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Multilevel Schoenberg-Marsden variation diminishing operator and related quadratures

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In this paper we propose an improvement of the classical Schoenberg-Marsden variation diminishing operator with applications to the construction of new quadrature rules that we show having better performances with respect to the already known ones based on the classical cited operator. We discuss convergence properties and error estimates. Numerical experiments are also carried out to confirm the presented theoretical results.

1. Introduction

Recently a kind of approximation technique, called *multilevel approximation* has been introduced [1] and studied [2–5] in the literature. It is a way of triggering an iterative procedure, acting on some kind of remainder (or error) in the chosen base method.

Induced by this idea, in this paper we are interested in investigating this technique when applied to the well known variation-diminishing Schoenberg-Marsden operator, also in order to define new quadrature formulas, based on it.

The reason to use the simplest among quasi-interpolating (QI) operators [6] is explained in [2,3], where, even if in $2D$, the best performance results are obtained right by this operator. The same is done in [5].

The reason of using such a *spline* quasi-interpolating operator [7,8] is to exploit the good properties of B-spline functions, as locality, shape smoothness and approximation accuracy [9].

The aim of using multilevel approximation techniques in the context of spline QI is to improve the performances of the base QI spline results, by generating a sequence of QI spline functions, whose sum approaches to the desired final spline.

As we know, interpolation has a well-developed theory and it is a very powerful tool for function approximation, but, since it needs to solve large systems of linear equations with possibly bad condition, the weaker form of quasi-interpolation becomes a very interesting alternative. In fact it can directly construct operators and does not require solution of any linear system. For this reason QI operators are very important in the study of approximation theory and its applications, e.g. in solving partial differential equations and integral equations, curve and surface fitting, integration, differentiation, approximation of zeros, and so on, and they have been widely studied in recent years [10–17].

Then in this paper we continue the study of spline QI operators on bounded domains, restricted to the C^1 quadratic variation-diminishing Schoenberg-Marsden operator, but suitably modified in order to satisfy the main interest that lies in the fact of providing the best approximation order, while being easy to compute.

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The paper is organized as follows.

In Section 2 we recall the definition and the properties of the classical Schoenberg-Marsden operator. In Section 3 we modify such an operator by multilevel techniques and use the new improved operators in Section 4 to approximate integrals. Polynomial reproduction and approximation order are also studied. Moreover in Section 5 some numerical results and comparisons between the classical and the improved operators are presented, showing the new operators provide better performances than the QI classical one and numerical convergence orders are consistent with theoretical ones. Finally some long and tedious computations, justifying some results in this paper, are reported in Appendix.

2. The variation-diminishing Schoenberg-Marsden operator

Let $I = [a,b] \subset \mathbb{R}$ and Λ_n be a uniform partition, dividing I in n subintervals, whose associated extended partitions are

$$x_{-2} < x_{-1} < x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b < x_{n+1} < x_{n+2}. \tag{1}$$

and

$$x_{-2} = x_{-1} = x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b = x_{n+1} = x_{n+2} \tag{2}$$

with $x_i = a + ih, \quad i = -2, \dots, n+2$ and $h = \frac{b-a}{n}$, except at multiple knots, where it is zero.

Let also $S_2^1(\Lambda_n)$ be the space of spline functions $s \in C^1(I)$ on Λ_n whose restriction on any subinterval is a polynomial in $\mathbb{P}_2$, space of polynomials of degree at most 2. The set of $n+2$ quadratic B-spline functions $\mathcal{B}_n = \{B_i := B_{i,2} : i = 0, \dots, n+1\}$ is a basis of $S_2^1(\Lambda_n)$. In case of partitions (1) they can be obtained as scaled translated of the quadratic C^1 B-spline B centered at $x = 0$ with support 'radius' $\frac{3}{2}$, i.e. $B_{i,2}(x) = B(nx - i + \frac{1}{2})$, while for partitions (2) the multiplicity at the interval extrema is to be taken into account. Such basis functions satisfy the usual good properties of non negativity, partition of unity, minimal compact support. In particular the support of B_i is $[x_{i-2}, x_{i+1}]$. Moreover they can be generated by the well known Cox-de Boor recurrence relation [9]

$$B_{i,d}(x) = \frac{x - x_{i-d}}{x_i - x_{i-d}} B_{i-1,d-1}(x) + \frac{x_{i+1} - x}{x_{i+1} - x_{i-d+1}} B_{i,d-1}(x),\ \ i = 0, \dots, n+d-1 \tag{3}$$

with

$$B_{i,0}(x) = \begin{cases} 1 & \text{if } x \in [x_i, x_{i+1}), \\ 0 & \text{otherwise} \end{cases}$$

for B-splines of degree d.

Let us consider the quadratic version of the variation-diminishing Schoenberg-Marsden operator $S : C(I^*) \to S_2^1(\Lambda_n)$, defined as

$$Sf = \sum_{i=0}^{n+1} f(s_i) B_i, \tag{4}$$

where I^* is an open interval containing I and where

$$s_i = \frac{x_{i-1} + x_i}{2} = a + \frac{(2i-1)}{2} h, \quad i = 0, \dots, n+1$$

for partitions (1) and

$$s_0 = x_0, \quad s_i = \frac{x_{i-1} + x_i}{2}, \quad \text{con} \quad i = 1, \dots, n, \quad s_{n+1} = x_n.$$

for partitions (2) are the so-called *Greville sites* [9]. Moreover S reproduces $\mathbb{P}_1$ and $\|S\| = 1$.

Theorem 2.1. *Let $f \in C(\overline{I})$, where $\overline{I}$ is the closure of I^*. Then*

$$\|f - Sf\|_I \le \omega\Big(f, \frac{3}{2}h\Big) \tag{5}$$

for sufficiently small h. If $f \in C^1(I)$, then

$$\|f - Sf\|_I \le h\omega\Big(f', \frac{h}{2}\Big)$$

and, if $f \in C^2(I)$, then

$$\|f - Sf\|_I \le \frac{1}{4} h^2 \|f''\|,$$

where $\|\cdot\|_I$ denotes the maximum norm on I and $\omega(f,\delta) = \max\{|f(x) - f(y)|, x, y \in \overline{I} : \|x - y\| \le \delta\}$ is the classical modulus of continuity of f on $\overline{I}$.

Proof. The proof of (5) is trivial by using the partition of unity property of B-splines. Note that $\frac{3}{2}h$ is the 'radius' of the support of B_i, for any i.

Suppose now f is $C^1(I)$. Let j, $j = 1, \dots, n$, such that $\|f - Sf\|_I = \|f - Sf\|_{[x_{j-1},x_j]}$. Then by the Mean Value Theorem it results

$$f(x) = p_1(x) + (f'(u) - f'(s_j))(x - s_j) \tag{6}$$

for some $u = tx + (1-t)s_j$, $0 \le t \le 1$, where $p_1(x) = f(s_j) + f'(s_j)(x - s_j)$. Since S reproduces $\mathbb{P}_1$ and $\|S\| = 1$, then from (6)

$$\|f - Sf\|_{[x_{j-1},x_j]} \le (1 + \|S\|)\|f - p_1\|_{[x_{j-1},x_j]} \le h\omega\Big(f', \frac{h}{2}\Big).$$

Finally, if $f \in C^2(I)$, by Taylor's formula we have $f(x) = p_1(x) + \frac{1}{2}f''(u)(x - s_j)^2$ and then, similarly to the previous case, it results

$$\|f - Sf\|_{[x_{j-1},x_j]} \le (1 + \|S\|)\frac{1}{2}\|f''\|\Big(\frac{h}{2}\Big)^2 = \frac{1}{4}h^2\|f''\|. \quad \square$$

3. The multilevel variation-diminishing Schoenberg-Marsden operator

The aim of modifying the operator (4) is to reach better performances by defining a multilevel operator starting from the corresponding base one.

3.1. The bilevel variation-diminishing Schoenberg-Marsden operator S^{1L}

In order to define S on two levels we start by setting $h_i^{(0)} = h$, $x_i^{(0)} = x_i$. Given an even integer n and a partition of type (1), we define $\Gamma^{(0)} := \{s_i^{(0)} = s_i, \quad i = 0, \dots, \frac{n}{2^0} + 1\}$ and $\Gamma^{(1)} = \{s_i^{(1)}, \quad i = -1, \dots, \frac{n}{2^1} + 2\}$ where

$$s_i^{(1)} = \frac{x_{i-1}^{(1)} + x_i^{(1)}}{2} = \frac{x_{2i-2} + x_{2i}}{2} \tag{7}$$

with $x_i^{(1)} = x_{2^1 i}$. Based on these new knots, we define

$$S^{(1)}f = \sum_{i=-1}^{\frac{n}{2^1}+2} f(s_i^{(1)})B_i^{(1)}, \tag{8}$$

with $B_i^{(1)}(x) = B(\frac{n}{2^1}x - i + \frac{1}{2})$, $i = -1, \dots, \frac{n}{2} + 2$ and the *first error function*

$$\Delta_1 f = f - S^{(1)}f. \tag{9}$$

Then we compute

$$S^{(0)}\Delta_1 f = \sum_{i=0}^{\frac{n}{2^0}+1} \Delta_1 f(s_i)B_i^{(0)}, \tag{10}$$

where $B_i^{(0)} = B_i$ and so $S^{(0)} = S$, in order to obtain the two-level spline function

$$S^{1L}f = S^{(1)}f + S^{(0)}\Delta_1 f. \tag{11}$$

From (8), (9), (10) it can be also written as

$$S^{1L}f = \sum_{i=-1}^{\frac{n}{2}+2} f(s_i^{(1)})B_i^{(1)} + \sum_{i=0}^{n+1} f(s_i)B_i^{(0)} - \sum_{i=0}^{n+1}\sum_{j=-1}^{\frac{n}{2}+2} f(s_j^{(1)})B_j^{(1)}(s_i)B_i^{(0)} = S^{(1)}f + S^{(0)}f - S^{(0)}S^{(1)}f. \tag{12}$$

Then in case of partitions (1), taking into account Table 1, obtained from (3) and where it is underlined the degree of B-splines with

$$B_{i,0}^{(1)}(x) = \begin{cases} 1 & \text{if } x \in [x_{2i}, x_{2i+2}) \\ 0 & \text{otherwise,} \end{cases}$$

it is possible to give the following explicit expression of (12), ready to be considered for numerical integration

$$\begin{aligned}
S^{1L}f = {} & \sum_{i=0}^{n+1} f(s_i)B_i^{(0)} + f(s_{-1}^{(1)})\Big[B_{-1}^{(1)} - \frac{1}{32}B_0^{(0)}\Big] + f(s_0^{(1)})\Big[B_0^{(1)} - \frac{22}{32}B_0^{(0)} - \frac{9}{32}B_1^{(0)} - \frac{1}{32}B_2^{(0)}\Big] \\
& + f(s_1^{(1)})\Big[B_1^{(1)} - \frac{9}{32}B_0^{(0)} - \frac{22}{32}B_1^{(0)} - \frac{22}{32}B_2^{(0)} - \frac{9}{32}B_3^{(0)} - \frac{1}{32}B_4^{(0)}\Big] \\
& + \sum_{i=2}^{\frac{n}{2}-1} f(s_i^{(1)})\Big[B_i^{(1)} - \frac{1}{32}B_{2i-3}^{(0)} - \frac{9}{32}B_{2i-2}^{(0)} - \frac{22}{32}B_{2i-1}^{(0)} - \frac{22}{32}B_{2i}^{(0)} - \frac{9}{32}B_{2i+1}^{(0)} - \frac{1}{32}B_{2i+2}^{(0)}\Big] \qquad (13) \\
& + f(s_{\frac{n}{2}}^{(1)})\Big[B_{\frac{n}{2}}^{(1)} - \frac{1}{32}B_{n-3}^{(0)} - \frac{9}{32}B_{n-2}^{(0)} - \frac{22}{32}B_{n-1}^{(0)} - \frac{22}{32}B_n^{(0)} - \frac{9}{32}B_{n+1}^{(0)}\Big]
\end{aligned}$$

Table 1
B-splines $B_i^{(1)}$, $i=-1,\dots,\frac{n}{2}+2$, on partitions (1).

$B_{-1,2}^{(1)}(x)=\frac{(x_0^{(1)}-x)^2}{8h^2}B_{-1,0}^{(1)}(x)$
$B_{0,2}^{(1)}(x)=\left[\frac{(x-x_{-2}^{(1)})(x_0^{(1)}-x)}{8h^2}+\frac{(x_1^{(1)}-x)(x-x_{-1}^{(1)})}{8h^2}\right]B_{-1,0}^{(1)}(x)+\frac{(x_1^{(1)}-x)^2}{8h^2}B_{0,0}^{(1)}(x)$
$B_{j,2}^{(1)}(x)=\frac{(x-x_{j-2}^{(1)})^2}{8h^2}B_{j-2,0}^{(1)}(x)+\left[\frac{(x-x_{j-2}^{(1)})(x_j^{(1)}-x)}{8h^2}+\frac{(x_{j+1}^{(1)}-x)(x-x_{j-1}^{(1)})}{8h^2}\right]B_{j-1,0}^{(1)}(x)+\frac{(x_{j+1}^{(1)}-x)^2}{8h^2}B_{j,0}^{(1)}(x)$, $j=1,\dots,\frac{n}{2}$
$B_{\frac{n}{2}+1,2}^{(1)}(x)=\frac{(x-x_{\frac{n}{2}-1}^{(1)})^2}{8h^2}B_{\frac{n}{2}-1,0}^{(1)}(x)+\left[\frac{(x-x_{\frac{n}{2}-1}^{(1)})(x_{\frac{n}{2}+1}^{(1)}-x)}{8h^2}+\frac{(x_{\frac{n}{2}+1}^{(1)}-x)(x-x_{\frac{n}{2}}^{(1)})}{8h^2}\right]B_{\frac{n}{2},0}^{(1)}(x)$
$B_{\frac{n}{2}+2,2}^{(1)}(x)=\frac{(x-x_{\frac{n}{2}}^{(1)})^2}{8h^2}B_{\frac{n}{2},0}^{(1)}(x)$

Table 2
B-splines $B_i^{(1)}$, $i=0,\dots,\frac{n}{2}+1$, on partitions (2).

$B_{0,2}^{(1)}(x)=\frac{(x_1^{(1)}-x)^2}{4h^2}B_{0,0}^{(1)}(x)$
$B_{1,2}^{(1)}(x)=\left[\frac{(x-x_0^{(1)})(x_1^{(1)}-x)}{4h^2}+\frac{(x_2^{(1)}-x)(x-x_0^{(1)})}{8h^2}\right]B_{0,0}^{(1)}(x)+\frac{(x_2^{(1)}-x)^2}{8h^2}B_{1,0}^{(1)}(x)$
$B_{j,2}^{(1)}(x)=\frac{(x-x_{j-2}^{(1)})^2}{8h^2}B_{j-2,0}^{(1)}(x)+\left[\frac{(x-x_{j-2}^{(1)})(x_j^{(1)}-x)}{8h^2}+\frac{(x_{j+1}^{(1)}-x)(x-x_{j-1}^{(1)})}{8h^2}\right]B_{j-1,0}^{(1)}(x)+\frac{(x_{j+1}^{(1)}-x)^2}{8h^2}B_{j,0}^{(1)}(x)$, $j=2,\dots,\frac{n}{2}-1$
$B_{\frac{n}{2},2}^{(1)}(x)=\frac{(x-x_{\frac{n}{2}-2}^{(1)})^2}{8h^2}B_{\frac{n}{2}-2,0}^{(1)}(x)+\left[\frac{(x-x_{\frac{n}{2}-2}^{(1)})(x_{\frac{n}{2}}^{(1)}-x)}{8h^2}+\frac{(x_{\frac{n}{2}}^{(1)}-x)(x-x_{\frac{n}{2}-1}^{(1)})}{4h^2}\right]B_{\frac{n}{2}-1,0}^{(1)}(x)$
$B_{\frac{n}{2}+1}^{(1)}(x)=\frac{(x-x_{\frac{n}{2}-1}^{(1)})^2}{4h^2}B_{\frac{n}{2}-1,0}^{(1)}(x)$

$$+f(s_{\frac{n}{2}+1}^{(1)})\left[B_{\frac{n}{2}+1}^{(1)}-\frac{1}{32}B_{n-1}^{(0)}-\frac{9}{32}B_{n}^{(0)}-\frac{22}{32}B_{n+1}^{(0)}\right]+f(s_{\frac{n}{2}+2}^{(1)})\left[B_{\frac{n}{2}+2}^{(1)}-\frac{1}{32}B_{n+1}^{(0)}\right].$$

In case of partition (2), where $s_0^{(1)}=a$, $s_{\frac{n}{2}+1}^{(1)}=b$ and $s_i^{(1)}$, $i=1,\dots,\frac{n}{2}$ is given by (7), taking into account Table 2, by a similar argument we get

$$\begin{aligned}
S^{1L}f &= \sum_{i=0}^{n+1} f(s_i)B_i^{(0)}+f(s_0^{(1)})\left[B_0^{(1)}-B_0^{(0)}-\frac{9}{16}B_1^{(0)}-\frac{1}{16}B_2^{(0)}\right]\\
&+f(s_1^{(1)})\left[B_1^{(1)}-\frac{13}{32}B_1^{(0)}-\frac{21}{32}B_2^{(0)}-\frac{9}{32}B_3^{(0)}-\frac{1}{32}B_4^{(0)}\right]\\
&+\sum_{i=2}^{\frac{n}{2}-1} f(s_i^{(1)})\left[B_i^{(1)}-\frac{1}{32}B_{2i-3}^{(0)}-\frac{9}{32}B_{2i-2}^{(0)}-\frac{22}{32}B_{2i-1}^{(0)}-\frac{22}{32}B_{2i}^{(0)}-\frac{9}{32}B_{2i+1}^{(0)}-\frac{1}{32}B_{2i+2}^{(0)}\right]\\
&+f(s_{\frac{n}{2}}^{(1)})\left[B_{\frac{n}{2}}^{(1)}-\frac{1}{32}B_{n-3}^{(0)}-\frac{9}{32}B_{n-2}^{(0)}-\frac{21}{32}B_{n-1}^{(0)}-\frac{13}{32}B_{n}^{(0)}\right]\\
&+f(s_{\frac{n}{2}+1}^{(1)})\left[B_{\frac{n}{2}+1}^{(1)}-\frac{1}{16}B_{n-1}^{(0)}-\frac{9}{16}B_{n}^{(0)}-B_{n+1}^{(0)}\right].
\end{aligned} \tag{14}$$

3.2. The three-level variation-diminishing Schoenberg-Marsden operator S^{2L}

Similarly to the bilevel operator, we introduce a positive integer n divisible by 2^2, a partition of type (1) and $\Gamma^{(2)}=\{s_i^{(2)},\quad i=-1,\dots,\frac{n}{2^2}+2\}$ where

$$s_i^{(2)}=\frac{x_{i-1}^{(2)}+x_i^{(2)}}{2}=\frac{x_{4i-4}+x_{4i}}{2},\quad i=-1,\dots,\frac{n}{4}+2 \tag{15}$$

with $x_i^{(2)}=x_{2^2 i}$. Based on these new knots, we define

$$S^{(2)}f=\sum_{i=-1}^{\frac{n}{2^2}+2} f(s_i^{(2)})B_i^{(2)},$$

with $B_i^{(2)}(x)=B(\frac{n}{2^2}x-i+\frac{1}{2})$, $i=-1,\dots,\frac{n}{2^2}+2$ and the error function $\Delta_2 f=f-S^{(2)}f$. Then we compute

$$S^{(1)}\Delta_2 f=\sum_{i=-1}^{\frac{n}{2^1}+2}\Delta_2 f(s_i^{(1)})B_i^{(1)},$$

Table 3
B-splines $B_i^{(2)}$, $i = -1, \dots, \frac{n}{4} + 2$, on partitions (1).

$B_{-1,2}^{(2)}(x) = \frac{(x_0^{(2)}-x)^2}{32h^2} B_{-1,0}^{(2)}(x)$
$B_{0,2}^{(2)}(x) = \left[\frac{(x-x_{-2}^{(2)})(x_0^{(2)}-x)}{32h^2} + \frac{(x_1^{(2)}-x)(x-x_{-1}^{(2)})}{32h^2} \right] B_{-1,0}^{(2)}(x) + \frac{(x_1^{(2)}-x)^2}{32h^2} B_{0,0}^{(2)}(x)$
$B_{j,2}^{(2)}(x) = \frac{(x-x_{j-2}^{(2)})^2}{32h^2} B_{j-2,0}^{(2)}(x) + \left[\frac{(x-x_{j-2}^{(2)})(x_j^{(2)}-x)}{32h^2} + \frac{(x_{j+1}^{(2)}-x)(x-x_{j-1}^{(2)})}{32h^2} \right] B_{j-1,0}^{(2)}(x) + \frac{(x_{j+1}^{(2)}-x)^2}{32h^2} B_{j,0}^{(2)}(x)$, $j = 1, \dots, \frac{n}{4}$
$B_{\frac{n}{4}+1,2}^{(2)}(x) = \frac{(x-x_{\frac{n}{4}-1}^{(2)})^2}{32h^2} B_{\frac{n}{4}-1,0}^{(2)}(x) + \left[\frac{(x-x_{\frac{n}{4}-1}^{(2)})(x_{\frac{n}{4}+1}^{(2)}-x)}{32h^2} + \frac{(x_{\frac{n}{4}+1}^{(2)}-x)(x-x_{\frac{n}{4}}^{(2)})}{32h^2} \right] B_{\frac{n}{4},0}^{(2)}(x)$
$B_{\frac{n}{4}+2,2}^{(2)}(x) = \frac{(x-x_{\frac{n}{4}}^{(2)})^2}{32h^2} B_{\frac{n}{4},0}^{(2)}(x)$

Table 4
B-splines $B_i^{(2)}$, $i = 0, \dots, \frac{n}{4} + 1$, on partitions (2).

$B_{0,2}^{(2)}(x) = \frac{(x_1^{(2)}-x)^2}{16h^2} B_{0,0}^{(2)}(x)$
$B_{1,2}^{(2)}(x) = \left[\frac{(x-x_0^{(2)})(x_1^{(2)}-x)}{16h^2} + \frac{(x_2^{(2)}-x)(x-x_0^{(2)})}{32h^2} \right] B_{0,0}^{(2)}(x) + \frac{(x_2^{(2)}-x)^2}{32h^2} B_{1,0}^{(2)}(x)$
$B_{j,2}^{(2)}(x) = \frac{(x-x_{j-2}^{(2)})^2}{32h^2} B_{j-2,0}^{(2)}(x) + \left[\frac{(x-x_{j-2}^{(2)})(x_j^{(2)}-x)}{32h^2} + \frac{(x_{j+1}^{(2)}-x)(x-x_{j-1}^{(2)})}{32h^2} \right] B_{j-1,0}^{(2)}(x) + \frac{(x_{j+1}^{(2)}-x)^2}{32h^2} B_{j,0}^{(2)}(x)$, $j = 2, \dots, \frac{n}{4} - 1$
$B_{\frac{n}{4},2}^{(2)}(x) = \frac{(x-x_{\frac{n}{4}-2}^{(2)})^2}{32h^2} B_{\frac{n}{4}-2,0}^{(2)}(x) + \left[\frac{(x-x_{\frac{n}{4}-2}^{(2)})(x_{\frac{n}{4}}^{(2)}-x)}{32h^2} + \frac{(x_{\frac{n}{4}}^{(2)}-x)(x-x_{\frac{n}{4}-1}^{(2)})}{16h^2} \right] B_{\frac{n}{4}-1,0}^{(2)}(x)$
$B_{\frac{n}{4}+1}^{(2)}(x) = \frac{(x-x_{\frac{n}{4}-1}^{(2)})^2}{16h^2} B_{\frac{n}{4}-1,0}^{(2)}(x)$

and $\Delta_1^2 f = \Delta_1(\Delta_2 f) = \Delta_2 f - S^{(1)}\Delta_2 f$ to evaluate

$$S^{(0)}\Delta_1^2 f = \sum_{1=0}^{\frac{n}{2^0}+1} \Delta_1^2 f(s_i) B_i^{(0)}$$

in order to obtain the three-level spline function

$$S^{2L} f = S^{(2)} f + S^{(1)}\Delta_2 f + S^{(0)}\Delta_1^2 f$$

that can be also written as

$$S^{2L} f = S^{(2)} f + S^{(1)} f + S^{(0)} f - S^{(1)}S^{(2)} f - S^{(0)}S^{(2)} f - S^{(0)}S^{(1)} f + S^{(0)}S^{(1)}S^{(2)} f. \tag{16}$$

Then in case of partitions (1), taking into account Table 3, obtained from (3) and where it is underlined the degree of B-splines with

$$B_{i,0}^{(2)}(x) = \begin{cases} 1 & \text{if } x \in [x_{2^2 i}, x_{2^2 i+2^2}) \\ 0 & \text{otherwise,} \end{cases}$$

it is possible to give an explicit expression of (16), as reported in Appendix A.1.

In case of partitions (2), where $s_0^{(2)} = a$, $s_{\frac{n}{4}+1}^{(2)} = b$ and $s_i^{(2)}$ is given by (15), but with $i = 1, \dots, \frac{n}{2}$, taking into account Table 4, by a similar argument we get the expression reported in Appendix A.2.

3.3. *The p-level variation-diminishing Schoenberg-Marsden operator S^{pL}*

Even if it is not interesting from a computational point of view for increasing computational cost [3], from a theoretical one we are interested in giving a general expression of multilevel spline operators.

We define $\Gamma^{(p)} = \{s_i^{(p)}, \quad i = -1, \dots, \frac{n}{2^p} + 2\}$, where $x_i^{(p)} = x_{2^p i}$ and

$$s_i^{(p)} = \frac{x_{i-1}^{(p)} + x_i^{(p)}}{2} = \frac{x_{2^p i - 2^p} + x_{2^p i}}{2}, \quad i = -1, \dots, \frac{n}{2^p} + 2.$$

If n is not even, then we can only define base, but not multilevel, operators.

If n is even, then p is at least 1, so it is possible to define a bilevel operator.

If n is an even integer decomposed as $n = u \cdot 2^p$, with $u, p \in \mathbb{N}$ and u odd, then it could be apparently possible to construct a multilevel spline operator at most with $p+1$ levels. In fact, since the indices in the set $\Gamma^{(p)}$, $i = 0, \dots, \frac{n}{2^p} + 1$ must be positive integers, then the quotient $\frac{n}{2^p}$ must be an integer, i.e. n must be a multiple of 2^p. However for partitions of type either (1) or (2) it seems reasonable to stop to $p = 2$, and in general to p=degree of B-splines, otherwise too many extra knots to the extended partitions should be added.

On $\Gamma^{(p)}$ we define the QI spline

$$S^{(p)}f=\sum_{i=-1}^{\frac{n}{2^p}+2} f(s_i^{(p)})B_i^{(p)},$$

with $B_i^{(p)}(x)=B(\frac{n}{2^p}x-i+\frac{1}{2})$, $i=-1,\dots,\frac{n}{2^p}+2$.

The first error function is $\Delta_p f=f-S^{(p)}f$, in order to get

$$S^{(p-1)}\Delta_p f=\sum_{i=-1}^{\frac{n}{2^{p-1}}+2}\Delta_p f(s_i^{(p-1)})B_i^{(p-1)}.$$

Let now $\Delta^2_{p-1}f=\Delta_{p-1}(\Delta_p f)=\Delta_p f-S^{(p-1)}\Delta_p f$ be the error function related to $S^{(p-1)}\Delta_p f$ for which we compute

$$S^{(p-2)}\Delta^2_{p-1}f=\sum_{i=-1}^{\frac{n}{2^{p-2}}+2}\Delta^2_{p-1}f(s_i^{(p-2)})B_i^{(p-2)}.$$

Going on this way, we perform the last steps where the error function is $\Delta_2^{p-1}f=\Delta_2(\Delta_3^{p-2}f)=\Delta_3^{p-2}f-S^{(2)}\Delta_3^{p-2}f$ for the approximation

$$S^{(1)}\Delta_2^{p-1}f=\sum_{i=-1}^{\frac{n}{2^1}+2}\Delta_2^{p-1}f(s_i^{(1)})B_i^{(1)}$$

and finally $\Delta_1^p f=\Delta_1(\Delta_2^{p-1}f)=\Delta_2^{p-1}f-S^{(1)}\Delta_2^{p-1}f$ for

$$S^{(0)}\Delta_1^p f=\sum_{i=0}^{\frac{n}{2^0}+1}\Delta_1^p f(s_i)B_i^{(0)}.$$

Then the $(p+1)$-level QI spline is given by

$$S^{pL}f=S^{(p)}f+S^{(p-1)}\Delta_p f+S^{(p-2)}\Delta^2_{p-1}f+\cdots+S^{(2)}\Delta_3^{p-2}f+S^{(1)}\Delta_2^{p-1}f+S^{(0)}\Delta_1^p f$$

or equivalently

$$S^{pL}f=\sum_{k=1}^{p+1}\sum_{\substack{i_1,i_2,\dots,i_k=0\\ i_1<i_2<\cdots<i_k}}^{p}(-1)^{k+1}S^{(i_1)}S^{(i_2)}\dots S^{(i_k)}f.$$

Moreover by the definition of error functions simple computations show it is possible to write a QI spline with $p+1$ levels, depending on the one with p levels as follows

$$S^{pL}f=S^{(p)}f+S^{(p-1)L}\Delta_p f, \tag{17}$$

with $\Delta_p f=f-S^{(p)}f$.

In general the following results hold.

Theorem 3.1. *The operator S^{pL}, $p\geq 0$, reproduces linear polynomials, i.e. $S^{pL}f=f$ if $f\in\mathbb{P}_1$.*

Proof. If $f\in\mathbb{P}_1$, as S reproduces $\mathbb{P}_1$, then $S^{(p)}f=f$. So $\Delta_p f(x)=0$, $\forall x$, and also $S^{(p-1)}\Delta_p f(x)=0$, $\forall x$. Therefore the error function related to $S^{(p-1)}\Delta_p f$, $\Delta^2_{p-1}f(x)=0$, $\forall x$. Thus also $S^{(p-2)}\Delta^2_{p-1}f(x)=0$, $\forall x$. Going on this way we see that from $p+1$ function terms, defining $S^{pL}f$, only the first term, i.e. $S^{(p)}f$ is non zero. So it results $S^{pL}f(x)=S^{(p)}f(x)=f(x)$, $\forall x$. $\square$

Theorem 3.2. *If $f\in C^\mu(\bar{I})$, $\mu=1,2$, then S^{pL}, $p\geq 0$, satisfies at least $\|f-S^{pL}f\|_I=O(h^\mu)$, $\quad\mu=1,2$.*

Proof. If $f\in C^\mu(\bar{I})$, $\mu=1,2$, then from Theorem 2.1 and the norm of S

$$\begin{aligned}\|f-S^{pL}f\|_I&=\|f-S^{(p)}f-S^{(p-1)}\Delta_p f-S^{(p-2)}\Delta^2_{p-1}f-S^{(p-3)}\Delta^3_{p-2}f-\cdots-S^{(2)}\Delta_3^{p-2}f-S^{(1)}\Delta_2^{p-1}f-S^{(0)}\Delta_1^p f\|_I\\ &=\|\Delta_p f-S^{(p-1)}\Delta_p f-S^{(p-2)}\Delta^2_{p-1}f-S^{(p-3)}\Delta^3_{p-2}f-\cdots-S^{(2)}\Delta_3^{p-2}f-S^{(1)}\Delta_2^{p-1}f-S^{(0)}\Delta_1^p f\|_I\\ &=\|\Delta^2_{p-1}f-S^{(p-2)}\Delta^2_{p-1}f-S^{(p-3)}\Delta^3_{p-2}f-\cdots-S^{(2)}\Delta_3^{p-2}f-S^{(1)}\Delta_2^{p-1}f-S^{(0)}\Delta_1^p f\|_I\\ &=\|\Delta^3_{p-2}f-S^{(p-3)}\Delta^3_{p-2}f-\cdots-S^{(2)}\Delta_3^{p-2}f-S^{(1)}\Delta_2^{p-1}f-S^{(0)}\Delta_1^p f\|_I.\end{aligned} \tag{18}$$

Iterating this procedure of collecting the first two terms of the sum in norm by the corresponding error function, we write (18) as $\|f-S^{pL}f\|_I=\|\Delta_1^p f-S^{(0)}\Delta_1^p f\|_I=\|(1-S^{(0)})\Delta_1^p f\|_I$. Now step after step we use the definition of error function:

$$\|f-S^{pL}f\|_I=\|(1-S^{(0)})\Delta_1^p f\|_I=\|(1-S^{(0)})(\Delta_2^{p-1}f-S^{(1)}\Delta_2^{p-1}f)\|_I=\|(1-S^{(0)})(1-S^{(1)})\Delta_2^{p-1}f\|_I,$$

obtaining

$$\|f - S^{pL}f\|_I = \|(1 - S^{(0)})(1 - S^{(1)})(1 - S^{(2)})\dots(1 - S^{(p-2)})(1 - S^{(p-1)})\Delta_p f\|_I$$

$$= \|(1 - S^{(0)})(1 - S^{(1)})(1 - S^{(2)})\dots(1 - S^{(p-2)})(1 - S^{(p-1)})(f - S^{(p)}f)\|_I$$

$$\le (1 + \|S^{(0)}\|)(1 + \|S^{(1)}\|)(1 + \|S^{(2)}\|)\dots(1 + \|S^{(p-2)}\|)(1 + \|S^{(p-1)}\|)\|f - S^{(p)}f\|_I = O(h^\mu),\ \ \mu = 1, 2. \quad \square$$

However partitions (1) have an unexpected effect on the multilevel operator S^{pL}, $p > 0$, stated in the following theorem.

Theorem 3.3. *On partitions (1) the operator S^{pL} reproduces $\mathbb{P}_2$, i.e. $S^{pL}f = f$ if $f \in \mathbb{P}_2$.*

Proof. We prove by induction. We start by showing that S^{1L} reproduces $\mathbb{P}_2$.

Indeed let us consider the function $f(x) = x^2$ and show $S^{1L}f = f$ on $[x_i, x_{i+1}]$, for any i. Let us evaluate (11) at x_i:

$$S^{1L}f(x_i) = S^{(1)}f(x_i) + S^{(0)}f(x_i) - S^{(0)}S^{(1)}f(x_i)$$

$$= \sum_{j=-1}^{\frac{n}{2}+2} f(s_j^{(1)})B_j^{(1)}(x_i) + \sum_{j=0}^{n+1} f(s_j)B_j^{(0)}(x_i) - \sum_{j=0}^{n+1}\sum_{k=-1}^{\frac{n}{2}+2} f(s_k^{(1)})B_k^{(1)}(s_j)B_j^{(0)}(x_i).$$

In computing $S^{(1)}f(x_i)$ only B-splines $B_j^{(1)}$, whose support contains x_i, are involved: $B_{\frac{i}{2}}^{(1)}$ with support $[x_{\frac{i}{2}-2}^{(1)}, x_{\frac{i}{2}+1}^{(1)}] = [x_{i-4}, x_{i+2}]$, $B_{\frac{i}{2}+1}^{(1)}$ with support $[x_{\frac{i}{2}-1}^{(1)}, x_{\frac{i}{2}+2}^{(1)}] = [x_{i-2}, x_{i+4}]$, $B_{\frac{i}{2}+2}^{(1)}$ with support $[x_{\frac{i}{2}}^{(1)}, x_{\frac{i}{2}+3}^{(1)}] = [x_i, x_{i+6}]$.

From the definition of $B_{j,2}^{(1)}$ in Table 1 we have

$$B_{\frac{i}{2}}^{(1)}(x_i) = \frac{1}{2}, \quad B_{\frac{i}{2}+1}^{(1)}(x_i) = \frac{1}{2} \quad \text{and} \quad B_{\frac{i}{2}+2}^{(1)}(x_i) = 0.$$

Therefore

$$S^{(1)}f(x_i) = \frac{1}{2}f(s_{\frac{i}{2}}^{(1)}) + \frac{1}{2}f(s_{\frac{i}{2}+1}^{(1)}).$$

In computing $S^{(0)}f(x_i)$ only B-splines $B_j^{(0)}$, whose support contains x_i, are involved: $B_i^{(0)}$ with support $[x_{i-2}, x_{i+1}]$, $B_{i+1}^{(0)}$ with support $[x_{i-1}, x_{i+2}]$, $B_{i+2}^{(0)}$ with support $[x_i, x_{i+3}]$.

From Cox-de Boor recurrence formula (3) it results

$$B_i^{(0)}(x_i) = \frac{1}{2}, \quad B_{i+1}^{(0)}(x_i) = \frac{1}{2} \quad \text{and} \quad B_{i+2}^{(0)}(x_i) = 0.$$

and then

$$S^{(0)}f(x_i) = \frac{1}{2}f(s_i) + \frac{1}{2}f(s_{i+1}).$$

In computing $S^{(0)}S^{(1)}f(x_i)$ only B-splines $B_j^{(0)}$, whose support contains x_i, are involved, i.e. $B_i^{(0)}(x_i)$, $B_{i+1}^{(0)}(x_i)$, $B_{i+2}^{(0)}(x_i) = 0$, already computed above.

So it follows

$$S^{(0)}S^{(1)}f(x_i) = \frac{1}{2}\sum_{k=-1}^{\frac{n}{2}+2} f(s_k^{(1)})B_k^{(1)}(s_i) + \frac{1}{2}\sum_{k=-1}^{\frac{n}{2}+2} f(s_k^{(1)})B_k^{(1)}(s_{i+1}).$$

Now let us check which B-spline $B_k^{(1)}$ supports contain s_i and which s_{i+1}.

For s_i we have $B_{\frac{i}{2}-1}^{(1)}$ with support $[x_{\frac{i}{2}-3}^{(1)}, x_{\frac{i}{2}}^{(1)}] = [x_{i-6}, x_i]$, $B_{\frac{i}{2}}^{(1)}$ with support $[x_{\frac{i}{2}-2}^{(1)}, x_{\frac{i}{2}+1}^{(1)}] = [x_{i-4}, x_{i+2}]$, $B_{\frac{i}{2}+1}^{(1)}$ with support $[x_{\frac{i}{2}-1}^{(1)}, x_{\frac{i}{2}+2}^{(1)}] = [x_{i-2}, x_{i+4}]$ and from Table 1 their values are

$$B_{\frac{i}{2}-1}^{(1)}(s_i) = \frac{1}{32}, \quad B_{\frac{i}{2}}^{(1)}(s_i) = \frac{22}{32} \quad \text{and} \quad B_{\frac{i}{2}+1}^{(1)}(s_i) = \frac{9}{32}.$$

For s_{i+1} we have $B_{\frac{i}{2}}^{(1)}$ whose support is $[x_{\frac{i}{2}-2}^{(1)}, x_{\frac{i}{2}+1}^{(1)}) = [x_{i-4}, x_{i+2})$, $B_{\frac{i}{2}+1}^{(1)}$ whose support is $[x_{\frac{i}{2}-1}^{(1)}, x_{\frac{i}{2}+2}^{(1)}) = [x_{i-2}, x_{i+4})$, $B_{\frac{i}{2}+2}^{(1)}$ whose support is $[x_{\frac{i}{2}}^{(1)}, x_{\frac{i}{2}+3}^{(1)}) = [x_i, x_{i+6})$ and from Table 1 their values are

$$B_{\frac{i}{2}}^{(1)}(s_{i+1}) = \frac{9}{32}, \quad B_{\frac{i}{2}+1}^{(1)}(s_{i+1}) = \frac{22}{32} \quad \text{and} \quad B_{\frac{i}{2}+2}^{(1)}(s_{i+1}) = \frac{1}{32}.$$

Therefore

$$S^{(0)}S^{(1)}f(x_i) = \frac{1}{2}\left[\frac{1}{32}f(s_{\frac{i}{2}-1}^{(1)}) + \frac{22}{32}f(s_{\frac{i}{2}}^{(1)}) + \frac{9}{32}f(s_{\frac{i}{2}+1}^{(1)})\right] + \frac{1}{2}\left[\frac{9}{32}f(s_{\frac{i}{2}}^{(1)}) + \frac{22}{32}f(s_{\frac{i}{2}+1}^{(1)}) + \frac{1}{32}f(s_{\frac{i}{2}+2}^{(1)})\right]$$

$$= \frac{1}{64} f(s^{(1)}_{\frac{i}{2}-1}) + \frac{31}{64} f(s^{(1)}_{\frac{i}{2}}) + \frac{31}{64} f(s^{(1)}_{\frac{i}{2}+1}) + \frac{1}{64} f(s^{(1)}_{\frac{i}{2}+2}).$$

Finally we obtain

$$S^{1L} f(x_i) = \frac{1}{2} f(s^{(1)}_{\frac{i}{2}}) + \frac{1}{2} f(s^{(1)}_{\frac{i}{2}+1}) + \frac{1}{2} f(s_i) + \frac{1}{2} f(s_{i+1}) - \frac{1}{64} f(s^{(1)}_{\frac{i}{2}-1}) - \frac{31}{64} f(s^{(1)}_{\frac{i}{2}}) - \frac{31}{64} f(s^{(1)}_{\frac{i}{2}+1}) - \frac{1}{64} f(s^{(1)}_{\frac{i}{2}+2})$$

$$= \frac{1}{2} f(s_i) + \frac{1}{2} f(s_{i+1}) - \frac{1}{64} f(s^{(1)}_{\frac{i}{2}-1}) + \frac{1}{64} f(s^{(1)}_{\frac{i}{2}}) + \frac{1}{64} f(s^{(1)}_{\frac{i}{2}+1}) - \frac{1}{64} f(s^{(1)}_{\frac{i}{2}+2}),$$

where $s^{(1)}_{\frac{i}{2}-1} = x_{i-3}$, $s^{(1)}_{\frac{i}{2}} = x_{i-1}$, $s^{(1)}_{\frac{i}{2}+1} = x_{i+1}$, $s^{(1)}_{\frac{i}{2}+2} = x_{i+3}$.

Having $f(x) = x^2$, it results

$$S^{1L} f(x_i) = \frac{1}{2}\left(\frac{x_{i-1}+x_i}{2}\right)^2 + \frac{1}{2}\left(\frac{x_i+x_{i+1}}{2}\right)^2 - \frac{1}{64}(x_{i-3})^2 + \frac{1}{64}(x_{i-1})^2 + \frac{1}{64}(x_{i+1})^2 - \frac{1}{64}(x_{i+3})^2$$

and, since $x_i = a + ih$,

$$S^{1L} f(x_i) = \frac{1}{2}\left(a + \frac{2i-1}{2}h\right)^2 + \frac{1}{2}\left(a + \frac{2i+1}{2}h\right)^2 - \frac{1}{64}(a+(i-3)h)^2$$

$$+\frac{1}{64}(a+(i-1)h)^2 + \frac{1}{64}(a+(i+1)h)^2 - \frac{1}{64}(a+(i+3)h)^2$$

that after some computations provides $S^{1L} f(x_i) = a^2 + i^2h^2 + 2aih = (a+ih)^2 = f(x_i)$. Similar argument can be carried out for x_{i+1} and s_{i+1} and we get

$$S^{1L} f(x_{i+1}) = a^2 + i^2h^2 + h^2 + 2ih^2 + 2aih + 2ah = \left(a + (i+1)h\right)^2 = f(x_{i+1}),$$

$$S^{1L} f(s_{i+1}) = a^2 + i^2h^2 + \frac{1}{4}h^2 + ih^2 + 2aih + ah = \left(a + \frac{2i+1}{2}h\right)^2 = f(s_{i+1}).$$

Now let us assume $S^{(p-1)L}$ reproduces $\mathbb{P}_2$, i.e. $S^{(p-1)L} f = f$ if $f \in \mathbb{P}_2$. Let us consider $f(x) = x^2$ and show $S^{pL} f(x) = f(x)$ at x_i, s_{i+1}, x_{i+1}, for any i. From (17) we have at x_i:

$$S^{pL} f(x_i) = S^{(p)} f(x_i) + S^{(p-1)L} \Delta_p f(x_i) = S^{(p)} f(x_i) + S^{(p-1)L} f(x_i) - S^{(p-1)L} S^{(p)} f(x_i).$$

Compute

$$S^{(p)} f(x_i) = \sum_{j=-1}^{\frac{n}{2^p}+2} f(s^{(p)}_j) B^{(p)}_j(x_i),$$

involving B-splines $B^{(p)}_j$ whose support contains x_i, i.e. $B^{(p)}_{\frac{i}{2^p}}$ with support $[x^{(p)}_{\frac{i}{2^p}-2}, x^{(p)}_{\frac{i}{2^p}+1}] = [x_{i-2^{p+1}}, x_{i+2^p}]$, $B^{(p)}_{\frac{i}{2^p}+1}$ with support $[x^{(p)}_{\frac{i}{2^p}-1}, x^{(p)}_{\frac{i}{2^p}+2}] = [x_{i-2^p}, x_{i+2^{p+1}}]$, $B^{(p)}_{\frac{i}{2^p}+2}$ with support $[x^{(p)}_{\frac{i}{2^p}}, x^{(p)}_{\frac{i}{2^p}+3}] = [x_i, x_{i+3\cdot 2^p}]$. From the expression of $B^{(p)}_{j,2}$, similar to the ones in Tables 1 and 3, we obtain

$$B^{(p)}_{\frac{i}{2^p}}(x_i) = \frac{1}{2}, \quad B^{(p)}_{\frac{i}{2^p}+1}(x_i) = \frac{1}{2} \quad \text{and} \quad B^{(p)}_{\frac{i}{2^p}+2}(x_i) = 0.$$

Then

$$S^{(p)} f(x_i) = \frac{1}{2} f(s^{(p)}_{\frac{i}{2^p}}) + \frac{1}{2} f(s^{(p)}_{\frac{i}{2^p}+1}),$$

where $s^{(p)}_{\frac{i}{2^p}} = x_{i-2^{p-1}}$, $s^{(p)}_{\frac{i}{2^p}+1} = x_{i+2^{p-1}}$, so becoming

$$S^{(p)} f(x_i) = \frac{1}{2}(x_{i-2^{p-1}})^2 + \frac{1}{2}(x_{i+2^{p-1}})^2.$$

It follows

$$S^{(p-1)L} S^{(p)} f(x_i) = S^{(p-1)L}\left(\frac{1}{2}(x_{i-2^{p-1}})^2 + \frac{1}{2}(x_{i+2^{p-1}})^2\right) = \frac{1}{2}(x_{i-2^{p-1}})^2 + \frac{1}{2}(x_{i+2^{p-1}})^2.$$

Then

$$S^{pL} f(x_i) = \frac{1}{2}(x_{i-2^{p-1}})^2 + \frac{1}{2}(x_{i+2^{p-1}})^2 + x_i^2 - \frac{1}{2}(x_{i-2^{p-1}})^2 - \frac{1}{2}(x_{i+2^{p-1}})^2 = x_i^2 = f(x_i).$$

Similar argument for x_{i+1} and s_{i+1}, obtaining

$$S^{pL} f(x_{i+1}) = f(x_{i+1}), \quad S^{pL} f(s_{i+1}) = f(s_{i+1}),$$

respectively. □

Table 5
Weights Ω_i, $i=-1,\dots,\frac{n}{2}+2$ in quadrature formula (19), where Ω_i, $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2$, are obtained by symmetry.

i	Ω_i
-1	$\omega_{-1}^{(1)}-\frac{1}{32}\omega_0$
0	$\omega_0^{(1)}-\frac{22}{32}\omega_0-\frac{9}{32}\omega_1-\frac{1}{32}\omega_2$
1	$\omega_1^{(1)}-\frac{9}{32}\omega_0-\frac{22}{32}\omega_1-\frac{22}{32}\omega_2-\frac{9}{32}\omega_3-\frac{1}{32}\omega_4$
$i=2,\dots,\frac{n}{2}-1$	$\omega_i^{(1)}-\frac{1}{32}\omega_{2i-3}-\frac{9}{32}\omega_{2i-2}-\frac{22}{32}\omega_{2i-1}-\frac{22}{32}\omega_{2i}-\frac{9}{32}\omega_{2i+1}-\frac{1}{32}\omega_{2i+2}$

Theorem 3.4. *If $f\in C^\mu(\overline{I})$, $\mu=1,2,3$, then S^{pL}, $p>0$, satisfies at least $\|f-S^{pL}f\|_I=O(h^\mu)$, $\quad\mu=1,2,3$.*

Proof. The proof follows the same algebraic computations carried out in Theorem 3.2. □

Theorem 3.3 shows that S^{pL} gains in polynomial reproduction in case of partitions (1) and we provide a numerical confirmation for the test function f_3 in Table 6. In this case data points outside $[a,b]$ are needed. However either they could be not known or f could be not defined outside $[a,b]$. For partitions (2) all evaluation points are either inside $[a,b]$ or at the endpoints, but in this case S^{pL} only reproduces linear polynomials. However, when n increases, partitions (1) and (2) tend to coincide so that S^{pL} with partitions (2) 'tends to reproduce' $\mathbb{P}_2$ (see also [2,3]).

4. Numerical integration based on S^{pL}

Approximating the integral $\mathcal{I}(f)=\int_a^b f$ of a function f on an interval $[a,b]$ by quadrature formulas based on QI splines is a well known topic in literature (see for example the wide bibliography in [18]). For the operator S we have

$$\mathcal{I}_S(f)=\sum_{i=0}^{n+1}f(s_i)\int_a^b B_i.$$

Since $\omega_i=\int_a^b B_i=\frac{x_{i+1}-x_{i-d}}{d+1}$, $i=0,\dots,n+1$, for degree $d=2$ we have $\omega_0=\omega_{n+1}=\frac{h}{3}$, $\omega_1=\omega_n=2\frac{h}{3}$, $\omega_j=h$, $j=2,\dots,n-1$ with $\omega_{n-i+1}=\omega_i$, $i=0,\dots,n+1$ in case of symmetric partitions. Then

$$\mathcal{I}_S(f)=\sum_{i=0}^{n+1}\omega_i f(s_i)=\frac{h}{3}\left[f(s_0)+f(s_{n+1})\right]+\frac{2h}{3}\left[f(s_1)+f(s_n)\right]+h\sum_{i=2}^{n-1}f(s_i).$$

4.1. Numerical integration based on S^{1L}

A quadrature formula $\mathcal{I}_{S^{1L}}(f)$ associated to S^{1L} is obtained by computing $\omega_i^{(1)}=\int_a^b B_i^{(1)}=\frac{x_{i+1}^{(1)}-x_{i-2}^{(1)}}{3}$, $i=-1,\dots,\frac{n}{2}+2$, where $x_i^{(1)}=x_{2^1 i}$, $h^{(1)}=2h$ and again $\omega^{(1)}_{\frac{n}{2}-i+1}=\omega_i^{(1)}$. They are: $\omega_{-1}^{(1)}=\omega^{(1)}_{\frac{n}{2}+2}=0$, $\omega_0^{(1)}=\omega^{(1)}_{\frac{n}{2}+1}=2\frac{h}{3}$, $\omega_1^{(1)}=\omega^{(1)}_{\frac{n}{2}}=4\frac{h}{3}$, $\omega_i^{(1)}=2h$, $i=2,\dots,\frac{n}{2}-1$.

In case of partitions (1) from (13) we can write

$$\mathcal{I}_{S^{1L}}(f)=\sum_{i=0}^{n+1}\omega_i f(s_i)+\sum_{i=-1}^{\frac{n}{2}+2}\Omega_i f(s_i^{(1)}) \tag{19}$$

with Ω_i given in Table 5.

Now, taking into account the symmetries of ω_i and $\omega_i^{(1)}$, we get

$$\begin{aligned}\mathcal{I}_{S^{1L}}(f)=&\frac{h}{3}\left[f(s_0)+f(s_{n+1})\right]+\frac{2h}{3}\left[f(s_1)+f(s_n)\right]+h\sum_{i=2}^{n-1}f(s_i)\\&-\frac{h}{96}\left[f(s_{-1}^{(1)})+f(s^{(1)}_{\frac{n}{2}+2})\right]+\frac{7h}{32}\left[f(s_0^{(1)})+f(s^{(1)}_{\frac{n}{2}+1})\right]-\frac{7h}{32}\left[f(s_1^{(1)})+f(s^{(1)}_{\frac{n}{2}})\right]+\frac{h}{96}\left[f(s_2^{(1)})+f(s^{(1)}_{\frac{n}{2}-1})\right].\end{aligned}$$

In case of partitions (2) from (14) we can write

$$\mathcal{I}_{S^{1L}}(f)=\sum_{i=0}^{n+1}\omega_i f(s_i)+\sum_{i=0}^{\frac{n}{2}+1}\Omega_i f(s_i^{(1)})$$

where Ω_i are the ones of Table 5, except for

$$\Omega_0=\omega_0^{(1)}-\omega_0-\frac{9}{16}\omega_1-\frac{1}{16}\omega_2,\qquad \Omega_1=\omega_1^{(1)}-\frac{13}{32}\omega_1-\frac{21}{32}\omega_2-\frac{9}{32}\omega_3-\frac{1}{32}\omega_4,$$

$$\Omega_{\frac{n}{2}}=\omega^{(1)}_{\frac{n}{2}}-\frac{1}{32}\omega_{n-3}-\frac{9}{32}\omega_{n-2}-\frac{21}{32}\omega_{n-1}-\frac{13}{32}\omega_n,\qquad \Omega_{\frac{n}{2}+1}=\omega^{(1)}_{\frac{n}{2}+1}-\frac{1}{16}\omega_{n-1}-\frac{9}{16}\omega_n-\omega_{n+1}.$$

Now, taking into account the symmetries of ω_i and $\omega_i^{(1)}$, we get

$$\mathcal{I}_{S^{1L}}(f)=\frac{h}{3}\left[f(s_0)+f(s_{n+1})\right]+\frac{2h}{3}\left[f(s_1)+f(s_n)\right]+h\sum_{i=2}^{n-1}f(s_i)$$
$$-\frac{5h}{48}\left[f(s_0^{(1)})+f(s^{(1)}_{\frac{n}{2}+1})\right]+\frac{3h}{32}\left[f(s_1^{(1)})+f(s^{(1)}_{\frac{n}{2}})\right]+\frac{h}{96}\left[f(s_2^{(1)})+f(s^{(1)}_{\frac{n}{2}-1})\right].$$

4.2. Numerical integration based on S^{2L}

In case of $p=2$ a quadrature formula associated to S^{2L} is obtained by computing $\omega_i^{(2)}=\int_a^b B_i^{(2)}=\frac{x_{i+1}^{(2)}-x_{i-2}^{(2)}}{3}$, $i=-1,\dots,\frac{n}{4}+2$, where $x_i^{(2)}=x_{2^2i}$, $h^{(2)}=2^2h$ with $\omega^{(2)}_{\frac{n}{4}-i+1}=\omega_i^{(2)}$, i.e.: $\omega_{-1}^{(2)}=\omega^{(2)}_{\frac{n}{4}+2}=0$, $\omega_0^{(2)}=\omega^{(2)}_{\frac{n}{4}+1}=4\frac{h}{3}$, $\omega_1^{(2)}=\omega^{(2)}_{\frac{n}{4}}=8\frac{h}{3}$, $\omega_i^{(2)}=4h$, $i=2,\dots,\frac{n}{4}-1$.

For partitions (1) from the expression of $S^{2L}f$ in Appendix A.1 we can write

$$\mathcal{I}_{S^{2L}}(f)=\sum_{i=0}^{n+1}f(s_i)\omega_i+\sum_{i=-1}^{\frac{n}{2}+2}f(s_i^{(1)})\Omega_i+\sum_{i=-1}^{\frac{n}{4}+2}f(s_i^{(2)})\Gamma_i \tag{20}$$

with Ω_i defined in the bilevel case and Γ_i given in Appendix A.3.

Now, taking into account all symmetries of ω_i, $\omega_i^{(1)}$ and $\omega_i^{(2)}$, it results

$$\Gamma_{-1}=\Gamma_{\frac{n}{4}+2}=-\frac{5}{768}h,\quad \Gamma_0=\Gamma_{\frac{n}{4}+1}=\frac{133}{256}h,\quad \Gamma_1=\Gamma_{\frac{n}{4}}=-\frac{133}{256}h,\quad \Gamma_2=\Gamma_{\frac{n}{4}-1}=\frac{5}{768}h,$$
$$\Gamma_{i+2}=0, i=1,\dots,\frac{n}{4}-4$$

from which we get

$$\mathcal{I}_{S^{2L}}(f)=\frac{h}{3}\left[f(s_0)+f(s_{n+1})\right]+\frac{2h}{3}\left[f(s_1)+f(s_n)\right]+h\sum_{i=2}^{n-1}f(s_i)$$
$$-\frac{h}{96}\left[f(s_{-1}^{(1)})+f(s^{(1)}_{\frac{n}{2}+2})\right]+\frac{7h}{32}\left[f(s_0^{(1)})+f(s^{(1)}_{\frac{n}{2}+1})\right]-\frac{7h}{32}\left[f(s_1^{(1)})+f(s^{(1)}_{\frac{n}{2}})\right]+\frac{h}{96}\left[f(s_2^{(1)})+f(s^{(1)}_{\frac{n}{2}-1})\right]$$
$$-\frac{5h}{768}\left[f(s_{-1}^{(2)})+f(s^{(2)}_{\frac{n}{4}+2})\right]+\frac{133h}{256}\left[f(s_0^{(2)})+f(s^{(2)}_{\frac{n}{4}+1})\right]-\frac{133h}{256}\left[f(s_1^{(2)})+f(s^{(2)}_{\frac{n}{4}})\right]+\frac{5h}{768}\left[f(s_2^{(2)})+f(s^{(2)}_{\frac{n}{4}-1})\right].$$

For partitions (2) from the expression of $S^{2L}f$ in Appendix A.2 we can write

$$\mathcal{I}_{S^{2L}}(f)=\sum_{i=0}^{n+1}f(s_i)\omega_i+\sum_{i=-1}^{\frac{n}{2}+2}f(s_i^{(1)})\Omega_i+\sum_{i=-1}^{\frac{n}{4}+2}f(s_i^{(2)})\Gamma_i \tag{21}$$

with Ω_i defined in the bilevel case and Γ_i given in Appendix A.4.

Now, taking into account all symmetries of ω_i, $\omega_i^{(1)}$ and $\omega_i^{(2)}$, it results

$$\Gamma_0=\Gamma_{\frac{n}{4}+1}=-\frac{7}{768}h,\ \Gamma_1=\Gamma_{\frac{n}{4}}=-\frac{5}{512}h,\ \Gamma_2=\Gamma_{\frac{n}{4}-1}=-\frac{5}{1536}h,\ \Gamma_{i+2}=0, i=1,\dots,\frac{n}{4}-4$$

from which we get

$$\mathcal{I}_{S^{2L}}(f)=\frac{h}{3}\left[f(s_0)+f(s_{n+1})\right]+\frac{2h}{3}\left[f(s_1)+f(s_n)\right]+h\sum_{i=2}^{n-1}f(s_i)$$
$$-\frac{5h}{48}\left[f(s_0^{(1)})+f(s^{(1)}_{\frac{n}{2}+1})\right]+\frac{3h}{32}\left[f(s_1^{(1)})+f(s^{(1)}_{\frac{n}{2}})\right]+\frac{h}{96}\left[f(s_2^{(1)})+f(s^{(1)}_{\frac{n}{2}-1})\right]$$
$$-\frac{5h}{768}\left[f(s_0^{(2)})+f(s^{(2)}_{\frac{n}{4}+1})\right]+\frac{5h}{512}\left[f(s_1^{(2)})+f(s^{(2)}_{\frac{n}{4}})\right]-\frac{5h}{1536}\left[f(s_2^{(2)})+f(s^{(2)}_{\frac{n}{4}-1})\right].$$

5. Numerical results

Now we present some numerical results, obtained both on approximation and numerical integration and based on the operators studied throughout this paper, applied to the following test functions:

- $f_1(x)=2x$, $x\in[0,1]$;
- $f_2(x)=\frac{1}{1+16x^5}$, $x\in[0,1]$;
- $f_3(x)=2x^2-5x+4$, $x\in[2,3.5]$;
- $f_4(x)=\frac{1}{3}e^{(-\frac{81}{16}(x-\frac{1}{2})^2)}$, $x\in[1.5,6]$;
- $f_5(x)=\sin(4.5x)$, $x\in[1.5,3]$;

Table 6
Maximum norm of error related to multilevel operators applied to test functions for partitions (1).

	n	E_S	$E_{S^{1L}}$	$E_{S^{2L}}$		E_S	$E_{S^{1L}}$	$E_{S^{2L}}$
f_1	12	6.66(−16)	1.11(−15)	1.33(−15)	f_5	3.93(−02)	9.53(−03)	1.11(−02)
	28	6.66(−16)	1.11(−15)	1.55(−15)		7.23(−03)	5.40(−04)	1.02(−03)
	56	4.44(−16)	8.88(−16)	1.11(−15)		1.82(−03)	6.18(−05)	1.34(−04)
	112	4.44(−16)	1.33(−15)	1.78(−15)		4.54(−04)	7.32(−06)	1.71(−05)
	224	4.44(−16)	1.11(−15)	1.33(−15)		1.14(−04)	9.32(−07)	2.15(−06)
f_2	12	8.17(−03)	2.51(−03)	2.82(−03)	f_6	1.42(−05)	1.45(−06)	3.09(−06)
	28	1.57(−03)	2.19(−04)	3.02(−04)		2.60(−06)	9.30(−08)	1.89(−07)
	56	3.95(−04)	2.20(−05)	4.19(−05)		6.49(−07)	1.15(−08)	2.42(−08)
	112	9.91(−05)	2.51(−06)	5.56(−06)		1.62(−07)	1.39(−09)	3.05(−09)
	224	2.48(−05)	3.07(−07)	6.95(−07)		4.06(−08)	1.68(−10)	3.83(−10)
f_3	12	7.81(−03)	5.32(−15)	8.88(−15)	f_7	2.73(−02)	2.67(−02)	2.67(−02)
	28	1.43(−03)	7.11(−15)	1.07(−14)		1.02(−02)	6.08(−03)	6.08(−03)
	56	3.58(−04)	8.88(−15)	1.24(−14)		2.61(−03)	7.61(−04)	6.59(−04)
	112	8.96(−05)	7.10(−15)	1.24(−14)		6.59(−04)	7.02(−05)	1.05(−04)
	224	2.24(−05)	1.07(−14)	1.07(−14)		1.66(−04)	7.14(−06)	1.56(−05)
f_4	12	3.92(−03)	1.34(−03)	2.05(−03)	f_8	2.97(−02)	2.66(−02)	2.61(−02)
	28	6.46(−04)	5.41(−04)	5.41(−04)		1.25(−02)	1.10(−02)	1.08(−02)
	56	1.58(−04)	3.97(−05)	4.89(−05)		5.93(−03)	5.13(−03)	5.13(−03)
	112	3.94(−05)	3.52(−06)	6.63(−06)		2.63(−03)	2.22(−03)	2.22(−03)
	224	9.84(−06)	4.04(−07)	8.41(−07)		1.00(−03)	7.98(−04)	7.98(−04)

Table 7
Observed approximation order related to multilevel operators applied to test functions for partitions (1).

	n	r_S	$r_{S^{1L}}$	$r_{S^{2L}}$		r_S	$r_{S^{1L}}$	$r_{S^{2L}}$
f_1	28	–	–	–	f_5	–	–	–
	56	–	–	–		1.99	3.13	2.93
	112	–	–	–		1.99	3.04	2.98
	224	–	–	–		1.99	3.01	2.99
f_2	28	–	–	–	f_6	–	–	–
	56	1.99	3.31	2.85		2.00	3.02	2.97
	112	1.99	3.13	2.91		2.00	3.04	2.98
	224	1.99	3.04	3.00		2.00	3.06	3.00
f_3	28	–	–	–	f_7	–	–	–
	56	2.00	–	–		1.96	3.00	3.21
	112	2.00	–	–		1.99	3.44	2.65
	224	2.00	–	–		1.99	3.30	2.76
f_4	28	–	–	–	f_8	–	–	–
	56	2.02	3.77	3.46		1.08	1.10	1.08
	112	2.00	3.49	2.88		1.17	1.21	1.21
	224	2.00	3.12	2.98		1.39	1.48	1.48

- $f_6(x) = \arctan(100(x-0.3)), \quad x \in [2,4]$;
- $f_7(x) = e^{-x}\sin(5\pi x), \quad x \in [2.8,5]$;
- $f_8(x) = \begin{cases} x^2, & 0 \le x < \frac{1}{2} \\ -\frac{x}{2}+\frac{1}{2} & \frac{1}{2} \le x \le 1 \end{cases}$.

5.1. Numerical results on approximation

For $Q = S, S^{1L}, S^{2L}$ we set $E_Q(f) = \|f - Q(f)\|_\infty$, evaluated on 500 equispaced points in $[a,b]$. Moreover we define the observed (or numerical) approximation order as follows:

$$r_Q = \log_2 \frac{E_Q|_{n=7\cdot 2^{l-1}}}{E_Q|_{n=7\cdot 2^l}} \quad \text{for} \quad Q = S, S^{1L}, S^{2L} \quad \text{and} \quad l = 3,4,5,$$

with $E_Q := E_Q(f)$, when it is not necessary to point out f.

As we can see in Tables 6–9, in general multilevel operators improve the results obtained with the corresponding base operators. We also remark that there are not big differences in the behaviour of S^{1L} e S^{2L}, so that, at least from a computational point of view, it is convenient to use S^{1L}, as already confirmed in [3]. Moreover $\mathbb{P}_1$ and $\mathbb{P}_2$ reproduction for multilevel operators with $p > 0$ on partitions (1) and (2) and on partitions (1), respectively, are confirmed. However we underline that in case of partitions (2) we note a sort of 'numerical reproduction' of $\mathbb{P}_2$ when n increases and partitions (1) and (2) tend to coincide (see also [2,3]).

Table 8
Maximum norm of error related to multilevel operators applied to test functions for partitions (2).

	n	E_S	$E_{S^{1L}}$	$E_{S^{2L}}$		E_S	$E_{S^{1L}}$	$E_{S^{2L}}$
f_1	12	4.44(−16)	6.66(−16)	1.11(−15)	f_5	3.93(−02)	1.17(−02)	1.42(−02)
	28	4.44(−16)	6.66(−16)	1.33(−15)		7.23(−03)	1.60(−03)	1.26(−03)
	56	4.44(−16)	8.88(−16)	1.33(−15)		1.82(−03)	3.73(−04)	1.84(−04)
	112	4.44(−16)	1.11(−15)	1.33(−15)		4.54(−04)	9.14(−05)	3.27(−05)
	224	4.44(−16)	1.11(−15)	1.33(−15)		1.13(−04)	2.27(−05)	6.83(−06)
f_2	12	8.17(−03)	2.51(−03)	2.80(−03)	f_6	1.07(−05)	3.42(−06)	2.20(−06)
	28	1.57(−03)	2.19(−04)	3.02(−04)		2.30(−06)	6.42(−07)	2.96(−07)
	56	3.95(−04)	2.21(−05)	4.19(−05)		6.10(−07)	1.61(−07)	5.90(−08)
	112	9.91(−05)	3.74(−06)	5.56(−06)		1.57(−07)	4.03(−08)	1.26(−08)
	224	2.48(−05)	9.18(−07)	6.95(−07)		3.97(−08)	1.01(−08)	2.84(−09)
f_3	12	7.81(−03)	1.95(−03)	4.88(−04)	f_7	2.60(−02)	2.56(−02)	2.57(−02)
	28	1.43(−03)	3.59(−04)	8.97(−05)		1.02(−02)	6.08(−03)	6.08(−03)
	56	3.59(−04)	8.97(−05)	2.24(−05)		2.61(−03)	8.15(−04)	6.53(−04)
	112	8.97(−05)	2.23(−05)	5.58(−06)		6.59(−04)	8.85(−05)	1.03(−04)
	224	2.24(−05)	5.55(−06)	1.39(−06)		1.66(−04)	1.11(−05)	1.42(−05)
f_4	12	5.23(−04)	4.59(−04)	4.55(−04)	f_8	2.97(−02)	2.66(−02)	2.61(−02)
	28	2.05(−04)	1.37(−04)	1.32(−04)		1.25(−02)	1.10(−02)	1.08(−02)
	56	8.36(−05)	3.87(−05)	3.41(−05)		5.93(−03)	5.13(−03)	5.13(−03)
	112	2.85(−05)	9.81(−06)	6.95(−06)		2.63(−03)	2.22(−03)	2.22(−03)
	224	8.34(−06)	2.44(−06)	1.25(−06)		1.00(−03)	7.98(−04)	7.98(−04)

Table 9
Observed approximation order related to multilevel operators applied to test functions for partitions (2).

	n	r_S	$r_{S^{1L}}$	$r_{S^{2L}}$		r_S	$r_{S^{1L}}$	$r_{S^{2L}}$
f_1	28	–	–	–	f_5	–	–	–
	56	–	–	–		1.99	2.10	2.77
	112	–	–	–		1.99	2.03	2.49
	224	–	–	–		1.99	2.01	2.26
f_2	28	–	–	–	f_6	–	–	–
	56	1.99	3.31	2.85		1.91	1.99	2.33
	112	1.99	2.56	2.91		1.96	2.00	2.23
	224	1.99	2.02	3.00		1.98	2.00	2.15
f_3	28	–	–	–	f_7	–	–	–
	56	2.00	2.00	2.00		1.96	2.90	3.22
	112	2.00	2.01	2.00		1.99	3.20	2.66
	224	1.99	2.00	2.00		1.99	3.00	2.86
f_4	28	–	–	–	f_8	–	–	–
	56	1.29	1.82	1.95		1.07	1.10	1.07
	112	1.55	1.98	2.30		1.17	1.21	1.21
	224	1.77	2.00	2.48		1.39	1.48	1.48

5.2. Numerical results on integration

For numerical integration we compare results obtained by quadratures $\mathcal{I}_S(f)$, $\mathcal{I}_{S^{1L}}(f)$, $\mathcal{I}_{S^{2L}}(f)$ to evaluate the integral of some functions whose exact value is known.

Let us set $E_Q(f) = \mathcal{I}(f) - \mathcal{I}_Q(f)$ and

$$r_Q = \log_2 \frac{|E_Q|_{n=2^{l-1}}}{|E_Q|_{n=2^l}} \quad \text{for} \quad Q = S, S^{1L}, S^{2L} \quad \text{with} \quad l = 8, 9, 10.$$

In the following tables (Tables 10–13) we approximate

$$\mathcal{I}(g_1) = \int_{-1}^{1} \frac{1}{1+16x^2}\,dx = \frac{1}{4}\left[\arctan(4) - \arctan(-4)\right] \approx 0.662908831834016,$$

$$\mathcal{I}(g_2) = \int_{-1}^{1} xe^x\,dx = \frac{2}{e} \approx 0.735758882342885,$$

$$\mathcal{I}(g_3) = \int_{0}^{1} |x^2 - 0.25|\,dx = \frac{1}{4},$$

$$\mathcal{I}(g_4) = \int_{0}^{1} e^{-x}\sin(5\pi x)\,dx = \frac{5\pi(e+1)}{e(25\pi^2+1)} \approx 0.086730404755780.$$

Table 10
Absolute errors for $\mathcal{I}(g_1)$ with partitions (2).

n	E_S	r_S	$E_{S^{1L}}$	$r_{S^{1L}}$	$E_{S^{2L}}$	$r_{S^{2L}}$
128	6.86(−06)		1.29(−07)		6.70(−06)	
256	1.70(−06)	2.00	1.57(−08)	3.00	1.56(−06)	2.10
512	4.24(−07)	2.00	1.93(−09)	3.02	3.77(−07)	2.04
1024	1.06(−07)	2.00	2.39(−10)	3.00	9.27(−08)	2.02

Table 11
Absolute errors for $\mathcal{I}(g_2)$ with partitions (2).

n	E_S	r_S	$E_{S^{1L}}$	$r_{S^{1L}}$	$E_{S^{2L}}$	$r_{S^{2L}}$
128	1.65(−04)		1.66(−06)		7.57(−05)	
256	4.13(−05)	1.99	2.10(−07)	2.98	1.96(−05)	1.95
512	1.03(−05)	2.00	2.63(−08)	2.99	4.99(−06)	2.00
1024	2.59(−06)	1.99	3.30(−09)	2.99	1.26(−06)	2.00

Table 12
Absolute errors for $\mathcal{I}(g_3)$ with partitions (2).

n	E_S	r_S	$E_{S^{1L}}$	$r_{S^{1L}}$	$E_{S^{2L}}$	$r_{S^{2L}}$
128	2.03(−05)		5.09(−06)		5.09(−06)	
256	5.09(−06)	1.99	1.27(−06)	2.00	1.27(−06)	2.00
512	1.27(−06)	2.00	3.18(−07)	2.00	3.18(−07)	2.00
1024	3.18(−07)	2.00	7.95(−08)	2.00	7.95(−08)	2.00

Table 13
Absolute errors for $\mathcal{I}(g_4)$ with partitions (2).

n	E_S	r_S	$E_{S^{1L}}$	$r_{S^{1L}}$	$E_{S^{2L}}$	$r_{S^{2L}}$
128	1.63(−04)		1.54(−06)		2.92(−04)	
256	4.09(−05)	1.99	1.27(−07)	3.60	4.36(−05)	2.74
512	1.02(−05)	2.00	1.18(−08)	3.42	6.98(−06)	2.64
1024	2.56(−06)	2.00	1.22(−09)	3.27	1.24(−06)	2.49

In conclusion, while for partitions (1) the results of base operator e multilevel operators on such test integrals are comparable, so that we do not report them here, in the above four tables we can see a good improvement in case of partitions (2), in particular using S^{1L}. Moreover the observed approximation order confirms the theoretical one, obtained by theorems on approximation, up to a multiplicative constant.

Data availability

Data will be made available on request.

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Appendix

A.1. $S^{2L}f$ On partitions (1)

$$S^{2L}f = \sum_{i=0}^{n+1} f(s_i)B_i^{(0)} + f(s_{-1}^{(1)})\Big[B_{-1}^{(1)} - \frac{1}{32}B_0^{(0)}\Big] + f(s_0^{(1)})\Big[B_0^{(1)} - \frac{22}{32}B_0^{(0)} - \frac{9}{32}B_1^{(0)} - \frac{1}{32}B_2^{(0)}\Big]$$

$$+ f(s_1^{(1)})\Big[B_1^{(1)} - \frac{9}{32}B_0^{(0)} - \frac{22}{32}B_1^{(0)} - \frac{22}{32}B_2^{(0)} - \frac{9}{32}B_3^{(0)} - \frac{1}{32}B_4^{(0)}\Big]$$

$$+ \sum_{i=2}^{\frac{n}{2}-1} f(s_i^{(1)})\Big[B_i^{(1)} - \frac{1}{32}B_{2i-3}^{(0)} - \frac{9}{32}B_{2i-2}^{(0)} - \frac{22}{32}B_{2i-1}^{(0)} - \frac{22}{32}B_{2i}^{(0)}$$

$$
\begin{aligned}
&-\frac{9}{32}B^{(0)}_{2i+1}-\frac{1}{32}B^{(0)}_{2i+2}\Big]+f(s^{(1)}_{\frac{n}{2}})\Big[B^{(1)}_{\frac{n}{2}}-\frac{1}{32}B^{(0)}_{n-3}-\frac{9}{32}B^{(0)}_{n-2}-\frac{22}{32}B^{(0)}_{n-1}-\frac{22}{32}B^{(0)}_{n}-\frac{9}{32}B^{(0)}_{n+1}\Big]\\
&+f(s^{(1)}_{\frac{n}{2}+1})\Big[B^{(1)}_{\frac{n}{2}+1}-\frac{1}{32}B^{(0)}_{n-1}-\frac{9}{32}B^{(0)}_{n}-\frac{22}{32}B^{(0)}_{n+1}\Big]\\
&+f(s^{(1)}_{\frac{n}{2}+2})\Big[B^{(1)}_{\frac{n}{2}+2}-\frac{1}{32}B^{(0)}_{n+1}\Big]+f(s^{(2)}_{-1})\Big[B^{(2)}_{-1}-\frac{9}{32}B^{(1)}_{-1}-\frac{1}{32}B^{(1)}_{0}+\frac{23}{1024}B^{(0)}_{0}+\frac{9}{1024}B^{(0)}_{1}+\frac{1}{1024}B^{(0)}_{2}\Big]\\
&+f(s^{(2)}_{0})\Big[B^{(2)}_{0}-\frac{22}{32}B^{(1)}_{-1}-\frac{22}{32}B^{(1)}_{0}-\frac{9}{32}B^{(1)}_{1}-\frac{1}{32}B^{(0)}_{2}-\frac{37}{1024}B^{(0)}_{0}\\
&+\frac{5}{1024}B^{(0)}_{1}+\frac{29}{1024}B^{(0)}_{2}+\frac{31}{1024}B^{(0)}_{3}+\frac{23}{1024}B^{(0)}_{4}+\frac{9}{1024}B^{(0)}_{5}+\frac{1}{1024}B^{(0)}_{6}\Big]\\
&+f(s^{(2)}_{1})\Big[B^{(2)}_{1}-\frac{1}{32}B^{(1)}_{-1}-\frac{9}{32}B^{(1)}_{0}-\frac{22}{32}B^{(1)}_{1}-\frac{22}{32}B^{(1)}_{2}-\frac{9}{32}B^{(1)}_{3}-\frac{1}{32}B^{(1)}_{4}\\
&+\frac{5}{1024}B^{(0)}_{0}-\frac{37}{1024}B^{(0)}_{1}-\frac{61}{1024}B^{(0)}_{2}-\frac{61}{1024}B^{(0)}_{3}-\frac{37}{1024}B^{(0)}_{4}+\frac{5}{1024}B^{(0)}_{5}\\
&+\frac{29}{1024}B^{(0)}_{6}+\frac{31}{1024}B^{(0)}_{7}+\frac{23}{1024}B^{(0)}_{8}+\frac{9}{1024}B^{(0)}_{9}+\frac{1}{1024}B^{(0)}_{10}\Big]\\
&+f(s^{(2)}_{2})\Big[B^{(2)}_{2}-\frac{1}{32}B^{(1)}_{1}-\frac{9}{32}B^{(1)}_{2}-\frac{22}{32}B^{(1)}_{3}-\frac{22}{32}B^{(1)}_{4}-\frac{9}{32}B^{(1)}_{5}\\
&-\frac{1}{32}B^{(1)}_{6}+\frac{9}{1024}B^{(0)}_{0}+\frac{23}{1024}B^{(0)}_{1}+\frac{31}{1024}B^{(0)}_{2}+\frac{29}{1024}B^{(0)}_{3}+\frac{5}{1024}B^{(0)}_{4}\\
&-\frac{37}{1024}B^{(0)}_{5}-\frac{61}{1024}B^{(0)}_{6}-\frac{61}{1024}B^{(0)}_{7}-\frac{37}{1024}B^{(0)}_{8}+\frac{5}{1024}B^{(0)}_{9}\\
&+\frac{29}{1024}B^{(0)}_{10}+\frac{31}{1024}B^{(0)}_{11}+\frac{23}{1024}B^{(0)}_{12}+\frac{9}{1024}B^{(0)}_{13}+\frac{1}{1024}B^{(0)}_{14}\Big]\\
&+\sum_{k=1}^{\frac{n}{4}-4}f(s^{(2)}_{k+2})\Big[B^{(2)}_{k+2}-\frac{1}{32}B^{(1)}_{2k+1}-\frac{9}{32}B^{(1)}_{2k+2}-\frac{22}{32}B^{(1)}_{2k+3}-\frac{22}{32}B^{(1)}_{2k+4}\\
&-\frac{9}{32}B^{(1)}_{2k+5}-\frac{1}{32}B^{(1)}_{2k+6}+\frac{1}{1024}B^{(0)}_{4k-1}+\frac{9}{1024}B^{(0)}_{4k}+\frac{23}{1024}B^{(0)}_{4k+1}\\
&+\frac{31}{1024}B^{(0)}_{4k+2}+\frac{29}{1024}B^{(0)}_{4k+3}+\frac{5}{1024}B^{(0)}_{4k+4}-\frac{37}{1024}B^{(0)}_{4k+5}-\frac{61}{1024}B^{(0)}_{4k+6}\\
&-\frac{61}{1024}B^{(0)}_{4k+7}-\frac{37}{1024}B^{(0)}_{4k+8}+\frac{5}{1024}B^{(0)}_{4k+9}+\frac{29}{1024}B^{(0)}_{4k+10}\\
&+\frac{31}{1024}B^{(0)}_{4k+11}+\frac{23}{1024}B^{(0)}_{4k+12}+\frac{9}{1024}B^{(0)}_{4k+13}+\frac{1}{1024}B^{(0)}_{4k+14}\Big]\\
&+f(s^{(2)}_{\frac{n}{4}-1})\Big[B^{(2)}_{\frac{n}{4}-1}-\frac{1}{32}B^{(1)}_{\frac{n}{2}-5}-\frac{9}{32}B^{(1)}_{\frac{n}{2}-4}-\frac{22}{32}B^{(1)}_{\frac{n}{2}-3}-\frac{22}{32}B^{(1)}_{\frac{n}{2}-2}\\
&-\frac{9}{32}B^{(1)}_{\frac{n}{2}-1}-\frac{1}{32}B^{(1)}_{\frac{n}{2}}+\frac{1}{1024}B^{(0)}_{n-13}+\frac{9}{1024}B^{(0)}_{n-12}\\
&+\frac{23}{1024}B^{(0)}_{n-11}+\frac{31}{1024}B^{(0)}_{n-10}+\frac{29}{1024}B^{(0)}_{n-9}+\frac{5}{1024}B^{(0)}_{n-8}\\
&-\frac{37}{1024}B^{(0)}_{n-7}-\frac{61}{1024}B^{(0)}_{n-6}-\frac{61}{1024}B^{(0)}_{n-5}-\frac{37}{1024}B^{(0)}_{n-4}\\
&+\frac{5}{1024}B^{(0)}_{n-3}+\frac{29}{1024}B^{(0)}_{n-2}+\frac{31}{1024}B^{(0)}_{n-1}+\frac{23}{1024}B^{(0)}_{n}+\frac{9}{1024}B^{(0)}_{n+1}\Big]\\
&+f(s^{(2)}_{\frac{n}{4}})\Big[B^{(2)}_{\frac{n}{4}}-\frac{1}{32}B^{(1)}_{\frac{n}{2}-3}-\frac{9}{32}B^{(1)}_{\frac{n}{2}-2}-\frac{22}{32}B^{(1)}_{\frac{n}{2}-1}-\frac{22}{32}B^{(1)}_{\frac{n}{2}}\\
&-\frac{9}{32}B^{(1)}_{\frac{n}{2}+1}-\frac{1}{32}B^{(1)}_{\frac{n}{2}+2}+\frac{1}{1024}B^{(0)}_{n-9}+\frac{9}{1024}B^{(0)}_{n-8}\\
&+\frac{23}{1024}B^{(0)}_{n-7}+\frac{31}{1024}B^{(0)}_{n-6}+\frac{29}{1024}B^{(0)}_{n-5}+\frac{5}{1024}B^{(0)}_{n-4}\\
&-\frac{37}{1024}B^{(0)}_{n-3}-\frac{61}{1024}B^{(0)}_{n-2}-\frac{61}{1024}B^{(0)}_{n-1}-\frac{37}{1024}B^{(0)}_{n}+\frac{5}{1024}B^{(0)}_{n+1}\Big]\\
&+f(s^{(2)}_{\frac{n}{4}+1})\Big[B^{(2)}_{\frac{n}{4}+1}-\frac{1}{32}B^{(1)}_{\frac{n}{2}-1}-\frac{9}{32}B^{(1)}_{\frac{n}{2}}-\frac{22}{32}B^{(1)}_{\frac{n}{2}+1}\\
&-\frac{22}{32}B^{(1)}_{\frac{n}{2}+2}+\frac{1}{1024}B^{(0)}_{n-5}+\frac{9}{1024}B^{(0)}_{n-4}+\frac{23}{1024}B^{(0)}_{n-3}\\
&+\frac{31}{1024}B^{(0)}_{n-2}+\frac{29}{1024}B^{(0)}_{n-1}+\frac{5}{1024}B^{(0)}_{n}-\frac{37}{1024}B^{(0)}_{n+1}\Big]\\
&+f(s^{(2)}_{\frac{n}{4}+2})\Big[B^{(2)}_{\frac{n}{4}+2}-\frac{1}{32}B^{(1)}_{\frac{n}{2}+1}-\frac{9}{32}B^{(1)}_{\frac{n}{2}+2}+\frac{1}{1024}B^{(0)}_{n-1}+\frac{9}{1024}B^{(0)}_{n}+\frac{23}{1024}B^{(0)}_{n+1}\Big].
\end{aligned}
$$

A.2. $S^{2L}f$ On partitions (2)

$$
\begin{aligned}
S^{2L}f &= \sum_{i=0}^{n+1} f(s_i^{(0)})B_i^{(0)} + f(s_0^{(2)})\Big[B_0^{(2)} - B_0^{(1)} - \frac{9}{16}B_1^{(1)} - \frac{1}{16}B_2^{(1)} \\
&+\frac{14}{512}B_1^{(0)} + \frac{30}{512}B_2^{(0)} + \frac{31}{512}B_3^{(0)} + \frac{23}{512}B_4^{(0)} + \frac{9}{512}B_5^{(0)} + \frac{1}{512}B_6^{(0)}\Big] \\
&+f(s_1^{(2)})\Big[B_1^{(2)} - \frac{13}{32}B_1^{(1)} - \frac{21}{32}B_2^{(1)} - \frac{9}{32}B_3^{(1)} - \frac{1}{32}B_4^{(1)} - \frac{42}{1024}B_1^{(0)} \\
&-\frac{90}{1024}B_2^{(0)} - \frac{92}{1024}B_3^{(0)} - \frac{60}{1024}B_4^{(0)} - \frac{4}{1024}B_5^{(0)} + \frac{28}{1024}B_6^{(0)} \\
&+\frac{31}{1024}B_7^{(0)} + \frac{23}{1024}B_8^{(0)} + \frac{9}{1024}B_9^{(0)} + \frac{1}{1024}B_{10}^{(0)}\Big] \\
&+f(s_2^{(2)})\Big[B_2^{(2)} - \frac{1}{32}B_1^{(1)} - \frac{9}{32}B_2^{(1)} - \frac{22}{32}B_3^{(1)} - \frac{22}{32}B_4^{(1)} - \frac{9}{32}B_5^{(1)} \\
&-\frac{1}{32}B_6^{(1)} + \frac{14}{1024}B_1^{(0)} + \frac{30}{1024}B_2^{(0)} + \frac{29}{1024}B_3^{(0)} + \frac{5}{1024}B_4^{(0)} \\
&-\frac{37}{1024}B_5^{(0)} - \frac{61}{1024}B_6^{(0)} - \frac{61}{1024}B_7^{(0)} - \frac{37}{1024}B_8^{(0)} + \frac{5}{1024}B_9^{(0)} \\
&+\frac{29}{1024}B_{10}^{(0)} + \frac{31}{1024}B_{11}^{(0)} + \frac{23}{1024}B_{12}^{(0)} + \frac{9}{1024}B_{13}^{(0)} + \frac{1}{1024}B_{14}^{(0)}\Big] \\
&+\sum_{k=1}^{\frac{n}{4}-4} f(s_{k+2}^{(2)})\Big[B_{k+2}^{(2)} - \frac{1}{32}B_{2k+1}^{(1)} - \frac{9}{32}B_{2k+2}^{(1)} - \frac{22}{32}B_{2k+3}^{(1)} - \frac{22}{32}B_{2k+4}^{(1)} \\
&-\frac{9}{32}B_{2k+5}^{(1)} - \frac{1}{32}B_{2k+6}^{(1)} + \frac{1}{1024}B_{4k-1}^{(0)} + \frac{9}{1024}B_{4k}^{(0)} + \frac{23}{1024}B_{4k+1}^{(0)} \\
&+\frac{31}{1024}B_{4k+2}^{(0)} + \frac{29}{1024}B_{4k+3}^{(0)} + \frac{5}{1024}B_{4k+4}^{(0)} - \frac{37}{1024}B_{4k+5}^{(0)} - \frac{61}{1024}B_{4k+6}^{(0)} \\
&-\frac{61}{1024}B_{4k+7}^{(0)} - \frac{37}{1024}B_{4k+8}^{(0)} + \frac{5}{1024}B_{4k+9}^{(0)} + \frac{29}{1024}B_{4k+10}^{(0)} + \frac{31}{1024}B_{4k+11}^{(0)} \\
&+\frac{23}{1024}B_{4k+12}^{(0)} + \frac{9}{1024}B_{4k+13}^{(0)} + \frac{1}{1024}B_{4k+14}^{(0)}\Big] \\
&+f(s_{\frac{n}{4}-1}^{(2)})\Big[B_{\frac{n}{4}-1}^{(2)} - \frac{1}{32}B_{\frac{n}{2}-5}^{(1)} - \frac{9}{32}B_{\frac{n}{2}-4}^{(1)} - \frac{22}{32}B_{\frac{n}{2}-3}^{(1)} \\
&-\frac{22}{32}B_{\frac{n}{2}-2}^{(1)} - \frac{9}{32}B_{\frac{n}{2}-1}^{(1)} - \frac{1}{32}B_{\frac{n}{2}}^{(1)} + \frac{1}{1024}B_{n-13}^{(0)} + \frac{9}{1024}B_{n-12}^{(0)} \\
&+\frac{23}{1024}B_{n-11}^{(0)} + \frac{31}{1024}B_{n-10}^{(0)} + \frac{29}{1024}B_{n-9}^{(0)} + \frac{5}{1024}B_{n-8}^{(0)} - \frac{37}{1024}B_{n-7}^{(0)} \\
&-\frac{61}{1024}B_{n-6}^{(0)} - \frac{61}{1024}B_{n-5}^{(0)} - \frac{37}{1024}B_{n-4}^{(0)} + \frac{5}{1024}B_{n-3}^{(0)} + \frac{29}{1024}B_{n-2}^{(0)} + \frac{30}{1024}B_{n-1}^{(0)} + \frac{14}{1024}B_n^{(0)}\Big] \\
&+f(s_{\frac{n}{4}}^{(2)})\Big[B_{\frac{n}{4}}^{(2)} - \frac{1}{32}B_{\frac{n}{2}-3}^{(1)} - \frac{9}{32}B_{\frac{n}{2}-2}^{(1)} - \frac{21}{32}B_{\frac{n}{2}-1}^{(1)} - \frac{13}{32}B_{\frac{n}{2}}^{(1)} \\
&+\frac{1}{1024}B_{n-9}^{(0)} + \frac{9}{1024}B_{n-8}^{(0)} + \frac{23}{1024}B_{n-7}^{(0)} + \frac{31}{1024}B_{n-6}^{(0)} + \frac{28}{1024}B_{n-5}^{(0)} \\
&-\frac{4}{1024}B_{n-4}^{(0)} - \frac{60}{1024}B_{n-3}^{(0)} - \frac{92}{1024}B_{n-2}^{(0)} - \frac{90}{1024}B_{n-1}^{(0)} - \frac{42}{1024}B_n^{(0)}\Big] \\
&+f(s_{\frac{n}{4}+1}^{(2)})\Big[B_{\frac{n}{4}+1}^{(2)} - \frac{1}{16}B_{\frac{n}{2}-1}^{(1)} - \frac{9}{16}B_{\frac{n}{2}}^{(1)} - B_{\frac{n}{2}+1}^{(1)} + \frac{1}{512}B_{n-5}^{(0)} \\
&+\frac{9}{512}B_{n-4}^{(0)} + \frac{23}{512}B_{n-3}^{(0)} + \frac{31}{512}B_{n-2}^{(0)} + \frac{30}{512}B_{n-1}^{(0)} + \frac{14}{512}B_n^{(0)}\Big] \\
&+f(s_0^{(1)})\Big[B_0^{(1)} - B_0^{(0)} - \frac{9}{16}B_1^{(0)} - \frac{1}{16}B_2^{(0)}\Big] + f(s_1^{(1)})\Big[B_1^{(1)} - \frac{13}{32}B_1^{(0)} - \frac{21}{32}B_2^{(0)} - \frac{9}{32}B_3^{(0)} - \frac{1}{32}B_4^{(0)}\Big] \\
&+\sum_{i=2}^{\frac{n}{2}-1} f(s_i^{(1)})\Big[B_i^{(1)} - \frac{1}{32}B_{2i-3}^{(0)} - \frac{9}{32}B_{2i-2}^{(0)} - \frac{22}{32}B_{2i-1}^{(0)} - \frac{22}{32}B_{2i}^{(0)} - \frac{9}{32}B_{2i+1}^{(0)} - \frac{1}{32}B_{2i+2}^{(0)}\Big] \\
&+f(s_{\frac{n}{2}}^{(1)})\Big[B_{\frac{n}{2}}^{(1)} - \frac{1}{32}B_{n-3}^{(0)} - \frac{9}{32}B_{n-2}^{(0)} - \frac{21}{32}B_{n-1}^{(0)} - \frac{13}{32}B_n^{(0)}\Big] \\
&+f(s_{\frac{n}{2}+1}^{(1)})\Big[B_{\frac{n}{2}+1}^{(1)} - \frac{1}{16}B_{n-1}^{(0)} - \frac{9}{16}B_n^{(0)} - B_{n+1}^{(0)}\Big].
\end{aligned}
$$

A.3. Weigths for quadrature formula (20)

$$\Gamma_{-1} = \omega_{-1}^{(2)} - \frac{9}{32}\omega_{-1}^{(1)} - \frac{1}{32}\omega_{0}^{(1)} + \frac{23}{1024}\omega_0 + \frac{9}{1024}\omega_1 + \frac{1}{1024}\omega_2;$$

$$\Gamma_{0} = \omega_{0}^{(2)} - \frac{22}{32}\omega_{-1}^{(1)} - \frac{22}{32}\omega_{0}^{(1)} - \frac{9}{32}\omega_{1}^{(1)} - \frac{1}{32}\omega_{2}^{(1)} - \frac{37}{1024}\omega_0 + \frac{5}{1024}\omega_1$$

$$+\frac{29}{1024}\omega_2 + \frac{31}{1024}\omega_3 + \frac{23}{1024}\omega_4 + \frac{9}{1024}\omega_5 + \frac{1}{1024}\omega_6;$$

$$\Gamma_{1} = \omega_{1}^{(2)} - \frac{1}{32}\omega_{-1}^{(1)} - \frac{9}{32}\omega_{0}^{(1)} - \frac{22}{32}\omega_{1}^{(1)} - \frac{22}{32}\omega_{2}^{(1)} - \frac{9}{32}\omega_{3}^{(1)} - \frac{1}{32}\omega_{4}^{(1)}$$

$$+\frac{5}{1024}\omega_0 - \frac{37}{1024}\omega_1 - \frac{61}{1024}\omega_2 - \frac{61}{1024}\omega_3 - \frac{37}{1024}\omega_4 + \frac{5}{1024}\omega_5$$

$$+\frac{29}{1024}\omega_6 + \frac{31}{1024}\omega_7 + \frac{23}{1024}\omega_8 + \frac{9}{1024}\omega_9 + \frac{1}{1024}\omega_{10};$$

$$\Gamma_{2} = \omega_{2}^{(2)} - \frac{1}{32}\omega_{1}^{(1)} - \frac{9}{32}\omega_{2}^{(1)} - \frac{22}{32}\omega_{3}^{(1)} - \frac{22}{32}\omega_{4}^{(1)} - \frac{9}{32}\omega_{5}^{(1)} - \frac{1}{32}\omega_{6}^{(1)}$$

$$+\frac{9}{1024}\omega_0 + \frac{23}{1024}\omega_1 + \frac{31}{1024}\omega_2 + \frac{29}{1024}\omega_3 + \frac{5}{1024}\omega_4 - \frac{37}{1024}\omega_5 - \frac{61}{1024}\omega_6 - \frac{61}{1024}\omega_7$$

$$-\frac{37}{1024}\omega_8 + \frac{5}{1024}\omega_9 + \frac{29}{1024}\omega_{10} + \frac{31}{1024}\omega_{11} + \frac{23}{1024}\omega_{12} + \frac{9}{1024}\omega_{13} + \frac{1}{1024}\omega_{14};$$

$$\Gamma_{i+2} = \omega_{i+2}^{(2)} - \frac{1}{32}\omega_{2i+1}^{(1)} - \frac{9}{32}\omega_{2i+2}^{(1)} - \frac{22}{32}\omega_{2i+3}^{(1)} - \frac{22}{32}\omega_{2i+4}^{(1)} - \frac{9}{32}\omega_{2i+5}^{(1)} - \frac{1}{32}\omega_{2i+6}^{(1)}$$

$$+\frac{1}{1024}\omega_{4i-1} + \frac{9}{1024}\omega_{4i} + \frac{23}{1024}\omega_{4i+1} + \frac{31}{1024}\omega_{4i+2} + \frac{29}{1024}\omega_{4i+3} + \frac{5}{1024}\omega_{4i+4}$$

$$-\frac{37}{1024}\omega_{4i+5} - \frac{61}{1024}\omega_{4i+6} - \frac{61}{1024}\omega_{4i+7} - \frac{37}{1024}\omega_{4i+8} + \frac{5}{1024}\omega_{4i+9} + \frac{29}{1024}\omega_{4i+10}$$

$$+\frac{31}{1024}\omega_{4i+11} + \frac{23}{1024}\omega_{4i+12} + \frac{9}{1024}\omega_{4i+13} + \frac{1}{1024}\omega_{4i+14}, \quad \text{with} \quad i = 1, \dots, \frac{n}{4} - 4;$$

$$\Gamma_{\frac{n}{4}-1} = \omega_{\frac{n}{4}-1}^{(2)} - \frac{1}{32}\omega_{\frac{n}{2}-5}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}-4}^{(1)} - \frac{22}{32}\omega_{\frac{n}{2}-3}^{(1)} - \frac{22}{32}\omega_{\frac{n}{2}-2}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}-1}^{(1)}$$

$$-\frac{1}{32}\omega_{\frac{n}{2}}^{(1)} + \frac{1}{1024}\omega_{n-13} + \frac{9}{1024}\omega_{n-12} + \frac{23}{1024}\omega_{n-11} + \frac{31}{1024}\omega_{n-10} + \frac{29}{1024}\omega_{n-9}$$

$$+\frac{5}{1024}\omega_{n-8} - \frac{37}{1024}\omega_{n-7} - \frac{61}{1024}\omega_{n-6} - \frac{61}{1024}\omega_{n-5} - \frac{37}{1024}\omega_{n-4}$$

$$+\frac{5}{1024}\omega_{n-3} + \frac{29}{1024}\omega_{n-2} + \frac{31}{1024}\omega_{n-1} + \frac{23}{1024}\omega_{n} + \frac{9}{1024}\omega_{n+1};$$

$$\Gamma_{\frac{n}{4}} = \omega_{\frac{n}{4}}^{(2)} - \frac{1}{32}\omega_{\frac{n}{2}-3}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}-2}^{(1)} - \frac{22}{32}\omega_{\frac{n}{2}-1}^{(1)} - \frac{22}{32}\omega_{\frac{n}{2}}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}+1}^{(1)}$$

$$-\frac{1}{32}\omega_{\frac{n}{2}+2}^{(1)} + \frac{1}{1024}\omega_{n-9} + \frac{9}{1024}\omega_{n-8} + \frac{23}{1024}\omega_{n-7} + \frac{31}{1024}\omega_{n-6} + \frac{29}{1024}\omega_{n-5}$$

$$+\frac{5}{1024}\omega_{n-4} - \frac{37}{1024}\omega_{n-3} - \frac{61}{1024}\omega_{n-2} - \frac{61}{1024}\omega_{n-1} - \frac{37}{1024}\omega_{n} + \frac{5}{1024}\omega_{n+1};$$

$$\Gamma_{\frac{n}{4}+1} = \omega_{\frac{n}{4}+1}^{(2)} - \frac{1}{32}\omega_{\frac{n}{2}-1}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}}^{(1)} - \frac{22}{32}\omega_{\frac{n}{2}+1}^{(1)} - \frac{22}{32}\omega_{\frac{n}{2}+2}^{(1)} + \frac{1}{1024}\omega_{n-5}$$

$$+\frac{9}{1024}\omega_{n-4} + \frac{23}{1024}\omega_{n-3} + \frac{31}{1024}\omega_{n-2} + \frac{29}{1024}\omega_{n-1} + \frac{5}{1024}\omega_{n} - \frac{37}{1024}\omega_{n+1};$$

$$\Gamma_{\frac{n}{4}+2} = \omega_{\frac{n}{4}+2}^{(2)} - \frac{1}{32}\omega_{\frac{n}{2}+1}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}+2}^{(1)} + \frac{1}{1024}\omega_{n-1} + \frac{9}{1024}\omega_{n} + \frac{23}{1024}\omega_{n+1}.$$

A.4. Weigths for quadrature formula (21)

$$\Gamma_0 = \omega_0^{(2)} - \omega_0^{(1)} - \frac{9}{16}\omega_1^{(1)} - \frac{1}{16}\omega_2^{(1)} + \frac{14}{512}\omega_1 + \frac{30}{512}\omega_2 + \frac{31}{512}\omega_3 + \frac{23}{512}\omega_4 + \frac{9}{512}\omega_5 + \frac{1}{512}\omega_6;$$

$$\Gamma_1 = \omega_1^{(2)} - \frac{13}{32}\omega_1^{(1)} - \frac{21}{32}\omega_2^{(1)} - \frac{9}{32}\omega_3^{(1)} - \frac{1}{32}\omega_4^{(1)} - \frac{42}{1024}\omega_1 - \frac{90}{1024}\omega_2 - \frac{92}{1024}\omega_3 - \frac{60}{1024}\omega_4 - \frac{4}{1024}\omega_5$$

$$+\frac{28}{1024}\omega_6 + \frac{31}{1024}\omega_7 + \frac{23}{1024}\omega_8 + \frac{9}{1024}\omega_9 + \frac{1}{1024}\omega_{10};$$

$$\Gamma_2 = \omega_2^{(2)} - \frac{1}{32}\omega_1^{(1)} - \frac{9}{32}\omega_2^{(1)} - \frac{22}{32}\omega_3^{(1)} - \frac{22}{32}\omega_4^{(1)} - \frac{9}{32}\omega_5^{(1)} - \frac{1}{32}\omega_6^{(1)}$$

$$+\frac{14}{1024}\omega_1 + \frac{30}{1024}\omega_2 + \frac{29}{1024}\omega_3 + \frac{5}{1024}\omega_4 - \frac{37}{1024}\omega_5 - \frac{61}{1024}\omega_6 - \frac{61}{1024}\omega_7$$

$$-\frac{37}{1024}\omega_8 + \frac{5}{1024}\omega_9 + \frac{29}{1024}\omega_{10} + \frac{31}{1024}\omega_{11} + \frac{23}{1024}\omega_{12} + \frac{9}{1024}\omega_{13} + \frac{1}{1024}\omega_{14};$$

$$\Gamma_{i+2} = \omega_{i+2}^{(2)} - \frac{1}{32}\omega_{2i+1}^{(1)} - \frac{9}{32}\omega_{2i+2}^{(1)} - \frac{22}{32}\omega_{2i+3}^{(1)} - \frac{22}{32}\omega_{2i+4}^{(1)} - \frac{9}{32}\omega_{2i+5}^{(1)} - \frac{1}{32}\omega_{2i+6}^{(1)}$$

$$+\frac{1}{1024}\omega_{4i-1} + \frac{9}{1024}\omega_{4i} + \frac{23}{1024}\omega_{4i+1} + \frac{31}{1024}\omega_{4i+2} + \frac{29}{1024}\omega_{4i+3} + \frac{5}{1024}\omega_{4i+4}$$

$$-\frac{37}{1024}\omega_{4i+5} - \frac{61}{1024}\omega_{4i+6} - \frac{61}{1024}\omega_{4i+7} - \frac{37}{1024}\omega_{4i+8} + \frac{5}{1024}\omega_{4i+9} + \frac{29}{1024}\omega_{4i+10}$$

$$+\frac{31}{1024}\omega_{4i+11} + \frac{23}{1024}\omega_{4i+12} + \frac{9}{1024}\omega_{4i+13} + \frac{1}{1024}\omega_{4i+14}, \quad \text{with} \quad i = 1, \dots, \frac{n}{4} - 4;$$

$$\Gamma_{\frac{n}{4}-1} = \omega_{\frac{n}{4}-1}^{(2)} - \frac{1}{32}\omega_{\frac{n}{2}-5}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}-4}^{(1)} - \frac{22}{32}\omega_{\frac{n}{2}-3}^{(1)} - \frac{22}{32}\omega_{\frac{n}{2}-2}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}-1}^{(1)}$$

$$-\frac{1}{32}\omega_{\frac{n}{2}}^{(1)} + \frac{1}{1024}\omega_{n-13} + \frac{9}{1024}\omega_{n-12} + \frac{23}{1024}\omega_{n-11} + \frac{31}{1024}\omega_{n-10} + \frac{29}{1024}\omega_{n-9}$$

$$+\frac{5}{1024}\omega_{n-8} - \frac{37}{1024}\omega_{n-7} - \frac{61}{1024}\omega_{n-6} - \frac{61}{1024}\omega_{n-5} - \frac{37}{1024}\omega_{n-4}$$

$$+\frac{5}{1024}\omega_{n-3} + \frac{29}{1024}\omega_{n-2} + \frac{30}{1024}\omega_{n-1} + \frac{14}{1024}\omega_n;$$

$$\Gamma_{\frac{n}{4}} = \omega_{\frac{n}{4}}^{(2)} - \frac{1}{32}\omega_{\frac{n}{2}-3}^{(1)} - \frac{9}{32}\omega_{\frac{n}{2}-2}^{(1)} - \frac{21}{32}\omega_{\frac{n}{2}-1}^{(1)} - \frac{13}{32}\omega_{\frac{n}{2}}^{(1)} + \frac{1}{1024}\omega_{n-9}$$

$$+\frac{9}{1024}\omega_{n-8} + \frac{23}{1024}\omega_{n-7} + \frac{31}{1024}\omega_{n-6} + \frac{28}{1024}\omega_{n-5} - \frac{4}{1024}\omega_{n-4} - \frac{60}{1024}\omega_{n-3} - \frac{92}{1024}\omega_{n-2} - \frac{90}{1024}\omega_{n-1} - \frac{42}{1024}\omega_n;$$

$$\Gamma_{\frac{n}{4}+1} = \omega_{\frac{n}{4}+1}^{(2)} - \frac{1}{16}\omega_{\frac{n}{2}-1}^{(1)} - \frac{9}{16}\omega_{\frac{n}{2}}^{(1)} - \omega_{\frac{n}{2}+1}^{(1)} + \frac{1}{512}\omega_{n-5} + \frac{9}{512}\omega_{n-4} + \frac{23}{512}\omega_{n-3} + \frac{31}{512}\omega_{n-2} + \frac{30}{512}\omega_{n-1} + \frac{14}{1024}\omega_n.$$

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