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Mean Curvature Flow of Singular Riemannian Foliations

Marcos M. Alexandrino[1] · **Marco Radeschi**[2]

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Abstract Given a singular Riemannian foliation on a compact Riemannian manifold, we study the mean curvature flow equation with a regular leaf as initial datum. We prove that if the leaves are compact and the mean curvature vector field is basic, then any finite time singularity is a singular leaf, and the singularity is of type I. This generalizes previous results of Liu–Terng and Koike. In particular, our results can be applied to study the orbits of an isometric action by a compact Lie group.

Keywords Mean curvature flow · Isometric action · Singular Riemannian foliation · Isoparametric foliation

Mathematics Subject Classification 53C12 · 53C44

1 Introduction

Given a Riemannian manifold M and an immersion $\varphi : L_0 \to M$, a smooth family of immersions $\varphi_t : L_0 \to M$, $t \in [0, T)$ is called a solution of the *mean curvature flow* (MCF for short) if φ_t satisfies the evolution equation

✉ Marco Radeschi
mrade_02@uni-muenster.de

Marcos M. Alexandrino
marcosmalex@yahoo.de; malex@ime.usp.br

1 Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508 090 São Paulo, Brazil

2 Mathematisches Institut, WWU Münster, Münster, Germany

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$$\frac{d}{dt}\varphi_t = H(t),$$

where $H(t)$ is the mean curvature of $L(t) := \varphi_t(L_0)$. We say that the MCF φ_t has *initial datum* L_0. By abuse of notation, we will often identify φ_t with its image $L(t)$, and we will talk about the MCF flow $L(t)$.

In [8] Liu and Terng studied the mean curvature flow equation in spheres and Euclidean spaces with an *isoparametric submanifold* as initial datum and they proved, among other things, that such an evolution moves through isoparametric submanifolds up to the (finite time) singularity. Later on, Koike [7] generalized Liu and Terng's results to the mean curvature flow on compact symmetric spaces, with isoparametric submanifolds with flat sections as initial datum.

Given an isoparametric submanifold L, one can partition the ambient manifold into the submanifolds "parallel" to L, which are all isoparametric unless they lie in the focal set of L, in which case they have lower dimension (and they are called *focal submanifolds*). Such a partition is a special example of a *singular Riemannian foliation* i.e., a foliation where every geodesic starting perpendicular to a leaf, stays perpendicular to all the leaves it meets (cf. [11, p. 189]). The results of Liu–Terng and Koike can then be restated by saying that given an isoparametric submanifold L of a sphere, Euclidean space or compact symmetric space, the MCF evolution with L as initial datum moves through the leaves of the foliation induced by L.

Singular Riemannian foliations induced by an isoparametric submanifold (also called *isoparametric foliations*) are characterized by the following two properties:

(i) The mean curvature form is *basic* (cf. Sect. 2).
(ii) The distribution orthogonal to the regular leaves (i.e., the leaves with maximal dimension) is integrable.

If a singular Riemannian foliation satisfies the former condition it is called *generalized isoparametric*. In this paper, we generalize the results of Liu–Terng and Koike to the class of generalized isoparametric foliations on compact Riemannian manifolds.

Despite the name, generalized isoparametric foliations are much more general than isoparametric ones. For example, the following foliations are generalized isoparametric:

(1) Any isometric group action of a connected Lie group G on a Riemannian manifold M induces a singular Riemannian foliation $(M, \mathcal{F})$ given by the orbits of G (*homogeneous foliation*) which is generalized isoparametric. By comparison, isoparametric foliations only appear when the group action is polar.
(2) Any singular Riemannian foliation in spheres or Euclidean space is generalized isoparametric; cf. [1, Remark 4.2]. This includes a new class of foliations, neither homogeneous nor polar, constructed using Clifford algebras; cf. [14]. By contrast, (irreducible) isoparametric foliations in spheres either have cohomogeneity one, or arise from a polar representation [17].
(3) Any singular Riemannian foliation $\mathcal{F}$ on $\mathbb{KP}^n$, ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) lifts to a foliation $\mathcal{F}^*$ on a sphere via the Hopf map $\mathbb{S}^m \to \mathbb{KP}^n$. Moreover, since Hopf fibrations have totally geodesic fibers the mean curvature vector field of the leaves

is preserved under the fibration, and in particular $\mathcal{F}$ is generalized isoparametric. Among these, the (irreducible) isoparametric foliations in $\mathbb{CP}^n$ were recently classified by Domínguez-Vázquez [4] (in this case, there are irreducible inhomogeneous isoparametric foliations if and only if $n+1$ is not prime).

Recall that a leaf of a singular Riemannian foliation is called *regular* if it has maximal dimension, and *singular* otherwise.

Theorem 1.1 *Let $(M, \mathcal{F})$ be a generalized isoparametric foliation with compact leaves on a compact manifold M. Let $L_0 \in \mathcal{F}$ be a regular leaf of M and let $L(t)$ denote the mean curvature flow evolution of L_0 with maximal interval of existence* $[0, T)$. *Then the following statements hold:*

(a) *$L(t)$, $t \in [0, T)$ are regular leaves of $\mathcal{F}$.*
(b) *The singular time T is finite if and only if $L(t)$ converges to a singular leaf L_T of $\mathcal{F}$ and, in this case, the singularity is of type I, i.e.,*

$$\limsup_{t \to T^-} \|A_t\|_\infty^2 (T - t) < \infty$$

where $\|A_t\|_\infty$ is the sup norm of the second fundamental form of $L(t)$.

Remark 1.2 The condition that M is compact can be replaced by the more general assumption that the MCF $L(t)$ with initial datum $L(0) = L_0$ stays at a bounded distance from L_0. This condition can be verified, for example, for any closed singular Riemannian foliation in Euclidean space.

The condition of having a finite time singularity holds in several situations. For example, if L_0 is a compact submanifold in Euclidean space, it follows from [16, Proposition 3.10] that the maximal time of existence T of the mean curvature flow is finite. For any generalized isoparametric foliation, we prove in Proposition 3.3 that there is a neighborhood around the singular leaves in which the MCF has always finite time singularities. If moreover the manifold is nonnegatively curved and the foliation is isoparametric, then we have the following stronger result.

Theorem 1.3 *For every isoparametric foliation on a compact nonnegatively curved space, the MCF with a regular leaf as initial datum has always finite time singularity.*

If we restrict ourselves to special cases, we obtain strengthenings of different already known results:

- Theorems 1.1 and 1.3 together generalize the main results of Liu–Terng [8] and Koike [7] to the case of isoparametric foliations on compact nonnegatively curved manifolds. Moreover, we also show that the flow has type I singularities independently of the singular leaf to which it converges, thus answering in the positive a question posed in [8, Remark].
- If $\mathcal{F}$ is a homogeneous foliation by a Lie group G acting on M, Pacini proved in [12, Theorem 2], among other things, that if a curve of principal orbits $t \to G(t)$ (where $t \in [0, T)$) is a solution of a MCF with finite time singularity, the limit is a singular orbit provided that such a limit exists. By our main result, such a limit always exists.

In general, one cannot expect Theorem 1.3 to hold without the curvature assumption; cf., for example, [12, Example 3]. Other examples of foliations whose MCF can have infinite time existence can be constructed as follows. Let Σ be a surface of revolution diffeomorphic to $\mathbb{S}^2$ with a "dumbbell metric", and an isoparametric (homogeneous) foliation $(\Sigma, \mathcal{F})$ induced by the isometric S^1-action. The central S^1-orbit is a geodesic, thus minimal, and it is easy to see that in this case any nearby orbit will converge to it in infinite time. Notice that for any such metric the foliation $(\Sigma, \mathcal{F})$ is isoparametric, and moreover we can choose the metric to be arbitrarily close to a nonnegatively curved one, in which case by Theorem 1.3 there would be no nonconstant solutions of the MCF with infinite time existence. Therefore, in a suitable sense, the property of a generalized isoparametric foliation to only admit short time MCF is not an open condition. The assumption on the foliation being isoparametric, however, is not crucial. In fact, in certain cases it suffices to have a foliation that is "close enough" to an isoparametric one (cf. Remark 5.4).

Like in the case of orbits of isometric actions in compact manifolds (cf. [12]), it is possible to prove that for generalized isoparametric foliations that the mean curvature of a singular leaf is tangent to the stratum that contains it; see [15, Proposition 2.10] for the case of singular Riemannian foliations in spheres. Moreover it is possible to check, using for example [15, Proposition 2.9], that the mean curvature is again basic in each stratum.

Therefore the restriction of a generalized isoparametric foliation to each stratum is again a generalized isoparametric foliation, and we immediately get the following result.

Corollary 1.4 *Let $(M, \mathcal{F})$ be a generalized isoparametric foliation with compact leaves on a compact manifold M. Let L be a singular leaf and let Σ be the stratum containing L. Let $L(t)$ be the MCF flow with initial datum L and let $[0, T)$ be the maximal interval of existence of the flow. Then the following statements hold.*

(a) *$L(t)$ is a singular leaf in Σ for every $t \in [0, T)$.*
(b) *If $T < \infty$ then L_T converges to a singular leaf L_T with $\dim L_T < \dim L$, and the singularity is of type I.*

This paper is divided as follows. In Sect. 2 we recall the definition and properties of singular Riemannian foliation, while in Sect. 3 we prove Theorem 1.1. The proofs rely on some, somewhat technical, estimates on the shape operator, which are proved in Sect. 4. Section 5 is devoted to the proof of Theorem 1.3.

2 Preliminaries

Given a compact Riemannian manifold M, a singular foliation $\mathcal{F}$ is called a *singular Riemannian foliation* if every geodesic that starts perpendicular to a leaf, stays perpendicular to all the leaves it meets; cf. [11, p. 189]. We denote by $\dim \mathcal{F}$ the maximal dimension of the leaves of $\mathcal{F}$, and call a leaf L *regular* if $\dim L = \dim \mathcal{F}$ and *singular* otherwise. The union of regular leaves is open and dense in M, it is called *regular stratum* and it is denoted by M_{reg}. The union of singular leaves of a fixed dimension is a disjoint union of (possibly non-complete) submanifolds, which we call *singular strata* of $\mathcal{F}$.

If the leaves of $\mathcal{F}$ are closed, the *leaf space* $M/\mathcal{F}$ inherits the structure of a Hausdorff metric space, where the distance between two points is defined as the distance between the corresponding leaves. Moreover, the subset $M_{reg}/\mathcal{F}$ of regular leaves is an orbifold, and the canonical map $\pi : M \to M/\mathcal{F}$ restricts to a (orbifold) Riemannian submersion $M_{reg} \to M_{reg}/\mathcal{F}$. A vector field X on M_{reg} is called *basic* if it projects via π_* to a well-defined vector field in $M_{reg}/\mathcal{F}$.

Given a singular Riemannian foliation $(M, \mathcal{F})$, we denote by A the shape operator of the leaves of $\mathcal{F}$. The *mean curvature* H of $\mathcal{F}$ at a point p is defined as the mean curvature of the leaf L_p through p. Since the regular part of the foliation is defined via a Riemannian submersion, the mean curvature H is smooth on M_{reg}. We will see, however, that the norm of H blows up as it approaches singular strata.

In the regular part of $\mathcal{F}$ the tangent bundle TM splits as $T\mathcal{F} \oplus \nu\mathcal{F}$, where $T\mathcal{F}$ is the bundle of the tangent spaces of the leaves in $\mathcal{F}$, and $\nu\mathcal{F}$ is its orthogonal complement. Moreover, for every regular point $p \in M$ it is possible to define the O'Neill tensor $ON : \nu_p\mathcal{F} \times \nu_p\mathcal{F} \to T_p\mathcal{F}$ as $(x, y) \mapsto ON_x\, y = \frac{1}{2}\mathrm{pr}_{T\mathcal{F}}[X, Y]$ where X, Y are local vector fields extending x, y, and $\mathrm{pr}_{T\mathcal{F}}$ denotes the orthogonal projection onto $T_p\mathcal{F}$. If $ON \equiv 0$ then the orthogonal distribution $\nu\mathcal{F}$ is integrable, and the foliation is called *polar*.

A singular Riemannian foliation is called *generalized isoparametric* if the mean curvature H on M_{reg} is basic. If moreover it is also polar, then it is called *isoparametric*.

2.1 Distinguished Tubular Neighborhoods

Let $(M, \mathcal{F})$ be a singular Riemannian foliation and let q be a point in a (possibly singular) leaf L. We recall (cf. [11, Theorem 6.1, Proposition 6.5], [3,9]) that it is possible to find a neighborhood P of q in L, a neighborhood O_ϵ of q in M and diffeomorphism $\varphi : O_\epsilon \to V \subseteq T_qM$ onto a neighborhood V of the origin, with the following properties:

(1) O_ϵ is the image of the normal exponential map $\exp^\perp : \nu_q^\epsilon P \to M$, where $\nu_q^\epsilon P = \{v \in \nu_q P \mid \|v\| < \epsilon\}$.
(2) The image of $\mathcal{F}|_{O_\epsilon}$ under φ is the restriction to V of a singular Riemannian foliation $(T_qM, \mathcal{F}_q)$.
(3) $\varphi(L \cap O_\epsilon) = T_qL \cap V$.
(4) Under the splitting $T_qM = T_qL \times \nu_qL$ the foliation $\mathcal{F}_q$ splits as well as $T_qL \times (\nu_qL, \mathcal{F}_q \cap \nu_qL)$, i.e., any leaf L' of $\mathcal{F}_q$ is of the form $(L' \cap \nu_qL) \times T_qL$.
(5) For any $\lambda \in (0, 1)$, the *homothetic transformation* $h_\lambda : O_\epsilon \to O_\epsilon$ given by $h_\lambda(\exp_p^\perp(v)) = \exp_p^\perp(\lambda v)$ for any $v \in \nu_p^\epsilon P$, corresponds under φ to the rescaling $(x, w) \mapsto (x, \lambda w)$ for any $(x, w) \in T_qL \times \nu_qL$.

We call O_ϵ a *distinguished tubular neighborhood* of q, and denote it simply with O if ϵ is understood. The map φ is a modification of the normal exponential map and in particular it sends radial geodesics around L to radial geodesics around T_qL.

3 Proof of Theorem 1.1

In this section we let $(M, \mathcal{F})$ be a generalized isoparametric foliation with closed leaves on a compact manifold. We also fix a regular leaf L_0 and assume that the MCF evolution $L(t)$ with initial datum L_0 has maximal interval of existence $[0, T)$, with T finite.

Since the mean curvature of $\mathcal{F}$ is basic, it projects via $\pi : M \to M/\mathcal{F}$ to a vector field on the orbifold $M_{reg}/\mathcal{F}$, and it is immediate to see that $L(t)$ is the preimage of the point $\gamma(t) \in M_{reg}/\mathcal{F}$, where γ is the integral curve of (the projection of) H. In particular $L(t)$ is a leaf of $\mathcal{F}$, and since the dimension of $L(t)$ is constant up to the singular time, we have that $L(t)$ is regular. We thus proved the following.

Proposition 3.1 *For any $t \in [0, T)$, $L(t)$ is a regular leaf of $\mathcal{F}$.*

The rest of the section is devoted to proving statement (b) of Theorem 1.1. In Sect. 4 we prove the following result (cf. Corollary 4.6):

Proposition 3.2 *Let $L_q \in \mathcal{F}$ be a singular leaf. For ϵ small enough, there exist constants δ, c such that in the regular part of $\mathrm{Tub}_\epsilon(L_q)$ the shape operator A of $\mathcal{F}$ satisfies:*

$$-(1+\delta)\frac{D}{r(x)} - c \le tr(A_{\nabla r})_x \le -(1-\delta)\frac{D}{r(x)} + c, \tag{3.1}$$

where $D = \dim \mathcal{F} - \dim L_q$ and $r(x) = \mathrm{dist}(x, L_q)$. Moreover, if $\epsilon' < \epsilon$ then one can choose constants $\delta' \le \delta$ and $c' \le c$ associated with ϵ', and $\lim_{\epsilon \to 0^+} \delta = 0$.

We can now prove the following:

Proposition 3.3 *Given a singular leaf L_q, there exists an $\epsilon = \epsilon(L_q)$ such that if $L(t_0)$ lies in $\mathrm{Tub}_\epsilon(L_q)$ for some $t_0 \in [0, T)$ then the following properties hold:*

(a) *For any $t > t_0$ the distance $r(t) = \mathrm{dist}(L(t), L_q)$ satisfies*

$$C_1^2(t - t_0) \le r^2(t_0) - r^2(t) \le C_2^2(t - t_0) \tag{3.2}$$

where C_1 and C_2 are positive constants that depend only on $\mathrm{Tub}_\epsilon(L_q)$.

(b) *$L(t) \subset \mathrm{Tub}_\epsilon(L_q)$ for all $t \in (t_0, T)$, and $T < t_0 + \frac{\epsilon}{C_1^2}$.*

(c) *If $L(t)$ converges to L_q at time T then for any $t \in (t_0, T]$,*

$$C_1\sqrt{T - t} \le r(t) \le C_2\sqrt{T - t}. \tag{3.3}$$

Proof Start with a tubular neighborhood $\mathrm{Tub}_{\epsilon_0}(L_q)$ in which the distance function $r = \mathrm{dist}_{L_q}$ is smooth away from L_q, and such that Proposition 3.1 holds for some δ and c. Fixing $p \in L(t_0)$, consider the curve $t \to \varphi_t(p)$ such that $\frac{d}{dt}\varphi_t(p) = H(t)$. Of course $\varphi_t(p) \in L(t)$ for all t, and $r(t) = \mathrm{dist}(\varphi_t(p), L_q)$ equals $\mathrm{dist}(L(t), L_q)$ by the equidistance of the leaves. Then we have

$$r'(t) = \langle \nabla r, \varphi_t'(p) \rangle = \langle \nabla r, H(t) \rangle = \mathrm{tr}(A_{\nabla r}).$$

From (3.1),

$$-(1+\delta)\frac{D}{r} - c \leq \mathrm{tr}A_{\nabla r} \leq -(1-\delta)\frac{D}{r} + c.$$

Now we choose $\epsilon < \min\{\epsilon_0, (1-\delta)\frac{D}{c}\}$ and define the constants C_1, C_2 by

$$\frac{C_1^2}{2} = (1-\delta)D - c\epsilon, \qquad \frac{C_2^2}{2} = (1+\delta)D + c\epsilon$$

The above equations imply

$$-\frac{C_2^2}{2r(t)} \leq r'(t) \leq -\frac{C_1^2}{2r(t)}$$

or, equivalently, $-C_2^2 \leq (r^2(t))' \leq -C_1^2$. Integrating this equation we get

$$C_1^2(t-t_0) \leq r^2(t_0) - r^2(t) \leq C_2^2(t-t_0) \tag{3.4}$$

for $t > t_0$ close to t_0. Since $r^2(t)$ is decreasing, $L(t)$ remains in $\mathrm{Tub}_\epsilon(L_q)$ for every $t > t_0$ and this concludes the proof of (a) and (b).

Statement (c) follows directly from (a). □

Remark 3.4 By Proposition 3.3 it immediately follows that if $L(t_0)$ lies in a tubular neighborhood defined as above, then T must be finite. This does not imply, for a generic M, that T is always finite when the initial datum is outside such a tube; see [12, Example 3]. Also note that in the proof of Proposition 3.3, ϵ has been chosen to be small, more precisely smaller than $(1-\delta)D/c$. This was necessary to ensure the existence of the constant C_1 (otherwise C_1^2 would be negative). The fact that ϵ cannot be chosen bigger (even when it would make sense to talk about tubular neighborhoods) is not a limitation of the proof, but it seems to have a geometrical meaning. In fact it is possible to see, e.g., in some examples of isoparametric foliations in Euclidean space, that if the radius of the tube is too big (although the tube is still well defined) then statement (b) of Proposition 3.3 is no longer true, i.e., the MCF of leaves in a tube of big radius can leave the tube after a finite time.

In the next proposition we prove that given a leaf L_0, if the MCF $L(t)$ with $L(0) = L_0$ has finite time singularity then it converges to a singular leaf L_q in the Hausdorff sense, i.e., the projection of $L(t)$ in the quotient space $M/\mathcal{F}$ converges to the projection of L_q. More precisely we prove the first part of statement (b) in Theorem 1.1.

Proposition 3.5 *Let $\mathcal{F}$ be a generalized isoparametric foliation with compact leaves on a complete manifold M and let L_0 be a regular leaf. Suppose that the MCF $L(t)$ with initial datum $L(0) = L_0$ stays in a bounded set, and that $L(t)$ has a finite time singularity. Then $L(t)$ converges in the Hausdorff sense to some singular leaf L_q.*

Proof Since $L(t)$ is contained in a bounded set and T is finite, it follows from [13, Proposition 9.1.4] that the limit set of $L(t)$ cannot be contained in the regular stratum and thus it must be contained in some singular stratum. When $\mathcal{F}$ is homogeneous this also follows from [12, Lemma 2.3].

Now consider a singular leaf L_q in the limit set, and take a sequence $\{t_n\} \subseteq [0, T)$ converging to T. For any arbitrarily small radius ϵ, we can find some t_ϵ such that $L(t_\epsilon) \in \mathrm{Tub}_\epsilon(L_q)$ and, by Proposition 3.3, $L(t) \in \mathrm{Tub}_\epsilon(L_q)$ for every $t \in (t_\epsilon, T)$. Due to the arbitrariness of ϵ we conclude that $L(t)$ converges to L_q. □

In what follows, we consider a singular leaf L_q which is the limit of the MCF $L(t)$ with initial datum L_0. We want to prove that this singularity is of Type I, this finishing the proof of Theorem 1.1.

Fixing a tubular neighborhood $\mathrm{Tub}_\epsilon(L_q)$, we consider the functions r_Σ, f : $\mathrm{Tub}_\epsilon(L_q) \to \mathbb{R}$ such that $r_\Sigma(x)$ is the distance between L_x and the singular strata, and $f(x)$ is the distance between L_x and its focal set. By abuse of notation, we also define $r_\Sigma(t) = r_\Sigma(L(t))$, $f(t) = f(L(t))$.

In Corollary 4.8 we prove the following.

Proposition 3.6 *There exists a constant C, depending on* $\mathrm{Tub}_\epsilon(L_q)$*, such that for any t close enough to the singular time T we have $r_\Sigma(t) \geq Cr(t)$, where $r(t) =$* $\mathrm{dist}(L(t), L_q)$.

Together with Proposition 3.3, we have that there is a constant $C_1' = C_1 C$ such that, close enough to the singular time T, one has

$$r_\Sigma(t) \geq C_1' \sqrt{T - t}. \tag{3.5}$$

Proposition 3.7 *There exists a constant $\sigma \in (0, 1)$ such that $f(p) \geq \sigma\, r_\Sigma(p)$ for every regular point $p \in M$.*

Proof The functions r_Σ and f are constant along the leaves of $\mathcal{F}$, and thus induce functions on the quotient, which we denote with the same letters. By Lytchak and Thorbergsson [9], the first focal point of a leaf L_p corresponds to either a singular leaf, or to a conjugate point in $M/\mathcal{F}$ of the projection of L_p. In the first case, $f(p) = r_\Sigma(p)$ and the proposition is proved.

Suppose now that the projection p^* of L_p into $M/\mathcal{F}$ has a conjugate point along some geodesic segment γ contained in the regular part of $M/\mathcal{F}$. Clearly $r_\Sigma(\gamma(s)) \geq r_\Sigma(p) - s$. From Lytchak and Thorbergsson [9, Remark 1.1], the supremum $\sup(\sec_{M/\mathcal{F}}(x^*))$ of the sectional curvatures at a point x^* in $U/\mathcal{F}$ satisfies

$$\sup(\sec_{M/\mathcal{F}}(x^*)) \leq \frac{K}{r_\Sigma(x^*)^2}, \tag{3.6}$$

for some constant K. Together with the previous equations,

$$\sec_{M/\mathcal{F}}(\gamma(s)) \leq \frac{K}{r_\Sigma(\gamma(s))^2} \leq \frac{K}{(r_\Sigma(p) - s)^2}. \tag{3.7}$$

By Rauch's Comparison Theorem, the first conjugate point along γ appears after the first conjugate point along a geodesic $\overline{\gamma}$ in a model space with curvature $\kappa(\overline{\gamma}(s)) = \frac{K}{(r_\Sigma(p)-s)^2}$.

To compute the conjugate point in such a model, it is enough to find the first positive zero of a solution h to the ODE

$$\begin{cases} h''(s) &= -\frac{K}{(r_\Sigma(p)-s)^2} h(s) \\ h(0) &= 0 \end{cases} \tag{3.8}$$

If we define $g(s) = h(r_\Sigma(p)s)$ then g satisfies the equation

$$\begin{cases} g''(s) &= -\frac{K}{(1-s)^2} g(s) \\ g(0) &= 0 \end{cases} \tag{3.9}$$

and if σ_0 is the first zero of g in $(0, 1)$, then the first zero of h is $\sigma_0 r_\Sigma(p)$ and $f(p) \geq \sigma_0 r_\Sigma(p)$.

On the other hand, if g does not admit any zeroes in $(0, 1)$, then h does not admit any zeroes in $(0, r_\Sigma(p))$ and therefore the first conjugate point along γ appears after $r_\Sigma(p)$. In either case, we proved that $f(p) \geq \sigma\, r_\Sigma(p)$, where

$$\sigma = \begin{cases} \sigma_0 & \text{if there exists a zero } \sigma_0 \text{ of } g \text{ in}(0, 1) \\ 1 & \text{otherwise} \end{cases}$$

Notice that σ does not depend on p. □

We can now prove the "if" statement (b) of Theorem 1.1. The "only if" statement, much simpler, is addressed later in Lemma 5.2.

Proposition 3.8 *Let $\mathcal{F}$ be a generalized isoparametric foliation with compact leaves on M. Let $L(t)$ be a MCF evolution with initial datum $L_0 \in \mathcal{F}$. Assume that the MCF $L(t)$ converges to a singular leaf L_q. Then the flow has a singularity in finite time and this singularity is of type I, i.e.,*

$$\limsup_{t \to T^-} \|A(t)\|_\infty^2 (T - t) < \infty \tag{3.10}$$

where $\|A(t)\|_\infty$ is the sup norm of the shape operator of $L(t)$.

Proof Fixing $q' \in L_q$, we consider a distinguished tubular neighborhood O_ϵ around q', with map $\varphi : O_\epsilon \to T_{q'}M$ as described in Sect. 2. We let $\overline{g}$ denote the pullback of the flat metric in $T_{q'}M$ via φ. We also denote by $\overline{A}, \overline{f}, \overline{r}_\Sigma$, etc., the quantities corresponding to A, f, r_Σ, etc., computed using the flat metric $\overline{g}$.

By calculations similar to those behind the proof of Eq. (4.4), we can prove that there exist constants C_1, C_2 (that depend only on $O_\epsilon(q')$ and φ) such that:

$$\|A_t\|_\infty \leq C_1 \|\overline{A}_t\|_\infty + C_2. \tag{3.11}$$

On the other hand we claim that

$$\|\overline{A}_t\|_\infty \sqrt{T - t} \leq C_3 \tag{3.12}$$

where C_3 is a constant that depends only on $O_\epsilon(q')$. In fact, by Lemma 4.1 we have $\|\overline{A}_t\|_\infty = 1/\bar{f}(t)$, where again $\bar{f}(t)$ is the distance between the submanifold $L(t)$ and its first focal point with respect to the flat metric. Moreover, from Eq. (3.5) we have $\bar{r}_\Sigma(t) > Cr_\Sigma(t) > C\sqrt{T-t}$. Applying Proposition 3.7 to the flat metric, $\overline{f}(t) > C_3\sqrt{T-t}$ and Eq. (3.12) follows.

Equations (3.12), (3.11) and the compactness of L_q imply (3.10). □

We have already discussed that if $L(t)$ is a MCF with initial datum $L_0 \in \mathcal{F}$ and there is a finite time singularity, then $L(t)$ converges to a singular leaf L_q in the Hausdorff sense. We now show that the convergence is in fact pointwise, i.e., for every $p \in L_0$ the integral curve $t \to \varphi_t(p)$ of H converges to a point in the singular leaf L_q as $t \to T^-$.

Proposition 3.9 *Let $(M, \mathcal{F})$ be a generalized isoparametric foliation with compact leaves, and let $L(t) = \phi_t(L_0)$ be the MCF evolution with $L(0) = L_0$ a regular leaf of $\mathcal{F}$. Assume that $L(t)$ converges to singular leaf L_q in a finite time T and let $p \in L(0)$. Then $\varphi_t(p)$ converges to a point of L_q.*

Proof Let $\gamma(t) = \varphi_t(p)$ be the integral curve of H starting at p. By Proposition 3.8 there exists a reparameterization $\sigma : [0, 1) \to [0, T)$ such that $\beta(s) := \gamma(\sigma(s))$ has $\|\beta'(s)\| < \infty$, consider for example $\sigma(s) = T - T(1-s)^2$.

In what follows we prove that β converges to a point of L_q.

Fixing some $\epsilon >$, let $\pi : \mathrm{Tub}_\epsilon(L_q) \to L_q$ be the orthogonal projection. Since $\|\beta'(s)\| < \infty$, $\pi \circ \beta : [0, 1) \to L_q$ is Lipschitz and thus $\lim_{s\to 1} \pi(\beta(s)) = p'$ for some $p' \in L_q$. Since $L(t)$ converges to the leaf L_q, this concludes the proof. □

4 Estimates on the Shape Operator

The goal of this section is to compute bounds for the shape operator of a singular Riemannian foliation, starting with foliations in Euclidean space. We start by recalling the following well-known fact.

Lemma 4.1 *Given a submanifold $L \subseteq \mathbb{R}^n$ and a normal vector x to L, tangent to the* stratum *Σ_L, let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of the shape operator A_x counted with multiplicity. Then the focal points of L along the geodesic $\gamma_x(t) = \exp tx$ are at distance $1/\lambda_1, \ldots, 1/\lambda_r$.*

Let $(M, \mathcal{F})$ be a singular Riemannian foliation, let $q \in M$ be a singular point, of $\mathcal{F}$, and let O_ϵ be a distinguished tubular neighborhood around q (cf. Sect. 2.1). Let g denote the restriction to O_ϵ of the metric of M and let $\overline{g}$ denote the pullback of the flat metric on T_qM under $\varphi : O_\epsilon \to T_qM$. Let $\nabla, \overline{\nabla}$ denote the Levi-Civita connections of g and $\overline{g}$ respectively, and let ω denote the connection difference tensor

$$\omega(X, Y) = \nabla_Y X - \overline{\nabla}_Y X.$$

We let G be the symmetric $(1, 1)$-tensor such that $g(x, y) = \overline{g}(Gx, y)$ for every $x, y \in T_qM|_U$. The splitting $T_qM = T_qL_q \times \nu_qL_q$ induces via φ a $\overline{g}$-orthogonal splitting $O_\epsilon = P \times S$ such that $L_q \cap O_\epsilon = P \times \{s\}$ for some $s \in S$. The submanifolds $S_{q'} = \{q'\} \times S$, $q' \in P$, are called *slices* of O_ϵ. Clearly, the slices are flat in the $\overline{g}$ metric and they contain all the $\overline{g}$-orthogonal spaces of the leaves in O_ϵ.

Any geometric quantity related to a flat metric will be denoted with a bar, e.g., $\overline{\mathrm{tr}}$, $\overline{A}$. Given a leaf L of $(\mathbb{R}^n, \mathcal{F})$, denote by $\overline{r}_L$ the distance function from L in the flat metric.

Remark 4.2 Given two distinguished tubular neighborhoods $O_\epsilon(q)$, $O_\epsilon(q')$ with $q' \in L_q$, the corresponding radial functions $\overline{r}(p) = \overline{\mathrm{dist}}(L_q, p)$ with respect to the two flat metrics agree on the intersection. Therefore, even though the flat metric $\overline{g}$ is only defined locally yet $\overline{r}$ can be uniquely defined on a neighborhood of the whole leaf L_q, and it makes sense to define

$$\overline{\mathrm{Tub}}_\epsilon(L_q) = \{p \in M \mid \overline{r}(p) < \epsilon\}.$$

Even more so, there exists a metric g_0 in $\overline{\mathrm{Tub}}_\epsilon(L_q)$ such that, for any distinguished neighborhood O_ϵ, g_0 has the same transverse metric of $\overline{g}$ (cf. [2]). In particular, for any leaf $L \subseteq \overline{\mathrm{Tub}}_\epsilon(L_q)$ it is possible to define a distance function $\overline{r}_L(p)$ in $\overline{\mathrm{Tub}}_\epsilon(L_q)$ whose restriction to any distinguished tubular neighborhood $O_\epsilon(q')$, $q' \in L_q$, coincides with $\overline{\mathrm{dist}}(L, p)$ in the flat metric.

Lemma 4.3 *Let* $(\mathbb{R}^n, \mathcal{F})$ *be a singular Riemannian foliation, and let* L *be a singular leaf. Then for every* ϵ_L *small enough there is a constant* C_L *such that*

$$-\frac{D_x}{\overline{r}_L(x)} - C_L \le \left(\overline{tr}\,\overline{A}_{\overline{\nabla}\overline{r}_L}\right)_x \le -\frac{D_x}{\overline{r}_L(x)} + C_L \qquad \forall x \in \mathrm{Tub}_{\epsilon_L}(L) \tag{4.1}$$

with $D_x = \dim L_x - \dim L$.

Proof Let ϵ be small enough that the normal exponential map $\exp : \nu^{\le\epsilon} L \to \mathrm{Tub}_\epsilon(L)$ is a diffeomorphism, and let $P : \mathrm{Tub}_\epsilon(L) \to L$ denote the metric projection. For every $p \in L$, $S_p = \exp_p(\nu_p^{\le\epsilon} L)$ is the slice of $\mathcal{F}$ at p. For ϵ small enough the distribution $V_1(x) = T_x L_x \cap T_x S_p$, $p = P(x)$, has dimension $D_x = \dim L_x - \dim L$ and hence codimension $\dim L$ in $T_x L_x$. Let $V_2(x) \subseteq T_x L_x$ denote the orthogonal complement of $V_1(x)$. Then the following are satisfied:

(1) $T_x L_x = V_1(x) \oplus V_2(x)$ is an orthogonal decomposition for every $x \in \mathrm{Tub}_\epsilon(L)$.
(2) V_2 is a regular distribution which coincides with $T_p L$ for every $p \in L$.
(3) By Lemma 4.1, $V_1(x)$ corresponds to the eigenspace of $\overline{A}_{\overline{\nabla}\overline{r}_L}$ with eigenvalue $-\frac{1}{\overline{r}_L}$. In particular, $V_2(x)$ consists of a sum of eigenspaces of $\overline{A}_{\overline{\nabla}\overline{r}_L}$.

It follows that $\overline{\mathrm{tr}}\,\overline{A}_{\overline{\nabla}\overline{r}_L} = \overline{\mathrm{tr}}\,\overline{A}_{\overline{\nabla}\overline{r}_L}\big|_{V_1} + \overline{\mathrm{tr}}\,\overline{A}_{\overline{\nabla}\overline{r}_L}\big|_{V_2}$ and, for every $x = \exp_p v$ in $\mathrm{Tub}_\epsilon(L)$,

$$\left(\overline{\mathrm{tr}}\,\overline{A}_{\overline{\nabla}\overline{r}_L}\big|_{V_1(x)}\right)_x = -\frac{D_x}{\overline{r}_L(x)}, \qquad \left|\left(\overline{\mathrm{tr}}\,\overline{A}_{\overline{\nabla}\overline{r}_L}|_{V_2(x)}\right)_x - \left(\overline{\mathrm{tr}}\,\overline{A}_v\right)_p\right| < \delta$$

where $\delta = \delta(\epsilon)$ is arbitrarily small. The result follows, by letting $C_L = \sup_{v \perp L, \|v\|=1} \left(\overline{\mathrm{tr}}\,\overline{A}_v\right) + \delta$. □

Remark 4.4 Suppose that every leaf L of $(\mathbb{R}^n, \mathcal{F})$ splits isometrically as $V \times L^{\perp}$, where V is a fixed totally geodesic leaf of $\mathcal{F}$ and $L^{\perp} \subseteq V^{\perp}$. The homothetic transformations h_λ at V act on $\mathbb{R}^n = V \times V^{\perp}$ by fixing V and rescaling the $V^{\perp}$ factor. In particular,

$$(h_\lambda)_* \left(\overline{A}_{\overline{\nabla} \overline{r}_L}\right) = \frac{1}{\lambda} \overline{A}_{\overline{\nabla} \overline{r}_{\lambda L}}$$

where $\lambda L = h_\lambda(L)$. In particular, if C_L satisfies Eq. (4.1) on $\mathrm{Tub}_\epsilon(L)$, then $C_{\lambda L} = \frac{1}{\lambda} C_L$ satisfies Eq. (4.1) for λL, in $\mathrm{Tub}_{\lambda\epsilon}(\lambda L)$. If we define $c_L = \frac{C_L}{\overline{r}(L)}$, where $\overline{r}(x) = \mathrm{dist}(x, V)$, then c_L becomes invariant under homothetic transformations ($c_L = c_{\lambda L}$) and Eq. (4.1) becomes

$$-\frac{D_x}{\overline{r}_L(x)} - \frac{c_L}{\overline{r}(x)} \leq \left(\mathrm{tr}\, \overline{A}_{\overline{\nabla} \overline{r}_L}\right)_x \leq -\frac{D_x}{\overline{r}_L(x)} + \frac{c_L}{\overline{r}(x)} \quad \forall x \in \mathrm{Tub}_{\epsilon_L}(L). \tag{4.2}$$

Clearly, if (L, ϵ_L, c_L) satisfy Eq. (4.2), then $(\lambda L, \lambda \epsilon_L, c_L)$ satisfy Eq. (4.2) as well, for every λ.

The next lemma holds for generic Riemannian metrics.

Lemma 4.5 *Let $(M, \mathcal{F})$ be a singular Riemannian foliation with compact leaves on a complete Riemannian manifold and let L_q be a singular leaf. Fix $\epsilon > 0$ small enough. Then for any L in $\mathrm{Tub}_\epsilon(L_q)$, there is a radius ϵ_L and a constant k_L such that in the regular part of $\mathrm{Tub}_{\epsilon_L}(L)$ the following holds*

$$-(1+\delta)\frac{D_L}{\overline{r}_L(x)} - \frac{k_L}{\overline{r}(x)} \leq tr(A_{\nabla \overline{r}_L})_x \leq -(1-\delta)\frac{D_L}{\overline{r}_L(x)} + \frac{k_L}{\overline{r}(x)}. \tag{4.3}$$

Here the constant δ only depends on L_q and ϵ, while k_L is homothety invariant (i.e., $k_L = k_{\lambda L}$).

Proof Fix a distinguished tubular neighborhood O_ϵ around some point in L_q, and let $\overline{g}$ denote the flat metric. In the following, every overlined geometrical quantity is computed with respect to $\overline{g}$. Using $\nabla = \overline{\nabla} + \omega$ and $g(x, y) = \overline{g}(Gx, y)$, it is not hard to prove that there are constants δ, c depending only on L_q and ϵ, with $\lim_{\epsilon \to 0} \delta = 0$, such that g and $\overline{g}$ are δ-close in the C^0-topology and

$$(1-\delta)\left|\mathrm{tr}\, \overline{A}_{\overline{\nabla} \overline{r}_L}\right| - c \leq \left|\mathrm{tr} A_{\nabla \overline{r}_L}\right| \leq (1+\delta)\left|\mathrm{tr}\, \overline{A}_{\overline{\nabla} \overline{r}_L}\right| + c. \tag{4.4}$$

Since the metric $\overline{g}$ splits as in Remark 4.4, we obtain that for every L there is a homothety invariant c_L and a small ϵ_L such that Eq. (4.2) applies. Using Eq. (4.4), we obtain

$$-(1+\delta)\frac{D_L}{\overline{r}_L(x)} - (1+\delta)\frac{c_L}{\overline{r}(x)} - c \leq \mathrm{tr}(A_{\nabla \overline{r}_L})_x \leq -(1-\delta)\frac{D_L}{\overline{r}_L(x)} + (1-\delta)\frac{c_L}{\overline{r}(x)} + c, \tag{4.5}$$

By setting $k_L = (1+\delta)c_L + \epsilon c$ we obtain the result. □

In the particular case of $L = L_q$, we can choose $\epsilon_L = \epsilon$ and follow the same steps as above, noticing that in this case $\overline{r}_L = r$ and thus $\nabla \overline{r}_L = \nabla r$. Moreover, in this case we get $C_L = c_L = 0$, thus from Eq. (4.5) we get the following

Corollary 4.6 *Let* $(M, \mathcal{F})$ *be a singular Riemannian foliation with compact leaves on a complete Riemannian manifold and let* L_q *be a singular leaf. For* $\epsilon > 0$ *small enough, there exist constants* δ, c *such that in the regular part of* $\mathrm{Tub}_\epsilon(L_q)$

$$-(1+\delta)\frac{D}{r(x)} - c \leq tr(A_{\nabla r})_x \leq -(1-\delta)\frac{D}{r(x)} + c. \tag{4.6}$$

where $D = \dim \mathcal{F} - \dim L_q$ *and* $r = \mathrm{dist}_{L_q}$.

Remark 4.7 The above corollary implies that there is no Riemannian metric on M, adapted to a singular Riemannian foliation $\mathcal{F}$ with compact leaves, for which all the leaves of $\mathcal{F}$ are minimal submanifolds; see also Miquel and Wolak [10].

Corollary 4.8 *Let* $M, \mathcal{F}, L_q$ *be as in Lemma 4.5 and assume that* $\mathcal{F}$ *is generalized isoparametric. Let* $\mathrm{Tub}_\epsilon(L_q)$ *be a tubular neighborhood of* L_q *with radius* ϵ *small enough and let* $\mathcal{M}$ *be the union of the singular leaves in* $\mathrm{Tub}_\epsilon(L_q)$*. Then there exists a foliated neighborhood* U *of* $\mathcal{M} \setminus L_q$ *with the following two properties:*

(1) *There exists a constant* C *such that for any* $x \in \mathrm{Tub}_\epsilon(L_q) \setminus U$*,* $\mathrm{dist}(x, \mathcal{M}) > C\ \mathrm{dist}(x, L_q)$.
(2) *for any regular leaf* $L_0 \in U$*, the MCF evolution* $L(t)$ *with* $L(0) = L_0$ *does not converge to* L_q.

Proof Let $\mathcal{L}$ denote the set of singular leaves in $\overline{\mathrm{Tub}_\epsilon}(L_q)$, and define

$$U = \bigcup_{L \in \mathcal{L}} \overline{\mathrm{Tub}}_{\epsilon_L}(L).$$

Here the tubes $\overline{\mathrm{Tub}_\epsilon}(L_q), \overline{\mathrm{Tub}_\epsilon}(L)$ are defined using the distance functions $\overline{r}(p) = \overline{\mathrm{dist}}(L_q, p)$ and $\overline{r}_L(p) = \overline{\mathrm{dist}}(L, p)$ (see Remark 4.2), while ϵ_L is some radius satisfying Lemma 4.5 and rescaling linearly under $\overline{g}$-homothetic transformations. In this way, for any distinguished tubular neighborhood $O_\epsilon = P \times S$ around L_q, the restriction $U \cap O_\epsilon$ has the form $P \times \{\text{conical open set in } S\}$. Clearly there is some constant C' such that $\overline{\mathrm{dist}}(x, \mathcal{M}) > C' \overline{r}_L(x)$ for every x in O_ϵ. Since the metrics $g, \overline{g}$ are equivalent, the first statement follows.

In order to prove the second statement, we choose $\epsilon_L < \frac{(1-\delta)k_L}{D_L}\overline{r}(L)$. Notice that the right-hand side of the inequality rescales linearly under $\overline{g}$-homothetic transformations, thus we can still choose ϵ_L with the same property. Let $L(t)$ be a MCF evolution with initial datum $L_0 \subseteq U$. Then L_0 belongs to $\mathrm{Tub}_{\epsilon_L}(L)$ for some singular leaf $L \subseteq U$. If we define $\overline{r}_L(t) = \overline{r}_L(L(t))$, by Lemma 4.5 we obtain

$$\overline{r}'_L(t) = \mathrm{tr} A_{\nabla \overline{r}_L} < -(1-\delta)\frac{D_L}{\overline{r}_L(x)} + \frac{k_L}{\overline{r}(x)}.$$

Since $\bar{r}_L < \epsilon_L < \frac{(1-\delta)k_L}{D_L}\bar{r}(L)$, we obtain $\bar{r}'_L(t) < 0$ and therefore $L(t)$ never leaves $\overline{\mathrm{Tub}_{\epsilon_L}}(L)$. □

5 Isoparametric Foliations in Nonnegative Curvature

The goal of this section is to prove Theorem 1.3 which we restate here.

Theorem 5.1 *Let $(M, \mathcal{F})$ be an isoparametric foliation (i.e., polar and generalized isoparametric) on a compact nonnegatively curved manifold. Then for every non-minimal regular leaf L_0, the MCF $L(t)$ with initial datum L_0 has finite time singularity.*

We start by proving a few lemmas.

Lemma 5.2 *Let $(M, \mathcal{F})$ be a closed, generalized isoparametric, singular Riemannian foliation on a compact manifold.*

(1) *If* $\mathrm{vol} : M_{reg} \to \mathbb{R}$ *denotes the volume function* $x \mapsto \mathrm{vol}(L_x)$ *then* $H = -\nabla(\log \mathrm{vol})$ *in* M_{reg}.
(2) *Fixing a regular leaf L_0, suppose that the MCF $L(t)$ with $L(0) = L_0$ does not have a finite time singularity. Then there exists a sequence of leaves L_i converging to a minimal regular leaf L' in the Hausdorff sense, such that* $\mathrm{vol}(L_i) > \mathrm{vol}(L')$.

Proof (1) Let ω denote the volume form of the regular leaves. By [6, Proposition 4.1.1], given a basic vector field X along a regular leaf L_p, we obtain

$$
\begin{aligned}
X(\mathrm{vol})(p) &= \int_{L_p} \mathcal{L}_X(\omega) \\
&= -\int_{L_p} \langle X, H\rangle \omega \\
&= -\langle X, H\rangle \mathrm{vol}(p)
\end{aligned}
$$

where the last equality holds because both X and H are basic, and therefore $\langle X, H\rangle$ is constant along L_p. Dividing the equation by $\mathrm{vol}(p)$ we obtain

$$
\langle X, \nabla(\log \mathrm{vol})(p)\rangle = X(\log \mathrm{vol})(p) = -\langle X, H\rangle
$$

hence the result.
(2) From Proposition 3.3, there is a neighborhood of the singular set U such that every MCF entering U has a finite time singularity, and therefore our flow $L(t)$ must lie in $M \setminus U$, which is a relatively compact subset of M_{reg} whose distance to the singular set is positive. Via the projection $\pi : M \to M/\mathcal{F}$, $L(t)$ is projected to an integral curve of the vector field $\pi_* H$. Since $(M \setminus U)/\mathcal{F}$ is relatively compact, there exists a sequence of times t_i going to infinity, such that $\pi(L(t_i))$ converges to some point $\pi(L') \in (\overline{M \setminus U})/\mathcal{F}$. Since $\log \mathrm{vol}(L(t))$ is decreasing, $\log \mathrm{vol}(L(t)) > \log \mathrm{vol}(L') > c$ for some $c \in \mathbb{R}$. On the other hand, from the previous result one has

$$
\frac{d}{dt}\Big(\log \mathrm{vol}(L(t))\Big) = H\Big(\log \mathrm{vol}\big(L(t)\big)\Big) = -\|H\|^2
$$

and since $\log \mathrm{vol}(L(t))$ is bounded from below, then (up to taking a subsequence) one has $\|H|_{L(t_i)}\|^2 \to 0$. By the continuity of the mean curvature in M_{reg}, $H|_{L'} = 0$ and therefore L' is minimal. On the other hand, L' is not a local maximum because it is obtained as a Hausdorff limit of leaves with bigger volume. □

Proof of Proposition 5.1 Suppose that there is a MCF $L(t)$ without a finite time singularity. By Lemma 5.2, there exists a sequence of leaves L_i converging to a minimal regular leaf L', such that $\mathrm{vol}(L_i) > \mathrm{vol}(L')$. This will provide a contradiction with the following result, which will then finish the proof. □

Proposition 5.3 *Let $(M, \mathcal{F})$ be a polar foliation on a compact nonnegatively curved manifold. Then for every regular minimal leaf L', there exists a tubular neighborhood U around L' such that, for every leaf L in U, $\mathrm{vol}(L) \leq \mathrm{vol}(L')$.*

Proof Fixing a unit-length, basic vector field X along L' and a point $p \in L'$ let $\gamma_X(s)$ denote the geodesic starting at p with initial velocity $X(p)$. We set

$$\delta(X) = \sup\{s \mid \mathrm{vol}(L_{\gamma_X(s)} \leq \mathrm{vol}(L')\}.$$

In order to prove the proposition, it is enough to show that $\delta(X) > c > 0$ for some c not depending on X.

Let $e_1, \ldots, e_n$ be an orthonormal frame of T_pL', let $E_1(s), \ldots, E_n(s) \in T_{\gamma_X(s)}L_{\gamma_X(s)}$ be the extension of $e_1, \ldots, e_n$ along $\gamma_X(s)$ by (vertical) parallel transport, which allow us to identify the tangent spaces $T_{\gamma_X(s)}L_{\gamma_X(s)}$ with T_pL'. Moreover, let $\omega_s(p) = E_1^*(s) \wedge \cdots \wedge E_n^*(s)$ denote the volume forms of $L_{\gamma_X(s)}$ at $\gamma_X(s)$.

The *holonomy map* $f_s : L' \to L_{\gamma_X(s)}$ defined by $f_s(q) = \exp_q sX(q)$ is a well-defined, smooth diffeomorphism between L' and $L_{\gamma_X(s)}$, whose differential at a point q is given by $f_{s*}(e_i) = J_i(s)$, where J_i is the unique *holonomy Jacobi field* starting at q with $J_i(0) = e_i$ (cf. [6, Sect. 1.4] for the definition and properties of holonomy Jacobi fields).

The volume function along $\gamma_X(s)$ then reads

$$\mathrm{vol}(L_{\gamma_X(s)}) = \int_{L_{\gamma_X(s)}} \omega_s = \int_{L'} f_s^*\omega_s = \int_{L'} j_s(q)\omega$$

where $j_s(q) = \det(J_1(s), \ldots, J_n(s))$. Since the curvature is nonnegative and the foliation is polar, by standard comparison theory (cf. [5]), $j_s(q)$ is bounded above by a corresponding function $\overline{j}_s(q)$ in Euclidean space. In other words, let $\overline{S}_q : [0, b] \to \mathrm{Sym}^2(T_qL')$ be the tensor satisfying

$$\overline{S}_q' + \overline{S}_q^2 = 0, \qquad \overline{S}_q(0) = -A_{X(q)}$$

and let $\overline{j}_s(q)$ be the function such that

$$\frac{d}{ds}\overline{j}_s(q) = \overline{j}_s(q) \cdot \mathrm{tr}(\overline{S}_q(s)), \qquad \overline{j}_0(q) = 1.$$

Then $j_s(q) \leq \overline{j}_s(q)$. It is easy to compute $\overline{j}_s(q)$:

$$\overline{j}_s(q) = (-1)^n \left(\det A_{X(q)}\right) \prod_i (s - \lambda_i(q)^{-1})$$

where $\lambda_1(q), \ldots, \lambda_n(q)$ are the eigenvalues of $A_{X(q)}$. Such a function has a local maximum at 0, where $\overline{j}_0(q) = j_0(q) = 1$. Moreover, this is a maximum in the interval $[\frac{1}{\lambda^-(q)}, \frac{1}{\lambda^+(q)}]$, where $\lambda^-(q)$ is the smallest (negative) eigenvalue of $A_{X(q)}$ and $\lambda^+(q)$ is the biggest (positive) eigenvalue. In particular, if $\lambda_X^+ = \max_q \lambda^+(q)$, then $j_s(q) \leq \overline{j}_s(q) \leq 1$ for all $q \in L'$ and $s \in [0, \frac{1}{\lambda_X^+}]$, and therefore

$$\mathrm{vol}(L_{\gamma_X(s)}) = \int_{L'} j_s(q)\omega \leq \int_{L'} \omega = \mathrm{vol}(L') \qquad \forall s \in [0, 1/\lambda_X^+].$$

Therefore, $\delta(X) \geq 1/\lambda_X^+$. By letting $c = 1/\|A\|_\infty$, we then have $\delta(X) > 1/\|A\|_\infty > 0$ for every X. $\square$

Remark 5.4 A weaker version of Theorem 5.1 can also be proved as follows. Given a minimal regular leaf L and a basic vector field X along L, the second variation of area in the direction of X reads

$$\frac{d^2}{dX^2}\mathrm{vol} = \int_L \left(\|ON_X\|^2 - \|A_X\|^2 - \mathrm{Ric}^v(X, X)\right) d\,\mathrm{vol},$$

where $\mathrm{Ric}^v(x, x)$ denotes the sum $\sum_i \langle R(x, e_i)e_i, x\rangle$ over an orthonormal basis of T_pL_p.

It follows from this formula that, whenever $\|ON_X\|^2 < \|A_X\|^2 - \mathrm{Ric}^v(X, X)$ then every minimal leaf is a local maximum among the nearby leaves, and by Lemma 5.2 the MCF with a regular leaf as initial datum cannot have infinite time singularity. This condition holds, for example, if $(M, \mathcal{F})$ is isoparametric and M is positively curved. Moreover, in the case of $M = \mathbb{R}^n$ or $\mathbb{S}^n$, we are not aware of any example where the inequality $\|ON_x\|^2 \leq \|A_x\|^2 + \mathrm{Ric}^v(x, x)$ does not hold, and in most examples such inequality is strict.

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