey monoids meet the sufficient conditions given in* [16]*?*

Hales-Jewett theorem [26] is a corollary of Corollary 3.6.2 for the special case of monoids M such that $ab = b$ for every $a, b \in M \setminus \{1\}$. In Ramsey theory, two of the strongest known results are a polynomial generalization [7] and a density generalization [20] of Hales-Jewett theorem for these monoids.

3.7.6 Open Problem *Do polynomial or density results hold for other finite Ramsey monoids?*

Ojeda-Aristizabal in [39] obtained upper bounds for the finite version of Gowers' FIN_k theorem, giving a constructive proof. It would be interesting to know if these upper bounds hold for other Ramsey monoids.

The work of Gowers on FIN_k and the related space $\mathrm{FIN}_{\pm k}$ was the key to his solution of an old problem in Banach spaces [24]. Also, the aforementioned example of Bartošova and Kwiatkowska found applications in metric spaces. Finally, a discussion about the connection between Ramsey spaces and Banach spaces can be found in Todorcevic's monograph. In this paper, we found new Ramsey monoids, and consequently new Ramsey spaces. Hence, it might be possible to find applications of these new results to metric spaces.

Recently various papers have found different common generalizations of Carlson's and Gowers' theorems, see [3], [29], [34]. Of particular interest is the context of adequate layered semigroups, introduced by Farah, Hindman, and McLeod [18] and recently studied by Lupini [34] and Barrett [3]. Barrett's paper [3] describes a framework which seems well suited for a connection between Ramsey monoids and layered semigroups. His work and ours are independent from each other and were written concurrently, so we did not investigate this research line. Nevertheless, in Example 3.7.7 and in the following paragraph we show a possible connection.

3.7.7 Example Let M be a monoid with linear $\mathbb{X}(M)$, and let $a_0 M \subseteq \cdots \subseteq a_n M$ be an increasing enumeration of $\mathbb{X}(M)$. Define $\ell : \mathrm{FIN}_M \to n+1$ by

$$\ell(w) = \min\{i : \mathrm{ran}(w) \subseteq a_i M\}.$$

Then, (FIN_M, ℓ) is an adequate partial layered semigroup as defined in [3, Definition 3.7]. Furthermore, the canonical action $\mathcal{F}_{\mathrm{cw}}$ of M on FIN_M is made of regressive maps, by Proposition 3.2.4.

This example shows that every monoid with linear $\mathbb{X}(M)$ generates an adequate partial layered semigroup, FIN_M, and a family of regressive functions $\mathcal{F}_{\mathrm{cw}}$. On the other hand, every family of regressive functions $\mathcal{F}$ on an adequate partial layered semigroup generates a monoid $M_{\mathcal{F}}$ with composition, acting on S by endomorphisms.

Chapter 4

Actions of infinite monoids

4.1 Introduction

In this chapter, we expand upon the results of Chapter 3, and we introduce two new notions for monoids: locally Ramsey and locally $\mathbb{Y}$-controllable.

In Chapter 3, we study Ramsey monoids and $\mathbb{Y}$-controllable monoids, but we focus on finite monoids. In particular, in Theorem 3.1.3 we characterize finite Ramsey monoids as those finite aperiodic monoids such that $\mathbb{X}(M)$ is linearly ordered by inclusion. In this chapter, when possible, we extend the results of Chapter 3 to infinite monoids.

One of our first results is the following.

4.1.1 Theorem (Theorem 4.3.7) *Let M be a Ramsey monoid. Then, M is finite.*

Together with Theorem 3.1.3, Theorem 4.1.1 leads to a full characterization of Ramsey monoids.

4.1.2 Corollary *A monoid M is Ramsey if and only if it is finite, aperiodic, and $\mathbb{X}(M)$ is linear.*

Theorem 4.1.1 forces us to study properties for infinite monoids that are weaker than Ramsey. Motivated by [47, Theorem 4.21], we introduce the notion of locally Ramsey monoid. Theorem 4.21 of [47] is an infinite version of Hales-Jewett Theorem and Carlson's theorem, and it is equivalent to saying that any countable monoid such that $ab = b$ for every $a, b \neq 1$ is locally Ramsey.

To give definitions of locally Ramsey and locally $\mathbb{Y}$-controllable, we work with sequences of pointed M-sets, as defined in the introduction of Chapter 3. One could also work with FIN_M, as in the introduction of this thesis (see section 1.2.4). The two approaches are equivalent. This can be shown with the arguments of Proposition 3.6.1.

4.1.3 Definition *Let M be a monoid acting on a partial semigroup S and let $\bar{s} \in S^\omega$ be a basic sequence. Let $(M_i)_{i\in\omega}$ be a sequence of finite subsets of M. The (M_i)-span of S is the set*

$$\langle \bar{s} \rangle_{(M_i)} = \{ m_0\, s_{i_0} \cdot \ldots \cdot m_n\, s_{i_n} : n \in \omega, i_0 < \cdots < i_n, m_i \in M_i, \text{ at least one } m_i \text{ is } 1_M \}.$$

4.1.4 Definition *We say that a monoid M is* locally Ramsey *if for every sequence of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly, for every finite coloring of $\langle (X_n)_{n\in\omega} \rangle$, and for every sequence $(M_i)_{i\in\omega}$ of finite subsets of M there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ such that s_n has a distinguished point for every $n \in \omega$ and such that $\langle \bar{s} \rangle_{(M_i)}$ is monochromatic.*

Notice that for every sequence $(M_i)_{i\in\omega}$ of finite subsets of M, the (M_i)-span of a sequence $\bar{s}$ is contained in the M-span of the same sequence, i.e. $\langle \bar{s} \rangle_{(M_i)} \subseteq \langle \bar{s} \rangle_M$. Hence, the following fact trivially holds.

4.1.5 Fact *If a monoid is Ramsey then it is locally Ramsey.*

One of the main results of this chapter is the following characterization.

4.1.6 Theorem (Theorem 4.5.5) *Let M be a monoid. Then, M is locally Ramsey if and only if M is aperiodic and $\mathbb{X}(M)$ is linear and finite.*

The notion of locally Ramsey monoid generalizes the notion of Ramsey monoid by limiting the action to finite sets of the monoid. The same idea leads to the notion of locally $\mathbb{Y}$-controllable monoid, which generalizes that of $\mathbb{Y}$-controllable monoid. We recall here the definition of $\langle \mathbb{Y}(M) \rangle$.

Given a monoid M, $\mathbb{Y}(M) \subseteq \mathscr{P}(\mathbb{X}(M))$ consists of the non-empty subsets of $\mathbb{X}(M)$ which are linearly ordered by inclusion. Given $x, y \in \mathbb{Y}(M)$, define $x \leq_{\mathbb{Y}(M)} y$ if $x \subseteq y$ and all elements of $y \setminus x$ are larger with respect to $\subseteq$ than all elements of x.

Let $\langle \mathbb{Y}(M) \rangle$, with operation $\vee$, be the semigroup freely generated by $\mathbb{Y}(M)$ modulo the relations

$$p \vee q = q = q \vee p \text{ for } p \leq_{\mathbb{Y}(M)} q.$$

4.1.7 Definition *Say that M is* locally $\mathbb{Y}$-controllable *if for every finite $F \subseteq \langle \mathbb{Y}(M) \rangle$, for every $\boldsymbol{y}$ maximal element in $\mathbb{Y}(M)$, for every sequence of pointed M-sets $(X_n)_{n\in\omega}$ on which M acts uniformly, for every finite coloring of $\langle (X_n)_{n\in\omega} \rangle$, and for every sequence $(M_i)_{i\in\omega}$ of finite subsets of M there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ such that s_n has a distinguished point for every $n \in \omega$ and such that for every $m, n \in \omega$ and for every $a_i, b_j \in M$ <u>if</u> $a_0\boldsymbol{y} \vee \cdots \vee a_n\boldsymbol{y} \in F$ and $a_0\boldsymbol{y} \vee \cdots \vee a_n\boldsymbol{y} = b_0\boldsymbol{y} \vee \cdots \vee b_m\boldsymbol{y}$, <u>then</u> $a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_n}$ has the same color of $b_0 s_{j_0} \cdot \ldots \cdot b_m s_{j_m}$, for every $i_0 < \cdots < i_n$, $j_0 < \cdots < j_m$ such that $a_k \in M_{i_k}$ and $b_k \in M_{j_k}$.*

The same remark as before leads to the following easy fact.

4.1.8 Fact *If a monoid is $\mathbb{Y}$-controllable then it is locally $\mathbb{Y}$-controllable.*

The following is the "local" version of Fact 3.4.5, which states that if a monoid M is $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear then M is Ramsey.

4.1.9 Fact *If M is locally $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear then M is locally Ramsey.*

Proof Notice that $\mathbb{X}(M)$ is linear if and only if $\mathbb{X}(M) \in \mathbb{Y}(M)$. Assume $\mathbb{X}(M)$ is linear and fix a sequence $(M_i)_{i\in\omega}$ of finite subsets of M. We want to use the definition of locally $\mathbb{Y}$-controllable with $\mathbf{y} = \mathbb{X}(M)$ and $F = \{\mathbf{y}\}$. It is enough to notice that for every $a \in M$ we have

$$a\mathbb{X}(M) = \{amM : m \in M\} = \{xM : xM \subseteq aM\}.$$

Hence, if $aM \subseteq bM$, then $a\mathbf{y} \vee b\mathbf{y} = b\mathbf{y} = b\mathbf{y} \vee a\mathbf{y}$. Therefore, $a_1\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = \mathbf{y} \in F$ for every $a_1, .., a_n \in M$ such that there is i such that $a_i = 1$. Hence, when $\mathbb{X}(M)$ is linear, the definition of locally $\mathbb{Y}$-controllable gives a basic sequence $\bar{s}$ which has monochromatic (M_i)-span. □

Our results on locally $\mathbb{Y}$-controllable are summarized in Theorem 4.5.3.

Finally, we improve the results of Chapter 3 on $\mathbb{Y}$-controllable monoids. In fact, in Theorem 4.5.4 we prove that there is a large class of infinite $\mathbb{Y}$-controllable monoids, which is not noted in Chapter 3.

4.2 Chain conditions in $\mathbb{X}(M)$

Towards an algebraic characterization of locally Ramsey monoids, our theorems can be divided in two sorts: necessary conditions and sufficient conditions.

In the case of locally $\mathbb{Y}$-controllable monoids, we do not reach an algebraic characterization, and our necessary conditions are different from our sufficient conditions. The absence of infinite chains in $\mathbb{X}(M)$ turns out to be a very relevant condition for locally $\mathbb{Y}$-controllable monoids. However, we do not know whether it is a necessary condition, and weaker conditions appear in our lemmas. In this section, we are going to list these weaker conditions, in Proposition 4.2.1.

We work with these weaker conditions because our lemmas may themselves be of interest. Also, this might help to reach an algebraic characterization of $\mathbb{Y}$-controllable monoids and locally $\mathbb{Y}$-controllable monoids, in some future work.

4.2.1 Proposition *Let M be a monoid. Assume that there are no infinite chains in $\mathbb{X}(M)$. Then, the following statements hold:*

1. *There are no $a \in M$ and maximal $\boldsymbol{y} \in \langle \mathbb{Y}(M) \rangle$ such that $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} a\boldsymbol{y}\}$ is infinite.*
2. *There is no maximal $\boldsymbol{y} \in \langle \mathbb{Y}(M) \rangle$ such that $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} \boldsymbol{y}\}$ is infinite.*
3. *There are no infinite decreasing chains in $\mathbb{X}(M)$.*

4. *For every $\mathbf{y} \in \langle \mathbb{Y}(M) \rangle$ the set $\{\mathbf{z} : \mathbf{z} \leq_{\mathbb{Y}} \mathbf{y}\}$ is finite.*

5. *There are no infinite increasing chains in $\mathbb{X}(M)$.*

Proof Let us prove 1. The other proofs are either trivial or similar to the proof of 1. Let us show that if 1 does not hold, then there is an infinite chain in $\mathbb{X}(M)$.

Assume there is an element $a \in M$ and a maximal $\mathbf{y} \in \langle \mathbb{Y}(M) \rangle$ such that the set $\{a'\mathbf{y} : a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is infinite. Then, $\{\mathbf{z} : \mathbf{z} \in \mathbb{Y}(M), \mathbf{z} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is infinite. Then, by definition of $\leq_{\mathbb{Y}}$, the set $\{bM : bM \in a\mathbf{y}\}$ is infinite. It is linearly ordered by inclusion since $a\mathbf{y} \in \mathbb{Y}(M)$. □

4.2.2 Remark Condition 4 is actually equivalent to the absence of infinite chains in $\mathbb{X}(M)$. In fact, assume there is an infinite chain in $\mathbb{X}(M)$, and let $\mathbf{y} \in \mathbb{Y}(M)$ be that chain. Then, $\mathbf{y}$ witnesses not 4.

Recall that a monoid is said $\mathcal{R}$-rigid if for every $a, b \in M$, if $ab \,\mathcal{R}\, b$, then $ab = b$.

Finite aperiodic monoids are exactly finite $\mathcal{R}$-rigid monoids. Aperiodic and $\mathcal{R}$-rigid monoids do not coincide in general. For example, $(\mathbb{N}, +, 0)$ is $\mathcal{R}$-rigid, since it is $\mathcal{R}$-trivial, but it is not aperiodic. The next proposition shows that the two classes coincide when there are no chains of a certain kind.

4.2.3 Proposition *Let M be a monoid. If a condition between 1, 2, 3, 4 of Proposition 4.2.1 hold, then the following are equivalent*

1. *M is aperiodic.*

2. *For every $g, a, g' \in M$, if $gag' = a$, then $ga = ag' = a$.*

3. *M is $\mathcal{R}$-rigid.*

Furthermore, 1 implies 2 and 2 implies 3 without any assumption on M.

Proof If M is aperiodic then point 2 holds, and if point 2 holds then M is $\mathcal{R}$-rigid, without any further assumption on M. This is shown in the proof of Proposition 3.2.2.

Assume that for every $a \in M$ the set $\{a^n M : n \in \omega\}$ is not an infinite chain, where we convene that $a^0 M = 1M = M$. This assumption is implied by any condition between 1, 2, 3, 4 of Proposition 4.2.1. We want to show that if M is $\mathcal{R}$-rigid then it is aperiodic.

Let $a \in M$ and let n be such that $a^{n+1} \,\mathcal{R}\, a^n$. This n exists since $\{a^n M : n \in \omega\}$ is finite. Since M is $\mathcal{R}$-rigid and $aa^n \,\mathcal{R}\, a^n$ we have $a^{n+1} = a^n$. □

We need the following technical proposition for the proof of Theorem 4.5.2. Notice that the second condition is the condition 1 of Proposition 4.2.1.

4.2.4 Proposition *The following are equivalent for a monoid M:*

1. *For every finite $F \subseteq \langle \mathbb{Y}(M) \rangle$, for every $\boldsymbol{y}$ maximal element in $\mathbb{Y}(M)$ there is a finite $B \subseteq \{a\boldsymbol{y} : a \in M\}$ such that if $a_0\boldsymbol{y} \vee \cdots \vee a_n\boldsymbol{y} \in F$ then $\{a_i\boldsymbol{y} : i \leq n\} \subseteq B$.*
2. *There are no $a \in M$ and maximal $\boldsymbol{y} \in \langle \mathbb{Y}(M) \rangle$ such that $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} a\boldsymbol{y}\}$ is infinite.*

Proof "not 2 implies not 1" Assume there are such a and $\mathbf{y}$. It is enough to choose $F = \{a\mathbf{y}\}$ to show that 1 does not hold.

"not 1 implies not 2" Suppose there is a finite F and a maximal $\mathbf{y}$ such that for every finite $B \subseteq \{a\mathbf{y} : a \in M\}$ there are $\{a_i\mathbf{y} : i \leq n\}$, such that $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} \in F$ and $\{a_i\mathbf{y} : i \leq n\} \nsubseteq B$. This implies that there are infinitely many $a\mathbf{y}$ such that there exist $a_0 \dots a_n \in M^{<\omega}$ and $i \leq n$ such that $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} \in F$ and $a_i\mathbf{y} = a\mathbf{y}$.

This implies that for some $f \in F$ there are infinitely many $a\mathbf{y}$ such that there exist $a_0 \dots a_n \in M^{<\omega}$ and $i \leq n$ such that $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} = f$ and $a_i\mathbf{y} = a\mathbf{y}$.

Let f be as above, say $f = a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y}$, we want to show that there is $i \leq n$ such that $\{a\mathbf{y} : a\mathbf{y} \leq_{\mathbb{Y}} a_i\mathbf{y}\}$ is infinite.

Let $g = b_0\mathbf{y} \vee \cdots \vee b_k\mathbf{y}$ be equal to f. It is enough to show that for every $j \leq k$ there is $i \leq n$ such that $b_j\mathbf{y} \leq_{\mathbb{Y}} a_i\mathbf{y}$. Suppose not and let j be such that $b_j\mathbf{y} \nleq_{\mathbb{Y}} a_i\mathbf{y}$, for every $i \leq n$. To reach a contradiction, we show that in this case every element equal to g has an element $\mathbf{x}$ such that $\mathbf{x} \nleq_{\mathbb{Y}} a_i\mathbf{y}$, for every $i \leq n$. In fact, assume that $\mathbf{z} \vee b_j\mathbf{y} \neq b_j\mathbf{y}$: then, $\mathbf{z} \vee b_j\mathbf{y} = \mathbf{z}$ and $b_j\mathbf{y} \leq_{\mathbb{Y}} \mathbf{z}$. Then, for every i we must have $\mathbf{z} \nleq_{\mathbb{Y}} a_i\mathbf{y}$. □

Notice that if B is as in condition 1, then the set $\{a : a\mathbf{y} \in B\}$ can be infinite, but it has elements from finitely many $\mathcal{R}$-classes.

4.3 Necessary conditions

In this section, we prove that all Ramsey monoids are finite. We also prove that locally Ramsey monoids are aperiodic, and have finite and linear $\mathbb{X}(M)$. Finally, we prove that locally $\mathbb{Y}$-controllable monoids are aperiodic and satisfy condition 1 of Proposition 4.2.1.

Our strategy is to show that if a monoid is infinite, then it is not Ramsey. Similarly, for the other classes of monoids.

To prove that some monoids are not Ramsey, we have to find an example of a sequence $(X_n)_{n\in\omega}$ of M-pointed sets and a coloring of $\langle (X_n)_{n\in\omega} \rangle$ such that there is no infinite sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^{\omega}$ such that s_n has a distinguished point for every n and such that $\langle \bar{s} \rangle_M$ is not monochromatic. Similarly, to show that a monoid is not locally Ramsey (or locally $\mathbb{Y}$-controllable), we have to choose a sequence $(X_n)_{n\in\omega}$ of M-pointed sets and a coloring of $\langle (X_n)_{n\in\omega} \rangle$.

In this section, as an example of sequence $(X_n)_{n\in\omega}$ as above, we always choose the constant sequence $X_n = M$. Namely, we always color the free semigroup $M^{<\omega}$. It can be seen as $\langle (X_n)_{n\in\omega} \rangle$, where $X_n = M$, with distinguished element 1. The action of M on X_n is given by multiplication. Elements of $\langle (X_n)_{n\in\omega} \rangle$ have distinguished points if and only if they are variable words.

The results of this section are stated for locally Ramsey monoids and locally $\mathbb{Y}$-controllable monoids. Since every Ramsey monoid is locally Ramsey, and every $\mathbb{Y}$-controllable monoid is locally $\mathbb{Y}$-controllable, these results also hold for Ramsey monoids and $\mathbb{Y}$-controllable monoids.

The following proposition is a straightforward generalization of Solecki's [43, Corollary 4.5 (ii)] to locally Ramsey monoids.

4.3.1 Proposition *If M is locally Ramsey then $\mathbb{X}(M)$ is linear.*

Proof Suppose $\mathbb{X}(M)$ is not linear, and let aM and bM such that $aM \not\subseteq bM$ and $bM \not\subseteq aM$. By $aM \not\subseteq bM$ we have $a \notin bM$, and by $bM \not\subseteq aM$ we have $b \notin aM$. Color $w \in M^{<\omega}$ with red if the letter a is in w and it appears before the first appearance of b. Otherwise, color w with blue. Let $(M_i)_{i\in\omega}$ be any sequence of finite subsets of M each containing $\{1, a, b\}$. Then, if $\bar{s}$ is any sequence of variable words, $a(s_1)s_2$ has color red and $b(s_1)s_2$ has color blue. Hence, the (M_i)-span of $\bar{s}$ is not monochromatic and M is not locally Ramsey. □

In the following, we prove that both locally $\mathbb{Y}$-controllable monoids and locally Ramsey monoids are aperiodic.

4.3.2 Proposition *Let M be a monoid. Then,*

1. *If M is locally $\mathbb{Y}$-controllable, then it is aperiodic.*

2. *If M is locally Ramsey, then it is aperiodic.*

Proof Suppose M is not aperiodic, and let $a \in M$ be such that $a^n \neq a^{n+1}$ for every $n \in \omega$. We want to prove that M is not locally $\mathbb{Y}$-controllable and is not locally Ramsey.

Let $A = \{a^n : n \in \omega\}$, let $C = \{m \in M : a^n m \in A \text{ for some } n \in \omega\}$. We convene that $a^0 = 1$. Notice that $m \in C$ if and only if $am \in C$.

Suppose first A is finite. Let $A \cup \{\text{black}, \text{red}, \text{blue}\}$ be the set of colors. Given a word $w \in M^{<\omega}$, color w with black if w contains no letter in C. Otherwise, let m be the first letter of w in C. Color w with m if $m \in A$, color w with red if $\min\{n \in \omega : a^n m \in A\}$ is even and with blue if it is odd.

If instead A is infinite, then $a^m \neq a^j$ for every $m \neq j$. Given a word $w \in M^{<\omega}$, color w with black if w contains no letter in C. Otherwise, let m be the first letter of w in C. Let $n = \min\{n \in \omega : a^n m \in A\}$ and let k be such that $a^n m = a^k$. Color w with red if $n - k$ is odd, and with blue if $n - k$ is even.

Let $(M_i)_{i\in\omega}$ be the sequence of finite subsets of M defined by $M_i = \{1, a\}$. Let $\mathbf{y} = \{a^n M : n \in \omega\}$, where we set $a^0 = 1$, and let $F = \{\mathbf{y}\}$. Then, $\mathbf{y}$ is a maximal element of $\mathbb{Y}(M)$, and $\mathbf{y} \vee \mathbf{y} = \mathbf{y} = a\mathbf{y} \vee \mathbf{y}$.

Let $(y_i)_{i<\omega}$ be any sequence of variable words. Both in the finite and infinite case, y_0 has at least a letter in C, since it has a 1. Then, $y_0 {}^\frown y_1$ and $a(y_0) {}^\frown y_1$ have different colors, hence M is not locally $\mathbb{Y}$-controllable. The same coloring shows that the (M_i)-span of $y_0 y_1$ is not monochromatic. Hence, M is not locally Ramsey. □

We notice a consequence of those properties already proved for locally Ramsey monoids.

4.3.3 Remark Let M be an aperiodic monoid such that $\mathbb{X}(M)$ is linear. Then, aM is a both-sided ideal for every $a \in M$ (and in particular $Ma \subseteq aM$).

In fact, consider $a, d \in M$. If $daM \subseteq aM$ we are done, so by linearity we may assume $aM \subseteq daM$. By induction this implies $aM \subseteq d^i aM$ for every $i < \omega$. Given n such that $d^n = d^{n+1}$, we have $a \in d^n aM$ and there is $m \in M$ such that $a = d^n am$. Hence,

$$da = d(d^n am) = (d^{n+1})am = d^n am = a.$$

In the following, we show that locally $\mathbb{Y}$-controllable monoids satisfy the condition 1 of Proposition 4.2.1.

4.3.4 Proposition *If a monoid M is locally $\mathbb{Y}$-controllable then there are no $a \in M$ and maximal $\mathbf{y} \in \mathbb{Y}(M)$ such that $\{a'\mathbf{y} : a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is infinite.*

Proof Towards a contradiction, assume there are $a \in M$ and a maximal $\mathbf{y}$ such that $A = \{a'\mathbf{y} : a' \in M, a'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}\}$ is infinite. Notice that if $a' \mathcal{R} a''$ and $\{a'\mathbf{y}, a''\mathbf{y}\} \subseteq A$, then $a'\mathbf{y} = a''\mathbf{y}$: in fact $a'\mathbf{y}$ and $a''\mathbf{y}$ are $\leq_{\mathbb{Y}}$-comparable and $a'M = a''M$ is their top element. Hence $A(M) = \{cM : c \in M, c\mathbf{y} \in A\}$ is infinite.

First, notice that $A(M) = \{cM : c\mathbf{y} \in A\}$ is linearly ordered by inclusion. In fact, if $c\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}$ and $c'\mathbf{y} \leq_{\mathbb{Y}} a\mathbf{y}$ then $c\mathbf{y}$ and $c'\mathbf{y}$ are $\leq_{\mathbb{Y}}$-comparable, and this implies that cM and $c'M$ are $\subseteq$-comparable, since $1M \in \mathbf{y}$.

We want to show that $A(M)$ contains no infinite ascending chain. Suppose not, and let $\{a_i M\}$ be ascending in $A(M)$ with $a_i M \subset a_{i+1} M$ and $a_i \mathbf{y} \in A$ for every $i \in \omega$. For each word $w \in M^{<\omega}$, define $i(w)$ to be the highest natural number such that a letter of w is in $[a_{i(w)}]_{\mathcal{R}}$, if there is such a letter in w. Color w with black if it contains no such letter, otherwise color w with red if $i(w)$ is odd and with blue if $i(w)$ is even. Let $(M_i)_{i\in\omega}$ be the sequence of finite subsets of M defined by $M_i = \{1, a, a_0, \dots, a_i\}$. Let $\bar{y} = (y_n)_{n\in\omega}$ be any sequence of variable words. If ay_0 has color black, then $ay_0 {}^\frown a_1 y_1$ has color blue or red, since y_1 has at least a letter 1. Otherwise, let $k = i(ay_0)$. Then,

$$i(ay_0 {}^\frown a_{k+1} y_{k+1}) = k + 1,$$

since $ay_0 \frown a_{k+1}y_{k+1}$ contains a_{k+1} as a letter, and each other letter of $a_{k+1}y_{k+1}$ belongs to $a_{k+1}M$. This contradicts that M is locally $\mathbb{Y}$-controllable, since

$$a\mathbf{y} \vee a_1\mathbf{y} = a\mathbf{y} \vee a_{k+1}\mathbf{y} = a\mathbf{y}.$$

Now we want to show that $A(M)$ contains no infinite descending chain. Suppose not, and let $\{a_iM\}$ be an infinite descending chain in $A(M)$, with $a_iM \supset a_{i+1}M$ and $a_i\mathbf{y} \in A$ for every $i \in \omega$. Set $a_0 = a$. For each word $w \in M^{<\omega}$, let H_w be the set of $i \in \omega$ such that there is a letter b in w such that $b \in [a_i]_{\mathcal{R}}$, there is no letter c in w such that $c \in [a_{i+1}]_{\mathcal{R}}$, and there exist $k \geq 2$ and a letter d in w such that $d \in [a_{i+k}]_{\mathcal{R}}$. We can think H_w as a set counting "holes". Color w by red if H_w has even cardinality, by blue otherwise. Let $(M_i)_{i\in\omega}$ be the sequence of finite subsets of M defined by $M_i = \{1, a_0, \ldots, a_i\}$. Let $\bar{y} = (y_n)_{n\in\omega}$ be any sequence of variable words. Given a word $w \in M^{<\omega}$ such that w has a letter in $\bigcup_{i\in\omega}[a_i]_{\mathcal{R}}$, let $l(w)$ be the maximum i such that w has a letter in $[a_i]_{\mathcal{R}}$. Let $i = l(ay_0)$ and let $k = l(ay_0 \frown a_{i+1}y_{i+1})$. If ay_0 has the same color of $ay_0 \frown a_{i+1}y_{i+1}$ then the cardinality of $H_{a_{i+1}y_{i+1}}$ is even. In this case, $ay_0 \frown a_{i+1}y_{i+1} \frown a_{k+1}y_{k+1}$ and $ay_0 \frown a_{k+1}y_{k+1}$ have different colors. This contradicts that M is locally $\mathbb{Y}$-controllable, since

$$a\mathbf{y} \vee a_{i+1}\mathbf{y} = a\mathbf{y} \vee a_{i+1}\mathbf{y} \vee a_{k+1}\mathbf{y} = a\mathbf{y} \vee a_{k+1}\mathbf{y} = a\mathbf{y}.$$

Hence, $A(M)$ is linearly ordered and does not contain infinite chains, in contradiction with the fact that $A(M)$ is infinite. □

Virtually the same proof shows the following. We write the proof for the ease of the reader.

4.3.5 Proposition *If a monoid M is locally Ramsey then there is no maximal $\boldsymbol{y} \in \mathbb{Y}(M)$ such that $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} \boldsymbol{y}\}$ is infinite.*

Proof Towards a contradiction, assume there is a maximal $\mathbf{y}$ such that $A = \{a'\mathbf{y} : a'\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}\}$ is infinite. Hence $A(M) = \{cM : c\mathbf{y} \in A\}$ is infinite.

Arguing as in the proof of Proposition 4.3.4, $A(M) = \{cM : c\mathbf{y} \in A\}$ is linearly ordered by inclusion.

We claim that $A(M)$ contains no infinite ascending chain. Suppose not, and let $\{a_iM\}$ be ascending in $A(M)$ with $a_iM \subset a_{i+1}M$ and $a_i\mathbf{y} \in A$ for every $i \in \omega$. For each word $w \in M^{<\omega}$, define $i(w)$ to be the highest natural number such that a letter of w is in $[a_{i(w)}]_{\mathcal{R}}$, if there is such a letter in w. Color w with black if it contains no such letter, otherwise color w with red if $i(w)$ is odd and with blue if $i(w)$ is even. Let $(M_i)_{i\in\omega}$ be the sequence of finite subsets of M defined by $M_i = \{1, a_0, \ldots, a_i\}$. Let $\bar{y} = (y_n)_{n\in\omega}$ be any sequence of variable words. If y_0 has color black, then $y_0 \frown a_1y_1$ has color blue or red, since y_1 has at least a letter 1. Otherwise, let $k = i(y_0)$. Then,

$$i(y_0 \frown a_{k+1}y_{k+1}) = k + 1,$$

since $y_0 {}^\frown a_{k+1}y_{k+1}$ contains a_{k+1} as a letter, and each other letter of $a_{k+1}y_{k+1}$ belongs to $a_{k+1}M$. This contradicts that M is locally Ramsey, since the span of y_0y_1 is not monochromatic.

Now we want to show that $A(M)$ contains no infinite descending chain. Suppose not, and let $\{a_iM\}$ be an infinite descending chain in $A(M)$, with $a_iM \supset a_{i+1}M$ and $a_i\mathbf{y} \in A$ for every $i \in \omega$. Set $a_0 = 1$. For each word $w \in M^{<\omega}$, let H_w be the set of $i \in \omega$ such that there is a letter b in w such that $b \in [a_i]_{\mathcal{R}}$, there is no letter c in w such that $c \in [a_{i+1}]_{\mathcal{R}}$, and there exist $k \geq 2$ and a letter d in w such that $d \in [a_{i+k}]_{\mathcal{R}}$. Color w by red if H_w has even cardinality, by blue otherwise. Let $(M_i)_{i\in\omega}$ be the sequence of finite subsets of M defined by $M_i = \{1, a_0, \dots, a_i\}$. Let $\bar{y} = (y_n)_{n\in\omega}$ be any sequence of variable words. Given a word $w \in M^{<\omega}$ such that w has a letter in $\bigcup_{i\in\omega}[a_i]_{\mathcal{R}}$, let $l(w)$ be the maximum i such that w has a letter in $[a_i]_{\mathcal{R}}$. Let $i = l(y_0)$ and let $k = l(y_0 {}^\frown a_{i+1}y_{i+1})$. If y_0 has the same color of $y_0 {}^\frown a_{i+1}y_{i+1}$, then the cardinality of $H_{a_{i+1}y_{i+1}}$ is even, hence $y_0 {}^\frown a_{i+1}y_{i+1} {}^\frown a_{k+1}y_{k+1}$ and $y_0 {}^\frown a_{k+1}y_{k+1}$ have different colors. This contradicts that M is locally Ramsey.

Hence, $A(M)$ is linearly ordered and does not contain infinite chains, in contradiction with the fact that $A(M)$ is infinite. □

The next corollary is a key step in the characterization of locally Ramsey monoids.

4.3.6 Corollary *Let M be a monoid. Then,*

1. *If M is locally $\mathbb{Y}$-controllable and $\mathbb{X}(M)$ is linear, then $\mathbb{X}(M)$ is finite.*
2. *If M is locally Ramsey, then $\mathbb{X}(M)$ is finite.*

Proof Let $\mathbf{y} = \mathbb{X}(M) \in \mathbb{Y}(M)$. If M is locally $\mathbb{Y}$-controllable, by Proposition 4.3.4 the set $\{a\mathbf{y} : a\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}\}$ is finite. Since $\mathbb{X}(M)$ is linear, $a\mathbf{y} \leq_{\mathbb{Y}} \mathbf{y}$ for every $a \in M$. Hence, the set $\{a\mathbf{y} : a \in M\}$ is finite. This implies that $\mathbb{X}(M)$ is finite.

The same proof works in the case of locally Ramsey monoids, using Proposition 4.3.1 and Proposition 4.3.5 □

4.3.7 Theorem *Let M be a monoid.*

1. *If M is locally Ramsey and $\mathbb{Y}$-controllable then it is finite.*
2. *If M is Ramsey, then it is finite.*

Proof By Proposition 4.3.1 and point 2 of Corollary 4.3.6 we know that if a monoid M is locally Ramsey then $\mathbb{X}(M)$ is linear and finite. Since a Ramsey monoid is locally Ramsey, the same holds for Ramsey monoids. To prove that M is finite it is enough to prove that every $\mathcal{R}$-class is finite. We are going to prove that if there is an infinite $\mathcal{R}$-class then M is not $\mathbb{Y}$-controllable and not Ramsey.

Let $[a]_{\mathcal{R}}$ be the maximal infinite $\mathcal{R}$-class, and notice that $M \setminus aM$ is finite. Let $B = M \setminus aM$. Fix an enumeration $\{a_i : i \leq \alpha\} = [a]_{\mathcal{R}}$ of $[a]_{\mathcal{R}}$ such that for infinitely many i the element a_i appears after all the elements of $a_i B$, i.e. if $a_j \in a_i B$ then $j \leq i$. This enumeration is possible since B is finite. This enumeration of $[a]_{\mathcal{R}}$ induces an enumeration of its subset C defined by

$$C = \{c \in [a]_{\mathcal{R}} : c \text{ appears after all the elements of } cB \setminus \{c\}\}.$$

Let $\mathbf{y} = \mathbb{X}(M)$, let $F = \{\mathbf{y}\} \subseteq \langle \mathbb{Y}(M) \rangle$. Then, $\mathbf{y}$ is a maximal element and for every $c \in [a]_{\mathcal{R}}$ we have $c\mathbf{y} \vee \mathbf{y} = \mathbf{y} \vee \mathbf{y} = \mathbf{y}$. Color a word $w \in M^{<\omega}$ by black if it has no letter in C, by red if the highest index of one of its letters in C is odd, and by blue if it is even.

Let $(y_i)_{i \in \omega}$ be a sequence of variable words. If $y_0 \frown y_1$ has color black, then $c(y_0) \frown y_1$ has color red or blue for any $c \in C$. Otherwise, let n be the maximal index of a letter in C occurring in $y_0 \frown y_1$. Then, $c_{n+1} y_0 \frown y_1$ and $y_0 \frown y_1$ have different colors. Either case, this shows that M is not $\mathbb{Y}$-controllable. The same coloring shows that M is not Ramsey, since the M-span of $(y_i)_{i \in \omega}$ is not monochromatic. □

4.3.8 Corollary *A monoid is Ramsey if and only if it is finite, aperiodic, and $\mathbb{X}(M)$ is linear.*

Proof By Theorem 3.4.7 and point 2 of Theorem 4.3.7. □

4.4 Compact right topological semigroups

In this section, we study actions of aperiodic monoids on compact right topological semigroups. The proofs contained in this section are similar to those of Chapter 3, but here we cannot use the hypothesis that the monoid is finite.

As mentioned in Section 4.2, we work with weaker assumptions than the absence of infinite chains in $\mathbb{X}(M)$ (see Proposition 4.2.1).

In this section, we work in conditions where $\mathcal{R}$-rigid monoids are exactly aperiodic monoids (see Proposition 4.2.3). We use this equivalence in what follows.

4.4.1 Lemma *Let M be an aperiodic monoid such that there are no infinite descending chains in $(\mathbb{X}(M), \subseteq)$, and such that $X_{\mathcal{R}}(M)$ is linear. Then, for every distinct $a, b \in M$ with $a \mathcal{R} b$ there are two distinct $g, h \in M$ such that $ag = b$, $bh = a$ and $gh = h$, $hg = g$. This in particular implies $gM = hM$.*

Proof Fix a non-trivial $\mathcal{R}$-class $[c]_{\mathcal{R}}$ and let $a, b \in [c]_{\mathcal{R}}$ with $a \neq b$.

For every $y, z \in M$, define

$$G_{y,z} = \{g_{y,z} \in M : y g_{y,z} = z\}.$$

Notice that if $y \mathcal{R} z$, then $G_{y,z}$ is non-empty. Let $g \in G_{a,b}$ be such that gM is minimal in $\{xM \mid x \in G_{a,b}\}$. Such g exists since there are no infinite descending chains in $\mathbb{X}(M)$. Similarly, let $h \in G_{b,a}$ be such that hM is minimal in $\{xM \mid x \in G_{b,a}\}$.

Notice that $hgh \in G_{b,a}$ since $bhgh = agh = bh = a$. Since $hghM \subseteq hgM \subseteq hM$, by minimality of hM we have $hghM = hgM = hM$, so $h \mathcal{R} hg$ and $h \mathcal{R} hgh$. Notice that $hg \in G_{b,b}$ and that $G_{b,a} \cap G_{b,b} = \emptyset$, so $h \neq hg$ and the class $[h]_{\mathcal{R}}$ is non-trivial. Similarly, $g \mathcal{R} gh \mathcal{R} ghg$, $gh \in G_{a,a}$ and so the class $[g]_{\mathcal{R}}$ is non-trivial. We want to show that $hM = gM$.

Since $\mathbb{X}_{\mathcal{R}}(M)$ is linear either $gM \subseteq hM$ or $hM \subseteq gM$. Suppose for example $gM \subseteq hM = hgM$. Then, $g = hgm$ for some $m \in M$, which implies $g = hg$, by point 2 of Proposition 4.2.3. Hence, $gM = hgM = hM$.

This implies that h, hg, g, gh, are all in the same $\mathcal{R}$-class, hence $gh = h$ and $hg = g$, by definition of $\mathcal{R}$-rigid. □

In the following lemma we do not need the hypothesis that there are no infinite chains in $\mathbb{X}(M)$.

4.4.2 Lemma *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear. Then, for every $a \in M$, if there are $b, c \in [a]_{\mathcal{R}}$ such that $bc = c$, then for every $b, c \in [a]_{\mathcal{R}}$ we have $bc = c$.*

Proof The case where $[a]_{\mathcal{R}} = \{a\}$ is easy, so we can assume that $aM \in \mathbb{X}_{\mathcal{R}}(M)$.

First, notice that if $xy = y$ for some $x, y \in M$, then $xz = z$ for every $z \in [y]_{\mathcal{R}}$, since $xzM = xyM = yM = zM$ and since M is $\mathcal{R}$-rigid by Proposition 4.2.3.

Hence, we just need to prove that given a non-trivial $\mathcal{R}$-class $[a]_{\mathcal{R}}$ such that $ax = x$ for every $x \in [a]_{\mathcal{R}}$, and given an element $b \in [a]_{\mathcal{R}}$ with $b \neq a$, then we have $ba = a$.

Let h be such that $bh = a$. Notice that $ha \mathcal{R} hb$ since $haM = hbM$, and also $ha \neq hb$ since $bha = a \neq b = bhb$. Then $haM \in \mathbb{X}_{\mathcal{R}}(M)$ and so $haM \subseteq aM$ or $aM \subseteq haM$.

If $haM \subseteq aM$ then

$$aM = aaM = bhaM \subseteq baM \subseteq bM = aM.$$

Hence, $ba \mathcal{R} a$ and $ba = a$.

If $aM \subseteq haM$ then $a = ham$ for some m, and by point 2 of Proposition 4.2.3, $a = ha$. Hence, $ba = bha = a$. □

The exact same proof of Theorem 3.3.5 works for the following, where Lemma 4.4.1 and Lemma 4.4.2 play the role of Lemma 3.3.3 and Lemma 3.3.4. In order to use the proof of Theorem 3.3.5, we need to assume that $\mathbb{X}_{\mathcal{R}}(M)$ is linear and finite.

4.4.3 Theorem *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is finite and linear, and such that there are no infinite descending chains in $(\mathbb{X}(M), \subseteq)$. Let U be a compact right topological semigroup on which M acts by continuous endomorphisms. Then, there exists a minimal idempotent $u \in E(U) \cap I(U)$ such that $a(u) = b(u)$ for all couples $a, b \in M$*

such that $a \mathcal{R} b$.

Let us recall a technical notion isolated by Solecki in [43].

4.4.4 Definition *A monoid M is called* good *if for every left action of M by continuous endomorphisms on a compact right topological semigroup U there exists a function $g\colon \mathbb{Y}(M) \to E(U)$ such that*

(i) g is M-equivariant;

(ii) g is order reversing with respect to $\leq_{\mathbb{Y}(M)}$ and $\leq_U$;

(iii) g maps maximal elements of $\mathbb{Y}(M)$ to $I(U)$.

The following lemma is stated in [43] for finite monoids, but the same proof holds for a larger class of monoids. In fact, the proof just uses that for every $\mathbf{y} \in \mathbb{Y}(M)$, the set $\{\mathbf{z} \in \mathbb{Y}(M) : \mathbf{z} \leq_{\mathbb{Y}} \mathbf{y}\}$ is finite. As proven in Remark 4.2.2, this condition is equivalent to asking that there are no infinite chains in $(\mathbb{X}(M), \subseteq)$.

4.4.5 Lemma [43, Lemma 2.5] *Let M be a monoid such that there are no infinite chains in $(\mathbb{X}(M), \subseteq)$. Assume that for every left action of M by continuous endomorphisms on a compact right topological semigroup U there is a M-equivariant f from $\mathbb{Y}(M)$ to U such that f maps maximal elements of $\mathbb{Y}(M)$ to $I(U)$. Then, M is good.*

The following lemma gives another sufficient condition for a monoid to be good. Notice that we need to ensure that every $\mathbf{y}$ has an element that is $\subseteq$-maximal. We assume that M is a monoid such that there are no infinite increasing chains in $\mathbb{X}(M)$.

4.4.6 Lemma *Let M be a monoid such that there are no infinite chains in $\mathbb{X}(M)$. Assume that for every action by continuous endomorphisms of M on a compact right topological semigroup U there exists a minimal idempotent $u \in E(U) \cap I(U)$ such that $a(u) = b(u)$ for all couples $a, b \in M$ such that $a \mathcal{R} b$. Then, M is good.*

Proof Let $\pi : \mathbb{Y}(M) \to \mathbb{X}(M)$ be the function that maps a set $y \in \mathbb{Y}(M)$ to the maximal element in y with respect to $\subseteq$. Here we use that $\mathbb{X}(M)$ does not have infinite increasing chains. Let $u \in E(U) \cap I(U)$ be given by hypothesis. The function $f : \mathbb{X}(M) \to E(U)$ that maps aM to $a(u)$ is well-defined, and maps $1M$ to $u \in E(U) \cap I(U)$. Also, notice that if y is a maximal element of $\mathbb{Y}(M)$, then $1M \in y$ and so $\pi \circ f(y) = u \in E(U) \cap I(U)$. Since both f and π are M-equivariant the map $f \circ \pi : \mathbb{Y}(M) \to E(U)$ satisfies the assumptions of Lemma 4.4.5, from which we get that M is good. □

4.4.7 Corollary *Let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear, and such that there are no infinite chains in $(\mathbb{X}(M), \subseteq)$. Then, M is good.*

Proof By Theorem 4.4.3 and Lemma 4.4.6. Notice that we can drop the hypothesis that $\mathbb{X}_{\mathcal{R}}(M)$ is finite. In fact, $\mathbb{X}_{\mathcal{R}}(M)$ is always finite under the assumptions that there are no infinite chains in $(\mathbb{X}(M), \subseteq)$ and $\mathbb{X}_{\mathcal{R}}(M)$ is linear. □

4.5 From good to locally $\mathbb{Y}$-controllable

In this section, we conclude the characterization of locally Ramsey monoids. Also, we find new classes of infinite $\mathbb{Y}$-controllable monoids and locally $\mathbb{Y}$-controllable monoids. We use the same model-theoretic setup of Section 3.5.

We need a technical definition.

4.5.1 Definition *Let F be a finite subset of the semigroup $\langle \mathbb{Y}(M) \rangle$, let $\boldsymbol{y}$ be a maximal element in $\mathbb{Y}(M)$, let $(M_i)_{i\in\omega}$ be a sequence of finite subset of M, and let c be a finite coloring of a semigroup S on which M acts. We say that a sequence $\bar{s} \in S^{\leq \omega}$ is $(M_i, F, \mathbf{y}, c)$-*controllable *if for every $m, n \leq |\bar{s}|$ and for every $a_i, b_j \in M$ <u>if</u> $a_0\boldsymbol{y} \vee \cdots \vee a_n\boldsymbol{y}$ belongs to F and $a_0\boldsymbol{y} \vee \cdots \vee a_n\boldsymbol{y} = b_0\boldsymbol{y} \vee \cdots \vee b_m\boldsymbol{y}$, <u>then</u> $a_0 s_{i_0} \cdot \ldots \cdot a_n s_{i_n}$ has the same color of $b_0 s_{j_0} \cdot \ldots \cdot b_m s_{j_m}$, for every $i_0 < \cdots < i_n$, $j_0 < \cdots < j_m$ such that $a_k \in M_{i_k}$ and $b_k \in M_{j_k}$.*

4.5.2 Theorem *Let M be a good monoid.*

1. *Assume that there are no $a \in M$ and maximal $\boldsymbol{y} \in \langle \mathbb{Y}(M) \rangle$ such that $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} a\boldsymbol{y}\}$ is infinite. Then, M is locally $\mathbb{Y}$-controllable.*

2. *Assume that for every finite $F \subseteq \langle \mathbb{Y}(M) \rangle$, for every $\boldsymbol{y}$ maximal element in $\mathbb{Y}(M)$ there is a finite $B \subseteq M$ such that if $a_0\boldsymbol{y} \vee \cdots \vee a_n\boldsymbol{y} \in F$ then $\{a_i : i \leq n\} \subseteq B$.*[1] *Then, M is $\mathbb{Y}$-controllable.*

Proof Let us show 1; the proof of 2 is essentially the same.

Let $(X_n)_{n\in\omega}$ be a sequence of pointed M-sets on which M acts uniformly, and let $\perp$ be not in $\bigcup_{n\in\omega} X_n$. Define $G = (\langle (X_n)_{n\in\omega} \rangle \cup \{\perp\}, ^\frown)$ to be the semigroup extending $(\langle (X_n)_{n\in\omega} \rangle, ^\frown)$ defining $x ^\frown y = \perp$ if $x ^\frown y$ is not defined in the partial semigroup $\langle (X_n)_{n\in\omega} \rangle$. In particular, $x ^\frown \perp = \perp ^\frown x = \perp$. We write $x \prec y$ if and only if $x ^\frown y \neq \perp$ or $x = \perp$.

It is enough to prove that for every finite subset F of $\langle \mathbb{Y}(M) \rangle$, for every maximal element $\mathbf{y}$ in $\mathbb{Y}(M)$, for every sequence $(M_i)_{i\in\omega}$ of finite subset of M, and for every c finite coloring of G there is a basic sequence $\bar{s} \in (\langle (X_n)_{n\in\omega} \rangle)^\omega$ that is $(M_i, F, \mathbf{y}, c)$-controllable and such that s_n has a distinguished point for every $n \in \omega$. By Proposition 4.2.4, there is a finite $B \subseteq \{a\mathbf{y} : a \in M\}$ such that if $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} \in F$ then $\{a_i\mathbf{y} : i \leq n\} \subseteq B$. Without loss of generality, we may assume that $M_i\mathbf{y} \supseteq B$ for every i.

[1] Notice that this condition is stronger than condition 1 of Proposition 4.2.4.

Let $L = \{^\frown, \prec\}$ and consider G with augmented signature L^+. Let

$$U = \{p \in S(G) : G \prec p\}$$

where $G \prec p$ is a shorthand for $\{g \prec x : g \in G\} \subseteq p(x)$. As we show in the proof of Theorem 3.4.4, in Section 3.5, U is a compact subsemigroup of $(S(G), \cdot_G)$ and M acts on U by continuous endomorphisms.

Let $u = g(\mathbf{y}) \in E(U) \cap I(U)$, where $g : \mathbb{Y}(M) \to E(U)$ is the function given by definition of good monoid. Let DP be the set of elements of $\langle (X_n)_{n\in\omega} \rangle$ that have at least one distinguished point, and let $J = \{p \in S(G) : DP \in p\}$. Since $J \cap U$ is a non-empty both-sided ideal of U, and $u \in I(U)$, we have that u is in J. Let $(u_n)_{n\in\omega}$ be a coheir sequence of u. We write $\bar{u}_{\restriction i}$ for the tuple $u_{i-1}, \dots, u_0$. Notice that since the map g is order-reversing and M-equivariant, for every $a_0, \dots, a_n, b_0, \dots, b_m \in M$ if $a_0\mathbf{y} \vee \dots \vee a_n\mathbf{y} = b_0\mathbf{y} \vee \dots \vee b_m\mathbf{y}$ then also $a_0 u \cdot_G \dots \cdot_G a_n u = b_0 u \cdot_G \dots \cdot_G b_m u$. Hence, it is straightforward to check that $\bar{u}_{\restriction i}$ is $(M_i, F, \mathbf{y}, c)$-controllable for every $i \in \omega$. Notice that $\bar{u}_{\restriction i}$ is a basic sequence since products stay in U and $\perp \notin U$. Finally, u_n are elements of DP since $u \in J$. Now, we use the sequence $\bar{u}$ to define $\bar{s} \in G^\omega$ with same properties as $\bar{u}$.

Let $k \in \omega$ be such that for every element $f \in F$ there are $k' < \kappa$ and $a_0, \dots, a_{k'} \in M$ such that $f = a_0\mathbf{y} \vee \dots \vee a_{k'}\mathbf{y}$. Notice that this implies that for every $f, f' \in \langle \mathbb{Y}(M) \rangle$ such that $f \vee f' \in F$ there are $c_0, \dots c_j \in M$ with $j < k$ such that $f' = c_0\mathbf{y} \vee \dots \vee c_j\mathbf{y}$. This follows from the property that the set of predecessors of any element of $\mathbb{Y}(M)$ is linearly ordered by $\leq_{\mathbb{Y}(M)}$.

Assume as induction hypothesis that the tuple obtained by concatenation $s_{\restriction i} {}^\frown \bar{u}_{\restriction k}$ is $(M_i, F, \mathbf{y}, c)$-controllable and is a basic sequence of elements of DP. Our goal is to find $s_i \in G$ such that the same properties hold for $s_{\restriction i+1}$.

From the induction hypothesis it follows that $s_{\restriction i} {}^\frown \bar{u}_{\restriction l}$ is $(M_i, F, \mathbf{y}, c)$-controllable for any $l \in \omega$. In fact, let $w = b_0 s_{i_0} {}^\frown \dots {}^\frown b_m s_{i_m} {}^\frown b_{m+1} u_{i_{m+1}} {}^\frown \dots {}^\frown b_n u_{i_n}$ be such that $b_0\mathbf{y} \vee \dots \vee b_m\mathbf{y} \vee b_{m+1}\mathbf{y} \vee \dots \vee b_n\mathbf{y} \in F$. Let $j < k$ and $c_0, \dots, c_j \in M$ be such that $b_{m+1}\mathbf{y} \vee \dots \vee b_n\mathbf{y} = c_0\mathbf{y} \vee \dots \vee c_j\mathbf{y}$. By assumption, $\{c_0, \dots, c_j\}\mathbf{y} \subseteq B \subseteq M_i\mathbf{y}$ for every i. The type over G of $b_0 s_{i_0} {}^\frown \dots {}^\frown b_m s_{i_m} {}^\frown c_0 u_j {}^\frown \dots {}^\frown c_j u_0$ is equal to the type over G of w. Therefore, we may use the induction hypothesis to conclude that $s_{\restriction i} {}^\frown \bar{u}_{\restriction l}$ is $(M_i, F, \mathbf{y}, c)$-controllable. Also, $s_{\restriction i} {}^\frown \bar{u}_{\restriction l}$ is a basic sequence by induction hypothesis and idempotence of u.

Let $\varphi(s_{\restriction i}, u_{i+1}, u_{\restriction i+1})$ say that $s_{\restriction i} {}^\frown \bar{u}_{\restriction i+2}$ is $(M_i, F, \mathbf{y}, c)$-controllable and that $s_{\restriction i} {}^\frown \bar{u}_{\restriction i+2}$ is a basic sequence of elements of DP. This is a formula since M_i are finite. As $\bar{u}$ is a coheir sequence we can find $s_i \in G$ such that $\varphi(s_{\restriction i+1}, u_{\restriction i+1})$. Hence, s_i has the desired properties. □

We summarize this chapter's results on locally-$\mathbb{Y}$-controllable monoids in the following theorem.

4.5.3 Theorem *Let M be a locally $\mathbb{Y}$-controllable monoid. Then, M is aperiodic. Also, for every $a \in M$ and maximal $\boldsymbol{y} \in \langle \mathbb{Y}(M) \rangle$ the set $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} a\boldsymbol{y}\}$ is finite.*

In the other direction, let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear and such that there are no infinite chains in $(\mathbb{X}(M), \subseteq)$. Then, M is locally $\mathbb{Y}$-controllable.

Proof The necessary conditions to be locally $\mathbb{Y}$-controllable are proved in point 1 of Proposition 4.3.2, and in Proposition 4.3.4. In the other direction, those monoids are good by Corollary 4.4.7. Since there are no infinite chains in $(\mathbb{X}(M), \subseteq)$, we can use Proposition 4.2.1 and apply Theorem 4.5.2. □

We summarize this chapter's results on $\mathbb{Y}$-controllable monoids in the following theorem. This result enlarges the class of known $\mathbb{Y}$-controllable monoids giving examples of infinite $\mathbb{Y}$-controllable monoids. See Proposition 4.6.1 for concrete examples.

4.5.4 Theorem *Let M be a $\mathbb{Y}$-controllable monoid. Then, M is aperiodic. Also, for every $a \in M$ and maximal $\boldsymbol{y} \in \langle \mathbb{Y}(M) \rangle$ the set $\{a'\boldsymbol{y} : a'\boldsymbol{y} \leq_{\mathbb{Y}} a\boldsymbol{y}\}$ is finite.*

In the other direction, let M be an aperiodic monoid such that $\mathbb{X}_{\mathcal{R}}(M)$ is linear, such that there are no infinite chains in $(\mathbb{X}(M), \subseteq)$ and there are no infinite $\mathcal{R}$-classes. Then, M is $\mathbb{Y}$-controllable.

Proof The necessary conditions to be $\mathbb{Y}$-controllable are proved for the larger class of locally $\mathbb{Y}$-controllable monoids in point 1 of Proposition 4.3.2, and in Proposition 4.3.4.

In the other direction, those monoids are good by Corollary 4.4.7. Since there are no infinite chains in $(\mathbb{X}(M), \subseteq)$, by Proposition 4.2.1 for every finite $F \subseteq \langle \mathbb{Y}(M) \rangle$, for every $\mathbf{y}$ maximal element in $\mathbb{Y}(M)$ there is a finite $B \subseteq \{a\mathbf{y} : a \in M\}$ such that if $a_0\mathbf{y} \vee \cdots \vee a_n\mathbf{y} \in F$ then $\{a_i\mathbf{y} : i \leq n\} \subseteq B$. Since there are no infinite $\mathcal{R}$-classes, there are finitely many $a \in M$ such that $a\mathbf{y} \in B$. Hence, we can apply Theorem 4.5.2. □

Finally, we can characterize locally Ramsey monoids. Notice that there are locally Ramsey monoids of any cardinality.

4.5.5 Corollary *Let M be a monoid. Then, M is locally Ramsey if and only if M is aperiodic and $\mathbb{X}(M)$ is linear and finite.*

Proof If M is locally Ramsey then M is aperiodic by point 2 of Proposition 4.3.2, and $\mathbb{X}(M)$ is linear and finite by Proposition 4.3.1 and point 2 of Corollary 4.3.6. If M is aperiodic and $\mathbb{X}(M)$ is linear and finite then M is locally $\mathbb{Y}$-controllable by Theorem 4.5.3. By Fact 4.1.9, every locally $\mathbb{Y}$-controllable monoid M such that $\mathbb{X}(M)$ is linear is locally Ramsey. □

Every Ramsey monoid is locally Ramsey and $\mathbb{Y}$-controllable. Also, every finite locally Ramsey monoid is finite, aperiodic, and has linear $\mathbb{X}(M)$: hence it is Ramsey, by Theorem 3.4.7. Hence, by point 1 of Theorem 4.3.7 we get the following.

4.5.6 Corollary *1. A monoid is Ramsey if and only if it is locally Ramsey and $\mathbb{Y}$-controllable.*

4.6 Final remarks

Here, we give a different proof that there are $\mathbb{Y}$-controllable monoids of any cardinality, as opposed to Ramsey monoids, which are finite. Proposition 4.6.1 is weaker than Theorem 4.5.4, since the class of monoids considered is included in the class of monoids of Theorem 4.5.4.

4.6.1 Proposition *Let $(M_i, *_i)$, $i \in I$, be a family of finite $\mathcal{R}$-trivial monoids, with a possible exception of one finite aperiodic monoid with linear $\mathbb{X}_{\mathcal{R}}(M)$, and assume that $M_i \cap M_j = \emptyset$ if $i \neq j$. Let $M = \bigcup_{i \in I} M_i \cup \{1, 0\}$, where $1, 0 \notin \bigcup_{i \in I} M_i$ and let $(M, \cdot, 1)$ be the monoid with neutral element 1 defined by $ab = a *_i b$ if $a, b \in M_i$ and by $ab = 0$ otherwise. Then, M is $\mathbb{Y}$-controllable.*

Proof It is straightforward to check that if $a \in M_i$ then $aM = aM_i \cup \{0\}$. Hence, for every element $\mathbf{y}$ of $\mathbb{Y}(M)$ there is $i \in I$ such that $\{a : aM \in \mathbf{y}\} \subseteq M_i \cup \{0, 1\}$. Also, if $b \in M_j$ then $\{a : aM \in b\mathbf{y}\} \subseteq M_j \cup \{0\}$; finally, $b\mathbf{y} = \{0M\}$ if and only if $b = 0$ or $\mathbf{y} = \{0M\}$.

For $J \subseteq I$, let $M_{\restriction J}$ be the submonoid of M defined by $M_{\restriction J} = \bigcup_{j \in J} M_j \cup \{1, 0\}$. Then, if F is a finite subset of $\langle \mathbb{Y}(M) \rangle$ there is a finite J such that if $b_0\mathbf{y} \vee \cdots \vee b_m\mathbf{y} \in F$ then $b_0, \ldots b_m \in M_{\restriction J}$.

Hence, in order to check that M is $\mathbb{Y}$-controllable it is enough to notice that $M_{\restriction J}$ is $\mathbb{Y}$-controllable, for every finite $J \subseteq I$. This is given by Theorem 3.4.6, since $M_{\restriction J}$ is aperiodic and has linear $\mathbb{X}_{\mathcal{R}}(M)$. □

We conclude with a remark which might be useful to reach an algebraic characterization of $\mathbb{Y}$-controllable monoids, in some future work.

Lemmas 4.4.1 and 4.4.2 are two key steps to prove that certain aperiodic monoids are (locally) $\mathbb{Y}$-controllable. They heavily use the hypothesis that $\mathbb{X}_{\mathcal{R}}(M)$ is linear. If one wants to prove that every finite aperiodic monoid is $\mathbb{Y}$-controllable, then a different method should be used. In fact, there are finite aperiodic monoids such that $\mathbb{X}_{\mathbb{R}}(M)$ is not linear for which the theses of lemmas 4.4.1 and 4.4.2 do not hold. An example of such monoid is given by the following Cayley table.

1	a	b	c	d	0
a	0	0	b	a	0
b	a	b	0	0	0
c	d	c	0	0	0
d	0	0	c	d	0
0	0	0	0	0	0

Table 4.1: Example of aperiodic monoid that does not satisfy the theses of lemmas 4.4.1 and 4.4.2

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Bibliography

[1] Claudio Agostini and Eugenio Colla, *Ramsey monoids*, arXiv preprint arXiv:2012.02506 (2020).

[2] Uri Andrews, Gabriel Conant, and Isaac Goldbring, *Definable sets containing productsets in expansions of groups*, J. Group Theory **22** (2019), no. 1, 63–82. MR3895638

[3] Jordan Mitchell Barrett, *Ramsey theory for layered semigroups*, The Electronic Journal of Combinatorics (2021), P2–10.

[4] Dana Bartošová and Aleksandra Kwiatkowska, *Gowers' Ramsey theorem with multiple operations and dynamics of the homeomorphism group of the Lelek fan*, J. Combin. Theory Ser. A **150** (2017), 108–136. MR3645569

[5] Vitaly Bergelson, *Ergodic Ramsey theory—an update*, Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994), 1996, pp. 1–61. MR1411215

[6] Vitaly Bergelson, Andreas Blass, and Neil Hindman, *Partition theorems for spaces of variable words*, Proc. London Math. Soc. (3) **68** (1994), no. 3, 449–476. MR1262304

[7] Vitaly Bergelson and Alexander Leibman, *Set-polynomials and polynomial extension of the Hales-Jewett theorem*, Ann. of Math. (2) **150** (1999), no. 1, 33–75. MR1715320

[8] Andreas Blass, *Ultrafilters: where topological dynamics = algebra = combinatorics*, Topology Proc. **18** (1993), 33–56. MR1305122

[9] Timothy J. Carlson, *Some unifying principles in Ramsey theory*, Discrete Math. **68** (1988), no. 2-3, 117–169. MR926120

[10] Timothy J. Carlson and Stephen G. Simpson, *A dual form of Ramsey's theorem*, Adv. in Math. **53** (1984), no. 3, 265–290. MR753869

[11] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups. Vol. I*, Mathematical Surveys, No. 7, American Mathematical Society, Providence, R.I., 1961. MR0132791

[12] Eugenio Colla and Domenico Zambella, *Ramsey's coheirs*, J. Symb. Log. **87** (2022), no. 1, 377–391. MR4404631

[13] W. W. Comfort, *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 417–455. MR454893

[14] Mauro Di Nasso, *Hypernatural numbers as ultrafilters*, Nonstandard analysis for the working mathematician, 2015, pp. 443–474. MR3409522

[15] Mauro Di Nasso, Isaac Goldbring, and Martino Lupini, *Nonstandard methods in Ramsey theory and combinatorial number theory*, Lecture Notes in Mathematics, vol. 2239, Springer, Cham, 2019. MR3931702

[16] Natasha Dobrinen and José G. Mijares, *Topological Ramsey spaces and metrically Baire sets*, J. Combin. Theory Ser. A **135** (2015), 161–180. MR3366475

[17] Pandelis Dodos and Vassilis Kanellopoulos, *Ramsey theory for product spaces*, Mathematical Surveys and Monographs, vol. 212, American Mathematical Society, Providence, RI, 2016. MR3469779

[18] Ilijas Farah, Neil Hindman, and Jillian McLeod, *Partition theorems for layered partial semigroups*, J. Combin. Theory Ser. A **98** (2002), no. 2, 268–311. MR1899627

[19] Hillel Furstenberg and Yitzchak Katznelson, *Idempotents in compact semigroups and Ramsey theory*, Israel J. Math. **68** (1989), no. 3, 257–270. MR1039473

[20] ______, *A density version of the Hales-Jewett theorem*, J. Anal. Math. **57** (1991), 64–119. MR1191743

[21] Gili Golan and Boaz Tsaban, *Hindman's coloring theorem in arbitrary semigroups*, J. Algebra **395** (2013), 111–120. MR3097232

[22] W. T. Gowers, *A Banach space not containing c_0, l_1 or a reflexive subspace*, Trans. Amer. Math. Soc. **344** (1994), no. 1, 407–420. MR1250820

[23] ______, *An infinite Ramsey theorem and some Banach-space dichotomies*, Ann. of Math. (2) **156** (2002), no. 3, 797–833. MR1954235

[24] W. Timothy Gowers, *Lipschitz functions on classical spaces*, European J. Combin. **13** (1992), no. 3, 141–151. MR1164759

[25] J. A. Green, *On the structure of semigroups*, Ann. of Math. (2) **54** (1951), 163–172. MR42380

[26] Alfred W. Hales and Robert I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222–229. MR143712

[27] Neil Hindman, *Finite sums from sequences within cells of a partition of N*, J. Combinatorial Theory Ser. A **17** (1974), 1–11. MR349574

[28] Karl Heinrich Hofmann and Paul S. Mostert, *Elements of compact semigroups*, Charles E. Merr ll Books, Inc., Columbus, Ohio, 1966. MR0209387

[29] Jamal K. Kawach, *Parametrized Ramsey theory of infinite block sequences of vectors*, Ann. Pure Appl. Logic **172** (2021), no. 8, Paper No. 102984, 22. MR4247793

[30] Ivan Khatchatourian and Micheal Pawliuk, *Almost $\mathcal{R}$-trivial monoids are almost never Ramsey*, arXiv preprint arXiv:1805.04921 (2018).

[31] Sabine Koppelberg, *The Hales-Jewett theorem via retractions*, Proceedings of the 18th Summer Conference on Topology and its Applications, 2004, pp. 595–601. MR2159746

[32] Mark V. Lawson, *Finite automata*, Chapman & Hall/CRC, Boca Raton, FL, 2004. MR2002708

[33] Martino Lupini, *Gowers' Ramsey theorem for generalized tetris operations*, J. Combin. Theory Ser. A **149** (2017), 101–114. MR3620731

[34] ______, *Actions on semigroups and an infinitary Gowers-Hales-Jewett Ramsey theorem*, Trans. Amer. Math. Soc. **371** (2019), no. 5, 3083–3116. MR3896106

[35] Robert McNaughton and Seymour Papert, *Counter-free automata*, MIT Research Monograph, No. 65, The M.I.T. Press, Cambridge, Mass.-London, 1971. With an appendix by William Henneman. MR0371538

[36] Keith R. Milliken, *Ramsey's theorem with sums or unions*, J. Combinatorial Theory Ser. A **18** (1975), 276–290. MR373906

[37] Ludomir Newelski, *Topological dynamics of definable group actions*, J. Symbolic Logic **74** (2009), no. 1, 50–72. MR2499420

[38] ______, *Topological dynamics of stable groups*, J. Symb. Log. **79** (2014), no. 4, 1199–1223. MR3343536

[39] Diana Ojeda-Aristizabal, *Finite forms of Gowers' theorem on the oscillation stability of C_0*, Combinatorica **37** (2017), no. 2, 143–155. MR3638339

[40] Jean-Éric Pin, *Mathematical foundations of automata theory*, Lecture notes LIAFA, Université Paris **7** (2010).

[41] M. P. Schützenberger, *On finite monoids having only trivial subgroups*, Information and Control **8** (1965), 190–194. MR176883

[42] Saharon Shelah, *Primitive recursive bounds for van der Waerden numbers*, J. Amer. Math. Soc. **1** (1988), no. 3, 683–697. MR929498

[43] Sławomir Solecki, *Monoid actions and ultrafilter methods in Ramsey theory*, Forum Math. Sigma **7** (2019), Paper No. e2, 40. MR3904213

[44] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245. MR369312

[45] Alan D. Taylor, *A canonical partition relation for finite subsets of* ω, J. Combinatorial Theory Ser. A **21** (1976), no. 2, 137–146. MR424571

[46] Katrin Tent and Martin Ziegler, *A course in model theory*, Lecture Notes in Logic, vol. 40, Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012. MR2908005

[47] Stevo Todorcevic, *Introduction to Ramsey spaces*, Annals of Mathematics Studies, vol. 174, Princeton University Press, Princeton, NJ, 2010. MR2603812

[48] Bartel van der Waerden, *Beweis einer baudetschen vermutung*, Nieuw Arch. Wisk. **19** (1927), 212–216.

[49] Domenico Zambella, *A crèche course in model theory*, AMS Open Math Notes. Linked to the github version.