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# An inverse approach to hyperspheres of prescribed mean curvature in Euclidean space

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**Abstract:** We construct families of smooth functions  $H: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  such that the Euclidean  $(n + 1)$ -space is completely filled by not necessarily round hyperspheres of mean curvature  $H$  at every point.

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## Introduction

This article deals with embedding  $n$ -dimensional hypersurfaces of prescribed mean curvature into Euclidean space. More precisely, considering a function  $H: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , one looks for compact, embedded hypersurfaces whose mean curvature at any point  $p$  is given by  $H(p)$ . We focus our attention to hypersurfaces which are diffeomorphic to the sphere  $\mathbb{S}^n$  and we call them *H-bubbles*.

When  $H$  is a nonzero constant, *H-bubbles* are exactly round spheres of radius  $|H|^{-1}$  centred at any point of the space. The case of nonconstant functions  $H$  is much more delicate and has been intensively investigated in the last years. In fact the behaviour of  $H$  crucially affects the existence of *H-bubbles*; see e.g. [3; 4; 5; 6; 7; 9]. Moreover, concerning the location of *H-bubbles*, some constraints occur. For example, if  $H$  depends just on one variable in a monotone way in some domain  $D$  of the space, then no *H-bubble* can be found in  $D$ ; see [4, Proposition 4.1]. In a perturbation setup, one sees that *H-bubbles* concentrate only at critical points of  $H$  or of other functions related to  $H$ ; see e.g. [4, Proposition 4.3], [2, Theorem 2.1].

In this paper, we are interested in the existence of nonconstant curvature functions  $H$  such that the whole Euclidean  $(n + 1)$ -space can be filled by *H-bubbles*, in other words, for every point  $p \in \mathbb{R}^{n+1}$  there exists an *H-bubble* passing through  $p$ . We construct families of nonconstant mappings  $H: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  of class  $C^1$  and with further optional properties (like periodicity or prescribed asymptotics at infinity), for which the desired filling property holds. In fact, we use a kind of inverse approach, starting with the construction of suitable bubbles before we exhibit the corresponding functions  $H$ .

This setup might have some importance in applications to weighted isoperimetric inequalities. For example, looking for area minimizing, disk-type surfaces with prescribed boundary and volume (enclosed within another fixed reference surface with the same boundary), a certain sphere-attaching mechanism is involved. Such construction (due to Wentz [10, p. 285 ff.], see also [1]) rests on the existence of spheres passing through an *a priori* unknown point of the space. The analogous issue in case of prescribed weighted volume would be quite interesting but no result similar to [10] seems to be still available. Since prescribing weighted volume is equivalent to prescribing mean curvature, up to a Lagrange multiplier, the presence (or not) of *H-bubbles* filling the space might have some impact with respect to the above described issue.

Let us spend a few words on the main idea developed in this paper. We firstly construct a smooth Jordan curve  $\Gamma$  lying in the half-plane of  $\mathbb{R}^{n+1}$  defined by  $x_1 = \dots = x_{n-1} = 0$  and  $x_{n+1} \geq 0$ , passing through the origin, and symmetric about the  $x_{n+1}$ -axis. Then we introduce a hypersurface of revolution  $S$  obtained by revolving the curve  $\Gamma$  around the last coordinate axis. Letting  $R = \max_{p \in S} |p|$ , we define a radial, positive

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mapping  $H = H(r)$  on the ball  $B_R = \{p \in \mathbb{R}^{n+1} \mid |p| \leq R\}$  taking the value of the mean curvature of  $S$  at a point  $p \in S$  with  $|p| = r$ . Since  $H$  is radially symmetric, every hypersurface obtained by rotation of  $S$  about the origin is an  $H$ -bubble. Hence the  $(n + 1)$ -dimensional ball  $B_R$  has the filling property with respect to  $H$ , that is, for every  $p \in B_R$  there is an  $H$ -bubble passing through  $H$ .

The main difficulty is to construct  $\Gamma$  in such a way that  $H$  is well defined, of class  $C^1$ , nonconstant and with null derivative at  $r = 0$  and  $r = R$ . This is achieved by taking  $\Gamma$  as a perturbation of a circle. Consequently, the curvature function  $H$  turns out to be a perturbation of a positive constant. We point out that the smallness of the perturbation can be suitably controlled; see Example 3.4.

Since  $H$  has null derivative on the boundary of  $B_R$ , we can extend it outside  $B_R$  in a  $C^1$  way with the constant value  $H(R)$ . Then the filling property is automatically satisfied on the whole Euclidean space, because the complement of the ball  $B_R$  can be filled by round hyperspheres with radius  $H(R)^{-1}$ . In fact, we can use the function  $H$  on  $B_{R'}$ , for some  $R' > R$ , as a fundamental block, and arrange infinitely many similar blocks, suitably spaced, to cover the whole space. Also in this case the resulting function  $H$  has the property that every point of  $\mathbb{R}^{n+1}$  is touched by an  $H$ -bubble. The arrangement of the blocks or even the form of  $H$  at every block can be arbitrarily adjusted in order to fulfil additional requirements on  $H$  (like periodicity). Hence, a very huge amount of examples, even with  $C^\infty$  regularity, can be built.

The main results are Theorems 3.1 and 3.2 and are displayed in the last section, together with an explicit example. The first two sections are devoted to the construction of the profile curve  $\Gamma$  and of the reference hypersurface  $S$ .

Lastly, we note that whereas, on one hand, the problem of the existence of  $H$ -bubbles under global or perturbative conditions on the prescribed mean curvature function requires rather sophisticated tools and arguments (see e.g. [8], [5] and the references therein), on the other hand, our inverse approach can be carried out by means of quite elementary methods and one needs just a basic knowledge of surface theory.

## 1 The profile curve

By a Jordan curve we mean a closed, simple curve. Here we aim to construct smooth (i.e.  $C^\infty$ ) Jordan curves  $\Gamma$  in the plane, with the following properties:

$$\Gamma \text{ is symmetric with respect to the vertical axis} \quad (1.1a)$$

$$0 \in \Gamma \quad \text{and} \quad \Gamma \subset \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \quad (1.1b)$$

$$\Gamma \text{ has positive curvature at every point} \quad (1.1c)$$

$$\text{for every } r \in [0, R] \text{ there exists a unique } z = (x, y) \in \Gamma \text{ with } x \geq 0 \text{ and } |z| = r \quad (1.1d)$$

where  $R = \max_{z \in \Gamma} |z|$ . More precisely, we prove the following result.

**Lemma 1.1.** *For every even and  $2\pi$ -periodic mapping  $h \in C^\infty(\mathbb{R})$  there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$  the parametric curve*

$$\Gamma_\varepsilon = \{(g_\varepsilon(t) \sin t, g_\varepsilon(\pi) + g_\varepsilon(t) \cos t) \mid t \in [-\pi, \pi]\} \quad \text{with} \quad g_\varepsilon := 1 + \varepsilon h \quad (1.2)$$

satisfies (1.1). In addition, setting

$$\gamma_\varepsilon(t) := (g_\varepsilon(t) \sin t, g_\varepsilon(\pi) + g_\varepsilon(t) \cos t), \quad (1.3)$$

the mapping  $|\gamma_\varepsilon|$  is strictly decreasing and of class  $C^1$  in  $[0, \pi]$ .

*Proof.* Fix  $h \in C^\infty(\mathbb{R})$  even and  $2\pi$ -periodic and consider the function  $g_\varepsilon$  and the parametric curve  $\Gamma_\varepsilon$  as in (1.2). For every  $\varepsilon \in \mathbb{R}$  with  $|\varepsilon| < \|h\|_\infty^{-1}$  we have that  $g_\varepsilon := 1 + \varepsilon h > 0$  everywhere and the parametric curve  $\Gamma_\varepsilon$  defined in (1.2) is a smooth Jordan curve satisfying (1.1a), with  $0 \in \Gamma_\varepsilon$ . Let us show that  $\Gamma_\varepsilon \subset \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$  if  $|\varepsilon|$  is sufficiently small. It is enough to prove that

$$\tilde{g}_\varepsilon(t) := g_\varepsilon(\pi) + g_\varepsilon(t) \cos t \geq 0 \quad \text{for all } t \in [0, \pi]. \quad (1.4)$$

One has  $\tilde{g}'_\varepsilon(t) = -\cos t + \varepsilon[h(t)\cos t]''$ . In particular there exists  $\varepsilon_0 > 0$  such that  $\tilde{g}_\varepsilon$  is strictly convex in  $[2\pi/3, \pi]$ . Since  $g_\varepsilon$  is even and  $2\pi$ -periodic,  $g'_\varepsilon(\pi) = 0$  and then also  $\tilde{g}'_\varepsilon(\pi) = 0$ . Therefore  $\tilde{g}_\varepsilon(t) > \tilde{g}_\varepsilon(\pi) = 0$  for  $t \in [2\pi/3, \pi]$ . In addition

$$\tilde{g}_\varepsilon(t) = -\int_t^\pi \tilde{g}'_\varepsilon(s)(t-s) ds = 1 + \cos t - \varepsilon \int_t^\pi [h(s)\cos s]''(t-s) ds \geq \frac{1}{2} - \varepsilon C \quad \text{for all } t \in [0, 2\pi/3]$$

for some constant  $C$  independent of  $\varepsilon$  and  $t$ . Hence, taking  $|\varepsilon|$  small, we obtain  $\tilde{g}_\varepsilon(t) > 0$  also for  $t \in [0, 2\pi/3]$ . Hence (1.4) holds.

Now let us consider the parameterisation  $\gamma_\varepsilon$  defined in (1.3) and let us study the mapping

$$|\gamma_\varepsilon|(t) = \sqrt{g_\varepsilon(t)^2 + g_\varepsilon(\pi)^2 + 2g_\varepsilon(\pi)g_\varepsilon(t)\cos t}. \quad (1.5)$$

We have

$$|\gamma_\varepsilon|'(t) = \frac{g_\varepsilon(t)g'_\varepsilon(t) + g_\varepsilon(\pi)g'_\varepsilon(t)\cos t - g_\varepsilon(\pi)g_\varepsilon(t)\sin t}{\sqrt{g_\varepsilon(t)^2 + g_\varepsilon(\pi)^2 + 2g_\varepsilon(\pi)g_\varepsilon(t)\cos t}} \quad \text{for all } t \in [0, \pi]. \quad (1.6)$$

At  $t = \pi$  one has to be careful since  $|\gamma_\varepsilon|(\pi) = 0$ . Writing the Taylor expansion about  $\pi$  yields

$$|\gamma_\varepsilon|'(t) = \frac{g_\varepsilon(\pi)^2(t-\pi) + o(t-\pi)}{g_\varepsilon(\pi)|t-\pi| + o(t-\pi)} \quad \text{as } t \rightarrow \pi.$$

Hence  $|\gamma_\varepsilon|$  is of class  $C^1$  in  $[0, \pi]$ . In particular the left derivative at  $\pi$  is

$$|\gamma_\varepsilon|'_-(\pi) = -g_\varepsilon(\pi) < 0. \quad (1.7)$$

We also need to control the behaviour of the derivative of  $|\gamma_\varepsilon|$  at 0. Writing the Taylor expansion about 0 yields

$$|\gamma_\varepsilon|'(t) = \frac{g_\varepsilon(0)g''_\varepsilon(0) + g_\varepsilon(\pi)g''_\varepsilon(0) - g_\varepsilon(0)g_\varepsilon(\pi)}{g_\varepsilon(0) + g_\varepsilon(\pi)}t + o(t) \quad \text{as } t \rightarrow 0. \quad (1.8)$$

In order that the mapping  $|\gamma_\varepsilon|$  is strictly decreasing we need  $|\gamma_\varepsilon|'_+(0) = 0_-$ . This occurs if

$$g_\varepsilon(0)g''_\varepsilon(0) + g_\varepsilon(\pi)g''_\varepsilon(0) - g_\varepsilon(0)g_\varepsilon(\pi) < 0. \quad (1.9)$$

Since  $g_\varepsilon = 1 + \varepsilon h$ , the left-hand side is  $-1 + O(\varepsilon)$ , and then (1.9) holds true taking  $|\varepsilon|$  small enough. Hence  $|\gamma_\varepsilon|' < 0$  in a right neighbourhood of 0 and taking a smaller  $\varepsilon_0 > 0$  if necessary, in view of (1.6)–(1.7), we obtain the desired monotonicity property for the mapping  $|\gamma_\varepsilon|$ . This immediately implies (1.1d).

Finally, let us evaluate the curvature of  $\Gamma_\varepsilon$ . This can be computed by

$$K(\gamma_\varepsilon(t)) = \frac{i\gamma'_\varepsilon(t) \cdot \gamma''_\varepsilon(t)}{|\gamma'_\varepsilon(t)|^3} = \frac{2g'_\varepsilon(t)^2 - g_\varepsilon(t)g''_\varepsilon(t) + g_\varepsilon(t)^2}{[(g'_\varepsilon(t))^2 + g_\varepsilon(t)^2]^{3/2}}, \quad (1.10)$$

where, in general, for  $z \in \mathbb{R}^2$ ,  $iz$  denotes the anticlockwise rotation of  $z$  through an angle  $\pi/2$ . Hence, for  $g_\varepsilon = 1 + \varepsilon h$ , (1.1c) is fulfilled by taking  $|\varepsilon|$  small, since the leading term is  $g_\varepsilon^2$ .  $\square$

For future convenience, let us study the regularity property of the curvature of  $\Gamma_\varepsilon$  as a function of the distance. More precisely, setting

$$R_\varepsilon := \max_{z \in \Gamma_\varepsilon} |z| = g_\varepsilon(0) + g_\varepsilon(\pi), \quad (1.11)$$

let us introduce the mapping  $k_\varepsilon: [0, R_\varepsilon] \rightarrow \mathbb{R}$  defined as

$$k_\varepsilon := K \circ \gamma_\varepsilon \circ |\gamma_\varepsilon|^{-1} \quad (1.12)$$

where  $|\gamma_\varepsilon|^{-1}$  is the inverse of  $|\gamma_\varepsilon|: [0, \pi] \rightarrow [0, R_\varepsilon]$ . Note that  $|\gamma_\varepsilon|^{-1}$  is well defined thanks to Lemma 1.1.

**Lemma 1.2.** *If  $h \in C^\infty(\mathbb{R})$  is even,  $2\pi$ -periodic and satisfies*

$$h''(0) = h''''(0) = 0, \quad (1.13)$$

*then the mapping  $k_\varepsilon$  defined in (1.12) is of class  $C^1$  in  $[0, R_\varepsilon]$  and  $k'_\varepsilon(0) = k'_\varepsilon(R_\varepsilon) = 0$ . Moreover, if  $h$  vanishes in neighbourhoods of 0 and  $\pi$ , then  $R_\varepsilon = 2$ ,  $k_\varepsilon \in C^\infty([0, 2])$  and  $k_\varepsilon = 1$  in neighbourhoods of 0 and 2.*

*Proof.* Since  $g_\varepsilon \in C^\infty(\mathbb{R})$ , (1.10) implies that also  $K \circ \gamma_\varepsilon \in C^\infty(\mathbb{R})$ . Moreover  $|\gamma_\varepsilon|^{-1}$  is of class  $C^1$  in  $[0, R_\varepsilon]$ . Hence also  $k_\varepsilon$  is so. Moreover, considering that

$$\frac{d}{dt}[K \circ \gamma_\varepsilon] = \frac{-3(g'_\varepsilon)^3 g''_\varepsilon - g_\varepsilon (g'_\varepsilon)^2 g'''_\varepsilon - 4g_\varepsilon (g'_\varepsilon)^3 + 3g_\varepsilon^2 g'_\varepsilon g''_\varepsilon - g_\varepsilon^3 g'''_\varepsilon - g_\varepsilon^3 g'_\varepsilon + 3g_\varepsilon g'_\varepsilon (g''_\varepsilon)^2}{[(g'_\varepsilon)^2 + g_\varepsilon^2]^{5/2}}, \quad (1.14)$$

and since  $|\gamma_\varepsilon|^{-1}(0) = \pi$  and  $g'_\varepsilon(\pi) = g'''_\varepsilon(\pi) = 0$  (because  $g_\varepsilon$  is even and  $2\pi$ -periodic), one obtains  $\frac{dk_\varepsilon}{dr}(0) = 0$ . Some care is needed at  $R_\varepsilon$  because  $|\gamma_\varepsilon|^{-1}(R_\varepsilon) = 0$ ,  $|\gamma_\varepsilon|'(0) = 0$ , whence  $\frac{d}{dr}|\gamma_\varepsilon|^{-1}(R_\varepsilon) = \infty$ . One has

$$\frac{dk_\varepsilon}{dr}(r) = \frac{\frac{d(K \circ \gamma_\varepsilon)}{dt}(|\gamma_\varepsilon|^{-1}(r))}{\frac{d|\gamma_\varepsilon|}{dt}(|\gamma_\varepsilon|^{-1}(r))}.$$

Taking (1.8) into account, we need that

$$\frac{d(K \circ \gamma_\varepsilon)}{dt}(t) = o(t) \quad \text{as } t \rightarrow 0. \quad (1.15)$$

Since  $g'_\varepsilon(t) = O(t)$ , by (1.14), equation (1.15) holds true if

$$3g_\varepsilon(t)^2 g'_\varepsilon(t) g''_\varepsilon(t) - g_\varepsilon(t)^3 g_\varepsilon(t)''' - g_\varepsilon(t)^3 g'_\varepsilon(t) + 3g_\varepsilon(t) g'_\varepsilon(t) g''_\varepsilon(t)^2 = o(t) \quad \text{as } t \rightarrow 0. \quad (1.16)$$

For  $g'_\varepsilon(0) = g'''_\varepsilon(0) = 0$ , (1.16) is fulfilled when  $g''_\varepsilon(0) = g_\varepsilon(0)'' = 0$ , too, which follows from (1.13). Concerning the last statement, if  $h$  vanishes in neighbourhoods of 0 and  $\pi$ , then  $g_\varepsilon$  takes the constant value 1 in the same sets. The same holds for  $K \circ \gamma_\varepsilon$  by (1.10). Hence  $\frac{dk_\varepsilon}{dr} \in C_c^\infty((0, R_\varepsilon))$ ,  $R_\varepsilon = 2$  and the assertion follows.  $\square$

## 2 The reference hypersurface

Now we introduce a family of hypersurfaces of revolution  $S_\varepsilon$  in  $\mathbb{R}^{n+1}$  obtained by revolving around the last coordinate axis the curves  $\Gamma_\varepsilon$  built in Section 1 and placed on the plane  $x_1 = \dots = x_{n-1} = 0$ . The symmetry of  $\Gamma_\varepsilon$  with respect to the vertical axis guarantees that  $S_\varepsilon$  is well-defined. We compute the principal curvatures  $K_i$  of  $S_\varepsilon$  and we impose conditions on the parameterisation of  $\Gamma_\varepsilon$  which guarantee that the functions  $K_i$  depend in a  $C^1$  way on the distance from the origin, with null derivative at the origin and at the maximum distance.

To carry out this plan, it is convenient to parameterise the  $n$ -dimensional unit sphere  $\mathbb{S}^n$  in the Euclidean  $(n + 1)$ -space in hyperspherical coordinates, as follows:

$$\sigma(\theta_1, \dots, \theta_n) = \begin{bmatrix} \sin \theta_1 \sin \theta_2 \cdots \sin \theta_n \\ \cos \theta_1 \sin \theta_2 \cdots \sin \theta_n \\ \cos \theta_2 \sin \theta_3 \cdots \sin \theta_n \\ \vdots \\ \cos \theta_{n-1} \sin \theta_n \\ \cos \theta_n \end{bmatrix} \quad \text{with } \theta_1 \in [0, 2\pi], \theta_i \in [0, \pi] \text{ for } i = 2, \dots, n.$$

Fixing a mapping  $g: \mathbb{R} \rightarrow (0, \infty)$  of class  $C^\infty$ , even and  $2\pi$ -periodic, let us introduce the hypersurface  $S_g$  parameterised by  $x(\theta_1, \dots, \theta_n) = g(\theta_n)\sigma(\theta_1, \dots, \theta_n)$ , where  $(\theta_1, \dots, \theta_n) \in [0, 2\pi] \times [0, \pi]^{n-1}$ .

**Lemma 2.1.** *For every  $p = x(\theta_1, \dots, \theta_n) \in S_g$  the principal curvatures of  $S_g$  at  $p$  are given by*

$$K_1 = \dots = K_{n-1} = \frac{g - \frac{\cos \theta_n}{\sin \theta_n} g'}{g \sqrt{g^2 + (g')^2}} \quad \text{and} \quad K_n = \frac{g^2 + 2(g')^2 - gg''}{[g^2 + (g')^2]^{3/2}} \quad (2.1)$$

where the primes denote derivatives with respect to  $\theta_n$ .

*Proof.* Let us fix a point  $p = x(\theta_1, \dots, \theta_n) \in S_g$ . For the sake of brevity, we suppress the variables  $\theta_1, \dots, \theta_n$  in our notation. Let us introduce the vectors  $\tau_i = \frac{\partial \sigma}{\partial \theta_i}$  for  $i = 1, \dots, n$ . One can check that

$$\sigma \cdot \tau_i = \tau_i \cdot \tau_j = 0 \quad \text{for all } i, j = 1, \dots, n, i \neq j. \quad (2.2)$$

Thanks to (2.2), the outward-pointing normal versor  $N$  at  $p$  can be expressed in the form  $N = \alpha_0 \sigma + \sum_{i=1}^n \alpha_i \tau_i$  where  $\alpha_0, \dots, \alpha_n \in \mathbb{R}$  satisfy

$$N \cdot \frac{\partial x}{\partial \theta_i} = 0 \quad \text{for all } i = 1, \dots, n. \quad (2.3)$$

For  $i = 1, \dots, n-1$  we have

$$N \cdot \frac{\partial x}{\partial \theta_i} = \left( \alpha_0 \sigma + \sum_{j=1}^n \alpha_j \tau_j \right) \cdot g \tau_i = \alpha_i g |\tau_i|^2, \quad \text{and } N \cdot \frac{\partial x}{\partial \theta_n} = \left( \alpha_0 \sigma + \sum_{j=1}^n \alpha_j \tau_j \right) \cdot (g' \sigma + g \tau_n) = \alpha_0 g' + \alpha_n g$$

since  $|\sigma| = |\tau_n| = 1$ . Hence the equations (2.3) imply  $\alpha_1 = \dots = \alpha_{n-1} = 0$ ,  $\alpha_n = -\frac{\alpha_0 g'}{g}$  and then we can write the outward-pointing normal versor in the form

$$N = \frac{1}{\sqrt{g^2 + (g')^2}} (g \sigma - g' \tau_n).$$

We compute

$$\begin{aligned} \frac{\partial(N \circ x)}{\partial \theta_i} &= \frac{g}{\sqrt{g^2 + (g')^2}} \tau_i - \frac{g'}{\sqrt{g^2 + (g')^2}} \frac{\partial \tau_n}{\partial \theta_i} \quad (i = 1, \dots, n-1) \\ \frac{\partial(N \circ x)}{\partial \theta_n} &= \frac{g}{\sqrt{g^2 + (g')^2}} \tau_n - \frac{g'}{\sqrt{g^2 + (g')^2}} \frac{\partial \tau_n}{\partial \theta_n} + \left[ \frac{g}{\sqrt{g^2 + (g')^2}} \right]' \sigma - \left[ \frac{g'}{\sqrt{g^2 + (g')^2}} \right]' \tau_n. \end{aligned}$$

Since  $\frac{\partial \tau_n}{\partial \theta_i} = \frac{\cos \theta_n}{\sin \theta_n} \tau_i$  for  $i = 1, \dots, n-1$  and  $\frac{\partial \tau_n}{\partial \theta_n} = -\sigma$ , we obtain

$$\begin{aligned} \frac{\partial(N \circ x)}{\partial \theta_i} &= \frac{g - \frac{\cos \theta_n}{\sin \theta_n} g'}{g \sqrt{g^2 + (g')^2}} \frac{\partial x}{\partial \theta_i} \quad (i = 1, \dots, n-1) \\ \frac{\partial(N \circ x)}{\partial \theta_n} &= \frac{g^2 + 2(g')^2 - g g''}{[g^2 + (g')^2]^{3/2}} \frac{\partial x}{\partial \theta_n}. \end{aligned} \quad (2.4)$$

By definition, the principal curvatures at  $p \in S_g$  are the eigenvalues of the shape operator  $L_p: T_p S_g \rightarrow T_p S_g$  given by  $v \mapsto L_p v = \frac{\partial N}{\partial v}(p)$  for  $v \in T_p S_g$ . Since for  $p = x(\theta_1, \dots, \theta_n)$  and for every  $i = 1, \dots, n$  one has  $L_p \frac{\partial x}{\partial \theta_i} = \frac{\partial(N \circ x)}{\partial \theta_i}$ , by (2.4) the principal curvatures of  $S$  are given by (2.1).  $\square$

Fixing  $h \in C^\infty(\mathbb{R})$  even and  $2\pi$ -periodic, for  $\varepsilon \in \mathbb{R}$  let  $g_\varepsilon$  be as in (1.2) and let  $S_\varepsilon$  be the  $n$ -dimensional hypersurface in  $\mathbb{R}^{n+1}$  parameterised by

$$x_\varepsilon(\theta_1, \dots, \theta_n) = g_\varepsilon(\theta_n) \sigma(\theta_1, \dots, \theta_n) + g_\varepsilon(\pi) \mathbf{e}_{n+1} \quad \text{where } (\theta_1, \dots, \theta_n) \in [0, 2\pi] \times [0, \pi]^{n-1}$$

and  $\mathbf{e}_{n+1} = \sigma(0, \dots, 0)$ . Observe that  $S_\varepsilon$  is obtained from  $S_{g_\varepsilon}$  via translation by an amount of  $g_\varepsilon(\pi)$  in the direction of the last coordinate axis and satisfies the following properties.

**Lemma 2.2.** *There exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ :*

- (i)  $S_\varepsilon$  is a compact, embedded  $n$ -dimensional hypersurface of revolution around the last coordinate axis of  $\mathbb{R}^{n+1}$ , diffeomorphic to  $\mathbb{S}^n$ ;
- (ii)  $0 \in S_\varepsilon$  and  $S_\varepsilon$  is contained in the half-space  $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \geq 0\}$ ;
- (iii)  $\max_{p \in S_\varepsilon} |p| = R_\varepsilon$  where  $R_\varepsilon > 0$  is given by (1.11);
- (iv) the principal curvatures of  $S_\varepsilon$  satisfy  $K_i(p) > 0$  for  $i = 1, \dots, n-1$  and for all  $p \in S_\varepsilon \setminus \{0, R_\varepsilon \mathbf{e}_{n+1}\}$ , and  $K_n(p) > 0$  for all  $p \in S_\varepsilon$ .

*Proof.* Properties (i)–(iii) follow from the definition of  $S_\varepsilon$  and from Lemma 1.1. Let us discuss (iv). According to (1.10) and (2.1), the  $n$ -th principal curvature  $K_n$  of  $S_\varepsilon$  equals the curvature of  $\Gamma_\varepsilon$ . Therefore  $K_n > 0$  on  $S_\varepsilon$ , again by Lemma 1.1. By (2.1), for  $i = 1, \dots, n-1$ , one has  $K_i > 0$  on  $S_\varepsilon \setminus \{0, R_\varepsilon \mathbf{e}_{n+1}\}$  when

$$g_\varepsilon(t) \sin t - g'_\varepsilon(t) \cos t > 0 \quad \text{for all } t \in (0, \pi). \quad (2.5)$$

One has  $g_\varepsilon(t) \sin t - g'_\varepsilon(t) \cos t = [1 + \varepsilon \bar{h}(t)] \sin t$  where  $\bar{h}(t) = h(t) - \frac{h'(t)}{\sin t} \cos t$ . Since  $h'(0) = h'(\pi) = 0$ ,

$$\lim_{t \rightarrow 0} \frac{h'(t)}{\sin t} = h''(0) \quad \text{and} \quad \lim_{t \rightarrow \pi} \frac{h'(t)}{\sin t} = h''(\pi),$$

the mapping  $\bar{h}$  is continuous in  $[0, \pi]$ . Therefore, for  $|\varepsilon|$  sufficiently small, (2.5) holds true.  $\square$

Let us study the regularity property of the principal curvatures of  $S_\varepsilon$  as functions of the distance from the origin. To this aim, let us introduce the mapping  $y_\varepsilon : [0, R_\varepsilon] \rightarrow S_\varepsilon$  defined by

$$y_\varepsilon(r) := x_\varepsilon(0, \dots, 0, |\gamma_\varepsilon|^{-1}(r)) = [g_\varepsilon(t)(\sin t)\mathbf{e}_n + (g_\varepsilon(t) \cos t + g_\varepsilon(\pi))\mathbf{e}_{n+1}]_{t=|\gamma_\varepsilon|^{-1}(r)}, \quad (2.6)$$

where  $y_\varepsilon$  is defined in (1.3) and  $|\gamma_\varepsilon|^{-1}$  is the inverse of the mapping  $|\gamma_\varepsilon| : [0, \pi] \rightarrow [0, R_\varepsilon]$ ; see Lemma 1.1.

**Lemma 2.3.** *If (1.13) and*

$$h''(\pi) = 0 \quad (2.7)$$

*hold, then the mappings  $K_i \circ y_\varepsilon : [0, R_\varepsilon] \rightarrow \mathbb{R}$  ( $i = 1, \dots, n$ ) are of class  $C^1$  in  $[0, R_\varepsilon]$  and with null derivatives at end points. Moreover, if  $h$  vanishes in neighbourhoods of 0 and  $\pi$ , then  $R_\varepsilon = 2$ ,  $K_i \circ y_\varepsilon \in C^\infty([0, 2])$  and  $K_i \circ y_\varepsilon = 1$  in neighbourhoods of 0 and 2 for  $i = 1, \dots, n$ .*

*Proof.* We observe that the mapping  $K_n \circ y_\varepsilon$  is the function  $k_\varepsilon$  defined in (1.12); then the regularity of  $K_n \circ y_\varepsilon$  immediately follows from Lemma 1.2. Let us consider  $K_i \circ y_\varepsilon$  with  $i = 1, \dots, n - 1$ . Setting

$$\bar{k}(t) := \frac{g_\varepsilon(t) \sin t - g'_\varepsilon(t) \cos t}{g_\varepsilon(t) \sin t \sqrt{g_\varepsilon(t)^2 + g'_\varepsilon(t)^2}} \quad (2.8)$$

we have that  $K_i \circ y_\varepsilon = \bar{k} \circ |\gamma_\varepsilon|^{-1}$  ( $i = 1, \dots, n - 1$ ). Moreover  $\bar{k} \in C^1((0, \pi))$ . Therefore  $K_i \circ y_\varepsilon \in C^1((0, R_\varepsilon))$ , with

$$\frac{d[K_i \circ y_\varepsilon]}{dr}(r) = \frac{\bar{k}'(|\gamma_\varepsilon|^{-1}(r))}{|\gamma_\varepsilon'|(|\gamma_\varepsilon|^{-1}(r))}.$$

We aim to show that  $\frac{d[K_i \circ y_\varepsilon]}{dr}(r) \rightarrow 0$  as  $r \rightarrow 0$  and as  $r \rightarrow R_\varepsilon$ . Let us study the limit as  $r \rightarrow 0$ . We have

$$\lim_{r \rightarrow 0} \frac{d[K_i \circ y_\varepsilon]}{dr}(r) = \lim_{t \rightarrow \pi} \frac{\bar{k}'(t)}{|\gamma_\varepsilon'|'(t)}.$$

Moreover

$$\bar{k}'(t) = \left[ \frac{g'_\varepsilon(t)}{\sin t} \right]^2 \frac{\sin t \cos t}{g_\varepsilon(t)^2 \sqrt{g_\varepsilon(t)^2 + g'_\varepsilon(t)^2}} + \frac{g'_\varepsilon(t) - g''_\varepsilon(t) \sin t \cos t}{(\sin t)^2 g_\varepsilon(t) \sqrt{g_\varepsilon(t)^2 + g'_\varepsilon(t)^2}} - \frac{g'_\varepsilon(t)[g_\varepsilon(t) + g''_\varepsilon(t)]}{(\sin t) g_\varepsilon(t) [g_\varepsilon(t)^2 + g'_\varepsilon(t)^2]^{3/2}}.$$

Since  $g'_\varepsilon(\pi) = 0$ , we have that  $g'_\varepsilon(t) - g''_\varepsilon(t) \sin t \cos t = O((t - \pi)^3)$ , and the rule of de l'Hôpital implies that  $\frac{g'_\varepsilon(t)}{\sin t} \rightarrow -g''_\varepsilon(\pi)$  as  $t \rightarrow \pi$ . In addition  $|\gamma_\varepsilon'|'(t) \rightarrow -g_\varepsilon(\pi) < 0$  as  $t \rightarrow \pi_-$ , see (1.7). Therefore

$$\lim_{r \rightarrow 0} \frac{d[K_i \circ y_\varepsilon]}{dr}(r) = -\frac{g''_\varepsilon(\pi)[g_\varepsilon(\pi) + g''_\varepsilon(\pi)]}{g_\varepsilon(\pi)^4} = 0$$

because  $g''_\varepsilon(\pi) = \varepsilon h''(\pi) = 0$  by (2.7).

Now let us study the limit as  $r \rightarrow R_\varepsilon$ . We have

$$\lim_{r \rightarrow R_\varepsilon} \frac{d[K_i \circ y_\varepsilon]}{dr}(r) = \lim_{t \rightarrow 0} \frac{\bar{k}'_1(t)}{|\gamma_\varepsilon'|'(t)}.$$

We know that

$$|\gamma_\varepsilon'|'(t) = \frac{[g_\varepsilon(0)g''_\varepsilon(0) + g_\varepsilon(\pi)g''_\varepsilon(0) - g_\varepsilon(0)g_\varepsilon(\pi)]}{g_\varepsilon(0) + g_\varepsilon(\pi)} t + o(t) \quad \text{as } t \rightarrow 0,$$

see (1.8). We also know that  $g'_\varepsilon(0) = g''_\varepsilon(0) = g'''_\varepsilon(0) = g''''_\varepsilon(0) = 0$  because of (1.13). Then, by the rule of de l'Hôpital,  $\frac{g'_\varepsilon(t)}{\sin t} \rightarrow -g''_\varepsilon(\pi) = 0$  as  $t \rightarrow 0$ . Moreover  $g'_\varepsilon(t) - g''_\varepsilon(t) \sin t \cos t = o(t^4)$  as  $t \rightarrow 0$ . Then

$$\frac{g'_\varepsilon(t) - g''_\varepsilon(t) \sin t \cos t}{(\sin t)^2 g_\varepsilon(t) \sqrt{g_\varepsilon(t)^2 + g'_\varepsilon(t)^2}} = o(t^2) \quad \text{as } t \rightarrow 0.$$

In addition also  $g'_\varepsilon(t) = o(t^4)$  as  $t \rightarrow 0$ . Therefore

$$\frac{g'_\varepsilon(t)[g_\varepsilon(t) + g''_\varepsilon(t)]}{(\sin t)g_\varepsilon(t)[g_\varepsilon(t)^2 + g'_\varepsilon(t)^2]^{3/2}} = o(t^2) \quad \text{as } t \rightarrow 0.$$

In conclusion, also

$$\lim_{r \rightarrow R_\varepsilon} \frac{d[K_i \circ y_\varepsilon]}{dr}(r) = 0.$$

For the last part of the lemma, one argues exactly as in the last part of the proof of Lemma 1.2, using (2.1).  $\square$

### 3 Families of curvature functions with the filling property

In this section we construct families of nonconstant mappings  $H : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  having the following properties:

- (H)<sub>1</sub>  $H \in C^1(\mathbb{R}^{n+1})$ ;
- (H)<sub>2</sub> there exist  $C_1, C_2 > 0$  such that  $C_1 \leq H(p) \leq C_2$  for every  $p \in \mathbb{R}^{n+1}$ ;
- (H)<sub>3</sub> for every  $p \in \mathbb{R}^{n+1}$  there exists an embedded hypersurface  $S$  diffeomorphic to  $\mathbb{S}^n$  with mean curvature  $H$  at every point and with  $p \in S$ .

Other optional properties on  $H$  like periodicity or some asymptotic behaviour can be added.

As a first result, we exhibit a family of radially symmetric curvature functions which satisfy (H)<sub>1</sub>–(H)<sub>3</sub> and are constant outside a ball.

**Theorem 3.1.** *Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a  $2\pi$ -periodic, even function of class  $C^\infty$  satisfying (1.13) and (2.7). For  $\varepsilon \in \mathbb{R}$  let  $g_\varepsilon, R_\varepsilon, K_i$ , and  $y_\varepsilon$  as in (1.2), (1.11), (2.1) with  $g = g_\varepsilon$ , and (2.6), respectively. Then there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$  the function  $H_\varepsilon : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  defined by*

$$H_\varepsilon(p) := \frac{g_\varepsilon(0)}{n} \sum_{i=1}^n (K_i \circ y_\varepsilon)(|p|) \quad \text{for } |p| \leq R_\varepsilon \quad \text{and } H_\varepsilon(p) := 1 \text{ for } |p| > R_\varepsilon \tag{3.1}$$

satisfies (H)<sub>1</sub>–(H)<sub>3</sub> and is radially symmetric. In addition, if  $\varepsilon \rightarrow 0$  then  $R_\varepsilon \rightarrow 2$  and  $H_\varepsilon \rightarrow 1$  in  $C^1(\mathbb{R}^{n+1})$ . Moreover, if  $h$  vanishes in neighbourhoods of 0 and  $\pi$  then  $H_\varepsilon$  is of class  $C^\infty$ .

*Proof.* Fixing  $h$  as in the statement of the theorem, and taking  $\varepsilon \in \mathbb{R}$  with  $|\varepsilon|$  small enough, according to Lemma 2.2 the hypersurface  $S_\varepsilon$  built in Section 2 is diffeomorphic to  $\mathbb{S}^n$  and, for every  $(\theta_1, \dots, \theta_n) \in [0, 2\pi] \times [0, \pi]^{n-1}$  the mean curvature of  $S_\varepsilon$  at  $p = x_\varepsilon(\theta_1, \dots, \theta_n) \in S_\varepsilon$  is given by  $M = \frac{1}{n} \sum_{i=1}^n K_i$  with  $K_i$  as in (2.1). Let

$$\widetilde{H}_\varepsilon(r) := (M \circ y_\varepsilon)(r) \quad \text{for all } r \in [0, R_\varepsilon]$$

with  $y_\varepsilon$  as in (2.6). Thus  $\widetilde{H}_\varepsilon(|p|)$  equals the mean curvature of  $S_\varepsilon$  at  $p$ . By radial symmetry, the same holds true for any hypersurface obtained by rotating  $S_\varepsilon$  about the origin of  $\mathbb{R}^{n+1}$ . Hence, for every  $p \in \mathbb{R}^{n+1}$  with  $|p| \leq R_\varepsilon$  there exists an embedded hypersurface diffeomorphic to  $\mathbb{S}^n$  with mean curvature  $H$  at every point and with  $p \in S$ . By Lemma 2.3,  $\widetilde{H}_\varepsilon \in C^1([0, R_\varepsilon])$  with  $\widetilde{H}'_\varepsilon(0) = \widetilde{H}'_\varepsilon(R_\varepsilon) = 0$  and setting  $\widetilde{H}_\varepsilon(r) := \widetilde{H}_\varepsilon(R_\varepsilon)$  for  $r > R_\varepsilon$ , we obtain a function of class  $C^1$  on  $[0, \infty)$ . In particular  $\widetilde{H}_\varepsilon(R_\varepsilon) = 1/g_\varepsilon(0)$ , and by Lemma 2.2 (iv) there exist  $C_1, C_2 > 0$  such that  $C_1 \leq \widetilde{H}_\varepsilon(r) \leq C_2$  for every  $r > 0$ . Moreover, for every point  $p \in \mathbb{R}^{n+1}$  with  $|p| \leq R_\varepsilon$  one can take a round hypersphere of radius  $g_\varepsilon(0)$  whose mean curvature equals  $\widetilde{H}_\varepsilon$ . Finally, because of the above discussion, the function  $H_\varepsilon(p) = g_\varepsilon(0)\widetilde{H}_\varepsilon(|p|)$ , as defined in (3.1), satisfies (H)<sub>1</sub>–(H)<sub>3</sub> and is radially symmetric. The last properties plainly follow, because of the definition of  $g_\varepsilon$  and by arguing as in the last part of the proof of Lemma 1.2.  $\square$

**Theorem 3.2.** Let  $h: \mathbb{R} \rightarrow \mathbb{R}$  be a  $2\pi$ -periodic, even function of class  $C^\infty$  satisfying (1.13) and (2.7). Then there exists  $\varepsilon_0 > 0$  such that for every set of numbers  $\{\varepsilon_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$  with  $|\varepsilon_j| < \varepsilon_0$  for all  $j \in \mathbb{N}$  and for every set of points  $\{p_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^{n+1}$  such that  $|p_i - p_j| \geq \max\{R_{\varepsilon_i}, R_{\varepsilon_j}\} + 2$ , the function  $H: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  defined by

$$H(p) = H_{\varepsilon_j}(p - p_j) \quad \text{if } |p - p_j| \leq R_{\varepsilon_j} \text{ for some } j \in \mathbb{N}, \quad \text{and } H(p) = 1 \text{ otherwise,} \quad (3.2)$$

with  $H_\varepsilon$  as in (3.1) satisfies  $(H)_1$ – $(H)_3$ . In particular, if  $\varepsilon_j = \varepsilon$  for all  $j$  and the set  $\{p_j\}$  is periodic (in all directions), then the mapping  $H$  is periodic. If  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow \infty$ , then  $H(p) \rightarrow 1$  as  $|p| \rightarrow \infty$ . Moreover, if  $h$  vanishes in neighbourhoods of 0 and  $\pi$  then  $H$  is of class  $C^\infty$ .

*Proof.* This follows from Theorem 3.1 and the fact that the gluing of the blocks defined by  $H(x) = H_{\varepsilon_j}(x - p_j)$  on  $B_{R_{\varepsilon_j}}(p_j)$  outside the region  $\bigcup_i B_{R_{\varepsilon_i}}(p_i)$  is nice because this function takes the common value 1 outside  $B_{R_{\varepsilon_j}}(p_j)$ . The filling property is satisfied because it holds in each ball  $B_{R_{\varepsilon_j}+2}(p_j)$  and these balls are pairwise disjoint by the lower bound for  $|p_i - p_j|$ . Hence the region  $\mathbb{R}^{n+1} \setminus \bigcup_i B_{R_{\varepsilon_i}}(p_i)$  can be filled by round unit spheres.  $\square$

**Remark 3.3.** Even more complicated mappings satisfying  $(H)_1$ – $(H)_3$  can be constructed by taking a sequence of  $2\pi$ -periodic, even functions  $h_j \in C^\infty(\mathbb{R} \rightarrow \mathbb{R})$  ( $j \in \mathbb{N}$ ), considering a corresponding sequence of maps  $H_{\varepsilon_j, h_j}$  defined as in (3.1) and then gluing them according to (3.2).

**Example 3.4.** As a function  $h$  which satisfies the assumptions of Theorems 3.1 and 3.2 one can take  $h(t) = (\sin t)^6$ . One can also give an estimate on the interval of admissible values for the smallness parameter  $\varepsilon$ . In fact, considering the proofs of lemmata of Sections 1 and 2, one needs  $|\varepsilon| < \|h\|_\infty^{-1}$  such that

$$g_\varepsilon(\pi) + g_\varepsilon(t) \cos t > 0 \text{ for } t \in [0, \pi], \text{ see (1.4),}$$

$$\text{the mapping } |\gamma_\varepsilon| \text{ defined in (1.5) is strictly decreasing in } [0, \pi],$$

$$\text{and the curvatures written in (1.10) and (2.8) are positive, respectively, in } [0, \pi] \text{ and in } (0, \pi).$$

Taking  $h(t) = (\sin t)^6$ , with elementary computations one can check that the previous conditions are fulfilled taking  $g_\varepsilon = 1 + \varepsilon h$  with  $\varepsilon \in (-1/7, 2/5)$ .

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