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IWASAWA MAIN CONJECTURE FOR THE CARLITZ CYCLOTOMIC EXTENSION AND APPLICATIONS

BRUNO ANGLÈS, ANDREA BANDINI, FRANCESC BARS, AND IGNAZIO LONGHI

Abstract. We prove an Iwasawa Main Conjecture for the class group of the $\mathfrak{p}$-cyclotomic extension $\mathcal{F}$ of the function field $\mathbb{F}_q(\theta)$ ($\mathfrak{p}$ is a prime of $\mathbb{F}_q[\theta]$), showing that its Fitting ideal is generated by a Stickelberger element. We use this and a link between the Stickelberger element and a $\mathfrak{p}$-adic L-function to prove a close analog of the Ferrero-Washington Theorem for $\mathcal{F}$ and to provide information on the $\mathfrak{p}$-adic valuations of the Bernoulli-Goss numbers $\beta(j)$ (i.e., on the values of the Carlitz-Goss ζ-function at negative integers).

1. Introduction

One of the main topics of modern number theory is the investigation of arithmetic properties of motives over a global field (in any characteristic) and their relation with (or interpretation as) special values of ζ-functions or L-functions. Iwasawa theory offers an effective way of dealing with various issues arising in this context, such as the variation of arithmetic structures in p-adic towers, and is one of the main tools currently available for the knowledge of ζ-values associated to an arithmetic object (see, e.g., [27]). One of the major outcomes of this theory is the construction of *p-adic L-functions*, which provide a good understanding of both the special values and the properties of the arithmetic object. In particular, the various forms of Iwasawa Main Conjecture (IMC) provide a link between the analytic side and the arithmetic side.

The prototype is given by the study of class groups in the extensions generated by p-power roots of unity. Let k be a totally real number field, fix a prime $p>2$ and consider the extensions $k_0 := k(\zeta_p)$ and $k_\infty := \cup_n k(\zeta_{p^n})$ (where ζ_m denotes a primitive m-th root of unity). We briefly recall the statement of IMC in this basic setting. Let $\Delta := Gal(k_0/k)$ (note that Δ is isomorphic to a subgroup of $\mathbb{F}_p^*$), then $k_{cyc} := k_\infty^\Delta$ is the *cyclotomic $\mathbb{Z}_p$-extension* of k. We put

- $G := Gal(k_{cyc}/k) \simeq \mathbb{Z}_p$, the Galois group;
- $\Lambda(k_{cyc}) := \mathbb{Z}_p[[G]]$, the associated *Iwasawa algebra*;
- $k_n := k_{cyc}^{G^{p^n}}$, the n-th layer of k_{cyc} with $Gal(k_n/k) \simeq \mathbb{Z}/p^n\mathbb{Z}$;
- $\mathcal{C}\ell_n$, the p-part of the class group of k_n and $\mathcal{C}\ell_{cyc} := \varprojlim \mathcal{C}\ell_n$ (limit with respect to the norm maps) the "pro-p class group" of k_{cyc}.

The group $\mathcal{C}\ell_{cyc}$ admits an action by Δ so we can consider its χ-part $\mathcal{C}\ell_{cyc}(\chi)$ for any character χ of Δ. Moreover $\mathcal{C}\ell_{cyc}$ is a finitely generated torsion $\Lambda(k_{cyc})$-module. Since $\Lambda(k_{cyc})$ is a noetherian Krull domain (it is noncanonically isomorphic to $\mathbb{Z}_p[[T]]$), a structure theorem allows to define a principal ideal

$$Ch_{\Lambda(k_{cyc})}(\mathcal{C}\ell_{cyc}(\chi)) = (p^{\mu_\chi} f^\chi) \subseteq \Lambda(k_{cyc})$$

(with f^χ a "polynomial" and $p \nmid f^\chi$), called the *characteristic ideal* of $\mathcal{C}\ell_{cyc}(\chi)$. By a celebrated theorem of Iwasawa, $Ch_{\Lambda(k_{cyc})}(\mathcal{C}\ell_{cyc}(\chi))$ gives information on the order of class groups: one

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has

$$|\mathcal{C}\ell_n(\chi)| = p^{\mu_\chi p^n + \deg_T(f^\chi)n + \nu_\chi} \tag{1}$$

for $n \gg 0$ and some $\nu_\chi \in \mathbb{Z}$. The class number formula provides another way of obtaining the information in (1), namely via special values of Dirichlet L-functions. This fact suggests a deeper relation between the algebraic and the analytic theory, expressed by the Iwasawa Main Conjecture. Let ω be the Teichmüller character, take an odd character $\chi \neq \omega$ of Δ and let κ be the cyclotomic character (so that $g\zeta = \zeta^{\kappa(g)}$ for any p-power root of unity ζ and any $g \in G$). Then there exists a p-adic L-function $L_p(\omega\chi^{-1}, s)$ which interpolates p-adically the special values of the (twisted) Dirichlet L-function, and an element $\ell^\chi \in \Lambda(k_{cyc})$, such that

$$L_p(\omega\chi^{-1}, s) = \ell^\chi(\kappa(g_0)^s - 1). \tag{2}$$

The *Iwasawa Main Conjecture* states that

$$(\ell^\chi) = (p^{\mu_\chi} f^\chi) \tag{3}$$

and was proved in this form by B. Mazur and A. Wiles in [29] for $k = \mathbb{Q}$ (K. Rubin provided a different and more general proof using Kolyvagin's method, see [42, Chapter 15] for an overview). Another major result was obtained by B. Ferrero and L. Washington in [18], where they showed that the invariant μ_χ in equation (3) satisfies

$$\mu_\chi = 0 \tag{4}$$

for any χ and any abelian number field k.

In the last decades Iwasawa theory grew enormously and found fruitful applications in different areas of number theory. In particular various instances of the IMC have been formulated and proved for elliptic curves (in relation with the Birch and Swinnerton-Dyer Conjecture, see, e.g., [26], [8] and [35]), for motives in general (connected with the Tamagawa Number Conjecture of S. Bloch and K. Kato [9]), for non-abelian extensions (where the characteristic ideals are substituted by elements in K-theory groups, see [14]) and so on. Nevertheless, most of these developments deal with global fields of characteristic 0.

Moving to the function field setting the situation is very different and much less understood (an excellent survey for an updated overview of the various aspects of Iwasawa theory over function fields is [12]). In characteristic p, an IMC like (3) has been proved for $\mathbb{Z}_p^d$-extensions (see [16] and/or [10] and [11]). However, geometric $\mathbb{Z}_p^d$-extension of function fields are somewhat artificial, if d is finite, when compared with the cyclotomic extension of $\mathbb{Q}$; and even the arithmetic extension is not very satisfactory. Hence, in our opinion, a true function field version of the basic Mazur-Wiles Theorem above was still missing. Providing it is the main goal of this paper: on the way we shall also obtain analogs of (2) (in many versions) and, as an application, of (4).

1.1. **The function field setting.** As usual in function field arithmetic, the field $F := \mathbb{F}_q(\theta)$ and the ring $A := \mathbb{F}_q[\theta]$ play the role of $\mathbb{Q}$ and $\mathbb{Z}$. Here and in the following, we assume that q is a power of p. For a lighter notation, we will usually write $\mathbb{F}$ for $\mathbb{F}_q$. The symbol ∞ will denote the place of F with uniformizer $\frac{1}{\theta}$. Let $\overline{F}$ be a fixed algebraic closure of F. The Carlitz module Φ (see Section 2 for a quick review of the relevant theory) is a morphism of $\mathbb{F}$-algebras $\Phi : A \to \mathrm{End}_{\mathbb{F}}(\overline{F})$ given by :

$$\Phi_\theta = \theta + \tau,$$

where $\tau : \overline{F} \to \overline{F}, x \mapsto x^q$. If $a \in A - \{0\}$, we set $\Phi[a] = \{x \in \overline{F} \,:\, \Phi_a(x) = 0\}$.

We fix a monic irreducible polynomial $\pi_{\mathfrak{p}}$ in A: $\mathfrak{p} = \pi_{\mathfrak{p}} A$ will correspond to $(p) \subset \mathbb{Z}$. The function field counterpart of $\mathbb{Q}(\zeta_{p^n})$ is obtained adding to F the $\mathfrak{p}^n$-roots of the Carlitz module Φ. The field $F_n := F(\Phi[\pi_{\mathfrak{p}}^{n+1}])$ is a Galois extension of F with Galois group isomorphic to

$(A/\mathfrak{p}^{n+1})^*$. We recall that F_n/F is unramified outside $\mathfrak{p}$ and ∞, and totally ramified at $\mathfrak{p}$. The "cyclotomic" extension of F we are going to consider is $\mathcal{F} := \cup_n F_n$.

1.1.1. *The algebraic side.* The group $Gal(\mathcal{F}/F)$ is isomorphic to the units of $A_\mathfrak{p}$ (the completion of A with respect to $\mathfrak{p}$). In characteristic p the 1-units in a local field form a free $\mathbb{Z}_p$-module of infinite rank: hence we get $Gal(\mathcal{F}/F) \simeq \Delta \times \mathbb{Z}_p^\infty$, where $\Delta \simeq \mathrm{Gal}(\mathcal{F}/F_0)$ is a cyclic group of order $q^d - 1$, $d = \deg_\theta \pi_\mathfrak{p}$. We will assume $q^d > 2$, in order to guarantee the existence of non-trivial characters of the group Δ. To include the values of these characters, we shall use the Witt ring W of the residue field $A/\mathfrak{p}$ and define our Iwasawa algebra as $\Lambda := W[[Gal(\mathcal{F}/F_0)]]$.

Observe that our Λ is isomorphic to a ring of power series over W in infinitely many variables: therefore it is not noetherian. In this situation, one cannot apply the usual structure theorem for modules over Iwasawa algebras; however, it is still possible to define the Fitting ideal $\mathrm{Fitt}_\Lambda(M)$ for a finitely generated Λ-module M.

1.1.2. *The analytic side.* An interesting feature of function field arithmetic is the presence of more L-functions than in the number field setting.

First of all, we have the usual, complex-valued L-functions as studied by Artin and Weil: to any continuous character $\psi\colon Gal(\mathcal{F}/F) \to \mathbb{C}^*$ one can attach $L(s,\psi)$. And we have p-adic L-functions arising from p-adic interpolation of $L(s,\psi)$.

The genuinely new phenomenon is the appearance of characteristic p L-functions. The first example was discovered by Carlitz already in 1935 ([13]); some decades later Goss developed a full theory around it (see [20] or [21, Chapter 8]). The Carlitz-Goss zeta function $\zeta_A(s)$ will be defined and discussed in detail in Section 3.2. We mention that the special values of this zeta function are given by:

$$\zeta_A(n) = \sum_{m\geqslant 0} \sum_{\substack{a\in A,\ a \text{ monic} \\ \deg_\theta a = m}} \frac{1}{a^n} \in \mathbb{F}((\theta^{-1})) \qquad (\text{with } n \in \mathbb{Z}).$$

In particular, for any positive integer n, $\zeta_A(-n) \in A$, and $\zeta_A(-n) = 0 \Leftrightarrow n \equiv 0 \pmod{q-1}$. Furthermore, for any $n \geqslant 1$, $\zeta_A(n)$ is transcendental over F ([43]). Goss also defined v-adic L-functions $L_v(s,\psi)$ where v can be any place of F, and ψ a "Dirichlet" character. As one would expect, when v is a finite place, $L_v(s,\psi)$ can be seen as a v-adic interpolation of $\zeta_A(s)$.

1.1.3. *Special values.* Evaluating $\zeta_A(s)$ at negative integers, one obtains the *Bernoulli-Goss numbers* $\beta(j) \in A$. If $j \in \mathbb{N}, j \geqslant 1, j \not\equiv 0 \pmod{q-1}$, then $\beta(j) = \zeta_A(-j) \in A - \{0\}$. For $j \in \mathbb{N}, j \geqslant 1, j \equiv 0 \pmod{q-1}$, $\zeta_A(-j) = 0$ and the precise formula for $\beta(j)$ will be given in Definition 3.13. Similarly to the classical ζ, special values of the the Carlitz-Goss zeta have relevant arithmetical interpretations and have been the object of many investigations in recent years (see, for example, L. Taelman's results in [36] and [37]).

1.2. **Our results.** In this paper we prove analogs of formulae (2), (3) and (4). We also provide some arithmetic information on the Bernoulli-Goss numbers.

1.2.1. *The analytic side.* If v is a place of F, we denote by $\mathbb{C}_v$ a v-adic completion of an algebraic closure of the v-adic completion of F. The extension $\mathcal{F}/F$ is ramified only at $S = \{\mathfrak{p}, \infty\}$.[1] Let $\mathcal{F}_S$ be the maximal abelian extension of F unramified outside S and put $G_S := Gal(\mathcal{F}_S/F)$. We consider the *Stickelberger series*

$$\Theta_{\mathcal{F}_S/F,S}(X) := \prod (1 - \mathrm{Fr}_\mathfrak{q}^{-1} X^{\deg(\mathfrak{q})})^{-1} \in \mathbb{Z}[G_S][[X]],$$

[1] We shall not distinguish between a prime ideal of A, like $\mathfrak{p}$, and the place of F corresponding to it.

where the product is taken over all places $\mathfrak{q}$ outside S and $\mathrm{Fr}_{\mathfrak{q}} \in G_S$ denotes the Frobenius at $\mathfrak{q}$. For any subextension of $\mathcal{F}_S$ a similar Stickelberger series is obtained by taking the projection of $\Theta_{\mathcal{F}_S/F,S}(X)$ to the appropriate algebra.

This series turns out to be a kind of universal object for L-functions attached to abelian characters unramified outside S. Evaluation at $X=1$ (convergence is ensured by Proposition 3.2) yields an element in the Iwasawa algebra $\mathbb{Z}_p[[G_S]]$ and hence one in Λ: these elements can be seen as p-adic L-functions. If $\mathbb{L}$ is any of the fields $\mathbb{C}$, $\mathbb{C}_\infty$ or $\mathbb{C}_{\mathfrak{p}}$ and ψ is a continuous character from G_S to $\mathbb{L}^*$, then there are interpolating relations between $\psi(\Theta_{\mathcal{F}_S/F,S})$ and

- the complex L-function $L_S(s,\psi)$ (for $\mathbb{L}=\mathbb{C}$);
- the Carlitz-Goss ζ-function $\zeta_A(s)$ (for $\mathbb{L}=\mathbb{C}_\infty$);
- the $\mathfrak{p}$-adic L-function $L_{\mathfrak{p}}(s,\psi)$ (for $\mathbb{L}=\mathbb{C}_{\mathfrak{p}}$).

Details and more precise formulations will be given after introducing the proper notations, in equation (11), Theorem 3.8 and Theorem 3.16 respectively.

1.2.2. *The algebraic side.* On the algebraic side we study $\mathcal{C}\ell^0(F_n)\{p\}$, that is, the p-part of the group of classes of degree zero divisors of F_n.

Let χ be any character defined on Δ: we recall that such characters are called *even* if $q-1$ divides $|\ker(\chi)|$ and *odd* otherwise. As usual, χ_0 will denote the trivial character. For any module M we shall denote by $M(\chi)$ the χ-component of M.

Let $\Theta_\infty(X,\chi)$ denote the χ-component of the projection of $\Theta_{\mathcal{F}_S/F,S}(X)$ to $\mathbb{Z}[\mathrm{Gal}(\mathcal{F}/F)][[X]]$ (see Definition 3.5) and put

$$\Theta^{\#}_\infty(X,\chi) := \begin{cases} \Theta_\infty(X,\chi) & \text{if } \ \chi \text{ is odd} \\ \\ \frac{\Theta_\infty(X,\chi)}{1-X} & \text{if } \ \chi \neq \chi_0 \text{ is even} \end{cases} .$$

Our first main result is the following: an *Iwasawa Main Conjecture* for the $\mathfrak{p}$-cyclotomic extension $\mathcal{F}/F$.

Theorem 1.1 (IMC). *For any $\chi \neq \chi_0$, $\mathcal{C}(\mathcal{F})(\chi) := \varprojlim_n \mathcal{C}\ell^0(F_n)\{p\}(\chi)$ is a finitely generated torsion Λ-module, and*

$$\mathrm{Fitt}_\Lambda\left(\mathcal{C}(\mathcal{F})(\chi)\right) = \left(\Theta^{\#}_\infty(1,\chi)\right).$$

Remark 1.2. In this paper we shall provide some results on the χ_0-component as well but we leave the precise statements to the following sections (see, in particular, Corollary 4.10 and Remark 4.14), since they require a few more notations and all the main arithmetical applications will involve only the χ-components for $\chi \neq \chi_0$.

The proof of Theorem 1.1 will be given in Section 5 (see Theorems 5.1 and 5.2). Here is a brief summary of the main ideas and steps.

The strategy is based on some results of C. Greither and C.D. Popescu (in [23] and [24]).

Let X_{F_n} be the curve associated with the field F_n and denote by $T_p(F_n) := T_p(Jac(X_{F_n})(\overline{\mathbb{F}}))$ the p-adic Tate module of its Jacobian (with $\overline{\mathbb{F}}$ a fixed algebraic closure of $\mathbb{F}$). Taking the limit over n, we get $T_p(\mathcal{F})$. Put $G_{\mathbb{F}} := \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$. Using Greither-Popescu results we can compute the Fitting ideal of $T_p(F_n)(\chi)$ and then, with some work to check the necessary compatibility conditions, we can compute the Fitting ideal of $T_p(\mathcal{F})(\chi)$ over $\Lambda[[G_{\mathbb{F}}]]$ (Theorem 4.16).

The group $\mathcal{C}\ell^0(F_n)\{p\}$ can be recovered as $G_{\mathbb{F}}$-coinvariants of $T_p(F_n)$ (Lemma 4.6). By specializing the arithmetic Frobenius (i.e., the generator of $G_{\mathbb{F}}$) to 1 in $\mathrm{Fitt}_{\Lambda[[G_{\mathbb{F}}]]}\left(T_p(\mathcal{F})(\chi)\right)$, we finally obtain Theorem 1.1.

1.2.3. *Special values.* Our second main result is the following.

Theorem 1.3. *For any $\chi \neq \chi_0$, one has $\Theta^{\#}_\infty(1,\chi) \not\equiv 0 \pmod{p}$.*

Thanks to Theorem 1.1, this can be seen as a close analog of the Ferrero-Washington Theorem. The proof (given in Theorem 6.3) is based on the following ideas. A map defined using a result by Sinnott provides a formula relating $\Theta_\infty(X,\chi)$ with a $\mathfrak{p}$-adic L-function $L_\mathfrak{p}(X,y,\chi)$ (Theorem 3.22). It turns out that the Bernoulli-Goss numbers $\beta(j)$ appear as special values of this $L_\mathfrak{p}(X,y,\chi)$ (see (51) and (52) for the precise statements). Theorem 1.3 then follows observing that the $\beta(j)$ are nonzero (Lemma 3.14).

In Section 2.4 we shall define a Teichüller character $\widetilde{\omega}_\mathfrak{p}$. Theorem 1.3 implies that the index

$$N_\mathfrak{p}(i) := \mathrm{Inf}\{n \geqslant 0 \,:\, \Theta_n^{\#}(1,\widetilde{\omega}_\mathfrak{p}^i) \not\equiv 0 \pmod{p}\}$$

is well defined for any $1 \leqslant i \leqslant q^d - 2$. In Corollary 6.9 we will show that $N_\mathfrak{p}(i)$ provides a lower bound for $v_\mathfrak{p}(\beta(j))$ (the $\mathfrak{p}$-adic valuations of the Bernoulli-Goss numbers) for $j \geqslant 1$, $j \equiv -i \pmod{q-1}$: a relation that, to our knowledge, seems to have no counterpart in the number field setting.

In this paper we are focused on the arithmetic of special values of the Carlitz-Goss zeta function at negative integers. If L/F is a finite extension and O_L denotes the integral closure of A in L, L. Taelman has introduced a finite A-module $H(\Phi/O_L)$ associated to the Carlitz module Φ and the extension L/F ([36]). This A-module is an analogue of the ideal class group of a number field. For $H(\Phi/O_{F_0})$, Taelman proved an analogue of the Herbrand-Ribet Theorem linking the isotypic components of the $\mathfrak{p}$-part of $H(\Phi/O_{F_0})$ to the arithmetic of the special values of the Carlitz-Goss zeta function at positive integers ([37]). It would be very interesting to study the projective limit (for the trace maps) $\varprojlim H(\Phi/O_{F_n})\otimes_A A_\mathfrak{p}$ in the spirit of Iwasawa Theory.

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2. Basic facts on the $\mathfrak{p}$-cyclotomic extension

We recall here some basic facts (and notations) about what we call the $\mathfrak{p}$-cyclotomic extension of the rational function field, including the corresponding Iwasawa algebra and the Iwasawa modules which will be relevant for our work.

2.1. **The $\mathfrak{p}$-cyclotomic extension.** Let $\mathbb{F}$, F, A and the place ∞ be as in the introduction, §1.1. Let Φ be the Carlitz module associated with A: it is an $\mathbb{F}$-linear ring homomorphism

$$\Phi : A \to F\{\tau\}\ ,\ \ \theta \mapsto \Phi_\theta = \theta\tau^0 + \tau\ ,$$

where $F\{\tau\}$ is the skew polynomial ring with $\tau f = f^q\tau$ for any $f \in F$.

We fix once and for all an algebraic closure of F, which shall be denoted by $\overline{F}$. For any ideal $\mathfrak{a}$ of A write

$$\Phi[\mathfrak{a}] := \{x \in \overline{F} \,:\, \Phi_a(x) = 0\ \forall\, a \in \mathfrak{a}\}$$

for the $\mathfrak{a}$-torsion of Φ. It is an A-module isomorphic to $A/\mathfrak{a}$; in particular, if $\mathfrak{a}|\mathfrak{b}$ as ideals of A, we have $\Phi[\mathfrak{a}] \subseteq \Phi[\mathfrak{b}]$.

Fix a prime ideal $\mathfrak{p} \subset A$ of degree $d > 0$ and, for any $n \in \mathbb{N}$, let

$$F_n := F(\Phi[\mathfrak{p}^{n+1}])$$

be the field generated by the $\mathfrak{p}^{n+1}$-torsion of the Carlitz module. It is well known (see [32, Chapter 12] or [21, §7.5]) that F_n/F is an abelian extension with Galois group

$$G_n := \mathrm{Gal}(F_n/F) = \Delta \times \Gamma_n \simeq (A/\mathfrak{p}^{n+1})^* \simeq (A/\mathfrak{p})^* \times (1+\mathfrak{p})/(1+\mathfrak{p}^{n+1})\ ,$$

where $\Delta \simeq \mathrm{Gal}(F_0/F) \simeq (A/\mathfrak{p})^*$ is a cyclic group of order q^d-1, and $\Gamma_n = \mathrm{Gal}(F_n/F_0)$ is the p-Sylow subgroup of G_n. (By a slight abuse of notation, we identify the prime-to-p part of G_n for all n, and denote it always as Δ.) The extension F_n/F is totally ramified at $\mathfrak{p}$ and tamely ramified at the place ∞, whose inertia group is cyclic of order $q-1$; in particular, F_n/F_0 is only ramified at $\mathfrak{p}$. In the isomorphism $\mathrm{Gal}(F_0/F) \simeq (A/\mathfrak{p})^*$, the inertia at ∞ corresponds to $\mathbb{F}^*$.

Definition 2.1. We define the $\mathfrak{p}$-*cyclotomic extension of* F as the field

$$\mathcal{F} := F(\Phi[\mathfrak{p}^\infty]) = \bigcup_n F(\Phi[\mathfrak{p}^n])$$

with abelian Galois group

$$G_\infty := \mathrm{Gal}(\mathcal{F}/F) = \varprojlim_n \mathrm{Gal}(F_n/F) = \Delta \times \varprojlim_n \Gamma_n =: \Delta \times \Gamma\ .$$

For any place v of F we denote by $I_{v,n}$ (resp. $\mathcal{I}_v$) its inertia group in G_n (resp. in G_∞). The set of ramified places in $\mathcal{F}/F$ is $S := \{\mathfrak{p}, \infty\}$ and, for any n, one has

$$I_{\mathfrak{p},n} = G_n \quad , \quad \mathcal{I}_\mathfrak{p} = G_\infty \quad \text{and} \quad I_{\infty,n} = \mathcal{I}_\infty \hookrightarrow \Delta\ .$$

Denote by $A_\mathfrak{p}$ the completion of A at $\mathfrak{p}$, $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$ and $\mathbb{F}_\mathfrak{p}$ the residue field of $A_\mathfrak{p}$. Readers who prefer a more "hands-on" approach might appreciate the equality $A_\mathfrak{p} = \mathbb{F}_\mathfrak{p}[[\pi_\mathfrak{p}]]$, where $\pi_\mathfrak{p}$ is the monic irreducible generator of $\mathfrak{p}$ in A. The group of units of the local ring $A_\mathfrak{p}$ has a natural filtration; we put $U_n := 1 + \mathfrak{p}^n A_\mathfrak{p}$. Let $\mathbb{C}_\mathfrak{p}$ be the completion of an algebraic closure of $F_\mathfrak{p}$; we also fix once and for all an embedding $\overline{F} \hookrightarrow \mathbb{C}_\mathfrak{p}$.

We have isomorphisms $G_\infty \simeq A_\mathfrak{p}^*$ and $\Gamma \simeq U_1$, which are induced by the $\mathfrak{p}$-*cyclotomic character* κ. To define $\kappa\colon G_\infty \to A_\mathfrak{p}^*$, we extend Φ to a formal Drinfeld module which we denote by the same symbol $\Phi\colon A_\mathfrak{p} \to A_\mathfrak{p}\{\{\tau\}\}$ (see [31]). Then for any $\sigma \in G_\infty$ and any $\varepsilon \in \Phi[\mathfrak{p}^\infty]$ we have

$$\sigma(\varepsilon) = \Phi_{\kappa(\sigma)}(\varepsilon)\ . \tag{5}$$

This action provides the isomorphisms mentioned above. In particular, Γ_n corresponds to $U_1/U_{n+1} \simeq (1+\mathfrak{p}A_\mathfrak{p})/(1+\mathfrak{p}^{n+1}A_\mathfrak{p})$. It is well known that $U_1 \simeq \mathbb{Z}_p^\infty$ (a product of countably many copies of $\mathbb{Z}_p$).

As mentioned earlier, we define $\mathbb{F}_\mathfrak{p}$ to be the residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$; it is the same as the residue field $A/\mathfrak{p}$. Since we are in positive characteristic, $\mathbb{F}_\mathfrak{p}$ can be canonically identified with a subring of $A_\mathfrak{p}$ (by lifting $x \neq 0$ to $\tilde{x}$, the unique root of 1 whose reduction mod $\mathfrak{p}$ is x) and in the rest of the paper we shall generally think of it as such.

2.2. **The Iwasawa algebra.** Let W be the Witt ring of $\mathbb{F}_\mathfrak{p}$, which is isomorphic to $\mathbb{Z}_p[\boldsymbol{\mu}_{q^d-1}]$ (where $\boldsymbol{\mu}_{q^d-1}$ denotes the (q^d-1)-th roots of unity). By definition of Witt ring, we have an identification $W/pW = \mathbb{F}_\mathfrak{p}$. Moreover, the projection $W \twoheadrightarrow \mathbb{F}_\mathfrak{p}$ has a partial inverse $\mathbb{F}_\mathfrak{p}^* \to \boldsymbol{\mu}_{q^d-1}$, the *Teichmüller character* (again by lifting x to $\hat{x}$, the unique root of 1 whose reduction mod p is x).

The Iwasawa algebra we shall be working with is the completed group ring

$$\Lambda := W[[\Gamma]] = \varprojlim W[\Gamma_n]\ .$$

For any $n \geqslant 0$, put $\Gamma^{(n)} := \mathrm{Gal}(\mathcal{F}/F_n)$. The exact sequence $\Gamma^{(n)} \hookrightarrow \Gamma \twoheadrightarrow \Gamma_n$ induces

$$\mathfrak{I}_n \hookrightarrow \Lambda \twoheadrightarrow W[\Gamma_n]\ .$$

We also put $\mathfrak{M}_n := p^n\Lambda + \mathfrak{I}_{n-1}$ for any $n \geqslant 1$.

We recall some other basic facts on this non-noetherian algebra:

- $\{\mathfrak{M}_n\}_{n\geqslant 0}$ is a basis of neighbourhoods of zero in Λ;
- $\Lambda/p\Lambda = \mathbb{F}_\mathfrak{p}[[\Gamma]]$;

- Λ is a compact W-algebra and a complete local ring with maximal ideal $\mathfrak{M}_1$, so that
$$\Lambda/\mathfrak{M}_1 \simeq W/pW \simeq \mathbb{F}_\mathfrak{p} \ .$$

2.3. **Consequences of a theorem of Sinnott.** Let $C^0(\mathbb{Z}_p, A_\mathfrak{p})$ be the space of continuous functions from $\mathbb{Z}_p$ to $A_\mathfrak{p}$, endowed with the topology of uniform convergence. More generally, we can consider $C^0(\mathbb{Z}_p, M)$ where M is any finitely generated $A_\mathfrak{p}$-module: it turns out that $C^0(\mathbb{Z}_p, A_\mathfrak{p})$ is the projective limit of $C^0(\mathbb{Z}_p, A/\mathfrak{p}^n)$ as n varies.

Following [34], we define the $A_\mathfrak{p}$-module of Dirichlet series $Dir(\mathbb{Z}_p, A_\mathfrak{p})$ as the closure in $C^0(\mathbb{Z}_p, A_\mathfrak{p})$ of the module generated by the functions $\vartheta_u : \mathbb{Z}_p \to A_\mathfrak{p}$, $y \mapsto u^y$, for all $u \in U_1$. (If F_v is the completion of F at a place v, the element $u \in F_v$ satisfies $|1-u| < 1$ and $y \in \mathbb{Z}_p$, we put
$$u^y := \sum_{n \geqslant 0} \binom{y}{n} (u-1)^n \in F_v^* \ ,$$
where $\binom{y}{n}$ is the reduction modulo p of the value of the usual binomial.)

The next theorem follows from Sinnott's results and ideas in [34] applied to our setting.

Theorem 2.2. *There is an injective morphism*
$$s \colon \Lambda/p\Lambda \hookrightarrow Dir(\mathbb{Z}_p, A_\mathfrak{p})$$
such that for any $\gamma \in \Gamma$, one has $s(\gamma) = \vartheta_{\kappa(\gamma)}$.

Proof. For any ring R, the algebra of R-valued distributions on Γ can be identified with $R[[\Gamma]]$. In [34, Theorem 1] Sinnott constructs an isomorphism $A_\mathfrak{p}[[U_1]] \to Dir(\mathbb{Z}_p, A_\mathfrak{p})$ by attaching to a measure μ the function $y \mapsto \int_{U_1} u^y d\mu(u)$. In particular, ϑ_u corresponds to the Dirac delta at u. To complete our proof, one just has to recall that $\Lambda/p\Lambda = \mathbb{F}_\mathfrak{p}[[\Gamma]]$ is a subring of $A_\mathfrak{p}[[\Gamma]]$ and compose Sinnott's isomorphism with the one $A_\mathfrak{p}[[\Gamma]] \simeq A_\mathfrak{p}[[U_1]]$ induced by the cyclotomic character κ. $\square$

It is clear from the proof that the image of s is exactly the closure of the $\mathbb{F}_\mathfrak{p}$-module generated by the functions ϑ_u.

Proposition 2.3. *The morphism s induces a ring homomorphism*
$$s_n \colon \mathbb{F}_\mathfrak{p}[\Gamma_n] \to C^0(\mathbb{Z}_p, A/\mathfrak{p}^{n+1}) \ .$$

Proof. It suffices to remark that $\kappa(\Gamma^{(n)}) = U_{n+1}$. Hence for $\gamma \in \Gamma^{(n)}$ and $y \in \mathbb{Z}_p$ we have
$$s(\gamma)(y) = \vartheta_{\kappa(\gamma)}(y) = \kappa(\gamma)^y \in 1 + \mathfrak{p}^{n+1} A_\mathfrak{p}$$
which implies that the ideal $(\gamma - 1 \, : \, \gamma \in \Gamma^{(n)})$ in $\mathbb{F}_\mathfrak{p}[[\Gamma]]$ is sent by s into $C^0(\mathbb{Z}_p, \mathfrak{p}^{n+1} A_\mathfrak{p})$. The kernel of the natural projection $\mathbb{F}_\mathfrak{p}[[\Gamma]] \to \mathbb{F}_\mathfrak{p}[\Gamma_n]$ is precisely the closure of this ideal (that is, the image in $\mathbb{F}_\mathfrak{p}[[\Gamma]]$ of $\mathfrak{I}_n \subset \Lambda$). $\square$

We get a commutative diagram
$$\begin{array}{ccc} \mathbb{F}_\mathfrak{p}[[\Gamma]] & \xrightarrow{\ s\ } & Dir(\mathbb{Z}_p, A_\mathfrak{p}) \\ \downarrow & & \downarrow \\ \mathbb{F}_\mathfrak{p}[\Gamma_n] & \xrightarrow{\ s_n\ } & C^0(\mathbb{Z}_p, A/\mathfrak{p}^{n+1}) \end{array}$$
where the vertical maps are the natural ones. By construction one has $s = \varprojlim_n s_n$.

Proposition 2.4. *If $q^d > 2$, the map s_n is not injective for $n > 0$.*

Proof. Fix $n>0$ and choose a_1, a_2 and a_3 in $A_{\mathfrak{p}}$ so that a_i and a_j are different modulo $\mathfrak{p}$ if $i\neq j$ (this is where we use $q^d>2$). Consider the elements $\gamma_i\in\Gamma$ defined by

$$\kappa(\gamma_i)=1+a_i\pi_{\mathfrak{p}}^n \qquad \text{for } i=1,2,3\ .$$

Our hypothesis on the a_i's implies that the γ_i's have $\mathbb{F}_{\mathfrak{p}}$-linearly independent images in $\mathbb{F}_{\mathfrak{p}}[\Gamma_n]$. We need to find $x_i\in\mathbb{F}_{\mathfrak{p}}$ so to have

$$x_1\kappa(\gamma_1)^y+x_2\kappa(\gamma_2)^y+x_3\kappa(\gamma_3)^y\in\mathfrak{p}^{n+1}A_{\mathfrak{p}}$$

for all $y\in\mathbb{Z}_p$. This is equivalent to

$$x_1+x_2+x_3+y(x_1a_1+x_2a_2+x_3a_3)\pi_{\mathfrak{p}}^n\equiv 0 \pmod{\mathfrak{p}^{n+1}}\ ,$$

i.e.,

$$\begin{cases} x_1+x_2+x_3=0 \\ x_1a_1+x_2a_2+x_3a_3\equiv 0 \pmod{\mathfrak{p}} \end{cases} .$$

For any nontrivial solution of this linear system, the image in $\mathbb{F}_{\mathfrak{p}}[\Gamma_n]$ of $x_1\gamma_1+x_2\gamma_2+x_3\gamma_3$ is a nontrivial element of the kernel of s_n. □

2.4. **Characters of Δ.** Let $\omega_{\mathfrak{p}}\colon A\to A_{\mathfrak{p}}$ be the morphism of $\mathbb{F}$-algebras obtained composing $A\twoheadrightarrow A/\mathfrak{p}=\mathbb{F}_{\mathfrak{p}}$ with the lift $\mathbb{F}_{\mathfrak{p}}\hookrightarrow A_{\mathfrak{p}}$ (i.e., the *Teichmüller character* in positive characteristic[2]). Then any $a\in A-\mathfrak{p}$ is uniquely decomposed as

(6) $$a=\omega_{\mathfrak{p}}(a)\langle a\rangle_{\mathfrak{p}}$$

where $\langle a\rangle_{\mathfrak{p}}\in U_1=1+\mathfrak{p}A_{\mathfrak{p}}$. The domain of $\omega_{\mathfrak{p}}$ can be extended to all of $A_{\mathfrak{p}}$ (note that then the restriction of $\omega_{\mathfrak{p}}$ to $\mathbb{F}_{\mathfrak{p}}$ is just the identity) and equality (6) holds for any $a\in A_{\mathfrak{p}}-\mathfrak{p}A_{\mathfrak{p}}$.

The restriction of $\omega_{\mathfrak{p}}\circ\kappa\colon G_\infty\to\mathbb{F}_{\mathfrak{p}}^*$ to Δ yields an isomorphism $\Delta\to\mathbb{F}_{\mathfrak{p}}^*$, which in the rest of the paper will be simply denoted $\omega_{\mathfrak{p}}$, by an abuse of notation meant to emphasize the "Teichmüller-like" quality of this characteristic p character. If, for any $a\in A-\mathfrak{p}$, we let $\sigma_a\in\Delta$ be the element such that $\sigma_a(\varepsilon)=\Phi_a(\varepsilon)\ \forall\,\varepsilon\in\Phi[\mathfrak{p}]$ (recall that $\Delta\simeq\mathrm{Gal}(F_0/F)$), then we have

$$\omega_{\mathfrak{p}}(\sigma_a)=\omega_{\mathfrak{p}}(a)\ .$$

Composition of $\kappa|_\Delta$ with the Teichmüller lift $\mathbb{F}_{\mathfrak{p}}^*\to\boldsymbol{\mu}_{q^d-1}$ yields a character $\widetilde{\omega}_{\mathfrak{p}}\colon\Delta\to W^*$ (the *Teichmüller character* in characteristic 0). It satisfies

$$\widetilde{\omega}_{\mathfrak{p}}(\sigma_a)\equiv\omega_{\mathfrak{p}}(a)\pmod{pW}\ .$$

A (p-adic) character χ on Δ is called *odd* if $\chi(\mathcal{I}_\infty)\neq 1$ and *even* if $\chi(\mathcal{I}_\infty)=1$. Since all such characters are powers of $\widetilde{\omega}_{\mathfrak{p}}$, this definition amounts to saying that $\widetilde{\omega}_{\mathfrak{p}}^i$ is even if and only if $q-1$ divides i.

2.4.1. *Decomposition by characters.* For any p-adic character $\chi\in\mathrm{Hom}(\Delta,W^*)=:\widehat{\Delta}$ we put

(7) $$e_\chi:=\frac{1}{|\Delta|}\sum_{\delta\in\Delta}\chi(\delta^{-1})\delta\in W[\Delta]$$

for the idempotent associated with χ. We recall a few basic relations:

- for any $\delta\in\Delta$,

(8) $$e_\chi\delta=\chi(\delta)e_\chi\,;$$

- for any $\psi\in\widehat{\Delta}$,

$$\psi(e_\chi)=\begin{cases}1 & \text{if } \psi=\chi\\ 0 & \text{if } \psi\neq\chi\end{cases}\ ;$$

[2] The map $\omega_{\mathfrak{p}}$ can also be defined as the morphism of $\mathbb{F}$-algebras such that $v_{\mathfrak{p}}(\theta-\omega_{\mathfrak{p}}(\theta))\geqslant 1$: it satisfies $\omega_{\mathfrak{p}}(a)\equiv a \pmod{\mathfrak{p}}$ and corresponds to the choice of a root of $\pi_{\mathfrak{p}}$ in $\overline{\mathbb{F}}$ (because $\pi_{\mathfrak{p}}=\pi_{\mathfrak{p}}(\theta)\in A=\mathbb{F}[\theta]$ and we have $\pi_{\mathfrak{p}}(\theta)\equiv\pi_{\mathfrak{p}}(\omega_{\mathfrak{p}}(\theta)) \pmod{\mathfrak{p}}$, therefore $\pi_{\mathfrak{p}}(\omega_{\mathfrak{p}}(\theta))\equiv 0$).

- $\sum_{\chi\in\widehat{\Delta}} e_\chi = 1$.

As usual, for any $W[\Delta]$-module M, we denote by $M(\chi)$ the χ-part of M (i.e., the submodule $e_\chi M$) and we have a decomposition

$$M \simeq \bigoplus_{\chi\in\widehat{\Delta}} M(\chi)\ . \tag{9}$$

3. $\mathfrak{p}$-adic interpolation of the Carlitz-Goss L-function

In this section we present the analytic side of our work, i.e., the Carlitz-Goss ζ-function ζ_A and the $\mathfrak{p}$-adic L-function we shall use to interpolate ζ_A at integers. Moreover we introduce the Stickelberger series which will appear also in the computation of Fitting ideals of Tate modules and class groups in Section 4. Actually, the Stickelberger series is the main hero of this section: as we shall see, it plays a universal role in interpolating L-functions attached to abelian characters with no ramification outside a prescribed locus. In the case of $\mathbb{C}$-valued characters and the complex L-functions attached to them, this will be clear from (11). In Theorems 3.8 and 3.16, we shall see how, taking characteristic p-valued characters, the Stickelberger series interpolates the L-functions defined by Goss. We also remark that in [28] the Stickelberger series is used as a p-adic L-function.

For the convenience of the reader we will recall different constructions and properties: some of them are known but we lack an explicit reference including all of them.

3.1. **The Stickelberger series.** Recall that $\mathscr{P}_F$ is the set of places of F. Places different from ∞ will be often identified with the correponding prime ideals of A.

The subset of $\mathscr{P}_F$ where the extension $\mathcal{F}/F$ ramifies is $S = \{\mathfrak{p},\infty\}$. Define G_S as the Galois group of the maximal abelian extension $\mathcal{F}_S$ of F which is unramified outside S. For any $\mathfrak{q}\in\mathscr{P}_F - S$, let $\mathrm{Fr}_\mathfrak{q}\in G_S$ denote the corresponding (arithmetic) Frobenius automorphism.

Definition 3.1. We define the *Stickelberger series* by

$$\Theta_{\mathcal{F}_S/F,S}(X) := \prod_{\mathfrak{q}\in\mathscr{P}_F-S}(1-\mathrm{Fr}_\mathfrak{q}^{-1}\,X^{\deg(\mathfrak{q})})^{-1}\in\mathbb{Z}[G_S][[X]]\ . \tag{10}$$

More generally, for any closed subgroup $U<G_S$, we define

$$\begin{aligned}\Theta_{\mathcal{F}_S^U/F,S}(X) &:= \pi^{G_S}_{G_S/U}(\Theta_{\mathcal{F}_S/F,S})(X)\\ &= \prod_{\mathfrak{q}\in\mathscr{P}_F-S}(1-\pi^{G_S}_{G_S/U}(\mathrm{Fr}_\mathfrak{q}^{-1})X^{\deg(\mathfrak{q})})^{-1}\in\mathbb{Z}[\mathrm{Gal}(\mathcal{F}_S^U/F)][[X]]\ ,\end{aligned}$$

where $\pi^{G_S}_{G_S/U}\colon\mathbb{Z}[G_S]\to\mathbb{Z}[\mathrm{Gal}(\mathcal{F}_S^U/F)]$ is the map induced by the projection $G_S\twoheadrightarrow G_S/U$.

The series in (10) is well-defined, since for any k there are only finitely many places of degree k.

3.1.1. *Convergence.* Let R be a topological ring, complete with respect to a non-archimedean absolute value. The algebra $R[[G_S]]$ is the inverse limit of $R[\mathrm{Gal}(E/F)]$ as E varies among finite subextensions of $\mathcal{F}_S/F$; as such, it is endowed with a topological structure. (Topologically each $R[\mathrm{Gal}(E/F)]$ is the product of $[E:F]$ copies of R and $R[[G_S]]$ has the coarsest topology such that all projections $R[[G_S]]\to R[\mathrm{Gal}(E/F)]$ are continuous.)

For any topological ring $\mathcal{R}$ the Tate algebra $\mathcal{R}\langle X\rangle$ consists of those power series in $\mathcal{R}[[X]]$ whose coefficients tend to 0. In particular, $R[[G_S]]\langle X\rangle$ contains all those power series whose image in $R[G_S/U][[X]]$ is a polynomial for any open subgroup $U<G_S$.

For any unitary R, the natural map $\mathbb{Z} \to R$ (by $1 \mapsto 1$) allows to think of $\Theta_{\mathcal{F}_S/F,S}(X)$ as an element in $R[[G_S]][[X]]$. Moreover, for any group homomorphism $\alpha\colon G_S \to R^*$, the extension by linearity to a map $\alpha\colon \mathbb{Z}[G_S] \to R$ yields a power series $\alpha\big(\Theta_{\mathcal{F}_S/F,S}\big)(X) \in R[[X]]$.

Proposition 3.2. *Let R be a unitary topological $\mathbb{Z}_p$-algebra, complete with respect to a non-archimedean absolute value. Then the series $\Theta_{\mathcal{F}_S/F,S}(X)$ defines an element in the Tate algebra $R[[G_S]]\langle X\rangle$.*

Proof. The proof is essentially the same as in [28, Proposition 4.1] (see also [6, §5.3]), so here we just sketch the basic ideas.

Let $\psi\colon G_S \to \mathbb{C}^*$ be a continuous character (G_S has the profinite topology, so ψ factors through a subgroup of finite index). Then

$$\psi\big(\Theta_{\mathcal{F}_S/F,S}\big)(q^{-s}) = \prod_{\mathfrak{q}\in\mathscr{P}_F - S}\left(1 - \frac{\psi(\mathrm{Fr}_{\mathfrak{q}}^{-1})}{(N\mathfrak{q})^s}\right)^{-1} =: L_S(s,\psi) \tag{11}$$

is (possibly up to the Euler factors from places in S) the classical complex L-function attached to ψ (here $N\mathfrak{q} := q^{\deg(\mathfrak{q})}$ is the order of the finite field $\mathbb{F}_{\mathfrak{q}}$ and we assume $Re(s) > 1$ to ensure convergence of the infinite product). More precisely, one has

$$L(s,\psi) := L_S(s,\psi)\cdot\prod_{v\in S}(1-\psi(v)q^{-s\deg(v)})^{-1}\ , \tag{12}$$

where $\psi(v)$ denotes the value of ψ on the inverse of the Frobenius of v (an element in $G_S/\ker(\psi)$ if v is not ramified in $\mathcal{F}_S^{\ker(\psi)}/F$), with the usual convention $\psi(v) = 0$ if $\mathcal{F}_S^{\ker(\psi)}/F$ is ramified at v.

A theorem of Weil (see, e.g., [38, V, Théorème 2.5]) implies that $L(s,\psi)$ is a polynomial in q^{-s}, unless $\psi = \psi_0$ is trivial, in which case one has

$$L(s,\psi_0) = \frac{1}{(1-q^{-s})(1-q^{1-s})}\ . \tag{13}$$

Thus $L_S(s,\psi)$ is a rational function of q^{-s}, with denominator bounded independently of ψ.

Choose an auxiliary place $\mathfrak{q}_0 \notin S$ and put

$$\Theta_{\mathcal{F}_S/F,S,\{\mathfrak{q}_0\}}(X) := (1 - q^{\deg(\mathfrak{q}_0)}\,\mathrm{Fr}_{\mathfrak{q}_0}^{-1}\,X^{\deg(\mathfrak{q}_0)})\Theta_{\mathcal{F}_S/F,S}(X)\ .$$

By Weil's theorem, for all ψ as above, $\psi\big(\Theta_{\mathcal{F}_S/F,S,\{\mathfrak{q}_0\}}\big)(q^{-s})$ belongs to $\mathbb{C}[q^{-s}]$ (more precisely, to $\mathbb{Z}[\psi(G_S)][q^{-s}]$). As a consequence, one gets

$$\pi^{G_S}_{G_S/U}(\Theta_{\mathcal{F}_S/F,S,\{\mathfrak{q}_0\}})(X) \in \mathbb{Z}[\mathrm{Gal}(\mathcal{F}_S^U/F)][X]$$

for all open subgroups $U < G_S$ and hence $\Theta_{\mathcal{F}_S/F,S,\{\mathfrak{q}_0\}}(X) \in \mathbb{Z}[[G_S]]\langle X\rangle$. It follows that also $\Theta_{\mathcal{F}_S/F,S}(X)$ is in $\mathbb{Z}_p[[G_S]]\langle X\rangle$, because the ratio between $\Theta_{\mathcal{F}_S/F,S,\{\mathfrak{q}_0\}}(X)$ and $\Theta_{\mathcal{F}_S/F,S}(X)$ is a unit in the Tate algebra.

Finally, for R as in the hypothesis, the natural map $\mathbb{Z}_p \to R$ is extended to a continuous homomorphism $\mathbb{Z}_p[[G_S]][[X]] \to R[[G_S]][[X]]$. Our proposition follows from the restriction $\mathbb{Z}_p[[G_S]]\langle X\rangle \to R[[G_S]]\langle X\rangle$. $\square$

Remark 3.3. We remind readers of two important properties of Stickelberges series (for details see [32, Chapter 15]). Let E/F be a finite subextension of $\mathcal{F}_S/F$ with $G = \mathrm{Gal}(E/F)$. Then one has

(1) $\Theta_{E/F,S}(X) \in \frac{1}{(1-qX)|G|}\overline{\mathbb{Z}}[G][X] \cap \mathbb{Z}[G][[X]]$ (where $\overline{\mathbb{Z}}$ is the integral closure of $\mathbb{Z}$ in the algebraic closure of $\mathbb{Q}$);

(2) the Brumer-Stark element $w_{E/F} = (q-1)\Theta_{E/F,S}(1) \in \mathbb{Z}[G]$ annihilates $\mathcal{C}\ell(E)$.

A different approach to the proof of (2) was proposed in [1].

Theorem 3.4. *Let R be as in Proposition 3.2 and $\alpha\colon G_S \to R^*$ a continuous group homomorphism. The power series $\alpha\big(\Theta_{\mathcal{F}_S/F,S}\big)(X)$ converges on the unit disk $\{x \in R : |x| \leqslant 1\}$.*

Proof. The ring homomorphism $R[[G_S]] \to R$ induced by α is continuous; hence it extends to a homomorphism of Tate algebras $R[[G_S]]\langle X\rangle \to R\langle X\rangle$. Thus, by Proposition 3.2, $\alpha\big(\Theta_{\mathcal{F}_S/F,S}\big)(X)$ is in $R\langle X\rangle$, which, by definition, consists exactly of those power series convergent on the unit disk. □

3.1.2. *Stickelberger series in the $\mathfrak{p}$-cyclotomic tower.* In the following, we shall be particularly interested in the image of the Stickelberger series along the $\mathfrak{p}$-cyclotomic tower. Define

$$\Theta_\infty(X) := \Theta_{\mathcal{F}/F,S}(X) \in \mathbb{Z}[G_\infty][[X]] \tag{14}$$

and, for all $n \in \mathbb{N}$,

$$\Theta_n(X) := \Theta_{F_n/F,S}(X) \in \mathbb{Z}[G_n][[X]] \ .$$

We shall think of Θ_∞ and Θ_n as power series with coefficients respectively in $W[[G_\infty]]$ and $W[G_n]$.

Any element in G_∞ can be uniquely written as $\delta\gamma$, with $\delta \in \Delta$ and $\gamma \in \Gamma$. Consequently, given $\chi \in \mathrm{Hom}(\Delta, W^*)$ we can define a group homomorphism $G_\infty \to \Lambda^*$ by $\delta\gamma \mapsto \chi(\delta)\gamma$. By linearity and continuity, this can be extended to a ring homomorphism (which, by abuse of notation, we still denote by the same symbol) $\chi\colon W[[G_\infty]] \to \Lambda$. The decomposition (9) applied to $W[[G_\infty]][[X]]$ then yields the following definition.

Definition 3.5. For any $\chi \in \mathrm{Hom}(\Delta, W^*)$, the *$\chi$-Stickelberger series* for the $\mathfrak{p}$-cyclotomic tower is

$$\Theta_\infty(X,\chi) := \chi(\Theta_\infty)(X) \in \Lambda[[X]] \ .$$

Similarly, we put

$$\Theta_n(X,\chi) := \chi(\Theta_n)(X) \in W[\Gamma_n][[X]] \ . \tag{15}$$

The series $\Theta_n(X,\chi)$ form a projective system: let $\pi_n^{n+1}\colon W[\Gamma_{n+1}][[X]] \to W[\Gamma_n][[X]]$ be the projection induced by the natural map $\Gamma_{n+1} \twoheadrightarrow \Gamma_n$, then we have

$$\pi_n^{n+1}(\Theta_{n+1}(X,\chi)) = \Theta_n(X,\chi)$$

and $\Theta_\infty(X,\chi) = \varprojlim \Theta_n(X,\chi)$ for all $\chi \in \widehat{\Delta}$. Moreover, (8) yields

$$e_\chi \Theta_\infty(X) = \Theta_\infty(X,\chi) e_\chi \tag{16}$$

and

$$\Theta_\infty(X) = \sum_{\chi \in \widehat{\Delta}} \Theta_\infty(X,\chi) e_\chi$$

(of course these relations descend to level n for all $n \in \mathbb{N}$). Finally, the proof of Proposition 3.2 shows that $\Theta_n(X,\chi) \in W[\Gamma_n][X]$ if χ is not the trivial character χ_0 and that $\Theta_n(X,\chi_0) \in \frac{1}{1-qX} W[\Gamma_n][X]$.

3.2. **Carlitz-Goss ζ-function and Bernoulli-Goss numbers.** We recall the construction of the Goss L-function and the main properties needed in our work (a general reference is [21, Chapter 8]).

As usual, F_∞ denotes the completion of F at ∞ and $\mathbb{C}_\infty$ is the completion of an algebraic closure of F_∞. The valuation on F_∞ extends to $v_\infty\colon \mathbb{C}_\infty \to \mathbb{Q} \cup \{\infty\}$. We also fix an embedding of $\overline{F}$ in $\mathbb{C}_\infty$. Finally, let $U_1(\infty)$ denote the group of 1-units in F_∞^*.

Since we are taking $F = \mathbb{F}(\theta)$, a somewhat natural choice of uniformizer at ∞ is θ^{-1}. Fixing a uniformizer establishes a sign function $sgn\colon F_\infty^* \to \mathbb{F}^*$, which sends $x \in F_\infty^*$ into the residue of $x\theta^{v_\infty(x)}$, and a projection

$$F_\infty^* \longrightarrow U_1(\infty) \ , \qquad x \mapsto \langle x\rangle_\infty := \frac{x\theta^{v_\infty(x)}}{sgn(x)} \ .$$

Note that one has $\ker(sgn) = \theta^{\mathbb{Z}} \times U_1(\infty)$.

Remark 3.6. These maps can be made more "concrete" by the following observation. Let $a \in A - \{0\}$ and write it as $a = a_0 + \ldots + a_n\theta^n$, with $n = \deg(a)$ and $a_i \in \mathbb{F}$. Then we have

$$sgn(a) = a_n \in \mathbb{F}^* \quad \text{and} \quad \langle a \rangle_\infty = \frac{a}{\theta^{\deg(a)} sgn(a)} = \frac{a}{a_n\theta^n} \in 1 + \theta^{-1}\,\mathbb{F}[[\theta^{-1}]] \ . \tag{17}$$

3.2.1. *The group* $\mathbb{S}_\infty$. Let $\mathbb{I}_F$ denote the group of ideles of F. Then we have

$$\mathbb{I}_F/F^* \simeq \ker(sgn) \times \prod_{\mathfrak{q}\in\mathscr{P}_F-\{\infty\}} A_{\mathfrak{q}}^* =: \mathcal{D} \ , \tag{18}$$

where $A_{\mathfrak{q}}$ denotes the completion of A with respect to $\mathfrak{q}$ and the isomorphism is given by the embedding of the right-hand side as a subgroup of $\mathbb{I}_F$.

The group of $\mathbb{C}_\infty$-valued principal quasi-characters on $\mathbb{I}_F/F^*$ is

$$\mathbb{S}_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p \ .$$

For $s = (x, y) \in \mathbb{S}_\infty$, we define a continuous homomorphism $\ker(sgn) \longrightarrow \mathbb{C}_\infty^*$ by

$$a \mapsto a^s := x^{-v_\infty(a)} \langle a \rangle_\infty^y \ . \tag{19}$$

This map is extended to all of $\mathbb{I}_F$ by the projection to $\ker(sgn)$ induced by the isomorphism (18).

The group structure on $\mathbb{S}_\infty$ is given by $(x_1, y_1) + (x_2, y_2) := (x_1x_2, y_1 + y_2)$. We have an injection $\mathbb{Z} \hookrightarrow \mathbb{S}_\infty$, by

$$j \mapsto c_j := (\theta^j, j) \ .$$

By (17) we get $a^{c_j} = a^j$ for all $j \in \mathbb{Z}$ and monic $a \in A$.

In analogy with the complex half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \Re(z) > 1\}$, we define a "half-plane"

$$\mathbb{S}_\infty^+ := \{(x, y) \in \mathbb{S}_\infty : |x| > 1\} \ .$$

3.2.2. *From* $\Theta_{\mathcal{F}_S/F,S}$ *to* ζ_A. Let A_+ be the set of monic polynomials in A. Thinking of A as a subset of F_∞, we have $A_+ = A \cap \ker(sgn)$.

Definition 3.7. The *Carlitz-Goss* ζ*-function* is defined as

$$\zeta_A(s) := \sum_{a\in A_+} a^{-s} \quad , \ s \in \mathbb{S}_\infty \ . \tag{20}$$

For $s = (x, y)$, we have $a^s = x^{\deg(a)}\langle a\rangle_\infty^y$, hence $|a^{-s}| = |x|^{-\deg(a)}$. It follows that the series (20) converges on $\mathbb{S}_\infty^+$. (Note the analogy with convergence of the series defining $L(s, \psi)$ for $Re(s) > 1$.)

Class field theory identifies the group $U_1(\infty)$ with a factor of G_S. Consequently, the construction in Section 3.2.1 can be used to define $\mathbb{C}_\infty$-valued characters on G_S. More precisely, for $y \in \mathbb{Z}_p$ let $\psi_y \colon G_S \to \mathbb{C}_\infty^*$ be the homomorphism obtained by composing the class field theoretic projection $\rho \colon G_S \twoheadrightarrow U_1(\infty)$ with $(1, y) \in \mathbb{S}_\infty$. Then Theorem 3.4 shows that $\psi_y(\Theta_{\mathcal{F}_S/F,S})(x)$ converges for all $x \in \mathbb{C}_\infty$ such that $|x| \leqslant 1$.

Theorem 3.8. *For all* $s = (x, y) \in \mathbb{S}_\infty^+$, *we have*

$$\psi_{-y}(\Theta_{\mathcal{F}_S/F,S})(x^{-1}) = (1 - \pi_{\mathfrak{p}}^{-s})\zeta_A(s) \ . \tag{21}$$

(Recall that $\pi_{\mathfrak{p}}$ is the monic irreducible generator of the ideal $\mathfrak{p}$ in A.)

Proof. For every place $\mathfrak{q}$ in $\mathscr{P}_F - S$, let $\pi_{\mathfrak{q}} \in A_+$ denote the monic generator of the corresponding prime ideal in A. Then (20) can be rewritten as an Euler product

$$\zeta_A(s) = \prod_{\mathfrak{q}\in\mathscr{P}_F-\{\infty\}} (1 - \pi_{\mathfrak{q}}^{-s})^{-1} = \prod_{\mathfrak{q}\in\mathscr{P}_F-\{\infty\}} (1 - \langle\pi_{\mathfrak{q}}\rangle_\infty^{-y} x^{-\deg(\mathfrak{q})})^{-1} \ . \tag{22}$$

By class field theory, we have a reciprocity map $rec\colon \mathbb{I}_F \to G_S$ with dense image isomorphic to $\ker(sgn) \times A_{\mathfrak{p}}^*$. Using (18), the composition $\rho \circ rec$ is just the projection

$$\mathcal{D} = \theta^{\mathbb{Z}} \times U_1(\infty) \times \prod_{\mathfrak{q} \in \mathscr{P}_F - \infty} A_{\mathfrak{q}}^* \longrightarrow U_1(\infty) \ .$$

For $\mathfrak{q}$ in $\mathscr{P}_F - S$, let $i_{\mathfrak{q}} \in \mathbb{I}_F$ denote the idele having $\pi_{\mathfrak{q}}$ as its $\mathfrak{q}$-component and 1 as component at all other places: then $\mathrm{Fr}_{\mathfrak{q}} = rec(i_{\mathfrak{q}})$. By the diagonal embedding $F^* \hookrightarrow \mathbb{I}_F$, we also get $rec(i_{\mathfrak{q}}) = rec(i_{\mathfrak{q}} a)$ for all $a \in F^*$. Since $\pi_{\mathfrak{q}}$ belongs to F^* and $i_{\mathfrak{q}}\pi_{\mathfrak{q}}^{-1}$ is in the fundamental domain $\mathcal{D}$, we finally obtain $\rho(\mathrm{Fr}_{\mathfrak{q}}) = \langle \pi_{\mathfrak{q}}^{-1} \rangle_\infty$.

Thus $\psi_{-y}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) = \langle \pi_{\mathfrak{q}} \rangle_\infty^{-y}$ and

$$\psi_{-y}(\Theta_{\mathcal{F}_S/F,S})(x^{-1}) = \prod_{\mathfrak{q} \in \mathscr{P}_F - S} (1 - \langle \pi_{\mathfrak{q}} \rangle_\infty^{-y} x^{-\deg(\mathfrak{q})})^{-1} \ .$$

Comparison with (22) completes the proof. □

Theorem 3.8 can be used to obtain analytical continuation of ζ_A on the "boundary" of $\mathbb{S}_\infty^+$ (that is, $\{s = (x, y) : |x| = 1\}$), since, by Theorem 3.4, the left-hand side of (21) converges if $|x| = 1$.

Remark 3.9. Let R be a topological ring: then the ring of R-valued distributions[3] on $\ker(sgn)$ is isomorphic to $R[[U_1(\infty)]][[X]]$. This suggests that equation (21) can be interpreted as providing an integral formula for the Carlitz-Goss zeta function (namely, integration of the quasi-character s against the distribution induced by the Stickelberger series; a variant of this will be made explicit in the proof of Theorem 3.22). Integral formulas for ζ_A and its generalizations were already known (starting with Goss's foundational paper [20]; see [39, §5.7] for a quick introduction to the topic), but (to the best of our knowledge) were all based on measures on some additive group; our approach instead stresses the role of the multiplicative group $\ker(sgn)$ and thus might provide some useful new insight.

3.2.3. *Bernoulli-Goss numbers.* Our final goal in this chapter is to interpolate the Carlitz-Goss zeta function at negative integers. Lacking a functional equation, we have to use more brute force techniques in order to extend the domain of ζ_A to all of $\mathbb{S}_\infty$.

For any $n \geqslant 1$ let $A_{+,n} := \{a \in A_+ \ : \ \deg(a) = n\}$ (note that $|A_{+,n}| = q^n$). For any $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ put

$$S_n(j) := \sum_{a \in A_{+,n}} a^j \ .$$

Note that we have $S_0(j) = 1$ for all $j \in \mathbb{Z}$.

Lemma 3.10. *If $1 \leqslant j < q^n - 1$, then $S_n(j) = 0$.*

Proof. This is due to Carlitz. See e.g. [21, Remark 8.12.1.1] for a proof. □

Remark 3.11. The statement in Lemma 3.10 is far from being the best possible. Necessary and sufficient criteria for the vanishing of $S_n(j)$ can be found in [41, Theorem 1], which also provides some information on the history of the subject.[4]

Reorganizing the terms in (20) we can also write the Carlitz-Goss ζ-function as

$$\zeta_A(x, y) := \sum_{n \geqslant 0} \left(\sum_{a \in A_{+,n}} \langle a \rangle_\infty^{-y} \right) x^{-n} \quad , \ (x, y) \in \mathbb{S}_\infty \ .$$

This second formula guarantees the convergence for all $s \in \mathbb{S}_\infty$ because of the following

[3] By R-valued distributions on a locally profinite group G we mean the linear functionals on the space of compactly supported locally constant functions $G \to R$.

[4] Readers are warned that the notation in [41] is different from ours: our $S_n(j)$ becomes $S_n(-j)$ in [41].

Lemma 3.12. *For any $y \in \mathbb{Z}_p$ and any $n \geqslant 1$, one has*

$$v_\infty\left(\sum_{a\in A_{+,n}} \langle a\rangle_\infty^y\right) \geqslant p^{n-1}\ .$$

Proof. The case $n=1$ is obvious. Now consider $n \geqslant 2$. If $y = p^{n-1}y'$, we can write $\langle a\rangle_\infty^{y'} = 1+\tilde{a}$ where $v_\infty(\tilde{a}) \geqslant 1$. The claim in this case then follows from

$$\sum_{a\in A_{+,n}} \langle a\rangle_\infty^y = \sum_{a\in A_{+,n}} \langle a\rangle_\infty^{y'p^{n-1}} = \sum_{a\in A_{+,n}} (1+\tilde{a})^{p^{n-1}} = \sum_{a\in A_{+,n}} (1+\tilde{a}^{p^{n-1}}) = \sum_{a\in A_{+,n}} \tilde{a}^{p^{n-1}}$$

(since we are in characteristic p and $|A_{n,+}| = q^n$).
If $y \not\equiv 0 \pmod{p^{n-1}}$, then take an integer $y_{n-1} \equiv y \pmod{p^{n-1}}$ with $1 \leqslant y_{n-1} \leqslant p^{n-1}-1$. Since $q \geqslant p$, we get $y_{n-1} < q^n - 1$ and Lemma 3.10 implies $S_n(y_{n-1}) = 0$. Therefore

$$\sum_{a\in A_{+,n}} \langle a\rangle_\infty^{y_{n-1}} = \sum_{a\in A_{+,n}} \left(\frac{a}{\theta^n}\right)^{y_{n-1}} = \frac{1}{\theta^{ny_{n-1}}} S_n(y_{n-1}) = 0\ .$$

Moreover

$$\langle a\rangle_\infty^y - \langle a\rangle_\infty^{y_{n-1}} = \langle a\rangle_\infty^{y_{n-1}}\left(\langle a\rangle_\infty^{p^{n-1}y'} - 1\right) = \langle a\rangle_\infty^{y_{n-1}}\left(\langle a\rangle_\infty^{y'} - 1\right)^{p^{n-1}} = \langle a\rangle_\infty^{y_{n-1}} c^{p^{n-1}}$$

(where $v_\infty(c) \geqslant 1$), so that

$$v_\infty(\langle a\rangle_\infty^y - \langle a\rangle_\infty^{y_{n-1}}) \geqslant p^{n-1}\ .$$

Hence

$$v_\infty\left(\sum_{a\in A_{+,n}} \langle a\rangle_\infty^y\right) = v_\infty\left(\sum_{a\in A_{+,n}} \left(\langle a\rangle_\infty^y - \langle a\rangle_\infty^{y_{n-1}}\right)\right) \geqslant p^{n-1}\ . \qquad \square$$

For any $j \in \mathbb{N}$ and $x \in \mathbb{C}_\infty^*$, we have the equality

$$\zeta_A\left(\frac{x}{\theta^j}, -j\right) = \sum_{a\in A_+} x^{-\deg(a)}\theta^{j\deg(a)}\left(\frac{a}{\theta^{\deg(a)}}\right)^j = \sum_{n\geqslant 0} S_n(j)x^{-n}\ , \tag{23}$$

which leads to the following

Definition 3.13.

(1) For any $j \in \mathbb{N}$ we put

$$Z(X,j) := \sum_{n\geqslant 0} S_n(j)X^n \in A[X] \tag{24}$$

(it is a polynomial because of Lemma 3.10).

(2) For any $j \in \mathbb{N}$, the *Bernoulli-Goss numbers* $\beta(j)$ are defined as

$$\beta(j) := \begin{cases} Z(1,j) & \text{if } j = 0 \text{ or } j \not\equiv 0 \pmod{q-1} \\ -\frac{d}{dX}Z(X,j)|_{X=1} & \text{if } j \geqslant 1 \text{ and } j \equiv 0 \pmod{q-1} \end{cases}\ .$$

By definition, for any $j \in \mathbb{N}$, we have

$$\zeta_A(-j) = Z(1,j)$$

(by an abuse of notation, we write $\zeta(-j)$ for $\zeta(c_{-j})$). It is known that, for $j \geqslant 1$ with $j \equiv 0 \pmod{q-1}$, we have $Z(1,j) = 0$, which corresponds to a trivial zero in this setting (see [21, Example 8.13.6]). Moreover it is clear that $\beta(j) \in A$ and $\beta(j) = 1$ for $0 \leqslant j \leqslant q-2$. We also have $\beta(q-1) = 1$, as can be deduced from the following lemma.

Lemma 3.14. *For any $j \in \mathbb{N}$, we have $\beta(j) \equiv 1 \pmod{\theta^q - \theta}$. In particular $\beta(j) \neq 0$.*

Proof. Recall that $\beta(0) = 1$ and $S_0(j) = 1$ for any $j \geqslant 0$. For any $\alpha \in \mathbb{F}$ we can write a polynomial $a \in A_{+,n}$ in terms of powers of $\theta - \alpha$, i.e., $a = a_0 + a_1(\theta - \alpha) + \cdots + (\theta - \alpha)^n$. Therefore, for any $j \geqslant 1$,

$$S_n(j) = \sum_{a \in A_{+,n}} a^j \equiv q^{n-1} \sum_{a_0 \in \mathbb{F}} a_0^j \pmod{\theta - \alpha} .$$

Thus $S_n(j) \equiv 0 \pmod{\theta - \alpha}$ for any $n \geqslant 2$ or for $n = 1$ and $j \not\equiv 0 \pmod{q-1}$. Moreover for $n = 1$ and $j \equiv 0 \pmod{q-1}$ one has

$$S_1(j) \equiv \sum_{a_0 \in \mathbb{F}} 1 \equiv -1 \pmod{\theta - \alpha} .$$

Hence

$$Z(X,j) \equiv \begin{cases} S_0(j) - X \equiv 1 - X \pmod{\theta - \alpha} & \text{if } j \geqslant 1 \text{ and } j \equiv 0 \pmod{q-1} \\ \\ S_0(j) \equiv 1 \pmod{\theta - \alpha} & \text{otherwise} \end{cases} .$$

The lemma follows by the definition of $\beta(j)$ (recalling that the terms $\theta - \alpha$ are relatively prime and their product is $\theta^q - \theta$). ☐

3.3. **$\mathfrak{p}$-adic L-function and interpolation.** The previous section dealt with the prime at infinity, now we focus on the other place in S. We give here the details of the construction of Goss's $\mathfrak{p}$-adic L-function (see [21] for more).

3.3.1. *The group $\mathbb{S}_\mathfrak{p}$.* Similarly to $\mathbb{S}_\infty$, we define a group of $\mathbb{C}_\mathfrak{p}$-valued quasi-characters on $\mathbb{I}_F/F^*$ by

$$\mathbb{S}_\mathfrak{p} := \mathbb{C}_\mathfrak{p}^* \times \mathbb{Z}_p \times \mathbb{Z}/|\mathbb{F}_\mathfrak{p}^*| .$$

However, in this case we shall be interested only in characters factoring through the compact group $A_\mathfrak{p}^*$. So we embed $\mathbb{Z}$ into $\mathbb{S}_\mathfrak{p}$ by $j \mapsto (1, j, j)$. (Note that the image of this map is dense in $\{1\} \times \mathbb{Z}_p \times \mathbb{Z}/(q^d - 1)$, in contrast with the discrete embedding $\mathbb{Z} \hookrightarrow \mathbb{S}_\infty$. This should be compared with the fact that $\mathbb{Z}$ is discrete in $\mathbb{C}$, but not in the p-adics.) [5]

For $s = (1, y, i) \in \mathbb{S}_\mathfrak{p}$ and $a \in A_\mathfrak{p}^*$, the decomposition (6) suggests to define

$$a^s := \omega_\mathfrak{p}^i(a)\langle a \rangle_\mathfrak{p}^y .$$

Then we obtain a continuous homomorphism $\xi_s \colon G_S \to \mathbb{C}_\mathfrak{p}^*$ as composition of the maps

$$G_S \xrightarrow{\sigma \mapsto \sigma|_\mathcal{F}} G_\infty \xrightarrow{\kappa} A_\mathfrak{p}^* \xrightarrow{a \mapsto a^s} \mathbb{C}_\mathfrak{p}^* . \tag{25}$$

3.3.2. *The $\mathfrak{p}$-adic L-function.* As with ζ_A, we first define a function by a certain power series and then interpret it as specialization of the Stickelberger series.

Definition 3.15. For any $0 \leqslant i \leqslant q^d - 2$ and any $y \in \mathbb{Z}_p$, we define the $\mathfrak{p}$-*adic L-function* as

$$L_\mathfrak{p}(X, y, \omega_\mathfrak{p}^i) := \sum_{n \geqslant 0} \left(\sum_{a \in A_{+,n} - \mathfrak{p}} \omega_\mathfrak{p}^i(a)\langle a \rangle_\mathfrak{p}^y \right) X^n . \tag{26}$$

Note that $L_\mathfrak{p}(X, y, \omega_\mathfrak{p}^i)$ is an element of $A_\mathfrak{p}[[X]]$: as such, it converges on the open unit disc of $\mathbb{C}_\mathfrak{p}$. We can think of it as a function defined on $\mathbb{S}_\mathfrak{p}^+ := \{(x, y, i) \in \mathbb{S}_\mathfrak{p} : |x| < 1\}$.

Theorem 3.16. *We have*

$$\xi_{-s}(\Theta_{\mathcal{F}_S/F,S})(X) = L_\mathfrak{p}(X, y, \omega_\mathfrak{p}^i) \tag{27}$$

for every $s = (y, i) \in \mathbb{Z}_p \times \mathbb{Z}/(q^d - 1)$.

[5] This definition of $\mathbb{S}_\mathfrak{p}$ - the same as in [39, §5.5(b)] - differs from the one in [21], where the factor $\mathbb{C}_\mathfrak{p}^*$ is missing. We decided to insert this factor in order to emphasize the symmetry with $\mathbb{S}_\infty$.

Proof. Equation (26) can be rewritten as an Euler product

$$L_{\mathfrak{p}}(X,y,\omega_{\mathfrak{p}}^i)=\prod_{\mathfrak{q}\in\mathscr{P}_F-S}\left(1-\omega_{\mathfrak{p}}^i(\pi_{\mathfrak{q}})\langle\pi_{\mathfrak{q}}\rangle_{\mathfrak{p}}^y X^{\deg(\mathfrak{q})}\right)^{-1}. \tag{28}$$

Thus, as in the proof of Theorem 3.8, we just need to check that the equality

$$\xi_{-s}(\mathrm{Fr}_{\mathfrak{q}}^{-1})=\omega_{\mathfrak{p}}^i(\pi_{\mathfrak{q}})\langle\pi_{\mathfrak{q}}\rangle_{\mathfrak{p}}^y=\pi_{\mathfrak{q}}^s$$

holds for every s and $\mathfrak{q}$. An element in G_∞ is completely determined by its action on $\Phi[\mathfrak{p}^\infty]$; since $\Phi_{\pi_{\mathfrak{q}}}(x)\in A[x]$ is monic and it satisfies

$$\Phi_{\pi_{\mathfrak{q}}}(\varepsilon)\equiv\varepsilon^{\deg(\mathfrak{q})}\pmod{\mathfrak{q}}$$

for every $\varepsilon\in\Phi[\mathfrak{p}^\infty]$, we get $\Phi_{\pi_{\mathfrak{q}}}(\varepsilon)=\mathrm{Fr}_{\mathfrak{q}}(\varepsilon)$. Then (5) implies that the restriction of $\mathrm{Fr}_{\mathfrak{q}}$ to $\mathcal{F}$ is exactly $\kappa^{-1}(\pi_{\mathfrak{q}})$. □

Theorem 3.16 implies that the series $L_{\mathfrak{p}}(X,y,\omega_{\mathfrak{p}}^i)$ converges on the closed unit disc. Actually, one can show that (26) defines an entire function on $\mathbb{C}_{\mathfrak{p}}$, by a reasoning similar to the one of Lemma 3.12. Since we are only interested in the specialization at $X=1$, we won't discuss the matter any further (see [21, Chapter 8] for more).

Corollary 3.17. *Let j be a natural number congruent to $i \pmod{q^d-1}$. Then*

$$L_{\mathfrak{p}}(X,j,\omega_{\mathfrak{p}}^i)=(1-\pi_{\mathfrak{p}}^jX^d)Z(X,j)\in A[X] \tag{29}$$

and, for any $y\in\mathbb{Z}_p$, we have

$$L_{\mathfrak{p}}(X,y,\omega_{\mathfrak{p}}^i)\equiv Z(X,i)\pmod{\mathfrak{p}}. \tag{30}$$

Proof. Since j is an integer and i is its reduction modulo q^d-1, we have

$$\xi_{(-j,-i)}(\mathrm{Fr}_{\mathfrak{q}}^{-1})=\omega_{\mathfrak{p}}^i(\pi_{\mathfrak{q}})\langle\pi_{\mathfrak{q}}\rangle_{\mathfrak{p}}^j=\pi_{\mathfrak{q}}^j=\langle\pi_{\mathfrak{q}}\rangle_\infty^j\cdot\theta^j=\psi_j(\mathrm{Fr}_{\mathfrak{q}}^{-1})\cdot\theta^j$$

for all places $\mathfrak{q}\notin S$. Therefore Theorem 3.16 gives an equality of power series in $F[[X]]$

$$L_{\mathfrak{p}}(X,j,\omega_{\mathfrak{p}}^i)=\xi_{(-j,-i)}(\Theta_{\mathcal{F}_S/F,S})(X)=\psi_j(\Theta_{\mathcal{F}_S/F,S})(\theta^jX).$$

It is convenient to extend the exponentiation in (19) by $a^{(xX,y)}:=\langle a\rangle_\infty^y(xX)^{-v_\infty(a)}$ (where x,y are as in (19) and X is a formal variable). Then Theorem 3.8 yields

$$\psi_j(\Theta_{\mathcal{F}_S/F,S})(\theta^jX)=(1-\pi_{\mathfrak{p}}^{-(\theta^jX,j)})\cdot\zeta_A\left(\frac{1}{\theta^jX},-j\right)=(1-\pi_{\mathfrak{p}}^jX^d)\,Z(X,j)$$

(the first equality is just a restatement of (21) in terms of Laurent series and the second one follows from (23)).
As for (30), it is enough to observe that one has $\langle a\rangle_{\mathfrak{p}}^y\equiv1\pmod{\mathfrak{p}}$ for any $a\in A_{\mathfrak{p}}^*$ and $y\in\mathbb{Z}_p$. Hence (26) shows that the variable y is irrelevant modulo $\mathfrak{p}$ and (29) yields

$$L_{\mathfrak{p}}(X,y,\omega_{\mathfrak{p}}^i)\equiv L_{\mathfrak{p}}(X,i,\omega_{\mathfrak{p}}^i)\equiv Z(X,i)\pmod{\mathfrak{p}}.$$ □

Remark 3.18. A more direct proof of (29) can be obtained from the equation

$$L_{\mathfrak{p}}(X,j,\omega_{\mathfrak{p}}^i)=\sum_{n\geqslant0}(S_n(j)-\pi_{\mathfrak{p}}^jS_{n-d}(j))X^n,$$

which is obvious from (26). However, the devious path we followed might be forgiven considering that it illustrates how (29) and (21) are essentially the same statement.

Regarding the special values of $L_{\mathfrak{p}}(X,i,\omega_{\mathfrak{p}}^i)$ we have the following

Lemma 3.19. *If $i\equiv0\pmod{q-1}$, then $L_{\mathfrak{p}}(1,y,\omega_{\mathfrak{p}}^i)=0$ for all $y\in\mathbb{Z}_p$.*

Proof. For any $y \in \mathbb{Z}_p$ and $m \geqslant 1$, take $j \in \mathbb{N} - \{0\}$ such that $j \equiv i \pmod{q^d - 1}$ and $j \equiv y \pmod{p^m}$. Then one has

$$\langle a \rangle_{\mathfrak{p}}^{y} \equiv \langle a \rangle_{\mathfrak{p}}^{y-j} \langle a \rangle_{\mathfrak{p}}^{j} \equiv \langle a \rangle_{\mathfrak{p}}^{j} \pmod{\mathfrak{p}^{p^m}}$$

for any $a \in A_{\mathfrak{p}}^{*}$ and hence, by (26) and (29),

$$L_{\mathfrak{p}}(X, y, \omega_{\mathfrak{p}}^{i}) \equiv L_{\mathfrak{p}}(X, j, \omega_{\mathfrak{p}}^{i}) = (1 - \pi_{\mathfrak{p}}^{j} X^d) Z(X, j) \pmod{\mathfrak{p}^{p^m}} \ . \tag{31}$$

Now, since $j \geqslant 1$ and $j \equiv i \equiv 0 \pmod{q-1}$, we have $Z(1, j) = 0$ (see [21, Example 8.13.6]) and the lemma follows taking the limit as m goes to infinity. □

We also recall one of the main results of [3].

Theorem 3.20. [3, Theorem E] *Let $0 \leqslant i \leqslant q^d - 2$ with $i \not\equiv 0 \pmod{q-1}$. Then*

$$L_{\mathfrak{p}}(1, -1, \omega_{\mathfrak{p}}^{i}) \neq 0 \ .$$

Remark 3.21. It would be interesting to investigate further the values of $L_{\mathfrak{p}}(1, y, \omega_{\mathfrak{p}}^{i})$ for odd i. From equation (30) one immediately has that

$$Z(1, i) \not\equiv 0 \pmod{\mathfrak{p}} \Longrightarrow L_{\mathfrak{p}}(1, y, \omega_{\mathfrak{p}}^{i}) \neq 0 \ \ \forall\, y \in \mathbb{Z}_p \ .$$

In general: is it true that for any $0 \leqslant i \leqslant q^d - 2$ with $i \not\equiv 0 \pmod{q-1}$ and for any $y \in \mathbb{Z}_p$, we have $L_{\mathfrak{p}}(1, y, \omega_{\mathfrak{p}}^{i}) \neq 0$?

We end this section by providing another formula for $L_{\mathfrak{p}}(X, y, \omega_{\mathfrak{p}}^{i})$. The Sinnott map s of Theorem 2.2 induces a map

$$s_X \colon \Lambda[[X]] \to Dir(\mathbb{Z}_p, A_{\mathfrak{p}})[[X]]$$

in the obvious way, sending $\sum_n c_n X^n \in \Lambda[[X]]$ into the function $y \mapsto \sum_n s(\bar{c}_n)(y) X^n$ (where $\bar{c}_n$ is the reduction of c_n modulo p).

Theorem 3.22. *For every $y \in \mathbb{Z}_p$ and $i \in \mathbb{Z}/(q^d - 1)\mathbb{Z}$, we have*

$$s_X(\Theta_\infty(X, \widetilde{\omega}_{\mathfrak{p}}^{-i}))(y) = L_{\mathfrak{p}}(X, -y, \omega_{\mathfrak{p}}^{i}) \ . \tag{32}$$

Proof. This is just an exercise in changing notations. For $y \in \mathbb{Z}_p$, let $\kappa^y \colon \Gamma \to \mathbb{C}_{\mathfrak{p}}^{*}$ be the character $\gamma \mapsto \kappa(\gamma)^y$. Any such character can be extended, by linearity and continuity, to a ring homomorphism $\kappa^y \colon \mathbb{F}_{\mathfrak{p}}[[\Gamma]] \to \mathbb{C}_{\mathfrak{p}}$, which is uniquely characterized by the following property: if μ_λ denotes the measure on Γ attached to $\lambda \in \mathbb{F}_{\mathfrak{p}}[[\Gamma]]$, then we have

$$\kappa^y(\lambda) = \int_\Gamma \kappa^y(\gamma) d\mu_\lambda = s(\lambda)(y) \tag{33}$$

(the last equality is the definition of s, as should be clear from the proof of Theorem 2.2). Let $\widetilde{\alpha}_i \colon \mathbb{Z}[[G_S]][[X]] \to \Lambda[[X]]$ be the homomorphism induced by composition of $G_S \twoheadrightarrow G_\infty$ with the ring homomorphism $\widetilde{\omega}_{\mathfrak{p}}^{i} \colon W[[G_\infty]] \to \Lambda$ (as explained in Section 3.1.2). Definition 3.5 then becomes $\Theta_\infty(X, \widetilde{\omega}_{\mathfrak{p}}^{i}) = \widetilde{\alpha}_i(\Theta_{\mathcal{F}_S/F,S})(X)$. Moreover, letting α_i denote the reduction of $\widetilde{\alpha}_i$ modulo p, (25) yields the equality $\xi_{(y,i)} = \kappa^y \circ \alpha_i$. For proving (32), one just has to check

$$L_{\mathfrak{p}}(X, -y, \omega_{\mathfrak{p}}^{i}) = \xi_{(y,-i)}(\Theta_{\mathcal{F}_S/F,S})(X) = \kappa^y\big(\alpha_{-i}(\Theta_{\mathcal{F}_S/F,S})\big)(X) = s_X(\Theta_\infty(X, \widetilde{\omega}_{\mathfrak{p}}^{-i}))(y) \ .$$

The first equality is Theorem 3.16 and the last one is an easy consequence of (33). □

4. Fitting ideals for Iwasawa modules

In this section we consider the algebraic aspect of the theory, i.e., Fitting ideals of Iwasawa modules associated with the $\mathfrak{p}$-cyclotomic extension. Here the Stickelberger element will appear as a generator of Fitting ideals of (χ-parts of) class groups; the final link between the algebraic and the analytic side will be provided by the Iwasawa Main Conjecture of Section 5.

Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$ and fix a topological generator γ of the Galois group $G_{\mathbb{F}} := \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ (the *arithmetic Frobenius*). For any field L we denote by L^{ar} the composition $\overline{\mathbb{F}}L$ (i.e., the arithmetic extension of L): if L/F is finite, then $\mathrm{Gal}(\overline{\mathbb{F}}L/L) \simeq G_{\mathbb{F}}$. The arithmetic extension F^{ar} is unramified at every prime and disjoint from $\mathcal{F}$ (which is a *geometric* extension), so $\mathrm{Gal}(\mathcal{F}^{ar}/F) \simeq G_\infty \times G_{\mathbb{F}}$.

4.1. Iwasawa modules in the $\mathfrak{p}$-cyclotomic extension.

For any finite extension L/F, we let $\mathcal{C}\ell^0(L)$ be the group of classes of degree zero divisors and we denote by X_L the projective curve (defined over $\mathbb{F}$) associated with L. Let

$$T_p(L) := T_p(Jac(X_L)(\overline{\mathbb{F}}))$$

be the p-Tate module of the $\overline{\mathbb{F}}$-points of the Jacobian of the curve X_L. A first task is to compute the Fitting ideals of the modules

$$\overline{C}_n := \mathcal{C}\ell^0(F_n^{ar})\{p\} \quad , \quad C_n := \mathcal{C}\ell^0(F_n)\{p\} \quad \text{and} \quad T_p(F_n)$$

as Iwasawa modules over some algebra containing $\mathbb{Z}_p[\Gamma_n]$ (the $\{p\}$ indicates the p-part of the module; since we shall mainly work with p-parts, the $\{p\}$ will often be omitted). Recall that $T_p(F_n) \simeq \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n)$ as $\mathbb{Z}_p[\Delta][\Gamma_n]$-modules.

Then we shall perform a limit on n in order to provide a Fitting ideal in the Iwasawa algebra Λ. This will be achieved by means of several maps induced by the natural norms and inclusions (see Section 4.4).

For any prime v of F, we let $F_n^{ar}(v)$ be the set of places of F_n^{ar} lying above v and we put

$$H_{v,n} := \mathbb{Z}_p[F_n^{ar}(v)] \ .$$

If v is unramified in F_n/F, then $H_{v,n}$ is a $\mathbb{Z}_p$-free module of rank $|F_n^{ar}(v)|$ and (which is more relevant) a $\mathbb{Z}_p[G_n]$-free module of rank $d_v := \deg(v)$.

Since we shall work with Fitting ideals we recall one of the equivalent definition of these ideals (the one more suitable for our computations).

Definition 4.1. Let M be a finitely generated module over a ring R. The *Fitting ideal* of M over R, $\mathrm{Fitt}_R(M)$ is the ideal of R generated by the determinants of all the (minors of the) matrices of relations for a fixed set of generators of M.

4.1.1. *Notation.* We remark that the integer $n \geqslant 1$ will always denote objects related with the n-th level F_n of the $\mathfrak{p}$-cyclotomic extension. We shall work at a fixed finite level n at first, and then, in Section 4.4, we let n vary to compute limits.
We recall that χ is a character in $\mathrm{Hom}(\Delta, W^*) =: \widehat{\Delta}$ and we shall denote by χ_0 the trivial character. To work with χ-parts we extend our coefficients to W by considering $W \otimes_{\mathbb{Z}_p} M$ for any module M. We will apply the decomposition (9) to $\varprojlim W \otimes_{\mathbb{Z}_p} C_n$, $\varprojlim W \otimes_{\mathbb{Z}_p} \overline{C}_n$ or $\varprojlim W \otimes_{\mathbb{Z}_p} T_p(F_n)$. To lighten notations we omit the ${}_{\mathbb{Z}_p}$; all tensor products will be defined over $\mathbb{Z}_p$ unless we specify otherwise. For the same purpose whenever we have a map η defined on a module M we shall still denote by η the induced map $\mathrm{id}_W \otimes \eta$ on the module $W \otimes M$.

Finally, for any finite group U, we put

$$n(U) := \sum_{h \in U} h \in \mathbb{Z}[U]$$

(this will mainly appear in the results on the χ_0-part).

4.2. **Fitting ideals for the Tate module (I): finite level.** Let Fr_v denote the Frobenius of v in G_n: it is the unique Frobenius attached to v if v is unramified, or any lift of the Frobenius $\mathrm{Fr}_v \in G_n/I_{v,n}$ to G_n if v is ramified (this construction is easily seen to be independent from the choice of the lift). In particular $\mathrm{Fr}_{\mathfrak{p}} = 1$ (because $\mathfrak{p}$ is totally ramified in F_n/F) and $\mathrm{Fr}_\infty = 1$ as well because ∞ is totally split in $F_n^{I_{\infty,n}}/F$. We define the *Euler factor* at v as

$$e_v(X) := 1 - \mathrm{Fr}_v^{-1} X^{d_v} \ ,$$

where X is a variable which will often be specialized to γ^{-1}, so we also put

$$e_v := e_v(\gamma^{-1}) = 1 - \mathrm{Fr}_v^{-1} \gamma^{-d_v} \ .$$

The next result is exactly [24, Lemmas 2.1 and 2.2] for F_n/F.

Lemma 4.2. *Let v be a place of F, then:*

(1) *if v is unramified, we have* $\mathrm{Fitt}_{\mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]}(H_{v,n}) = (e_v)$*;*
(2) $\mathrm{Fitt}_{\mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]}(H_{\infty,n}) = (e_\infty, Aug_{\infty,n}) = (1-\gamma^{-1}, Aug_{\infty,n})$*;*
(3) $\mathrm{Fitt}_{\mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]}(H_{\mathfrak{p},n}) = (e_{\mathfrak{p}}, Aug_{\mathfrak{p},n}) = (1-\gamma^{-d}, Aug_{\mathfrak{p},n})$,

where $Aug_{v,n}$ is the augmentation ideal associated to $I_{v,n}$, i.e., $Aug_{v,n} := (\tau - 1\,,\ \tau \in I_{v,n}\,)$.

Remark 4.3. If the prime v is unramified, then the module $H_{v,n}$ is cyclic over the ring $\mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]$ and one has

$$H_{v,n} \simeq \mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]/\,\mathrm{Fitt}_{\mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]}(H_{v,n}) = \mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]/(e_v) \ .$$

If we consider ramified primes then the same holds over the ring $\mathbb{Z}_p[G_n/I_{v,n}][[G_{\mathbb{F}}]]$. Hence

$$H_{\infty,n} \simeq \mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]/(e_\infty, Aug_{\infty,n}) = \mathbb{Z}_p[\Delta/\mathbb{F}^* \times \Gamma_n][[G_{\mathbb{F}}]]/(e_\infty)$$

and

$$H_{\mathfrak{p},n} \simeq \mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]/(e_{\mathfrak{p}}, Aug_{\mathfrak{p},n}) = \mathbb{Z}_p[[G_{\mathbb{F}}]]/(e_{\mathfrak{p}}) \ .$$

Let Σ be a finite set of places of F disjoint from S and, for any n, put $\overline{S}_n := F_n^{ar}(S)$ (resp. $\overline{\Sigma}_n := F_n^{ar}(\Sigma)$) for the set of places of F_n^{ar} lying above places in S (resp. Σ). Consider the Deligne's Picard 1-motive $\mathcal{M}_{\overline{S}_n,\overline{\Sigma}_n}$ associated to F_n^{ar}, $\overline{S}_n$ and $\overline{\Sigma}_n$; it is represented by a group homomorphism

$$Div^0(\overline{S}_n) \longrightarrow Jac_{\overline{\Sigma}_n}(X_{F_n})(\overline{\mathbb{F}}) \ ,$$

where $Div^0(\overline{S}_n)$ is the kernel of the degree map $\mathbb{Z}[\overline{S}_n] \to \mathbb{Z}$ and $Jac_{\overline{\Sigma}_n}(X_{F_n})$ is the extension of the Jacobian of X_{F_n} by a torus (for more details on the definition of $\mathcal{M}_{\overline{S}_n,\overline{\Sigma}_n}$ and its properties we refer the reader to [23, Section 2]).

We shall be working with the p-part of class groups, hence (by [23, Remark 2.7]) there is no contribution from the toric part of $Jac_{\overline{\Sigma}_n}(X_{F_n})$. Therefore we can neglect the set Σ (i.e., assume it is empty) in what follows and focus simply on our $S = \{\mathfrak{p}, \infty\}$. The multiplication by p map

$$Div^0(\overline{S}_n) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m \longrightarrow Div^0(\overline{S}_n) \otimes_{\mathbb{Z}} \mathbb{Z}/p^{m-1}$$

induces a surjective map on the p^m-torsion of $\mathcal{M}_{\overline{S}_n} := \mathcal{M}_{\overline{S}_n,\emptyset}$. Thus one defines the *$p$-adic Tate module* of $\mathcal{M}_{\overline{S}_n}$ as

$$T_p(\mathcal{M}_{\overline{S}_n}) := \varprojlim_m \mathcal{M}_{\overline{S}_n}[p^m]$$

(see [23, Definitions 2.5 and 2.6]). With this notations, in our setting, the main result of [23] reads as

Theorem 4.4. (Greither-Popescu [23, Theorem 4.3]) *One has*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]}(T_p(\mathcal{M}_{\overline{S}_n})) = (\Theta_{F_n/F,S}(\gamma^{-1})) := (\Theta_n(\gamma^{-1})) \tag{34}$$

where $\Theta_n(X)$ *is the Stickelberger element*

$$\Theta_n(X) = \prod_{v\notin S}(1-\mathrm{Fr}_v^{-1}\,X^{d_v})^{-1} \in \mathbb{Z}[G_n][[X]] \tag{35}$$

(see Definition 3.1, with $d_v := \deg(v)$*).*

The relation between $\mathcal{M}_{\overline{S}_n}$ and degree zero divisors with support in $\overline{S}_n$ leads to an exact sequence

$$0 \to T_p(F_n) \to T_p(\mathcal{M}_{\overline{S}_n}) \to Div^0(\overline{S}_n)\otimes_{\mathbb{Z}}\mathbb{Z}_p \to 0\ , \tag{36}$$

(see [23, after Definition 2.6]).
Our first task is to compute the Fitting ideal (over $\mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]$) of $T_p(F_n)$ and then project into $\mathbb{Z}_p[G_n]$ by specializing at $\gamma^{-1}=1$. To do this we have to study the χ-parts of the module $D_n := Div^0(\overline{S}_n)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ using the fact that it is contained in $\mathbb{Z}_p[\overline{S}_n] = \oplus_{v\in S}H_{v,n}$. In most cases we will be able to compute Fitting ideals using short exact sequences, while for the "trivial" component we will have to look for a resolution

$$0 \to D_n(\chi_0) \to X_3 \to X_4 \to 0$$

that will fit in the sequence (36) transforming it in a 4-term exact sequence to which we can apply [24, Lemma 2.4].

4.2.1. *The χ-parts of D_n.* As seen in Remark 4.3, we have

$$H_{\infty,n} \simeq \mathbb{Z}_p[G_n/I_{\infty,n}][[G_{\mathbb{F}}]]/(1-\gamma^{-1}) \simeq \mathbb{Z}_p[\Delta/\mathbb{F}^*\times\Gamma_n]\ .$$

Therefore the χ-parts depend on the values of χ on the elements of $\mathbb{F}^* = I_{\infty,n}$ and we have

$$(W\otimes H_{\infty,n})(\chi) \simeq \begin{cases} 0 & \text{if } \chi \text{ is odd} \\ W[\Gamma_n] & \text{if } \chi \text{ is even} \end{cases}\ . \tag{37}$$

Since there is no action of Δ on $H_{\mathfrak{p},n} \simeq \mathbb{Z}_p[[G_{\mathbb{F}}]]/(e_{\mathfrak{p}})$, we have

$$(W\otimes H_{\mathfrak{p},n})(\chi) \simeq \begin{cases} 0 & \text{if } \chi\neq\chi_0 \\ W[[G_{\mathbb{F}}]]/(e_{\mathfrak{p}}) & \text{if } \chi=\chi_0 \end{cases}\ . \tag{38}$$

For any $\chi\neq\chi_0$ we can also observe that

$$(W\otimes D_n)(\chi) = \ker\{e_\chi(W\otimes(H_{\mathfrak{p},n}\oplus H_{\infty,n})) \longrightarrow e_\chi(W\otimes\mathbb{Z}_p)=0\} \simeq (W\otimes H_{\infty,n})(\chi)\ ,$$

so we are left with the trivial component $(W\otimes D_n)(\chi_0)$.

The degree map on $H_{\mathfrak{p},n}$ provides a decomposition

$$H_{\mathfrak{p},n} \simeq (1-\gamma^{-1})H_{\mathfrak{p},n}\oplus\mathbb{Z}_p\ ,$$

where $(1-\gamma^{-1})H_{\mathfrak{p},n}$ is obviously in the kernel of the degree map on D_n as well. For the trivial component we have $e_{\chi_0}(W\otimes H_{\mathfrak{p},n}) = e_{\chi_0}(W\otimes(1-\gamma^{-1})H_{\mathfrak{p},n})\oplus W\cdot 1_{H_{\mathfrak{p}}}$ (where $1_{H_{\mathfrak{p}}}$ is the unit element of $H_{\mathfrak{p},n}$), and the map

$$e_{\chi_0}(W\otimes(1-\gamma^{-1})H_{\mathfrak{p},n})\oplus e_{\chi_0}(W\otimes H_{\infty,n}) \longrightarrow e_{\chi_0}(W\otimes D_n)$$

given by

$$(\alpha,\beta) \to (\alpha-\deg(\beta)1_{H_{\mathfrak{p}}},\beta) \in e_{\chi_0}(W\otimes(H_{\mathfrak{p},n}\oplus H_{\infty,n}))$$

is easily seen to be an isomorphism of $W[\Gamma_n][[G_{\mathbb{F}}]]$-modules. Hence we obtain

$$(W\otimes D_n)(\chi)\simeq\begin{cases} 0 & \text{if } \chi \text{ is odd}\\ W[\Gamma_n] & \text{if } \chi\neq\chi_0 \text{ is even}\\ (1-\gamma^{-1})W[[G_{\mathbb{F}}]]/(1-\gamma^{-d})\oplus W[\Gamma_n] & \text{if } \chi=\chi_0\end{cases}\ . \tag{39}$$

4.2.2. *Computation of Fitting ideals (I).* The previous descriptions of the $(W\otimes D_n)(\chi)$ allow the first computation of Fitting ideals for the Tate modules.

Proposition 4.5. *We have*

$$\mathrm{Fitt}_{W[\Gamma_n][[G_{\mathbb{F}}]]}((W\otimes T_p(F_n))(\chi))=\begin{cases}(\Theta_n(\gamma^{-1},\chi)) & \text{if } \chi \text{ is odd}\\ \left(\frac{\Theta_n(\gamma^{-1},\chi)}{1-\gamma^{-1}}\right) & \text{if } \chi\neq\chi_0 \text{ is even}\end{cases}$$

and

$$\mathrm{Fitt}_{W[\Gamma_n][[G_{\mathbb{F}}]]}((W\otimes T_p(F_n))(\chi_0)^*)=\frac{\Theta_n(\gamma^{-1},\chi_0)}{1-\gamma^{-1}}\left(1,\frac{n(\Gamma_n)}{\nu_d}\right)$$

(where * denotes the $\mathbb{Z}_p$-dual and $\nu_d:=\frac{1-\gamma^{-d}}{1-\gamma^{-1}}$).

Proof. We split the proof in three parts, depending on the type of the character $\chi\in\widehat{\Delta}$.
Case 1: χ is odd. Since $e_\chi D_n=0$, we have an isomorphism

$$(W\otimes T_p(F_n))(\chi)\simeq(W\otimes T_p(M_{\overline{S}_n}))(\chi)\ .$$

Hence, by Theorem 4.4 above,

$$\mathrm{Fitt}_{W[\Gamma_n][[G_{\mathbb{F}}]]}((W\otimes T_p(F_n))(\chi))=(\Theta_n(\gamma^{-1},\chi))$$

(because, by equation (16), $e_\chi\Theta_n(X)=\Theta_n(X,\chi)e_\chi$).
Case 2: $\chi\neq\chi_0$ is even. In this case $(W\otimes D_n)(\chi)\simeq W[\Gamma_n][[G_{\mathbb{F}}]]/(1-\gamma^{-1})$ is a cyclic $W[\Gamma_n][[G_{\mathbb{F}}]]$-module. We have an exact sequence

$$(W\otimes T_p(F_n))(\chi)\hookrightarrow(W\otimes T_p(M_{\overline{S}_n}))(\chi)\twoheadrightarrow W[\Gamma_n][[G_{\mathbb{F}}]]/(1-\gamma^{-1})$$

to which we can apply [15, Lemma 3] to get

$$\mathrm{Fitt}_{W[\Gamma_n][[G_{\mathbb{F}}]]}((W\otimes T_p(F_n))(\chi))(1-\gamma^{-1})=(\Theta_n(\gamma^{-1},\chi))\ .$$

Case 3: $\chi=\chi_0$. Consider the resolution for $D_n(\chi_0)$ provided by the sequences

$$\mathbb{Z}_p[\Gamma_n]\hookrightarrow\mathbb{Z}_p[\Gamma_n][[G_{\mathbb{F}}]]/(e_\infty)\twoheadrightarrow\mathbb{Z}_p[\Gamma_n][[G_{\mathbb{F}}]]/(e_\infty,n(I_{\infty,n})) \tag{40}$$

and

$$(1-\gamma^{-1})\mathbb{Z}_p[[G_{\mathbb{F}}]]/(e_{\mathfrak{p}})\hookrightarrow(1-\gamma^{-1})\mathbb{Z}_p[\Gamma_n][[G_{\mathbb{F}}]]/(e_{\mathfrak{p}})\twoheadrightarrow(1-\gamma^{-1})\mathbb{Z}_p[\Gamma_n][[G_{\mathbb{F}}]]/(e_{\mathfrak{p}},n(I_{\mathfrak{p},n})) \tag{41}$$

where the map on the left is given by $1_{H_v}\to n(I_{v,n})$ ($v=\infty,\ \mathfrak{p}$). To check exactness one simply observes that all the modules involved are $\mathbb{Z}_p$-free modules and counts ranks. Joining the sequences (40) and (41) with the sequence (36) and tensoring with W (limiting ourselves to the χ_0-part), we find

$$\begin{array}{c}(W\otimes T_p(F_n))(\chi_0)\hookrightarrow(W\otimes T_p(M_{\overline{S}_n}))(\chi_0)\to W[\Gamma_n][[G_{\mathbb{F}}]]/(e_\infty)\oplus W[\Gamma_n][[G_{\mathbb{F}}]]/(\nu_d)\\ \big\downarrow\\ W[\Gamma_n][[G_{\mathbb{F}}]]/(e_\infty,n(I_{\infty,n}))\oplus W[\Gamma_n][[G_{\mathbb{F}}]]/(\nu_d,n(I_{\mathfrak{p},n}))\ .\end{array}$$

We note that the assumptions of [24, Lemma 2.4] hold for the previous sequence (actually they hold before tensoring with W but the computation of Fitting ideals is not affected by that, moreover we use the full ring $R=\mathbb{Z}_p[\Gamma_n][[G_{\mathbb{F}}]]$ instead of the R' of the original paper but the lemma still holds as the authors mention right before stating it). Indeed $T_p(M_{\overline{S}_n})(\chi_0)$ is finitely generated and free over $\mathbb{Z}_p$ (so it has no nontrivial finite submodules) and it is Γ_n-cohomologically trivial by the proof of [23, Theorem 3.9]: hence it is of projective dimension 1 over $\mathbb{Z}_p[\Gamma_n][[G_{\mathbb{F}}]]$ by [30, Proposition 2.2 and Lemma 2.3]. The other 3 modules are finitely generated and free over $\mathbb{Z}_p$ and, obviously, $\mathbb{Z}_p[\Gamma_n][[G_{\mathbb{F}}]]/(e_\infty) \oplus \mathbb{Z}_p[\Gamma_n][[G_{\mathbb{F}}]]/(\nu_d)$ has projective dimension at most 1. Therefore we can apply [24, Lemma 2.4] which immediately yields the final statement of the proposition. ☐

4.3. **Fitting ideals for class groups.** There are deep relations between $T_p(F_n)$ and the modules $C_n := \mathcal{C}\ell^0(F_n)\{p\}$ (the ones we are primarily interested in). Indeed, as noted at the beginning of [24, Section 3], the $\mathbb{Z}_p$-dual $T(F_n)^*$ of $T(F_n)$ verifies

$$(T_p(F_n)^*)_{G_{\mathbb{F}}} \simeq \mathcal{C}\ell^0(F_n)\{p\}^\vee = C_n^\vee \ , \tag{42}$$

i.e., its $G_{\mathbb{F}}$-coinvariants are isomorphic to the Pontrjagin dual of C_n. Another one is provided by the following

Lemma 4.6. *We have an isomorphism of $\mathbb{Z}_p[G_n]$-modules*

$$C_n \simeq T_p(F_n)/(1-\gamma^{-1})T_p(F_n) = T_p(F_n)_{G_{\mathbb{F}}} \ .$$

Proof. We recall that $\overline{C}_n$ is the p-Sylow of $\mathcal{C}\ell^0(F_n^{ar})$, hence it is divisible and isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$ for some $r \leqslant g_n$ (where g_n is the genus of X_{F_n}). Obviously $\overline{C}_n$ is a $\mathbb{Z}_p[G_n][[G_{\mathbb{F}}]]$-module and we have

$$T_p(F_n) = \varprojlim_m \overline{C}_n\,[p^m] \simeq \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n)$$

(the Galois action on the module on the right is the usual one $(\sigma \cdot f)(y) := \sigma f(\sigma^{-1}y)$ for any $f \in \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n)$).
By Lang's theorem (see, for example, [33, Chapter VI, §4]) we have an exact sequence

$$0 \to C_n \to \overline{C}_n \xrightarrow{1-\gamma^{-1}} \overline{C}_n \to 0 \ .$$

Applying the functor $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, *)$ (a similar argument can be found in [2, Lemma 4.1]) one gets

$$0 \to T_p(F_n) \xrightarrow{1-\gamma^{-1}} T_p(F_n) \to \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \to 0$$

because $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C_n) = 0$ (C_n is finite) and $\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n) = 0$ ($\overline{C}_n$ is divisible). Now from the usual short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0 \ ,$$

applying $\mathrm{Hom}(*, C_n)$, we obtain

$$\mathrm{Hom}(\mathbb{Q}_p, C_n) = 0 \to \mathrm{Hom}(\mathbb{Z}_p, C_n) \simeq C_n \to \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \to \mathrm{Ext}^1(\mathbb{Q}_p, C_n) = 0 \ .$$

Therefore we have an isomorphism

$$C_n \simeq \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \simeq T_p(F_n)/(1-\gamma^{-1})T_p(F_n) = T_p(F_n)_{G_{\mathbb{F}}} \ .$$ ☐

Remark 4.7. Equation (42) and Lemma 4.6 together yield

$$(T_p(F_n)^*)_{G_{\mathbb{F}}} \simeq C_n^\vee \simeq (T_p(F_n)_{G_{\mathbb{F}}})^\vee \ .$$

A general statement of this type appears in [23, Lemma 5.18].

Definition 4.8. Let $\chi \in \mathrm{Hom}(\Delta, W^*)$ with $\chi \neq \chi_0$ and $n \in \mathbb{N} \cup \{\infty\}$, we define the *modified Stickelberger series* as

$$\Theta_n^\#(X,\chi) := \begin{cases} \Theta_n(X,\chi) & \text{if } \chi \text{ is odd} \\[2ex] \dfrac{\Theta_n(X,\chi)}{1-X} & \text{if } \chi \text{ is even} \end{cases} \ .$$

Consider the projection map $\pi_{G_{\mathbb{F}}}\colon W[\Gamma_n][[G_{\mathbb{F}}]] \to W[\Gamma_n]$ which maps γ to 1. The properties of Fitting ideals and Lemma 4.6 yield

Corollary 4.9. *For $\chi \neq \chi_0$ we have*

$$\mathrm{Fitt}_{W[\Gamma_n]}((W \otimes C_n)(\chi)) = (\Theta_n^{\#}(1,\chi)) \ .$$

While the isomorphism (42) leads to

Corollary 4.10. *For the trivial character χ_0 we have*

$$\mathrm{Fitt}_{W[\Gamma_n]}((W \otimes C_n^{\vee})(\chi_0) \otimes_W Q(W)) = \frac{\Theta_n(X,\chi_0)}{1-X}|_{X=1}\left(1, \frac{n(\Gamma_n)}{d}\right)$$

(where $Q(W)$ is the quotient field of W).

4.4. **Fitting ideals for Tate modules (II): infinite level.** Consider the Iwasawa tower $\mathcal{F}/F$ and let $\varphi_n^{n+1}\colon X_{F_{n+1}} \to X_{F_n}$ be the morphism of curves corresponding to the field extension F_{n+1}/F_n; it is a $\Gamma_n^{n+1} := \mathrm{Gal}(F_{n+1}/F_n)$ Galois cover totally ramified at $\mathfrak{p}$. As before χ denotes an element of $\mathrm{Hom}(\Delta, W^*)$.
We have a morphism $i_{n+1}^n\colon T_p(F_n) \hookrightarrow T_p(F_{n+1})$ (induced by the natural map from $\overline{C}_n$ to $\overline{C}_{n+1}$) and a map $N_n^{n+1}\colon T_p(F_{n+1}) \to T_p(F_n)$ induced by the norm map from $\overline{C}_{n+1}$ to $\overline{C}_n$. Observe that, for any n, $N_n^{n+1} \circ i_{n+1}^n = q^d$.
In this section we shall meet various other maps induced by norms (resp. inclusions) on different modules/objects: by abuse of notations we shall denote all of them by N_n^{n+1} (resp. i_{n+1}^n), when we need some distinction between them we shall write $N(\bullet)_n^{n+1}$ (resp. $i(\bullet)_{n+1}^n$) to denote the map defined on the objects $\bullet$ or $T_p(\bullet)$.

4.4.1. *Norm and inclusion maps.* We have an inclusion $i_{n+1}^n\colon T_p(\mathcal{M}_{\overline{S}_n}) \hookrightarrow T_p(\mathcal{M}_{\overline{S}_{n+1}})$ such that $T_p(\mathcal{M}_{\overline{S}_{n+1}})^{\Gamma_n^{n+1}} = i_{n+1}^n(T_p(\mathcal{M}_{\overline{S}_n}))$ by [23, Theorem 3.1]. We also have a natural norm map $N(\mathcal{M})_n^{n+1}\colon T_p(\mathcal{M}_{\overline{S}_{n+1}}) \to T_p(\mathcal{M}_{\overline{S}_n})$.

Lemma 4.11. *The norm map $N(\mathcal{M})_n^{n+1}$ is surjective and its kernel is $I_{\Gamma_n^{n+1}}T_p(\mathcal{M}_{\overline{S}_{n+1}})$ where $I_{\Gamma_n^{n+1}}$ is the augmentation ideal (i.e., generated by $\{\sigma - 1 \,:\, \sigma \in \Gamma_n^{n+1}\}$).*

Proof. By [23, Theorem 3.9] (in particular, its proof) we have that $T_p(\mathcal{M}_{\overline{S}_{n+1}})$ is Γ_n^{n+1}-cohomologically trivial, i.e.,

$$\widehat{H}^i(\Gamma_n^{n+1}, T_p(\mathcal{M}_{\overline{S}_{n+1}})) = 0 \quad \forall\, i \ .$$

For $i = 0$ we have that

$$T_p(\mathcal{M}_{\overline{S}_{n+1}})^{\Gamma_n^{n+1}} = N(\mathcal{M})_n^{n+1}(T_p(\mathcal{M}_{\overline{S}_{n+1}})) \ ,$$

but, as recalled above, $T_p(\mathcal{M}_{\overline{S}_{n+1}})^{\Gamma_n^{n+1}} = T_p(\mathcal{M}_{\overline{S}_n})$, therefore the norm map is surjective.
With $i = -1$ we obtain that the kernel of $N(\mathcal{M})_n^{n+1}$ is given by the augmentation module $I_{\Gamma_n^{n+1}}T_p(\mathcal{M}_{\overline{S}_{n+1}})$. $\square$

We have a commutative diagram of short exact sequences

(43)
$$\begin{tikzcd}
0 \arrow[r] & T_p(F_{n+1}) \arrow[r] \arrow[d, "N(F)_n^{n+1}"] & T_p(\mathcal{M}_{\overline{S}_{n+1}}) \arrow[r] \arrow[d, "N(\mathcal{M})_n^{n+1}"] & D_{n+1} \arrow[r] \arrow[d, "N(D)_n^{n+1}"] & 0 \\
0 \arrow[r] & T_p(F_n) \arrow[r] & T_p(\mathcal{M}_{\overline{S}_n}) \arrow[r] & D_n \arrow[r] & 0
\end{tikzcd}$$

where all vertical maps are induced by norms: in particular note that $N(D)_n^{n+1}$ corresponds to the natural map on divisors

$$\mathbb{Z}_p[F_{n+1}^{ar}(\infty)] \oplus \mathbb{Z}_p[F_{n+1}^{ar}(\mathfrak{p})] \to \mathbb{Z}_p[F_n^{ar}(\infty)] \oplus \mathbb{Z}_p[F_n^{ar}(\mathfrak{p})] \ .$$

Lemma 4.12. *Let $\chi \neq \chi_0$ and $n \geqslant 1$. Then*

$$\ker(N(D)_n^{n+1})(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is odd} \\ I_{\Gamma_n^{n+1}}(W \otimes D_{n+1})(\chi) & \text{if } \chi \text{ is even} \end{cases} .$$

Moreover, the map $N(D)_n^{n+1}$ is surjective.

Proof. The last assertion immediately follows from the surjectivity of $N(\mathcal{M})_n^{n+1}$ and the snake lemma sequence of diagram (43). Now consider the same diagram but with χ-parts and tensored with W and note that the maps $N(\mathcal{M})_n^{n+1}$ and $N(D)_n^{n+1}$ remain surjective.
From the computations in Section 4.2.1 the case χ odd is obvious. When $\chi \neq \chi_0$ is even we have $e_\chi(W \otimes D_{n+1}) \simeq W[\Gamma_{n+1}]$ and the norm corresponds to the projection $W[\Gamma_{n+1}] \to W[\Gamma_n]$ which has kernel $I_{\Gamma_n^{n+1}}$. ☐

The previous two lemmas lead to similar statements for the map $N(F)_n^{n+1}(\chi)$.

Proposition 4.13. *Let $\chi \neq \chi_0$, then $(W \otimes T_p(F_n))(\chi)$ is Γ_n and Γ_n^{n+1}-cohomologically trivial and a free $W[\Gamma_n]$-module. In particular*

$$(W \otimes T_p(F_{n+1}))(\chi)^{\Gamma_n^{n+1}} = (W \otimes T_p(F_n))(\chi) \ ,$$

$N(F)_n^{n+1}(\chi)$ is surjective and $\ker(N(F)_n^{n+1})(\chi) = I_{\Gamma_n^{n+1}}(W \otimes T_p(F_{n+1}))(\chi)$. Moreover it is also a $W[\Gamma_n][[G_{\mathbb{F}}]]$-module of projective dimension less than or equal to one.

Proof. Consider the short exact sequence

$$0 \to W \otimes T_p(F_n) \to W \otimes T_p(\mathcal{M}_{\overline{S}_n}) \to W \otimes D_n \to 0 \ .$$

Since $D_n(\chi)$ is 0 or $W[\Gamma_n]$, it is $W[\Gamma_n]$-free and cohomologically trivial, while $W \otimes T_p(\mathcal{M}_{\overline{S}_n})$ is also $W[\Gamma_n]$-free and cohomologically trivial by [23, Theorem 3.9]. Thus $(W \otimes T_p(F_n))(\chi)$ is projective over $W[\Gamma_n]$ and cohomologically trivial. Now, since Γ_n is a p-group, $W[\Gamma_n]$ is a local ring and projective modules coincide with free modules. The cohomological triviality over Γ_n^{n+1} is similar and straightforward.
The assertion on $\ker(N(F)_n^{n+1})(\chi)$ comes from the triviality of the $\widehat{H}^1$. Now take Γ_n^{n+1}-invariants in the sequence for even characters (for odd ones there is nothing to prove)

$$(W \otimes T_p(F_{n+1}))(\chi) \hookrightarrow (W \otimes T_p(\mathcal{M}_{\overline{S}_{n+1}}))(\chi) \twoheadrightarrow (W \otimes D_{n+1})(\chi) \simeq W[\Gamma_{n+1}] \ ,$$

to get

$$(W \otimes T_p(F_{n+1}))(\chi)^{\Gamma_n^{n+1}} \hookrightarrow (W \otimes T_p(\mathcal{M}_{\overline{S}_n}))(\chi) \twoheadrightarrow (W \otimes D_n)(\chi) \simeq W[\Gamma_n]$$

(using Lemmas 4.11 and 4.12). Thus $(W \otimes T_p(F_{n+1}))(\chi)^{\Gamma_n^{n+1}} = (W \otimes T_p(F_n))(\chi)$ and, since the $\widehat{H}^0$ is trivial, $N(F)_n^{n+1}(\chi)$ is surjective.
The last statement of the lemma follows from [30, Proposition 2.2 and Lemma 2.3] (see also [17, Proposition 5.3]) because $(W \otimes T_p(F_n))(\chi)$, being free, has no nontrivial finite $W[\Gamma_n]$-submodule. ☐

Remark 4.14. For $\chi = \chi_0$ the modules $(W \otimes T(\mathcal{M}_{\overline{S}_n}))(\chi_0)$ and $W[\Gamma_n]$ are still Γ_n-cohomologically trivial and we have the short exact sequence

$$(W \otimes T_p(F_n))(\chi_0) \hookrightarrow (W \otimes T(\mathcal{M}_{\overline{S}_n}))(\chi_0) \twoheadrightarrow \frac{(1-\gamma^{-1})W[[G_{\mathbb{F}}]]}{(1-\gamma^{-d})} \oplus W[\Gamma_n] \ .$$

Since F_n/F is disjoint from $\overline{\mathbb{F}}/\mathbb{F}$, the norm acts on $G_{\mathbb{F}}$ as multiplication by $[F_n : F]$. For any subextension E/K and for any $k \in \mathbb{Z}$, we have

$$\widehat{H}^k\left(\mathrm{Gal}(E/K), \frac{(1-\gamma^{-1})W[[G_{\mathbb{F}}]]}{(1-\gamma^{-d})}\right) \simeq \widehat{H}^{k+1}(\mathrm{Gal}(E/K), (W \otimes T_p(F_n))(\chi_0))$$

and, in particular,

$$\widehat{H}^0(\mathrm{Gal}(E/K),(W\otimes T_p(F_n))(\chi_0))\simeq\frac{(1-\gamma^{-1})W[[G_{\mathbb{F}}]]}{(1-\gamma^{-d})}/|\,\mathrm{Gal}(E/K)|\frac{(1-\gamma^{-1})W[[G_{\mathbb{F}}]]}{(1-\gamma^{-d})}\ .$$

Therefore $(W\otimes T_p(F_n))(\chi_0)$ is not Γ_n-cohomologically trivial (and not not necessarily $W[\Gamma_n]$-free).

4.4.2. *Computation of Fitting ideals (II).* We define

$$T_p(\mathcal{F})(\chi):=\varprojlim_n (W\otimes T_p(F_n))(\chi)$$

(the limit is on the norm maps studied above). We recall that $T_p(F_n)=T_p(Jac(X_{F_n})(\overline{\mathbb{F}}))$, so $T_p(\mathcal{F})(\chi)$ is a $\Lambda[[G_{\mathbb{F}}]]$-module, where $\Lambda=W[[\Gamma]]$.

Proposition 4.15. *For $\chi\neq\chi_0$, $T_p(\mathcal{F})(\chi)$ is a finitely generated torsion $\Lambda[[G_{\mathbb{F}}]]$-module.*

Proof. The ideals $\mathfrak{I}_n$ (defined in Section 2.2) form an open filtration for Λ and we note that $\mathfrak{I}_n=\varprojlim_m I_{\Gamma_n^{m+n}}$ (where $I_{\Gamma_n^{m+n}}$ denotes the augmentation ideal associated to $\Gamma_n^{m+n}=\mathrm{Gal}(F_{n+m}/F_n)$). From Proposition 4.13 we have that

$$T_p(\mathcal{F})(\chi)/(\Lambda[[G_{\mathbb{F}}]]\otimes_\Lambda\mathfrak{I}_n)T_p(\mathcal{F})(\chi)\simeq(W\otimes T_p(F_n))(\chi)$$

for all $n\geqslant 1$. The module on the right is finitely generated over

$$W[\Gamma_n][[G_{\mathbb{F}}]]=\Lambda[[G_{\mathbb{F}}]]/(\Lambda[[G_{\mathbb{F}}]]\otimes_\Lambda\mathfrak{I}_n)\ ,$$

so, by a generalized Nakayama Lemma (see [4, Corollary p. 226]), we have that $T_p(\mathcal{F})(\chi)$ is finitely generated as a $\Lambda[[G_{\mathbb{F}}]]$-module. Moreover $\Theta_n^{\#}(\gamma^{-1},\chi)(W\otimes T_p(F_n))(\chi)=0$ for any n (by Proposition 4.5), hence

$$\Theta_\infty^{\#}(\gamma^{-1},\chi)T_p(\mathcal{F})(\chi)=0\ ,$$

i.e., the module $T_p(\mathcal{F})(\chi)$ is torsion. □

Therefore the Fitting ideal of the $\Lambda[[G_{\mathbb{F}}]]$-module $T_p(\mathcal{F})(\chi)$ is well defined and we have the following formula for it (for a similar result see [22, Theorem 2.1], but note the particular case of [22, Remark 2.2 (2)] which fits our setting).

Theorem 4.16. *For $\chi\neq\chi_0$ we have*

$$\mathrm{Fitt}_{\Lambda[[G_{\mathbb{F}}]]}(T_p(\mathcal{F})(\chi))=(\Theta_\infty^{\#}(\gamma^{-1},\chi))\ .$$

Proof. By the previous proposition we can find an $r\in\mathbb{N}$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & B_{n+1} \arrow[r] \arrow[d, "b_n^{n+1}"] & W[\Gamma_{n+1}][[G_{\mathbb{F}}]]^r \arrow[r] \arrow[d, "\pi_n^{n+1}"] & (W\otimes T_p(F_{n+1}))(\chi) \arrow[r] \arrow[d, "N(F)_n^{n+1}(\chi)"] & 0 \\ 0 \arrow[r] & B_n \arrow[r] & W[\Gamma_n][[G_{\mathbb{F}}]]^r \arrow[r] & (W\otimes T_p(F_n))(\chi) \arrow[r] & 0 \end{tikzcd} \tag{44}$$

(where the central map is the canonical projection). The maps π_n^{n+1} and $N(F)_n^{n+1}(\chi)$ are surjective and that their kernels are $\left(I_{\Gamma_n^{n+1}}W[\Gamma_{n+1}][[G_{\overline{\mathbb{F}}}]]\right)^r$ and $I_{\Gamma_n^{n+1}}(W\otimes T_p(F_{n+1}))(\chi)$. The map between these two kernels is obviously surjective, hence, by the snake lemma sequence, we have that b_n^{n+1} is surjective as well.
Taking the inverse limit in diagram (44) (which verifies the Mittag-Leffler condition), we obtain the exact sequence

$$0\to B_\infty:=\varprojlim_n B_n\to\Lambda[[G_{\mathbb{F}}]]^r\to\varprojlim_n (W\otimes T_p(F_n))(\chi)=T_p(\mathcal{F})(\chi)\to 0\ . \tag{45}$$

Recall that we can use any $\beta_1,\dots,\beta_r \in B_n$ as rows of a matrix $M_{\beta_1,\dots,\beta_r} \in Mat_r(W[\Gamma_n])$ and $\mathrm{Fitt}_{W[\Gamma_n]}((W\otimes T_p(F_n))(\chi))$ is generated by the $\det(M_{\beta_1,\dots,\beta_r})$. The surjectivity of the maps b_n^{n+1} implies the same property for the induced maps $b_n : B_\infty \twoheadrightarrow B_n$, i.e., the "relations" at the infinite level are all induced by "relations" already existing at lower levels (the technical arguments of the final parts of [22, Theorem 2.1] are not necessary here because of the presence of just one ramified prime and our previous computations on kernels of norm maps). Using the characterization of the Fitting ideal in Definition 4.1, it is easy to see that the surjectivity of the b_n and the sequence (45) yield the desired result, i.e.,

$$\begin{aligned}\mathrm{Fitt}_{\Lambda[[G_{\mathbb{F}}]]}(T_p(\mathcal{F})(\chi)) &= (\det(B_\infty)) = \varprojlim_n (\det(B_n))\\ &= \varprojlim_n \mathrm{Fitt}_{W[\Gamma_n][[G_{\mathbb{F}}]]}((W\otimes T_p(F_n))(\chi))\\ &= \varprojlim_n (\Theta_n^{\#}(\gamma^{-1},\chi)) = (\Theta_\infty^{\#}(\gamma^{-1},\chi))\ . \qquad \square\end{aligned}$$

5. Iwasawa main conjecture for the $\mathfrak{p}$-cyclotomic extension

Consider now $W\otimes C_n = W\otimes Jac(X_{F_n})(\mathbb{F}) = W\otimes Pic^0(X_{F_n})(\mathbb{F})$ as a $W[G_n]$-module and the natural maps

$$i(C)^n_{n+1}\colon W\otimes C_n \to W\otimes C_{n+1} \quad \text{and} \quad N(C)^{n+1}_n\colon W\otimes C_{n+1}\to W\otimes C_n\,.$$

Denote by $\mathcal{C}$ the $W[[G_\infty]]$-module $\varprojlim W\otimes C_n$ (defined, as usual, with respect the norm maps $N(C)^{n+1}_n$).
The main results of this section are the following

Theorem 5.1. *Let $\chi\neq\chi_0$, then the module $\mathcal{C}(\chi) := \varepsilon_\chi\mathcal{C}$ is a finitely generated torsion Λ-module.*

Therefore the Fitting ideal $\mathrm{Fitt}_\Lambda(\mathcal{C}(\chi))$ is well defined and we have

Theorem 5.2 (Iwasawa Main Conjecture)**.** *Let $\chi\neq\chi_0$, then*

$$\mathrm{Fitt}_\Lambda(\mathcal{C}(\chi)) = (\Theta_\infty^{\#}(1,\chi))\ .$$

Remark 5.3. The theorem above allows us to compute the Fitting ideal of $\mathcal{C}(\chi)$ as the inverse limit of the Fitting ideals appearing in the (natural) filtration of $\mathcal{F}$ given by the fields F_n. A different approach to the same problem is provided in [6, Section 5] where the authors use a filtration of $\mathbb{Z}_p^d$-extensions (a more general approach and the fact that the limit is independent from the filtration are shown in [7]). In that paper the statement of the Main Conjecture involves characteristic ideals but (for Iwasawa modules) they coincide with Fitting ideals whenever the Fitting is principal (see, for example, [5, Lemma 5.10]).

Before going into the proofs of the above theorems, we need a crucial lemma.

Lemma 5.4. *Let $F_0\subset K\subset E\subset\mathcal{F}$, where E/F is a finite extension and the group $G := \mathrm{Gal}(E/K)$ is a p-group. For any field $L\subset\mathcal{F}$ we let $\mathfrak{p}_L$ be the unique prime of L lying above $\mathfrak{p}$, we recall that $\mathcal{C}\ell^0(L)$ denotes the group of classes of degree zero divisors. We have the following properties:*

1. *the map $i_E^K\colon \mathcal{C}\ell^0(K)\to\mathcal{C}\ell^0(E)$ is injective;*
2. *there is an equality $\mathcal{C}\ell^0(E)^G = i_E^K(\mathcal{C}\ell^0(K)) + \langle r\frac{|G|}{p^t}\mathfrak{p}_E - \frac{d}{p^t}i_E^K(v)\rangle$, where $p^t := (|G|,d)$, v is a place of K lying above a prime of A of degree r (prime with p) and which is totally split in E. The second term disappears when we consider χ-parts for nontrivial characters, in particular, for $\chi\neq\chi_0$, we have*
$$((W\otimes\mathcal{C}\ell^0(E))(\chi))^G = i_E^K((W\otimes\mathcal{C}\ell^0(K))(\chi))\ ;$$
3. *the norm map $N_K^E\colon \mathcal{C}\ell^0(E)\to\mathcal{C}\ell^0(K)$ is surjective;*

(4) *for* $\chi \neq \chi_0$*, we have* $\ker(N_K^E(\chi)) = I_G(W \otimes \mathcal{C}\ell^0(E))(\chi)$.

Proof. For simplicity we write N and i for N_K^E and i_E^K respectively.
(1) For any field L write P_L for the principal divisors of L. Consider the exact sequences

$$0 \to \mathbb{F}^* \to E^* \to P_E \to 0 \tag{46}$$

and

$$0 \to P_E \to Div^0(E) \to \mathcal{C}\ell^0(E) \to 0\ . \tag{47}$$

Taking G-cohomology in (46) one finds

$$0 \to \mathbb{F}^* \to K^* \to P_E^G \to 0 \to 0 \to H^1(G, P_E) \to 0$$

(because of Hilbert 90 and because G is a p-group so $H^i(G, \mathbb{F}^*) = 0$ for any $i \geqslant 1$), so, in particular, $P_E^G = P_K$ and $H^1(G, P_E) = 0$. The G-invariants of the sequence (47) then fit into the diagram

$$\begin{tikzcd} P_K \arrow[r, hook] \arrow[d, equal] & Div^0(K) \arrow[r, two heads] \arrow[d, "i"] & \mathcal{C}\ell^0(K) \arrow[d, "i_E^K"] & \\ P_E^G = P_K \arrow[r, hook] & Div^0(E)^G \arrow[r] & \mathcal{C}\ell^0(E)^G \arrow[r] & H^1(G, P_E) = 0\ . \end{tikzcd} \tag{48}$$

The injectivity of the central vertical map and the snake lemma sequence yield the desired injectivity of i_E^K.
(2) We need several steps:

(a) *The group* $Div(E)^G/i(Div(K))$ *is cyclic of order* $[E:K]$ *and is generated by the class of* $\mathfrak{p}_E$.
Write $Div(K) = \oplus_v \mathbb{Z}v$ (v runs through all the primes of K) and $Div(E) = \oplus_v H_v$ with $H_v = \oplus_{w|v} \mathbb{Z}w$. If $v \neq \mathfrak{p}_K$ is unramified, then $H_v = \mathbb{Z}[G/G_v]w$, where G_v is the decomposition subgroup of v in G, and obviously

$$H_v^G = \mathbb{Z}i(v) \quad \text{with} \quad i(v) = \sum_{\sigma \in G/G_v} \sigma w\ .$$

For the ramified place we have $\sigma(\mathfrak{p}_E) = \mathfrak{p}_E$ and $|G|\mathfrak{p}_E = i(\mathfrak{p}_K)$: this yields the statement.

(b) *The group* $Div^0(E)^G/i(Div^0(K))$ *is killed by* $p^t := (|G|, d)$.
Take $D \in Div^0(E)^G$, by part (a) we can write $D = n\mathfrak{p}_E + D'$ with $D' \in i(Div(K))$. Since D has degree zero we have $-nd = \deg_E(D') = |G| \deg_K(D')$ and $\frac{|G|}{p^t}$ divides $n = \frac{|G|}{p^t} n'$. Therefore

$$\begin{aligned} p^t D &= p^t n \mathfrak{p}_E + p^t D' = |G| n' \mathfrak{p}_E + p^t D' \\ &= n' \mathfrak{p}_K + p^t D' \in i(Div(K))\ . \end{aligned}$$

Moreover

$$\begin{aligned} \deg_K(n'\mathfrak{p}_K + p^t D') &= n'd + p^t \deg_K(D') = \frac{p^t n}{|G|} d + p^t \deg_K(D') \\ &= \frac{p^t}{|G|}(nd + |G| \deg_K(D')) = 0\ , \end{aligned}$$

hence $p^t D \in i(Div^0(K))$.

(c) *There are isomorphisms* $\mathcal{C}\ell^0(E)^G/i_E^K(\mathcal{C}\ell^0(K)) \simeq Div^0(E)^G/i(Div^0(K)) \simeq \mathbb{Z}/p^t$.
The first isomorpshim is a consequence of the snake lemma sequence of diagram (48). For the second, by part (b) it is enough to build a divisor of exact order p^t. Let r be prime with p; by Chebotarev density theorem there exists a (monic) irreducible

polynomial Q in A such that $\deg(Q) = r$ and the prime (Q) is totally split in E. Take a prime v of K dividing (Q) so that, in particular, $\deg_K(v) = r$. Put

$$\widetilde{D} := r\frac{|G|}{p^t}\mathfrak{p}_E - \frac{d}{p^t}i(v) \ ;$$

it is easy to check that (by construction) $\widetilde{D} \in Div^0(E)^G$ and its order is p^t.

(d) *If $\chi \neq \chi_0$, then $e_\chi(\widetilde{D}) = 0$.*
Recall that $\Delta = \mathrm{Gal}(F_0/F)$ has order prime to p. The previous steps can be proved exactly in the same way for the field extension E^Δ/K^Δ, i.e., we have

$$\frac{Div^0(E^\Delta)^G}{i_{E^\Delta}^{K^\Delta}(Div^0(K^\Delta))} \simeq \frac{\mathcal{C}\ell^0(E^\Delta)^G}{i_{E^\Delta}^{K^\Delta}(\mathcal{C}\ell^0(K^\Delta))} \simeq \mathbb{Z}/p^t$$

and a generator is the class of

$$\widetilde{D'} := r\frac{|G|}{p^t}\mathfrak{p}_{E^\Delta} - \frac{d}{p^t}i_{E^\Delta}^{K^\Delta}(\tilde{v}) \in Div^0(E^\Delta)^G$$

(where $\tilde{v}$ is a prime of K^Δ lying below v). Note that the image of $\widetilde{D'}$ in $Div^0(E)$ is

$$i_E^{E^\Delta}(\widetilde{D'}) = r\frac{|G|}{p^t}|\Delta|\mathfrak{p}_E - \frac{d}{p^t}i_E^{K^\Delta}(\tilde{v})$$

and it still has order p^t because $(|\Delta|, p) = 1$. Therefore the class of $i_E^{E^\Delta}(\widetilde{D'})$ generates $\mathcal{C}\ell^0(E)^G/i(\mathcal{C}\ell^0(K))$ and, by construction, Δ acts trivially on it. Hence for $\chi \neq \chi_0$ we obtain

$$e_\chi(W \otimes \mathcal{C}\ell^0(E))^G = e_\chi i(W \otimes \mathcal{C}\ell^0(K))$$

and for the trivial character we have

$$e_{\chi_0}\left(\frac{(W \otimes \mathcal{C}\ell^0(E))^G}{i(W \otimes \mathcal{C}\ell^0(K))}\right) \simeq W/p^tW \ .$$

(3) This is just class field theory. Let v be a place of K which divides ∞ and write B for the ring of elements in K which are regular outside v; since v is of degree 1, we have $\mathcal{C}\ell(B) \simeq \mathcal{C}\ell^0(K)$. Let $H(K)$ be the maximal abelian unramified extension of K in which v is totally split. By class field theory, the Artin map provides an isomorphism $\mathrm{Gal}(H(K)/K) \simeq \mathcal{C}\ell(B)$ and, because of the ramification in E/K, we have $H(K) \cap E = K$. Denote by C the integral closure of B in E (i.e., the elements in E which are regular outside any $w|v$); there is a natural map $\mathcal{C}\ell^0(E) \to \mathcal{C}\ell(C) \simeq \mathrm{Gal}(H(E)/E)$ which preserves Galois action and is surjective because $\deg_E(w) = 1$ ($H(E)$ is the analog of $H(K)$, now totally split at w). It only remains to prove that the natural norm map $\mathcal{C}\ell(C) \to \mathcal{C}\ell(B)$ is surjective. By construction $EH(K) \subset H(E)$, hence the restriction map

$$Res : \mathrm{Gal}(H(E)/E) \to \mathrm{Gal}(EH(K)/E) \simeq \mathrm{Gal}(H(K)/K)$$

is surjective. The well known diagram of class field theory

$$\begin{tikzcd}
\mathcal{C}\ell(C) \arrow[r, "\simeq"] \arrow[d, "N"] & \mathrm{Gal}(H(E)/E) \arrow[d, "Res"] \\
\mathcal{C}\ell(B) \arrow[r, "\simeq"] & \mathrm{Gal}(H(K)/K)
\end{tikzcd}$$

concludes the proof.

(4) Consider the sequence (exact by part (3))

$$0 \to (W \otimes \ker(N))(\chi) \to (W \otimes \mathcal{C}\ell^0(E))(\chi) \xrightarrow{N} (W \otimes \mathcal{C}\ell^0(K))(\chi) \to 0 \ ,$$

which yields

$$|(W\otimes \ker(N))(\chi)| = \frac{|(W\otimes \mathcal{C}\ell^0(E))(\chi)|}{|(W\otimes \mathcal{C}\ell^0(K))(\chi)|}$$

(we recall that for any W-module M one has $|M| = p^{u\cdot \ell_W(M)}$ with ℓ_W the length and $u := [W\otimes \mathbb{Q}_p : \mathbb{Q}_p]$).
We first assume that $G = \langle\delta\rangle$ is cyclic. Then, for $\chi \neq \chi_0$, the sequence (exact by part (2))

$$0 \to (W\otimes \mathcal{C}\ell^0(K))(\chi) \to (W\otimes \mathcal{C}\ell^0(E))(\chi) \xrightarrow{1-\delta} (1-\delta)(W\otimes \mathcal{C}\ell^0(E))(\chi) \to 0$$

yields $|(1-\delta)(W\otimes\mathcal{C}\ell^0(E))(\chi)| = |(W\otimes\ker(N))(\chi)|$ (simply by counting cardinalities). Since $(1-\delta)(W\otimes \mathcal{C}\ell^0(E))(\chi) \subseteq (W\otimes \ker(N))(\chi)$, we have the equality between them.
For the general case G we use an induction argument on $|G|$. If $|G| = 1$ there is nothing to prove (or, if $|G| = p$, then G is cyclic and we have the proof above). Consider now $K \subset E' \subset E$, where $G_1 := \mathrm{Gal}(E/E')$ is a cyclic group and we put $G_2 := \mathrm{Gal}(E'/K)$. By the inductive step and the cyclic case we have

$$(W\otimes \ker(N_K^{E'}))(\chi) = I_{G_2}(W\otimes \mathcal{C}\ell^0(E'))(\chi)$$

and

$$(W\otimes \ker(N_{E'}^{E}))(\chi) = I_{G_1}(W\otimes \mathcal{C}\ell^0(E))(\chi) \ .$$

By part (3) all norms are surjective and, since $N = N_K^{E'} \circ N_{E'}^{E}$, we have

$$\begin{aligned}(W\otimes \ker(N))(\chi) &= (N_{E'}^{E})^{-1} I_{G_2}(W\otimes \mathcal{C}\ell^0(E'))(\chi)\\ &= I_G(W\otimes \mathcal{C}\ell^0(E))(\chi) + (W\otimes \ker(N_{E'}^{E}))(\chi)\\ &= I_G(W\otimes \mathcal{C}\ell^0(E))(\chi)\ ,\end{aligned}$$

because $(W\otimes \ker(N_{E'}^{E}))(\chi) = I_{G_1}(W\otimes \mathcal{C}\ell^0(E))(\chi) \subset I_G(W\otimes \mathcal{C}\ell^0(E))(\chi)$. □

Proof of Theorem 5.1. Recall that $C_n := \mathcal{C}\ell^0(F_n)\{p\}$: by Corollary 4.9, we know that for $\chi \neq \chi_0$

$$\mathrm{Fitt}_{W[\Gamma_n]}((W\otimes C_n)(\chi)) = (\Theta_n^{\#}(1,\chi)) \ .$$

By the previous lemma the kernel of $N(C)_n^{n+1} : (W\otimes C_{n+1})(\chi) \twoheadrightarrow (W\otimes C_n)(\chi)$ is $I_{\Gamma_n^{n+1}}(W\otimes C_{n+1})(\chi)$ and we know that $\mathfrak{I}_n = \varprojlim_m I_{\Gamma_n^{m+n}}$. Hence

$$\mathcal{C}(\chi)/\mathfrak{I}_n\mathcal{C}(\chi) \simeq (W\otimes C_n)(\chi)$$

as $W[\Gamma_n]$-modules. The generalized version of Nakayama Lemma implies that $\mathcal{C}(\chi)$ is a finitely generated Λ-module because $(W\otimes C_n)(\chi)$ is a finitely generated $W[\Gamma_n]$-module. Now simply recall that $\Theta_n^{\#}(1,\chi)((W\otimes C_n)(\chi)) = 0$ and that $\Theta_\infty^{\#}(1,\chi) = \varprojlim_n \Theta_n^{\#}(1,\chi)$ to get

$$\Theta_\infty^{\#}(1,\chi)\mathcal{C}(\chi) = \varprojlim_n \Theta_n^{\#}(1,\chi)\left(\varprojlim_n (W\otimes C_n)(\chi)\right) = 0\ ,$$

i.e., $\mathcal{C}(\chi)$ is a torsion Λ-module. □

Proof of Theorem 5.2 (IMC). By Theorem 5.1 the Fitting ideal $\mathrm{Fitt}_\Lambda(\mathcal{C}(\chi))$ is well defined. The statement is equivalent to the equality

$$\mathrm{Fitt}_\Lambda\left(\varprojlim_n (W\otimes C_n)(\chi)\right) = \varprojlim_n \left(\mathrm{Fitt}_{W[\Gamma_n]}((W\otimes C_n)(\chi))\right)\ .$$

Take $r \in \mathbb{N}$ such that $e_\chi \mathcal{C}$ is generated by r elements, consider the following commutative diagram

$$\begin{tikzcd} B_{n+1} \arrow[r, hook] \arrow[d, "b_n^{n+1}"] & W[\Gamma_{n+1}]^r \arrow[r, two heads] \arrow[d, "\pi_n^{n+1}"] & (W\otimes C_{n+1})(\chi) \arrow[d, "N(C)_n^{n+1}"] \\ B_n \arrow[r, hook] & W[\Gamma_n]^r \arrow[r, two heads] & (W\otimes C_n)(\chi) \end{tikzcd}$$

(where the central vertical map is the natural projection) and note that the kernels of π_n^{n+1} and $N(C)_n^{n+1}$ are respectively $\left(I_{\Gamma_n^{n+1}} W[\Gamma_{n+1}]\right)^r$ and $I_{\Gamma_n^{n+1}}(W\otimes C_{n+1})(\chi)$ (by Lemma 5.4 part (4)). Therefore the induced map between the kernels is surjective and this, together with the surjectivity of π_n^{n+1}, yields the surjectivity of b_n^{n+1} by the snake lemma.
Now the diagram above verifies the Mittag-Leffler condition, so, taking the limit, we have an exact sequence

$$B_\infty := \varprojlim_n B_n \hookrightarrow \Lambda^r \twoheadrightarrow \mathcal{C}(\chi)\ . \tag{49}$$

Working as in Theorem 4.16, one sees that the surjectivity of the b_n and the sequence (49) yield

$$\begin{aligned} \mathrm{Fitt}_\Lambda(\mathcal{C}(\chi)) &= \varprojlim_n \left(\mathrm{Fitt}_{W[\Gamma_n]}((W\otimes C_n)(\chi))\right) \\ &= \varprojlim_n \left(\Theta_n^{\#}(1,\chi)\right) = (\Theta_\infty^{\#}(1,\chi))\ . \end{aligned}$$ □

We end this section with a remark on the module structure of $\mathcal{C}(\chi)$ which depends on the injectivity of the inclusion maps (i.e., part (1) of Lemma 5.4); the proof is similar to [42, Proposition 13.28] (which depends on the injectivity of [42, Proposition 13.26]).

Proposition 5.5. *The Λ-module $\mathcal{C}(\chi)$ has no nontrivial finite Λ-submodule.*

Proof. Let M be a finite Λ-submodule of $\mathcal{C}(\chi)$ of order $|M| = s$ (obviously a power of p). It is enough to prove that there is no p-torsion in M, so let $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in M$ be such that $p\alpha = 0$ (so that $p\alpha_n = 0$ for any $n \gg 0$). Fix n and take a $\gamma_{i,j}$ among the generators of Γ which acts trivially on F_n. Denote by $L_\infty = \cup L_m$ the $\mathbb{Z}_p$-extension topologically generated by $\gamma_{i,j}$ over F_0. The $s+1$ elements of M

$$\alpha\ ,\ \gamma_{i,j}^{p}\alpha\ ,\ \gamma_{i,j}^{p^2}\alpha\ ,\ \dots\ ,\ \gamma_{i,j}^{p^s}\alpha$$

cannot be distinct, hence there exist $0 \leqslant r < t \leqslant s$ such that

$$\gamma_{i,j}^{p^r}\alpha = \gamma_{i,j}^{p^t}\alpha \quad , \text{ i.e.,} \quad \gamma_{i,j}^{p^r}\left(1-\gamma_{i,j}^{p^t-p^r}\right)\alpha = 0\ .$$

This yields $\gamma_{i,j}^{p^r(p^{t-r}-1)}\alpha = \alpha$: since $\gamma_{i,j}$ and $\gamma_{i,j}^{p^{t-r}-1}$ generate the same $\mathbb{Z}_p$-extension we can assume from the beginning that there exists an $r \geqslant 0$ such that $\gamma_{i,j}^{p^r}\alpha = \alpha$. By construction $F_n \cap L_\infty = F_0$ and, for any $m \geqslant r$, $\mathrm{Gal}(L_{m+1}/L_m)$ (generated by $\gamma_{i,j}^{p^m}$) acts trivially on α. Take ν big enough to have $p\alpha_\nu = 0$ and $L_{m+1}F_n \subset F_\nu$, and consider the tower of extensions

$$F_n \subset L_m F_n \subset L_{m+1}F_n \subset F_\nu\ .$$

From the surjectivity of the norm maps proved in Lemma 5.4 one has

$$\mathcal{C}(\chi) = \varprojlim_{[L:F_0]<\infty} (W\otimes C_L)(\chi)\ ,$$

so we can compute

$$i_\nu^n(\alpha_n) = i_\nu^n(N_n^\nu(\alpha_\nu)) = i_\nu^n(N_{F_n}^{L_mF_n}(N_{L_mF_n}^{L_{m+1}F_n}(N_{L_{m+1}F_n}^{F_\nu}(\alpha_\nu))))\ .$$

Since $\mathrm{Gal}(L_{m+1}F_n/L_mF_n)$ acts trivially on α, the norm $N_{L_mF_n}^{L_{m+1}F_n}$ is just multiplication by p and we get $i_\nu^n(\alpha_n)=0$. But Lemma 5.4 part (1) shows that the maps like i are injective so $\alpha_n=0$ and, eventually, $\alpha=0$ as well. $\square$

6. Application to Bernoulli-Goss numbers and $\mathfrak{p}$-adic L-functions

We define an arithmetic invariant related to our $\mathfrak{p}$-adic L-function.

Definition 6.1. For any i, define

$$m_{\mathfrak{p}}(i):=\begin{cases}\mathrm{Inf}\left\{v_{\mathfrak{p}}(L_{\mathfrak{p}}(1,y,\omega_{\mathfrak{p}}^i))\ :\ y\in\mathbb{Z}_p\right\} & \text{for } i\not\equiv 0 \pmod{q-1}\\ \mathrm{Inf}\left\{v_{\mathfrak{p}}\left(\frac{d}{dX}L_{\mathfrak{p}}(X,y,\omega_{\mathfrak{p}}^i)_{|X=1}\right)\ :\ y\in\mathbb{Z}_p\right\} & \text{for } i\equiv 0 \pmod{q-1}\end{cases}. \tag{50}$$

Obviously the value of $m_{\mathfrak{p}}(i)$ depends only on the class of i modulo q^d-1.

Lemma 6.2. *We have the following equality*

$$m_{\mathfrak{p}}(i)=\mathrm{Inf}\left\{v_{\mathfrak{p}}(\beta(j))\ :\ j\geqslant 1,\ j\equiv i \pmod{q^d-1}\right\}\ .$$

Proof. Let $j\equiv i\pmod{q^d-1}$, then, by Corollary 3.17,

$$L_{\mathfrak{p}}(X,j,\omega_{\mathfrak{p}}^i)=(1-\pi_{\mathfrak{p}}^jX^d)Z(X,j)\ .$$

By Definition 3.13, if $j\not\equiv 0\pmod{q-1}$, we have

$$(1-\pi_{\mathfrak{p}}^j)\beta(j)=(1-\pi_{\mathfrak{p}}^j)Z(1,j)=L_{\mathfrak{p}}(1,j,\omega_{\mathfrak{p}}^i)\ , \tag{51}$$

while, if $j\geqslant 1$ with $j\equiv 0\pmod{q-1}$, we have

$$\frac{d}{dX}L_{\mathfrak{p}}(X,j,\omega_{\mathfrak{p}}^i)_{|X=1}=(1-\pi_{\mathfrak{p}}^j)\frac{d}{dX}Z(X,j)_{|X=1}=-(1-\pi_{\mathfrak{p}}^j)\beta(j) \tag{52}$$

(recall $Z(1,j)=0$ in this case). The lemma follows noting that the set $\{j\geqslant 1\,,\ j\equiv i\pmod{q^d-1}\}$ is dense in $\mathbb{Z}_p$. $\square$

We can now prove a function field version of the Ferrero-Washington Theorem (see, e.g., [42, Theorem 7.15]), but its statement is limited to nontrivial characters.

Theorem 6.3. *For any $1\leqslant i\leqslant q^d-2$, one has $\Theta_\infty^\#(1,\widetilde{\omega}_{\mathfrak{p}}^i)\not\equiv 0\pmod p$.*

Proof. We consider two cases depending on the type of the character $\widetilde{\omega}_{\mathfrak{p}}^i$. Recall that, by Lemma 3.14, $\beta(j)\neq 0$ for any $j\geqslant 0$.

Case 1: *$i\not\equiv 0\pmod{q-1}$, i.e., $\widetilde{\omega}_{\mathfrak{p}}^i$ is odd.*

In this case $\Theta_\infty^\#=\Theta_\infty$. Take $j\equiv -i\pmod{q^d-1}$. Then (51) shows that $L_{\mathfrak{p}}(1,j,\omega_{\mathfrak{p}}^{-i})$ is nonzero, hence, by Theorem 3.22, $s_X(\Theta_\infty(X,\widetilde{\omega}_{\mathfrak{p}}^i))(-j)_{|X=1}$ is nonzero as well. It follows that $s(\Theta_\infty(1,\widetilde{\omega}_{\mathfrak{p}}^i))\neq 0$ and therefore

$$\Theta_\infty(1,\widetilde{\omega}_{\mathfrak{p}}^i)\not\equiv 0\pmod p\ ,$$

since the Sinnott map s has domain $\Lambda/p\Lambda$.

Case 2: *$i\equiv 0\pmod{q-1}$, i.e., $\widetilde{\omega}_{\mathfrak{p}}^i$ is even but $\neq\chi_0$.*

In this case $\Theta_\infty^\#=\dfrac{\Theta_\infty}{1-X}$, hence

$$\Theta_\infty^\#(1,\widetilde{\omega}_{\mathfrak{p}}^i)=-\frac{d}{dX}\Theta_\infty(X,\widetilde{\omega}_{\mathfrak{p}}^i)_{|X=1}\ .$$

Again, take $j\equiv -i\pmod{q^d-1}$. Then (52) shows

$$\frac{d}{dX}L_{\mathfrak{p}}(X,j,\omega_{\mathfrak{p}}^{-i})_{|X=1}\neq 0$$

and Theorem 3.22 yields

$$s_X\left(\frac{d}{dX}\Theta_\infty(X,\widetilde{\omega}_{\mathfrak{p}}^i)\right)(-j)=\frac{d}{dX}s_X(\Theta_\infty(X,\widetilde{\omega}_{\mathfrak{p}}^i))(-j)=\frac{d}{dX}L_{\mathfrak{p}}(X,j,\omega_{\mathfrak{p}}^{-i})\ .$$

From here we get the claim by the same reasoning as in case 1. □

As a consequence we find that the p^n-torsion of $\mathcal{C}(\chi)$ looks like a pseudo-null module in the non-noetherian Iwasawa algebra Λ.

Corollary 6.4. *For any character $\chi\neq\chi_0$, p does not divide $\mathrm{Fitt}_\Lambda(\mathcal{C}(\chi))$ and the p^n-torsion modules $\mathcal{C}(\chi)[p^n]$ have at least two relatively prime annihilators.*

Proof. Easy consequences of the previous theorem and Theorem 5.2 □

Remarks 6.5.

1. Since we are working in the non-noetherian algebra Λ, the module $\mathcal{C}(\chi)[p^\infty]$ might be not finitely generated on W. The last statement (recalling pseudo-nullity for noetherian Iwasawa algebras) might be false if we consider the whole set of p-power torsion points $\mathcal{C}(\chi)[p^\infty]$. However a combination of Proposition 5.5 and the previous corollary suggests to investigate the possibility that $\mathcal{C}(\chi)[p^\infty]=0$.
2. In [25, page 4446] the authors provide a formula for the class number growth in subextensions of the $\mathfrak{p}$-cyclotomic extension and note that the growth can be exponential, i.e., the direct analog of the Ferrero-Washington Theorem ($\mu=0$) does not hold for function fields.

An estimate for $m_{\mathfrak{p}}(i)$ is provided by the following

Lemma 6.6. *For any positive integer m let $\ell(m)$ be the sum of the digits of the q-adic expansion of m (i.e., writing $m=\sum m_iq^i$ with $0\leqslant m_i\leqslant q-1$, one has $\ell(m)=\sum m_i$). For any $1\leqslant i\leqslant q^d-2$ with $i\not\equiv 0 \pmod{q-1}$, one has*

$$m_{\mathfrak{p}}(i)\leqslant\frac{i}{d}\cdot\frac{\ell(i)}{q-1}\ . \tag{53}$$

Proof. By [19, Corollary 2.12], one has

$$S_n(j)=\sum_{a\in A_{+,n}}a^j=0\quad\text{if}\ \ n>\frac{\ell(j)}{q-1}\ .$$

Hence

$$\beta(j)=\sum_{n\geqslant 0}S_n(j)=1+\sum_{n=1}^{\lfloor\frac{\ell(j)}{q-1}\rfloor}\sum_{a\in A_{+,n}}a^j$$

(where $\lfloor *\rfloor$ means the integral part of $*$). Clearly we have

$$\deg(\beta(j))\leqslant\left\lfloor\frac{\ell(j)}{q-1}\right\rfloor j$$

and the result follows from Lemma 6.2. □

Remark 6.7. Using the bounds on $\deg S_n(j)$ provided in [40, Section 6] (please be aware that the notations in that paper differ from ours, in particular our $\deg S_n(j)$ corresponds to $-s_n(-j)$ there), it is possible to improve the bound (53) and also to find a lower bound for $m_{\mathfrak{p}}(i)$. We decided to stop here and just provide (53) as an easy example of what can be achieved. Computations using the bounds of [40] can become quite cumbersome and our techniques (depending on the IMC) are of a completely different nature.

In the classical setting of p-adic L-functions defined for the cyclotomic $\mathbb{Z}_p$-extension of a number field (see, for example, [42, Chapter 5]), there is the following natural problem on the p-adic valuation of values of p-adic L-functions (related to p-adic valuations of generalized Bernoulli numbers)

Open Question 6.8. *Let χ be an even character in $\mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_p)/\mathbb{Q}), \mathbb{Z}_p^*)$, then is it true that*

$$\mathrm{Inf}\,\{v_p(L_p(y,\chi))\ :\ y\in\mathbb{Z}_p\,\}\leqslant 1\ ?$$

In the following we still consider non-trivial characters only.

Corollary 6.9 (Arithmetic properties of Bernoulli-Goss numbers)**.** *Let $1\leqslant i\leqslant q^d-2$ and define*

$$N_{\mathfrak{p}}(i) := \mathrm{Inf}\{n\geqslant 0\ :\ \Theta_n^{\#}(1,\widetilde{\omega}_{\mathfrak{p}}^i)\not\equiv 0 \pmod p\,\}$$

(it is well defined because of Theorem 6.3). Then

$$N_{\mathfrak{p}}(i)\leqslant \mathrm{Inf}\,\big\{v_{\mathfrak{p}}(\beta(j))\ :\ j\geqslant 1,\ \ j\equiv -i \pmod{q^d-1}\,\big\} = m_{\mathfrak{p}}(q^d-1-i) = m_{\mathfrak{p}}(-i)\ .$$

Proof. Assume $i\not\equiv 0 \pmod{q-1}$: by definition of $m_{\mathfrak{p}}(i)$ (or by Lemma 6.2), there exists $y_0\in\mathbb{Z}_p$ such that $L_{\mathfrak{p}}(1,y_0,\omega_{\mathfrak{p}}^{-i})\not\equiv 0 \pmod{\mathfrak{p}^{m_{\mathfrak{p}}(-i)+1}}$, while for any $y\in\mathbb{Z}_p$ we have $L_{\mathfrak{p}}(1,y,\omega_{\mathfrak{p}}^{-i})\equiv 0 \pmod{\mathfrak{p}^{m_{\mathfrak{p}}(-i)}}$.
In Section 2.3 we saw that the map s can be computed by taking the limit on the (induced) maps

$$s_n : W/pW[\Gamma_n]\to C^0(\mathbb{Z}_p, A_{\mathfrak{p}}/\mathfrak{p}^{n+1})\ .$$

Therefore, for $n< m_{\mathfrak{p}}(-i)$, we obtain that $s_n(\Theta_n^{\#}(1,\widetilde{\omega}_{\mathfrak{p}}^i))$ is the zero function, while

$$s_{m_{\mathfrak{p}}(-i)}(\Theta^{\#}_{m_{\mathfrak{p}}(-i)}(1,\widetilde{\omega}_{\mathfrak{p}}^i))(-y_0)\neq 0\ .$$

Since the maps s_n are not injective in general (see Proposition 2.4) we only obtain an inequality

$$N_{\mathfrak{p}}(i) = \mathrm{Inf}\{n\geqslant 0\ :\ \Theta_n^{\#}(1,\chi)\not\equiv 0 \pmod{pW[\Gamma_n]}\,\}\leqslant m_{\mathfrak{p}}(-i)\ .$$

The proof for even nontrivial characters (i.e., for $i\equiv 0 \pmod{q-1}$, $i\neq 0$) is similar. □

Remark 6.10. We can define similar arithmetic invariants for the $\mathbb{Z}_p$-cyclotomic extension of a number field k. For simplicity we just consider $k=\mathbb{Q}$ with $p\neq 2$, the generalization is straightforward. Take a character χ in $\mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_p)/\mathbb{Q}),\mathbb{Z}_p^*)$ and let

$$L_p(y,\chi) = f((1+p)^y-1,\chi)\ ,\quad f(T,\chi)\in\mathbb{Z}_p[[T]]$$

be the associated p-adic L-function. Let $\mathbb{Q}_n:=\mathbb{Q}(\boldsymbol{\mu}_{p^{n+1}})$ be the layers of the $\mathbb{Z}_p$-cyclotomic extension and let $\mathcal{C}\ell(\mathbb{Q}_\infty):=\varprojlim \mathcal{C}\ell(\mathbb{Q}_n)\otimes_{\mathbb{Z}}\mathbb{Z}_p$. As mentioned in the introduction, the Iwasawa Main Conjecture in this setting reads as

$$\mathrm{Fitt}_{\mathbb{Z}_p[[T]]}(\mathcal{C}\ell(\mathbb{Q}_\infty)(\omega_p\chi^{-1})) = (f(T,\chi))\ ,$$

and we can define

$$m_p(\chi):=\mathrm{Inf}\{\,v_p(L_p(y,\chi))\ :\ y\in\mathbb{Z}_p\,\}\in\mathbb{N}\ .$$

Finally let $\Theta_{\mathbb{Q}_n/\mathbb{Q},p}(\chi)$ be the Stickelberger element (see [42, Chapter 6]): again via the Main Conjecture, we have

$$\mathrm{Fitt}_{\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})]}(\mathcal{C}\ell(\mathbb{Q}_n)(\omega_p\chi^{-1})) = (\Theta_{\mathbb{Q}_n/\mathbb{Q},p}(\chi))\ .$$

The Ferrero-Washington Theorem implies that

$$N_p(\chi):=\mathrm{Inf}\{\,n\geqslant 0\ :\ \Theta_{\mathbb{Q}_n/\mathbb{Q},p}(\chi)\not\equiv 0 \pmod p\,\}$$

is well defined.
At present it is not clear whether there is any kind of relation between $m_p(\chi^{-1})$, the p-adic valuations of generalized Bernoulli numbers and $N_p(\chi)$ in this setting.

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Bruno Anglès: LMNO, Université de Caen BP 5186, 14032 Caen Cedex. France
E-mail address: bruno.angles@unicaen.fr

Andrea Bandini: Dipartimento di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze, 53/A - 43124 Parma (PR), Italy
E-mail address: andrea.bandini@unipr.it

Francesc Bars: Departament de Matemàtiques, Facultat de Ciencies, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Catalonia
E-mail address: francesc@mat.uab.cat

Ignazio Longhi: Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, 111 Ren Ai Road, Dushu Lake Higher Education Town, Suzhou Industrial Park, Suzhou, Jiangsu, 215123, China
E-mail address: Ignazio.Longhi@xjtlu.edu.cn