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On spline quasi-interpolation through dimensions

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Abstract
The approximation of functions and data in one and high dimensions is an important problem in many mathematical and scientific applications. Quasi-interpolation is a general and powerful approximation approach having many advantages. This paper deals with spline quasi-interpolants and its aim is to collect the main results obtained by the authors, also in collaboration with other researchers, in such a topic through spline dimension, i.e. in the 1D, 2D and 3D setting, highlighting the approximation properties and the reconstruction of functions and data, the applications in numerical integration and differentiation and the numerical solution of integral and differential problems.

Keywords Quasi-interpolation · Spline approximation · Multivariate splines

Mathematics Subject Classification 65D07 · 41A15

1 Introduction

The approximation of functions and data in one and high dimensions is an important problem in many mathematical and scientific applications. Interpolation and quasi-interpolation are both highly useful tools used in such a context.

The interpolation technique requires that the approximant exactly matches the data at certain points and this requirement could be a problem if we are dealing with noisy

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data. Moreover, there is often no guarantee that such interpolants exist. So it is more appropriate to use other constructive techniques and one of them is quasi-interpolation.

Quasi-interpolation is a general and powerful approximation approach introduced by Schoenberg several decades ago for function approximation [73, 74] (see also [75]). Its advantages are manifold: quasi-interpolants are able to approximate in any number of dimensions, they are efficient and relatively easy to formulate for scattered and meshed nodes and for any number of data, they can be computed without solving linear systems of equations.

The importance of this subject in the current literature is proved by the publication of the recent book [19] focused on the topic.

If we search keywords related to *Quasi-Interpolation* in the Scopus database [78] (accessed on June 24, 2022), we obtain the following results:

- searching for *quasi-interpolation*, *quasi-interpolating*, *quasi-interpolant*, *quasi-interpolants* in the "Article Title" field, Scopus returns 389 papers, 351 of which from the year 2000 and the first one published in 1974;
- searching for *quasi-interpolation*, *quasi-interpolating*, *quasi-interpolant*, *quasi-interpolants* in the "Article Title/Abstract/Keywords" field, Scopus returns 722 papers, 643 of which from the year 2000 and the first one published in 1987.

It is also worthwhile to recall that Schoenberg referred to this kind of approximation as smoothing interpolation, that is what now we call quasi-interpolation.

The choice of the function space that is behind the quasi-interpolant construction is of fundamental importance. Important features are its approximation power, its applicability in high space dimension and its simplicity in formulating and stating the approximation problem.

In this paper we focus on spline quasi-interpolants, i.e. the approximants here considered are piecewise polynomials [18, 19, 21, 76, 77, 82]. In particular, the aim of this paper is to collect the main results obtained by the authors, also in collaboration with other researchers, in such a topic, highlighting the approximation properties and the reconstruction of functions and data [3, 5, 6, 8–10, 26, 32–34, 36, 43, 45, 47, 57, 58, 67, 69–71], the applications in numerical integration and differentiation [22, 25, 27–31, 35, 37, 39–42, 44, 56, 68] and the numerical solution of integral and differential problems [2, 13, 24, 38, 46]. The above results can also be considered through spline dimension: 1D [2, 13, 25, 28, 29, 35–37, 39–42, 46, 57, 68], 2D [6, 8, 9, 22, 24, 26, 27, 30–34, 38, 43, 45, 56–58, 67, 70] and 3D [3, 5, 10, 44, 47, 69, 71].

Moreover, if we narrow our Scopus Research in the "Article Title/Abstract/Keywords" field, adding also the word *spline* (accessed on June 24, 2022) we obtain 367 papers and if we consider the papers published from 2017, we can also mention the following works, subdividing them into different topics. In particular, [62, 79] deal with the construction and study of new quasi-interpolating operators, in [12, 48, 59] generalized spline quasi-interpolants are proposed, in [1, 20, 49–51, 53] new integration formulas based on spline quasi-interpolants are constructed, in [15, 54, 61, 63, 64, 80, 81, 84] and [4, 11] quasi-interpolants are used for the numerical approximation of the solution of differential and integral equations, respectively. Furthermore, in [7, 16, 17, 23, 52, 65, 66] quasi-interpolants are used in different areas of science and engineering: imaging, Computer Aided Geometric Design, industry, etc. Finally, in [19], we find some other interesting references to papers on the above topics.

The paper is organized as follows. In Sect. 2 we give a general definition of spline quasi-interpolant operators, shared by the cited papers, and we recall their important properties and features. Then, in Sect. 3 we report some results about quasi-interpolation in the field of approximation of functions and data, in Sect. 4 the applications in numerical integration and differentiation and in Sect. 5 in differential and integral equations.

2 Spline quasi-interpolating operators in $C\left(\mathbb{R}^s\right)$

Although there are many possibilities to express quasi-interpolating splines, for example constructing local and stable minimal determining sets (see e.g. [55] and references therein) or by setting their Bernstein–Bézier coefficients to appropriate combinations of the given data values (see e.g. [6, 8, 9] and references therein), in the following we consider the use of locally supported spanning functions (see e.g. [14, 21, 55, 60, 72, 82] and references therein).

Therefore, we consider a linear quasi-interpolating operator

$$Q : C\left(\Omega\right) \longrightarrow \mathcal{S}, \qquad \Omega \subset \mathbb{R}^s,\ s \geq 1,$$

where $\mathcal{S}$ is a suitable spline space, spanned by a set of non-negative compactly supported functions. It is supposed that they form a convex partition of unity. Usually quasi-interpolation operators are constructed to be exact on the space of polynomials of maximum degree included in $\mathcal{S}$ and have the following form

$$Qf(u) = \sum_{\alpha \in A} \lambda_\alpha\left(f\right) B_\alpha(u), \qquad u \in \Omega, \tag{1}$$

where

- $A \subset \mathbb{Z}^s$ is a (finite or infinite) set of indices usually closely connected to the information about the function that is available for the approximation;
- $\{B_\alpha,\ \alpha \in A\}$ is the set of non-negative compactly supported functions spanning $\mathcal{S}$ with support Σ_α and called *B-splines*;
- $\{\lambda_\alpha,\ \alpha \in A\}$ is a set of continuous linear forms, called *coefficient functionals*. They can be of different types, chosen according to the provided information about the function f to be approximated. Usually they are point, derivative or integral linear functionals. In the first case, $\lambda_\alpha(f)$ is a finite linear combination of values of f at some points in a neighbourhood of Σ_α. In the second case, $\lambda_\alpha(f)$ is a finite linear combination of values of f and some of its (partial) derivatives at some points in a neighbourhood of Σ_α. Finally, in the third case, $\lambda_\alpha(f)$ is a finite linear combination of weighted mean values of f.

We will refer to Q as quasi-interpolation operator (QIO) and to Qf as quasi-interpolant (QI), provided by Q for the given function f.

Among the different methods known in literature about spline QIOs of the above type, our contribution to this topic is related to point QIOs, i.e., given a set of *quasi-interpolation knots* $\{P_\alpha,\ \alpha \in M\}$, $M \subset \mathbb{Z}^s$, the coefficient functionals λ_α in (1) have the following form

$$\lambda_\alpha(f) = \sum_{\beta \in F_\alpha} \sigma_\alpha(\beta) f(P_\beta), \tag{2}$$

where the finite set of points $\left\{P_\beta,\ \beta \in F_\alpha\right\}$, $F_\alpha \subset M$, lies in some neighbourhood of Σ_α and the $\sigma_\alpha(\beta)$'s are convenient real coefficients that provide a suitable polynomial reproduction. We recall that a point QIO can also be written in the quasi-Lagrange form

$$Qf(u) = \sum_{\alpha \in M} f\left(P_\alpha\right) L_\alpha(u), \qquad u \in \Omega \subset \mathbb{R}^s, \tag{3}$$

where $\{L_\alpha,\ \alpha \in M\}$ is the set of so called *fundamental functions* obtained as linear combinations of B_α, according to the definition of the point linear functionals λ_α in (2).

The main advantage of QIs is that they have a direct construction without solving any system of linear equations. Moreover they are local, in the sense that the value of $Qf(u)$ depends only on values of f in a neighbourhood of u.

Now, we want to recall some general results on the approximation properties of such operators. We require that the QIOs reproduce at least all constant functions. The majority of them are at least exact on linear polynomials too, as for example the well-known Schoenberg variation-diminishing operator. However, usually we are interested in operators that are exact on polynomials of higher degree, possibly on the space of polynomials of maximum degree included in $\mathcal{S}$. Increasing the order of polynomial reproduction is one possible option to get better error estimates. Being h the maximum of the step size used in Ω to construct the knot vectors in the definition of $\mathcal{S}$, we say that a QI has approximation order k if

$$\|f - Qf\|_\infty \le Ch^k, \quad f \in C^k(\Omega),$$

i.e. the maximum error is $O(h^k)$ for $h \to 0$, with an h-independent constant C. The maximum value of k we can obtain, that provides *optimal approximation*, is related to the polynomial reproduction properties of Q. Finally, we recall that, if the operator is exact on $\mathcal{S}$, it is called *quasi-interpolating projector* (QIP).

3 Approximation of functions and data

Since they entered the numerical analysis scene, splines have been used in approximation of functions and data.

In this context, in [57] 1D and 2D spline QI schemes, having tension and shape preserving properties, are presented. The point coefficient functionals of (1) are determined in order to ensure the QI spline reproduces constants and/or polynomials of first degree. Then two tension parameters with values in (0, 1] are introduced to generate a family of C^2 non-negative compactly supported functions, so called B-spline-like,

Table 1 Some point QIOs of type (1) with $s = 2$ and the corresponding coefficient functionals, where $M_{i,j} = \left(\frac{1}{2}(x_{i-1} + x_i), \frac{1}{2}(y_{j-1} + y_j)\right)$, $A_{\gamma,\tau} = (x_\gamma, y_\tau)$, $-1 \le \gamma \le m+1$, $-1 \le \tau \le n+1$ and b_{ij}, $a_i, c_i, \overline{a}_j, \overline{c}_j$ depending on step sizes $h_i = x_{i+1} - x_i$, $k_j = y_{j+1} - y_j$

QI	$\lambda_\alpha(f)$, $\alpha = (i, j)$
S_1	$f(M_{ij})$
W_2	$2f(M_{ij}) - \frac{1}{4}\sum_{h=-1}^{0}\sum_{k=-1}^{0} f(A_{i+h,j+k})$
S_2	$b_{ij} f(M_{ij}) + a_i f(M_{i-1,j}) + c_i f(M_{i+1,j}) + \overline{a}_j f(M_{i,j-1}) + \overline{c}_j f(M_{i,j+1})$

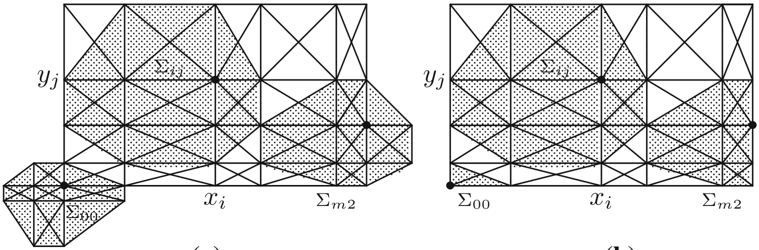


Fig. 1 Criss-cross triangulation of Ω with some supports Σ_α, $\alpha = (i, j)$, of B-splines in case of **(a)** simple knots, **(b)** multiple knots on the boundary $\partial\Omega$ of Ω

that move from hat functions to classical C^2 cubic B-splines. Thus, on one hand the corresponding QI spline approaches the piecewise linear function interpolating the data and on the other hand it reproduces quadratic polynomials.

Taking into account [82], with $s = 2$ and $u = (x, y)$ in (1), C^1 quadratic spline QIs on a criss-cross triangulation of a bounded domain $\Omega = [a, b] \times [c, d]$ (Fig. 1(a)) are considered and studied in [32, 33]. They are defined by the knots

$$\begin{aligned} &x_{-2} < x_{-1} < a = x_0 < x_1 < \ldots < x_m = b < x_{m+1} < x_{m+2}, \\ &y_{-2} < y_{-1} < c = y_0 < y_1 < \ldots < y_n = d < y_{n+1} < y_{n+2}, \end{aligned} \tag{4}$$

for general point coefficient functionals λ_α. Their approximation power both in case of uniform [32] and non uniform [33] partitions is studied. Such an approach is interesting since it provides the approximation of a real function and its partial derivatives up to an optimal order with local and global upper bounds both for the errors and for the spline partial derivatives, in case the spline is more differentiable than the function.

In particular, for any $f \in C(\Omega)$ the spline QIOs S_1 and W_2 are introduced and studied in the above two papers. Their approximation order is nearly optimal and optimal, respectively, and their coefficient functionals are reported in Table 1. We recall that S_1 is the well-known Schoenberg variation-diminishing operator. However, the above QIOs are defined by B-splines having supports not completely included in Ω and some QI knots are outside the domain, so that the function f has to be defined in an open set containing Ω (Fig. 1(a)).

In order to have all QI knots inside Ω or on $\partial\Omega$, a possible approach consists: *i)* in defining C^1 quadratic B-splines with supports completely contained in Ω (Fig. 1(b)), i.e. assuming

$$\begin{aligned} x_{-2} \equiv x_{-1} \equiv a, \quad b \equiv x_{m+1} \equiv x_{m+2}, \\ y_{-2} \equiv y_{-1} \equiv c, \quad d \equiv y_{n+1} \equiv y_{n+2}, \end{aligned} \tag{5}$$

in (4), and *ii)* in choosing coefficient functionals based on QI knots lying inside Ω or on $\partial\Omega$, for which extra values outside the domain are not necessary.

Therefore with such a new partition S_1, S_2 and W_2 operators (see Table 1) are proposed, taking into account both boundary conditions [26] (see also [36] for 1D case) and the presence of multiple knots [43]. Moreover some computational aspects of their construction are presented in [34] and an error analysis for f and its derivatives is provided in [45], making a particular effort to give error bounds in terms of the smoothness of f and the characteristics of the triangulation, also in the case of functions that are not regular enough. In Fig. 2(a) we show the quadratic C^1 B-spline surface $W_2 f$ approximating the function

$$f(x, y) = e^{-\frac{(5-10x)^2}{2}} + \frac{3}{4}e^{-\frac{(5-10y)^2}{2}} + \frac{3}{4}e^{-\frac{(5-10x)^2}{2}}e^{-\frac{(5-10y)^2}{2}}, \text{ on } \Omega = [0, 1] \times [0, 1]. \tag{6}$$

The presence of multiple knots is also exploited in [24], where NURBS (Non-Uniform Rational B-splines), based on quadratic B-splines on criss-cross triangulations with supports inside Ω, are investigated and applications related to the modeling of objects are presented. In particular, given a set of control points $\{\mathbf{C}_\alpha\}_{\alpha\in A}$ in $\mathbb{R}^3$ and a set of positive weights $\{W_\alpha\}_{\alpha\in A}$, the corresponding quadratic NURBS surface $\mathbf{S}$ has the form

$$\mathbf{S} = \sum_{\alpha\in A} \mathbf{C}_\alpha R_\alpha, \quad \text{with} \quad R_\alpha = \frac{W_\alpha \ B_\alpha}{\sum_{\beta\in A} \ W_\beta \ B_\beta}. \tag{7}$$

The functions $\{R_\alpha\}_{\alpha\in A}$ are quadratic NURBS on criss-cross triangulations. In Fig. 2(b) a quadratic NURBS surface reproducing a goblet is reported.

The above approach of multiple knots on the boundary $\partial\Omega$ implies the definition of boundary B-spline of first and second layer, in addition to the classical ones with octagonal support (see Fig. 1(b)). So, in order to avoid these further constructions and use only octagonal support B-splines, in [67] spline QIs, based on C^1 quadratic B-splines on criss-cross triangulations, with supports not completely included in Ω (Fig. 1(a)), but with all QI knots inside Ω or on $\partial\Omega$ are proposed. In this case the main problem consists in finding good coefficient functionals associated with boundary generators (i.e. generators with support not completely inside the domain), giving the optimal approximation order 3, small infinity norm of the operator and using QI knots inside Ω or on $\partial\Omega$. For inner generators (i.e. generators with support inside Ω) the coefficient functionals are those defining S_2 in Table 1. The boundary coefficient functionals are constructed in two different ways: either by minimizing an upper bound for the QIO infinity norm, or by inducing superconvergence at some specific points. In particular in the first case, for $\|f\|_\infty \leq 1$ and $\alpha \in A$, then $|\lambda_\alpha(f)| \leq \|\sigma_\alpha\|_1$, where σ_α is the vector with components $\sigma_\alpha(\beta)$ in (2), and we deduce immediately

$$|Qf| \leq \sum_{\alpha\in A} |\lambda_\alpha(f)| B_\alpha \leq \max_{\alpha\in A} |\lambda_\alpha(f)| \leq \max_{\alpha\in A} \|\sigma_\alpha\|_1 ,$$

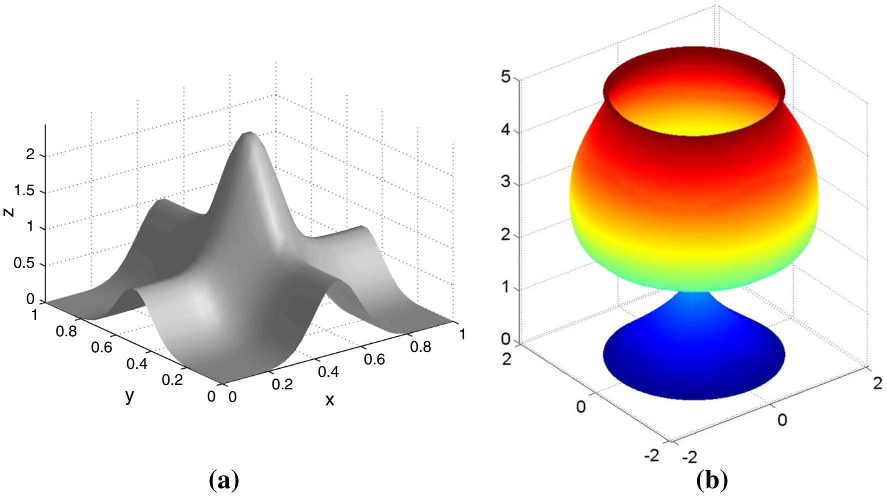


Fig. 2 (**a**) $W_2 f$ with f defined in (6), $m = n = 16$ and $\|f - W_2 f\|_\infty = 9.72 \cdot 10^{-3}$ on a 55×55 uniform rectangular grid of evaluation points in the domain Ω [34]; **(b)** a quadratic NURBS surface reproducing a goblet, defined by 63 control points [24]

concluding $\|Q\|_\infty \le \max_{\alpha \in A} \|\sigma_\alpha\|_1$. Therefore, we find $\sigma_\alpha{}^* \in \mathbb{R}^{\mathrm{card}(F_\alpha)}$ as solution of the minimization problem

$$\left\|\sigma_\alpha{}^*\right\|_1 = \min\left\{\|\sigma_\alpha\|_1 \,;\, \sigma_\alpha \in \mathbb{R}^{\mathrm{card}(F_\alpha)},\ \text{s.t. } Q \text{ is exact on quadratic polynomials}\right\}.$$

In the second case, we impose superconvergence of the operator at some specific points of Ω: we require that the quasi-interpolation error at such points is $O(h^4)$, beside a global error $O(h^3)$. In [70] the same approaches are used in the space of C^2 cubic splines on uniform Powell-Sabin triangulations of a rectangular domain.

Bringing together the ideas of spline QIOs and multilevel techniques [83], recently new spline QIOs with $s = 2$ in (1) and $p + 1$ levels, defined by

$$Q^{pL} f = Q^{(p)} f + \sum_{r=1}^{p} Q^{(r-1)} \Delta_r^{p+1-r} f, \tag{8}$$

are studied in [58], with

- $\Omega = [0, 1] \times [0, 1]$ endowed with a criss-cross triangulation based on (4) uniform and (5) inside-uniform partitions;
- $0 \le p \le \min\{\gamma, \tau\}$ with $m = \epsilon \cdot 2^\gamma$, $n = \eta \cdot 2^\tau$, $\epsilon, \eta, \gamma, \tau \in \mathbb{N}$ and ϵ, η odd numbers;
- $\Delta_r^{p+1-r} f = \Delta_r^1(\Delta_{r+1}^{p-r} f) = \Delta_{r+1}^{p-r} f - Q^{(r)} \Delta_{r+1}^{p-r} f$, $r = p - 1, \ldots, 1$, with $\Delta_p^1 f = f - Q^{(p)} f$ the $(p + 1 - r)$-th error function;
- $Q^{(\ell)} f$ the spline QI defined by the knots $\left(2^\ell x_i, 2^\ell y_j\right)$.

They provide some improvement in the performances of the corresponding classical spline QIOs Qf with 1 level ($p = 0$), especially in case of the operator S_1 that for $p > 0$ reaches the optimal approximation order 3. In fact in [58] the unexpected result of quadratic polynomial reproduction for S_1^{pL}, $p > 0$, is proved, while S_1 usually reproduces bilinear polynomials.

Concerning quasi-interpolation in the three-dimensional setting, we recall that the reconstruction of volume data is an active area of research, due to its relevance to many applications, such as scientific visualization, medical imaging and computer graphics. Indeed, volume data sets typically represent some kind of density acquired by special devices that often require structured input data, so that the samples are arranged on a regular three-dimensional grid. In classical approaches the underlying mathematical models use local trivariate tensor-product polynomial splines, defined as linear combinations of univariate B-spline products.

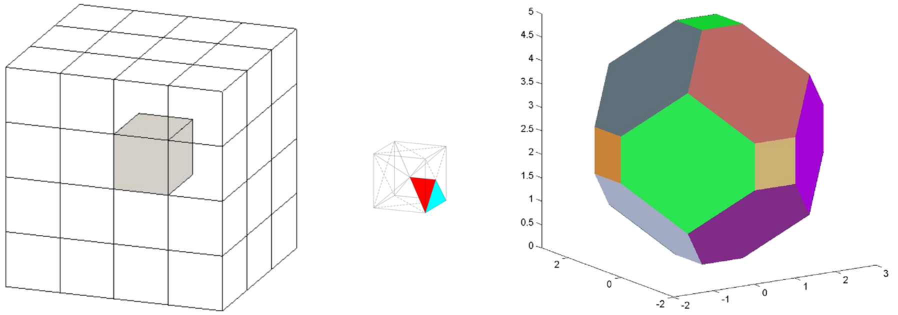
A possible 3D spline model, beyond the classical tensor product schemes, is represented by blending sums of univariate and bivariate spline QIs. This technique allows to combine 1D and 2D QIOs as boolean sum

$$R = S_1\overline{T} + T\overline{S}_1 - S_1\overline{S}_1,$$

where $\overline{S}_1$ and S_1 denote the univariate and bivariate Schoenberg variation-diminishing operator exact on linear polynomials, respectively, while $\overline{T}$ and T represent univariate and bivariate optimal approximation operators, respectively. In particular, in [5] univariate and bivariate C^1 quadratic spline QIs are considered and a trivariate QI of near-best type is constructed, i.e. the coefficients functionals are determined by minimizing an upper bound of its infinity norm, derived from the Bernstein-Bézier coefficients of its Lebesgue function. Moreover, an alternative method that combines the blending sum of 1D and 2D QIOs and the near-best approach is proposed in [10]. The above methods allow oversampling. If we have to use only QI knots inside Ω or on $\partial\Omega$, it is necessary to construct coefficient functionals associated with boundary generators. In [71], the problem is faced proposing two blending sums of univariate and bivariate C^1 quadratic spline QIs having optimal approximation order and a reasonable infinite norm.

An alternative 3D spline model consists in the construction of QIOs of type (1) with $s = 3$, where B_α is the trivariate C^2 quartic box spline defined on a type-6 tetrahedral partition of the domain Ω (see Fig. 3) and the point coefficient functionals $\lambda_\alpha(f)$ have their support in some neighbourhood of Σ_α.

In this case, as in the bivariate setting, firstly spanning functions with supports not completely included in Ω and QI knots also outside the domain are considered and studied. In particular, in [69], starting from a differential QI, whose coefficient functionals are defined as linear combinations of values of f with its partial derivatives at the center of the support of B_α, by convenient discretizations, three different kinds of point QIs are defined, all of them achieving the optimal approximation order 4. The first one, Q_1, is constructed so that it is exact on the space of all polynomials contained in $\mathcal{S}$, as the differential one. The second one, Q_2, is exact only on the space of cubic polynomials and it minimizes an upper bound for its infinity norm. Finally, the third one, Q_3, is constructed so that it is exact on cubic polynomials and in addition it shows



(a) **(b)**

Fig. 3 **(a)** Cube partition and uniform type-6 tetrahedral partition obtained by subdividing each cube into 24 tetrahedra. **(b)** Support of the trivariate C^2 quartic box spline

some superconvergence properties at specific points of the domain. Their expression is reported in Table 2. Moreover, the construction of new QIs based on the same trivariate C^2 quartic box spline, having optimal approximation order and small infinity norm is addressed in [3]. Such near-best QIs are obtained imposing exactness on the space of cubic polynomials and minimizing an upper bound of their infinity norm which depends on a finite number of free parameters. This problem has always a unique solution, which is explicitly given. Then, in order to deal with bounded domain, using only QI knots inside Ω or on $\partial\Omega$, a new class of quartic quasi-interpolating splines is proposed in [47], where the support of $\lambda_\alpha(f)$ is in some neighbourhood of $\Sigma_\alpha \cap \Omega$. In particular, QIOs of near-best type and achieving the optimal approximation order 4 are constructed, with coefficient functionals for boundary generators obtained by minimizing an upper bound for their infinity norm and some interesting results are obtained about reconstruction of medical imaging. Indeed, starting from a discrete set of data, we obtain a non-discrete model of a real object with C^2 smoothness: in Fig. 4(a) we show two isosurfaces, corresponding to the isovalues $\rho = 60, 90$, of the near-best C^2 quartic QI spline approximating a gridded volume data set consisting of $256 \times 256 \times 99$ data samples, obtained from a CT scan of a cadaver head. Similarly, in Fig. 4(b), we show the spline, corresponding to the isovalue $\rho = 40$, approximating a gridded volume data set of $256 \times 256 \times 99$ data samples, obtained from a MR study of head with skull partially removed to reveal brain. In order to visualize the above isosurfaces we evaluate the splines at $N \approx 8,6 \times 10^6$ points.

4 Numerical integration and differentiation

This section deals with numerical methods for integration and differentiation of functions, based on QI splines. In the approximation of integrals and derivatives of a function f, the choice of the method for their numerical evaluation is not a secondary consideration.

A problem that arises in many physical applications is the evaluation of the one-dimensional integral

Table 2 Some point QIOs of type (1) with $s = 3$ and the corresponding coefficient functionals, where M_α are the centres of the cubes in the partition given in Fig. 3(*a*), $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, $e_4 = (1, 1, 1)$, $e_5 = (-1, 1, 1)$, $e_6 = (1, -1, 1)$, $e_7 = (-1, -1, 1)$

QI	$\lambda_\alpha(f), \quad \alpha = (i, j, k)$
Q_1	$\frac{191}{64} f(M_\alpha) - \frac{107}{288}\left(f(M_{\alpha\pm e_1}) + f(M_{\alpha\pm e_2}) + f(M_{\alpha\pm e_3})\right) + \frac{47}{1152}\left(f(M_{\alpha\pm 2e_1}) + f(M_{\alpha\pm 2e_2}) + f(M_{\alpha\pm 2e_3})\right)$
Q_2	$\frac{21}{16} f(M_\alpha) - \frac{5}{96}\left(f(M_{\alpha\pm 2e_1}) + f(M_{\alpha\pm 2e_2}) + f(M_{\alpha\pm 2e_3})\right)$
Q_3	$\frac{16871}{4416} f(M_\alpha) - \frac{507}{736}\left(f(M_{\alpha\pm e_1}) + f(M_{\alpha\pm e_2}) + f(M_{\alpha\pm e_3})\right) + \frac{47}{1152}\left(f(M_{\alpha\pm 2e_1}) + f(M_{\alpha\pm 2e_2}) + f(M_{\alpha\pm 2e_3})\right) + \frac{1435}{13248}(f(M_{\alpha\pm(e_1+e_2)}) + f(M_{\alpha\pm(e_1-e_2)}) + f(M_{\alpha\pm(e_1+e_3)}) f(M_{\alpha\pm(e_1-e_3)}) + f(M_{\alpha\pm(e_2+e_3)}) f(M_{\alpha\pm(e_2-e_3)})) - \frac{2}{69}(f(M_{\alpha\pm e_4}) + f(M_{\alpha\pm e_5}) + f(M_{\alpha\pm e_6}))$

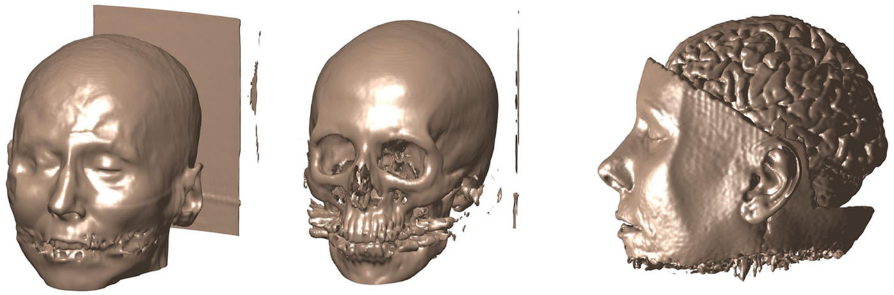


Fig. 4 Isosurfaces of the C^2 trivariate quartic spline approximating: **(a)** the *CT Head data set* (courtesy of University of North Carolina) with isovalue $\rho = 60$ and $\rho = 90$, respectively; **(b)** the *MR brain data set* (courtesy of University of North Carolina) with isovalue $\rho = 40$

$$I(kf) = \int_a^b k(x) f(x) \mathrm{d}x, \quad [a, b] \subset \mathbb{R}, \tag{9}$$

or of the Cauchy Principal Value (CPV) integral

$$J(wf; \lambda) = ⨍_a^b w(x) \frac{f(x)}{x - \lambda} \mathrm{d}x, \qquad a < \lambda < b, \tag{10}$$

where k is a singular, but absolutely integrable function, f is a bounded function for the case (9) and w, f are such that $J(wf; \lambda)$ exists for the case (10). Univariate point QIOs are useful tools to construct quadrature formulas both for (9) and for (10). In [25, 28, 29, 35, 39–42] we generate integration rules based on the approximation of f in (9) or in (10) by point QI splines, we prove a very satisfactory error theory and we provide experimental results.

By means of product of quadratures such as those obtained for (10), in [27] we construct and study cubature formulas for the numerical evaluation of the following

CPV integral:

$$\tilde{J}(w_1 w_2 f; \lambda, \mu) = \fint_{-1}^{1}\fint_{-1}^{1} w_1(x) w_2(y) \frac{f(x, y)}{(x-\lambda)(y-\mu)} \mathrm{d}x\mathrm{d}y$$

where $w_1(x) = (1-x)^{\alpha_1}(1+x)^{\beta_1}$, $w_2(y) = (1-y)^{\alpha_2}(1+y)^{\beta_2}$, $\alpha_i, \beta_i > -1$, $i = 1, 2$ and $-1 < \lambda, \mu < 1$.

For $\Omega = [a, b] \times [c, d]$ and $s = 2$ in (1) and (3), cubatures for the evaluation of integrals

$$I(f) = \int_{\Omega} f(x, y)\mathrm{d}x\mathrm{d}y, \quad f \in C(\Omega), \ (x, y) \in \mathbb{R}^2 \tag{11}$$

are generated in [30, 56] by approximating f with point QI splines, defined on a criss-cross triangulation with partitions of kind (4) and (5), as follows

$$I(f) \approx I(Qf) = \int_{\Omega} Qf(x, y)\mathrm{d}x\mathrm{d}y = \begin{cases} \sum_{\alpha \in A} \omega_\alpha \lambda_\alpha(f) \\ \sum_{\alpha \in M} w_\alpha f(P_\alpha) \end{cases},$$

where $\alpha = (i, j)$ and

$$\omega_\alpha = \int_{\Omega} B_\alpha(x, y)\mathrm{d}x\mathrm{d}y, \qquad w_\alpha = \int_{\Omega} L_\alpha(x, y)\mathrm{d}x\mathrm{d}y. \tag{12}$$

Since the B_α's are known, we can compute ω_α in (12). Moreover, for $Q = S_1, S_2, W_2$ (see Table 1) we get a closed expression of the cubature weights w_α in (12), for which we prove some interesting computational features: for example some symmetry properties and the local support of B-splines lead to cubature formulas with reduced number of weights. An application of the above rules to 2D finite part integral evaluation is presented in [31], where cubature convergence properties are also proved.

In [22] multilevel spline QIs (8) are used to get new efficient cubature formulas for (11). This procedure is carried out for all three QIOs S_1, S_2, W_2 of Table 1 on both uniform (4) and inside-uniform (5) criss-cross triangulations and with weights $w_{ij}^{(\ell)} := 4^\ell w_{ij}$, $\ell = 0, \ldots, p$ for a $(p+1)-$level QIO and for suitable function evaluation sums, instead of $f\left(P_{ij}\right)$ of a classical $1-$level QIO.

Adding again one dimension, i.e. assuming $s = 3$ in (1), we use a QI spline in a similar way to approximate the integrand function defined on a volume domain. This is done in [44] with the trivariate C^2 quartic spline QIs proposed in [69] and introduced in Sect. 3.

We remark, in case of evaluation of proper integrals, the convergence order can be easily deduced from the approximation order of the spline QI sequence for $h \to 0$: if $||f - Qf||_\infty$ is $O(h^k)$, then also $|I(f) - I(Qf)|$ is $O(h^k)$.

In the case of differentiation, a local spline method based on an optimal non-uniform C^1 quadratic QI spline of the form (1), with $s = 1$, is proposed in [68] and differentiating it the pseudo-spectral derivative at the QI knots and the corresponding differentiation matrix are constructed. Indeed, since

$$Q' f(P_\beta) = \sum_{\alpha \in A} \lambda_\alpha (f) \, B'_\alpha (P_\beta), \quad \beta \in M,$$

the pseudo-spectral derivative at the QI knots $\{P_\beta\}$ can be computed using only the values of f and B'_α at such points. The values of B'_α at the QI knots can be stored in the matrix $D \in \mathbb{R}^{\mathrm{card}(M) \times \mathrm{card}(M)}$ and, defining $\mathbf{v}$ as the vector of components $\mathbf{v}(\beta) = f(P_\beta)$, $\mathbf{v}'$ as the vector of components $\mathbf{v}'(\beta) = Q' f(P_\beta)$, then $\mathbf{v}' = D\mathbf{v}$. Moreover, in [37], for the particular case of uniform knot vectors, the pseudo-spectral derivative at the QI knots and the corresponding differentiation matrices are computed, considering local optimal QI splines of degree 3, 4 and 5 and applications in collocation methods for the solution of some univariate boundary-value problems are given. Regarding the global error, $||f' - Q' f||_\infty$ is $O(h^{k-1})$ if $||f - Qf||_\infty$ is $O(h^k)$, $h \to 0$. We remark a superconvergence phenomenon for odd case degrees is present at the inner QI knots.

5 Integral and differential problems

QI spline models can be very useful for the construction of approximating solution in problems governed by either differential or integral equations.

In this context, the application of NURBS, based on quadratic B-splines, to the solution of partial differential equations with mixed boundary conditions on a given physical domain is provided in [24]. Let $\Lambda \subset \mathbb{R}^2$ be an open, bounded and Lipschitz domain, whose boundary $\partial\Lambda$ is partitioned into two relatively open subsets, Λ_D and Λ_N, i.e. $\emptyset \subseteq \Lambda_D, \Lambda_N \subseteq \partial\Lambda$, $\Lambda_D \neq \emptyset$, $\Lambda_D \cap \Lambda_N = \emptyset$ and $\partial\Lambda = \bar{\Lambda}_D \cup \bar{\Lambda}_N$ and let

$$\begin{cases} -\nabla \cdot (X \nabla \psi) = f, & \text{in } \Lambda, \\ \psi = g, & \text{on } \Lambda_D, \text{ (Dirichlet condition)} \\ \dfrac{\partial \psi}{\partial \mathbf{n}_X} = g_N, & \text{on } \Lambda_N, \text{ (Neumann condition)} \end{cases} \tag{13}$$

be the differential problem, where $X \in \mathbb{R}^{2\times 2}$ is a symmetric positive-definite matrix, $\mathbf{n}_X = X\mathbf{n}$ is the outward conormal vector on Λ_N, $f \in L^2(\Lambda)$, $g_N \in L^2(\Lambda_N)$ and $g \in H^{1/2}(\Lambda_D)$, having denoted by $H^{1/2}(\Lambda_D)$ the space of functions of $L^2(\Lambda_D)$ that are traces of functions of $H^1(\Lambda)$, with $H^1(\Lambda) := \{v \in L^2(\Lambda) : \ D^\alpha v \in L^2(\Lambda), |\alpha| \le 1\}$. Since many domains of interest in applications are often described by conic sections, they can be exactly represented by such NURBS in the form (7). Furthermore, in order to avoid the heavy computations related to their derivatives and integrals, since the computation with B-splines is strictly related to the corresponding NURBS, the same above B-splines are used to get the basis for the solution space of the differential problem. In this way, a unique description of the geometry is kept, while avoiding the use of rational functions in the discretization of the solution. Moreover, to impose non-homogeneous Dirichlet boundary conditions, several spline approximation schemes, also based on quasi-interpolation, are considered. The problem (13) is solved using Galerkin procedure and the numerical solution is able to approximate the exact one achieving the optimal approximation order 3.

Concerning the solution of integral equations, we mention the papers [2, 13, 38, 46]. In particular, spline QIPs of the form (1), with $s = 1$ and Ω a bounded interval, are used for the numerical solution of linear [46] and non linear [13] integral equations of the second kind

$$\varphi - K(\varphi) = f, \tag{14}$$

where K is defined as

$$K(\varphi)(x) := \int_0^1 k(x, y)\varphi(y)\mathrm{d}y, \qquad x \in [0, 1], \qquad \varphi \in C[0, 1],$$

in the linear case, with $k \in C([0, 1]^2)$ and as the Urysohn integral operator

$$K(\varphi)(x) := \int_0^1 k(x, y, \varphi(y))\mathrm{d}y, \qquad x \in [0, 1], \qquad \varphi \in C[0, 1],$$

in the non linear case, with $k(x, y, \varphi)$ a real valued function defined on $[0, 1] \times [0, 1] \times \mathbb{R}$. We assume that, for $f \in C[0, 1]$, (14) has a unique solution φ in both cases. In the linear case collocation, Kantorovich, Sloan and Kulkarni schemes based on QI projectors of degree 2 and 3 are considered and studied, showing that higher orders of convergence can be obtained by Kulkarni scheme. Similarly, in the non linear case collocation and Kulkarni schemes, based on spline QIPs, are proposed. Given a QIP Q, in the collocation method K is approximated by $K^c = QKQ$ and the right hand side f by Qf. The approximate equation is then $\varphi^c - QKQ(\varphi^c) = Qf$. Instead, in Kulkarni's type method K is approximated by $K^k = QK + KQ - QKQ$ and the approximate equation is $\varphi^k - K^k(\varphi^k) = f$. Regarding the convergence of the methods, in both linear and non linear case, the collocation method achieves order 3 for quadratic QIPs and order 4 for the cubic ones, while Kulkarni method achieves order 7 and 8, respectively. In the non linear case Green's function type kernels are also considered and the convergence of collocation and Kulkarni schemes is studied. In this case we have a reduction of the convergence order, according to the smoothness of the kernel.

In [2] spline QIOs, which are not projectors, are applied to solve linear Fredholm integral equations of second kind by using superconvergent Nyström and degenerate kernel methods. Also in this case the presence of Green's function type kernels is investigated and the corresponding error analysis is studied.

In the 2D setting, spline methods for the numerical solution of integral equations

$$\varphi(\boldsymbol{\Gamma}_1) - \int_{\mathbf{S}} k(\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2)\varphi(\boldsymbol{\Gamma}_2)\mathrm{d}\mathbf{S}_{\boldsymbol{\Gamma}_2} = f(\boldsymbol{\Gamma}_1), \quad \boldsymbol{\Gamma}_1 \in \mathbf{S}, \tag{15}$$

on a connected surface $\mathbf{S}$ in $\mathbb{R}^3$, described by a sufficiently smooth map $\mathbf{F} : \Omega \to \mathbf{S}$, with Ω a polygonal domain in $\mathbb{R}^2$, and the kernel $k(\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2)$ continuous for $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2 \in \mathbf{S}$, are proposed in [38], by using optimal superconvergent QIs of the form (1) with $s = 2$, defined on the space of C^1 quadratic splines on uniform criss-cross triangulations.

Therefore, the integral Eq. (15) can be written in the form (14) with

$$K(\varphi)(\mathbf{F}(x, y)) = \int_{\Omega} k(\mathbf{F}(x, y), \mathbf{F}(v, z))\varphi(\mathbf{F}(v, z))|(D_v\mathbf{F} \times D_z\mathbf{F})(v, z)| \, \mathrm{d}v\mathrm{d}z,$$

where $(x, y) \in \Omega$ and $|(D_v\mathbf{F} \times D_z\mathbf{F})(v, z)|$ is the Jacobian of the map $\mathbf{F}(v, z)$. We remark that (15) has a unique solution $\varphi \in C(\mathbf{S})$ for any given $f \in C(\mathbf{S})$. The problem is faced by proposing a modified version of the classical collocation method (achieving convergence order 3) and two spline collocation methods with high order of convergence (achieving convergence order 7). In particular, given the superconvergent QI Q, in the collocation method the integral equation is approximated by $\varphi^c - QK(\varphi^c) = Qf$ and in the spline collocation methods with high order of convergence K is approximated by one of the following finite rank operators $K_i = QK + K_i^* - QK_i^*, i = 1, 2,$ where K_1^* is the degenerate kernel operator defined by

$$K_1^*(\varphi)(\mathbf{F}(x, y)) = \int_{\Omega} Q\left(k(\mathbf{F}(x, y), \mathbf{F}(v, z))|(D_v\mathbf{F} \times D_z\mathbf{F})(v, z)|\right) \varphi(\mathbf{F}(v, z)) \, \mathrm{d}v\mathrm{d}z$$

and K_2^* is the Nyström operator based on Q. Since for many surfaces $\mathbf{S}$, getting the derivatives of $\mathbf{F}$ can be a major inconvenience, both to specify and to program, so surface approximations based on quasi-interpolation (for which the Jacobians are more easily computed) are also considered and the effects on the spline modified collocation method are investigated.

6 Concluding remarks

This work means to be a sum up of the main results obtained by the authors, also in collaboration with other researchers, framed in the literature on spline quasi-interpolation, highlighting the approximation properties and the reconstruction of functions and data, the applications in numerical integration and differentiation and in the numerical solution of integral and differential problems. As proved also by the other cited references, such a technique is still *avant-garde* and it is a useful tool for the construction of new approximation operators, providing good results in several fields of science and engineering for the solution of real problems (imaging, Computer Aided Geometric Design, scientific computing, industry, etc.). Indeed several open problems, regarding extension to higher dimensional problems, adaptive refinement schemes, multiresolution, higher order singularities in quadratures with applications to the numerical solution of integral equations and other interesting issues are currently under investigation.

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Declarations

Conflicts of Interest The authors declare no conflict of interest.

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