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Decay estimates for the double dispersion equation with initial data in real Hardy spaces

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Abstract

We study the Cauchy problem for the linear double dispersion equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u - \Delta u_t = 0, \quad t \geq 0,\ x \in \mathbb{R}^n$$

and we derive long time decay estimates for the solution in L^p spaces and in real Hardy spaces. We employ the obtained results to study the equation with nonlinearity $\Delta f(u)$ and nonsmooth f.

Keywords Double dispersion equation · Cauchy problem · Fourier multiplier estimates · Real Hardy spaces · Decay estimates · Global small data solutions

Mathematics Subject Classification 35L15 · 35L30 · 42B15 · 42B30 · 42B37

1 Introduction

In this paper, we study the Cauchy problem for the generalized double dispersion equation

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u - \Delta u_t = \Delta f(u), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases} \tag{1.1}$$

where f is a Lipschitz-continuous function verifying

$$|f(u)| \leq C|u|^{1+\sigma}, \qquad |f(u) - f(v)| \leq C|u-v|(|u|^\sigma + |v|^\sigma), \tag{1.2}$$

for some $\sigma > 0$ and $C > 0$. For instance, $f(u) = |u|^{1+\sigma}$.

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The structure of the equation in (1.1) may be better understood by writing it as a wave equation with a Bessel potential and a Laplace operator applied to a damping term and a nonlinearity

$$u_{tt} - \Delta u = (1-\Delta)^{-1}\Delta\big(u_t + f(u)\big).$$

As a consequence, problem (1.1) shares some properties of the Cauchy problem for the damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = f(u), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases} \tag{1.3}$$

but the action of the operator $(1-\Delta)^{-1}\Delta$ to the damping term and to the nonlinearity produces completely new effects. Problem (1.3) has been deeply investigated in recent years, in particular, global small data solutions to (1.3) exist if $\sigma > 2/n$ (see [36], see also [19,22]). On the other hand, problem (1.1) shares some properties of the Cauchy problem for the strongly damped wave equation with nonlinearity $\Delta f(u)$:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = \Delta f(u), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x). \end{cases} \tag{1.4}$$

The asymptotic profile of the solution to the corresponding linear model, i.e. (1.4) with $f = 0$, has been recently investigated (see, in particular, [18]). Partial results for the problem with power nonlinearity $f(u)$ instead of $\Delta f(u)$ in (1.4), have been obtained in [14].

In this paper, we exploit the special structure of (1.1) to study the influence of assuming initial data u_0, u_1 in real Hardy spaces $\mathcal{H}^p(\mathbb{R}^n)$ on the existence of global small data solutions to (1.1) (the definition and some properties of real Hardy spaces are collected in Sect. 6). In order to do that, we derive $\mathcal{H}^q - \mathcal{H}^p$ estimates, $0 < q \leq p \leq 2$, for the solution to the linear homogeneous problem

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u - \Delta u_t = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x). \end{cases} \tag{1.5}$$

If $p > 1$, then $\mathcal{H}^p = L^p$. On the other hand, $\mathcal{H}^1 \hookrightarrow L^1$, with a proper inclusion. As a consequence, our estimates are also $L^q - L^p$ estimates when $1 < q \leq p \leq 2$. In this paper, we use the notation $\mathcal{H}^p$ instead of the classical notation H^p to avoid possible confusion with the Sobolev space $W^{p,2}$.

Assuming initial data in real Hardy spaces $\mathcal{H}^q$, $q \in (0,1]$, or in L^q spaces, $q \in [1,2]$, and in the energy space, one may easily derive the following long time decay estimates for the solution and its derivatives, on L^2 basis.

Theorem 1.1 *Let $n \geq 1$, $q_0, q_1 \in (0,2]$, and $j \in \mathbb{N}$. Assume that $u_0 \in \mathcal{H}^{q_0} \cap W^{j+1,2}$ and $u_1 \in \mathcal{H}^{q_1} \cap W^{j,2}$. Then the solution $u \in \mathcal{C}([0,\infty), W^{j+1,2}) \cap \mathcal{C}^1([0,\infty), W^{j,2})$ to (1.5) verifies the decay estimate*

$$\begin{aligned} \|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q_0}-\frac{1}{2}\right)+k+|\alpha|\right)} \|u_0\|_{\mathcal{H}^{q_0}} \\ &\quad + C(1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q_1}-\frac{1}{2}\right)-1+k+|\alpha|\right)} \|u_1\|_{\mathcal{H}^{q_1}} \\ &\quad + Ce^{-ct}\left(\|u_0\|_{W^{k+|\alpha|,2}} + \|u_1\|_{W^{k+|\alpha|-1,2}}\right), \end{aligned} \tag{1.6}$$

for any $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$, with $1 \leq k + |\alpha| \leq j + 1$ and $t \geq 0$, for some $C, c > 0$, independent of the initial data. Decay estimate (1.6) is also valid for $k = |\alpha| = 0$, provided that

$$n\left(\frac{1}{q_1} - \frac{1}{2}\right) \geq 1.$$

Moreover, if $q_j = 1$, we may replace $\mathcal{H}^1$ by L^1 in (1.6), provided that $n \geq 3$, if $q_1 = 1$ and $k = |\alpha| = 0$.

In particular, setting $k + |\alpha| = 1, 2$ in Theorem 1.1, we find that the energy for (1.5) given by

$$E(t) = \|u_t(t,\cdot)\|_{L^2}^2 + \|\nabla u_t(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2 + \|\Delta u(t,\cdot)\|_{L^2}^2,$$

verifies the decay estimate

$$E(t) \leq C(1+t)^{-\min\left\{n\left(\frac{1}{q_1}-\frac{1}{2}\right),\, n\left(\frac{1}{q_0}-\frac{1}{2}\right)+1\right\}} \left(E(0) + \|u_0\|_{\mathcal{H}^{q_0}}^2 + \|u_1\|_{\mathcal{H}^{q_1}}^2\right).$$

By using a Mikhlin–Hörmander type multiplier theorem, which provides $\mathcal{H}^p$ boundedness of parameter-dependent operators, we are also able to estimate the solution in real Hardy spaces $\mathcal{H}^p$ with $p < 2$ (we recall that $\mathcal{H}^p = L^p$ for $p > 1$). However, in this case the oscillations coming from the wave part of the equation produces two issues: a loss of regularity which is known from the theory of damped wave equations when one works in L^p spaces with $p \in [1, 2)$ and in real Hardy spaces $\mathcal{H}^p$ with $p \in (0, 1]$ (see [26]), and a loss of decay rate, which is known from the theory of strongly damped wave equations.

The reason behind the fact that the oscillations produce both bad effects related to two class of damped waves is that the profile of the solution to the double dispersion equation contains oscillations both at low and high frequencies. On the one hand, in classical damped wave equations as in (1.3) with $f = 0$, oscillations only appear at high frequencies, so that they do not influence the decay rate of the solution, but only its regularity (see, in particular, [25,29,30]). On the other hand, in strongly damped wave equations as

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases} \tag{1.7}$$

oscillations only appear at low frequencies, so that they do not influence the regularity of the solution, but only its decay rate. This loss of regularity and of decay is determined by the distance of $\mathcal{H}^p$ from L^2, in particular, by the quantity

$$\theta = \theta(n, p) = n\left(\frac{1}{p} - \frac{1}{2}\right). \tag{1.8}$$

This phenomenon of a behavior related to a partial influence of oscillations at low or high frequencies also appear in wave equations with so-called structural damping $(-\Delta)^\theta u_t$, $\theta \in [0, 1]$, as in

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^\theta u_t = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x),\ u_t(0, x) = u_1(x). \end{cases} \tag{1.9}$$

or more general damped evolution equations, see [1–4,6,8,9,11,31]. A classification of models with time-dependent coefficients for which oscillations influence either the decay rate of the solution or its regularity is given in [10].

With respect to the study carried on in these works, the novelty of the equation in (1.1) is that the oscillations influence the whole extended phase space, not only its low frequency or high frequency region. At the opposite of this scenario, we find the wave equation with double dissipation, for which the influence of oscillations is neglected at both low and high frequencies, see [7,20], and the damped Klein–Gordon equation, for which the influence of oscillations at low frequencies may be ignored [13].

We are now ready to state the analogous of Theorem 1.1, for solutions in L^p or $\mathcal{H}^p$ spaces, in which the loss $\theta(n, p)$, both of regularity and decay rate, due to the special structure of the oscillations for the equation in (1.1), comes into play.

Theorem 1.2 *Let $n \geq 1$, $p \in (0, 2]$, $q_0, q_1 \in (0, p]$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Let $\theta(n, p)$ be as in (1.8). Assume that $u_0 \in \mathcal{H}^{q_0}$ with $(1-\Delta)^{\frac{\theta+k+|\alpha|}{2}} u_0 \in \mathcal{H}^p$, and $u_1 \in \mathcal{H}^{q_1}$ with $(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}} u_1 \in \mathcal{H}^p$. Moreover, assume that*

$$n\left(\frac{1}{q_1} - \frac{1}{p}\right) \geq 1,$$

if $k = |\alpha| = 0$. Then the solution to (1.5) verifies the estimate

$$\begin{aligned} \|\partial_t^k \partial_x^\alpha u(t, \cdot)\|_{\mathcal{H}^p} \leq{} & C(1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q_0} - \frac{1}{p}\right) - \theta + k + |\alpha|\right)} \|u_0\|_{\mathcal{H}^{q_0}} \\ & + C(1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q_1} - \frac{1}{p}\right) - \theta - 1 + k + |\alpha|\right)} \|u_1\|_{\mathcal{H}^{q_1}} \\ & + Ce^{-ct} \|(1-\Delta)^{\frac{\theta+k+|\alpha|}{2}} u_0\|_{\mathcal{H}^p} \\ & + Ce^{-ct} \|(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}} u_1\|_{\mathcal{H}^p}, \end{aligned} \tag{1.10}$$

for any $t \geq 0$ and for some $C, c > 0$, independent of the initial data.

Taking $p = 2$ in (1.10), we find estimate (1.6) in Theorem 1.1, except for the special case of $u_0, u_1 \in L^1$.

Remark 1.3 We don't expect the optimality of estimates (1.10), in general. The possible improvement of the decay estimates in Theorem 1.2 will be object of future investigations.

The decay estimates provided in Theorem 1.1 are useful to study the global existence of small data solutions to (1.1).

Assuming initial data in the (weak) energy space $W^{1,2}\times L^2$, it is easy to show that local (weak) energy solutions $u\in\mathcal{C}([0,T),W^{1,2})\cap\mathcal{C}^1([0,T),L^2)$ to (1.1) exist in space dimension $n=1,2$, for any $\sigma>0$, and in space dimension $n\geq 3$, for any $\sigma\in(0,2/(n-2)]$. The upper bound is due to the embedding $W^{1,2}\hookrightarrow L^{2(1+\sigma)}$.

Assuming small initial data in a suitable space, global existence of small data solutions to (1.1) have been proved in [23, Theorem 4.2] for smooth f, in particular, with $f(u)=u^2$. Our aim is to prove global existence of small data (weak) energy solutions for nonsmooth f, assuming $\sigma>\bar\sigma(n)$ in (1.2), where $\bar\sigma$ is the positive solution to

$$n\sigma^2+(n+2)\sigma-2=0,$$

and $\sigma\leq 2/(n-2)$ if $n\geq 3$. Explicitly, we have

$$\bar\sigma(n)=\frac{\sqrt{(n+2)^2+8n}-(n+2)}{2n}, \tag{1.11}$$

so that

$$\frac{2}{n+4}<\bar\sigma(n)<\frac{2}{n+2},\qquad \bar\sigma(n)=\frac{2}{n+4}+O(n^{-3}),\quad \text{as } n\to\infty.$$

In particular, $\bar\sigma(n)<1$, for any $n\in\mathbb{N}$, that is, we may deal with nonsmooth nonlinearities in any space dimension $n\geq 1$. We also notice that $\bar\sigma(n)$ is smaller than Fujita exponent $2/n$, the critical exponent for global small data solutions to (1.3).

Theorem 1.4 *Let $n\geq 1$ and assume that $\sigma>\bar\sigma$ in (1.2), where $\bar\sigma(n)$ is as in (1.11). Moreover, let $\sigma\leq 2/(n-2)$ if $n\geq 3$. We fix $m=1$ if $\sigma\geq 1$ or $m=2/(1+\sigma)$ otherwise, and we define*

$$m^*=\frac{mn}{n+m},\qquad m^{**}=\frac{mn}{n+2m}.$$

Then there exists $\varepsilon>0$ such that for any initial data

$$(u_0,u_1)\in\mathcal{A},\quad \textit{with } \|(u_0,u_1)\|_{\mathcal{A}}\leq\varepsilon,$$

where

$$\mathcal{A}=(\mathcal{H}^{m^*}\cap W^{1,2})\times(\mathcal{H}^{m^{**}}\cap L^2),$$

there exists a unique solution $u \in \mathcal{C}([0,\infty), W^{1,2}) \cap \mathcal{C}^1([0,\infty), L^2)$ *to* (1.1). *Moreover, the solution verifies the decay estimate*

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^2} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1+k+|\alpha|}{2}} \|(u_0,u_1)\|_{\mathcal{A}}, \tag{1.12}$$

for $k+|\alpha| = 0, 1$, *and for some* $C > 0$, *independent of the initial data.*

Remark 1.5 Decay estimate (1.12) corresponds to take $q_0 = m^*$ and $q_1 = m^{**}$ in Theorem 1.1. In particular, $q_1 < 1$ in space dimension $n \leq 4$, for any $\sigma > \bar{\sigma}$, that is, assuming initial data in real Hardy spaces may bring real benefits to study (1.1). On the other hand, $q_0 > q_1 > 1$ in space dimension $n \geq 7$, for any $\sigma \leq 2/(n-2)$, that is, it is sufficient to assume small initial data in L^p spaces.

In general, the best decay rate obtained for the solution to (1.3), with f as in (1.2) and $\sigma > 2/n$, corresponds to take initial data in L^1 and in the energy space $W^{1,2} \times L^2$ (unless special assumption are taken on f). The extra decay rate produced for the solution to (1.1) in low space dimension, is due to the special structure of the equation [see later, the property of the operator $G^\sharp$ in (2.6)], which allows to derive benefit assuming initial data in real Hardy spaces.

Indeed, linear estimates with initial data in real Hardy spaces for some damped evolution equations [which generalize (1.3)] have been previously considered in [12], but they could not be effectively applied to study nonlinear problems, due to the lack of the special structure of the nonlinearity, as in (1.1) (see Sect. 2).

1.1 Discussion about the critical exponent $\bar{\sigma}(n)$

An alternative strategy to derive an existence result for the semilinear problem (1.1) for nonsmooth f ($\sigma \in (0,1)$) could be the use of $L^1 - L^{1+\sigma}$ estimates, instead of $L^{\frac{2}{1+\sigma}} - L^2$ estimates, at low frequencies. This strategy has been effectively employed in the study of semilinear damped wave Eq. (1.3) by Narazaki [29]. However, the presence of oscillations imposes a upper bound on the highest possible space dimension that can be considered, i.e. $n \leq 5$. This bound on the space dimension can be relaxed for wave equations with effective, structural, damping, when $\theta \in (0, 1/2)$ in (1.9), with power nonlinearity $|u|^p$, as shown in [9,11]. The upper bound disappears in the special case $\theta = 1/2$ (see [6]).

For problem (1.1) a similar upper bound would appear (whereas we can deal with any space dimension $n \geq 1$ in Theorem 1.4), due to the presence of high frequencies oscillations. The bound is expected to be $\sigma \geq (n-2)/(n+2)$, obtained by the solution to $\theta(n, 1+\sigma) \leq 1$, which corresponds to the $\mathcal{H}^{1+\sigma}$-boundedness of the solution operator G and of the operator $G^\sharp$, as well (see later, estimate (4.1) in Lemma 4.1). As one may easily checks, the critical exponent $\bar{\sigma}$ given by (1.11) is greater than $(n-2)/(n+2)$ in any space dimension $n \geq 4$. As a consequence, this strategy may be effective to lower the critical exponent $\bar{\sigma}$ only in space dimension $n \leq 3$.

However, the presence of oscillations in (1.1) also at low-frequencies, caused a loss of decay rate in Theorems 1.2 and 2.2. This loss makes the decay rate of $L^1 - L^{1+\sigma}$ estimates (1.10) worse than the decay rate of the $L^{\frac{2}{1+\sigma}} - L^2$ estimates. Indeed,

$$n\left(1-\frac{1}{1+\sigma}\right)-\theta(n,1+\sigma)=n\left(\frac{3}{2}-\frac{2}{1+\sigma}\right)<n\left(\frac{2}{1+\sigma}-\frac{1}{2}\right).$$

For the above reasons, the estimates in Theorems 1.1 and 2.1 appear better than the estimates in Theorems 1.2 and 2.2, to study the semilinear problem (1.1). As a consequence, we cannot improve our global existence result (namely obtaining existence for some power $\sigma \leq \bar{\sigma}(n)$), only by relying on the $L^1 - L^{1+\sigma}$ estimates derived in Theorem 2.2, as one may do in the case of classical damped waves.

It remains open the possibility that other strategies may bring an improvement on the exponent $\bar{\sigma}(n)$. In particular, the possible improvement of the estimates in Theorems 1.2 and 2.2, and the use of the new estimates to lower the critical exponent in space dimension $n = 1, 2, 3$, will be object of future investigations.

1.2 Initial data in weighted L^1 spaces

The assumption of initial data in real Hardy spaces, allows us to produce additional decay rate in Theorem 1.1, with respect to the assumption of data in L^1. This additional decay rate is essential to prove Theorem 1.4. However, instead of assuming initial data in real Hardy spaces, it is possible to assume initial data in weighted L^1 spaces, namely, in

$$L^{1,\gamma} = \{u \in L^1 : |x|^\gamma u \in L^1\},$$

and assume that they verify a suitable moment condition. In particular, if

$$\int_{\mathbb{R}^n} u_0(x)\,dx = \int_{\mathbb{R}^n} u_1(x)\,dx,$$

then the spaces $\mathcal{H}^{q_0}$ and $\mathcal{H}^{q_1}$ may be replaced by the space $L^{1,\gamma}$ in Theorem 1.1, provided that

$$n\left(\frac{1}{q_j}-1\right) \leq \gamma \leq 1.$$

The proof relies on the pointwise estimate for the Fourier transform of a function $f \in L^{1,\gamma}$, with $\gamma \in (0, 1]$, employed in [17]:

$$|\hat{f}(\xi)| \leq |\xi|^\gamma \, \|f\|_{L^{1,\gamma}} + |P|, \qquad P = \int_{\mathbb{R}^n} f(x)\,dx.$$

Combining the previous estimate with Plancherel's theorem, it is possible to prove Theorem 1.1 assuming initial data in weighted spaces, verifying the moment conditions. Analogous estimates may be derived for functions in $L^{1,\gamma}$, with $\gamma > 1$, which involve higher order moments. The corresponding pointwise estimate for the Fourier transform of a function in real Hardy space $\mathcal{H}^q$, $q \in (0, 1]$, is (see Chapter III, Corollary 7.21 in [16]):

$$|\hat{f}(\xi)| \lesssim |\xi|^{n\left(\frac{1}{q}-1\right)} \|f\|_{\mathcal{H}^q}.$$

In view of this alternative for initial data in Theorem 1.1, the smallness assumption of initial data in real Hardy spaces in Theorem 1.4, may be replaced by the assumption of small initial data in suitable weighted L^1 spaces, verifying the moment conditions. We address the interested reader to [4,18,20,21], and the reference therein, for a series of results in which the assumption of initial data in weighted L^1 spaces is used to derive linear estimates for dissipative evolution equations and to determinate the asymptotic profile of the solution.

However, the assumption of initial data in real Hardy spaces, and not in weighted L^p spaces, has the advantage that it allows to derive $\mathcal{H}^p - \mathcal{H}^q$ estimates, relying on Fourier multiplier theorems (in particular, we will use Theorems 6.3 and 6.4 in Sect. 6).

1.3 Physical derivation of the problem

The main interest of this paper is in the study of new qualitative properties of problem (1.1) from a mathematical point of view. However, even if problem (1.1) is interesting by itself from a theoretical mathematical point of view, it is originated by a real world physical problem.

A presentation of the model is provided in [38]: in some problems of nonlinear wave propagation in waveguides, in case of energy exchange between the surface of nonlinear elastic rod in material (e.g., the Murnaghan material) and an external medium, the following double dispersion equation (DDE)

$$u_{tt} - \Delta u = \frac{1}{4}(6\Delta u^2 + a\Delta u_{tt} - b\Delta^2 u) \tag{1.13}$$

and the general cubic DDE (CDDE)

$$u_{tt} - \Delta u = \frac{1}{4}(c\Delta u^3 + 6\Delta u^2 + a\Delta u_{tt} - b\Delta^2 u + d\Delta u_t) \tag{1.14}$$

can be derived from Hamilton Principle. Here $u(t,x)$ is proportional to strain $\frac{\partial U}{\partial x}$, where $U(t,x)$ is the longitudinal displacement, $a > 0$, $b > 0$, and $d \neq 0$ are some constants depending on the Young modulus, the shear modulus μ, density of waveguide ρ and the Poisson coefficient ν. Equations (1.13) and (1.14) were studied in literature, the travelling wave solutions, depending upon the phase variable $z = x \pm ct$ were studied by Samsonov [33,34], the strain solutions of Eqs. (1.13) and (1.14) were observed in [24,35].

Setting $a = b = d = 4$ for simplicity, Eq. (1.14) with $c = 0$ is a special case of (1.1) with $f(u) = 3u^2/2$, so that $\sigma = 1$ in (1.2).

The double dispersion equation has been well investigated in recent times, in particular see [5,32,38,39].

Notation

In this paper, we use the following.

Notation We denote by $\mathscr{F}$ the Fourier transform with respect to the space variable x,

$$\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^n} \varphi(x)e^{-ix\xi}\,dx,$$

and we write $\hat{\varphi}(\xi) = \mathscr{F}f(\xi)$, and $\hat{\varphi}(t,\xi) = (\mathscr{F}\varphi(t,\cdot))(\xi)$.

Notation In this paper differential operators are denoted by $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, where $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the length of α.
With the symbol Δ we denote the Laplace operator as $\Delta = \sum_{i=1}^n \partial_{x_i}^2$. Fractional powers $s > 0$ of $-\Delta$ and $1-\Delta$ are intended as defined by their action

$$(-\Delta)^s\varphi = \mathscr{F}^{-1}(|\xi|^{2s}\hat{\varphi}), \qquad (1-\Delta)^s\varphi = \mathscr{F}^{-1}(\langle\xi\rangle^{2s}\hat{\varphi}),$$

where

$$\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}.$$

Similarly, we define the Riesz and Bessel potentials (see also Sect. 6) for $s > 0$:

$$I_s\varphi = \mathscr{F}^{-1}(|\xi|^{-s}\hat{\varphi}), \qquad (1-\Delta)^{-s}\varphi = \mathscr{F}^{-1}(\langle\xi\rangle^{-2s}\hat{\varphi}).$$

By $W^{m,p}$, $p \in [1,\infty]$, we denote the usual Sobolev space of L^p functions with derivatives up to the order m in L^p, recalling that $W^{m,p} = (1-\Delta)^{-\frac{m}{2}}L^p$ if $p > 1$.

Notation Let $f, g\colon \Omega \to \mathbb{R}$ be two functions. We use the notation $f \approx g$ if there exist two constants $C_1, C_2 > 0$ such that $C_1 g(y) \le f(y) \le C_2 g(y)$ for all $y \in \Omega$. If the inequality is one-sided, namely, if $f(y) \le Cg(y)$ (resp. $f(y) \ge Cg(y)$) for all $y \in \Omega$, then we write $f \lesssim g$ (resp. $f \gtrsim g$).

The definition of real Hardy spaces $\mathcal{H}^p$ and some of their properties are collected in Sect. 6.

2 Fundamental solution and decay estimates

We consider the linear problem

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u - \Delta u_t = \Delta f(t,x), \quad t \ge 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases} \tag{2.1}$$

which corresponds to (1.1), by replacing the nonlinearity $f(u)$ with a inhomogeneous term $f(t,x)$.

We introduce the t-dependent Fourier multiplier $\hat{G}(t,\xi)$ given by

$$\hat{G}(t,\xi) = e^{-\eta t}\,\frac{\sin(\omega t)}{\omega}\,, \tag{2.2}$$

$$\eta = \eta(\xi) = \frac{|\xi|^2}{2\langle\xi\rangle^2}\,, \tag{2.3}$$

$$\omega = \omega(\xi) = |\xi|\sqrt{1-\frac{|\xi|^2}{4\langle\xi\rangle^4}}\,, \tag{2.4}$$

recalling that $\langle\xi\rangle^2 = 1+|\xi|^2$. The solution to (2.1) may be written as

$$\begin{aligned} u(t,x) &= (H(t,\cdot)*u_0)(x) + (G(t,\cdot)*u_1)(x) \\ &\quad + \int_0^t \big(G^\sharp(t-s,\cdot)*f(s,\cdot)\big)(x)\,ds, \end{aligned} \tag{2.5}$$

where the fundamental solution is $G(t,x) = \mathscr{F}^{-1}\hat{G}(t,\xi)$ and

$$G^\sharp(t,x) = (1-\Delta)^{-1}\Delta G(t,x), \tag{2.6}$$

$$H(t,x) = \big(\partial_t - (1-\Delta)^{-1}\Delta\big)G(t,x) = G_t(t,x) - G^\sharp(t,x). \tag{2.7}$$

Indeed,

$$\hat{H}(t,\xi) = e^{-\eta t}\left(\cos(\omega t) + \eta\frac{\sin(\omega t)}{\omega}\right) = (\partial_t + 2\eta)\left(e^{-\eta t}\,\frac{\sin(\omega t)}{\omega}\right),$$

and $2\eta = |\xi|^2\langle\xi\rangle^{-2}$. From the point of view of regularity, the kernels $G(t,x)$ and $G^\sharp(t,x)$ behave in the same way. In particular, $G(t,x)*_{(x)}$ and $G^\sharp(t,x)*_{(x)}$ are L^p-bounded operators if $n|1/p-1/2|\leq 1$ when $n\geq 3$. On the other hand, $G(t,x)$ has the following asymptotic expression (see Remark 4.3 in [37])

$$G(t,x) \sim \mathcal{G}(t,x) *_{(x)} \mathcal{E}(t,x), \quad \text{“in some sense”},$$

where $\mathcal{G}(t,x)$ and $\mathcal{E}(t,x)$ are the fundamental solution to the heat equation and the wave equation:

$$\begin{aligned} &(\partial_t - \Delta)\mathcal{G} = 0, \qquad \mathcal{G}(0,\cdot) = \delta, \\ &(\partial_{tt} - \Delta)\mathcal{E} = 0, \qquad \mathcal{E}(0,\cdot) = 0, \quad \mathcal{E}_t(0,\cdot) = \delta. \end{aligned}$$

As a consequence, in space dimension $n\geq 3$, we have the asymptotic profile [37]

$$\|\partial_t^k\partial_x^\alpha G(t,\cdot)*\varphi\|_{L^2} \sim t^{-\frac{n}{4}-\frac{|\alpha|+k-1}{2}}\left(\|\varphi\|_{L^1} + \|\varphi\|_{W^{k+|\alpha|-1,2}}\right), \tag{2.8}$$

$$\|\partial_t^k\partial_x^\alpha G^\sharp(t,\cdot)*\varphi\|_{L^2} \sim t^{-\frac{n}{4}-\frac{1+|\alpha|+k}{2}}\left(\|\varphi\|_{L^1} + \|\varphi\|_{W^{k+|\alpha|-1,2}}\right), \tag{2.9}$$

provided that we assume $n \geq 3$ if $k = |\alpha| = 0$ in the first one. As a consequence,

$$\|\partial_t^k \partial_x^\alpha H(t,\cdot) * \varphi\|_{L^2} \sim t^{-\frac{n}{4}-\frac{|\alpha|+k}{2}} \left(\|\varphi\|_{L^1} + \|\varphi\|_{W^{k+|\alpha|-1,2}}\right), \tag{2.10}$$

provided that

$$\int_{\mathbb{R}^n} \varphi(x)\, dx \neq 0. \tag{2.11}$$

On the other hand, estimates (2.8), (2.9) and (2.10) may be improved if φ verifies the moment condition, i.e.

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 0. \tag{2.12}$$

Indeed, if one is interested in the asymptotic profile of $\|G(t,\cdot) * \varphi\|_{L^2}$ and $\|G^\sharp(t,\cdot) * \varphi\|_{L^2}$ as $t \to \infty$, the oscillations coming from the wave part of the fundamental solution have no influence, so that, by Plancherel theorem, one may, in general, estimate $|\sin(\omega t)|/\omega$ by $1/\omega \approx |\xi|^{-1}$ in (2.2), obtaining

$$\|\partial_t^k \partial_x^\alpha G(t,\cdot) * \varphi\|_{L^2} \lesssim \|\partial_t^k \partial_x^\alpha \mathcal{G}(t,\cdot) * I_1\varphi\|_{L^2}, \tag{2.13}$$

where I_1 denotes the Riesz potential.

The presence of the Riesz potential leads to the loss of $t^{-\frac{1}{2}}$ in the profile in (2.8), with respect to the corresponding profile of

$$\|\partial_t^k \partial_x^\alpha \mathcal{G}(t,\cdot) * \varphi\|_{L^2} \sim t^{-\frac{n}{4}-\frac{|\alpha|+k}{2}} \|\varphi\|_{L^1},$$

valid under the assumption (2.11).

In [37], it has been proved global existence of small data solutions to (1.1) with $f = O(|u|^{1+\sigma})$, for any $\sigma \geq 1$ in space dimension $n \geq 3$.

By assuming $u_1 \in \dot{W}^{-1,1}$, that is, $I_1 u_1 \in L^1$, global existence of small data solutions has been proved in [23, Theorem 4.2] for $f(u) \sim u^m$, with $m \in \mathbb{N}$, $m \geq 2$, in space dimension $n \geq 1$. We emphasize that $I_1 u_1 \in L^1$ implies the moment condition (2.12) for u_1, that is,

$$\int_{\mathbb{R}^n} u_1(x)\, dx = 0,$$

due to $|\xi|^{-1}\hat{u}_1(\xi) \in \mathcal{C}$ by Riemann–Lebesgue theorem, so that $\hat{u}_1(0) = 0$. Indeed, assuming the moment condition for u_1 invalidates the asymptotic profile given in (2.8) and it may improve the corresponding decay estimates for the solution to (1.1).

In particular, moment condition (2.12) is a necessary condition for measurable functions φ to be in real Hardy spaces $\mathcal{H}^p$, with $p \in (0,1]$. Real Hardy spaces may be defined in several ways (see Sect. 6) and $\mathcal{H}^p = L^p$, for $p \in (1,\infty]$, whereas $\mathcal{H}^1 \subset L^1$ with a proper inclusion. The Riesz potential has the following mapping property in real Hardy spaces (see Theorem 6.5 in Sect. 6):

$$I_s : \mathcal{H}^p \to \mathcal{H}^{p^*}, \qquad \frac{1}{p} = \frac{1}{p^*} + \frac{s}{n},$$

for any $s > 0$ and $p, p^* \in (0, \infty)$. In particular, as a consequence, $\mathcal{H}^p \hookrightarrow \dot{W}^{-s,1}$, for $p = n/(n+s)$, due to

$$\|\varphi\|_{\dot{W}^{-s,1}} = \|I_s\varphi\|_{L^1} \lesssim \|I_s\varphi\|_{\mathcal{H}^1} \lesssim \|\varphi\|_{\mathcal{H}^p}.$$

Having this in mind, in view of (2.13) it appears natural to study problem (1.1) with initial data in $\mathcal{H}^p$.

In order to prove our results, we shall study the decay and mapping properties of the operators $G(t,x)*_{(x)}$ and $G^\sharp(t,x)*_{(x)}$, and their derivatives.

Theorem 2.1 *Let $n \geq 1$, $q \in (0, 2]$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Assume that $\varphi \in \mathcal{H}^q \cap W^{k+|\alpha|-1,2}$. Then we have the decay estimate*

$$\begin{aligned}\|\partial_t^k \partial_x^\alpha G^\sharp(t,\cdot) * \varphi\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q}-\frac{1}{2}\right)+1+k+|\alpha|\right)} \|\varphi\|_{\mathcal{H}^q} \\ + e^{-ct}\|\varphi\|_{W^{k+|\alpha|-1,2}},\end{aligned} \tag{2.14}$$

for any $t \geq 0$ and for some $c > 0$. Moreover, if $k + |\alpha| \geq 1$, or

$$n\left(\frac{1}{q} - \frac{1}{2}\right) \geq 1, \tag{2.15}$$

then we have the decay estimate

$$\begin{aligned}\|\partial_t^k \partial_x^\alpha G(t,\cdot) * \varphi\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q}-\frac{1}{2}\right)-1+k+|\alpha|\right)} \|\varphi\|_{\mathcal{H}^q} \\ + e^{-ct}\|\varphi\|_{W^{k+|\alpha|-1,2}},\end{aligned} \tag{2.16}$$

for any $t \geq 0$ and for some $c > 0$. If $q = 1$, we may replace $\mathcal{H}^1$ by L^1 in (2.14) and (2.16), provided that $n \geq 3$ in (2.16) if $k = |\alpha| = 0$.

We recall that $\mathcal{H}^q = L^q$ if $q \in (1, 2]$.

Theorem 2.2 *Let $n \geq 1$, $p \in (0, 2)$, $q \in (0, p]$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Let $\theta(n, p)$ be as in (1.8). Assume that $\varphi \in \mathcal{H}^q$ with $(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}}\varphi \in \mathcal{H}^p$. Then we have the estimate*

$$\begin{aligned}\|\partial_t^k \partial_x^\alpha G^\sharp(t,\cdot) * \varphi\|_{\mathcal{H}^p} \lesssim (1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q}-\frac{1}{p}\right)-\theta+1+k+|\alpha|\right)} \|\varphi\|_{\mathcal{H}^q} \\ + e^{-ct}\|(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}}\varphi\|_{\mathcal{H}^p},\end{aligned} \tag{2.17}$$

for any $t \geq 0$ and for some $c > 0$. Moreover, if $k + |\alpha| \geq 1$, or

$$n\left(\frac{1}{q} - \frac{1}{p}\right) \geq 1, \tag{2.18}$$

then we have the estimate

$$\begin{aligned}\|\partial_t^k\partial_x^\alpha G(t,\cdot)*\varphi\|_{\mathcal{H}^p} \lesssim &(1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q}-\frac{1}{p}\right)-\theta-1+k+|\alpha|\right)}\|\varphi\|_{\mathcal{H}^q}\\ &+e^{-ct}\|(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}}\varphi\|_{\mathcal{H}^p},\end{aligned} \tag{2.19}$$

for any $t\geq 0$ *and for some* $c>0$.

We recall that $\mathcal{H}^p = L^p$ if $p\in(1,2]$, and $\mathcal{H}^q = L^q$ if $q\in(1,2]$. Moreover, we recall that $\mathcal{H}^1 \hookrightarrow L^1$ [that is, (2.19) is also valid replacing $\mathcal{H}^p$ by L^p when $p=1$].

In Theorem 2.2, the term $\theta(n,p)$ produces a loss of decay rate and of regularity, which is due to the oscillations caused by the wave part of the fundamental solution. For the wave equation, it is known that the corresponding loss is reduced to $\theta = (n-1)(1/p-1/2)$ (see [26]). The possibility to reduce the loss of decay rate in (2.19) will be object of future investigations.

It is clear that, setting $p=2$, Theorem 2.2 reduces to Theorem 2.1, except for the fact that we cannot replace $\mathcal{H}^1$ by L^1 in Theorem 2.2, in general.

As a consequence of Theorems 2.1, 2.2 and (2.7), the proofs of Theorems 1.1 and 1.2 follow. In Sects. 3 and 4, we will prove Theorems 2.1 and 2.2.

3 Proof of Theorem 2.1

In order to prove Theorem 2.1, it is convenient to localize φ at low frequencies and at high frequencies.

Proof of Theorem 2.1 We fix $\chi\in\mathcal{C}_c^\infty$, a (compactly supported) cut-off function, with $\chi=1$ in a neighborhood of the origin. For a given function φ, we define

$$\varphi_0 = \mathscr{F}^{-1}(\chi\hat{\varphi}), \qquad \varphi_1 = 1-\varphi_0.$$

We first prove that

$$\|\Delta^j(1-\Delta)^{-j}\partial_t^k\partial_x^\alpha G(t,\cdot)*\varphi_1\|_{L^2} \lesssim e^{-ct}\|\varphi\|_{W^{k+|\alpha|-1,2}}, \tag{3.1}$$

for some $c>0$ which depends on χ. By Plancherel theorem, we may estimate

$$\begin{aligned}&\|\Delta^j(1-\Delta)^{-j}\partial_t^k\partial_x^\alpha G(t,\cdot)*\varphi_1\|_{L^2}\\ &\approx \|(1-\chi)(i\xi)^\alpha|\xi|^{2j}\langle\xi\rangle^{-2j}\partial_t^k\hat{G}(t,\cdot)\hat{\varphi}\|_{L^2}\\ &\leq \||1-\chi|\langle\xi\rangle^{1-k}\partial_t^k\hat{G}(t,\cdot)\|_{L^\infty}\|\langle\xi\rangle^{|\alpha|+k-1}\hat{\varphi}\|_{L^2}.\end{aligned}$$

If $\chi=1$ for any $|\xi|\leq\delta$, then, recalling the definition of $\hat{G}$ in (2.2), it is sufficient to notice that

$$\sup_{|\xi|\geq\delta}\langle\xi\rangle^{1-k}|\partial_t^k\hat{G}(t,\cdot)| \lesssim \sup_{|\xi|\geq\delta}\langle\xi\rangle^{1-k}\frac{1}{\omega}(\eta+\omega)^k e^{-\eta t} \lesssim e^{-ct},$$

with $c = \delta/(2\langle\delta\rangle) > 0$, for $k \geq 0$, where we used $\eta \approx 1$ and $\omega \approx |\xi|$.

In order to complete the proof of Theorem 2.1, it is sufficient to estimate

$$\|(1-\Delta)^{-j}\partial_t^k\partial_x^\alpha G(t,\cdot) * \varphi_0\|_{L^2} \lesssim t^{-\frac{1}{2}\left(n\left(\frac{1}{q}-\frac{1}{2}\right)-1+k+|\alpha|\right)}\|\varphi\|_{\mathcal{H}^q},$$

for $j = 0, 1$, provided that $n(1/q - 1/2) \geq 2$ if $k = |\alpha| = 0$. Let $\varepsilon > 0$ be such that χ is compactly supported in $B = \{|\xi| \leq \varepsilon\}$. For any $|\xi| \leq \varepsilon$, we may estimate

$$\frac{|\xi|^2}{2(1+\varepsilon)^2} \leq \eta \leq \frac{|\xi|^2}{2},$$

whereas $\omega \approx |\xi|$. In particular, $\eta + \omega \approx |\xi|$, as well. Therefore, in B we may estimate

$$|\chi\langle\xi\rangle^{-2j}|\xi|^{|\alpha|}\partial_t^k\hat{G}(t,\cdot)| \lesssim |\chi|\frac{(\eta+\omega)^k}{\omega}e^{-\eta t} \lesssim |\xi|^{|\alpha|+k-1}e^{-c|\xi|^2 t},$$

with $c = 1/(2(1+\varepsilon)^2)$.

If we define

$$\beta = n\left(\frac{1}{q} - \frac{1}{2}\right), \tag{3.2}$$

then, by Plancherel theorem and Riesz potential mapping properties (Theorem 6.5 in Sect. 6), we get

$$\||\xi|^{-\beta}\hat{\varphi}\|_{L^2} \approx \|I_\beta\varphi\|_{L^2} \lesssim \|\varphi\|_{\mathcal{H}^q}.$$

Therefore, we may estimate

$$\begin{aligned}\|(1-\Delta)^{-j}\partial_t^k\partial_x^\alpha G(t,\cdot) * \varphi_0\|_{L^2} &\lesssim \sup_{|\xi|\leq\varepsilon}|\xi|^{|\alpha|+k-1+\beta}e^{-c|\xi|^2 t}\,\||\xi|^{-\beta}\hat{\varphi}\|_{L^2}\\ &\lesssim (1+t)^{-\frac{|\alpha|+k-1+\beta}{2}}\|\varphi\|_{\mathcal{H}^q},\end{aligned}$$

provided that $|\alpha| + k - 1 + \beta \geq 0$, that is, $k + |\alpha| \geq 1$ or $\beta \geq 1$.

To replace $\|\varphi\|_{\mathcal{H}^1}$ by $\|\varphi\|_{L^1}$ we shall modify the proof, due to the fact that the Riesz potential $I_{2/n}$ does not map L^1 into L^2. By Plancherel theorem, Hölder inequality, and Riemann–Lebesgue theorem ($\|\hat{\varphi}\|_{L^\infty} \leq \|\varphi\|_{L^1}$), we may estimate

$$\|(1-\Delta)^{-j}\partial_t^k\partial_x^\alpha G(t,\cdot) * \varphi_0\|_{L^2} \lesssim \||\chi||\xi|^{|\alpha|+k-1}e^{-c|\xi|^2 t}\|_{L^2}\|\varphi\|_{L^1}.$$

The function $|\chi||\xi|^{|\alpha|+k-1}e^{-c|\xi|^2 t}$ is in L^2 for any $t \geq 0$, provided that $|\alpha| + k \geq 1$ or $n \geq 3$. Then we immediately get

$$\||\chi||\xi|^{|\alpha|+k-1}e^{-c|\xi|^2 t}\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-\frac{|\alpha|+k-1}{2}}.$$

This concludes the proof. □

4 Proof of Theorem 2.2

We will separately prove Theorem 2.2 at low and high frequencies. We fix $\chi \in \mathcal{C}_c^\infty$ a (compactly supported) cut-off function, with $\chi = 1$ in a neighborhood of the origin, as in the proof of Theorem 2.1. For a given function φ, we define

$$\varphi_0 = \mathscr{F}^{-1}(\chi\,\hat{\varphi}), \qquad \varphi_1 = 1 - \varphi_0.$$

Then we will prove the following two results which, together, lead to the statement in Theorem 2.2.

Lemma 4.1 *Let $n \geq 1$, $p \in (0,2)$, $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and θ as in (1.8). Assume that $(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}}\varphi \in \mathcal{H}^p$. Then we have the estimate*

$$\|\Delta^j(1-\Delta)^{-j}\partial_t^k\partial_x^\alpha G(t,\cdot) * \varphi_1\|_{\mathcal{H}^p} \lesssim e^{-ct}\|(1-\Delta)^{\frac{\theta+k+|\alpha|-1}{2}}\varphi\|_{\mathcal{H}^p}, \tag{4.1}$$

for any $t \geq 0$ and for some $c > 0$, with $j = 0, 1$.

Lemma 4.2 *Let $n \geq 1$, $p \in (0,2)$, $q \in (0,p]$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Assume that $\varphi \in \mathcal{H}^q$. Moreover, assume that $k+|\alpha| \geq 1$, or that (2.18) holds. Then we have the estimate*

$$\|(1-\Delta)^{-j}\partial_t^k\partial_x^\alpha G(t,\cdot) * \varphi_0\|_{\mathcal{H}^p} \lesssim (1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q}-\frac{1}{p}\right)-\theta-1+k+|\alpha|\right)}\|\varphi\|_{\mathcal{H}^q}. \tag{4.2}$$

for any $t \geq 0$ and for some $c > 0$, with $j = 0, 1$ and θ as in (1.8).

In order to prove $\mathcal{H}^p$ estimates with $p \in (0,2)$, the derivatives of $\hat{G}(t,\xi)$ come into play. We notice that we may estimate

$$|\partial_\xi^\gamma \eta(\xi)| \lesssim |\xi|^{2-|\gamma|}\langle\xi\rangle^{-2}, \tag{4.3}$$

$$|\partial_\xi^\gamma \omega(\xi)| \lesssim |\xi|^{1-|\gamma|}. \tag{4.4}$$

To prove Lemma 4.1, we rely on Theorem 6.3 in Sect. 6.

Proof of Lemma 4.1 We consider the Fourier multiplier (see Definition 6.2)

$$m(t,\xi) = (1-\chi)|\xi|^{2j}\langle\xi\rangle^{-2j-\theta-k-|\alpha|+1}(i\xi)^\alpha\partial_t^k\hat{G}(t,\xi),$$

and we prove that the operator T_m is $\mathcal{H}^p$-bounded, with

$$\|m(t,\cdot)\|_{\mathcal{M}(\mathcal{H}^p)} \lesssim e^{-ct}, \tag{4.5}$$

for some $c > 0$. Due to the fact that $|\xi| \geq \varepsilon$, for some $\varepsilon > 0$ (since $1-\chi$ vanishes in a neighborhood of the origin), it holds $\langle\xi\rangle \approx |\xi|$. Taking into account of (4.3) and (4.4), together with

$$\partial_{\xi_j} e^{-\eta t} = t e^{-\eta t} \partial_{\xi_j} \eta,$$
$$\partial_{\xi_j} \sin(\omega t) = t \cos(\omega t) \partial_{\xi_j} \omega,$$
$$\partial_{\xi_j} \cos(\omega t) = -t \sin(\omega t) \partial_{\xi_j} \omega,$$

we may estimate

$$|\partial_\xi^\gamma \partial_t^k \hat{G}(t,\xi)| \lesssim |\xi|^{k-|\gamma|-1}(1+t+t|\xi|)^{|\gamma|} e^{-\frac{|\xi|^2}{2\langle\xi\rangle^2}t}$$
$$\lesssim |\xi|^{k-1}(1+t)^{|\gamma|} e^{-\frac{\varepsilon^2}{2(1+\varepsilon^2)}t}.$$

In the last estimate, we used that $|\xi| \geq \varepsilon$ to control $|\xi|^{-1}(1+t+t|\xi|) \lesssim 1+t$. Therefore,

$$|\partial_\xi^\gamma m(t,\xi)| \lesssim |\xi|^{-\theta}(1+t)^{|\gamma|} e^{-\frac{\varepsilon^2}{2(1+\varepsilon^2)}t}.$$

By applying Theorem 6.3 in Sect. 6, with $a = 1$ and $A = 1+t$, we obtain

$$\|m(t,\cdot)\|_{\mathcal{M}(\mathcal{H}^p)} \lesssim (1+t)^\theta \, e^{-\frac{\varepsilon^2}{2(1+\varepsilon^2)}t}.$$

Therefore, it is sufficient to take $c < \varepsilon^2/(2(1+\varepsilon^2))$ to obtain (4.5) and conclude the proof. □

To prove Lemma 4.2, we rely on Theorem 6.4 in Sect. 6.

Proof of Lemma 4.2 We define

$$\beta = n\left(\frac{1}{q} - \frac{1}{p}\right),$$

[setting $p = 2$ we find the definition in (3.2)], we consider the Fourier multiplier (see Definition 6.2)

$$m(t,\xi) = \chi |\xi|^\beta \langle\xi\rangle^{-2j} (i\xi)^\alpha \partial_t^k \hat{G}(t,\xi),$$

and we prove that the operator T_m is $\mathcal{H}^p$-bounded, with

$$\|m(t,\cdot)\|_{\mathcal{M}(\mathcal{H}^p)} \lesssim (1+t)^{-\frac{1}{2}\left(n\left(\frac{1}{q}-\frac{1}{p}\right)-\theta-1+k+|\alpha|\right)}. \tag{4.6}$$

By Theorem 6.5 in Sect. 6, we have that

$$\|I_\beta \varphi\|_{\mathcal{H}^p} \leq \|\varphi\|_{\mathcal{H}^q},$$

so the proof of Lemma 4.2 follows from (4.6).

Due to the fact that $|\xi| \leq \delta$ for some $\delta > 0$ (since χ is compactly supported), it holds $\langle \xi \rangle \approx 1$. Thanks to (4.3) and (4.4), we may estimate

$$|\partial_\xi^\gamma \partial_t^k \hat{G}(t,\xi)| \lesssim |\xi|^{k-|\gamma|-1}(1+t|\xi|^2+t|\xi|)^{|\gamma|} e^{-\frac{|\xi|^2}{2\langle\xi\rangle^2}t}$$
$$\lesssim |\xi|^{k-|\gamma|-1}(1+(t|\xi|)^{|\gamma|}) e^{-\frac{|\xi|^2}{2(1+\delta^2)}t}$$

In the last estimate, we used that $|\xi| \leq \delta$ to control $1+t|\xi|^2+t|\xi| \lesssim 1+t|\xi|$. Therefore,

$$|\partial_\xi^\gamma m(t,\xi)| \lesssim |\xi|^{k+|\alpha|+\beta-|\gamma|-1}(1+(t|\xi|)^{|\gamma|}) e^{-\frac{|\xi|^2}{2(1+\delta^2)}t}.$$

Due to $k+|\alpha|+\beta-1 \geq 0$ and $|\xi| \leq \delta$, we may estimate

$$|\xi|^{k+|\alpha|+\beta-1} e^{-\frac{|\xi|^2}{2(1+\delta^2)}t} \lesssim (1+t)^{-\frac{k+|\alpha|+\beta-1}{2}},$$
$$|\xi|^{k+|\alpha|+\beta-1}(t|\xi|)^{|\gamma|} e^{-\frac{|\xi|^2}{2(1+\delta^2)}t} \lesssim (1+t)^{-\frac{k+|\alpha|+\beta-1}{2}} t^{\frac{|\gamma|}{2}}.$$

Estimating $1+t^{\frac{|\gamma|}{2}} \lesssim (1+t)^{\frac{|\gamma|}{2}}$, we obtain

$$|\partial_\xi^\gamma m(t,\xi)| \lesssim (1+t)^{-\frac{k+|\alpha|+\beta-1}{2}} (\sqrt{1+t}\,|\xi|^{-1})^{|\gamma|}.$$

By applying Theorem 6.4 in Sect. 6, with $a=0$ and $A=\sqrt{1+t}$, we obtain

$$\|m(t,\cdot)\|_{\mathcal{M}(\mathcal{H}^p)} \lesssim (1+t)^{-\frac{k+|\alpha|+\beta+\theta-1}{2}},$$

that is, we obtain (4.6) and this concludes the proof. □

5 Proof of Theorem 1.4

Thanks to Theorems 1.1 and 2.1, we may prove Theorem 1.4.

Proof of Theorem 1.4 We define the solution space

$$X = \mathcal{C}\big([0,\infty), W^{1,2}\big) \cap \mathcal{C}^1\big([0,\infty), L^2\big) \tag{5.1}$$

equipped with norm given by

$$\|u\|_X = \sup_{t\in[0,\infty)} (1+t)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{1}{2}} \Big\{\|u(t,\cdot)\|_{L^2} + (1+t)^{\frac{1}{2}} \|(u_t,\nabla u)(t,\cdot)\|_{L^2}\Big\}.$$

Then a function $u \in X$ is a solution to (1.1) if, and only if, it verifies in X the integral equality

$$u(t,x) = \psi(t,x) + Fu(t,x). \tag{5.2}$$

where ψ is the solution to the linear problem (1.5), that is,

$$\psi(t,x) = H(t,x) *_{(x)} u_0(x) + G(t,x) *_{(x)} u_1(x),$$

and $F : u \mapsto Fu$ is the integral operator defined by

$$Fu(t,x) = \int_0^t G^\sharp(t-s,x) *_{(x)} f(u(s,x))\,dx.$$

Thanks to Theorem 1.1, the function ψ verifies the estimate

$$\|\psi\|_X \leq C\,\|(u_0,u_1)\|_{\mathcal{A}}.$$

Indeed, setting $q_0 = m^*$ and $q_1 = m^{**}$, we obtain

$$\|\partial_t^k \partial_x^\alpha \psi(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1+k+|\alpha|}{2}}\,\|(u_0,u_1)\|_{\mathcal{A}},$$

for $k+|\alpha| = 0, 1$. We now want to prove the estimates

$$\|Fu\|_X \leq C\|u\|_X^{1+\sigma}, \tag{5.3}$$

$$\|Fu - Fv\|_X \leq C\|u-v\|_X\left(\|u\|_X^\sigma + \|v\|_X^\sigma\right). \tag{5.4}$$

By standard arguments, from (5.3) it follows that $F+\psi : u \mapsto \psi + Fu$ maps balls of X into balls of X, for small data in $\mathcal{A}$, and that estimates (5.3), (5.4) lead to the existence of a unique solution $u \in X$ to (5.2). Moreover,

$$\|u\|_X \leq C\,\|(u_0,u_1)\|_{\mathcal{A}},$$

that is, the solution to (1.1) verifies estimate (1.12). We simultaneously gain a local and a global existence result. Therefore, we prove (5.3) and (5.4).

We preliminary notice that a function $u \in X$ verifies the following decay estimates:

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^2} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1+k+|\alpha|}{2}}\,\|u\|_X, \tag{5.5}$$

for $k+|\alpha| = 0, 1$. As a consequence of (5.5) and Gagliardo–Nirenberg inequality, a function $u \in X$ also verifies the estimates

$$\|u(t,\cdot)\|_{L^r} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{r}\right)-\frac{1}{2}}\,\|u\|_X, \tag{5.6}$$

for any $r \in [2,\infty)$ if $n = 1, 2$, and for any $r \in [2, 2n/(n-2)]$ if $n \geq 3$.

We now plan to estimate

$$\|\partial_t^k \partial_x^\alpha Fu(t,\cdot)\|_{L^2} \leq \int_0^t \|G^\sharp(t-s,\cdot) * f(u(s,\cdot))\|_{L^2} ds.$$

We split the integral into $[0,t/2]$ and $[t/2,t]$. In $[0,t/2]$, we set $q=m$ in estimate (2.14) in Theorem 2.1, and in $[t/2,t]$, we set $q=2$ in estimate (2.14) in Theorem 2.1. Therefore, we get

$$\begin{aligned}&\|\partial_t^k \partial_x^\alpha Fu(t,\cdot)\|_{L^2}\\ &\quad\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1+k+|\alpha|}{2}} \left(\|f(u(s,\cdot))\|_{L^m} + \|f(u(s,\cdot))\|_{L^2}\right) ds\\ &\qquad + \int_{t/2}^t (1+t-s)^{-\frac{1+k+|\alpha|}{2}} \|f(u(s,\cdot))\|_{L^2}\, ds.\end{aligned}$$

Due to the fact that $u\in X$, thanks to (5.6), we may estimate

$$\begin{aligned}\|f(u(s,\cdot))\|_{L^m} &\leq \|u(s,\cdot)\|_{L^{m(1+\sigma)}}^{1+\sigma}\\ &\leq C\|u\|_X^{1+\sigma}(1+s)^{-\frac{n}{2m}\sigma-\frac{1+\sigma}{2}}, \qquad (5.7)\\ \|f(u(s,\cdot))\|_{L^2} &\leq \|u(s,\cdot)\|_{L^{2(1+\sigma)}}^{1+\sigma}\\ &\leq C\|u\|_X^{1+\sigma}(1+s)^{-\frac{n}{2}\left(\frac{1+\sigma}{m}-\frac{1}{2}\right)-\frac{1+\sigma}{2}}\\ &= C\|u\|_X^{1+\sigma}(1+s)^{-\frac{n}{2m}\sigma-\frac{1+\sigma}{2}-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}. \qquad (5.8)\end{aligned}$$

We used that $m(1+\sigma)\geq 2$ and that $2(1+\sigma)\leq 2n/(n-2)$, that is, $\sigma\leq 2/(n-2)$, in space dimension $n\geq 3$, to apply Gagliardo–Nirenberg inequality.

We now claim that

$$\kappa = \frac{n}{2m}\sigma + \frac{1+\sigma}{2} > 1. \qquad (5.9)$$

If $\sigma\geq 1$, so that $m=1$, (5.9) trivially holds. Now let $\sigma<1$, so that $m=2/(1+\sigma)$. Then

$$\frac{n}{2m}\sigma + \frac{1+\sigma}{2} = \frac{n}{4}\sigma(1+\sigma) + \frac{1+\sigma}{2} = \frac{n}{4}\sigma^2 + \frac{n+2}{4}\sigma + \frac{1}{2},$$

so that (5.9) holds if, and only if, $\sigma>\bar{\sigma}$, where $\bar{\sigma}$ is as in (1.11).

Therefore, using that $1+t-s\approx 1+t$ for $s\in[0,t/2]$ and $1+s\approx 1+t$ for $s\in[t/2,t]$, thanks to (5.7), (5.8), (5.9), we derive

$$\begin{aligned}\|\partial_t^k \partial_x^\alpha Fu(t,\cdot)\|_{L^2} &\lesssim \|u\|_X^{1+\sigma}(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1+k+|\alpha|}{2}} \int_0^{t/2}(1+s)^{-\kappa}\, ds\\ &\quad + \|u\|_X^{1+\sigma}(1+s)^{-\kappa-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \int_{t/2}^t (1+t-s)^{-\frac{1+k+|\alpha|}{2}}\, ds\end{aligned}$$

$$\lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1+k+|\alpha|}{2}} \|u\|_X^{1+\sigma},$$

that is, we proved (5.3). By (1.2) and Hölder inequality, due to the fact that $u \in X$, thanks to (5.6), we may estimate

$$\begin{aligned}
&\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^m} \\
&\quad \le \|(u-v)(|u|^\sigma + |v|^\sigma)(s,\cdot)\|_{L^m} \\
&\quad \le \|(u-v)(s,\cdot)\|_{L^{m(1+\sigma)}} \|(|u|^\sigma + |v|^\sigma)(s,\cdot)\|_{L^{m\left(1+\frac{1}{\sigma}\right)}} \\
&\quad \le C\|(u-v)(s,\cdot)\|_{L^{m(1+\sigma)}} \left(\|u(s,\cdot)\|^\sigma_{L^{m(1+\sigma)}} + \|v(s,\cdot)\|^\sigma_{L^{m(1+\sigma)}}\right) \\
&\quad \le C\|u-v\|_X \left(\|u\|_X^\sigma + \|v\|_X^\sigma\right)(1+s)^{-\frac{n}{2m}\sigma - \frac{1+\sigma}{2}},
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2} \\
&\quad \le C(1+s)^{-\frac{n}{4}}\|u-v\|_X \left(\|u\|_X^\sigma + \|v\|_X^\sigma\right)(1+s)^{-\frac{n}{2m}\sigma - \frac{1+\sigma}{2} - \frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}.
\end{aligned}$$

Following the proof for $\|\partial_t^k \partial_x^\alpha Fu(t,\cdot)\|_{L^2}$, we are then able to obtain

$$\begin{aligned}
&|\partial_t^k \partial_x^\alpha (Fu - Fv)(t,\cdot)\|_{L^2} \\
&\quad \lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1+k+|\alpha|}{2}} \|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^m \cap L^2}\, ds \\
&\qquad + \int_{t/2}^t (1+t-s)^{-\frac{1+k+|\alpha|}{2}} \|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2}\, ds \\
&\quad \lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1+k+|\alpha|}{2}} \|u-v\|_X \left(\|u\|_X^\sigma + \|v\|_X^\sigma\right).
\end{aligned}$$

This proves (5.4), and so we concluded the proof. □

6 Some known results about Fourier multipliers in real Hardy spaces

We recall how the Hardy spaces $\mathcal{H}^p(\mathbb{R}^n)$ are presented by Fefferman and Stein [15]. We use the notation $\mathcal{H}^p$ instead of the classical notation H^p to avoid possible confusion with the Sobolev space $W^{p,2}$.

Fix, once for all, a radial nonnegative function $\phi \in C_c^\infty(\mathbb{R}^n)$ supported in the unit ball with integral equal to 1. For $u \in \mathcal{S}'(\mathbb{R}^n)$ we define the *maximal function* $M_\phi u$ by

$$M_\phi u(x) = \sup_{0<t<\infty} |(u * \phi_t)(x)|,$$

where $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 6.1 Let $0 < p < \infty$. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $\mathcal{H}^p(\mathbb{R}^n)$ if and only if $M_\phi u \in L^p(\mathbb{R}^n)$, i.e.,

$$\|u\|_{\mathcal{H}^p} = \|M_\phi u\|_{L^p} < \infty.$$

For $p = \infty$, we set $\mathcal{H}^\infty(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

The spaces $\mathcal{H}^p(\mathbb{R}^n)$ are independent of the choice of $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)\, dx \neq 0$. For $p = 1$, $\|u\|_{\mathcal{H}^1}$ is a norm and $\mathcal{H}^1(\mathbb{R}^n)$ is a normed space densely contained in $L^1(\mathbb{R}^n)$. For $p > 1$, $\|u\|_{\mathcal{H}^p}$ is a norm equivalent to the usual L^p norm and we denote $\mathcal{H}^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, by abusing notation. For $0 < p \leq 1$, the space $\mathcal{H}^p(\mathbb{R}^n)$ is a complete metric space with the distance

$$d(u, v) = \|u - v\|^p_{\mathcal{H}^p}, \qquad u, v \in \mathcal{H}^p(\mathbb{R}^n).$$

Although $\mathcal{H}^p(\mathbb{R}^n)$ is not locally convex for $0 < p < 1$ and $\|u\|_{\mathcal{H}^p}$ is not truly a norm, we will still refer to $\|u\|_{\mathcal{H}^p}$ as the "norm" of u, as it is customary.

The property $f \in \mathcal{H}^p$ can be characterized by appropriate singular integrals in a way that has some analogy with the earlier maximal characterization [28, Theorem C]: a function $f \in L^2$ belongs to $\mathcal{H}^p$ when $p \in (0, 1]$, if and only if $f \in L^p$ and $R_\alpha f \in L^p$, for $|\alpha| \leq k$, where $k = 1 + [(n-1)(1/p - 1)]$, and $R_\alpha f$ denotes the Riesz transform of f, defined via the Fourier transform by

$$\widehat{R_\alpha f}(\xi) = (i\xi|\xi|^{-1})^\alpha \hat{f}(\xi).$$

Moreover,

$$\|f\|_{\mathcal{H}^p} \approx \sum_{|\alpha| \leq k} \|R_\alpha f\|_{L^p}.$$

A number similar to k also fixes the order of moment conditions which functions in Hardy spaces shall verify. Indeed,

$$\int_{\mathbb{R}^n} x^\alpha f(x)\, dx = 0, \qquad \forall\, |\alpha| \leq [n(1/p - 1)]$$

for any function $f \in \mathcal{H}^p \cap \mathcal{C}_c^\infty$.

In this paper, we use a variant of the celebrated Mikhlin–Hörmander multiplier theorem for Hardy spaces (see [27]) to obtain the boundedness of operators acting on $\mathcal{H}^p(\mathbb{R}^n)$, in Theorem 2.2.

Definition 6.2 Let m be a bounded function on $\mathbb{R}^n$ and consider the operator T_m defined by

$$T_m f = \mathscr{F}^{-1}\big(m(\xi)\hat{f}(\xi)\big). \tag{6.1}$$

We say that m is a Fourier multiplier for $\mathcal{H}^p$ if $T_m f \in \mathcal{H}^p$ for all $f \in \mathcal{H}^p$ and

$$\|T_m f\|_{\mathcal{H}^p} \leq C\|f\|_{\mathcal{H}^p}; \tag{6.2}$$

in other words, if T_m can be extended to a bounded operator from $\mathcal{H}^p$ to $\mathcal{H}^p$.

In this context, $\mathcal{M}(\mathcal{H}^p)$ denotes the set of all the Fourier multipliers for $\mathcal{H}^p$. The norm $\|m\|_{\mathcal{M}(\mathcal{H}^p)}$ is defined to be the operator norm of T_m in $\mathcal{H}^p$, i.e.

$$\|m\|_{\mathcal{M}(\mathcal{H}^p)} = \sup_{f\in\mathcal{H}^p,\, f\neq 0} \frac{\|T_m f\|_{\mathcal{H}^p}}{\|f\|_{\mathcal{H}^p}}. \tag{6.3}$$

Theorem 6.3 *Let $p \in (0, 2)$, and $\theta = \theta(n, p) = n(1/p-1/2)$, as in (1.8). Assume that $m \in C^k(\mathbb{R}^n)$, with $m(\xi) = 0$ in a neighborhood of the origin, and $k = \max\{[\theta], [\frac{n}{2}]\}+1$. If*

$$|\partial_\xi^\gamma m(\xi)| \leq |\xi|^{-a\theta}(A|\xi|^{a-1})^{|\gamma|},\ \ |\gamma| \leq k,$$

for some constant $a \geq 0$ and $A \geq 1$, then $m \in \mathcal{M}(\mathcal{H}^p(\mathbb{R}^n))$ and

$$\|m\|_{\mathcal{M}(\mathcal{H}^p(\mathbb{R}^n))} \leq CA^\theta,$$

where $C > 0$ is a constant independent of A.

Theorem 6.4 *Let $p \in (0, 2)$, and $\theta = \theta(n, p) = n(1/p - 1/2)$, as in (1.8). Assume that $m \in C^k(\mathbb{R}^n\backslash\{0\})$, with $m(\xi) = 0$ for $|\xi| \geq 1$, and $k = \max\{[\theta], [\frac{n}{2}]\} + 1$. If*

$$|\partial_\xi^\gamma m(\xi)| \leq |\xi|^{a\theta}(A|\xi|^{-a-1})^{|\gamma|},\ \ |\gamma| \leq k,$$

for some constant $a \geq 0$ and $A \geq 1$, then $m \in \mathcal{M}(\mathcal{H}^p(\mathbb{R}^n))$ and

$$\|m\|_{\mathcal{M}(\mathcal{H}^p(\mathbb{R}^n))} \leq CA^\theta,$$

where C is a constant independent of A.

Let I_r be the Riesz potential with order $r > 0$, defined by means of $I_r f = \mathscr{F}^{-1}(|\xi|^{-r}\hat{f}(\xi))$. If $r \in (0, n)$, the Riesz potential may be represented for sufficiently smooth f by

$$I_r f(x) = c_{n,r} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-r}}\, dy,$$

for suitable $c_{n,r}$. Real Hardy spaces have the property that the Hardy–Littlewood–Sobolev theorem for Riesz potential, valid in L^p spaces, with $p > 1$, extend to $\mathcal{H}^p$, with $p \in (0, \infty)$, see [28, Theorem F].

Theorem 6.5 *Consider $r > 0$ and $0 < q < n/r$. Then, there exists $C = C(r, q) > 0$ such that*

$$\|I_r f\|_{\mathcal{H}^p(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{H}^q(\mathbb{R}^n)}, \quad \frac{1}{q} = \frac{1}{p} + \frac{r}{n}.$$

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