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Rationality in mathematics teaching: the emergence of emotions in decision-making

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Contents

Introduction

During this writing process, many emotions were evoked: tension, curiosity, enjoy, exultation as well as frustration, loss, failure. The latter emotions cannot be avoided, rather they were the most significant. Surely, without them I never would end this work. In writing this sentence as the first page of my work, I could also discover one of my beliefs: the failure is the necessary condition to be able to express our passion and to realize our dreams. This work is the proof that emotions have a very significant role in success, no matter what the domain.[1]

The sentence in the preamble of the introduction expresses the essence of my work: emotions play a relevant role in all human activities, mathematical ones included.

I started my research aiming to study the *rationality* of the mathematics teacher in her decision-making processes, while she was teaching linear equations. It is important to consider the decision-making of the teacher, because the peculiarity of the teacher is the fact that she has to make decisions within the classroom. Indeed, many authors have always recognized a crucial role to the decision-making in relation to the work of mathematics teacher. For example, Bishop consider decision-making as the activity that is "...at the heart of the teaching process" (Bishop, 1976, p.42).

I adopted, as a theoretical lens, the theory of rationality developed by Habermas in 1998, according to the interpretation given by Boero and his collaborators in mathematics education (Boero et al., 2010), (Boero and Planas, 2014). The central issue of this philosophical speculation is the idea of the "discursive rationality" of a "rational being". From one side, it seems to be suitable to analyse the decision-making of the teacher: in general, Habermas focuses not on beliefs or knowledge — which is what most of the literature on teachers does — but on decision-making through the different kinds of rationality. From the other one, the study of the discursive activity of the teacher and not of the individual learners is something new because it is

[1]Inspired by Sinclair (2004), p. 281.

an uncommon analysis through this lens. Indeed, the theory of rationality has been adapted to mathematics education by a group of researchers who mainly used it to examine students (Boero et al., 2010), (Boero and Planas, 2014).
In order to better understand the philosophy of Habermas I read many commentaries and critiques, the latter of which confirmed my sense that his theory does not account for the emotional component of the human being. Many times I thought that he spoke of an "ideal" world in which all emotions are avoided. Indeed, I found many philosophers who recognized this lack in Habermas' speculation. Hence, I chose to pursue the idea of extending Habermas' theory. In particular, my research attempts to highlight how *emotionality* is actually present and intertwined with the rationality in the decision-making processes of the teacher. This led to propose the term *ræmotionality* as a neologism to describe this idea.
Entering in the structure of the thesis, in **Chapter 1**, I will illustrate the theory of rationality of Habermas (Habermas, 1998) and the critical debates about his philosophy (Habermas, 1982). I do so because the theory of rationality has triggered my research and the critical issues have acted as its catalysts.
In **Chapter 2**, I will present a panorama of the neuroscientific research that has proven the deep relationship between the emotional and the rational aspects of the human being (Damasio, 1994), (Damasio, 1999). Moreover, I will show how the neuroscientific research has been transported in the educational world in general (Immordino-Yang and Damasio, 2007). Then, entering into the specific of our research field, I will consider the research on mathematics-related affect that has extensively developed the intertwinement between the affect and the cognitive domains (e.g. (Zan et al., 2006), (Hannula, 2012)).
In **Chapter 3**, I will deepen the concept of "emotional orientation" (Brown and Reid, 2006) and I will adapt it to my context in order to solve the methodological problem of how it is possible to speak of the intertwinement of emotion and rationality. In particular, I will speak of the emotional orientation of the teacher as the set of expectations she has from her teaching. Then, I will illustrate the methodology used to conduct the analysis and I will present the emotional orientations of each teacher involved in my research.
Chapter 4 aims to prove the interaction between emotion and rationality in the decision-making processes of the teachers. Indeed, I will analyse the "fabric" of rationality and emotion in each teacher. In particular, I will show how their decisions are made visible through their emotionalities. I will try to say something about the *emotionality* of each teacher, because it seemed to be less ambiguous than emotion. In fact, the term "emotionality" is referred to somatic aspects that are observable and it is linked to the underpinning

emotion. This analysis has led in **Chapter 5** to introduce the idea of what we have called *ræmotionality*: it can allow researchers to consider the rationality and the emotion of the subject as an *unicum*. Lastly, I will present how ræmotionality has allowed me to explain that the differences among teachers are not mainly due to the decisions they take, but to the reason why they act in that way and not in another.

Chapter 1

Rationality

1.1 The starting point: Habermas' theory of rationality

The starting aim of my PhD thesis is the study of the discursive activity of the teacher when she is involving in the introduction of linear equations. The starting point to accomplish this goal is the theory of rationality developed, at the end of the twentieth century, by Jürgen Habermas.
Since Habermas speaks about a rational being involved in a discursive activity, the choice of this theoretical framework seems to be suitable for this context.
Moreover, in the last years, this philosophical and sociological framework has been re-elaborated and adjusted to mathematics education.
This research on the teacher becomes challenging because many of the previous studies led in terms of habermasian rationality were centred on the students in the mathematics classroom (Boero et al., 2010).
Habermas wrote a book called "Wahrheit und Rechtfertigung" in which he develops the notion of discursive rationality related to a rational being (Habermas, 1998). In 1998, it was published the essay "On the Pragmatics of Communicative Rationality" in which there is the English translation of the German chapter of the theory of rationality (Habermas, 1998).
In this chapter, Habermas introduces the notion of *discursive rationality* proper of a rational being involved in a discursive activity. Within this frame, he states that a rational being is a human being who "can give account for his orientation toward validity claims" (Habermas, 1998, p. 310). Habermas explains that the discursive rationality not only referred to the discourse (as it could seem to be from the term), but it has three different roots: the knowledge, the action and the speech, or in a different manner,

knowing, acting and speaking.
It becomes clear that discursive rationality is strictly connected to the decision-making space of the teacher, because when a rational being is involved in a discursive activity is a decisional-agent.

1.1.1 Discursive rationality

As I said above, Habermas states that the rationality of a subject is proportionate to his ability "to give account for" his expression in a reflexive stance towards validity claims. He calls this rationality "accountability" and, for him, "accountability presupposes a reflected self-relation on the part of the person to what she believes, says, and does" (Habermas, 1998, p. 310). There are three types of self-relation linked to the three types of rationality. The epistemic self-relation implies a reflexive attitude concerning the beliefs and convictions of the subject, the technical-practical self-relation implies a reflexive attitude concerning the acting of the subject to achieve a goal and the last one, but not the least, the moral-practical self-relation that implies a reflexive attitude to "her own norm-regulated actions".
Then, starting from Habermas' assumption that for a rational being the discourse and the reflection on it (not necessary explicit) are entwined, on this integrative level of reflection and discourse, "the three rationality components-knowing, acting, and speaking- combine, that is, form a syndrome" (Habermas, 1998, p. 311). Moreover, the structure of discourse determines an intertwinement among the knowledge, the action and the speech "by bringing together" the three types of rationality (Habermas, 1998, p. 309). These three types of rationality are inseparable, since a rational being acts in a specific manner to achieve a goal, on the basis of a specific knowledge, communicating in a precise way. Hence, the three types of rationality are always present in the discursive activity; the only thing an external observer can say is the preponderance of one of the three types on the others.
In the following part, I will show what Habermas means with the three types of the discursive rationality.

1.1.2 Three roots of rationality

Epistemic rationality

Habermas states that our knowledge is constituted by propositions or judgments that cannot be correct or incorrect in absolute sense, but they can be valid or invalid within a reference context. As Habermas affirms, the epis-

temic rationality is connected with a specific idea of knowledge that is not intended in a trivial sense as a sequence of facts. In fact, Habermas writes that "in order to know something in an explicit sense, it is not, of course, sufficient merely to be familiar with facts that could be represented in true judgements. We *know* facts and have a knowledge of them at our disposal only when we simultaneously know why the corresponding judgments are true'. Otherwise we speak of intuitive or implicit knowledge — of a "practical" knowledge of how one does something" (Habermas, 1998, p. 311-312).
We face an epistemic rationality when simultaneously we can give an account of the justification of the knowledge at play. Conversely, someone is irrational when she puts forward her beliefs in a dogmatic way, "clinging to them although she sees that she cannot justify them" (Habermas, 1998, p. 312). By its very nature, the epistemic rationality is strictly entwined with the action and the use of language. For this reason the epistemic rationality is not a self-supporting structure, but it is always combined with the action and the discourse.

Teleological rationality

Habermas starts his discussion about the teleological rationality from the fact that "all action is intentional", so every action is originated from an intention of the subject with the aim of the realization of a set of goals. Habermas speaks about teleological rationality not when the state that actually occurs is the same what the subject intended in her mind from the beginning and it corresponds to conditions of success, but when "the actor has *achieved* this result on the basis of the deliberately selected and implemented means" (Habermas, 1998, p. 313).
Deepening this perspective, Habermas affirms that a successful actor has acted rationally when, first, he knows and he is conscious of the *why* he was successful, and then "this knowledge motivates the actor (at least in part) in such a way that he carries out his action for reasons that can at the same time explain its possible success". There is an analogy between the epistemic rationality and the teleological rationality: for the former it is important to develop a reflexive "having" of knowledge that refers to possible justifications and for the latter, correspondingly, the action related to the intention "requires a reflexive "having" of the decisive action-intention, that is, a calculation of the success of the action" (Habermas, 1998, p. 314).
Once again, there is a mutual relationship among the action, the discourse and the knowledge that constitutes the basis on which the actor puts at play his decisive reasons for making decisions. Habermas explains in a very clear way this entanglement: "for the practical considerations by means of which

a rational plan of action is carried out are dependent on the input of reliable information — even if, in general, the actors acting in a purposive-rational way have to be satisfied with highly incomplete information. On the other hand, such information can be processed intelligently [...] only in the medium of linguistic representation" (Habermas, 1998, p. 314).

Communicative rationality

For Habermas the role of the communicative rationality is very relevant, in fact the title of this essay is "On the pragmatics of communicative rationality" and that of the chapter about the types of rationality is "Some further clarifications of the concept of communicative rationality". In particular, Habermas states that the communicative rationality "is expressed in the unifying force of speech oriented toward reaching understanding" (Habermas, 1998, p. 315). The communicative use of linguistic expressions not only give a "life" to the intentions of the subject, but also they represent the state that actually occurs and they establish interpersonal relations with a second person. Hence, it comes from a triangle in the vertex of which we have an actor who reaches understanding, with someone, about something on which they are reflecting. Once again, the communicative rationality is strictly connected to the epistemic and the teleological ones: "what the speaker wants to say with an expression is connected both with what is literally said in it and with the action as which it should be understood" (Habermas, 1998, p. 315). If the aim of communication is to reach understanding with a second person about something and, consequently, the acceptance and sharing about this "something", "the rationality of the use of language oriented toward reaching understanding depends on whether the speech acts are sufficiently comprehensible and acceptable for the speaker to achieve illocutionary success with them" (Habermas, 1998, p. 315). In this sense, Habermas highlights that this illocutionary aim cannot be separated from the linguistic means of reaching understanding, so the reaching understanding can be achieved just trough argumentative means.

Within this perspective, there is another significant point that is the fact that in the discursive activity the hearer enjoys the freedom of being able to say freely what he is thinking about that "something" of which he speaks with his interlocutor. Once again, we don't speak about rational speech acts as "valid", "but rather all comprehensible speech acts for which the speaker can take on a *credible* warranty in the given circumstances to the effect that the validity claims raised could, if necessary, be vindicated discursively" (Habermas, 1998, p. 315-316). By its very nature, a rational speech acts involved its possible justification.

Summarizing the discourse made above about the communicative rationality, Habermas states that it constitutes an internal connection between the conditions that make a speech act valid, the claim raised by the speaker that these conditions are satisfied and the credibility of the warranty issued by the speaker to the effect that he could, if necessary, discursively vindicate the validity claim.

The intertwinement among the epistemic, the teleological and the communicative rationality

As I pointed out above, Habermas states that "the three rationality components-knowing, acting, and speaking — combine, that is, form a syndrome" (Habermas, 1998, p. 311). Also in other passages, he clearly states that the rationality cannot be referred just to one of the three roots (knowledge, action and speech) separated from the others. Indeed, Habermas says, for example, that it does not make sense to speak just of the rationality referring to knowledge isolated from the action and the speech. Rather, it is meaningful to ascribe "the predicate "rational" to the *use* of knowledge in linguistic utterances and in actions" (Habermas, 1982, p. 234). Moreover, the author declares that "subjects capable of speech and action employ knowledge in speaking and acting, and connect with their utterances, at least implicitly, claims to validity (or to success)" (Habermas, 1982, p. 234). In other words, the three roots are strictly intertwined because a subject acts in a certain way to achieve a goal, drawing on a specific knowledge, communicating in a precise manner. Habermas stresses the same concept also in 1998, stating that the communicative rationality does not constitute the structure that embraces both the epistemic and the teleological rationality. Rather, it is just one of the different roots of rationality that is strictly interwoven with them "by way of the discursive rationality that emerges out of communicative rationality" (Habermas, 1998, p. 309).
In his speculation, Habermas very often joins knowledge, action and speech. For example, one of his well-known ideas is the "communicative action", "in which actors in society seek to reach common understanding and to coordinate actions by reasoned argument, consensus, and cooperation rather than strategic action strictly in pursuit of their own goals (Habermas, 1984, p. 86)" (Bolton, 2005, p. 1). The distinction that Habermas operates between the communicative and the strategic actions highlights the connection between action and speech, indeed they are different by virtue of the diverse role of language. In the communicative action "the language serves to build consensus within the community, instead in the strategic action, the language serves as a mean of influencing aimed to the own success" (Habermas, 1982,

p.236). Habermas develops the theory of communicative action based on an analysis of the use of language orientated to reach understanding.

1.1.3 The concept of freedom

As I pointed out in Paragraph 1.1.1, Habermas relates the rationality with "a reflected self-relation on the part of the person to what she believes, says, and does" (Habermas, 1998, p.310). The ability of the subject to distance himself in this way according to these three different perspectives from himself is a necessary condition of his *freedom*. Habermas distinguishes between two types of freedom according to "the different self-relations of the knowing and acting subject". In the specific, we can read that "reflexive freedom in the sense of cognitive openness requires liberation from the egocentric perspective of a participant deeply involved in action contexts", while the "freedom of choice consists in the capacity for rationally choosing to act in one way or another, or for making new start in the chain of events" (Habermas, 1998, p.311).
The second type of freedom seems to be strictly related to the decision-making of the subject.

The concept of the freedom of choice is strictly connected to how Habermas begins to speak of the teleological rationality, namely, to the statement "all action is intentional" (Habermas, 1998, p.313). It could be plausible to think that he supposed that the action of the subject draws on her beliefs, but surely this seems not as related to the emotional sphere. In this sense, there are some critiques of the work of Habermas about the delicate issue of the role affect in his speculation. In the next section, I will go deep this part, being one of the significant and crucial point of the research.

1.2 Debates about Habermas' philosophy

When I looked for more explications about the types of rationality of Habermas, I found many debates about his work due to the fact that Habermas doesn't take in account the emotional sphere of the subject.
Habermas seems to lack any reference to emotion or passion in his discussion of reason. The philosopher thinks of us to be more logicians and less human beings. He avoids any emotion by claiming that the force of the better argument must be free of emotional overtones. Sometimes, instead, it happens

that only emotion will drive us to accept what reason requires. When discussing an external truth, saying that emotion has no part in determining what is true or how we validate truth is counter-intuitive. It would be very difficult to investigate something un-emotionally. Possibly, to be involved in something is to feel it. Moreover, to be engaged also means to have a point of view. Having a point of view entails having a will, a desire on the basis of own expectations. The search for truth is a passion, an emotion, and a desire. The perception of truth and untruth is an involvement with it, so it could be plausible to think of it as a feeling. At every moment emotions are connected to reasoning, in discerning what is true and what isn't, providing justifications of what we are stating. Saying that, I want to avoid to state that emotion rules the rationality, but, surely, there exists a mutual relationship between them.

Briefly speaking, our judgments about what we consider "rational" is surely affected by emotions and expectations. This way the criteria for determining truth should take emotions into account as evidence. Habermas does not discuss emotions because his interest is in redeeming validity claims, not in certain sociology-of-science exploration of why a person X believes a theory and a person Y doesn't. Habermas does not account for an exploration of this source, because his interest is the building consensus within a community.

In fact, Habermas states that, in the discursive activity, the participants establish interpersonal relations "through intersubjective recognition of criticizable validity-claims". In this discursive activity, the subjects are orientated of reaching understanding and one has to obtain the agreement with his interlocutor(s). Talking of this fact, Habermas writes that "what is peculiar about this mechanism of reaching understanding is that ego can, in a certain sense, rationally motivate alter to accept its offer; that is, ego can motivate alter by its readiness to cover the claim it has raised through providing grounds or reasons. What stands behind the reciprocally raised validity-claims in communicative action are potential reasons as a (kind of) security reserve, rather than sanctions or gratifications, force or money, with which one can influence the choice situation of another strategic action. What counts in a given case as a reason or ground also depends of course on the background cultural knowledge that the participants in communication share as members of a particular life-world" (Habermas, 1982, pp. 269-270).

And then he continues saying that "jokes, fictional representations, irony, games, and so on, rest on intentionally of validity-claims and corresponding modes (being/illusion, is/ought, essence/appearance), are seen through as category mistakes" (Habermas, 1982, p. 271), p. 271.

Many authors embraced this perspective in which emotions are considered a significant aspect within the discursive rationality of a subject. I will discuss them in the paragraphs below.

1.2.1 Heller's critique

Rienstra and Hook, quoting the philosopher Heller, highlight the question that "Habermas leaves no room for "sensuous experiences of hope and despair, of venture and humiliation" accusing him of completely avoiding the "creature-like" aspects of human beings" (Rienstra and Hook, 2006, p. 13).
The essay in which Heller speaks about the lack of something in the philosophy of Habermas is "Habermas and marxism". This work is part of a consistent collection of essays, "Habermas: critical debates", written by many philosophers and social theorists from Europe and the United States (Habermas, 1982, p. 21). The essays contained in this volume represent an attempt for a sustained critical discussion of Habermas' ideas, in view of the relevance and significance of his philosophy.
If we deepen the essay by Heller, we find that the avoiding of accounted for the emotional aspects is the consequence of the fact Habermas doesn't construct his own philosophy for a particular addressee. This way he does not consider the subjectivity and reality of human beings. Heller continues her thought saying that he "always rejected the philosophy of hope and despair". If we better analyse the critique of Heller about the philosophy of Habermas, we find that, indeed, for her, "habermasian man has, however, no body, no feelings: the "structure of personality" is identified with cognition, language and interaction" (Heller, 1982, p. 22). But in one of his response about these critiques, Habermas himself writes that "the philosophy of despair is 'not binding' ", but as Heller highlighted one has to add that "non-binding philosophies are irresponsible" (Heller, 1982, p. 21).
It seems that, for Habermas, a "good life" consists only of rational communication and "that needs can be argued for without being felt" (Heller, 1982, p. 22). An emblematic example of this is the analysis made by Habermas of one chapter in the "Economic-Philosophical Manuscripts", written by Marx in 1844, in which the "odyssey of human suffering exemplifies for Habermas the structure of instrumental rationality" (Heller, 1982, p. 23). If Habermas speaks about a human beings without feelings, consequently, he constructs his unique theory as universal for each considered addressee. In do that, he supposes that the "interests of one particular social group are *identical* with the emancipatory interest as such, that universality is inherent in its particularity" (Heller, 1982, p. 30).
For Heller, human beings do not accept social theories because they belong

to a specific group with specific interests, but they accept them from the point of view of their own lives “as a whole, as a needing, wanting, feeling being. The system of need is identical with the form of living. If we accept the plurality of ways of life, we have to accept the plurality of theories as well. Consensus regarding *one* theory would mean consensus in one single way of life. To exchange pluralism for consensus would be a bad bargain (not only for me, but for Habermas as well)” . In the response of the Heller’s critique, Habermas says that “’reason’...has no body, cannot suffer, and also arouses no passion.” (Habermas, 1982, p. 221)

The perspective of Habermas is very different from the one of Marx. Drawing upon the differences between the two authors, Heller sketches the lacks in the philosophy of Habermas.

Continuing to read her paper in the book “Habermas: critical debates” she says that Habermas doesn’t take in account the motivational system of human beings, while Marx attributes many kind of motivations to the proletariat, because, for him, the subjects that constitute the proletariat are, first of all, people who suffer and who feel unhappy in their alienation.

1.2.2 Flyvbjerg’s critique

Flyvbjerg moves his critique towards the philosophy of Habermas, reviewing what the author states about the communicative action. As I also stressed in the previous part, communicative action is an action based upon a deliberative process, where two or more individuals interact and coordinate their actions based upon agreed interpretations of the situation. Communicative action is distinguished by Habermas from other forms of action, such as instrumental action, which is pure goal-oriented behaviour, dealt with primarily in economics, by taking all functions of language into consideration. That is, communicative action has the ability to reflect upon language used to express propositional truth, normative value, or subjective self-expression. In this perspective it becomes clear that, for Habermas, the communicative rationality is not referred to the language as such, but to the use of linguistic expressions. The mean of linguistic expression has the global goal of reaching understanding, while the speaker can give expression to his intentions, to the state of affairs of that moment and last, but not least while the speaker can establish an interrelation with the his interlocutor. Indeed, Habermas situates rationality as a capacity inherent within language, especially in the form of argumentation. The philosopher defines the term “argumentation” as “that type of speech in which participants thematize contested validity claims and attempt to vindicate or criticize them through argumentation” (Haber-

mas, 1984, p. 18). Hence, Flyvbjerg points out that the philosopher does not account for all of the other means of communication as the language of the body, the prosody and so on, that are the indicators that spread out the feeling of the subject. According to Habermas, the structures of argumentative speech, which he identifies as the absence of coercive force, the mutual search for understanding, and the compelling power of the better argument, form the key features from which intersubjective rationality can make communication possible. Action undertaken by participants through a process of such argumentative communication can be assessed as to their rationality to the extent which they fulfill those criteria. As Flyvbjerg states, the consequence is that, for Habermas, human beings are defined as democratic beings, as *homo democraticus.* As for the validity claims, Habermas explains that validity is defined as consensus without force: "a contested norm cannot meet the consent of the participants in a practical discourse unless...all affected can *freely* [*zwanglos*] accept the consequences and the side effects that the *general* observance of a controversial norm can be expected to have for the satisfaction of the interests of *each individual*" (Flyvbjerg, 1998, p. 213). Similarly, when Habermas speaks about truth says that argumentation "insures that all concerned in principle take part, freely and equally, in a cooperative search for truth where nothing coerces anyone except the force of the better argument" (Flyvbjerg, 1998, p.213). "The only 'force' which is active in the ideal speech situation and in communicative rationality is thus this "force of the better argument".
Habermas shows a model whereby validity and truth of statements of the participants, involved in a given discourse, are ensured. This model is constituted by five components: 1) "the requirement of generality: no party affected by what is being discussed should be excluded from the discourse"; 2) "autonomy: all participants should have the equal possibility to present and criticize validity claims in the process of discourse"; 3) "ideal role taking: participants must be willing and able to empathize which each other's validity claims"; 4) "power neutrality: existing power differences between participants must be neutralized such that these differences have no effect on the creation of consensus": "transparence: participants must openly explain their goals and intentions and in this connection desist from strategic action". The last component of the model is a theoretical one because it is unlimited time (Flyvbjerg, 1998, p. 213). Flyvbjerg critiques this model, saying that in a society where rules this model "participation is *discursive* participation and participation is *detached* participation, in as much as communicative rationality requires ideal role taking, power neutrality, etc." (Flyvbjerg, 1998, p. 213). The adjective "ideal" this in an interesting adjective that suggests that he knows that not all speech situations are ideal! Also from the expla-

nation of the model that rules the communication between a subject and his interlocutor(s) it seems clear that, for Habermas, the only means to achieve consensus are arguments.

1.2.3 Steinhoff's critique

In the same perspective, ten years later, Steinhoff highlighted the fact that in a discursive activity there is not only the argumentation to obtain consensus from the interlocutor about the statements, but there are all sort of non-verbal means that play a significant role in the discursive activity. In particular, Steinhoff says that when people conduct discourses "they do try to convince the listener (and for this purpose not all listeners have to be participants in the discourse), but not just with arguments; they also use emotion, rhetoric and all means of achieving strategic influence, and the conflict resolution that discourse might be aimed at does not have to be consensual either (why not, for example, through majority decision?)" (Steinhoff, 2009, p. 147). In another part of his book, Steinhoff continues, stating that: "as soon as not just purely theoretical questions but practical ones are concerned questions that bring values, attitudes and emotions into play agreement will not be reached exclusively through arguments, as Habermas demands of all agreement reached communicatively but rather through all sorts of non-argumentative means of influence, such as the way arguments are presented, affection or dislike for the one presenting the argument, unconscious group dynamics, etc. There is not, in point of fact, any agreement in practical questions where such factors do not play a role" (Steinhoff, 2009, p. 205).

From the critiques about the Habermas' theory it emerges that it lacks the taking in account of the affective sphere as a significant character of the discursive activity. The next step of my research was finding some studies in different fields that support this possible connection between affect and reason. As a matter of fact there are many important studies, not only in the field of mathematics education, that highlight the issue that emotion and rationality are inseparable.

In the following sections, I will present the research on affect in the neuroscientific field and the research on affect in mathematics education: from different points of view they explain why when we talk of the rational part of the subject we cannot avoid to take into account also her emotional counterpart. They have to consider entwined as the two faces of the same coin.

Interlude

As shown in Chapter 1, starting from the study of Habermas' philosophy, I found philosophers and social theorists who recognized the same lack in Habermas' speculation: he does not account for the emotional aspects of the human being in her discursive activity. However, in these critical debates, especially in that of Heller, there is no suggestion for how to address the affective dimension within Habermas, so, as a pioneer, I tried to do this. Hence, the first challenge has been to discover if there are studies, in some research fields, that combine both the rational and the emotional side. In particular, I will focus the attention on the studies on affect in mathematics education and in the neurological field in order to consider them as a support of the grounded hypothesis that rationality is deeply linked with the emotional sphere.

I will present first the neuroscientific research and, then, the research on mathematics-related affect. I chose to follow this order because many of the studies developed in mathematics education are based on the neuroscientific results.

Chapter 2

Emotion

2.1 Research in the neuroscientific field

2.1.1 The neurological research of Damasio

Damasio is a professor of Neurology and Head of the Department of Neurology at the University of Iowa city. He is internationally recognized for his research on the neurology of memory, language and vision. In particular, he and his wife, Hanna, created a centre of research in which they investigate the neurological disorders of mind and behaviour. He writes many books, but, in this thesis, I would like to concentrate attention on two books that are "Descartes' error: emotion, reason and the human brain" (Damasio, 1994) and "The feeling of what happens: body and emotion in the making of consciousness" (Damasio, 1999).
In his wide research Damasio has scientifically proved that emotions and feeling play a significant role in the decision-making of the subject. Damasio writes that "certain levels of emotion processing probably point us to the sector of the decision-making space where our reason can operate most efficiently" (Damasio, 1999, p. 42).
First of all, it is convenient to speak of Damasio's idea about reason and rationality: he defines rationality as the quality of thought and behaviour coming from adapting reason to a persona and social contexts[1].

[1] This is not in any contrast with Habermas, provided that we interpret the adaptation process in light of the evolutionary and dynamic nature of reasoning in the decision-making space

2.1.2 Emotions and feelings

Drawing on his neurological results, Damasio writes that "the apparatus of rationality, traditionally presumed to be *neo*cortical, does not seem to work without that of biological regulation, traditionally presumed to be *sub*cortical. Nature appears to have built the apparatus of rationality not just on top of the apparatus of biological regulation, but also *from* it and *with* it. The mechanisms for behaviour beyond drives and instincts use, I believe, both the upstairs and the downstairs: the neocortex becomes engaged *along with* the older brain core, and rationality results from their concerted activity" (Damasio, 1994, p. 128). Hence, Damasio decides to speak and develop discourse about emotion and feeling, the central aspects of biological regulation, to suggest that they provide the link between rational and non rational processes, and so between cortical and subcortical structures.
Damasio affirms that all human beings have emotions and govern their lives with the aim to search an emotion, in general happiness, avoiding unpleasant emotions. At the first glance, it seems that emotions are not the distinctive character of the human being because also nonhuman creatures have emotions in abundance. Indeed, specific features of stimuli in the world or in our body are perceived, like "size (as in a large animals), large span (as in flying eagles), type of motion (as in reptiles), certain sounds (such as growling); certain configurations of body state (as in the pain felt during a heart attack)." (Damasio, 1994, p. 131). Nevertheless, there is something of distinctive in the way "emotions have become connected to the complex ideas, values, principles and judgements the only humans can have, and in that connection lies our legitimate sense that human emotion is special." (Damasio, 1994, p. 131).
In fact the emotions are not only the sexual pleasure or the fear of snakes, but for example, "the thick beauty of words and ideas in Shakespeare's verse...and about the harmony that Einstein sought in the structure of an equation" (Damasio, 1999, p. 36). These causes of emotions have on the human beings the effect of triggering feelings, indeed Damasio writes "it is through feelings, which are inwardly directed and private, that emotions, which are outwardly directed and public, begin their impact on the mind; but the full and lasting impact of feelings requires consciousness, because only along with the advent of a sense of self do feelings become known to the individual having them". (Damasio, 1999, p. 36)
Hence, Damasio distinguishes between the "feeling" and the "knowing that we have a feeling". He explains his position, saying that an organism can represent in neural and mental patterns the state that the couscous creatures call a feeling, without the awareness that the feeling is taking place.

This is a difficult distinction to understand since we "*tend* to be conscious of our feelings" (Damasio, 1999, p. 36). Conversely, there is no evidence that human beings are conscious of all own feelings. Damasio gives an example in order to explain better and more clear what he expressed: "we often realize, suddenly, in a given situation, that we feel anxious or uncomfortable, pleased or relaxed, and it is apparent that the particular state of feeling we know then has not begun on the moment of knowing but rather sometime before. Neither the feeling state nor the emotion that led to it have been "in consciousness", and yet they have been unfolding as biological processes (Damasio, 1999, p. 36).
What Damasio concludes is that consciousness has to be present if feelings are to influence the subject having them beyond the "hic et nunc". Damasio pays attention to divide the emotion and the feeling, because the former is characterized by a relative publicness, while the latter by "the complete privacy". So the feeling is reserved for the private, mental experience of an emotion, while the emotion "should be used to designate the collection of responses, many of which are publicly observable" (Damasio, 1999, p. 42). The definition of emotion comes from directly from the etymology of the word "emotion" (from Latin, *ex movere* means literally "movement out") that suggests an external direction, from the body.
The consequence of these facts is that a subject cannot observe the feelings in another person, although she can see a feeling in herself when she is conscious of it, perceiving her own emotional states. Conversely, some of the aspects of emotions that give rise to her feelings become observable to others.
Damasio proved with many scientific experiments on a person who had serious damages in different parts of the brain that emotions are not dependent on consciousness, that is human beings don't be to be aware of the inducer of an emotion and often are not. For this reason, subjects cannot control emotions willfully. In fact, we can find ourself in a happy or sad state, without knowing why we stay in that particular state. We can hypothesize the cause of that state, but without the absolute certainty that it is the actual one. For example, the actual cause may have been the image of an event or a transient change in the biological profile. Surely, the effects are observable, since they alter the body state of the person, in fact Damasio says that "all emotions use the body as their theater (internal milieu, visceral, vestibular, and musculoskeletal systems)" (Damasio, 1999, p. 51).
For Damasio, emotions and feelings can be seen as the beginning and the end of a progression and he explains very well how it works. In a emotion certain regions of the brain send commands both to other regions of the brain and to almost all parts of the body, through two paths: one is the bloodstream end the other one consists of neuron pathways. The global result of the action

of these commands is a change in state of the organism (both the body and also the human brain). Beyond the emotion, described above as a collection of responses, there are two additional steps that have to take place before "an emotion is *known*. The first is feeling, the imaging of the changes we just discussed. The second is the application of core consciousness to the entire set of phenomena. Knowing an emotion — feeling a feeling — only occurs at that point." (Damasio, 1999, p. 68).

2.1.3 Somatic markers

Damasio, after proving the role of the emotion in the decision-making processes, explains well how this interplay works, introducing the hypothesis of somatic markers.

Damasio proposes a realistic situation which calls for a choice: "imagine yourself as the owner of a large business, faced with the prospect of meeting or not with a possible client who can bring valuable business but also happens to be the archenemy of your best friend, and proceeding or not with a particular deal" (Damasio, 1994, p. 170).

The brain of a normal subject reacts providing some possible situations of possible response options and related outcomes. For example, in that specific case offered by Damasio, the meeting with the possible client, the best friend who sees you with the client (his enemy), losing the good business but safeguarding the friendship and so on. At this point, there are two different possibilities to proceed: the first one comes from a traditional notion of "high reason" view of decision-making, while the second one comes from the hypothesis of somatic markers. "The "high-reason" view, which is none other than the common sense view, assumes that when we are at our decision-making best, we are the pride and joy of Plato, Descartes and Kant" (Damasio, 1994, p. 171). Just formal logic get us for the best solution of any problem. The important aspect in this mechanism is that "emotions must be *kept out*." (Damasio, 1994, p. 171). In this perspective, the different scenarios are considered one by one and you perform a cost/benefit analysis of each of them. For instance, you consider the consequences of each option at different points in the projected future and the possible losses and gains. Even if the problem has just two choice options, the analysis is not so simple. In fact, "gaining a client may bring immediate reward and also a substantial amount of future reward. How much reward is unknown and so you must estimate its magnitude and rate, over time, so that you can pit it against the potential losses among which you must now count the consequences of losing a friendship. Since the latter loss will vary over time, you must also figure its "depreciation" rate!". This strategy is very complex to manage, indeed

"at best, your decision will take an inordinately long time, far more than acceptable if you are to get anything else done that day. At worst, you may not even end up with a decision at all because you will get lost in the byways of your calculation" (Damasio, 1994, p. 172).
Hence, Damasio invites to consider the same situation explained above. He affirms that before you make any kind of cost/benefit analysis of the problem, something of quite special happens: "when the bad outcome connected with a given response options comes into mind, however fleetingly, you experience an unpleasant gut feeling. Because the feeling is about the body, I gave the phenomenon the technical term "*somatic*" state ("soma" is Greek for body); and because it "marks" an image, I called it a "marker"" (Damasio, 1994, p. 173).
Now, we can pose the problem of how somatic marker works. It forces the attention on the negative outcome of a given action and functions "as an automated alarm signals which says: beware of danger ahead if you choose the option which leads to this outcome" (Damasio, 1994, p. 173). This signal permits to you to abandon immediately that action in order to choose among the other remained alternatives. Conversely, a positive somatic marker works as a "beacon of incentive" (Damasio, 1994, p. 174).
So the somatic marker allows the subject "*to choose from among fewer alternatives*". Now it may convenient to apply the cost/benefit analysis, but only after this automatic step that has reduced drastically the possibilities of choice.
Damasio highlighted very well that the hypothesis of somatic markers doesn't concern the reasoning steps after the action of the somatic marker. In this sense, somatic markers can be considered feeling triggering from emotions that are linked, by learning, to predicted future results of a certain situation.

2.1.4 Emotion and learning

Immordino-Yang and Damasio (2007) have considered the connection among emotion, social functioning and decision-making as turning point in understanding the role of emotion in decision-making, the relationship between learning and emotion, and how culture shapes learning. While educational research often treated decision-making, reasoning and processes related to reading, language and mathematics as detached from emotion and body, these authors have stated that "learning, in the complex sense in which it happens in schools or in the real world, is not a rational or disembodied process; neither is it a lonely one" (Immordino-Yang and Damasio, 2007, p. 4). Many neuroscience research results stress that there is a deep connection

between emotion, social functioning and decision making. Immordino-Yang and Damasio went beyond, interpreting this relationship in the field of education. In fact, they suggested that "the aspects of cognition that we recruit most heavily in schools, namely learning, attention, memory, decision making, and social functioning, are both profoundly affected by and subsumed within the processes of emotion; we call these aspects emotional thought" (Immordino-Yang and Damasio, 2007, p. 3). Moreover they stated that the emotion-related processes are necessary for skills and knowledge to be transferred from the world of the school to the daily life. They talk of these processes in terms of "emotional rudder of judgement and action".
However yet research in the field of education considers reasoning, decision making and processes related to reading, language, mathematics detached from emotion and body.
The authors specified that they didn't state that "emotions rule our cognition, nor that rational thought does not exist" (Immordino-Yang and Damasio, 2007, p. 3). It is, rather, that this connection is due to the fact that a subject has to deal not only with one's own self, but he also manages social interactions and relationships, producing a very complex society.
Immordino-Yang and Damasio clarified their thought, answering to the issue of why a high school student solves a math problem. The reasons vary from the personal reward of having found the solution, to getting a good mark by the teacher, to avoiding punishment, to helping a friend, to pleasing her parents or the teacher. All of these reasons are constituted of "powerful emotional component and relate both to pleasurable sensations and to survival within our culture, our brains still bear evidence of their original purpose: to manage our bodies and minds in the service of living, and living happily, in the world with other people" (Immordino-Yang and Damasio, 2007, p. 4).
The relationship among reasoning, decision making and emotion has been proved with many evidences from patients with brain damage: their social behaviour was compromised, making them oblivious to the consequences of their actions. Moreover, they were insensitive to others' emotions, and, also, unable to learn from their mistakes, breaking often social and ethical rules.
Furthermore, these defects in the social conduct came along with the inability to make advantageous decisions in rational matter in real life, even if their knowledge and skills were not compromised and they were able to take appropriate choices in a decontextualized set. Therefore, the disturbances in the realm of emotion could provide a better account of their lacking decision making. Due to the impossibility of evoking emotions associated with particular past situations, choices and outcomes, the patients became unable to choose the best response based on their past experiences. This is the proof of the link between emotion and decision-making. In other words, for the au-

thors, “emotion is a basic form of decision making, a repertoire of know-how and actions that allows people to respond appropriately in different situations”.
Immordino-Yang and Damasio highlighted that every actions, even the most simple, has an emotional goal, namely, “emotions and the mechanisms that constitute them as behaviours, which humans experience as resulting in punishment or reward, pain or pleasure, are, in essence, nature’s answer to one central problem, that of surviving and flourishing in an ambivalent world.” (Immordino-Yang and Damasio, 2007, p. 6).
Moreover, the author highlight the crucial role of creativity in constituting the bridge between cognition and emotion, indeed they stated that “out of these processes of recognizing and responding, the very processes that form the interface between cognition and emotion, emerge the origins of creativity - the artistic, scientific, and technological innovations that are unique to our species.” (Immordino-Yang and Damasio, 2007, p. 7), p. 7).
The consequences of these insights in the educational context led the authors to postulate two important hypothesis. The first is that the emotional processes are necessary to transfer the knowledge acquired in school to the real life situations. The second is that the strictly relationship among decision making, emotion, and social functioning may provide a new point of view for the connection between biology and culture. In the specific, it may be via the emotional process that the social influences of culture come to shape learning, thought, and behaviour. In fact, “new neurobiological evidence regarding the fundamental role of emotion in cognition holds the potential for important innovations in the science of learning and the practice of teaching.” (Immordino-Yang and Damasio, 2007, p. 9). In conclusion, we as human beings cannot detach us from our biology, nor can we neglect our social and cognitive aspects that can distinguish use within the animal realm. Transferring in the education world, “when we educators fail to appreciate the importance of students’ emotions, we fail to appreciate a critical force in students’ learning. One could argue, in fact, that we fail to appreciate the very reason that students learn at all.” (Immordino-Yang and Damasio, 2007, p. 9).

2.2 Research on mathematics-related affect

For giving an insight of which has been the “story” of the research on “mathematics - related affect” from the beginning to nowadays, I considered the paper of Hannula (2012) in which he clearly analyses its development. Moreover, in this same paper, the author proposes a new metatheoretical foundation for linking different aspects of research in mathematics-related affect. I will use the term "affect" as Hannula suggests, namely, as "an umbrella concept for those aspects of human thought which are other than cold cognition, such as emotions, beliefs, attitudes, motivation, values, moods, norms, feelings and goals" (Hannula, 2012, p. 138).

Early research in this field generally does not account for emotions. For example, it focused on the anxiety towards mathematics (Zan et al., 2006). The pioneers of the research on emotions were Buxton (Buxton, 1981) and Mason, Burton and Stacey (Mason et al., 1982). In particular, they considered the role of emotions in problem solving. Then, from the end of the '80s, many theories for the mathematics related-affect were appeared: the self-efficacy in mathematics (Bandura and Schunk, 1981), affect in mathematical problem solving (Schoenfeld, 1985), (McLeod and Adams, 1989) and mathematics anxiety (Hembree, 1990).

After few years, there was one of the turning point in the research on mathematics - related affect. McLeod (1992) had the aim of building "an overall framework of mathematics-related affect that would be consistent with research that is cognitively oriented" (Hannula, 2012, p. 138). In particular, the research of McLeod had highlighted three constituting elements: beliefs, attitude and emotions. Beliefs were the more stable aspects among them, emotions the opposite and attitudes in the middle point between them with respect to the stability. McLeod saw repeated emotional reactions as the origin of attitudes, while he considered culture and personal background that as the origin of beliefs (Hannula, 2012, p. 138). He considered emotions as "to be unstable, or at least less stable than beliefs and attitudes" (Hannula, 2012, p. 141). This framework became one of the cornerstone of the research on mathematics-related affect.

Then, there was much research that developed different issue of this field; for example, the research on the relationship between the mathematical affect and achievement (Ma and Kishor, 1997a), (Ma and Kishor, 1997b), (Ma, 1999), (Ma and Xu, 2004), (Minato and Kamada, 1996). Putting together, these studies surface that there is a not an actual causality between affect and achievement, rather there exists a bidirectional relationship between them.

Another issue that has been analysed is the role of gender. From one side, Finnish studies have shown that there is not "gender difference in how much students like mathematics or how useful they perceive mathematics (Mattila, 2005), (Hannula, 2010)" (Hannula, 2012, p. 139). From the other one, they have found a quite strong correlation between the gender and students' self-confidence. For example, Hannula et al. (2002) has pointed out that a lower self-confidence is generally typical of females both concerning correct and incorrect answers.

Unfortunately the theoretical framework of McLeod didn't accomplish its primary aim, namely, to construct a general framework that could embrace all of the research on different issues. The problem was and still is the ambiguity of the terminology. There is not exist an homogeneity in the definitions of the various elements treating in the research on mathematics-related affect. For example, the major problem in the framework of McLeod is the concept of attitude. He defines attitudes as "affective responses that involve positive or negative feelings of moderate intensity and reasonable stability (McLeod, 1992, p.581)" (Hannula, 2012, p. 140). Others define attitudes "as positive or negative degree of affect" or "as consisting of cognitive (beliefs), affective (emotions) and conative (behaviour) dimensions (Di Martino and Zan, 2010)" (Hannula, 2012, p. 140).
Concerning the more recent research on mathematics-related affect, there were introduced new concepts beyond those of McLeod, such as values (DeBellis and Goldin, 1997), (Bishop, 2001), (Wong and Lee, 2011), identity (Beijaard et al., 2004), (Frade et al., 2010), (Sfard and Prusak, 2005), motivation (Hannula, 2006), (Middleton and Spanias, 1999), (Yates, 2000) and norms (Yackel and Cobb, 1996), (Partanen, 2011). The concept of motivation has assumed a crucial role. For example, it had shown that there is a positive correlation between motivation and achievement and that male students are more motivated to study mathematics rather than females.
Returning to the theoretical framework of McLeod, his aim is still a difficult target to achieve, because there exists a lot of research in this field and each study has its complexity and its own structure that don't allow a synthesis, without losing important pieces of them. Hence, Hannula in the paper of 2012 presented a metatheory (Wagner and Berger, 1985), where the term metatheories are interpreted as "overarching frameworks that link, separate and contextualize other theories (Edwards, 2008, p. 63)" (Hannula, 2012, p. 143). This original and new framework has incorporated emotions, attitudes and beliefs and also its dimensions, namely intensity, stability and cognitive-affective (Hannula, 2012, p. 143). Instead of ascribe the stability to beliefs and the instability to emotions, this framework consider stability as an independent dimension. Moreover, motivation is not treated through beliefs and emotions. Rather it is considered having a "distinctive influence on choices, which cannot be exhaustively analysed through cognitive and emotional processes" (Hannula, 2012, p. 143). Concerning emotions, there is a general agreement that they consist of three processes: physiological processes regulating body, subjective experience regulating behaviour and expressive processes regulating social coordination (Buck, 1999), (Power and Dalgleish, 2007). These three levels can be recognized both in cognition and motivation. For this reason, the metatheory developed by Hannula for

mathematics-related affect is grounded on three distinct dimensions: "1) cognitive, motivational and emotional aspects of the affect; 2) rapidly changing affective states versus relatively stable affective traits; 3) the physiological (or embodied), psychological and social nature of affect" (Hannula, 2012, p. 143). In this article (2012) he explains deeply these three dimensions. Furthermore, he tests the metatheory to analyse and compare three different theoretical frameworks (Hannula, 2012, p. 154).
In the last years, Sinclair (2004), (2008), (2009) has developed a lot of research on the "interplay between the aesthetic, cognitive, and affective processes involved in mathematical inquiry" (Sinclair, 2004, p. 261). In the mathematics classroom, Sinclair has suggested that the aesthetic capacity of a student is not only her ability to identify "formal qualities as economy, unexpectedness, or inevitability in mathematical entities" (Sinclair, 2004, p. 262). Rather her aesthetic capacity is referred to her "sensibility in combining information and imagination when making purposeful decisions regarding meaning and pleasure" (Sinclair, 2004, p. 262). Sinclair asserted that the most significant aspect of aesthetics is that related to the motivations of students. In general, she structures the different points of view about aesthetic in Western mathematics, identifying three major roles of aesthetics: the evaluative role, the generative role and the motivational role. Through these different roles that aesthetics can have in creating mathematics she wanted to surface values and emotions underlying aesthetic behaviours, disclosing aspects of the mathematical emotional orientation (Drodge and Reid, 2000), that connect the affective, cognitive and aesthetic dimensions of mathematics. I will consider the concept of mathematical emotional orientation in Chapter 3. She deeply discusses these three roles of aesthetic, making explicit the connection between the aesthetic responses and emotions, attitudes, beliefs and values (Sinclair, 2004). For example, in the case of the mathematical activity of problem solving, the author recalled that "problem-solver becomes alert to aesthetic response *through* affective states" (Sinclair, 2009, p. 54). Moreover, Sinclair recalled also that "decisions or evaluations based aesthetic considerations are often made because the problem solver "feels" he or she should do so because he or she is satisfied or dissatisfied with a method or result" (Sinclair, 2009, p. 55). Then, from a perception, an awareness or something like that positive or negative feeling can be triggered. Hence, there is an interplay between the aesthetic and the affective domains, but they have different functions: "the aesthetic draws the attention of the perceiver to a phenomenon (a pattern, a relationship, a contradiction), while the affective can bring these perceptions to the conscious attention of the perceiver" (Sinclair, 2009, p. 55). Hence, the affective domain makes conscious the perceptions to the perceiver.

2.2.1 Cognition, emotion and motivation

As I pointed out above, the terms as cognition, emotion, motivation and so on have been used in different manner within the research on mathematics-related affect. These three important elements are linked to different aspects. For example, cognition involves information (self and the environment), motivation drives behaviour (goals and choices). The positive or negative result of a goal-oriented action is disclosed by emotions. Quoting Hannula, "these emotions, in turn, act as a feedback system to cognitive and motivational processes" (Hannula, 2012, p. 144).
As Hannula highlights theoretical frameworks on mathematics-related affect are grounded on just one of these categories, while the other two are in a subordinate position with respect to the first one. Nevertheless, in mathematics education there are some theories that combine these three aspects and that treat them in the same level of importance and significance.
As Hannula recalled in his article, DeBellis and Goldin (DeBellis and Goldin, 1997), (DeBellis and Goldin, 2006) have proposed a framework constituted by emotions, attitudes, beliefs and values. In their paper (2006), the authors stated that the research on mathematical learning and problem solving generally studied cognition, rather than affect or cognitive-affect interactions. DeBellis and Goldin explained this fact stating that mathematics is normally seen as ""purely rational", with emotion playing no role" (DeBellis and Goldin, 2006, p. 131). Moreover, they raised the problem that it is very difficult in the research to design and carry out "reliable empirical studies of affect" (DeBellis and Goldin, 2006, p. 131) also due to the well-known problem of the existence of a shared terminology of affect. DeBellis and Goldin presented a theoretical framework in the context of individual mathematical problem solving. From behaviour of individual children's mathematical problem solving they infer affect. The term "affect" "includes changing states of emotional feeling during mathematical problem solving -feelings of which the individual may be consciously aware, as well as unconscious or preconscious emotional states (Damasio, 1994)" (DeBellis and Goldin, 2006, p. 133). The grounded hypothesis of the authors is that affect is not an evanescent aspect of the human being, rather it is representational, without being a system constituted of mostly involuntary, physiological side-effects of cognition. It means that "the states of emotional feeling carry meanings for the individual. They encode and exchange information in interaction with other internal systems of representation, in a way essential to mathematical understanding and problem-solving performance" (DeBellis and Goldin, 2006, p. 133). Affect considered as representational shapes a system of communication through intonation, eye movements, facial expression, body language, prosody and

so on. This system is very ambiguous to analyse, but it works efficiently in the mutual interactions among subjects. As I said previously, the authors also accounted for the domain of values. In the specific that refer them "to the deep, 'personal truths' or commitments cherished by individuals. They help motivate long-term choices and shorter-term priorities" (DeBellis and Goldin, 2006, p. 136). Finally each of the affective domains (emotions, attitudes, beliefs and values) interacts dynamically with others in a subject and they interacts with those of other individuals. In the case of a student, his individual's affect does not only encounter that of the teacher or of the other students, but also interacts with that of institutions, namely, it interplays "over time with shared, normative emotional expectations, attitudes, beliefs and values gel by peers, school authorities, etc" (DeBellis and Goldin, 2006, p. 136).

Moreover, Op 't Eynde, De Corte and Verschaffel (2006), affirm that it is necessary to "stay aware of the close interactions between affective, motivational, and cognitive processes within emotional processes and mathematics learning" (Hannula, 2012, p. 145). In particular, they look at "emotions as being constituted by the dynamic interplay of cognitive, physiological, and motivational processes in a specific context" (Op't Eynde et al., 2006, p. 193). Being able to speak of emotions within the mathematics classroom conducts to understanding the nature of the classroom processes and also the relationship between these processes and the behaviour of students involved in problem solving activities.

Schoenfeld (2010) developed a theoretical framework for the decision making of the teacher, constituted by three different elements: knowledge, goals and beliefs. Actually, Schoenfeld considered the decision making in general. On the one hand, in the category of beliefs are included the emotional aspects, on the other hand goals are linked to motivation. In this book, the author focused on how and why people make the choices they make within their activity. In particular, he concentrated attention on teachers. His aim was to offer "a theoretical account of the (not necessarily conscious) decisions that teachers make amidst the extraordinary complexity of classroom interactions" (Schoenfeld, 2010, p. 2).

2.2.2 Discourse and affect

There also has been some focus on affect in relation to discourse, which seems to be an important aspect of communication, and, hence, it has some relation to Habermas.

Hannula states that "to be social is an essential characteristic of human nature", so for example, the mathematics classroom is a social community in which the relationships among the students are mediated by interpersonal affective aspects like love, hate, friendship, loyalty and so on and "they generate their own shared and understanding discourse" (Hannula, 2012, p. 151). This way, "social phenomena such as social order, discourse, and division of labour, emerge" (Hannula, 2012, p. 150). The author recalled the thought of Luhmann who spoke about the communication as the mode of autopoietic reproduction of social system: "Societies are encompassing systems in the sense that they include all events which, for them, have the quality of communication [...] Interactions, on the other hand, form their boundaries by the presence of people who are well aware that communication goes on around them without having contact with their own actual interaction. [...] interactions also are closed systems, in the sense that their own communication can be motivated and understood only in the context of the system (Luhmann, 1986)" (Hannula, 2012, p. 151).

When people interact in a social groups shaping it, they negotiate "about shared norms, values and understandings, i.e., learning in a community of practice (Wenger, 1998)" (Hannula, 2012, p. 151). To achieve this negotiation "it is not necessary that the norms and values are explicit, rather, norms and values become established as participants enact them. In this process of negotiation, both the individual and the social system change (Bandura, 1978)" (Hannula, 2012, p. 151).

Evans (2006), considering the discourse as a set of signs that regulates social practices, states that it "provides resources for participants to construct meanings and identities, experience emotions, and account for actions" (Evans et al., 2006, p. 210). Discourses identify what concepts and objects the interlocutors have to consider and which position, namely, which role they may adopted in the practice. In other words, the concept of "positioning" is referred to discursive positions participants take up from those available (Evans, 2000). Positioning is very significant in understanding emotions of participants, indeed "it affects how individual's identities are constructed within a power structure of social relationships" (Evans et al., 2006, p. 210). Positioning is not a free choice, in fact subjects are conditioned by their own history and also by the discursive resources at their disposal. The authors speak of "emotional experience" drawing on both psychoanalytic idea and post-structuralist theories of discourse (Henriques et. al, 1984). They define emotion as "a 'charge' attached to signifiers" (Evans, 2000). As the authors highlight in 2006, this metaphor "captures the energy and the intensity of emotions, and supports a unified approach to cognition and affect, seeing emotions as 'attached' to (chains of) signifiers representing ideas" (Evans et al., 2006, p. 211). They define emotional experience of a subject the result of "the interaction between the personal history of involvement in discursive practices, and present discursive positioning(s)" (Evans et al., 2006, p. 211). The history of the participants is strictly related to their own background.

Concluding this brief overview on the research on mathematics-related affect, as I anticipated in the interlude of this Chapter, the human neuropsychology (Damasio, 1994), (Damasio, 1999), (LeDoux, 1998) is always more applying in mathematics education. For example, Brown and Reid (2006) have developed the hypothesis of "somatic markers" offered by Damasio in 1994, as a sort of unconscious "driven" in the decision-making processes of the teacher. I will return on this crucial concept for my research in Chapter 3.

Interlude

As presented in Chapter 2, there exists a lot of research in different fields that puts forward the idea of an entanglement between rationality and emotion. In particular, one of the most important result that I consider is that cognition and affect are not separable and, within this perspective, the emotion plays a significant role. In particular, any decision-making process is not detached from body and emotion.

Then, if this research aims to study the rationality of the teacher in the mathematics classroom, I cannot avoid to consider the emotional aspects intertwined with her rationality. I am especially interested in the entanglement of rationality and emotion that can be identified in the decision-making processes of the teacher within the social context of the mathematics classroom. At this point, it arises a methodological problem of *h*ow is possible to speak about this entanglement.

In the following chapter, I will develop the notion of the emotional orientation, which Brown and Reid (2006) offer as means to study the teacher's decision-making, to identify a source for talking about the entanglement.

In addition, speaking of the mathematics teacher, I will make an overview of the research on mathematics teaching in mathematics education, developing the discourse through different variables that intervene in the analysis of the teacher.

Lastly, attempting to understand better what Habermas defines as "rationality", I found many researchers in mathematics education who use the term "rationality" (through which they develop their theoretical frameworks), but in a different manner with respect to Habermas. Nevertheless, it is important to relate, in this context, these different studies that are not in contrast with the theory of rationality of Habermas, rather they could have several intersection points with it.

Chapter 3

Making the "invisible" visible

Research on mathematics teaching

The focus of my research is on the teacher and, in particular, I analyse "the work of being a teacher". Then, I cannot avoid to account for many variables that intervene in these types of analysis. The first variable is the issue related to teacher education. There is a wide overview on this research in the 15th ICMI study on teacher education (Even and Ball, 2009) and in the volumes of the *International Handbook of Mathematics Teacher Education* (Boero and Guala, 2008). In particular, a lot of research focuses on content matters for teaching. The earliest studies date at the mid-1980s. In that period, Shulman (1986) and his colleagues suggested a precise domain of teacher knowledge called "pedagogical content knowledge" (PCK). They proposed a content unique for teaching, namely, a specific professional knowledge for teaching. Shulman and colleagues define it as "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Arzarello et al., 2014, p. 349). As the terms "pedagogical content knowledge" suggests, the authors had proposed to construct a bridge between the knowledge and the practice of teaching, in relation to different ways of teaching and learning. Then, starting from Shulman, Ball and Bass (2003) suggest a finer notion: the *mathematical knowledge for teaching* (MKT). Bass defines it as "the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed by the actual work of teaching" (Bass, 2005, p. 429), namely, "the daily tasks in which teachers engage, and the responsibilities they have to teach mathematics, both inside and outside the classroom" (Arzarello et al., 2014, p. 349). In (2008), Ball and colleagues points out that mathematics is related to the capacity of "compressing the information into abstract forms" (Arzarello et al., 2014, p. 349), while mathematics for teaching "requires a sort of decompression, in that the main ideas

pertaining to the mathematical content is made more explicit" (Arzarello et al., 2014, p. 349). In order to inquiry what teaching itself requires, they adopt an empirical approach, unlike starting from the school curriculum or from a list of topics that teachers should know. In other words, they focus "on the *work* teachers do in teaching mathematics" (BB, p. 390). This way they are able to identify a set of "testable hypotheses about the nature of mathematical knowledge for teaching" (Ball et al., 2008, p. 390).
The authors structure their notion of mathematical knowledge for teaching as in Fig. (3.1). In particular, they identify the major categories of MKT as Shulman's subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Then, they divided each categories in sub-domains.
SMK is divided into common content knowledge (CCK), specialized content knowledge (SCK) and horizon content knowledge. The first is "the mathematical knowledge known in common with others who know and use mathematics" (Ball et al., 2008, p. 403); the second is "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400); the third is "is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403).
PCK is divided into knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of content and curriculum.
In (2005), Bass explains the role of SCK in this model: "contrary to popular belief, the purely mathematical part of MKT is not a diminutive subset of what mathematicians know. It is something distinct, and, without dedicated attention, it is not something likely to be part of the instruction in content courses for teachers situated departments" Bass (2005, p. 429).

However, the very recent research focused on teaching and, in particular, on teacher education, complemented the notion of mathematical knowledge for teaching. In fact, in this notion the authors didn't account for the institutional aspects that represent another important variable playing in the analysis of the teacher and they didn't include the historical dimension either.
For example, in (2014), Arzarello and colleagues propose a new model for framing teacher education projects that takes both the aspects highlighted by Ball et al. and the institutional ones into account. In particular, they recognize a significant role that institutions plays "in the school context, including the national curriculum, national assessment tools and the constraints of teachers' time and space, and textbooks" (Arzarello et al., 2014, p. 348). Hence, they consider a theoretical framework that takes into account these aspects. In particular, they draw upon the "Antropological Theory of Didactics" (ATD) developed by Chevallard (1985, 1992, 1999) and the notion of "didactic transposition". They suggested the term "Meta-Didactical Transpo-

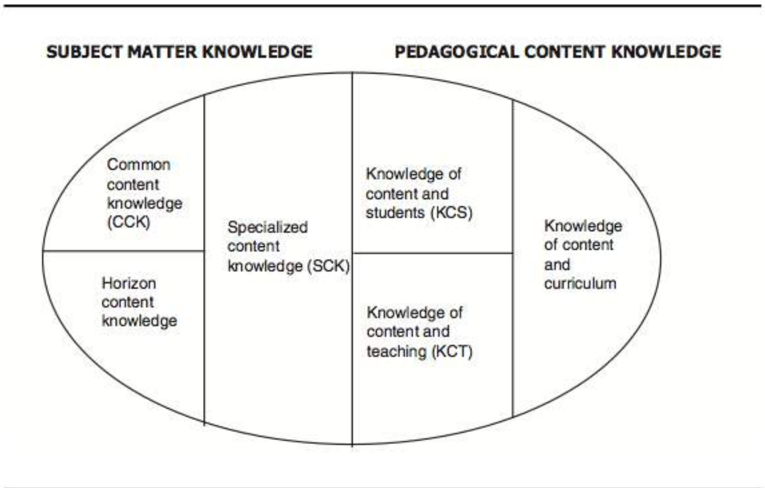


Figure 3.1: Domains of Mathematical Knowledge for teaching in Ball et al. (2008, p. 403)

sition" (MDT), in order to analyse the complexity of teacher education "as a research problem that involves a transposition from the practice of research to that of teaching" (Arzarello et al., 2014, p. XX) (see Fig. 3.2). The protagonists of this model are the community of teachers, the community of researchers, the institutions. MDT analyses "(i) the complex dynamic interplay, which develops in activities involving different communities (e.g. between teachers and the mathematics educators); (ii) the constraints imposed by institutions that promote such activities (including schools and Ministry of education) in view of some specific goals (e.g. promoting teachers' knowledge of new curricula or of new technologies); (iii) other "institutional" constraints, including the tradition of the school(s), the related (intended, implemented, attained) curricula and the textbooks used by the teachers" (Arzarello et al., 2014, p. 351). The underpinning ideas of this theoretical framework are the concept of didactical transposition and praxeology both developed by Chevallard. The didactical transposition is referred to the transformation of knowledge through different levels: the scholarly knowledge, the knowledge to be taught, the taught knowledge and the learnt knowledge. The latter is constituted by four intertwined elements: the type of task, the techniques to solve that task, the technology and the theory. The interplay among these elements is the following: in front of a task there is a set of techniques that solve it. The justification of the employed techniques is the technology and, in turn, the justification of the technology corresponds to the theory.
When the researchers' and teachers' communities work together, within a

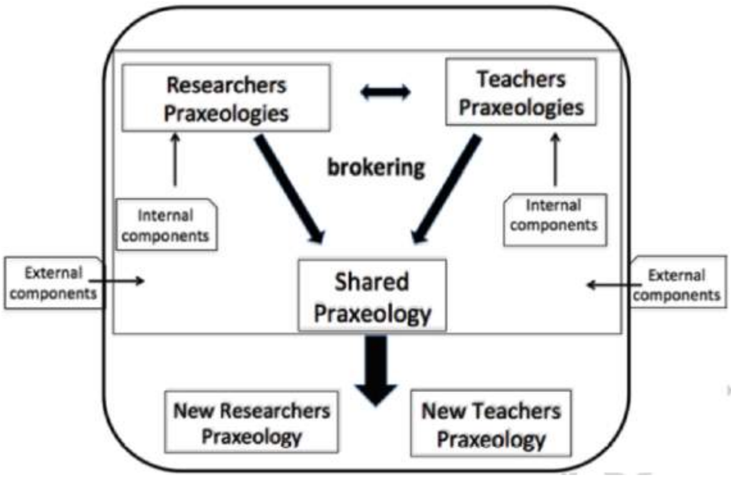


Figure 3.2: Meta-didactical transposition in (Arzarello et al., 2014, p. 355)

teacher education program, it surfaces a double level of transposition. From one side, the didactical transposition emerges in the teachers' activities in classroom and, from the other one, the predicate "meta-didactical" has been introduced "to stress that the processes under scrutiny are, in this case, the practices and the theoretical reflections developed within teacher education activities" (Clark-Wilson, 2014, p. 91). As shown in Fig. 3.2, at the beginning of the MDT process, the two communities of researchers and teachers have their own praxeologies, related to specific tasks. A priori, they could be similar or they could also be different. During this dynamic process, both the praxeologies of the researchers' and the teachers' communities change, because of the dynamic interactions between them. This change aims to develop new praxeologies, blending of the starting ones of the two communities, in order to create shared praxeologies, thanks to which teachers acquire new competences on the methodological-didactical level, "which led to activate in their classrooms, a didactical transposition in line with recent educational trends" (Arzarello et al., 2014, p. 355). The researchers employ some aspect that characterize their praxeologies (such as the use of GeoGebra, the open-ended problem, etc.). They constitute internal components for the researchers' community, while for the teachers' community they could be external. The goal of a teacher education program, based on the MDT process, is transforming the teachers' external components into internal ones. In turn, obviously, also new researchers' praxeologies may emerge: introducing new components that are internal for teachers' community that become internal also to the researchers' one. In this dialectical interplay between the two communities, intermediary figures intervene: the brokers. They help fostering the "double dialectic", which is one of the main inner processes of the meta-didactical transposition.

In the overview made above, there is no mention about the methodological problems that actually arise in the analysis of the work of teacher. However, there is also other research that suggests solutions to these problems.
For example, Barwell (2013) proposes the discursive psychology as a mean to study the teacher in interaction with her students within the mathematics classroom. In doing that, he offers also a way of defining mathematical teacher knowledge different, for example, from studies I mentioned above. This work has a double interest for the context of my research: he presents a way to analyse the teacher in classroom through the discourse analysis of her. This has a strictly connection with what I want to observe of the teacher, namely, her discursive activity. I recall that discursive activity is one of the central core of Habermas' philosophy.
As Barwell explains, discursive psychology has a double function: it is both a theoretical framework and a methodological perspective on human cognition that "seeks to understand the locally produced methods through which participants deal with each other's mental processes in interaction" (Barwell, 2013, p. 599). The crucial idea of discursive psychology is to develop an approach to inquiry cognition that started from the everyday life in which it is recognized a fundamental role to the interactions among human beings. As I already recalled this approach employs discourse analysis to examine the discourse of cognition and related discourse practices. The aim of such perspective is not to investigate what mental processes are really like, rather "how these mental processes are understood and interpreted by participants in interaction, and how they are discursively organized" (Barwell, 2013, p. 599). Applying this approach on the research on mathematics teacher knowledge, what becomes relevant is not so much what teachers know as when and how "what teachers know" constitutes a significant question for participants in mathematics classroom. Adopting this perspective, it can be solved the problem that Barwell highlighted in the the model of teacher knowledge proposed by Shulman, namely, the fact that the model of Shulman "does not necessarily correspond to how teachers or students make sense of each other's knowledge, thinking or learning, and hence does not adequately explain how such metal processes are constructed in classroom interaction" (Barwell, 2013, p. 599).
Underpinning the discursive psychology there is one of the crucial idea of ethnomethodology, "the aim of which is to understand how social actors (i.e. people) interpret, construct and orient to social norms that are rarely pre-

given and are frequently not explicitly articulated by participants" (Barwell, 2013, p. 600).

Summarizing, the work of Barwell is interested for my research because he offers a effective way to analyse the teacher within the mathematics classroom making a discourse analysis, but it does not consider the affective domain. Then, I cannot use it to speak of the intertwinement between the rational and the emotional aspects. As I previously explained, my focus is on the discursive activity of the teacher in order to say something about her rationality, in the sense offered by Habermas. Actually, in mathematics education, there is a lot of research on rationality in mathematics teaching.

For example, Herbst and colleagues (2003), (2013) talk about rationality, but it is not exactly the same as Habermas. Indeed, they offer the concept of "practical rationality of mathematics teaching". The authors hypothesize that "teacher' s actions in an instructional situation are modulated by a practical rationality, a feel for the game, which an outside observer might describe as dispositions to act in certain ways [...]. Practical rationality consists also of the principles or values that members of a practice use to justify or otherwise discard possible actions in an instructional situations" (Herbst et al., 2013, p. 218-219). Herbst and colleagues stresses that they don't want to assert that practical rationality drives and determines a certain course of actions, rather "this rationality describes what is perceived in instruction and outlines boundaries between what is reasonable, or customary, for a teacher to do and what is deemed as "out of bounds" (Herbst et al., 2013, p. 219). In particular, they study the rationality behind the customary practices of teaching, in order to understand "what principles could possibly justify or rebuff actions that deviate from what is expected within the discourse of a collective (such as the teachers of a particular course of studies, taken as a group), rather than the personal motives or reasons for individual teachers to choose some actions over others" (Herbst et al., 2013, p. 213)[1]. Their methodology of work is based of the notion of "breaching experiment" that comes directly from the ethnomethodology (Garfinkel,). They use breaching experiments on a group of teachers. Teachers discuss together and with researcher about a video episode of lessons in which they reconstruct what they have done. Through this activity, the authors are be able to highlight usually silent elements of rationality of teachers.

However, broadly speaking, rationality is not a well-defined concept as Vinner points out. The author stresses that the notion of rationality has become the focus of research in many different fields like philosophy, psychology, eco-

[1]This perspective is different from mine: I will focus on singles teachers and I will try to highlight the reasons of their choices through their emotional component

nomics, game theory and so on. Hence it is extremely difficult to uniquely characterize this notion. On the contrary, for ordinary people, namely, people who use an ordinary language, the concept of rationality appears quite clear. As Vinner points out "in everyday situations, people, quite often, recommend to each other to behave rationally. When they do it, it is quite clear what they mean" (Vinner, 2013, p. 478). Vinner recalls the definition of rationality of the Merriam-Webster dictionary to propose it as an educational goal of mathematics teaching. In this definition, "rationality is the quality or state of being agreeable to reason. Rationality is applied to opinions, beliefs and practices" (Vinner, 2013, p. 478). In addition, "about being reasonable, the dictionary adds that reasonable is not extreme or excessive and it is moderate and fair" (Vinner, 2013, p. 478). Hence, he is suggesting as educational goal a rationality that includes also the moral aspects.
This perspective already enlarges the Habermas' rationality, because it seems to admit elements different from the "pure" rational ones. It can be considered another suggestion that Habermas didn't account for the aspects of the affective domain of the subject.
All of the studies I mentioned above do not seem operatiionally useful to analyse practically the entanglement between emotion and rationality of the mathematics teacher. As you could see, some of them are concentrated just on the discursive activity of the teacher or just on the rationality in mathematics teaching. They don't account for the emotional sphere of teachers.
However, in the research on mathematics-related affect, I found that Brown, Drodge and Reid (2000), (2006), who developed the notion of emotional orientation that will be revealed useful for my aim.

3.1 Emotional orientation

In mathematics education, Brown, Drodge and Reid proposes their theoretical framework grounded on the notion of *emotional orientation* in order to study the decision-making processes both of the teacher and the students.
Their study is significant for my research because of its interests in that particular emotional aspect of human behaviour, which the Brown and Reid feel as neglected and see as "related to the decision-making that happens before conscious awareness of the decision to be made occurs." (Brown and Reid, 2006, p. 179). For this reason, they seem to be a useful tool of analysis because they compenetrate the two aspects, that of rationality and that of emotion in the decision-making processes of the teacher .

Drawing on Maturana (1988a, 1988b)[2], they refer the notion of emotional orientation to the criteria for acceptance of an explanation by members of a community, and consider emotions as being the foundation of such criteria.
Drodge and Reid states that close to these criteria there exist other two aspects: one is constituted by "the activities that are considered appropriate" (Drodge and Reid, 2000) and the other one is shaped by the "shared experience and assumptions of a community" (Drodge and Reid, 2000, p. 250).
Hence, an emotional orientation defines a domain of explanations, of which mathematics is one.
The authors consider emotions as the basis of these criteria, because they referred to the research by Damasio (1994) that revealed that emotions (in this case we speak of secondary emotions) constitute a significant part of all human decision-making. (We recall that secondary emotions are based on the simple emotional reactions (primary emotions: fear, pleasure, desire, etc.)).
Furthermore, the authors adapted this concept to the mathematics field and they defined the notion of mathematical emotional orientation. The criteria for accepting an explanation in the particular case of the mathematical emotional orientation include "the use of deductive reasoning, a basis in agreed upon premises, and a formal style of presentation" (Reid, 1999, p. 1). Moreover, there are many shared experiences and assumptions in mathematics, like the language used to talk about it. In the end, there are also many actions when someone does mathematics, like "drawing diagrams, generalizing statements, making conjectures" (Reid, 1999, p. 1).
They also recall the somatic markers hypothesis of (1994), proposing to consider somatic markers proper as a basis for mathematical emotional orientations. Somatic markers are structures that inform our action and decision-making, pushing us to decide something because "It feels right" in terms of its acceptance in a community. So, emotions related to being right are attached to somatic markers, sets of which constitute emotional orientations.
In decision-making, "many possibilities are rejected because they are associated with negative somatic markers" (Brown and Reid, 2006, p. 180), while positive somatic markers imply possible behaviours, being revealed by the decisions of the teacher in the activity.
The concept of "emotional orientation" allows me to speak of the interconnection between rationality and emotion, in fact, as the words themselves suggest, the "orientation" of a subject oriented towards validity claims is

[2]Maturana used the term *emotional orientation* "to describe the bodily predisposition that underlies individuals' decisions to accept some things as explanations and to reject others" (Reid, 1999)

"emotional", that is, affected by the emotions in a certain way. But there is still a methodological problem of how, practically, I can analyse this entanglement. Hence, I will present an adaptation of the theoretical framework of the emotional orientation in order to speak practically about these two sides of the same coin. I define the "emotional orientation" of a subject (e.g. a teacher) in terms of "the set of the expectations" of her, where the term "expectation" is connected to her "emotions of being right" when she uses specific criteria for accepting an explanation by a community (e.g. a class) rather than other ones (Ferrara and De Simone, 2014).

3.2 Rethinking the concept of emotional orientation

As Habermas underlined, for talking about the teacher's rationality in her decision-making, we need to consider reflection on personal activity, what involves values and beliefs of the acting subject. This is a key feature for our research because it entails a true complexity for it, since the beliefs and the background of the teacher contribute to her choices. The frame is even more complex whether we accept that beliefs are not agent-neutral, which means that the affective sphere of the teacher also affects her beliefs. So, *what happens* in the mathematics classroom is only a part of the story. We need a wider perspective, which leaves room for *the feeling of what happens* that the teacher brings to activity. Briefly speaking, if we remain just to what happens we might lose the reasons for which that specific "what" happens in the way it does. There is here an implicit assumption: what happens in the classroom is entangled with feelings of what happens that can be ascribed to individual teachers. Our perspective to study rationality in mathematics teaching expands to include also the affective domain, especially the "emotion side".

As we spoke above, Brown and Reid refer the idea of emotional orientation to the criteria for acceptance of an explanation by members of a community and emotions to the foundation of such criteria. The criteria for accepting an explanation (xs) cannot be the same as the criteria for accepting the criteria ("meta-criteria" ys). We can draw on this distinction to interpret emotions as being at the subtlest degree, that is, as moving those ys for accepting the xs, figuring out emotional orientation as set of meta-criteria. We stop here in order to avoid an infinite regress.

So, the teacher's rationality will allow us to talk about what happens-in terms of xs, while her emotional orientation will inform us of the feeling of what happens-in terms of ys. The two aspects are fully intertwined and joined in a unique frame that intends to speak directly to mathematics teaching in contextual situations. But we need to identify the emotional orientation of the teacher. Following Brown and Reid in seeing "the being right" in terms of the acceptance in a community as crucial, we characterize the emotional orientation as follows. We focus on the teacher's beliefs concerning the context, the content, the subject matter and her experiential background-beliefs that she declares in an a-priori interview. We identify her expectations concerning the activity-expectations that are attached to the beliefs and that we recover from videos of her actual activity in the classroom. The word "expectation" is used for its positive meaning of wait and anticipation, which we can refer back to emotions of being right. In other words, the cluster of expectations shapes the teacher's emotional orientation, which entails belief-related actions that reveal the rationality of her decision-making.

3.3 Research questions

As I anticipated in the Interlude of this chapter, I would like to investigate the intertwinement between the emotional and rational aspects in the decision-making processes of mathematics teacher and develop a method of identifying this intertwinement. In particular, I would investigate the relationships between emotions and decisions. *Is it that emotions rule decisions or that decisions are made visible through the emotions?*

3.4 Methodology

As already highlighted in previous chapters, research in different domains has highlighted the fact that the rational and the emotional spheres coexist and are deeply intertwined in the subject. In particular, given that I was interested in studying this entanglement in the mathematics teacher, I decided to follow this structure: from an *a-priori* interview of the teacher I identified the expectations from her teaching, then, I saw if they are actually reflected in her teaching. Lastly, I analysed her lessons in order to investigate the intertwinement between emotion and rationality and I attempted to highlight how teacher's decisions are made visible through her emotions.

Participants

This research involved the study of teachers explaining linear equations at secondary school. In particular, the participants were 3 teachers and their grade 9 classrooms, in a scientifically oriented secondary school in Western Italy. In Italy, teachers involved in such types of research are volunteers. Among them, there can be those who already collaborate with the university research team (teachers-researchers) or those who are interesting in this type of research and who are fully available for teaching experiments. Generally, regarding Italian teachers' gender, there are many more female teachers than male ones. Hence, it is more probable to work with female teachers, as in my case. In addition, I chose three different teachers in order to have a potential range of expectations and choices. In particular, I chose to plan the teaching experiments while they explained linear equations because they constitute a crucial mathematical topic in the Italian national curriculum at high school. Indeed, for example, it is one of the first mathematical content in which there is the delicate "shift" from the arithmetical world to the algebraic one. Hence, it is a topic that should require particular attention. The teachers have different backgrounds and they have had a diverse development of their carrier. Lorenza works just at school, namely she is the most "traditional" teacher among them; Sara is a teacher-researcher and, in turn, she has taken the master for training teachers who, in turn, will train other teachers; Carla is not a teacher-researcher, but she also has taken the same master of Silvia. The course Silvia and Carla have attended covered two years. The choice of considering 3 teachers has been made for having 3 different "macro-categories" of teachers: the traditional teacher, the teacher attending training courses at the university and the teacher-researcher. Assuming that, in general, people have their own rationality and emotional orientation, I chose to work with different kinds of teachers for appreciating better this diversity.

Data collection

A-priori interview

Each teacher was first interviewed and asked about her personal beliefs on the use of the didactical material, on the topic of linear equations and on Algebra in general. Each interview was roughly twenty minutes long and was videotaped with the camera facing the interviewer and the subject. The interviews were transcribed for the analysis. In particular, the questions of the interview were:

1) What is the rationale in the adoption of the mathematics textbook at school?

2) Do you follow the adopted mathematics textbook[3]?

3a) If you use it, for which reason do you follow it? and, how do you follow it?

3b) If you don't use it, why don't you use it?

4 Do you use other didactical material more than the mathematics textbook?

5a) If yes, for which aspects do you retain that it is more adequate to your needs and/or to those of your students, with respect to the adopted mathematics textbook?

5a) If no, for which reasons do you judge it in order to satisfy completely your needs?

[3]The adopted mathematical textbook is equal for all the three teachers. In particular is "Matematica.blu", Bergamini M., Trifone A., Barozzi G., Zanichelli, (2011)

6) Which view of Algebra do you want to transmit to your students?

7a) How do you introduce the study of Algebra?

8) If and when do you use the algebraic language to formalize problematic situations?

9) In particular, do you try to link the introduction of Algebra to other mathematics topics already faced in the past? If yes, how do you do it?

10) Which obstacles did you feel and do you feel as teacher in the passage from Arithmetic to Algebra?

11) Did you try to modify your didactical practices in order to face them?

12) When you have introduced Algebra, do you return to Arithmetics? If yes, in which way?

13) In particular, how do you introduce the topic of equations?

14) Do you use preparatory or complementary/transversal activities?

15) Which prerequisites related to equations, do you require to your students?

16) How to you link Algebra and, in particular, the equations to the concept of function?

18) Which relationship do you establish among examples, syntactic manipulations and theory? In this path, do you go just in one direction?

19) Do you justify the formal passages you do related to the pure technical aspects of equations?

20) Do you require the same attitude to students?

The activity in the classroom

Teachers' normal lectures in the classroom and students' group work were also videotaped. I videotaped more or less 5 lessons for each teacher and I proposed to their classrooms 3 different activities. One of them was a test and the other two were working group activities. The lessons were roughly 2 hours long. All voice and bodily movement during the classroom activities were recorded. The videos were transcribed for data analysis.

Structure of analysis

First, I considered the a-priori interview and, from what the teachers explicitly declared to me, I was able to identify some of their expectations from their teaching. In particular, I detected the expectations from the beliefs the teachers explicitly stated in the a-priori interview. Having beliefs and personal background the teachers develop expectations with respect to their students. Then, being potential expectations, I looked at what actually happened in the classroom in order to see if there was a correspondence between what the teachers stated a priori and how they actually behaved in classroom. For determining whether the expectations are actually reflected in their activity classrooms, I looked at the "emotional indicators". In other words, for determining the expectations of the teachers actually reflected in the classroom activities, I highlighted the gestures, the facial expressions, the emphasis of the words, the repetition, the rhetorical questions, the pauses, the tone of voice and so on. For example, if a teacher has a very insistent rhythm of the tone of voice in asking examples, then she actually has the expectation that students are able to make examples.
This way I outlined in the emotional orientation for each teacher. Then, in Chapter 4, I went deeper into the lessons of the teachers in order to identify the intertwinement between the rationality and the emotion. I observed the decisions of the teachers through the three components of rationality (epistemic, teleological and communicative) and, simultaneously, looking at the emotional indicators, expressions of their expectations, I was able to say something about why teachers took those decisions and not others.
In Chapter 3, I present excerpts of the lectures of the teachers in which their expectations become clear. I tried to collect several examples for each expectation for giving more reliability to my work. In Chapter 4, I analyze both some of the excerpts showed in Chapter 3 and some totally new. I chose excerpts in which it is relevant the entanglement between the rationality and the emotions of the teacher.

Typographic rules

In Chapter 3 and Chapter 4, it will be the following legend: the letter "T" is for the teacher and the letter "S" followed by a number (e.g. S1, S2,..., Sn) is for students.

3.5 Emotional orientation of Lorenza

Before entering in the specific of each expectation of the teacher, I summarize them in the following list:

1. Expectation that the authority ensures acceptability, where the word "authority" is referred to the didactical material and to Algebra as discipline.

2. Expectation that the classroom culture is valid, where for "classroom culture"[4] I mean both that constructed with her and that constructed in middle school.

3. Expectation that example are suitable for that precise context.

4. Expectation that the justifications are made through what Algebra states or the mathematical textbook state.

5. Expectation that students are able to coordinate different representation registers.

Expectation 1.

During the a-priori interview, in relation of the questions concerning the didactical material, Lorenza explicitly declares:

> "For choosing a textbook I look at the didactical procedure, I look at how it divides the topics, I look at which is the logic order. I see if the line of the treatment of the topic is more o less similar to mine, then I look at the exercises. I prefer the exercises divided with respect to the difficulty level from the simpler to the most complex (she mimes the range of the exercises). I choose on the basis of my feeling (smiling). I like my current mathematical textbook. If I have to be honest, I'm happy to use it."

> "Almost always, I make the theoretical part on the blackboard, such that (pause) students (hesitating and playing with the ring) don't, (speeding up) yes, yes, at home I look at the line chosen by the textbook and then, in classroom, I propose it (pause and again hesitating) more freely. Students find all of I say, obviously (speeding up and gesturing) I don't say strange or wrong things (smiling). They find the line on the text, yes, yes."

[4]For "classroom culture" I mean just the previous contents.

> "I and this textbook have a very similar way of doing things, I find in it things that I would say, I find them in the same order in which I propose them and, then, (speeding up) I like it, both for the theory and for the exercises. For me, it is very precise, rigorous."

> "Sometimes, I give them summaries in which I write the rules of Algebra or other important things we have already said in classroom. It is very important for me that they know these rules. Many times in their notes they are misinterpreted. They could find them also on the textbook, but they are not still autonomous to go to see them for their own."

For Lorenza, the roles of the didactical materials (e.g. the mathematics textbook) and that of Algebra as discipline are very important. In a certain way, they ensure acceptability of what she decides to do. Then, from the interview, I could infer that Lorenza has the *expectation that the authority ensures acceptability, where the word "authority" is referred to the didactical material and to Algebra as discipline.* In the excerpts below, you can see many segments of Lorenza's lessons in which one could recover some hints about this expectation.

1	T: now, I write the definition [of equation] (pause) I try to formalize well this part. Then, let's take the definition of equation (*simultaneously, she takes the mathematics textbooks and she starts to dictate from it*, Fig. 3.3). An equation is an equality where there are literal expressions for which we search for values that make true the equality (*pause*)	 Figure 3.3
2	*after few minutes*	

3	T: the solutions of an equation are called solutions of, obviously (*she looks at the textbook*) roots of the equation. You will find also this terminology on the textbook (*she looks again at the textbook to be sure*, Fig. 3.4) and sometimes I will use this terminology.	 Figure 3.4
4	*after few minutes*	
5	S4: normal form or?	(S4 is referring to the canonical form of an equation)
6	T: or (*reading from the textbook*)it is called (*pronouncing*) canonical form. You will find it also called this way and it is absolutely correct.	
7	*after few minutes*	
8	T: then, when two equations are equivalent? (*she reads from the textbook*) if they admit the same solutions set, then (*she stops to read from the textbook*) two equations in the same unknown are equivalent if they contain or admit the same solutions set (Fig. 3.5).	 Figure 3.5

At the end of the same lesson, she introduces the second principle of equivalence:

9	T: Let's see also the second principle of equivalence (*she reads from the textbook*). The second principle of equivalence says that we obtain an equivalent equation to the given one if we multiply or divide each side of the equation by the same quantity, as long as it is not equal to zero.	
10	*after few minutes*	
11	T: then, now, we see a consequence of the second principle of equivalence, it is called (*she reads from the textbook*) "rule of changing the sign" (pause). This rule says that we can change signs to both sides of the equation, right?	

Discussion

In this specific moment of the lesson, Lorenza introduces the definition of a linear equation. In general, she is used to give a definition, dictating it to the class from the mathematics textbook. Also in this case, it happens the same thing as showed in Fig. 3.3, Fig. 3.4. The mathematics textbook seems to represent for her an authority that gives acceptability and accuracy to what she is explaining. Indeed in the excerpt above, she dictates the definition of equation as the mathematics textbook proposes: "an equation is an equality where there are literal expressions for which we search for values that make true the equality" (#1). After few minutes, Lorenza wants to say to her students that there are many ways to indicate the solutions of an equation: for example, she anticipates that it will be possible that she will use the term "roots" to indicate the solutions of an equation. The interesting thing is that Lorenza refers always to the textbook as one can see in Fig. 3.4. She also continues to read from the textbook for explaining that two equations are equivalent when they have the same solution set (#8). Furthermore, she has the same behaviour when she introduces the "rule of changing the sign" as a consequence of the second principle of equivalence (#11).

In the excerpt below, Lorenza explains what the first principle of equivalence "says", reading and referring always to the mathematical textbook.

12	T: Let's start from the first, the first principle, there are two of them, ok? from the first principle will derive some calculation rules (*speeding up and sure*) that are those you apply (*gesture for miming the "mechanically"*) mechanically, (*speeding up*) you have already learnt them and from the second principle will derive some rules of calculation. What does the first principle say? it says (*in the meanwhile she is reading from the textbooks and she is dictating to her students*): given an equation if we add, oh sorry, I didn't say that this is called first side (*she circles the first side of* $ax = b$) and this one is called second side of the equation, (*speeding up*), but, probably we have already spoken about that, yes? you know that an equation, being an equality between two expressions, the expression that is on the left of the equal sign is called first side, while that on the right is called second side. (*dictating from the textbooks*) so, (*loudly*) given an equation if we add (*she mimes the two sides and then she returns on the textbooks*) a given number to the two sides	
13	S9: a given number?	
14	T: yes, (*speeding up*) then I write it in symbols, ok? now, let's try to understand	

15	T: This principle says that (*pronouncing*) if we add to both sides the same quantity or expression, the equation we obtain must result equivalent to the given one, ok? Then (*referring to the example just made*) which quantity do we decide to add? (*waiting for feedback, looking at the class*)	

Discussion

Also in the excerpt above, it can be noticed that Lorenza refers to the mathematics textbook when she has to give formal definitions. It seems that she wants to be sure that the definition is correct and, then, she dictates it from the mathematics textbook (#12). This way, she expects that students can accept the formal definition. She sees a sort of reliability in the mathematics textbook, indeed she dictates the definition of the first principle increasing the tone of voice (#12). Moreover, she personifies the "first principle of equivalence". Indeed, she uses the verb "to say" for referring to what the first principle is (#12, #15). In general, Lorenza personifies Algebra as discipline and she uses it as a possible justification for what she does. It seems that she "is giving voice" to Algebra in order to have the acceptability of what she is doing. Personifying Algebra and reading from the textbook are probably ways through which Lorenza can be more sure of the acceptability of the definition. It seems that since the first principle (an entity within the field of Algebra seen as an "authority") "says" something, then this "something" is automatically correct and acceptable.

In the lesson of the passage below she is explaining why the transportation rule is considered a consequence of the first principle of equivalence:

16	T: We change the sign. But why do we make it? Why do we change the sign? Right? Actually, it is a consequence of the first principle, because if we didn't change the sign, we would obtain an equation that isn't equivalent to the given one, right? (*smiling*). Let's consider the previous example (*catching it in the air with her hand*) $x - 2 = 8x - 4$, I have said that I carry to the first side $8x$ and I write $-8x$. Practically, (*loudly*) what did I do? I have added, let's say, to both sides the value $-8x$, then I have applied the first principle. I have applied to both sides the same value $-8x$.	
17	T: Algebra says that $+8x - 8x$ is	
18	Ss: zero	
19	T: zero (*she deletes the* $8x$ *and the* $-8x$)	

Discussion

Also in the examples she uses Algebra for justifying the operations she makes. For example, when she is solving the equation $x - 2 = 8x - 4$ she says that she adds the term 8x to both sides of the equation and then she is legitimated to delete the terms $8x$ and $-8x$, stating that "Algebra says that $+8x - 8x$ is" (#17). The verb "to say" ascribed to Algebra gives to it the guarantee of what she is doing.

These are just few examples among many others that could testify that for Lorenza it is important to consider the didactical materials and Algebra as means for having the shared acceptability of the mathematics at play within the mathematics classroom.

Expectation 2.

During the a-priori interview, in relation of the questions concerning the topic of linear equations and Algebra, Lorenza explicitly declares:

> "Usually, I begin to treat linear equations starting from their previous knowledge in order to see whether it is valid, or whether the students have misinterpreted the various procedures that they have been taught in the previous years. Anyway, I begin a new topic starting from the knowledge that the students already have"

> "At the beginning of the first year we make the review of the several numerical sets and we review in a deepen way what they know. We start with the natural numbers, we analyse them quickly because it is a set they know well from middle school and, then, from here, we reflect upon which is the operation we cannot do in that set. This way we begin to treat also the relative numbers. Students know very well also them"

> "I introduced the letter in physics, but they are able already to manage it a little bit. When we speak of direct or inverse proportionality they have to substitute the numeric value to the x. In mathematics, following the national curriculum, I introduce it when we speak of monomial or also when we speak of sets the letter represents already something for students. In mathematics I recall what we have said in physics concerning the letter."

> "[...] yes, yes, yes, I treated the topic of functions. I already use some mathematical functions also in physics, but very basic, because they serve to me to work with inverse formulas. Then, when we consider them in mathematics, I started saying "what do you remember about them?" and, from here, we arrive also to link the linear equation with the straight line"

The role of the classroom culture is very relevant in the teaching of Lorenza. As we can see from the pieces of the a-priori interview above, she starts a new topic considering valid the previous work of the students done also in middle school. Then, from what the teacher stated in the interview, I identified the *expectation of Lorenza that the classroom culture is valid, where for "classroom culture"* [5] *I mean both that constructed with her and thatpronouncing constructed in middle school.* In other words the classroom culture is the totality of mathematical ideas developed and learnt in classroom. In the excerpts below, you can see many segments of Lorenza's lessons in which this expectation is actually reflected.

[5] In this context, classroom culture concerns just previous knowledge.

20	T: before holidays, I hope that someone remembers just something, we have spoken about (*pronouncing*) identities, then is there someone who wants to give, for now, the definition of identity and to do only an example (*tone of voice of a statement and not of a question*) of identity? Don't be shy! and she smiles (*she lifts up her chin, she smiles, she is waiting for an answer, biting her lips*, Fig. 3.6, Fig. 3.7, Fig. 3.8, Fig. 3.9). Please (*referring to a student who raises up his hands*)	 Figure 3.6  Figure 3.7

21		 Figure 3.8  Figure 3.9
22	S1: it is an equality that is verified for each value that replacing to the letter	

23	T: Fine, it is an equality between two expressions that contain letters that is verified for each value we go to ascribe to the unknown. One example, we have done an example within the classical ones (*she smiles*, Fig. 3.10)	 Figure 3.10
24	S1: $(a+b)^2 = a^2 + b^2 + 2ab$	
25	T: for example, the development of a special product is an equality between two expressions that contain letters and then it can be considered an identity and each value we go to give to the unknown a or to the unknown b, the result on the left and on the right of the equality sign must be the same and, conversely, what can be considered as an equation, do you remember? (*speeding up*) you have already seen them in middle school partly, right? we have already reviewed them in physics since at the beginning of the year, they serve us for working with formula etcetera, so we have already given indications but in the light of the path we have done, any of you would like to hazard a definition of equation (*tone of voice of a statement and not of a question*, Fig. 3.11), let's try to hazard, Andrea	 Figure 3.11

26	S2: it is an equality between two literal expressions in which the value of x is replaced by a unique value to make it true	
27	T: we say that it is satisfied just f(or) (Fig. 3.12)	 Figure 3.12
28	S2: for a single value of x	

29	T: always?! (Fig. 3.13), do we always find it?(Fig. 3.14) for you this value or, let's try to think a little bit	 Figure 3.13  Figure 3.14
30	S2: sometimes it's impossible	
31	T: it could be right.	

Discussion

In this passage of the lesson, Lorenza wants to recall the concept of identity in order to introduce that of equation. Her expectation about the validity of the previous knowledge is disclosed by many indicators: she uses the verb "to hope"; she spells what she wants students recall; she requires the concept of identity of it with an affirmative tone of voice as it is quite sure that it will come naturally from the class; she smiles to put students at ease to answer and she incites them into doing ("Don't be shy"), waiting for a feedback (Fig. 3.8, Fig. 3.9) (#20). Moreover, she recalls also the knowledge they learnt in middle school. In fact, she requires the definition of equation, being sure that they have already known it in middle school. She is so sure that she uses a tone of voice proper of a statement and not of a question and she is waiting for an answer as in Fig. 3.11 (#25. Furthermore, she discusses with the class if it can exist always the value that satisfies the equation and she tries to "catch" her previous knowledge in order to answer to the problem (Fig. 3.14).

32	T: Where does the concept of equivalence come from? We have already studied it, who remembers when we have spoken of equivalence, do you remember? (*tone of voice proper of a statement not of a question*) Do you remember (Fig. 3.15) the equivalence relation, never (*she shakes her head*), never (*nervously smiling*, Fig. 3.16), we have done the relations, do you remember? We have defined the equivalence relations, those of admitted	 Figure 3.15  Figure 3.16
33	S19: S19: those symmetric	
34	T: yes	
35	S19: reflexivity, symmetry, transitivity	

36	T: Has still meaning to link that concept to the equivalent equations, for you?! (*rhetorical question*). (*nervously*) If we take into account two equations and we suppose that they are equivalent, the property, that is reflexivity, every equation is equivalent to itself, (*rhetorical question*) ok?	
37	S3: yes	
38	T: sure, because it admits the same solution, (*rhetorical question*) symmetric?	
39	S4: yes	
40	T: if an equation is equivalent to a second one, then the second one is equivalent to the first, because it admits the same solution, yes! (*rhetorical question*) transitivity?	
41	S5: no	
42	T: (*disappointed*) why not? If this (*she is pointing the example done in the previous session of the lesson*) is equivalent to a second one and the second is equivalent to a third one also the first and the third are equivalent each other, you see that the concept of equivalence relation returns, right? because actually the relation among equivalent equations gives us an equivalence relation among the different equations, (*annoyed*) it's nothing of new.	

Discussion

In the excerpt above she wants to link the equivalence relation with the equivalent equations, then she needs that students remember what is an equivalence relation. It seems that she is a little bit nervous when, miming the past (Fig. 3.15), none gives feedback to her request. Indeed she says "never" shaking her head and nervously smiles (Fig. 3.16). It seems that she is expecting that students remember the equivalence relation in order to relate it to equivalent equations (#32). After students recall the properties of equivalence relation applied to the relation among equivalent equations, it seems disappointed when a student responds “no” concerning the property of transitivity (#42) and she explicitly states that all of this knowledge “is nothing of new” (#42).

43	T: we change the sign. But why do we make it? Why do we change the sign? Right? Actually, it is a consequence of the first principle, because if we didn't change the sign, we would obtain an equation that isn't equivalent to the given one, right? (*smiling*). Let's consider the previous example (*catching it in the air with her hand*) $x-2=8x-4$, I have said that I carry to the first side $8x$ and I write $-8x$. Practically, (*loudly*) what did I do? I have added, let's say, to both sides the value $-8x$, then I have applied the first principle. I have applied to both sides the same value $-8x$.	
44	T: Algebra says that $+8x-8x$ is	
45	Ss: zero	
46	T: zero (*she deletes the* $8x$ *and the* $-8x$), and then, actually, skipping this intermediate passage, I have carried the value, the term $8x$ that was on the right of the equal sign to the first member that is on the left of the equal sign. You apply this rule mechanically because you have already learnt it, (*rhetorical question*) right? (*miming the past with her hand*) (*pronouncing*) in middle school almost of you (*gesture for encompassing all go her students*), (*rhetorical question*) right?	
47 l	S3: I didn't see it	

48	T: (*imperative tone of voice*) Who says "no"?	
49	S3: I didn't know it	
50	T: never, (*quite astonished*) you never do that!	
51	S3: Yes, I did it, but I didn't know this passage	
52	T: (*astonished again*) The fact of transporting a term from one side to the other one, changing the sign?!	

Discussion

In this passage of the lesson, Lorenza shows to her student that the well-known "transportation rule" can be seen as a consequence of the first principle of equivalence. Hence, she needs and hopes that students remember the "transportation rule" indeed she makes rhetorical questions to her students to verify it (#46). Moreover, she remains astonished when a student seems to deny to know the "transportation rule" (#50, #52).

Expectation 3

In the a-priori interview, concerning the role of the examples, Lorenza explicitly declares:

> "(self confident tone of voice) sometimes we start from the example, (speeding up) rather very often because from the example students have immediately the perception of where we want to arrive. They understand it and they see it with their own eyes; hardly I start from the theory, usually I start from an example or a situation that is relevant in order to arrive where we want to go."
>
> "when students make errors in procedures of algebraic calculus as the simplification in the algebraic fraction, I consider examples to clarify. Can I make an example?[referring to the interviewer] for example, in the fraction $\frac{(2+a)+3}{(2+a)}$ students simplify $(2+a)$ up and down. Then, I make the numerical example. If we have 5 instead of $(2+a)$, we would obtain $\frac{8}{5}$. If, incorrectly, I simplify 5 up and down, I would obtain 3. Is it true that $\frac{8}{5}$ is equal to 3?"

During her lessons, many times Lorenza constructs suitable examples in order to improve the understanding of her students. She explicitly declares in her a priori interview that many times she starts a new mathematical topic from an example. Doing this Lorenza thinks of the fact that “students have immediately the perception of where we want to arrive, at least they understand and they see it with their own eyes”. From her beliefs, it seems that Lorenza has *the expectation that examples are suitable for that precise context*. On the basis of this expectation, Lorenza makes many examples during her explanations, especially when she wants to highlight a difference between two cases (like for example the difference between an indeterminate equation and an impossible one). In the excerpts below there are some evidences of this possible expectation of Lorenza.

53	T: What do you have in mind when we speak of equation?	
54	S3: uhm	
55	T: one randomly	
56	S4: $5x + 2 = 10$	

57	T: equal to 10 (*she writes on the blackboard this equation*), this is an equality in which it appears the unknown. It is not an identity. Why is it not an identity? because for example (Fig. 3.17) if I give to x (Fig. 3.18) the value 1, what do I obtain? If x is equal to 1 I would obtain 5 times 1 plus 2; is it equal to 10? no, because I obtain 7 equal to 10. Obviously, for the value 1 it is not satisfied, then it is not an identity because we have said that identities are satisfied for any numeric values (Fig. 3.19) we give to the letter, then for which value will it be satisfied? let's think a little bit	 Figure 3.17  Figure 3.18  Figure 3.19
58	S6: $\frac{8}{5}$	

59	T: then surely it is not an integer, then there is not a single integer value of x such that multiplied by 5 and added to 2 is equal to 10 and, then, at which value do we think? someone said $\frac{8}{5}$, then if x is $\frac{8}{5}$ we would have $5 \cdot \frac{8}{5} + 2 = 10$ and, then, yes it is right. Are there other values?	
60	S3: no	
61	T: then, (*self-confident tone of voice*) this one is an equation in the sense we have just defined that is satisfied for the unique value equal to $\frac{8}{5}$.	

Discussion

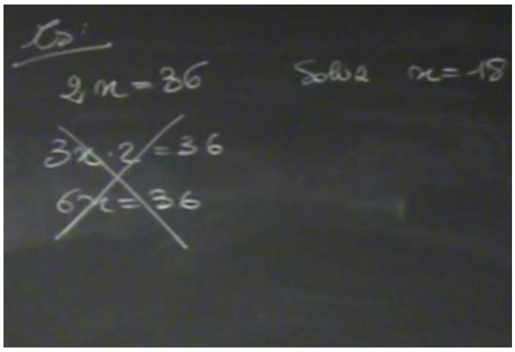
Lorenza asks to the class to construct an adapted example of an equation. A student suggests $5x + 2 = 10$. The teacher "catches" this suggestion to show that, surely, this is not an identity. It seems that she is quite confident in working on example, indeed she tries first to substitute the value 1 (Fig. 3.18) and, then, she generalizes for any numeric values (Fig. 3.19) (#57). At the end, with a self-confident tone of voice she declares that, actually, that examples is an equation.

62	T: ok, then the solution, the root or the solution is for x equal to 18, now we try to invent, to create another equation that has as solution 18	
63	S7: $3x$ times 2	

<table>
<tr>
<td>64</td>
<td>T: What does it mean? (Fig. 3.20) $3x$ times 2 and stop (Fig. 3.21) where is the equation (Fig. 3.22, Fig. 3.23)?</td>
<td>
Figure 3.20

Figure 3.21</td>
</tr>
</table>

65		 Figure 3.22  Figure 3.23
66	S7: equal to 36	

67	T: equal to 36 (Fig. 3.24)	 Figure 3.24
68	S6: or $6x = 36$	
69	T: How does $6x = 36$ have the same solution? 6 times 6 is 36, then this one is not equivalent to the starting one, it doesn't have the same solutions set. We have already said that they are equivalent if they have the same solutions set, namely if they are satisfied for the same value we give to the x, right? This not (*she puts a cross on* $6x = 36$, Fig. 3.25)	 Figure 3.25

70	S10: $x - 18$ (*facial expression of Lorenza*, Fig. 3.26)	 Figure 3.26
71	S11: x divided by 2 equal to 9	
72	T: x divided by 2 equal to 9, then this yes because if we substitute instead of x the value 18, 18 divided by 2 is 9, but there are also some others, hence there is a family of equations, for example, Alberto	
73	S12: $x + 2 = 20$	
74	T: $x + 2 = 20$, also this one is another equation equivalent to the others, then we have understood what is the concept, right? two or more equations are equivalent if they admit the same solution, right?	
75	S12: Yes	

Discussion

In this segment of a lesson, Lorenza prompts the attention of the students on the meaning of equivalent equations: starting from a particular equation she asks to her students to create an equation equivalent to the given one. The interesting thing is that, through this way, students grasp the fact that there exist infinite equations equivalent to the starting one. The teacher hopes and needs that students construct suitable examples, namely examples of equivalent equations to the starting one that has as solution the value 18. In fact, when a student says "$3x$ times 2", she is disturbed remaining as in Fig.

3.21. Then she asks for explanation to the students in a very nervous way, as testified from the Fig. 3.22 (#64). Furthermore, when the same student adds "equal to 36" (#66), she looks at him in a very disappointed way as revealed by the Fig. 3.24. Then she explains that $6x = 36$ cannot as solution 18 and she puts a big cross on it to underline that is wrong (#69).

Expectation 4

In the a-priori interview, at the question about the role of justification, Lorenza answers as in the first excerpt below:

> "At the beginning, let's say yes, often I make an example and on it I ask what we have done, which principle we have applied, namely I resume well all the points. Then, after working a little bit on this things, I go more quickly in order to conduct them to an autonomy."

Then the interviewer asks to her: "For the students, is it difficult to explicit always what principle they applied on it?" and Lorenza answers as you can see below.

> "Sometimes yes, because they don't study a lot. Hence, they are not able to justify associating the [technical] passage to the rule they have used. They solve equations mechanically without recall what they have learnt in Algebra. Hence, they make errors, sometimes it happens that they write $x = 3$ or $x = -3$ for the equation $3x = 0$, because they don't justify with what the second principle says. Instead, if they know that the rule says that if I divide to multiply, namely if they understand what the rule says, the justification comes mechanically but correct."

From the passages of interview above, it seems that Lorenza has the *expectation that justifications are made through what Algebra or the mathematical textbook state*. Indeed, she declares that students "are not able to justify associating the [technical] passage to the rule they have used", but "if they know that the rule says that if I divide to multiply, namely if they understand what the rule says, the justification comes mechanically but correct". It seems that, for the teacher, students can justify well if they are understood what algebraic rules state. In addition, it seems that for the teacher understanding a rule means to understand *what* it "orders" to do technically and not *why* it actually works. The excerpts of lessons below are examples in which we can see reflected this expectation.

76	T: we change the sign. But why do we make it? Why do we change the sign? Right? Actually, it is a consequence of the first principle, because if we didn't change the sign, we would obtain an equation that isn't equivalent to the given one, right? (*smiling*). Let's consider the previous example (*catching it in the air with her hand*) $x-2 = 8x-4$, I have said that I carry to the first side $8x$ and I write $-8x$. Practically, (*loudly*) what did I do? I have added, let's say, to both sides the value $-8x$, then I have applied the first principle. I have applied to both sides the same value $-8x$.	
77	T: Algebra says that $+8x - 8x$ is	
78	Ss: zero	
79	T: zero (*she deletes the* $8x$ *and the* $-8x$), and then, actually, skipping this intermediate passage, I have carried the value, the term $8x$ that was on the right of the equal sign to the first member that is on the left of the equal sign. You apply this rule mechanically because you have already learnt it, [rhetorical question] right? (miming the past with her hand) [pronouncing] in middle school almost of you (*gesture for encompassing all go her students*), (*rhetorical question)* right?	

Discussion

In the excerpt above, Lorenza is explaining why the "transportation rule"[6] is a consequence of the first principle of equivalence. In particular, she justifies

[6] "In an equation, if you carry a term from one side to the other one you have to change the sign of it."

that the transportation rule is a consequence of the first property because if we don't change the sign, we would not to obtain an equivalent equation. It seems a justification made by what Algebra and, in particular, the first principle of equivalence, states. It is quite clear her need that Algebra can be seen as an authority that, implicitly, justifies what they are doing. In fact she uses many rhetorical questions and she smiles being sure of receiving consensus from students given that it is difficult that they go against what Algebra says (#76).
Moreover, in the resolution of the equation she explicitly says "Algebra says $+8x - 8x$ is", personifying Algebra. Given Algebra "says" that $+8x - 8x = 0$, she uses this fact to justify the direct use of the transportation rule skipping the first principle of equivalence, namely, skipping the fact of adding $-8x$ to both sides of the equation (#79).

80	T: An equation is said to be of first degree, of second degree, of third degree depending (Fig. 3.27) on whether the letter or letters that appear, we have to calculate the maximum degree of the monomial that constitutes the equation, if we consider an equation of this type (*she pointing the previous example* $5x + 1 = 10$), what degree will it have?	 Figure 3.27  Figure 3.28
81	Ss: 1	
82	T: degree 1 because the unknown appears of first degree, if we consider, I don't know, this one (*she writes* $3x^2 + 8x + 5 = 0$)	
83	S7: 2	

84	T: why 2?	
85	S3:1	
86	T: you have to think to the degree of the polynomial (*referring to S7*), how did you determine it? in your textbook the definition of the degree of a polynomial states that it was the maximum degree of the monomial that was a term of the polynomial, the maximum degree is 2 (Fig. 3.29), so it's 2. If we had $3a^3x^2 + 2xa + 5$, in this case you should have seen with respect to which letter we want to compute the degree, if I ask you what is the degree with respect to the letter a, you tell me	 Figure 3.29
87	Ss:3	
88	T: 3, What is that with respect to the letter x?	
89	Ss: 2	
90	T: you tell me 2, then what is the global degree?	
91	Ss: 5	
92	T: you tell me fifth, however we will treat first degree equations, that is those in which the unknown will appear with exponent 1. Is it all clear? Is it simple, right? we partly have already seen first degree equations. (*highest pitch*) It is important that you are able to justify through what we have already studied on the textbook.	

Discussion

In the excerpt above, Lorenza discusses with her class which are the degrees of the equations $5x+1 = 10$ and $3x^2+8x+5 = 0$ and that of this polynomial $3a^3x^2 + 2xa + 5$. Lorenza justifies the degrees of the equations and of the polynomial recalling the definition of degree of a polynomial written on the mathematical textbook (Fig. 3.28). After recalling that, she affirms that the degree of $3x^2 + 8x + 5 = 0$ is 2 with an emblematic facial expression (3.29). She seems that thinks that for students is obvious that the degree of that equation is 2 because this answers fits perfectly with the definition of the textbook. She is self-confident that students have accepted this justification, in fact she makes many rhetorical questions (#92: "Is it all clear? Is it simple, right?"). At the end she stresses with highest pitch that it is important that students justify what they do through what they have already studied on the textbook.

Expectation 5

From the a-priori interview, Lorenza explicitly declares:

> When I have the possibility of teaching within the class the disciplines of both Mathematics and Physics, I try always of making links and of coordinating the different registers of representation. Actually, I do that also when I teach just one of the two disciplines, but in that cases I believe that my interventions result less efficient. Often, the students reason through "compartments" (they are narrow-minded), they reason through "drawers". Many times, they say the statement:"but we are doing Physics, not Mathematics" and vice versa. Then, the care and the attention in underlying parallelisms between the two disciplines it is always constant in my lessons in order that students overcome this limitation and understand how one discipline attends upon the other one and how a same topic (for example the straight line) can be represented in several and variegated applications, but it's always a matter of a straight line!

From what she stated in the interview, it seems that Lorenza has *Expectation that students are able to coordinate different representation registers*. On the basis of this expectation, Lorenza makes many coordinations among the different registers, especially when she wants to introduce the geometrical interpretation of the linear equations. In the excerpts below there are some evidences that confirm this potential expectation of the teacher.

93	T: both k and y_0 were numbers, real, that represented something. Do you remember (posture as in Fig. 3.30)? It is just to make (*gesture to mime the "box" of the class culture*) the review of what we have already known, uhm (ok?)? k represents	 Figure 3.30
94	S1: proportionality	

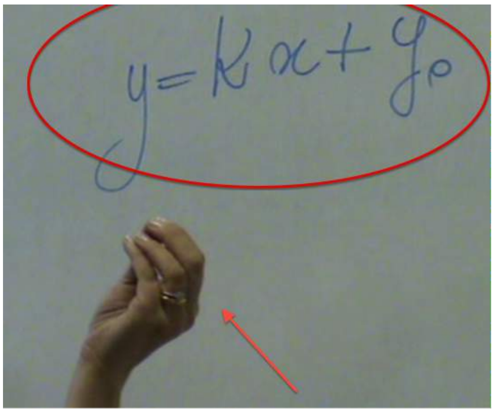
<table>
<tr>
<td>95</td>
<td>T: (gesture as in Fig. 3.31, Fig. 3.32, Fig. 3.33 and insistent rhythm) But, how did we repre(sent it), dra(w), did we draw a sketch?</td>
<td>


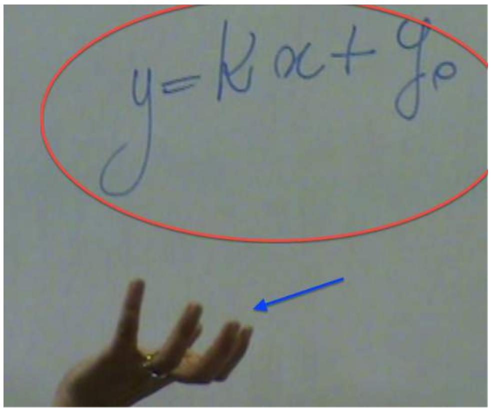
Figure 3.31


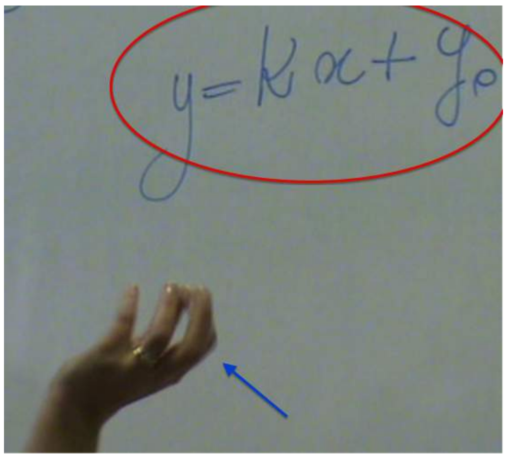
Figure 3.32


Figure 3.33
</td>
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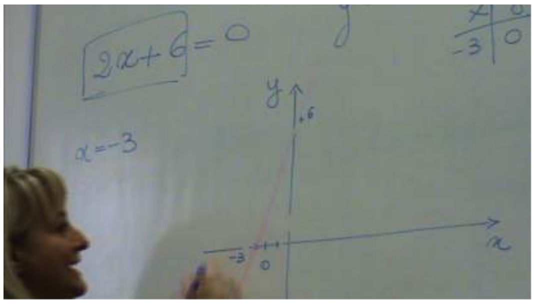
</table>

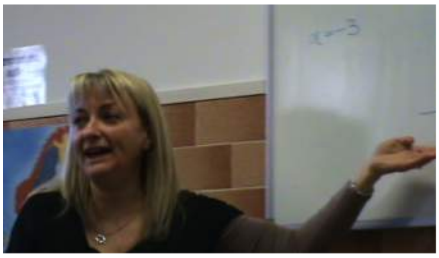
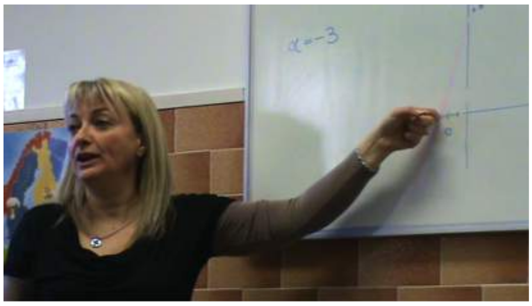
96	S3: a straight line that passes...	
97	T: that is (*nodding*), it was a straight line, it was a random straight line of the plane and then k and y_0 gave us particular values of the straight line, so, Riccardo	
98	R: k is the gradient	
99	T: it is called gradient, but what does it mean in short? (*facial expression in* Fig. 3.34), so, Riccardo	 Figure 3.34
100	S2: the slope (S2 mimes the slope, Fig. 3.35)	 Figure 3.35

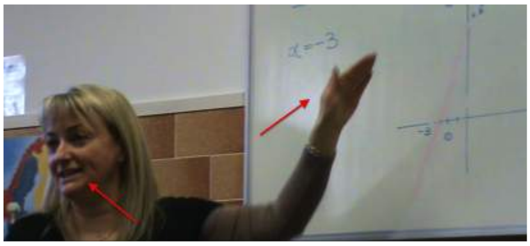
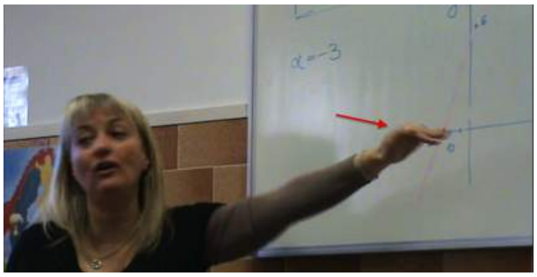
101	T: the slope with respect to the x-axis and y_0 gave us another information	
102	R: the intersection point between the x-axis	
103	T: y_0 was the point that corresponded to the intersection of the straight line and the y-axis.	

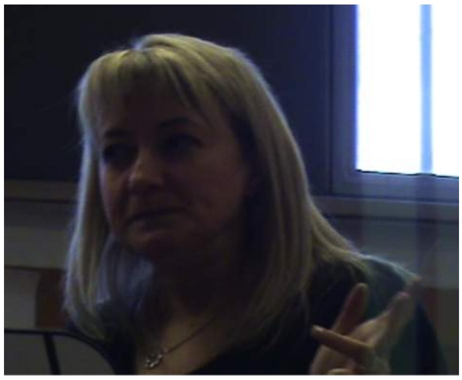
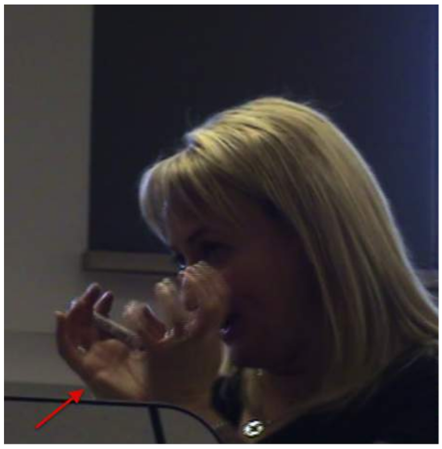
Discussion

Lorenza wants to coordinate what the letters in the equation of a straight line represents on the graph. She is expecting that students are able to do this coordination because, first, they have already spoken about it in physics and, then, because she thinks to having insisted on it. Indeed, she nervously moves her hand to coordinate the algebraic equation of a straight line with its graph (Fig. 3.31, Fig. 3.32, Fig. 3.33). In particular, she is waiting for a feedback as in Fig. 3.34 when she asks what represents k on the graph.

104	(*increasing the tone of voice*) then, the value $x = -3$, namely the solution of this equation, what does it represent (*pronouncing*) geometrically, on the cartesian plane? Does some of you see it? understand it?	 Figure 3.36
105	noise	

106	T: some of you have any idea? it represents (*smiling and pointing to the cartesian plane*, Fig. 3.37), (*pointing with the knuckle to* $x = -3$ *on the graph*, Fig. 3.38) the abscissa of the point [pauses] of intersection between the straight line (*smiling, miming the straight line, pausing*, Fig. 3.39) with the x-axis (*she mimes the x-axis*, Fig. 3.40)	 Figure 3.37  Figure 3.38

107		 Figure 3.39  Figure 3.40
108	S3: x	

109	T: it represents the point of the straight line that has 0 as ordinate, ok? (*she stops and she remains as in* Fig. 3.41) are you able to see it? (Fig. 3.42)	 Figure 3.41  Figure 3.42
110	noise	
111	T: hence the solution of the associated equation is the value of the abscissa of the intersection point between the straight line and the x-axis, (*tone of voice of a statement*) do you understand?	
112	S7: yes	

Discussion

In the excerpt above, Lorenza wants to show that the solution of the equation $2x + 6 = 0$ is the abscissa of intersection point between the straight line $y = 2x + 6$ and the x-axis. She increases the tone of voice to draw the whole the attention of the class on the example (#104). Then she asks on that example if someone has some ideas of what it represents the solution of the equation. She smiles because probably she hopes that students easily see what is the solution of the equation on the graph, as shown in Fig. 3.37. Moreover, when none answers she responds herself (#109) and, then, she stops and she is waiting for a consensus remaining as in Fig. 3.41.

3.6 Emotional orientation of Carla

Carla declared many expectations that could be seen in many passages of her lessons. Before entering in the specific of each expectation of Carla, I summarize them in the list below:

1. Expectation of constructing new knowledge from what has been already done in the classroom

2. Expectation that the justifications are necessary to give sense to what she and her students do

3. Expectation that also her students feel the need of justification to give sense to what they do

4. Expectation that students see analogies

5. Expectation that students learn to use the mathematical textbook in a critical way [7], referring to other didactical materials when it's necessary [8], underlying analogies and differences.

6. Expectation that students are able to make examples, because she thinks that examples are an useful tool to construct procedures or to review properties.

[7] to interpret correctly the definitions, to reflect upon its examples and so on
[8] e.g. the worksheets she prepares for the class

Expectation 1.

During the interview, to the question: "Which obstacles did you feel and do you feel as teacher in the passage from Arithmetic to Algebra?", Carla answered:

> "For me (*sighing*), the greatest problem I'm trying to solve-I realised and it's becoming dramatic in the last years-is the problem of the stability of knowledge, in that I feel that in many classrooms, (*speeding up*) a part from the good ones, students don't remember what we did and, for me, this is serious. That is, for example, in grade 10, I would like to refer to something that I did before, on which I have even insisted, without having to repeat it entirely (...) the big problem to solve, in that I persist a lot, is being able to find a way for constructing a core (*miming a base with her hand*), a base of knowledge (*miming a list with her open hand*), of abilities that stay. For me, aside from time economy-'cause, maybe, it's a bit annoying having always to recall-it's really a matter that has to do with cognitive science, I don't know, I wouldn't know how to face it, but it's becoming a generalized problem, then (.) we should look for (...) the problem is looking for meaningful activities that allow (...) fixing things."

For Carla, classroom culture is very important for having knowledge stability. On this belief she develops her *expectation of constructing new knowledge from what has been already done in the classroom.* The expectation determines choices, for example when she introduces the "properties of linear equations" by calling back the "leggi di monotonia" for equalities thpronouncingat were explained at the start of the year (laws, related to the substitution property, according to which adding/subtracting the same number to, or multiplying/dividing by it, both sides of an equality does not change the equality). For each new topic that she introduced in the classroom, Carla gives to the students worksheets that they have to fill in the blank spaces. All the times these worksheets don't have the title, because Carla wanted that it is given by the students at the end of activity, when they become aware of the aim of that worksheet. In all of them, both the theoretical and the practical part are interlocked and Carla requests her students to explain their reasoning after all of the activities. An example of these worksheets is illustrated in 4.84 in which she wanted to recall the "leggi di monotonia" for equalities in order to construct the "properties of linear equations". The transcription of the part of the lesson, in which they have already finished the activity and they have to put the title to the worksheet, is in the following table.

113	T: At the beginning, we said that there is no title (on the Worksheet 4.84) and we will put it at the end. Hence, how could we title the worksheet? (highest pitch) let's think of that, ok? (*pause*) For the moment, I don't say anymore eh? (*she frowns and nods looking at the class*, Fig. 3.43) (.) but this question has the aim to review some properties that we have already meet, let's try a bit to think of them, (*highest pitch*) ok?	 Figure 3.43
114	*Noise in the classroom*	
115	Do you hear my suggestion? I have already said it, but these questions have the aim to recall some properties of equalities that we (*self confident tone of voice*) have already meet. If you reflect upon them you should also understand which properties are	
116	S4: the transportation rule	
117	T: I never heard the transportation rule (annoyed). I never heard the transportation rule. (pause) Hence, do you come to mind which properties I'm referring to? (*after 10 minutes none have still answered*) So, have you finished the activity? Did you come to mind which these properties are? They are properties that people call in different manner, but you textbooks calls them "leggi di monotonia"	
118	T: Hence, let's quickly recall them, if it is given a true equality we will obtain again a true equality, doing what?	

119	*students answer*	
120	T: How do the laws for equalities translate into properties for equations? (*forward-facing, with a hand on the desk, and raising her eyebrows*) What can you say? (pause) That, if you have an equation, right? What do you do?	
121	S3: If we multiply or divide both sides of an equation by the same value, we will get an equivalent equation, yeah	
122	T: Then, we say: For the first law, given an equation, if we add the same number to both sides or if we subtract (...) Remember how addition is defined, it means to sum the opposite, right? So we can speak of sum. Then, if we sum both sides of the equation (*miming them with both hands*) we get (*looking at the students and nodding, waiting for them to speak*)	
123	S6: An equivalent equation	
124	T: An equation equivalent (*nodding*) to the given one. Instead, for the second law (*nodding and biting her close lips, gesturing a fist in the air*; Fig. 3.44)	 Figure 3.44

125	S3: if we multiply or divide (*Carla nods, remaining with lips as in* Fig. 3.44)	
126	S5: by a number not equal to zero (*Carla nods, remaining with lips as in* Fig. 3.44)	
127	S7: both sides (*Carla nods, remaining with lips as in* Fig. 3.44)	
128	S3: we obtain an equivalent equation to the given one (*Carla nods, remaining with lips as in* Fig. 3.44)	
129	T: (*nodding*) Do you all agree?	
130	Ss: Yeah!	

Discussion

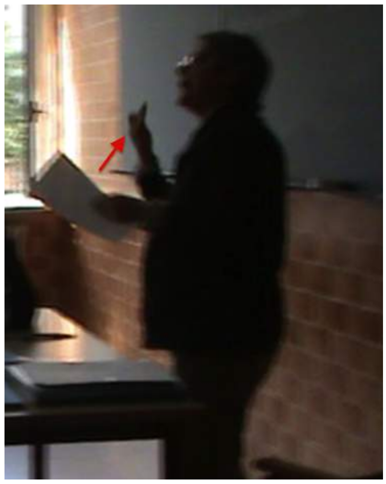
As we underlined above, Carla's expectation of constructing new knowledge starting from the previous one that students developed with her, determines the choices of introducing the "properties of linear equations" by calling back the "leggi di monotonia" for equalities that were explained at the start of the year and she does it through a worksheet that students have to complete. The expectation of Carla is visible in many passages of her lessons: for example, she makes many times questions and, often, the same questions waiting that students answer to them. In different moments, she uses the exclamation "right?", because she expects that they remember what they did in the lessons before. In the specific of the transcription, when she says "let's think of that, eh?" (#113), she expects that students remember the "leggi di monotonia" for equalities that they have seen the previous lesson. Furthermore, when she says "For the moment, I don't say anymore eh?" (#113), it is interesting the facial expression that accompanies this phrase, that is she frowns and nods looking at the class (Fig. 3.43). She seems to be sure that students understood what she wants to recall. Many times, Carla says "ok" speaking up and this seems to be a manner of encourage the students to think of, in this case, these properties of equalities and then to answer to her question. This can be noticed when she says "but this question has the aim to review some properties that we have already meet, let's try a bit to think of them, ok?" (#113)
Moreover, Carla is used to ask rhetorical questions because she expect that what she said is "ascertained" for the students. For example she says "Do you hear my suggestion?" (115): this is a rhetorical question because it is

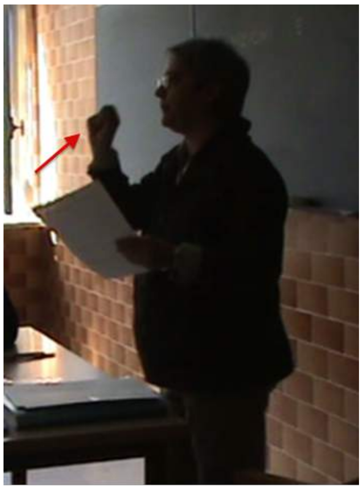
obvious that all of the students in the classroom has heard her suggestion (also because she repeats it many times), but maybe she hopes that someone says something about the "leggi di monotonia" that they already known. The students don't answer to her question, but the interesting thing is that she doesn't directly answer to the question, but she speaks about "some properties of equalities" (#115) without specifying the exact name. This seem to be another attempt of Carla to receive the answer from her students. Moreover, she invites them explicitly to "reflect upon them". A student answers the "transportation rule" that is a rule mathematically incorrect[9], that many student have learnt in the previous years of school. A clue that the expectation of Carla is to recall the "leggi di monotonia" in order to construct the properties of equation, she answers to this student in a very irritating manner, first because she doesn't expect that type of answer and second because she never speak of this "transportation rule", in fact she says "I never heard the transportation rule". For her it is mathematically incorrect (see the Footnote 10).
A t the end, she has to answer, but she specifies that she recall them "quickly". The use of this adverb seems to indicate the fact that, even if the students don't immediately remember them, it is sufficient for them to recall the name in order to come to mind all of discourses about them. When she decides to recall explicitly the "leggi di monotonia" she prompts however the attention of the students on the new topic that is the linear equations, in fact she asks to the classroom "What can you say?" and the she stops with the expectation that someone answers. Then she continues the discourse considering the equation in order to adapt the "leggi di monotonia" for the equalities to the equation. Indeed, she says "That, if you have an equation, right?" (#120), using the adverb right because she seems at least a bit sure that they are able to adapt what they did before to the equation. A student answers to her and then she summarizes both laws with very impressive facial expression as you can see in Fig. 3.44. When she expects something from her students she is used to look at the students and nod, waiting for them to speak. Moreover she makes many gestures that seem to be linked to her expectation of the fact that students can construct/adapt "old" knowledge to the new one, in fact when she recalled the second law she nods and bits her close lips, expecting the answer from the students and she mimes the classroom culture in her fist . At the end of the summarizing and at the same time adapting the "leggi

[9]The transportation rule can be seen a consequence of the principle that is that if we add the same number to an equation, we will obtain an equivalent equation. The transportation rule doesn't specify why one can transport a term from one side of the equation to the other one. It seems to be like a trick, in which it doesn't explain the fact that a term "disappear" in one side because we add that term to both sides.

di monotonia" to the equations in order to construct the principles of linear equation, she says "Do you all agree?" (#129), nodding. The fact that she nods seems to indicate the fact that she is a bit sure that the construction of the principle of equivalence is clear for all the students in the classroom.

131	T: that (*pronouncing*) must be true or false, then I have called propositions those that were in the first part of the activity then we have seen that the elementary propositions have a predicate and here the predicate is "to be equal to" and, every day, we have to do with propositions of this type and they are called (*gesture and facial expression*, Fig. 3.45)	 Figure 3.45
132	S3: equations	
133	T: no, there is also written here. how are they called?	
134	S8: open statements	

135	T: (*irritated*) we are still at the beginning (*miming the past*, Fig. 3.46), those propositions (Fig. 3.47: *irritated gesture for recalling just those propositions*)	 Figure 3.46  Figure 3.47
136	S7: equalities	

137	T: equalities. the equalities are (*gesture*: Fig. 3.48) particular propositions and as for all the propositions will can be true or false and within those I proposed what did you say?	 Figure 3.48

Discussion

Carla wants to recall equalities to remind to students that they are open statements. In this way, she can introduce equations as open statements. She needs that students remember equalities, indeed she ask to her students about them with the open hand as to expect a feedback from them (Fig. 3.45). Moreover, she hopes that they remember it, indeed when a student answers “open statements” instead of “equalities”, she seems very irritated for the answer (#135). Instead, when a student answers “equalities” she specifying with finger his answer to underline the correctness of it (#137).

138	T: then, we considered open statements. In general what is an open statement? (*pause, gesture:* Fig. 3.49)	 Figure 3.49
139	S10: a proposition that	

140	T: it is a phrase in which it appears a variable for which, first of all, we have to precise (*pause and waiting* Fig. 3.50) what is the universe set, right? try to answer to the questions I make in order to find of (Fig. 3.51) summarizing. Can we say for an open statement if it is true or false?	 Figure 3.50  Figure 3.51
141	Ss: no, we can say	
142	T: we can ask ourselves?	
143	S19: what is the truth set	
144	T: (*nodding*) in general, what do we call truth set?	
145	S3: solution	
146	S7: the set of solutions	

147	T: (*nodding*) in general, the truth set of an open statement is constituted by all the elements of the universe set (pause, she is waiting with arms on her hip, Fig. 3.52)	 Figure 3.52
148	S5: that make true the statement	
149	T: that make true it, then the equations, and this fact had to emerge here, are particular open statements in which the predicate is “to be equal to”, then, as for all the open statements, of an equation (*raising her eyebrows*), given obviously the universe set, we ask ourselves what is the truth set, that we can continue to call this way, but for tradition it is called set of solutions, and an element of it, it is called solution.	

Discussion

Carla recalls the concept of open statements to introduce that of equation. She hopes to have in her fist the knowledge of students, as shown in Fig. 3.49. She needs that students say what is an open statement, indeed she pauses and she remains as in Fig. 3.50 waiting for an answer. The teacher has the same attitude when she asks how is the truth set of an open statement. In particular, she remains with her arms on hip, waiting for an answer (Fig. 3.52). At the end, she hopes that all of the students remember the concept of open statements and she tries, raising eyebrows, to introduce for the first time the definition of equation.

Expectation 2.

During the a-priori interview, concerning how she introduces Algebra, Carla explicitly says:

> "(gesture with joined fingers and she beats on the table) I insist very much on why the letters instead of numbers. I use an activity of M@t.abel[10] on the use of letters to prove. It was a thing that (smiling and gesturing) homemade I tried to do always. It is always a discourse to give sense, a meaning to that we are doing."
>
> "often I justify the use of letters with the fact that we work with infinite sets. I try to let them understand (smiling) that if they should make (gesture to indicate many) all of the examples with all numbers with three digits, ok? then to justify the advantage of using letters and to require it also from them."
>
> "I justify always the use of letters, I anticipate it a lot, for example when we speak of problems I say that with letter we can solve a class of problem, instead of solving a single numerical problem [...] When I treat the resolution of equations, I signal and I am very careful to the conscious use of the principle of equivalence. I care a lot about that it is not a mechanical thing."

For Carla the aspect of the justification is very significant, indeed from the interview it seems that Carla has the *expectation that the justifications are necessary to give sense to what she and her students do*. For this reason, first she is very careful to justify all of what she does [11].

---

[10]M@t.abel is a national teacher education programme. I will speak of it in the next section.

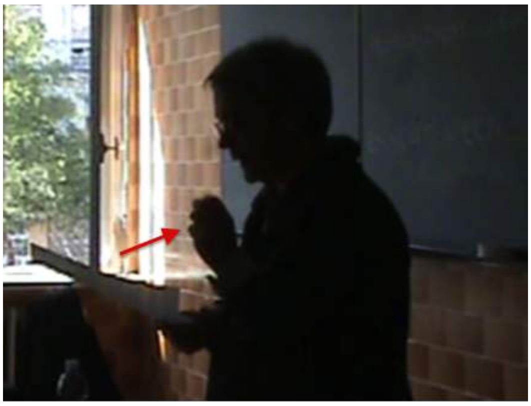
[11]The next expectation concerns the fact that she expects the same attitude from students

150	T: equivalent (*nodding*), let's remember this definition, when do we say that two open statements are equivalent, (*pronouncing*) within a given universe set, we underline it (Fig. 3.53)?	 Figure 3.53
151	S2: they are the same	
152	S3: when they have the same truth values set	
153	T: when they have the same truth value set and why, we have seen it in the second part of the activity (4.83), (*highest pitch, justification*) why is it important to specify (*pronouncing*) "within a given universe set"?	
154	S4: the truth set can change with respect to the universe set	

155	T: in the second part of the activity, we have verified it on some examples and that one was the conclusion that we have to write at the end that two open statements can be equivalent in a given universe set, but not in another one, this means that when we speak of (*pronouncing and highest pitch*) equivalent open statements we to specify within what set. Then, also today we work, you will work on worksheet that I have prepared and also here, as you will see, the worksheet has not title, and, at the end of the activity, you all together we will title the worksheet.	

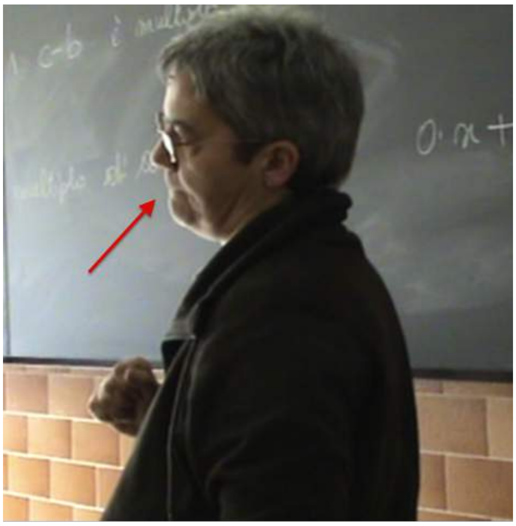
Discussion

Carla explains in a very accurate way when open statements are equivalent to each other. She justifies this definition, pointing out the fact that the solution set of an equation depends on the universe set. For example, she specifying the importance of the universe set with a gesture as in Fig. 3.53 (#168) and, then, she spells "within a given universe set" (#153). At the end of this brief segment, she repeats why it is important to explicit the universe set and she explains the reason increasing the tone of voice and pronouncing as to underline another time this fact (#155).

156	T: the unique way for seeing who is right is trying to substitute. Try to substitute, then pay attention, I have said you and this fact has to be always keep in mind, you have to (*posture as in* Fig. 3.54, *then, she inclines her body towards the class*) give a (*pronouncing and highest pitch*) sense of what you read, then solving the equation $(x+1)^2 = 81$ means asking ourselves if it exist a value x such that doing $x+1$ and squaring it is 81, working in (*highest pitch and raining eyebrows*) $\mathbb{Z}$, this should make understand to you that (Fig. 3.55, *indicating* $x+1$) $x+1$, (*she inclines her body towards the class*) how it has to be?	 Figure 3.54  Figure 3.55
157	Ss: 9 o -9	

Discussion

Carla insists many time on the giving sense to what students do. For example, in this brief excerpt, she increases the tone of voice pronouncing “sense”, underlying with gesture in Fig. 3.54 this fact (#156). Moreover, she shows students an example of she is saying. Solving the equation $(x+1)^2 = 81$ in $\mathbb{Z}$ means asking ourselves if it exist a value x such that doing $x+1$ and squaring it is 81. She raises eyebrows for the universe set because it is a determinant element in the searching of the meaning of that equation. Changing it, the solution set can vary.

158	T: the hours of today and those of tomorrow are very important because now we have developed the topic of equations and obviously it is trivial to solve equations. Why do we give so much importance to equations? (Fig. 3.56, *she is waiting for an answer*)	 Figure 3.56
159	Ss: noise	

160	T: They are a (*raising eyebrows*) fundamental tool for solving problems (Fig. 3.57), ok? These are (*raising eyebrows*) very important activities because problem solving, you should already realize it, is one of (*raising eyebrows*) the fundamental application fields of mathematics. Now we will face with problems that we can solve through the tools you already have, namely with problems that can be solved through first grade equations, ok? It is a discourse that will be continued during all of the years of high school, ok? then, these are the basis for the next developments.	 Figure 3.57
161	*after few minutes*	
162	S15: what is the void?	
163	T: it is the cost of the container. Also now, for example, you can buy the mineral water in glass bottles, ok? and you pay the void, the void is the container, the bottle in this case	
164	*students individually work on the problem*	

165	T: all of this was to let you (*pronouncing*) understand (Fig. 3.58) that it is important to use equations. Let's say that in this problem if someone does not reflect upon it and answers immediately (Fig. 3.59), it can happen often that the answer is not correct and some of you immediately have answered (Fig. 6, *waiting for a feedback*), the content (*nodding while she is speaking with the hand as in* Fig. 3.60, Fig. 3.61)	 Figure 3.58  Figure 3.59

166		 Figure 3.60  Figure 3.61
167	Ss: 1 euro	

168	T: that it cost 1 euro, if you have checked the answer as required you will realize that this answer does not work because we have a contradiction. Then, walking through desks, I heard that many of you, reasoning with calm, have found the correct newer. Then, let's see (*gesture as in* Fig. 3.62) how we can translate this problem in equation or, as it is said, how we can (*pronouncing*) formalize the problem. We will use this verb in the sense of make something formal, namely, in the sense of (*pronouncing*) translate from the language we use to communicate each others, that is the natural language, and the language that use the mathematical symbols. In this case this problem can be translated in an equation.	 Figure 3.62

Discussion

The problem the teacher proposed in the part of the lesson above is the "Oil problem" that it is shown in the worksheet 4.86: "A drink costs 1.10 euro, included the empty; the content costs one euro more with respect to the empty. How much is the content?". In the part above, Carla markedly stresses the justification of why treating equations. She wants to specify that they are not useful alone, but they become full of meaning when they are used as tool for solving problems. She specifies that in many hours of her lessons they will give meaning to why they have studied linear equations. She explains that they are fundamental tool for solving problem, raising eyebrows as stressing tho justification and staying with posture with open hand as in Fig. 3.57 (#160). Another time she repeats the same concept again with raising eyebrows in order to underline it a second time. Moreover, during the reading of the text of the problem she is very careful in explaining the meaning of the element involved in it (#163). For a third time, she insists on the fact that problems are used to give a meaning to the resolution of equation and she accompanies this statement with posture as in Fig. 3.58, in Fig. 3.59, in Fig. 3.60 that could be revealed her hope that students understand this

meaning.
At the end of this brief passage she explains that when they solve problems through equation they are formalizing the problem and she spells "formalize" to underline its importance in this context (#168). Then, she justifies what it means "formalize" and she spells the key-word "translate" in (#168). Summarizing she seems very careful to explain and give sense to all of what she considers in her teaching.

Expectation 3.

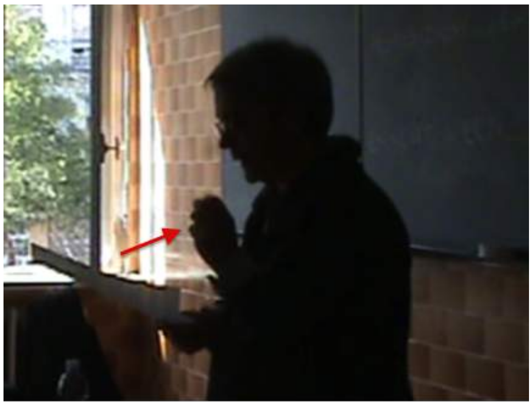
During the a-priori interview, the interviewer asks to Carla if she is careful about justification in her teaching and the teacher answers as follows:

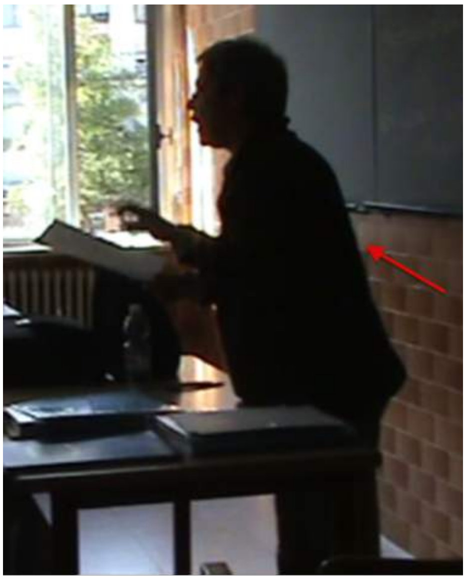
> "(self-confident tone of voice) ah yes sure, I ask always "let's justify your reasoning" (she begins to gesture), "let's justify your answer", yes sure (pause), both during the work within the classroom both in the exams in class" [...] (Carla speaks about the national assessment for the middle school, called "Invalsi") For me, in middle school, something is moving (pause) many people say that this assessment "falls on the head", but they serve to give a meaning to what it is doing. Many tests of this assessment have the question "Explain why", then students has to expose and justify their reasoning [...] I make many tests of the type "know" and "know-how"; the "know" is the set of justifications that students give in relation to what they do. Concerning equations, for example, I gave a problem of the type "given an equation, write an equivalent one to it applying just the first principle of equivalence and explain why".

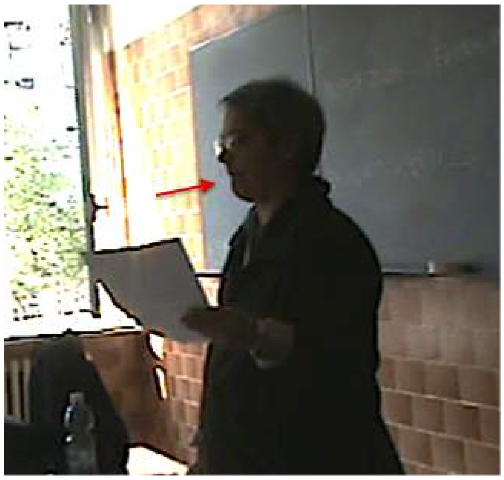
For Carla it is very important to highlight the aspects of justification. Hence, she gives reasons herself to what she does as you could see in the previous expectation, but also she seems to have *the expectation that also her students feel the need of justification to give sense to what they do*. I found this expectation reflected in her classroom activity in the passages below. As you will see, very often, Carla makes questions to her students like "How do you justify your answer", or "Let's justify your answer".
Carla gives to her students many equations to solve and she explicitly declares that in order to solve them, that is to find the solutions set of them she says: "you have to give a meaning to what you read", so it seems that for Carla it is very important to think about what they do and at the same time she expects that also her students feel this need. As you have noticed, the next example is the same of one considered for the Expectation 2. of Carla. I

want to put the same excerpt concerning this third expectation for showing that simultaneously Carla gives sense to what she does and she expects the same behaviour from students.

169	T: In the central part of the worksheet there were equations in which you had to find the truth set applying, treating them as they are that is as open statements, without making particular calculation, trying to see what elements make true the statement. Then, in the case *a* what did you find? (Fig. 3.63: *she is waiting*)	 Figure 3.63
170	S3:4	
171	T: (*nodding*) in the case *b*?	
172	S5: 4	
173	T: because we work within $\mathbb{N}$, ok? then we will return on it, in the case *c*?	
174	S5: 8	
175	S9: -10	
176	Ss: -9	

177	T: the unique way for seeing who is right is trying to substitute. Try to substitute, then pay attention, I have said you and this fact has to be always keep in mind, you have to (*posture and gesture as in* Fig. 3.64, *then she inclines her body towards the class with open hand*) give a (*pronouncing and highest pitch*) sense of what you read, then solving the equation $(x+1)^2 = 81$ means asking ourselves if it exist a value x such that doing $x + 1$ and squaring it is 81, working in (*highest pitch and she raises eyebrows*) $\mathbb{Z}$, this should make understand to you that (Fig. 3.65: *she mimes* $x + 1$) $x + 1$, (*she inclines her body towards the class*) how it has to be?	 Figure 3.64  Figure 3.65
178	Ss: 9 o -9	

179	T: How $x+1$ has to be? if its square must be (*she inclines her body towards the class*: Fig. 3.66) 81? $x+1$ must be equal to 9 or -9 because of (*pronouncing and raising eyebrows*: Fig. 3.67) integers whose square is 81 there is not only 9 but also -9 and then what are the elements?	 Figure 3.66  Figure 3.67
180	Ss: 8 e -10	

181	T: 8 e -10 (*nodding*), d, what have you found?	
182	Ss: 2 e -5	
183	T: and how have you found them? (Fig. 3.68: *she bites her lips waiting for an answer*)	 Figure 3.68
184	S6: putting one equals to zero	
185	T: we have applied an important property	
186	S3: "annullativo"	

187	T: (*gesture in* Fig. 3.69, *for saying that what S3 said is wrong*) it is called property of	 Figure 3.69
188	S2: of equality	
189	T: (*she lowers the arm, annoyed, but she retries*) property of	
190	S9: zero-product property	

191	T: Zero-product property, ok?try always to give a (*highest pitch and then she inclines her body towards the class*: Fig. 3.70) sense, to give (*raising eyebrows*) a meaning of what we read, solving the equation $(x+5)(x-2)=0$ means understanding when that product is (*raising eyebrows*) 0 and id we consider that the product of two factors is zero (Fig. 3.71: *she is waiting for an answer*) try to complete in order to review also the zero-product property, when the product of two factors is 0?	 Figure 3.70  Figure 3.71
192	Ss: when one of them is 0	
193	T: (*nodding and raising eyebrows*) when at least one of them is 0 and (*pause*) vice versa.	

Discussion

Carla asks questions about the justification waiting for an answer from students. For example, in the excerpt above, the teacher is correcting a worksheet in which students have to search for the elements that make true the open statement. She waits answers from students remaining as in Fig. 3.63. Then, when they consider the equation $x + 1 = 81$ in $\mathbb{Z}$, first she inclines her body towards the class as to communicate to the them the need she has of justifying (#177). Then she begins herself the justification and then she inclines her body another time towards the class to request feedback from it (#177). Actually, students react (#178) and the teacher shows students what kind of questions they make to justify what they are doing, inclining her body another time towards the class as to pass her need of justifying (#179). After insisting on how they should behave with the justification and inclining her body towards the class as to transfer her way of doing, she begins to require justifications directly from them. Indeed, she remains as in Fig. 3.68 biting her lips to not answer herself waiting for feedback from students. Unfortunately, she is not satisfied about the answer of S6 (#184) and, in the specific, she asks which property they have applied, pointing students with her fingers (Fig. 3.69). She seems very annoyed after discovering that students don't remember the zero-product property. In fact she lowers her arm (#189) and, very annoyed, she retries to recall the same property. Actually, this time a student recall the zero-product property and she takes this opportunity to insist on the fact that they feel the need of giving a sense of what they are doing. Another time she inclines her body towards the class as to pass this necessity (#191). At the end, she asks to students what it means the zero-product property and she remains as in Fig. 3.71 with an eloquent facial expression of who is hoping that her interlocutors are able to answer. After a right feedback, the teacher nods and raises eyebrows to signal her satisfaction (#193).

194	T: Then, in these exercises served to (*pronouncing*) review the "leggi di monotonia". You should identify, supposed true the given equality, those that are surely true applying the "leggi di monotonia". let's start from the first one, a?	
195	Ss: True	
196	T: What did you do in a? (*she bites her lips, waiting for an answer*)	

197	S3: we have transported b	
198	T: the transporting is not an operation	
199	S3: we have added b to both sides	
200	T: adding b to both sides, starting from the given one we obtain the a. Number b?	
201	Ss: False	
202	T:the c?	
203	Ss: True	
204	T: what did you do? (*she bites her lips, waiting an answer*)	
205	S3: we have subtracted minus, no, minus, it was subtracted $2c$	
206	T: you can see as subtracting $2c$ to both sides or as adding $-2c$, then d?	
207	Ss: it's true	
208	T: both in a and in c it was applied the "first legge di monotonia", in d?	
209	S3: the second one	
210	T: not only	
211	S3 and T: both	
212	T: in the sense that first it was subtracted 5 to both sides or adding -5, as you prefer ok?, but then we also have to	
213	Ss: dividing by 2	
214	T: dividing by 2, then the d is true, e?	
215	Ss: true	
216	S3: no it is false	
217	T: then who answers true, how did justify it?	
218	S10: I transport 5 to the other side of the equal sign	
219	T: (*nervously*) transporting here and there, I repeat that it is not a mathematical operation, you have (*pronouncing*) to justify it with the "leggi di monotonia"	

220	S9: with the "first legge di monotonia", I have subtracted $-3a$ to both sides and then I divided by -1	
221	T: yes, then you divided by -1	
222	...	
223	T: the g?	
224	Ss: true	
225	T: Did you understand the justification?	
226	S3: we have multiplied both sides by -1	
227	T: we have multiplied both sides by -1 (*she repeats, speeding up*), yes, h?	
228	Ss: true	
229	T: (*she nods and raises eyebrows because she is satisfied*) we have multiplied both sides by?	
230	S9: 4	
231	T: (*nodding*) and multiplying a polynomial by 4 applying the distributive property is equal to multiply all the terms by 4, ok?then, i?	
232	Ss: true	
233	T: what did you do?	
234	Ss: we have added 2 to both sides	
235	T: we have added 2 to both sides, j?	
236	Ss: false	
237	T: (*marked nodding*), k?	
238	Ss: true	
239	...	
240	T: l?	
241	Ss: false	
242	Ss: true	
243	T: then, who said true, how did justify it?	
244	S13: we have added the opposite of the first side	

245	T: yes sure, it can be seen like that, so I don't know if you heard what your classmate said, he said that he added to both sides the opposite on the first side, that is -3a+b, a way to see it is this one. Then, *n*?	
246	Ss: true	
247	T: how do you justify the fact that it is true?	
248	S3: we have subtracted first -2 and then we have divided all by 5	

Discussion

Carla pays very much attention to the justification aspects of what she and her students do. For example, in the excerpt above she was collectively discussing the activities she gave to her students (Worksheet 4.84). In the activity, students have to recognize which of the given equalities are true and which are false. Under this discover they should know the "leggi di monotonia" for equalities that they did in the previous months with Carla. This activity is preparatory to introduce the principle of equivalence starting from the "leggi di monotonia for equalities" (this is linked with the expectation of Carla that she wants to construct new knowledge starting from what they did with her mathematics classroom before).
The activity is structured by many equalities and Carla decides to ask to her students all the equalities one by one saying the corresponded letter. For example, first of all she asks to her students if the specific equality is true or false and after this she asks always questions like: "What did you do in *a*? (she bites her lips, waiting an answer)" and she expects the answer with eloquent facial expression, because she seems to create in her students the sense and the need of justification. She repeated all the times the same questions in order to make the justifying natural for students. The important thing is that she requires to her students mathematical justification. Obviously, it is not sufficient for Carla to justify the technique they use at a simple mechanical level. Indeed, when a student answers "it is transported the b", she says: "Shifting is not an operation", with an annoyed tone of voice. From the beginning of the year, she was trying to let understand her students that the "rule of transportation" (shifting a term from one side to the other one in

an equation, changing the sign) is not a mathematical operation, because it does not explain that when it shifts a term from one side to the other one in an equation, we add the same quantity to both sides. Students have learnt this rule in the middle school without the awareness of the mathematical justification behind it, so Carla seems to break up this misconception of her students. She requires to her students that they feel the need to justify their operations with the "leggi di monotonia" for equalities. As I said above, she asks questions like "then who answers true, how did justify it?" (#217) and all the times some of students answers to her and then she is very careful to show to them that there exist different ways in which they could see the justification of that operation. For example, Carla says: "yes sure, it can be seen like that, so I don't know if you heard what your classmate said, he said that he added to both sides the opposite on the first side, that is -3a+b, a way to see it is this one" (#245).

Expectation 4.

During the a-priori interview, Carla spontaneously adds the following fact:

> "A discourse that for me it has to be made is constructing the different numeric sets through successive enlargements, from $\mathbb{N}$ to $\mathbb{Z}$. I want that students see the analogy between the different numeric sets. It is something of abstract[12].For example, in the literal calculus, we start from monomial and then (gesture with hand of the enlargement) we enlarge to polynomial. We see the structural analogies. Obviously I don't speak of polynomial ring, but I show students that the structural properties are those of the set $\mathbb{Z}$ and so on. I try always to underline analogies between what we do."

As we spoke above, for Carla it is very important the classroom culture in order to develop new knowledge, starting from the previous one that students have done with her. Linked with this expectation there is another one that is the *expectation that students see analogies.*

Carla wants to construct the concept of equivalent equations starting from that of equivalent open statements:

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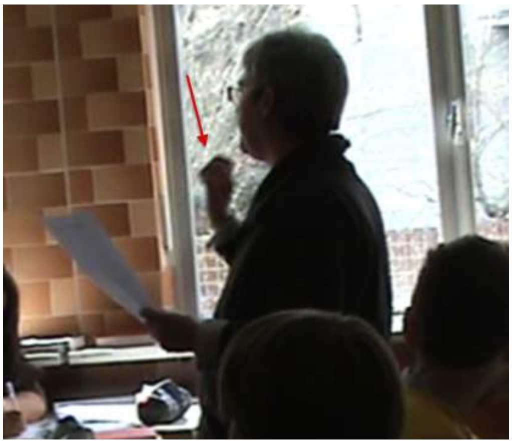
[12]she seems to intend that seeing analogies is something at a meta-level

249	S3: two equations defined in the same universe set U are equivalent in U if they have the same truth set or solution set	

250	T: that is (*nodding*), more or less, then remember (Fig. 3.72) the fundamental concept emerged in the Activities 4.82, 4.83 was that of equivalent open statements, that is, statements that within (gesture: Fig. 3.73) a given universe set have the (*highest pitch and raising eyebrows*) same truth set. Now, given that equations are particular open statements we will speak of (Fig. 3.74)	 Figure 3.72  Figure 3.73  Figure 3.74

251	none answers	
252	T: of equivalent equation when in a given set they have the same truth set. This was the (*pronouncing and gesture for clarifying:* Fig. 3.75) very important definition that we wanted to construct, try to continue with the worksheet in the final part of the first page.	 Figure 3.75
253	T: our goal is being able to (*raising eyebrows*) solve equations, then (*raising eyebrows*) to find the truth set (*pause*). For equalities the question can be if I have a true equality what can I do for obtaining again a true equality? speaking of (*highest pitch*) equations, instead, what should be the question? If I have an equation ok? what can I do for obtaining an equation (*raising eyebrows*)	
254	S12: equivalent	

Discussion

In this brief part of the lesson, the teacher wants that students see the analogies between equivalent open statement and equivalent equations. She recalls the meaning of equivalent open statements, underlying it with gesture in Fig. 3.73 and, then, she hopes that students see naturally what happens in the case of equations. Indeed she waits for feedback as in Fig. 3.74. Unfortunately, none answers. Hence, she says that, in the case of equation, they have to speak of equivalent equations, stressing that this was the important definition she wanted to construct through seeing the analogies with equalities (#252). Another time she hopes that students see the analogy between the concept of equivalence in equalities and in equations. She expects that students are able to respond to her question, indeed she stops in speaking and she raises eyebrows waiting for feedback (#253).

255	T: then how are the "leggi di monotonia" translated (*pause*) in properties on equations (*she inclines her body towards the class and she raises eyebrows*)? (*pause waiting for feedback*) What can you say if you have an equation ok? and what do you do?	
256	S3: if we multiply or divide both sides of an equation for the same value we will obtain an equivalent equation.	
257	S8: also subtracting we obtain	
258	S3: yes	
259	T: then say, given an equation, for the first "legge di monotonia", if we add the same number to both sides or we subtract it, working in groups ok?, let's remember how the addition is defined, it means adding the opposite ok? then we can speak of sum. Then if we add to both sides of the equations the same number we obtain (*she looks at the class, she stops in speaking and nodding as waiting for feedback*)	
260	S6: an equivalent equation	

261	T: an equivalent equation (*she repeats nodding*) to the given one. Instead for the second "legge di monotonia" (*she stops in speaking and she nods, biting her lips:* Fig. 3.76)?	 Figure 3.76
262	S3: if we multiply or divide (*Carla nods, remaining with lips as in* Fig. 3.76)	
263	S5: by a number not equal to zero (*Carla nods, remaining with lips as in* Fig. 3.76)	
264	S7: both sides (*Carla nods, remaining with lips as in* Fig. 3.76)	
265	S3: we obtain an equivalent equation to the given one (*Carla nods, remaining with lips as in* Fig. 3.76)	
266	T: (*nodding*) do you all agree?	
267	Ss: Yes	
268	*after few minutes*	

269	T: will it be the goal? The goal ok? (*pause, gesture and very relevant face in* Fig. 3.77) the goal is arriving to an equation that is equivalent to the starting one, but that is so simple that it can be possible to read (*long pause and gesture for inviting the class to continue:* Fig. 3.78) which is the solution set.	 Figure 3.77  Figure 3.78

Discussion

In the excerpt above, the teacher hopes that students understand the analogy between the "leggi di monotonia" they have already done for equalities and the principles of equivalence for the equations. Her hoping is made visible by her inclining her body towards the class raising eyebrows to having feedback from it, by her pauses waiting that someone notices the analogy (#255). In particular, she expects that students understand the analogy between the first "legge di monotonia" and the first principle of equivalence, indeed she stops in speaking, looking at the class and nodding as to be quite sure that someone will be able to see it (#259). Actually when a student answers in a right way, she nods another time because now she is satisfied of the feedback (#261). She repeats the same scheme for the second "legge di monotonia". Carla stops in speaking and she looks at the class and she nods, biting her lips (Fig. 3.76). She seem again quite sure that students are able to see the analogy for the second principle of equivalence and, then, she is waiting for answer (#261). Actually, students construct as an orchestra the second principle of equivalence and the teacher remains in the same position of Fig. 3.76, biting her lips for not to intervene and nodding for the satisfaction that students have seen the analogy and they are constructing themselves the second principle. Hence, at the end of the passage, Carla is expecting that students have understand that the goal is finding an equivalent equation so simple such that it can be read immediately the solution. Her hope is testified by her relevant gesture in Fig. 3.77 in which she mimes the action of coming out this observation from the class.

Expectation 5.

In the a-priori interview, Carla declares what she thinks about the use of the mathematical textbook:

> "Let's say that this mathematical textbook[13] has a (pronouncing) traditional approach, let's say that what (raising up her shoulders) I appreciate of it is not the theoretical exposition, rather the richness of exercises. Then using this textbook I must not search for exercises as I did previously when I had to see other exercises in other textbooks or something like that. This is positive, but what I don't appreciate of the textbook is the theoretical approach (speeding up) that however I want that students study. It is very traditional in the sense that topics are not introduced with problematic situations. I try to integrate it from this point of view".

From what the teacher declared in the a-priori interview, I infer her *expectation that students learn to use the mathematical textbook in a critical way*

[13](Bergamini et al., 2011)

[14], *referring to other didactical materials when it's necessary* [15], *underlying analogies and differences.* In the excerpt below, it is quoted a pieces of the last lesson about linear equations. After treating them on her worksheets in which the teacher combines theory and practice together, she is very careful to take with her students the mathematical textbook in order to analyse how it treats that mathematical topic.

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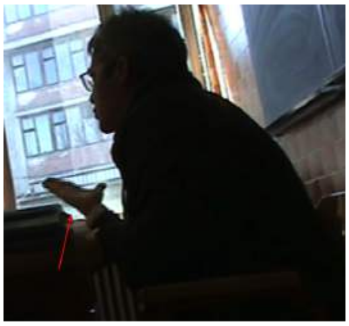
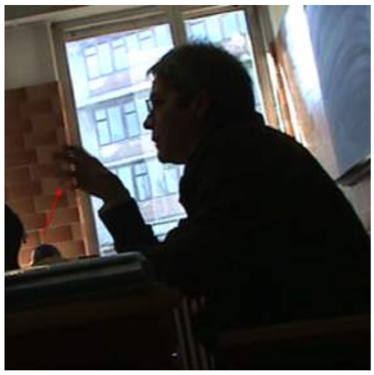
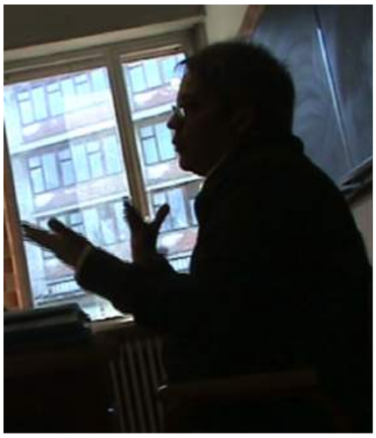
[14]to interpret correctly the definitions, to reflect upon its examples and so on

[15]e.g. the worksheets she prepares for the class

270	T: Now, follow on the textbook at page 491. Then, your textbook dedicates the chapter number 7 to equations. We have already encountered the definition of identity, that there is in the paragraph 1 (*gesture to mime the past*). Then, pay attention! the equations start from the paragraph 2 (*increasing the tone of voice and pronouncing*) but (*simultaneously to the "but" she raises up her index and she nods closing her mouth:* Fig. 3.79) you have to refer for the definition of equation to the (pronouncing) worksheet on which we have worked (*gesture in* Fig. 3.80), in which equations are presented as (*pronouncing*) open statements. On this discourse on your book there is (*in the meanwhile she flips through the pages*) just a rapid observation (*gesture as in* Fig. 3.81) in the frame that is at the beginning of page 494 (while she is speaking she nods). In this frame, there is something about equations as open statements, but you must (*gesture:* Fig. 3.82) work as we have worked on the worksheets. Then (*she turns page*), in the paragraph 3 there are the (*pronouncing*) principles of equivalence. At page 495 you find the definition of equivalent equations (*gesture as in* Fig. 3.83) that, however, you have to frame as the particular case of the general case of the equivalent open statements (*increasing the tone of voice and pronouncing*) and then at page 496 you have (*pronouncing*) the first principle of equivalence that comes from the "first legge di monotonia".	 Figure 3.79  Figure 3.80  Figure 3.81

<table>
<tr><td>271</td><td></td><td>
Figure 3.82

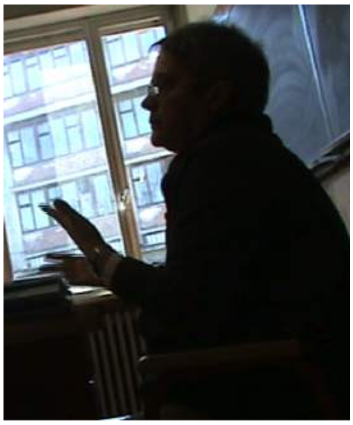



Figure 3.83</td></tr>
</table>

272	T: for this reason this principle is called also principle of addition and subtraction and you can see the two examples (*she raises up her face looking at the class, raising eyebrows and nodding*) that you find at page 496. Concerning the second principle of equivalence (*speeding up*) called also principle of multiplication and division, you can find it at the bottom of page 497, the (*she looks at the class, nodding*) you can also see the examples in next page, uhm? Let's say, however, that (*raising up eyebrows and gesturing*) all said until now there is, uhm, you should (*she put her hand on the book*) having (*gesture for miming the construction*, Fig. 3.84) constructed uhm? working on (*gesture and she inclines her body towards students:* Fig. 3.85) the worksheet I proposed you.	 Figure 3.84  Figure 3.85

Discussion

After explaining linear equations through worksheets, the teacher looks with her class how their mathematics textbook treats this mathematical topic. She describes how the textbook organizes the treatment of equations. She is very careful to underline that she wants that students refer to the definition of equation they already saw on the worksheet. In fact she raises up her index as in Fig. 3.79, as "to intimidate" students to consider her definition of equation. She pronouncing "worksheets" in order to underline where they can find the definition of equation as an open statement. She repeats that they have to consider the definition of the equation given in the worksheet even if there is a brief consideration on their textbook about it. Moreover, she focuses the attention of the students, increasing the tone of voice and pronouncing, on the fact that the first principle of equivalence has to be considered as a consequence of the first "legge di monotonia", even if the textbook does not specify it (#270). At the end, she inclines her body towards the class (Fig. 3.85) to involve the whole class for summarizing that they however have to refer to what they "constructed" through worksheets (#272).

273	T: (*she summarizes the types of equations*) in this case we speak of (*pronouncing*) identity, on your book (*gesture, raising up her index and moving it to mime the negation:* Fig. 3.86) you don't find written (*she points to the textbook to say the page and she peers up in speaking*) in the schema you have at the bottom of page 501, (*pronouncing*) identity, but what do you find? (*she makes the questions to understand if students are following her and she bites her lips*)	 Figure 3.86
274	Ss: undetermined	
275	T: (*nodding*) then (*she returns to the blackboard and she makes a gesture as it said that now she is going to explain the difference*)	
276	S4: is it the same thing?	
277	T: not really (*she shakes her head*) but your book, let's say, considering first grade equations in one unknown is the same thing, but we will see that it is not always in this way	

Discussion

In the segment above, she summarizes the different types of equations and she underlines that their mathematics textbook does not speak of "identity"

(Fig. 3.86), but of “undetermined equations”. She stresses that, in general, it is not the same thing, shaking her head. However, she adds that the textbook does not make a mistake, because for the first grade equations “identity” and “undetermined equation” can be considered as synonyms.

Expectation 6.

Concerning the role of the example, Carla explicitly declares that:

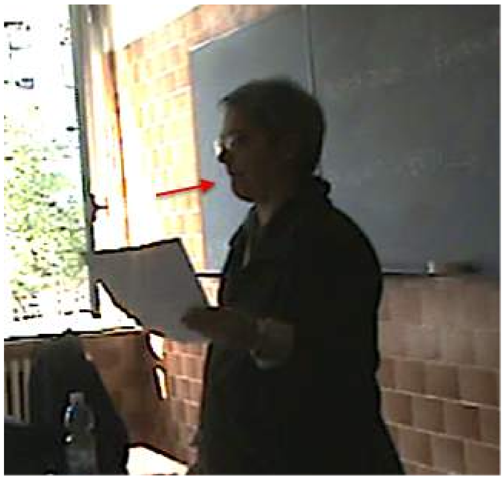
> “Examples can be the starting point to introduce a new topic. It is necessary to choose them accurately such that they are significant, they trigger discussion and they make students feel the need of constructing new tool or procedure. Then there are routine examples, that can be useful to illustrate a definition, to reinforce a procedure, to go deep into. They are that ones that traditionally appear on the mathematical textbook, ok? The problem is that students don’t use examples in a correct way, indeed they use them to generalize properties, without feeling the need of proving. For students is not trivial to produce examples to sustain an argumentation or justify an answer. Hence, I try always to start from examples, examples required to them, ok?”

From what Carla explicitly declared in the segment of the interview above, I infer her expectation that *students are able to make examples, because she thinks that examples are an useful tool to construct procedure or to review a property.* In particular, she specifies that examples can have different functions: they can help them to construct a tool or a procedure or they can be useful to understand better the theory. She seems that she expects that students are able to do examples, because she stresses the fact that students often use examples in a wrong way. In fact, they use examples to generalize properties instead of proving them.

278	T: then I have written also (*increasing the tone of voice and nodding*) "help you with examples", take an equation, ok? (*gesture for miming a random example*) try to write it on a sheet ok? (*she points to the sheet of a student in the first line*) then you have to ask yourself what it is possible to do for obtaining an equivalent equation from this one (*she is referring to the "potential example" that she expects that students construct. For this reason she looks again to the sheet of the same student in the first line*), the equation is (*raising eyebrows*) an open statement.	

Discussion

In the excerpt above, students are working on another worksheet for constructing how the first principle of equivalence works. In particular, they have to invent an equation and starting from it they have to find a way to obtain an equivalent equation to it. The interesting thing is that she has written in the worksheet "help you with examples" and she, simultaneously, increases the tone of voice and she nods. This could reveal her expectation that actually students are able to construct examples themselves. Moreover, it seems that Carla hopes that students can construct example insomuch as she refers to an "invisible example" pointing to the worksheet of a student, as if she has already considered it.

279	T: and how have you found them?(Fig. 3.87: *biting her lips and waiting for an answer*)	 Figure 3.87
280	S6: putting one equals to zero	
281	T: we have applied an important property	
282	S3: "annullativo"	
283	T: (*gesture of* Fig. 3.88, *for saying that what S3 said is wrong*) it is called property of	 Figure 3.88
284	S2: of equality	
285	T: (*annoyed*) property of	

286	S9: zero-product property	
287	T: Zero-product property, ok?try always to give a (*highest pitch and then she inclines her body towards the class:* Fig. 3.89) sense, to give (*raising eyebrows*) a meaning of what we read, solving the equation $(x+5)(x-2)=0$ means understanding when that product is (*raising eyebrows*) 0 and if we consider that the product of two factors is zero (Fig. 3.90) try to complete in order to review also the zero-product property, when the product of two factors is 0?	 Figure 3.89  Figure 3.90
288	Ss: when one of them is 0	
289	T: (*nodding and raising eyebrows*) when at least one of them is 0 and vice versa.	

Discussion

In the part of the lesson above she discusses with the class the equation $(x+5)(x-2)=0$ in $\mathbb{Z}$. Students have correctly found solutions, namely 2 and -5, but the teacher wants to make a step forward, using example to go deep into which property they have applied to solve thus equation. She hopes that example is helpful to students for remembering the special product property. At the beginning, students seem to remember just an improbable "annullativo" property, producing a visible disappointment in Carla. But, then, a student recalls the special product property and the teacher takes the opportunity from that example to review this property, being at the end satisfied (she nods: #289) because she has understood that students finally have recalled it.

Worksheets

Nome e Cognome .... Classe .... Data ....

ATTIVITA' 1A – TITOLO: ....

ACTIVITY 1A - TITLE:

Considera le seguenti coppie di enunciati aperti, definiti nell'insieme universo U indicato, ed elenca tutti gli elementi dei rispettivi insiemi di verità A e B:

Consider the following couples of open statements, defined in the indicated universe set U and list the elements of the corresponding truth sets A and B:

a) $U = \{x \in N / x \leq 20\}$ $A(x)$: x è pari ; $B(x)$: x^2 è multiplo di 4

A(x): x is even ; B(x): x^2 is multiple of 4

A = ...

B = ...

b) $U = \{x \in N / x \leq 100\}$ $A(x)$: la cifra delle unità di x è 0 ; $B(x)$: x è multiplo di 10

A(x): the unit digit of x is 0 ; B(x): x is multiple of 10

A = ...

B = ...

c) $U = \{x \in N / x \leq 20\}$ $A(x)$: x è dispari ; $B(x)$: x^3 è dispari

A(x): x is odd ; B(x): x^3 is odd

A = ...

B = ...

d) $U = Z$ $A(x)$: $|x| \leq 5$; $B(x)$: $x^2 \leq 25$

A = ...

B = ...

e) $U = Z$ $A(x)$: $|x| < 5$; $B(x)$: $-4 < x + 1 < 6$

A = ...

B = ...

Che cosa puoi osservare? ....

What can you observe?

....

Come definiresti gli enunciati aperti di ciascuna delle coppie considerate? ....

How would you define the open statements for each couple you considered?

Figure 3.91: Activity 1a

ATTIVITA' 1B

ACTIVITY 1B

a) Nell'insieme N sono definiti i seguenti enunciati aperti: $A(x):\ x < 6$; $B(x):\ x \leq 5$

In N are defined the following open statements:

Elenca tutti gli elementi dei rispettivi insiemi di verità:

List all of the elements of the corresponding truth sets:

A = ……………………………………………………………………………………………

B = ……………………………………………………………………………………………

Che cosa puoi concludere? ………………………………………………………………

What can you conclude?

b) Considera nell'insieme universo Q gli enunciati aperti definiti in a). Che cosa osservi? ……

In the universe set Q consider the open statements defined in a). What do you observe?

……………………………………………………………………………………………………

……………………………………………………………………………………………………

c) Considera ora i seguenti enunciati aperti: $A(x):\ x \leq 4$; $B(x):\ x^2 \leq 16$

Consider now the following open statements:

Nell'insieme N: ……………………………………………………………………………

In N:

……………………………………………………………………………………………………

……………………………………………………………………………………………………

Nell'insieme Z: ……………………………………………………………………………

In Z:

……………………………………………………………………………………………………

……………………………………………………………………………………………………

Quale importante osservazione ti suggeriscono gli esempi precedenti? ……………………

Which important observation do the previous examples suggest to you?

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

Figure 3.92: Activity 1b

Nome e Cognome .... Classe .... Data ....

ATTIVITA' 2 – TITOLO ....

ACTIVITY 2 - TITLE

Considera le seguenti proposizioni e stabilisci quali sono vere e quali false:

Consider the following propositions and establish which are true and which are false:

a) $3 \cdot 5 + 1 = 18$ ....

b) $3^5 \cdot 3^2 = 3^7$ ....

c) $(2+1)^3 = 2^3 + 1$ ....

d) $30 + 10 + 5 = 5 \cdot (6+2)$ ....

e) $(7+3)^2 = 7^2 + 6 \cdot 7 + 3^2$ ....

Nelle precedenti proposizioni compare il predicato "essere uguale", esse sono esempi di uguaglianze e possono quindi essere vere o false.

In the previous propositions, it appears the predicate ``to be equal to", they are examples of equalities and they can be true or false.

Nei seguenti enunciati aperti compare il predicato "essere uguale", essi vengono detti equazioni.

Risolvere un'equazione significa determinare l'insieme di verità di un enunciato aperto in un insieme universo fissato. Sapresti risolvere le seguenti equazioni?

In the following open statements, it appears the predicate ``to be equal to", they are called equations. Solving an equation means determining the truth set of an open statement in a given universe set. Can you solve the following equations?

a) $3x + 6 = 18$ $U = N$ ....

b) $x(x+1) = 20$ $U = N$ ....

c) $(x+1)^2 = 81$ $U = Z$ ....

d) $(x+5)(x-2) = 0$ $U = Z$ ....

e) $5x + 3 = 2$ $U = Q$ ....

Nel caso delle equazioni, l'insieme di verità è detto insieme delle soluzioni ed i suoi elementi sono le soluzioni dell'equazione.

Ricordando gli esempi e la definizione incontrati nell'attività 1, completa la definizione:

Due equazioni definite, nello stesso insieme universo U, si dicono equivalenti in U se ....

....

Considera le seguenti equazioni: $2x + 1 = 9$; $x^2 - 16 = 0$ e risolvile

nell'insieme N: ....

nell'insieme Z: ....

Che cosa è quindi importante osservare? ....

....

Figure 3.93: Activity 2a

L'insieme delle soluzioni S di un'equazione è un sottoinsieme dell'insieme universo U.
Risolvi le seguenti equazioni nell'insieme indicato:

a) $3x + 6 = 20$ $U = N$ ....

b) $3x + 6 = 20$ $U = Z$ ....

c) $3x + 6 = 20$ $U = Q$ ....

d) $2x + 8 = 6$ $U = N$ ....

e) $2x + 8 = 6$ $U = Z$ ....

f) $x^2 + 1 = 0$ $U = Q$ ....

g) $3(x - x) = 0$ $U = Q$ ....

h) $(x + 1)^2 = x^2 + 1$ $U = Q$ ....

i) $3(x + 2) = 3x + 6$ $U = Q$ ....

j) $\frac{3}{x} = 0$ $U = Q$ ....

Osservando i risultati ottenuti, quali osservazioni puoi fare riguardo all'insieme delle soluzioni di un'equazione? ....

....

....

....

....

....

Considera ora la seguente equazione, nell'incognita x, $ax + b = c$ con $a \neq 0$ e stabilisci quali condizioni devono soddisfare a, b e c affinché essa sia risolubile.........

In N ....

....

In Z ....

....

In Q ....

....

Figure 3.94: Activity 2b

Nome e Cognome ....Classe.............. Data............................

ATTIVITA' 1

Problema 1

Una bibita costa 1,10 euro, vuoto compreso; il contenuto costa un euro più del vuoto.
Quanto costa il contenuto?

....

Spiega come hai ottenuto il risultato che hai proposto:

....

....

VERIFICA SE IL TUO RISULTATO È COMPATIBILE CON I DATI DEL PROBLEMA:

La bibita senza vuoto costa........
Il vuoto costa......
La bibita, vuoto compreso costa....
La differenza tra il costo del contenuto e del vuoto è.......

LA VERIFICA APPENA COMPIUTA TI PERMETTE DI CONFERMARE IL RISULTATO CHE HAI PROPOSTO ALL'INIZIO? **SI** **NO**

Se hai risposto **SI**, puoi passare alla fase successiva dell'attività.
Se hai risposto **NO**, allora proponi un nuovo risultato………………………………………………………….

TRADUCI NEL LINGUAGGIO ALGEBRICO
Rileggi attentamente il testo del problema e rispondi alle domande che seguono, completando a parole la colonna che corrisponde al linguaggio naturale e, coi simboli dell'algebra, la colonna che si riferisce al linguaggio algebrico:
Quali sono i dati, ossia gli elementi noti desunti dal testo?
Quali sono le incognite?
Quali relazioni legano i dati alle incognite o le incognite tra di loro?

	Linguaggio naturale	Linguaggio algebrico
Richiesta o richieste		
Incognita o incognite		
Dati e relazioni fornite dal problema		
Equazione risolvente		
Risoluzione dell'equazione		
Controllo se la soluzione è accettabile		
Risposta		

Figure 3.95: Oil problem

3.7 Emotional orientation of Sara

Before entering into the identification of Sara's expectation, I would like to briefly present how Sara introduces linear equations. As already highlighted, the topic of linear equations is crucial in the curriculum of the first years of high school. For example, the M@t.abel Project develop this topic through several different mathematical activities. The M@t.abel Project that is an Italian teacher education programme for in-service mathematics teacher supported by the Ministry of Education. Within this project have been developed several mathematical activities aiming "toward the construction of meanings, in which the students can learn by doing, seeing, imitating and communicating with each other, under the guidance of the teacher" (Arzarello et al., 2014, p. 359). Sara uses many times in her teaching the activities proposed in the M@t.abel project. Concerning linear equations she decides to use the activity called "Equation and Inequation of first grade". The authors of this activity are P. Accomazzo, M.Ajello, D. Paola, F. Turiano. This activity is divided into several phases: each of them is constituted by worksheets. Fo example, the first phase aims to formalize a problem written in natural language through an equation like $ax = b$. In particular, the first problem is *Luca has decided to participate in a foot race that takes place every year in his city; today he will train on the path of the race. From the starting line, he starts to walk in a regular way, namely with a constant velocity. He began walking at* $14:00$ *(2 p.m.), now it is* $14:15$ *(2:15 p.m.) and he covered* $3km$*. There still remains* $5km$*; at what time will he reach the arrival? How do you find the answer? Write the steps you did to arrive to the solution.*
Then, the second problem is *The organizers of the foot race have put on the street some pickets that indicate the progressive distances from the starting point. At which distance will Luca be after walking for 5'? and after 10'? and after 18'? and after 20'? and after 25'? When will he arrive to the picket of* $2km$*?*. From this activity, it becomes quite clear to introduce the function distance ($dist$) – time (t) expressed by the formula $dist = \frac{1}{5}t$. The next question, should be how much time Luca uses to cover $8km$, as Sara actually will do.
Lastly, the third problem is *Luca trains always on the same path of the race; today he has decided to start from the picket at 500 m from the starting line. He starts to walk pronouncingin a regular way. He began to walk always at* $14:00$ *(2 p.m.) now it is* $14:15$ *(2:15 p.m.) and he covered* $3km$*. There still remains* $5km$*; at what time will he cover* $8km$*?*. Students have to solve the equation $\frac{1}{5}x + \frac{1}{2} = 8$. The solution of it is the time Luca uses to cover 8 km, starting from 500 m from the starting line. Hence, knowing that he uses 15 minutes to cover 3 km, namely his velocity is $\frac{1}{5}km/min$, the time he uses

is the solution of $\frac{1}{5}x + \frac{1}{2} = 8$. From here, this problem can be generalized asking *"If the organizers change of the same distance both the arrival and the starting point of Luca, what will be the time?"* in order to introduce the principle of equivalence of equations.
The activity of M@t.abel involves also the exploration of these problems with a dynamic geometry software as GeoGebra, in order to reason on equations also from a geometrical point of view.

From what Sara explicitly declared during the a-priori interview, I identified many expectations that could be seen reflected in many passages of her lessons. Before entering in the specific of each expectation, I summarize them in the list below:

1. Expectation that students learn "to see" through the graphic register in order to reason (think of) on equations
2. Expectation that Algebra becomes for students a thinking tool
3. Expectation that students learn to pass from one representation register to the other one
4. Expectation that students learn to use the algebraic language as an extension of the arithmetical one
5. Expectation that examples are useful for students to understand the meaning of what they are doing [16]
6. Expectation that justifications serve to go deep in the meaning of what they are doing

Expectation 1.

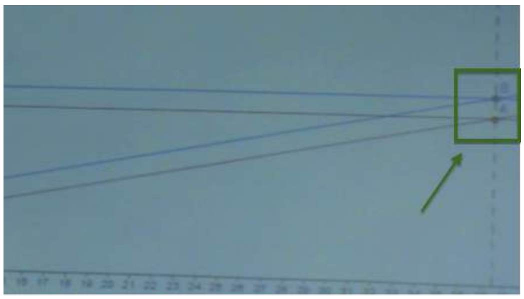
In the a-priori interview, Sara explains her way of introducing linear equations:

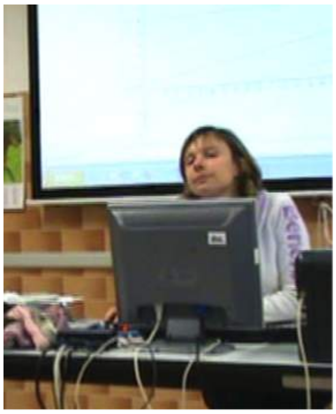
> I introduce linear equations through an activity of M@t.abel. In particular, we start to see a real situation, a pseudo-real situation of a boy who walks with constant velocity and we ask, knowing the velocity, how many kilometers he covers while the time passes. "How many kilometers while the time passes", it is a linear function, then we consider a table and we start to see

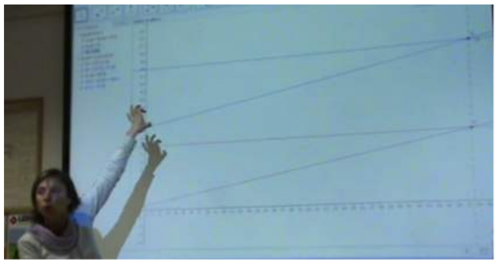
[16] with examples she can also understand if students actually understand what they are doing

> after how much time he will cover $300m$ and then we go to see (she mimes the solution on the graph) the answer on the graph. We don't go to read the answer as the zero of the function, but we put ourselves on the graph and we try to read the different answers. (pronouncing) We start, from that, to talk of equations because, after we have the straight line (she mimes the straight line), we can read and then we have the intersection between the oblique straight line that represents the velocity and the horizontal straight line that represents for example $300m$. Then we are able to see the intersection point as the solution of an equation. Then, always working on the graph I try to highlight that if I translate the graph up or down (she mimes the translation) the solution is simply translated up or down. Hence, I can add or subtract the same term to both sides of the equation and I will obtain the same solution.

From what she declared above, it can be plausible to say that Sara has the *expectation that students learn "to see" through the graphic register in order to reason on linear equations.* During the interview, she uses herself the verb "to see" and, as showed in the following, this expectation is actually reflected in classroom. After introducing both the concept of linear equation as an open statement and that of equivalent equation, Sara prompts the attention of her students to reviewing them from a geometrical point of view, using GeoGebra.

290	T: 9. In fact, it is what happens. We take the point B as the intersection point and then we colour blue these two new straight lines and B (Fig. 3.96). Now we could try to ask what happens if I vary the value of k, ok? Varying k something happens that is interesting. Then, let's start to see what happens on the graph varying k (*she moves the slider* k). On the graph, what happens while varying k (*she continues to move the slider* k *and she looks at the class*)?	 Figure 3.96

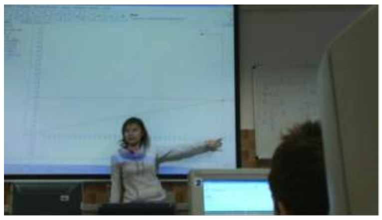
291	S2: (*perplexed*) uhm, it happens that the oblique straight lines remain parallel and the horizontal straight line remains parallel	
292	T: Ok, then the straight line are parallel (*pause*, Fig. 3.97)	 Figure 3.97
293	S3: They change position varying k	
294	T: How do they change position (*she continues to move the slider*)? What kind of movement they do (*she is referring to S3,* Fig. 3.98)	 Figure 3.98
295	S3: vertical	

296	T: (*nodding in marked way*) there is a vertical translation (*gesture for miming the vertical translation*) that can be up (*gesture for miming "up"*) or down (*gesture for miming "down"*), but they have simply a vertical translation (*miming the translation*) and they have not an horizontal translation (*miming it*). In terms of the time used by Luca in his foot race, the problem is "if I change of the same distance both the arrival and the starting point of Luca, what will be the time?". Namely, what the solution given by the blue straight lines (Fig. 3.99) will coincide with? (*pause and she remains in posture as in* Fig. 3.100)	 Figure 3.99  Figure 3.100
297	S3: 37.5	
298	T: hence the time is again 37.5, (*highest pitch*) because the number of km he has to cover is the same.	

Discussion

In the excerpt above, Sara hopes that students are able to see that varying the slider k on GeoGebra produces a vertical translation of the two straight line considered to solve the equation. In fact, she remains as in Fig. 3.97 waiting for feedback, just because she needs that students go beyond to the fact that the straight line remain parallel. Moreover, she explicitly asks to students, what kind of movement the two straight lines make and she remains in a posture of waiting as in Fig. 3.98. At the end she pushes students to see that the abscissa of the intersection point between the two straight lines is again 37.5. Actually, what happens is that Sara remains as in Fig. 3.100, looking at the class and waiting for feedback of students (#296).

299	T: In the previous lesson (*she mimes the past and she raises eyebrows:* Fig. 3.101), before easter holidays, we have said that the first principle of equivalence said us that we could add the same number to both sides and that the result of the equation continued to not change, ok? then I could add or subtract the same number to both terms and have (*pronouncing*) always equivalent equations. Then, what does it mean (*returning on the "Algebra view"*)? It means that I can add to both sides (*moving k*), see that the blue straight lines have the same movement, they have the same translations (*she mimes the translation:* Fig. 3.102), namely they have exactly the same movement, then we add or subtract to both sides exactly the same quantity, our result doesn't change. If I wanted to obtain the result of the equation, I would take k, I would do such that B coincide (*pronouncing*) exactly with the $x-axis$ (*she is doing it on GeoGebra*). To let coincide B exactly with the $x-axis$, what value I have to give to k?	 Figure 3.101  Figure 3.102
300	S11: -8	

301	T: -8. If I give -8 to k, what happens is that B belongs to the $x - axis$ (Fig. 3.103). The second side of the equation (*pronouncing*) takes the value 0. The first side of our equation has a certain expression and I, actually, go (*pronouncing*) to see where the blue equation intersects the $x - axis$ (Fig. 3.104). I go to find what it is called the (*pronouncing*) zero of function (*gesture to accompany the pronouncing*) because it is the point in which the straight line touches the $x - axis$, ok?	 Figure 3.103  Figure 3.104

Discussion

In this passage of the lesson, the teacher needs again that students are able to see that adding the same term to both sides of an equation produces the same movement of the straight lines on the graph, and, then, the result does not change. Moreover, she underlines how they could find the solution on the graph (#299). In particular, she wants that students see that for having the solution of the equation, the intersection point B has to belong to the x-axis. This operation means to have a side of a function equal to 0. Hence, she takes the opportunity to see the solution of an equation as a zero of the straight line.

Expectation 2.

After the question "Which view of Algebra do you want to transmit to your students?", Sara answers this way:

> "Oh, that big question! ok (she takes time) what view of Algebra, ok (she takes time) I would like that Algebra becomes a thinking tool in the sense that I would like that students use Algebra to

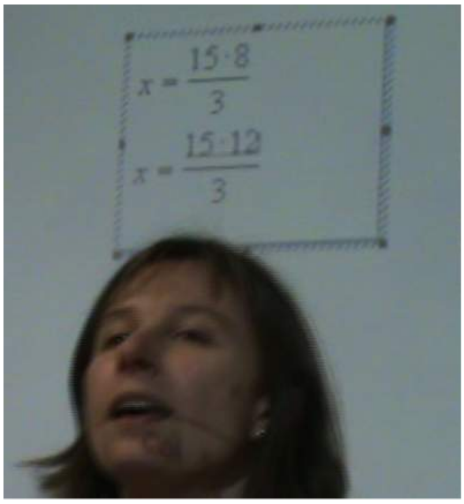
> understand a little bit more what is around them. For example, if there is a spot in which an item, I don't know, lasts twice with respect to another one, I would like that students are able to translate things of this type under the algebraic aspect. This way you are able to understand better the world around. I would like that they are able to use Algebra to generalize, to prove and also a little bit to count, because if you understand well Algebra, it can help you to computes also with numbers. I think that what it is taught independently from Mathematics has to be useful in the real world. In the real word knowing just the special products is not so helpful if you don't transform them to make counts more quickly, to understand if someone is saying the truth or not. (self confident) Speaking of percentages, if I want to see if there is the same thing if a product increases by 20% and then decreases of the 20%, we uses letters. This way we are able to answer (pronouncing) in general and then here we have the algebraic language. It serves to speak of results in which there is not the starting value."

This way of thinking of the teacher is the product of an evolution in her didactical methodology. In fact, Sara continues, stating:

> "In these years, I tried to do a thousand different things, I began to work with the textbook making the definition of monomial and polynomial and then I discovered that it was a bad thing, because they memorized all things and then they don't learn and they don't have the sense of that we did. When they were in the real world, it does not work well, then little by little I I tried to take the worksheet of M@t.abel. Then I tried to apply them very slowly, because actually, at the beginning, a person is afraid of making mistakes, she is also afraid of detaching too much from the textbook. But then I discovered that this way was good and I continued on it. [...] I introduced linear equations with an activity of M@t.abel. In particular, we start to see a real situation, a pseudo-real situation of a boy who walks with a constant velocity and we ask ourselves, knowing the velocity, how many kilometers he covers while time passes (gesture Fig. 4). How many kilometers while time passes is a linear function, then we make the table of values and we start to work on it and then we see after how much time he will cover $300m$ and then we go to see (she mimes the solution on the graph) the answer on the graph."

From what Sara declared in the a-priori interview, I infer that Sara has the *expectation that Algebra becomes for students a thinking tool.* In fact she would like that students will learn to use Algebra in everyday life situations. She sees Algebra also a powerful tool to make arithmetical calculations: she makes the examples of the special products that treating just from a technically point of view are not so useful. To accomplish her aim, she uses several didactical materials that should push students towards having such view of Algebra. For example, she uses the M@t.abel Project that is an Italian teacher education programme for in-service mathematics teacher supported by the Ministry of Education. Within this project have been developed several mathematical activities aiming "toward the construction of meanings, in which the students can learn by doing, seeing, imitating and communicating with each other, under the guidance of the teacher" (Arzarello et al., 2014, p. 359).

302	T: (*highest pitch*) but instead of calculating this way, (*very slowly and pronouncing all the words*) instead of calculating how much time Luca uses in 8 *km*, namely, how much time I use to make 8 *km*, I wanted to know how much time he uses to make 12 *km*, what should I do? (*gesturing, speeding up and smiling*) always thinking of the fact that Luca isn't ever tired and that he walks always at the same velocity?	
303	S10: 15 times 12 divided by 5	
304	T: 15 times 12 divided by 5	

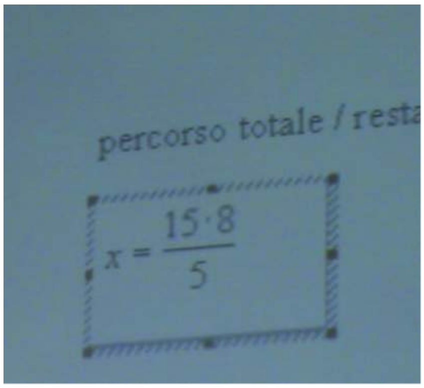
305	T: where x is always the time, right? (*with a tone of voice indicating certainty*) let's write it somewhere (Fig. 3.105)	 Figure 3.105

Discussion

In the excerpt above, Sara uses Algebra as a thinking tool, in fact she hopes that students employs the algebraic language to find the time Luca uses to cover 12 km. She accompanies this request going very slowly in speaking and pronouncing all the words to draw the attention of students on the problem (#302). After the correct answer of a student, she recall that what they have called x is always the time of the real problem from which they have started, smiling as in Fig. 3.105.

306	T: If I wanted to (*pronouncing*) put in formula (*pause*, Fig. 3.106) (*highest pitch*) what I'm saying, how could I do it (Fig. 3.107)? If I wanted to put it in formula, how could I do it (*she remains in the same manner for many seconds:* Fig. 3.108)? We have seen that a formula serves to speak in mathematics (*smiling*) a little bit, then if I wanted to put in a formula (*pronouncing*) the time used to cover the entire path, what can I do to put it in formula?	 Figure 3.106  Figure 3.107  Figure 3.108

307	S1: but, in words?	
308	T: no, with a formula (*smiling:* Fig. 3.109). We have already written it in words, you have told me that we have done the proportion and then you have solved the proportion or you have calculated the time necessary for doing one kilometer and then you have multiplied the time necessary for doing one kilometer times 8 (*she is waiting again on an answer as in* Fig. 3.110, *playing a little bit nervously with her ring*).	 Figure 3.109  Figure 3.110
309	S4: they are proportional	

310	T: they are directly proportional yes, because we have made a proportion and then they are surely directly proportional	
311	S6: x is equal to the minutes already done times the total kilometers divided by the time for doing one kilometer	
312	T: Then if I had to write it, how can I do it?	
313	S6 dictates to the teacher	 Figure 3.111
314	T: (*she repeats what S6 says*) 15 times the total kilometers divided by the time used for one kilometer. Ok this one could be (*smiling*) right.	

Discussion

In this excerpt, Sara pushes her students toward generalization. She is solving a problem that comes from the activities of the M@t.abel project. In particular, it is the problem of a boy who has to cover a certain distance and it is asked to express a formula that express how much time he uses to cover that distance. In particular, she hopes that students find a formula for the time employed by Luca for covering any distance. She thinks that, in this case, Algebra could be an helpful thinking tool, indeed she is waiting for a formula (Fig. 3.109) to calculate the time necessary for doing a specific distance (#308), playing a little bit nervously with her ring (Fig. 3.110).

Actually, a student uses correctly Algebra to express the time to cover a certain distance (#313) and teachers seems to be satisfied of her reasoning, repeating it and smiling (#314).

Expectation 3.

Concerning the coordination among different registers of representation, the teacher explicitly declares:

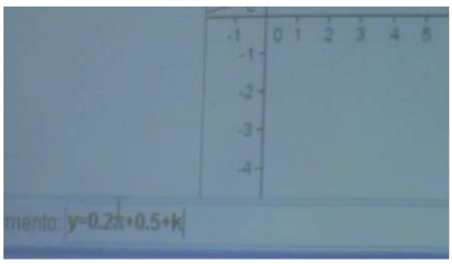
> "We introduce function, initially, through the verbal register and the table of values in which we go to put different values. This is the first way which I introduce functions, then, from here we pass to the graph, to the equation of the graph and then we put the different things all together and we try to see that they are exactly (pronouncing) the same thing."

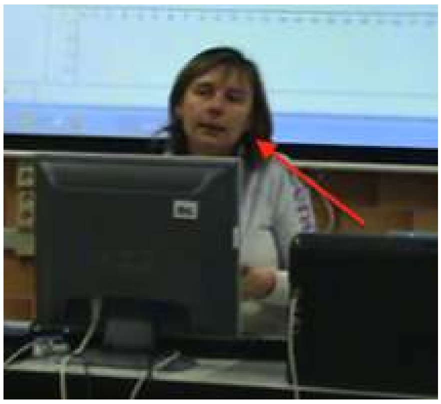
In another passage of the interview concerning linear equations:

> "We make the graph and we go to search for the solution (pronouncing) on the graph , then they look at the graph and we see what means translated up or down and we see that the solution is translated or is shifted, then we consider the balance, we play a little bit with the balance adding and subtracting. After playing with the balance, we see the technical aspect. Naturally, the formalization of the technical aspect serves and it serves also a little bit of training for the technical aspect. Summarizing, we reason on the graph, then we play with the balance, then we see the technical aspect, then we return on the graph to see that I can use equations to see the zeros of functions. I treat equations considering all of the registers and I stress in a very marked way that they are all equivalent (pronouncing) different ways to speak of the (pronouncing) same thing"

From what teacher explicitly declared in the interview, I infer that Sara has the *expectation that students learn to pass from one representation register to the other one*, underlining that she is very careful to stress that, when she passes from one register to the other, they are all equivalent ways to speak of the same thing. In the passages of the lessons below, this expectation is actually reflected.

315	T: Actually, we have already seen the first part of the activity. The activity asked us to prepare a slider, to call it k and to vary it from -15 to 15 and (blaring tone of voice) then it told us to take into account two equations: one we have already solved in the previous time and the other one was established, if you remember it. The equation already solved previously was $\frac{1}{5}x + \frac{1}{2} = 8$. To solve this equation we have already said that, actually, we could work on two different (pronouncing) functions, precisely on two (pronouncing) straight lines: one was this straight line (*she draws on GeoGebra the function* $y = \frac{1}{5}x + \frac{1}{2}$) (*pause and she looks at the screen*) and the other one was $y = 8$. Then, you have told me, if you remember, that the solution to the equation was the intersection point between these two straight lines. (rhetorical question) Do you remember it? There was Elena who said "(*highest pitch*) I go to see where I intersect and then I read the solution". Actually the solution we have to read is not on the $y - axis$, but it is on the $x - axis$, because it is the value of x that is of interest to us as solution and (*blaring tone of voice*) then the fact of asking to draw the perpendicular line to the $x - axis$ passing through A served simply to say that (*highest pitch*) I can go to read the solution.	

316	T: I can go to read the solution here (*she stands up and she goes on the screen, pointing to the abscissa of the intersection point and then she looks at the class as in* Fig. 3.112,). (highest pitch) Going to read this number or (*speeding up*) given that I cannot be sure of the value of this number because GeoGebra has limits (Fig. 3.113), I can read it here (*she points to the "Algebra view" of GeoGebra*). In the "Algebra view" the point A has coordinates 37.5 and 8 and, then, the solution of the equation is the number 37.5. If instead of x I put 37.5 the two straight lines intersect and they have the same value (Fig. 3.114), ok? (*she nods and she returns to the pc*). Then in the activity it was asked to have two different colours, namely to colour red this one (*she colours of red* $y = \frac{1}{5}x + \frac{1}{2}$, $y = 8$ *and* A). Then we were asked to draw another two straight lines. The other two straight lines are: one is $y = 0.2x + 0.5 + k$ (Fig. 3.115). In this case, k is equal to 1 and we see that GeoGebra writes (*pointing to the "Algebra view"*) $y = 0.2x + 1.5$. Why 1.5? (*pause, a student try to say something but he does not finish the sentence*) because k	 Figure 3.112  Figure 3.113  Figure 3.114  Figure 3.115

317	S1: it is 1	
318	T: it is 1and, then, 1 plus 0.5 is 1.5, it has already calculated (*referring to GeoGebra*). The other straight line is $y = 8{+}k$ (*blaring tone of voice and she looks at the class*) if I write $y = 8{+}k$, in the "Algebra view" it will write $8 + k$? (*facial expression in* Fig. 3.116, *long pause and she continues to look at the class*) (*smiling*) I don't hear answers (*she looks at the class smiling*).	 Figure 3.116
319	Ss: no	
320	T: No, What will it write? (*facial expression as in* Fig. 8)	
321	S1: It puts the value of k	
322	T: k is 1 now, then, will it be equal to?	
323	Ss: 9	

Discussion

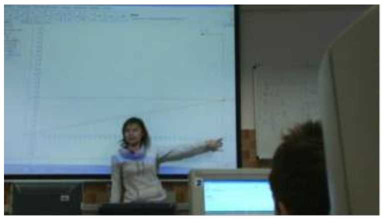
Sara works on GeoGebra in order to introduce the graphical solution of an equation. In the activity they have to solve the equation $\frac{1}{5}x + \frac{1}{2} = 8$. The solution of it is the time Luca uses to cover 8 km, starting from 500 m from the starting line. Hence, knowing that he uses 15 minutes to cover 3 km, namely his velocity is $\frac{1}{5}km/min$, the time he uses is the solution of $\frac{1}{5}x + \frac{1}{2} = 8$. The teacher reminds the students that they could work on two different straight lines: $y = \frac{1}{5}x + \frac{1}{2}$ and $y = 8$. She spells both "functions" and "straight lines" (#315). She recalls that the solution of the equation is related to the intersection point between them. Moreover, she clarifies that they have constructed the perpendicular line to the $x - axis$ passing through A, because the solution of the equation can be "read" on the $x - axis$. Stressing that the software has limits, she invites students to read the solution not directly on the graph, but on the "Algebra view" of GeoGebra. Hence, she seems hoping that students coordinating the "Algebra view" of GeoGebra and the abscissa on the graph. She accompanies this statement with an eloquent facial expression in Fig. 3.113. After that, she continues with the activity in which they are requested to draw another couple of straight lines depending on k: $y = 0.2x + 0.5 + k$ and $y = 8 + k$. The teacher highlights that GeoGebra gives automatically the value of k to the first function ($y = 0.2x + 0.5 + k$) and then she seems to want from the students the response for the second one ($y = 8 + k$). In fact, after asking with a blaring tone of voice what happens for $y = 8 + k$, she pauses as in Fig. 3.116. In this case, she hopes that students understand that GeoGebra makes immediately calculations in the algebraic part. Then, she smiles when she says that she isn't hearing any answers, probably, in order to keep the mood light (#316).

324	T: (*pronouncing*) What are we doing (Fig. 3.117)?	 Figure 3.117
325	S3: equivalent equations	

326	(*repeating and nodding*) we are constructing many equivalent equations. You remember that in the previous lesson we have said that we have equivalent equations (Fig. 3.118), namely equations written (*pronouncing*) in a different way, but that they have (*pronouncing*) always (*pausing*) the same result. (*highest pitch*) Do we have equivalent equations just for $k = 7.5$, for $k = -3$ (speeding up) that are the equations we have just seen? or do we have equivalent equations for many values of k (*she has returned on the pc and she moves k, smiling to students:* Fig. 3.119)?	 Figure 3.118  Figure 3.119
327	Ss: Many	
328	T: For many or for each value of k (*she continues to move k*)?	
329	Ss: for all of them	
330	T: for each value of k. For each value of k I obtain however equivalent equations. The filling of the table was just to write equivalent equations. For example, when I write $0.2x + 1.5$, what value has k to have 1.5? (*pause and she lifts her chin*)	

331	Ss: noise	
332	S2: 1	
333	T: 1. Then, If I give the value 1 (*she returns on GeoGebra to put k equal to 1*) I see that the equation is (*pointing*) $0.2x + 1.5 = 9$. (*pronouncing*) What happened to the sides of the equations? What did we do the sides of the equation (*she lifts up her chin:* Fig. 3.120)?	 Figure 3.120
334	S1: We have added 1	
335	T: we have added 1 (*pausing*)	
336	S1: to both sides	

337	T: (*smirking*) We have added 1 to both sides. In the previous lesson (*gesture to mime the past and she raises eyebrows:* Fig. 3.121), before easter holidays, we have said that the first principle of equivalence said us that we could add the same number to both sides and that the result of the equation continued to not change, ok? then I could add or subtract the same number to both terms and have (*pronouncing*) always equivalent equations. Then, what does it mean (*returning on the "Algebra view"*)? It means that I can add to both sides (*moving k*), see that the blue straight lines have the same movement, they have the same translations (*she mimes the translation:* Fig. 3.122), namely they have exactly the same movement, then we add or subtract to both sides exactly the same quantity, our result doesn't change. If I wanted to obtain the result of the equation, I would take k, I would do such that B coincide (*pronouncing*) exactly with the $x - axis$ (*she is doing it on GeoGebra*). To let coincide B exactly with the $x - axis$, what value I have to give to k?	 Figure 3.121  Figure 3.122
338	S11: -8	

339	T: -8. If I give -8 to k, what happens is that B belongs to the $x - axis$ (Fig. 3.123). The second side of the equation (*pronouncing*) takes the value 0. The first side of our equation has a certain expression and I, actually, go (*pronouncing*) to see where the blue equation intersects the $x - axis$ (Fig. 3.124). I go to find what it is called the (*pronouncing*) zero of function (*gesture to accompany the pronouncing*) because it is the point in which the straight line touches the $x - axis$, ok? More or less it was to try to remember the first principle of equivalence, so, if you pay attention to what we have done, we should have seen the first principle of the equivalence working with balance (*she mimes the balance:* Fig. 3.125), because we added (*she mimes the balance also with all the body*) or subtracted small weights from the balance in equivalent manner to both sides and it remains in balance (*pronouncing*) or we can obtain the first principle starting from a situation of this type (*she moves again k and she looks at the class in an emblematic way:* Fig. 3.126).	 Figure 3.123  Figure 3.124  Figure 3.125  Figure 3.126

Discussion

After working on the graph, Sara prompts the students to link the vertical translation of the straight lines to the concept of equivalent equations. In particular, she explicitly asks to her class what they are doing. She accompanies this action with pause as in Fig. 3.117, in which she is waiting for feedback from the class (#324). Then, satisfied, she repeats, nodding, what a student answers (#326). To accomplish her goal, Sara remembers what is the definition of equivalent equations. She spells "in a different way" and "always" (#326). Moreover, Sara shifts the attention of the students on for how many values of k they can have equivalent equations. This question comes along with an increasing tone of voice and her emblematic posture in Fig. 3.119, in which she seems quite relaxed that students are able to response. Actually, while Sara moves the slider k, students become aware that they can have equivalent equations for infinite values of k. She explains how the first principle works, showing that if k is 1, GeoGebra adds automatically 1 on both sides (#330, #333). She accompanies this discussion with many questions to her students, pauses and facial expressions with the chin up (#330, #333). It is quite clear that she is waiting answers from the class. Furthermore, this expectation is proven also by her smirk in #337 when a student says that they have added 1 to both sides. Then she repeats what the first principle says, with the facial expression in Fig. 3.121 and pronouncing another time "always" (#337). In terms of what happens on the graph, she highlights that the straight lines are translated of the same value, hence the result doesn't change. To explain what happens she uses a specific example: adding 0 to both sides. In fact, she invites her students to move the intersection point of the straight lines on the $x - axis$. She stresses this fact pronouncing "exactly with the $x - axis$" (#337). At this moment, Sara introduces the concept of the zero of a function and she spells both "zero of function" and "because it is the point in which the straight line touches the $x - axis$" (#339). At the end, she explicitly links the first principle of equivalence using the balance and what they are doing now with GeoGebra. She accompanies this fact gesturing the balance and pronouncing the sentence of the geometrical interpretation of the principle of equivalence (#339). At the end, she finishes looking at the class remaining in posture as in Fig. 3.126.

Expectation 4

In the a- priori interview, Sara explains what is, for her, the bound between Arithmetic and Algebra. The interviewer explicitly asked her: "do you try to link the introduction of Algebra with previous mathematical topics?"

> "Yes, particularly to numbers, yes exactly this way, it is a thing that I try to do. I want that they understand that they (Arithmetic and Algebra) are not two (gesture, Fig. 1) different things, but that they are (pronouncing) exactly the same thing. For me when they understand that they are the same thing when I return to speak of numbers at the second year in which they have to radicals, they understand that we don't speak with different languages (gesture), but that we are speaking of the same language and they are facilitated in understanding. In a sense I want to pass the idea that Algebra can be considered an extension of Arithmetics".

The interviewer asks to Sara if she perceives obstacles in the passage from Arithmetics to Algebra and she states that:

> "For me, it is necessary to pay attention and to go very slowly in order to not force the passage. They must have all things clear, they have to having understand what it means the employ of a letter and when employing it. I don't force students to see this passage, I wait that they make it spontaneously."

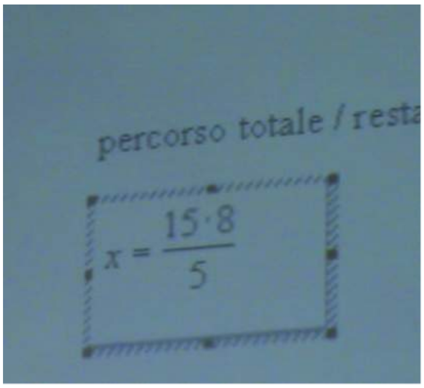
In another passage of the interview, when it was asked to Sara if she returns to Arithmetics after introduced Algebra, the teacher explicitly declares:

> "Yes, I return to Arithmetics when I want that they see that we can use Algebra to return to Arithmetics, for example, in the fast operations, for example if I have to calculate 12^2", I can use the square of the binomial $(10 + 2)^2$. This way it becomes simpler. Moreover, I return to Arithmetics when I start from arithmetics problem and then we use Algebra to create a model of it and then we return to Arithmetics for seeing if the model is correct or not."

From what the teacher declares in the a-priori interview, I infer that Sara has the *expectation that students learn to use the algebraic language as an extension of the arithmetical one.* This expectation is reflected in many excerpts of her lessons. Below, you can see some examples of them.

340	T: If I wanted to (*pronouncing*) put in formula (*pause*, Fig. 4.87) (*highest pitch*) what I'm saying, how could I do it (Fig. 4.88)? If I wanted to put it in formula, how could I do it (*she remains in the same manner for many seconds:* Fig. 4.89)? We have seen that a formula serves to speak in mathematics (*smiling*) a little bit, then if I wanted to put in a formula (*pronouncing*) the time used to cover the entire path, what can I do to put it in formula?	 Figure 3.127  Figure 3.128  Figure 3.129

341	S1: but, in words?	
342	T: no, with a formula (*smiling* Fig. 4.90). We have already written it in words, you have told me that we have done the proportion and then you have solved the proportion or you have calculated the time necessary for doing one kilometer and then you have multiplied the time necessary for doing one kilometer times 8 (*she is waiting again on an answer as in* Fig. 4.91, *playing a little bit nervously with her ring*).	 Figure 3.130  Figure 3.131
343	S4: they are proportional	

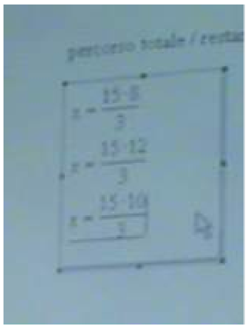
344	T: they are directly proportional yes, because we have made a proportion and then they are surely directly proportional	
345	S6: x is equal to the minutes already done times the total kilometers divided by the time for doing one kilometer	
346	T: Then if I had to write it, how can I do it?	
347	S6 dictates to the teacher	 Figure 3.132
348	T: (*she repeats what S6 says*) 15 times the total kilometers divided by the time used for one kilometer. Ok this one could be (*smiling*) right.	

Discussion

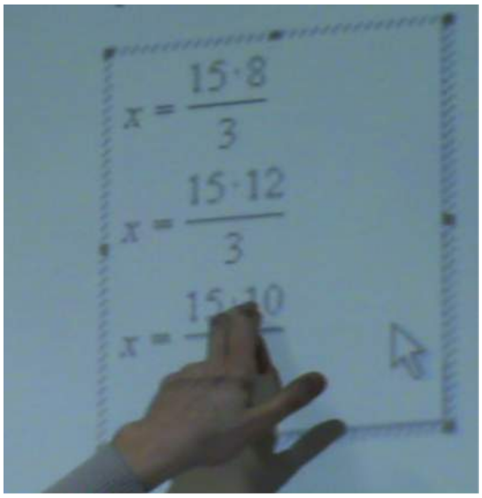
Sara declares to her class that she wants to make a step forward, putting in the form of a formula what they have explored in Arithmetics[17]: namely to make a formula for the time used to cover the entire path of Luca. She emphasize the words "to put in a formula" making a little bit long pause after the. It seems quite clear that, after exploring the problem with different numbers, she inclines her body to conduct the class to use the algebraic language as an extension of the arithmetical one. The most interesting thing

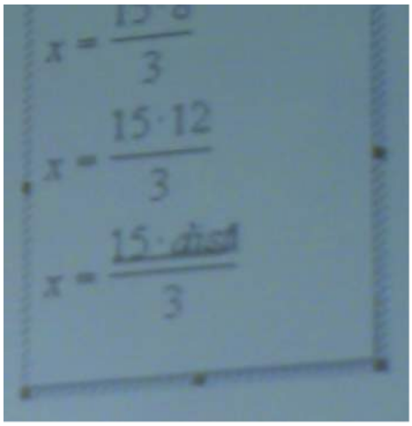
[17]with words and proportions

is that she repeats two times, one after the other, the same words "if I wanted to put it in a formula, how could I do it?" (#340). Then, she waits for an answer as she remains in the posture shown in Fig. 3.129. She recalls that they have seen formulas as ways of speaking of mathematics. Simultaneously, she ironically smiles, probably, because she uses the adverb "a little bit", even if she is perfectly aware that formulas are one of the fundamental mathematics tools. In particular, Sara wants to put in a formula what they have found with the proportion constructed to solve the starting problem.

349	T: good, then, if I went ahead this way, if I tried to continue this way (*gesture with hand to indicate the repetition and she is referring substituting* 8 *km with another number*) actually what am I doing? namely, if instead of having 12 *km* I would have uhm (*gesture for indicating the randomness*) 10 *km* what should I do?	
350	Ss: 15 times 10 divided by 3?	
351	T: x equal to	
352	Ss: 15 times 10 divided by 3? (Fig. 3.133)	 Figure 3.133
353	T: ok, 15 times 10 divided by 3 and if we wanted to put somewhere some letters, what could we put? because she told me "lets' put letters, ok?" (*referring to a student*), ok?	
354	Ss: noise	
355	S3: instead of 10 I put	

356	T: instead of 10 I put (*waiting for an answer:* Fig. 3.134)?	 Figure 3.134
357	S3: a letter	
358	T: a letter	
359	S3: for every length that I covered	
360	T: for every length that I covered, I can put a number of any length of path and calculate what?	
361	S4: x, the time	
362	S5: the time	
363	T: the time (*nodding*)	
364	S3: the total time to cover that length	

365	T: oh (*as if to say "well done!"*) I could put here (*pointing the kilometers in the formula*) a measure, a letter that represents the distance and calculates the time used to cover that corresponding distance, right? (Fig. 3.135)	 Figure 3.135

366	T: instead of putting 10 that is the distance already covered, I could put a letter, I could put something and I'm able to calculate the distance (Fig. 3.135). In so doing you calculate the time that I use to cover a distance, obviously the time (*pause*) depends on the distance. (*satisfied tone of voice*) Good job!In writing this (*she returns to the screen and she points:* Fig. 3.136), is there someone that notices something in mathematics, a mathematical object that you have already seen somewhere (Fig. 3.137: *she inclines her body towards the class, then she pauses and she looks at the class*)?	 Figure 3.136  Figure 3.137
367	Ss: noise	
368	T: when you write 15 times 8 divided by 3, 15 times 10 divided by 3, 15 times 12 divided by 3.	
369	S6: they are equations	
370	T: (*satisfied tone of voce*) they are equations	

Discussion

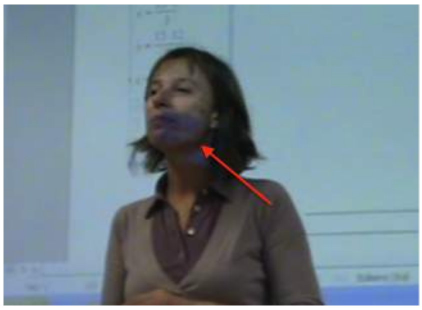
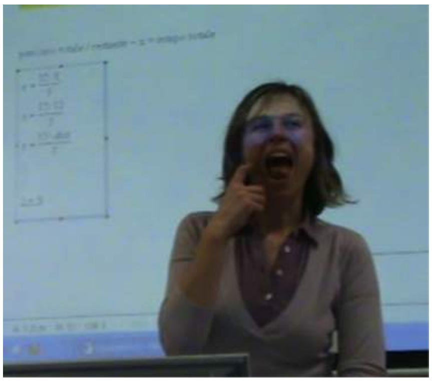
Sara continues to push her students towards generalization, inviting them to consider, now, $10km$ as distance. A student answered: "x is equal to 15 times 10 divided by 3". But, Sara notices that they have suggested to put some letters in the formula. Hence, she is waiting for an answer (Fig. 3.134) and, actually, a student proposes to put the letter for the distance. The teacher exclaims "oh" to express her satisfaction for the answer (#365). In addition, justifying the introduction of the letter, she explicitly says "Good job" (#366). It is quite clear that she is expecting that students are able to use Algebra as a generalization of Arithmetics. At the end she asks students which objects are the formula they have constructed. When a students answers "equations", she repeats the same thing with a satisfied tone of voice. This could be seen as another hint of her expectation of using Algebra (and in this particular case equations) as a generalization of Arithmetics.

Expectation 5

In the a-priori interview, concerning the role of examples, Sara explicitly declares:

> "I would not know if students are able to understand with examples, but it seems to me that examples can be a way for helping them in understanding. Surely, when they have to propose an example they have to think to the meaning. Let's say that from the example I expect to understand if they are understanding. The choice of an example can be useful to understand. In unusual cases, someone understands simply from the definition."

Hence, I could identified the *expectation that examples are useful to understand the meaning of what they are doing.* For the teacher, the use of the example is also a means to know if students have understood what they have already done, because, if they have to make an example they have to think to the meaning of the thing of which they want to exemplify. In the segment below, one can see this expectation reflected.

371	T: Could you tell me examples of other equations? (*waiting, eloquent facial expression:* Fig. 3.138) (*she pauses for a while*) let's say an equation (*doing knee-bends*) an any one, simple (*smiling*).	 Figure 3.138
372	S7: 3 equal to 5	
373	T: (*in the meanwhile she writes*) 3 equal to 5? (Fig. 3.139, Fig. 3.140)	 Figure 3.139  Figure 3.140
374	S11: 5 minus 2	

375	T: 5 minus 2. Is it an equation?	
376	Ss: No	
377	T: No	
378	noise	
379	T: There must be an unknown	
380	S9: 8 equal to 2 times x	
381	S10: $x = 4$	
382	T: ok (*nodding*), If I write this way (*she writes* $x+45 = 4-3x$) is it an equation or not?	
383	S2: there isn't the equal sign	
384	Ss: yes there is the equal sign	
385	T: then, let's restart from the beginning. You have told me that the equation is an equality. Then when I have asked you to tell me an equation, you have said $3 = 5$ and $3 = 5 - 2$. Are these two equalities?	
386	S2: the second	
387	T: 3 equal to 5 is false, it is a false statement, but 3 equal to 5 is a statement, ok? I can perfectly say that 3 is equal to 5, but then it is not true, but I can say that 3 is equal to 5. Rightly, you have told me that for having an equation there must be an unknown, an unknown that I have called x, it could be called?	
388	Ss: any one	

389	T: (*nodding*) any letter of the alphabet, but there must be an unknown because if there isn't an unknown then the things don't work, we aren't speaking of equations, then probably saying that an equation is an equality between two terms is not completely correct. Let's say that we define (*gesture with the hand with the joined index and thumb moved up and down:* Fig. 3.141), let's say it immediately such that we are ok. We define (*again the same gesture:* Fig. 3.141) equation (*pronouncing*) as a statement in mathematics in which there is the verb, and that verb is (*pronouncing*) "to be equal to" (.) (*pause looking at the class*) ok? then we define equation as a statement in mathematics in which there is the verb "to be equal to". Obviously, in this statement there has to be at least one unknown, at least an unknown because there could be also more ok? (*she bites her lips and she looks at the class*). Listen! If I defined the equation as a statement with the verb "to be equal to", (*speeding up*) then it doesn't change too much if, instead of saying that an equation is an equality, we say that is a statement that has the verb "to be equal to" and we say that there in an unknown when I search (*pronouncing*) for a solution of that equation.	 Figure 3.141

Discussion

Sara decides to ask her class for other examples of equation with the aim of formalizing well what is an equation. This action is accompanied by a pause with the eloquent facial expression in Fig. 3.138, in which she seems to have

a searching mood. Then, she changes her attitude: doing knee-bends and smiling, she declares that she is satisfied just by a simple example (#706). She wants, probably, to put students at ease to answer. Actually, a student proposes $3 = 5$ and, first, she ironically smiles (Fig. 3.139). She is aware that this is not an equation but, probably, she doesn't want to say it. Then, she makes another interesting face (Fig. 3.140) in which she smiles again. She remains with the open mouth touching it with her finger. This posture could express her hope that someone asserts that $3 = 5$ is not an equation, even if, probably, it is hard, for her, to resist in saying it.
After the interventions of some students, the teacher states that there must be an unknown in an equation. She chooses the equation $x + 45 = 4 - 3x$f which she discusses with her class. Then, she returns to the definition of equation as equality. At this point, she recalls equalities on the two examples made by the students ($3 = 5$, $3 = 5 - 2$) to notice the difference between them and an equation. In an equality there are just numbers, while in an equation there must be at least one unknown. She stresses this fact and she defines formally an equation. She accompanies this definition with a gesture of moving up and down her hand joining her thumb and her index (Fig. 3.141). Probably, she wants to signal the fact that, now, she is giving a formal mathematical definition. Moreover, she spells the word "statement" and "to be equal to", repeating for two times, one after the other, the definition of equation (#389). This, probably, because, after the first time, while she is pausing and looking at the class, she is not sure that the whole class understands the definition. In addition, after clarification about how many unknowns there could be in an equation, she repeats for the third time its definition (#389).

Expectation 6

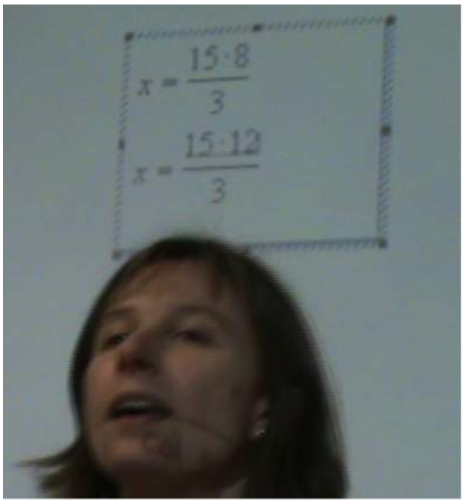
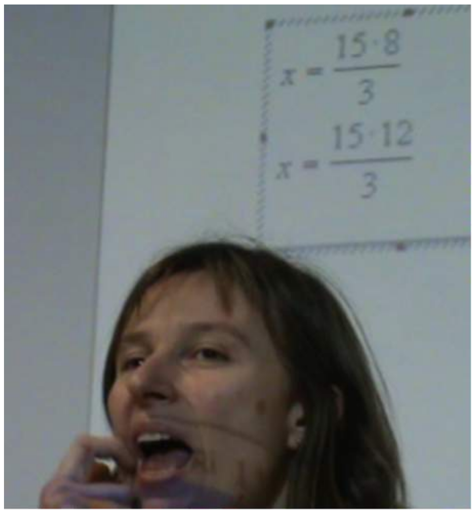
Asking to Sara which role she ascribes to the justification, she answers this way:

> "I give a very relevant role to justification in the sense that I insist a lot, I "hammer" students in why they have done certain things. Often, I seem that a simple problem, a very trivial one with adding of an "explain why" becomes more beautiful and articulate. This allows to go deep into the problem. I am very insistent on this fact that students continue to justify their answers even when I don't ask it in an explicit way. At the beginning, I ask in an explicit way to justify and then I realize they justify always

> what they do both in written and in oral. I pay very attention to the fact they discuss and prove what they are doing. I don't pay more attention to the fact that they justify in the most correct way from a mathematical point of view, namely, from the mathematical language point of view, especially, if they are in the first years of high school. But, obviously, going on I give also the attention on the formalism. In the last year of high school they are able to speak of mathematics in a correct way. In the first years, I does not accounted for the formalism in order to arrive to the content. The content, instead, has to be right, that is I pay attention that it is correct, but I don't care if it is said with a correct formalism."

From what she stated in the a-priori interview, I identified that Sara has *the expectation that justifications serve to go deep in the meaning of what they are doing.* the verb "to hammer" give well the idea of the fact that she hopes, through her teaching, that students develop the need of justifying to go deep in the meaning of what they are doing. This expectation can be seen in the following excerpts.

390	T: (*highest pitch*) but instead of calculating this way, (*very slowly and pronouncing all the words*) instead of calculating how much time Luca uses in 8 km, namely, how much time I use to make 8 km, I wanted to know how much time he uses to make 12 km, what should I do? (*gesturing, speeding up and smiling*) always thinking of the fact that Luca isn't ever tired and that he walks always at the same velocity?	
391	S10: 15 times 12 divided by 5	
392	T: 15 times 12 divided by 5	

393	T: where x is always the time, right? (*with a tone of voice indicating certainty*) let's write it somewhere (Fig. 3.142)	 Figure 3.142
394	S6: divided by 3	
395	T: good, why divided by 3 and not divided by 5 (*facial expression as in* Fig. 3.143)?	 Figure 3.143
396	noise	

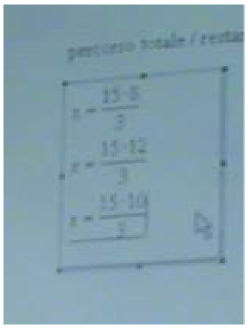
397	T: (*nodding*) not only, (*smiling:* Fig. 3.144) from a conceptual point of view, what does it mean 15 divided by 5? Make (*gesture:* Fig. 3.145)15 minutes divided by 5 kilometers, what does it mean (*she moves her fingers as if grasping for a justification, then opens her hand towards the students:* Fig. 3.146)	 Figure 3.144  Figure 3.145  Figure 3.146

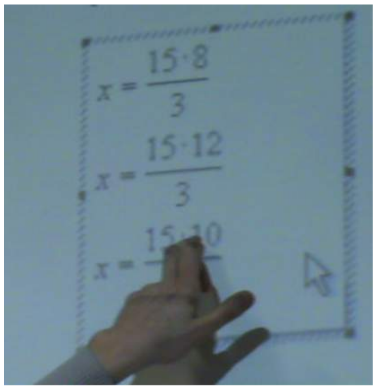
398	noise	
399	T: yes, it means that you have used 15 minutes to do 5 kilometers and not 15 minutes to do 3 kilometers, ok? And then Luca walks more slowly or more quickly?	
400	Ss: more quickly	
401	T: yes, he goes faster, then we would make Luca (*smiling*) a little bit too fast	

Discussion

A student answers that the time requested to make $12km$ is 15 times 12 divided by 5. Sara writes it on the pc, when another student notices that it is divided by 3 and not by 5. The teacher seems satisfied by the observation and she asks why it is divided by 3 and not by 5. The request of this justification comes along with a particular facial expression of Sara in Fig. 3.143. After an overlapping of answers of the class, Sara wants to focus the attention of her students on the meaning of the operation 15 divided by 5. This more specific request of justification is accompanied by the smirking of Sara in Fig. 3.144, probably, because that someone realized the error; by her gesture for highlighting the two protagonists of the ratio; by her moving the fingers as grasping the justification from her students Fig. 3.146 (#397). At the end of this passage, the teacher, smiling, uses the error to make an important consideration: if Luca had employed 15 minutes to do $5km$ and not $3km$, he would have gone too fast.

402	T: good, then, if I went ahead this way, if I tried to continue this way (*gesture with hand to indicate the repetition and she is referring substituting* 8 km *with another number*) actually what am I doing? namely, if instead of having 12 km I would have uhm (*gesture for indicating the randomness*) 10 km what should I do?	
403	Ss: 15 times 10 divided by 3?	
404	T: x equal to	

405	Ss: 15 times 10 divided by 3? (Fig. 3.147)	 Figure 3.147
406	T: ok, 15 times 10 divided by 3 and if we wanted to put somewhere some letters, what could we put? because she told me "lets' put letters, ok?" (*referring to a student*), ok?	
407	Ss: noise	
408	S3: instead of 10 I put	
409	T: instead of 10 I put (Fig. 3.148)?	 Figure 3.148
410	S3: a letter	
411	T: a letter	
412	S3: for every length that I covered	

413	T: for every length that I covered, I can put a number of any length of path and calculate what?	
414	S4: x, the time	
415	S5: the time	
416	T: the time (*nodding*)	
417	S3: the total time to cover that length	
418	T: oh (*as if to say "well done!"*) I could put here (*pointing the kilometers in the formula*) a measure, a letter that represents the distance and calculates the time used to cover that corresponding distance, right? (Fig. 3.149)	 Figure 3.149
419	T: instead of putting 10 that is the distance already covered, I could put a letter, I could put something and I'm able to calculate the distance (Fig. 3.149). In so doing you calculate the time that I use to cover a distance, obviously the time (*pause*) depends on the distance. (*satisfied tone of voice*) Good job!	

Discussion

The teachers wants to generalize the way to find the time to cover a given distance. She wants that students are able to put a letter instead of the number of the given distance. She seems quite satisfied because students have recognized what is the advantage to use the letter (#418). Then, she justifies that she can put a measure to indicate a generic distance in order to calculate the time used to cover that distance. At the same time, she indicates the number that represents the distance (Fig. 3.149) (#418). To stress the meaning of putting a letter instead of a number, she repeats for a second time that the use of the letter serves to consider any distance to cover (#419). Lastly, if before she could seem satisfied (#418), now she declares her satisfaction for the fact that students have understood why it is important to use the letter, explicitly saying "Good job" ((#419).

Chapter 4

The “fabric” of Rationality and Emotion

In Chapter 3, I attempted to highlight how the processes of the classroom strictly depend on the teacher involved. When we make an analysis on mathematics teaching, we have to be aware of the complexity of the teacher. This complexity depends on her beliefs and on her personal background that are reflected in her own activity. Drawing on her personal beliefs, the teacher develops, necessarily, expectations for her teaching. This emotional counterpart is always merged with the rational one. Hence, it becomes interesting to investigate how emotional aspects are intertwined with those of rationality. Using a metaphor, rationality and emotions of the teacher can be seen as the weave and the warp of the fabric. As the weave and warp entwined constitute the fabric, the rationality and the emotions entwined shape the teacher as she actually is.

I'm going to present several examples of Lorenza, Carla and Sara, in which it becomes clear the coexistence of the emotional aspects and the rational ones in mathematics teaching. Moreover, I will try to explain their interplay. Mostly, in a single excerpt of a lesson, there are more of just one expectation of the teacher. Then, to analyse the discursive activity of the teacher in its whole complexity, I will discuss pieces of the lessons surfacing also this intricacy. In particular, I will show how this interplay outlines the decision-making processes of the teacher. All the teachers' decisions – about knowing, acting and speaking – are “visible” in language, but, mostly, in her emotional aspects. This doesn't mean that emotions explain the decisions, but, rather, that decisions are very often “visible” through emotions. These emotions originate from expectations of the teacher developed on her own beliefs. As the literature in mathematics education highlights, “emotion has been used less in mathematics education research – so far – despite being arguably the most fundamental concept" (Zan & al., p. 116). In fact, the scientific research suggests its cruciality suggesting "how repeated experience of emotion may be seen as the basis for more "stable" attitudes and beliefs" (Zan & al., p.116). The most difficulty encountered in studying emotions is their "visibility" and, then, their "certain" identification. Bypassing the problem, I will speak of "emotionality" of the teacher, where the definition of it is taken from "The Penguin Dictionary of Psychology" (Reber and Reber, 2001):

> *because of all the confusion surrounding the connotations of the term EMOTION many writers favour this term. Their point is to try to avoid the surplus meanings of emotion by operationalizing the term. In this sense emotionality is defined in terms of behaviors that are observable and theoretically linked to the (hypothetical) underlying emotion. Note that this meaning of the term is actually not far removed from the ordinary sense in which it is used; i.e. to refer to the degree with which an individual reacts to emotive situations, with the connotation that such displays are often excessive given the circumstances. In both the behaviorist's technical sense and the layperson's common meaning the underlying notion is that it is the behavioral manifestations that are taken as the critical component in assessment of the emotion experienced.*

In our case, the emotionality of the teacher will be disclosed by her prosody, her gestures, her facial expressions, her postures and so on. As I will attempt to present in this chapter, the emotionality will be always intertwined with the rationality of the teacher. Then, we will give to the emotionality of the teacher the adjectives of the habermasian rationality. For this reason, we talk of *epistemic emotionality*, of *teleological emotionality* and of *communicative emotionality*.

For example, the epistemic emotionality surfaces when the teacher decides to draw on the notion of the concept of identity they have already introduced previously and, simultaneously, remaining in a waiting posture, she nods after knowing that students actually remembers it. This way, it is not just which kind of knowledge she chooses to consider (epistemic rationality), but why she decides to draw on that particular knowledge. The reason is connected to her expectation that the previous knowledge is valid for students and this expectation is made visible through the posture of waiting for something and her nodding after the positive feedback of the students.

The teleological emotionality could be highlighted when the teacher justifies which is the meaning of solving a linear equation and, at the same time, she has highest pitches of the tone voice corresponding to the word "sense". There is not just the action the teacher makes to accomplish a goal, namely justifying the resolution of a linear equation for finding the solution, but also the fact that she has the expectation that the justifications serve to give meaning to what they are doing. This expectation can make be visible through the highest pitches of her tone of voice when she says the word "sense".

The communicative emotionality can be surfaces when the teacher has in insistent rhythm of the tone of voice in asking examples. Then, there is not just matter of the speech oriented toward reaching understanding within the classroom, but also why she decides to communicate with an insistent rhythm. The reason is connected to her expectation that students are able to make examples and the expectation become clear through the insistent rhythm of the tone of voice.

Overall, I consider the *epistemic emotionality* as related to *why* the teacher uses that specific justification of the knowledge at play; the *teleological emotionality* as related to *why* the teacher makes that actions to achieve a goal and the *communicative emotionality* as related to *why* the teacher uses that speech oriented towards reaching understanding within the classroom.

For pragmatic necessities of analysis, these three types of emotionality could appear separated. Nevertheless, it is important to stress that they are always intertwined and present in the discursive activity of the teacher.

As you are going to see in the analysis, often, I will highlight aspects of the emotionality of the three teachers that, at a first glance, could appear similar. For example, all of them pronounce key-words, smile, waiting for answers and so on. But, as I will stress in Chapter 5, the meaning of these "common" emotional aspects is deeply different and it is reflected in the diversity of the reasons of their decisions taken within the classroom.

4.1 The “fabric” of Rationality and Emotion in Lorenza

This example is taken from the first lesson after Easter holidays, during which the teacher was recalling the concept of identity – explained in the last lesson before Easter – with the aim of introducing, formally, the concept of equation.

420	T: before holidays, I hope that someone remembers just something, we have spoken about (*spelling*) identities, then is there someone who wants to give, for now, the definition of identity and to do only an example (*tone of voice of a statement and not of a question*) of identity? Don’t be shy! (*she lifts up her chin, she smiles, she is waiting for an answer, biting her lips*, Fig. 4.1, Fig. 4.2, Fig. 4.3, Fig. 4.4). Please (*referring to a student who raises up his hands*)	 Figure 4.1  Figure 4.2

421		 Figure 4.3  Figure 4.4
422	S1: it is an equality that is verified for each value that replacing to the letter	

423	T: Fine, it is an equality between two expressions that contain letters that is verified for each value we go to ascribe to the unknown. One example, we have done an example within the classical ones (*she smiles:* Fig. 4.5)	 Figure 4.5
424	S1: $(a+b)^2 = a^2 + b^2 + 2ab$	
425	T: for example, the development of a special product is an equality between two expressions that contain letters and then it can be considered an identity and each value we go to give to the unknown a or to the unknown b, the result on the left and on the right of the equality sign must be the same and, conversely, what can be considered as an equation, do you remember? (*speeding up*) you have already seen them in middle school partly, right? we have already reviewed them in physics since at the beginning of the year, they serve us for working with formula etcetera, so we have already given indications but in the light of the path we have done, any of you would like to hazard a definition of equation (*tone of voice of a statement and not of a question*, Fig. 4.6), let's try to hazard, Andrea	 Figure 4.6

The action of Lorenza of asking something that students have already known[1] is aimed to construct the concept of equation[2]. This action comes along with a particular tone of voice proper not of a question, rather of a statement (#420). The affirmative tone of voice of the question and that facial expression (#420: she lifts up her chin after speaking, Fig. 4.1) could show her expectation that someone remembers the concept of identity and will answer to her, because the class has already seen it a short time before. In particular, the tone of voice is affirmative instead the facial expression is proper of a question (Fig. 4.1). Hence, I could highlight a sort of mismatch between the tone of voice and the facial expression. Probably, the teacher does not want to markedly stress the request, because she is expecting that the definition of identity comes naturally from the class. Then, waiting for an answer, she laughs (Fig. 4.2) and she bites her lips (Fig. 4.3, Fig. 4.4), probably, because she wants feedback from the class. The action of asking something that students should know is full of emotional hues linked to her expectation about the validity of the previous knowledge. Hence, the *teleological emotionality* of Lorenza involves both her action to recall the concept of identity in order to introduce, after, the concept of equation (rational key) and her need that students are able to respond (emotional key). It is not just a matter of what she is doing, but rather of how and why she is acting in that way. The emotional key is shown by her prosody, her facial expressions I have already highlighted above. From the beginning, her speech seems to be charged by emotions (1: “I hope”, “just something”, “Don’t be shy!”). These emotions are related to her expectation (“I hope”) that students remember the concept of identity, even “just something” (she can be easily satisfied, as long as, they are able to say something). Lorenza seems quite sure that she explained well in the past. Then, she seems quite confident also about the knowledge of students, thinking they don’t answer because they are shy. Hence, she incites them into doing, using the imperative phrase “Don’t be shy!”. Having hopes and needs, Lorenza cannot be neutral in her speech. This “emotion-soaked” speech constitutes a *communicative emotionality* of Lorenza related to her expectation about classroom culture. There is not only what she is communicating, but also how and why she is doing it that way. Requiring again the example of an identity (#423), after the answer of S1, could be interpret as a way of involving more students in the discussion and to evoke the classroom culture as much as possible (#423: “One example, we have done an example within the classical ones”). Another time the teacher’s speech comes along with an emotional element (she smiles, Fig. 4.5), because she seems to

[1] The definition of identity and an example of it

[2] This global aim will disclose in the remaining of the lesson.

feel that students need to be comfortable for answering, even if they already should know the example. Lorenza recalls just the term of identity to shift easily to that of equation: the former is an equality always true for every value of the unknown; the latter is an equality that can be impossible or, if it is possible, it can be undetermined or determined. This epistemic shifting comes along with an insistence of Lorenza on the fact that they have already seen equations both in middle school and with her (#425). In addition, she asked her students "to hazard" a definition of equation, but, another time, with the tone of voice of a statement and not of a question (#425). Probably, she does not want, with the voice, to show students her expectation that they are able to construct new knowledge from the previous one. But, her facial expression with the chin up (Fig. 4.6) looking at the class is proper of a question. Hence, there is another mismatch between the tone of voice and the facial expression. It could testify her hope that students can construct the definition of equation themselves. In addition, in Fig. 4.6 she plays a little bit nervously with her ring waiting for feedback. Then, her *epistemic emotionality* is identified both by the definition of identity and an example of it (rational key) and by her hope that students are able to reconstruct them (emotional key). The insistence in the speech, the facial expression, the tone of voice linked to the request of the definition and the example of identity show the emotional involvement of the teacher.
Then, the discussion goes on as you can see below:

426	S2: it is an equality between two literal expressions in which the value of x is replaced by a unique value to make it true	
427	T: we say that it is satisfied just f(or) (Fig. 4.7)	 Figure 4.7

428	S2: for a single value of x	
429	T: always?! (Fig. 4.8), do we always find it?(Fig. 4.9) for you this value or, let's try to think a little bit	 Figure 4.8  Figure 4.9
430	S2: sometimes it's impossible	

431	T: it could be right (Fig. 4.10). Let's try to make an example of one thing of this type, we have given a definition that then we will write, so the fundamental difference is that an equation is satisfied, if it is possible, for a single value, let's see to make an example of equation, who wants to make an example of equation? (*tone of voice proper of a statement and not of a question*, Fig. 4.11), Oscar	 Figure 4.10  Figure 4.11

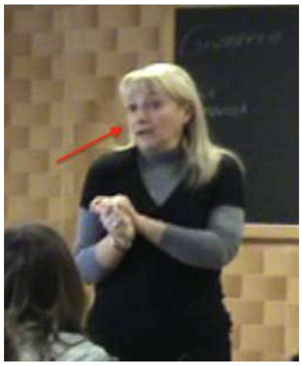
After the answer of S2 (#426), Lorenza clarified his definition of an equation. This action is aimed to begin to introduce the different types of equation (determined, undetermined, impossible) that she will develop in the next lesson. The rhetorical questions (#429: "always?!", "do we always find it?!"), the facial expressions in Fig. 4.8 and Fig. 4.9 and the general involvement in the discussion (#429: "let's try to think a little bit") accompany the actions she made to treat the classification of the types of equations. Particularly, in

Fig. 4.9, she seems to "catch" with her hands what they are in mind about the concept of equation. Hence the *teleological emotionality* of the teacher is constituted both by her actions made to introduction the classification (rational key) and by her hope of recovering the previous knowledge from the students. This emotional key is revealed by the rhetorical questions, her gesture to "catch" her knowledge and so on. After the answer of S2 (#430), the teacher incites students to make a suitable example in order to construct the theory. She repeats two times the fact of making examples: the first time she involves also herself in the discussion (#431: "Let's try to make an example"). Then, she recalls what changes in an equation from an equality and, for a second time, she invites students to make an example (#431: "who wants to make an example of equation"). Moreover, she does not use the tone of voice of a question, probably because it is quite sure that students know what she is referring to. Another time, there is a mismatch between the tone of voice of the question and the facial expression accompanied to it. Indeed, the tone of voice is affirmative, while in the facial expression in Fig. 4.11 she is waiting for an answer from a student. Probably, she does not want to underline her request of an example of equation, even if she is expecting it. Hence, not being able to hide her hope, she show it with the facial expression in Fig. 4.11. Hence the *epistemic emotionality* of Lorenza is constituted both by the example of equation (rational key) and by her expectation that students are able to make adapted examples (emotional key). In fact, the teacher hopes that students are used to make examples and that they are illuminating for them, also as a sort of validation of what they are saying. This emotional key is revealed by her insistent rhythm of the request of examples, by the mismatch off her tone of voice and her facial expression. Lorenza has established a communication channel between her and the class. Her speech is full of eloquent facial expressions, gestures and specific tone of voice that students seize and answer also grounding on how she is speaking. In other words, students understand what she is expecting from them because she cannot hide her hopes and needs. For this reason we talk of *communicative emotionality* of Lorenza.

432	T: Where does the concept of equivalence come from, eh?(*she frowns:* Fig. 4.12), we have already studied it, who remembers when we have spoken of equivalence, do you remember? (*tone of voice proper of a statement not of a question*) Do you remember (Fig. 4.13) the equivalence relation, never (*she shakes her head*), never (*nervously smiling:* Fig. 4.14), we have done the relations, do you remember? We have defined the equivalence relations, those of admitted	 Figure 4.12  Figure 4.13  Figure 4.14

433	S19: S19: those symmetric	
434	T: yes	
435	S19: reflexivity, symmetry, transitivity	

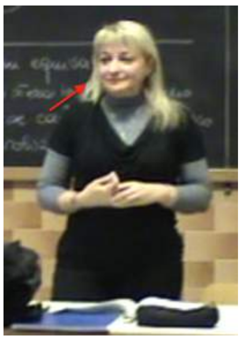
Lorenza has just explained the concept of equivalent equations, as those that admit the same set of solutions. The action of Lorenza of linking equivalent equations to equivalence relations is aimed to show that the relation among equivalent equations is an equivalence one. At the beginning, she decides not to introduce immediately the term "equivalence relation", rather she gives to students some clues about it (#432: "Where does the concept of equivalence come from?", "we have already studied it", "who remembers when we have spoken of equivalence?"). Unfortunately, none seems to remember what she is asking. Hence, Lorenza becomes more explicit, introducing the term "equivalence relation". This action comes along with a particular tone of voice proper of a statement and not of a question, even if she is asking if students remember what is an equivalence relation. Simultaneously, she says "equivalence relation" (# 432) and she makes the gesture in Fig. 4.13, miming the past. Hence, the *teleological emotionality* of Lorenza involves both her actions to recall equivalence relations (rational key) and her hope and need that students are able to remember them (emotional key). This emotional key is revealed by the affirmative tone of voice of the question "Do you remember?" (#432) that could testify her expectation that students actually remember it; by her gesture of miming the past to give students a hint about when they have already spoken about it. As you noticed, from the first row of the discussion (# 432), her speech is full of emotions, indeed, beyond the involvement of the whole class in the discourse, she makes many questions one after the other and she repeats many times the verb "to remember". Always in the #432, when none responses to her, she repeats for two times the adverb "never" smiling in a quite nervous way, as it is showed in Fig. 4.14. This emotional speech constitutes the *communicative emotionality* of the teacher. She cannot be plain in her discourse, then her expectation that students remember and re-elaborate the previous knowledge cannot be hidden. She expresses it through her somatic language and her prosody.

436	T: Has still meaning to link that concept to the equivalent equations, for you?! (*rhethorical question*). [nervously] If we take into account two equations and we suppose that they are equivalent, the property, that is reflexivity, every equation is equivalent to itself, (*rhetorical question*, Fig. 4.15)?	 Figure 4.15
437	S3: yes	
438	T: sure, because it admits the same solution, (*rhetorical question*) symmetric?	
439	S4: yes	
440	T: if an equation is equivalent to a second one, then the second one is equivalent to the first, because it admits the same solution, yes? (*rhetorical question, another time she raises eyebrows and shoulders as in* Fig. 4.15) transitivity?	
441	S5: no	
442	T: (*nervously disappointed*) why not? If this (*she is pointing the example done in the previous session of the lesson*) is equivalent to a second one and the second is equivalent to a third one also the first and the third are equivalent each other, you see that the concept of equivalence relation returns, right? because actually the relation among equivalent equations gives us an equivalence relation among the different equations, (*annoyed*) it's nothing of new.	

In this brief excerpt, Lorenza justifies the previous recalling equivalence relations to prove that the relation among equivalent equations is an equivalence one. At the beginning, she decides to ask her students if it “has still meaning to link that concept to the equivalent equations, for you?!” (# 436). This is a rhetorical question probably because Lorenza is quite sure that students are able to give the answer themselves. After any feedback from the class, she is not so sure anymore of it. Then, she, explicitly, asks them if two equivalent equations satisfy the first property of equivalence relation, namely the reflexivity. In the brief discussion with her students, Lorenza verifies with them the other two properties, namely, the symmetric and the transitivity. When she justifies the fact that the relation among equivalent equations is an equivalence one, she uses many rhetorical questions because, probably, she is expecting that students know these properties (#436). They, actually, response “yes” two times (#437, #439), but, concerning the transitivity, one student answers “no” (#441). Lorenza appears disappointed, probably, because she is quite sure that they are able to construct this analogy from the previous knowledge, being able also to justify why it is satisfied also the transitivity. At the end of this brief excerpt, she recalls them that what they are doing is “nothing of new” (#442). The choice of recalling the particular epistemic of “equivalence relation” is could be justified by her expectation that students are able to remember it, in order to link it to the new concept of “equivalent equations” they have just introduced.
Hence, the *teleological emotionality* of the teacher is constituted both by her asking students if there is a sense in linking the equivalence relations to the equivalent equations (rational key) and by her hope that students have already seen this link before her question (emotional key). This emotional key is revealed by her rhetorical question in #436. In addition, when none answers, she nervously starts with the reflexivity and she makes another rhetorical question (#436). This is another expression of her expectation that they are able to use the equivalence relation. Her *epistemic emotionality* involves both the proof that the relation is an equivalence one (rational key) and by her hope that students, after her incipit, are able to understand why it is an equivalence relation (emotional key). This emotional aspect is revealed by the fact that she makes always rhetorical questions (#436, #438, #440); by the fact that she become nervous and disappointed when a student answers “no” concerning the transitivity of the relation (#442). In addition, it could appear quite clear the need of the teacher that students remember what is an equivalence relation also at the end from her words “it’s nothing of new” said in annoyed manner (#442). Lorenza has need and hopes from their students and, in general, from her teaching. This does not allow her to be neutral in the communication. Hence, what she is expecting is expressed

in her somatic involvement, in her prosody and so on. In this short passage of the lesson, she uses many rhetorical questions and she changes the tone of voice. This changing surfaces when she is disappointed from the answer of students, when she is quite sure that they have grasped the answer, when she stresses in an annoyed manner that they have just seen what an equivalence relation is. For this reason, I speak of the *communicative emotionality* of the teacher.

In the lesson below, Lorenza is introducing the principles of equivalence. As one will see in the analysis, her rationality is always entwined with her expectations that are spread along her discursive activity.

443	T: Let's start from the first, the first principle, there are two of them, ok? from the first principle will derive some calculation rules (*speeding up and sure*) that are those you apply (*gesture for miming the “mechanically"*) mechanically, (*speeding up and tone of voice proper of a statement, even if, at the beginning, she seems to be a question*) you have already learnt them (*she mimes the past*, Fig. 4.16). And from the second principle will derive some rules of calculation. What does the first principle say? it says (*in the meanwhile she is reading from the textbooks and she is dictating to her students*, Fig. 4.17): given an equation if we add, oh sorry, I didn't say that this is called first side (*she circles the first side of $ax = b$*) and this one is called second side of the equation, (*speeding up*), but, probably (*she smiles nervously*, Fig. 4.18), we have already spoken about that, yes? you know that an equation, being an equality between two expressions, the expression that is on the left of the equal sign is called first side, while that on the right is called second side. (*dictating from the textbooks*) so, (*loudly*) given an equation if we add to the two sides (*she mimes the two sides and then she returns on the textbooks*) a given number	 Figure 4.16  Figure 4.17  Figure 4.18
444	S9: a given number?	

445	T: yes, (*speeding up*) then I write it in symbols, ok? now, let's try to understand	
446	S8: Can you repeat?	
447	T: So, given an equation if we add to the the both sides of it the same number or the same expression we obtain an equivalent equation to the given one.	
448	S7: we obtain?	
449	S10: after the same number?	
450	T: or the same expression, we obtain again an equivalent equation to the given one.	

Lorenza starts her discussion with the class from the first principle of equivalence and she anticipates the existence of two rules she wants to introduce as consequences of it. As you can see after, they are the transportation rule[3] and the cancellation rule[4]. From the beginning she stresses, speeding up her speech and in affirmative tone of voice, that students have already learnt these calculation rules (Fig. 4.16). Her way of recalling the two rules could testify her certainty about what is the students' previous knowledge (in this case, that constructed in middle school). [5] So, it emerges a *teleological emotionality* of Lorenza that involves both her action of recalling something of known by the students in order to use it for the new mathematical topics (rational key). Moreover, it is constituted by her need that students remember what they have already done in middle school that remains valid for her (emotional key). This emotional key is disclosed by her certainty, by her speed in speaking and by her nervously smiling (Fig. 4.18) . She seems that she would take for granted the fact that they have already seen the rules of equations and she wouldn't recall it, but, at the end, she prefers to say it. Summarizing, her emotional involvement is the expression of her expectation about the validity of the previous knowledge.
Lorenza begins to explain the first principle of equivalence giving to the class the definition of it, dictating from the mathematical textbook. The defini-

[3]In an equation, if a term is "carried" from one side to the other one, it has to change the sign

[4]In an equation, if there are two equal terms in the two sides of it, they can be deleted each others

[5]From the beginning of the year, she never recalls the two rules with the class.

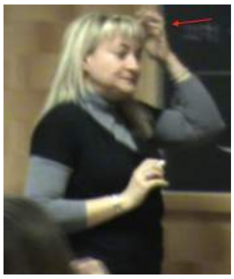
tion of the textbook is: “given an equation if we add to the two sides of it a given number or the same expression, we obtain an equivalent equation to the given one”. The knowledge that she puts into play comes along with the rhetorical question (#443: “What does the first principle say?” and then she responses immediately to herself), with the loud tone of voice when she gives the definition (she seems quite sure of what she is saying) and with, very often, her gazes on the textbook while she is speaking. These emotional aspects could testify that Lorenza justifies the knowledge at play through the authority of Algebra (#443: “What does the first principle say? it says”) and that of the textbook (because she dictates the definition from it). Hence the *epistemic emotionality* of the teacher is constituted both by the definition of the first principle of equivalence (rational key) and by her need of referring to what Algebra “says” in the mathematical textbook (emotional key). This emotional key is revealed by the rhetorical question in which she personified the first principle of equivalence; by increasing the tone of voice to give the formal definition of the textbook; by her many gazes to what is written on the textbook (Fig. 4.17). These emotional aspects could reveal her expectation that authority (in this case, “the first principle” and the mathematical textbook) ensures the acceptability by the class of what she is explaining. Then, the justification of the knowledge into play is linked to the expectation about the authority. Moreover, Lorenza stresses, speeding up (it’s just an anticipation) that after the “verbal” definition of the first principle she will shift to symbols, even if she wants that students understand initially the definition (#445: “yes, (speeding up) then I write it in symbols, ok? now, let’s try to understand). This moment of the discussion could reveal her expectation about the coordination among different registers of representation. Lorenza has hopes and needs in her teaching, hence it is very difficult for her to avoid emotional overtones. Then, the words of her speech are always accompanied by emotional timbres. In this excerpt, there is more prosody than body language of the teacher. Indeed I signaled just the features of her speaking, like the rhetorical questions, the changing of the tone of voice, the loud tone of voice when she gives the formal definitions and so on. These elements don’t make the speech of Lorenza plain and they are expressions of what she is feeling in that precise moments. For this reason, I talk of *communicative emotionality* of Lorenza.

451	T: Then, (*loudly*) let's suppose to have an equation expressed this way, let's say that it is represented this way: $IM = IIM$. Think as we had a balance, (*in a whisper and speeding up*) now I will make an horrible sketch. Then let's suppose that first and second sides are in balance, that is, what is on the left is equal to what is on the right, namely balance in balance. If we go to add a quantity or an expression on one side, let's suppose to add a small square here (*she draws a small square on the side of the balance that contains the first side of the equation*) we have to add the same small square also there (*smiling*). This way, the balance returns to being in balance (*she mimes the balance smiling*, Fig. 4.19), otherwise it is unbalanced (*she mimes the unbalance and she smiles*, Fig. 4.20), right? If we add a small square on one side, what will happen? It will happen that the balance will lean, then, in order to return to have the balance situation, what we add on one side has to be added also on the other. This is the representation of the first principle of equivalence, indeed if I add on the first side a certain quantity, let's call it, uhm	 Figure 4.19  Figure 4.20
452	S11: 2	
453	T: Not 2, general, let's call it	
454	S12: circle	
455	T: circle (*nodding*),	
456	S13: (*smiling*) circle	

457	T: we have to add the same circle also on the other side (*writing on the blackboard IM + ○= IIM + ○*), otherwise the equation is not equivalent to the given one. In doing this, this one, that we obtain, instead, results equivalent to the starting one (*she draws an arrow that linking the two equations*). Is it difficult?	
458	S1: No	
459	T: No (*self-confident tone of voice*)	
460	S3: I understood, then no	

Lorenza justifies how the first principle of equivalence works through the metaphor of the balance. Then, she decides to consider an iconic representation of the situation. The action of considering the balance comes along with her worry of making an "horrible" sketch of it (# 451). Then, the explanation of how the first principle works is accompanied by her smile when she says to put a "small square" on the other side of the balance for balancing; by her miming of the balancing and unbalancing of the system (Fig. 4.19, Fig. 4.20); by the rhetorical questions in #451. In particular, in the previous piece of lesson, she started the explanation with the definition of the first principle of equivalence in words. Now, she wants to give sense to it through something that students know well already, namely the balance. Hence, she returns on the equation written on the blackboard (#457 IM+ ○=IIM+ ○), saying that if it is added a "circle" on one side, it has to be added the same "circle" to the other one. Moreover, she repeats for two times the conclusion of the use of the balance. Probably she wants to clarify better what she has already said (# 451) and she wants to be sure that, with a second time, almost all of students understand how the first principle works. Finally, Lorenza concludes her explanation, with the rhetorical question to her students "Is it difficult?" (#457). One student answers "no" and she confirms, also, "no" with a particular tone voice that could reveal her belief that this explanation is quite easy.

The *epistemic emotionality* of the teacher involves the justification of how the first principle works through the system of the balance (rational key). Moreover, it is constituted by her expectation that this explanation is quite simple for students because they have already seen it in words and, now, they have just to coordinate the verbal and iconic representations (emotional key). In particular, she seems to hope that they understand that adding the same term to both sides corresponds to adding the same weight to both plates of the balance. This emotional key is revealed by her repeated smiling in #451 when she adds the same circle to both sides and the balance remains in balance. Her smiling could testify her satisfaction that, actually, she is showing to students that there is a correspondence between different representations of the same concept. In addition, the emotional key of the teacher is disclosed by her rhetorical questions in #451, in which she does not give time to students for answering because she is quite sure that they know the answers. The *teleological emotionality* of the teacher involves both her action of considering the balance to explain how the first principle works (rational key) and by her hope that for students are quite clear understanding it through the balance, already knowing what is the first principle in words (emotional key). In other words, she hopes that students are able to coordinate different representation. This emotional key is revealed first by the fact that she wants to put students in the best conditions for understanding the metaphor, indeed she is worried that the sketch of the balance could be "horrible"; by the repetition for two times of how the principle works; by, at the end of the explanation, the rhetorical questions like "Is it difficult?" (#457). When a student answers "no" and she confirms "no" with a self-confident tone voice that could reveal her belief that the coordination between the verbal representation and the iconic one of the principle is quite easy for them. Furthermore, her discourse is full of the personal pronoun "we" and of the exhortations that involve both the teacher and the class (# 451 let's suppose, let's say, let's suppose, let's call it...). This is a way of Lorenza for constructing new knowledge together with her students and not alone, because she aims that students participate all together to the explanation. In addition, she smiles as saying to students "all it is working" and students notice this way of communicating. In fact, a student, also smiling, proposes to put the same circle to the both plates (#456). Hence, it can be highlighted the *communicative emotionality* of Lorenza.

461	T: Let's make an example (*affirmative tone of voice*). Let's suppose that the starting equation is $x-2$ equal to (*she lifts up her chin as meaning that she doesn't know what puts on the other side*) $8x-4$ (*she scratches head,* Fig. 4.21, *and she makes a gesture as to mime that she invented it,* Fig. 4.22), ok? (she pauses and, then, she looks the equation for few seconds) ok? (in a whisper, but referring to the class: it seems a comment to herself as to mean that if it doesn't work is because it is invented) I have invented it, ok? (she turns her hand in order to mime the "invention")	 Figure 4.21  Figure 4.22
462	T: What does this principle say? In the meanwhile, let's see if we are able to understand what is the value that solves this equation. Make your reasonings. This equation is not in normal form, it should be transformed in normal form, etcetera, etcetera. Let's try to solve it. What we have to do? You already know it.	
463	S14: 2/7	
464	T: (*self-confident tone of voice*) this (*pointing* $8x$) is carried here (*pointing the first side*), then, it is added to that one (*pointing* x), it would result $-7x$, this (*pointing* -2) would be carry there (*pointing the second side*) and it would be added to this one (*pointing* -4) and we obtain	
465	S2: -6	

466	T: -2	
467	S2: No	
468	T and S14: I change the signs	
469	T: This way, we obtain $\frac{2}{7}$	

At this point, Lorenza makes an example of an equation, aiming to both the better understanding of how, practically, the first principle works and the introduction of the two calculation rules that derive from it[6]. She proposes the equation $x - 2 = 8x - 4$ and she stresses, in different ways, that she has invented it in that specific moment. For example, she lifts up her chin, she makes the gesture of the "invention" (Fig. 4.22), she scratches her head (Fig. 4.21), but at the same time, she looks for a few seconds the equation as to be sure, probably, that is suitable for her goals (# 461). Initially, Lorenza solve it with her class through the transportation rule, expecting that students are able to use it. Actually, they know it to solve equation "mechanically". At this point, she doesn't justify why they have to "carry" a term to the other side, changing its sign, rather she wants to find the solution that is $\frac{2}{7}$. Hence, the *teleological emotionality* of the teacher is constituted both by the action of making an example of equation (rational key) and by her hope that example is suitable for a double function (emotional key). From one side, Lorenza hopes that the example is useful to apply the theory about the first principle. From the other side, the teacher needs an example to introduce on it the transportation rule, explaining why it works. The emotional key is revealed by her insistence in stressing the fact that she invented the example (facial expression in Fig. 4.21, gesture in Fig. 4.22, tone of voice: #461)), but, at the same time, she looks it many times to be sure that it is a helpful example and she stops in speaking to reflect by herself. In fact, she says, as in a whisper, "I have invented it, ok?" (#461). In a sense, she hopes of having constructed an adapted example, but, simultaneously, she justifies herself in the case in which it does not help them. The *epistemic emotionality* of Lorenza is constituted both by solving the equation (rational key) and by her need that students remember the transportation rule they have already learnt in middle school (emotional key). She needs that they are used, even if mechanically, to apply it for the resolution of an equation. In fact, she would like that students apply it mechanically such that she has the occasion to explain why it works through the first principle of equivalence. The emotional key is revealed by the fact that she applies the transportation rule to solve the equation without calling it, speaking with a self-confident tone of voice without pauses, as if she is sure that students understand what she is doing

[6]the transportation and the cancellation rules.

(#464). Hence, she decides to make the example to explain the properties the equations, but there are several emotional elements of the teacher that disclose her hope that the example is actually useful. For example, at the beginning, she speaks whispering almost to herself because she is not so sure of the example she choices (#461). Then, when she becomes awareness that the example works for her aims, she changes the tone of voice becoming sure that it is suitable for applying the transportation rule (#464). Hence, these aspects constitute the *communicative emotionality* of the teacher.

470	T: This principle says that (*spelling*) if we add to both sides the same quantity or expression, the equation we obtain must result equivalent to the given one, ok? Then (*referring to the example just made*) which quantity do we decide to add? (*waiting for feedback, looking at the class,* Fig. 4.23)	 Figure 4.23
471	S5: +3	
472	T: +3	
473	T: $x-2+3 = 8x-4+3$. Let's see what happens to this equation. Let's try also here to make our passages, we obtain again $\frac{2}{7}$ as solution. (*speeding up*) Then, I would carry this one (*pointing* $8x$) here, $-7x$, I would carry this one there, etcetera etcetera, -2, then x is equal to $\frac{2}{7}$. Does it correspond, right? It corresponds with respect to the the things we have studied? (*affirmative tone of voice*) It corresponds to what we have already studied.	

Lorenza justifies how the first principle of equivalence works through the starting example solved with the transportation rule[7]. She personifies the first principle of equivalence, using the verb "to say" referring to it (#470).

[7]In the remaining of the lesson, she will show that, in this sense, the transportation rule can be considered as a consequence of the first principle of equivalence.

Moreover, she states the definition of the first principle, spelling it and she uses "must" with respect to what happens to the equation if we add the same term (#470). It seems that she considers Algebra as an authority. At the beginning of this excerpt, she involves students in deciding which quantity to add to both sides of the equation, waiting for feedback from students (Fig. 4.23). She seems quite careful to deal students in the discussion (# 473: "Let's see", "Let's try"). But, then, she solves alone the equation, probably, because she is expecting that students already know how the terms can "be transported" from one side to the other one, indeed she talks of "our passages" (# 473). The action of solving the "new" equation, in which they add +3 is aimed to show her students that the first principle, actually, works as explained in the definition. This resolution comes along with an hurry of Lorenza in solving this new equation through the transportation rule, probably, because she is expecting that students already know how mechanically solve it. Then, after finding the same solution of the previous one, she employs two rhetorical questions, using also the pronoun "we" (# 473). Without waiting for an answer from the students, she answers herself with an affirmative tone of voice. It could be that the teacher is quite sure that this example clarifies and corresponds with they already learnt. Hence, the *teleological emotionality* of the teacher draws on the action of solving the example in order to explain how the first principle works (rational key). Furthermore, it involves also her hope and need that example are illuminating for understanding the first property of equation (emotional key). This emotional key is revealed by her hurry in solving the equation applying the transportation rule, that could be seen as a way to arrive immediately to the end for seeing that the first principle actually works. Moreover, at the end of the passage, she makes two rhetorical questions justifying that what they have found in the example is coherent with what they have already studied about the first property of equations. Actually, she responds herself to them with affirmative tone of voice, because, probably, she expects that all of the students agree with her (#473). The *epistemic emotionality* of Lorenza is constituted both by the knowledge of the transportation rule (rational key) and by her expectation that Algebra could be seen an authority to recall for justifying what they are doing (emotional key). The emotional aspect is revealed by her speed and self-confidence in which she solves the equation, applying mechanically what the transportation rule considers. It seems that she is expecting to see Algebra as if is something that none can contradict (#470). Summarizing, she decides to use the example for justifying how the first principle of equivalence works, reaching the final understanding by the students with rhetorical questions. Lastly, I talk of *communicative emotionality* of Lorenza, because her speech is full of emotional tones. It cannot be plain because the teacher has hopes and needs. In fact, she spells the first principle, she uses rhetorical questions, she waits for feedback from the class, she speeds up when she is sure about the knowledge of the students and so on.

474	T: we change the sign. But why do we make it? Why do we change the sign? Right? Actually, it is a consequence of the first principle, because if we didn't change the sign, we would obtain an equation that isn't equivalent to the given one, right? (*smiling*). Let's consider the previous example (*catching it in the air with her hand*) $x-2 = 8x-4$, I have said that I carry to the first side $8x$ and I write $-8x$. Practically, (*loudly*) what did I do? I have added, let's say, to both sides the value $-8x$, then I have applied the first principle. I have applied to both sides the same value $-8x$.	
475	T: Algebra says that $+8x - 8x$ is	
476	Ss: zero	
477	T: zero (*she deletes the* $8x$ *and the* $-8x$), and then, actually, skipping this intermediate passage, I have carried the value, the term $8x$ that was on the right of the equal sign to the first member that is on the left of the equal sign. You apply this rule mechanically because you have already learnt it, (*rhetorical question*) right? (*miming the past with her hand*) (*spelling*) in middle school almost of you (*gesture for encompassing all go her students*), (*rhetorical question*), right?	
478	S3: I didn't see it	
479	T: (*imperative tone of voice*) Who says "no"?	
480	S3: I didn't know it	

481	T: never, (*quite astonished*) you never do that!	
482	S3: Yes, I did it, but I didn't know this passage	
483	T: (*astonished again*) The fact of transporting a term from one side to the other one, changing the sign?!	
484	S3: Yes, but I didn't make that passage in which I had to put to both sides the same quantity	
485	T: (*more relaxed tone of voice*) No, sure, you never do that because, probably, you have learnt the rule to make the passage from one side to the other one of the equal sign. It is sufficient changing the sign, without passing through the first principle of equivalence, right?	
486	S3: Yes	
487	T: (*satisfied*) But now we have understood from that rule we apply mechanically comes from. The transportation rule is still valid, you can apply it in the resolution of the equations.	

Lorenza explains why the transportation rule is a consequence of the first principle of equivalence. At the beginning of this passage, she seems to ask this justification from the class. This request comes along with a particular insistence of Lorenza in asking it, but, then, she answers herself to the question (# 474). In particular, she explains that if we didn't change the sign we would go against the first principle, not obtaining an equivalent equation to the given one. It accompanies this justification with a rhetorical question and a smile that could testify her self-confidence that students understand what she is saying (#474).
Then, she shows what she has already stated on the starting example $x-2 = 8x-4$. In the specific, she explains that the term $8x$, carried on the other side, becomes $-8x$, because she adds $-8x$ to both sides, namely, she has applied correctly the first principle. The actions of referring to the example

and of showing how the first principle works on it come along with her gesture of having the starting example in her hand and with the loud tone of voice of the "false" question: "Practically, what did I do? I have added..." (#474). I used the adjective "false", not being an out-and-out question. In fact, she makes the question and immediately she explains how the first property works on the example. Adding $-8x$ to both sides of the equation produces that one side becomes equal to zero because she says that Algebra ensures that $+8x-8x$ is zero. Then, the teacher shows to her students that, skipping this passage, they apply what the transportation rule says. This fact is accompanied by many rhetorical questions (#477) as she is expecting that students see what she is referring to. Moreover, she stresses, spelling, that they have already seen the transportation rule in middle school and, simultaneously, she mimes the past. Hence the *teleological emotionality* of Lorenza is constituted both by the action of taking the previous example to show why the transportation rule is a consequence of the first principle (rational key) and by her hope that employing the example ensures that students understand what she wants to explain (emotional key). This emotional key is revealed by her gesture of catching the previous example because, another time, it could be illuminating for students; by the loud tone of voice of the "non-question" made, probably, just to give the rhythm to what she has to do on the example (#474). The *epistemic emotionality* of the teacher involves the justification of the knowledge of the first principle of equivalence and of the transportation rule (rational key) and, also, her need that Algebra can be seen as an authority that, implicitly, justifies what they are doing (emotional key). This emotional aspect is disclosed by her rhetorical question and her smiling when she justifies that the transportation rule is a consequence of the first property because if we don't change the sign, we would not to obtain an equivalent equation. It seems a justification made by what Algebra states. Lorenza needs that students know the transportation rule to explain why it is a consequence of the first principle and she hopes that working on the example is illuminating for them. Hence, she cannot have a neutral speech. Indeed, during this passage, she smiles, she uses rhetorical questions. Especially at the end, she uses two rhetorical questions in which she says that they have already learnt it in middle school. When a student answers "I didn't see it" (#478), she wants knowing, in an imperative tone of voice, who says not. Then, with astonished facial expression and tone of voice, she explicitly asks clarifications to the student #483. When she discovers that student wanted to say that he never makes the passage of adding the same term, she speaks in a more relaxed way, stating that it is normal that in middle school they don't see where the rule comes from. Finally, with a satisfied tone of voice, she states that now they understand where the rule comes from. All of these facts express again her hope and need that students remember the transportation rule. These elements constitute what we have called *communicative emotionality* of the teacher.

488	T: today we see to make a forward step, let's see an interpretation of the linear equations, let's make a bit of work and a bit of reasonings and if we manage we make something on the problems, in order to reduce a little the times.	
489	T: let's think of an equation written in normal form, a random (*she seems puzzled and, at the same, she scratches her head:* Fig. 4.24) one . $2x + 6 = 0$, for example, right? (*affirmative tone of voice*) we have said that an equation, a linear equation of the first grade, is the classical equation we have analysed up to now, already expressed in normal form, we say also that it is solved for what value of x?	 Figure 4.24
490	S3: -3	
491	T: ok, if we substitute -3 in place of x we obtain that that equality is true, but the word (*spelling*) linear, (*facial expression to catch her knowledge:* Fig. 4.25)do you come into mind something that we have already seen?	 Figure 4.25
492	A: the linear relation	
493	T: that is, the linear relation, what was the linear relation?	

494	Ss: noise	
495	S2: k times x plus a number	
496	T: first of all, let's remember when we have spoken of it	
497	S2: with functions	
498	T: when we have treated the functions in physics and we have analysed the type of the dependence among the physical quantities, (*tone of voice of a statement rather than a question*) do you remember we have seen the different proportionality, at a certain point we have met the dependence of the linear type and we have written an equation that represented that function, that link between two variables into play, which were the x variable and the y variable, do you remember? they represented two generic variables, in physics we spoke of physical quantities, who does remember that linear equation?(*she specifies with the gesture:* Fig. 4.26) which that expressed the linear dependence, Riccardo	 Figure 4.26
499	R: $y = kx + y_0$	
500	T: it's ok, right, do you remember? what was k?	
501	S2: a constant	

In this lesson, the goal of Lorenza is to introduce the geometrical interpretation of the solution of an equation as the abscissa of the intersection point between the straight line associated to the equation and the x-axis. She starts with an example, choosing the linear equation $2x+6 = 0$. She uses the example with the aim of introducing this geometrical interpretation. The action of making an example comes along with a worry of Lorenza of choosing an apt example for working on it in order accomplish her goal (# 489: "...a random (she seems puzzled and, at the same, she scratches her head: Fig. 4.24) one."). Moreover, after invented it, she seems to require the sharing by the class, using a rhetorical question (# 489: "$2x + 6 = 0$, for example, right?"). The action of making the example (rational key) together with the emotional

aspects of it (emotional key revealed by the fact that she seems puzzled, she uses rhetorical questions) can highlight a *teleological emotionality* of Lorenza. The emotional counterpart could be related to her expectation that her examples are suitable for her goal of that precise moment. It could be that emotions, then, explain why she decides to act in that way (e.g. starting from an example for working on it, in order to develop a new interpretation of the solution of a linear equation) and not in another one. Indeed, she needs to consider the suitable example, because the students have immediately the perception of where we have to go. At the same time, she is communicating speaking with the pronoun "we" and, from her tone of voice, she seems quite sure that students know what she is referring to, indeed she repeats what the students should already know without asking questions about it (# 489). In this sense, it is not just a matter of what she is saying in order to reach understanding by the class, but it is also a fact of why she is speaking in that way. In fact, her "emotional" speech could be related to the expectation of Lorenza that students can remember the previous knowledge to construct the new one starting from and upon it.
She started with an example of a linear equation to intend to recall to the students the term "linear". The action of recalling something that they already know comes along with: a spelling way in saying "linear", her facial expression in Fig. 4.25 in which she seems to "catch" the knowledge of students, the rhetorical question "do you come into mind something that we have already seen?" (# 491), the question "what was the linear relation?"(# 493) in which she uses the simple past to remark that it is a concept they have already faced with. It is not just an action oriented to a goal (recalling what they have already learnt in physics with her in order to link linear equations to the mathematical functions), but it is accompanied by different emotional aspects and all together can be reveal an *teleological emotionality* of Lorenza related to her expectation that, within the class, she can construct new knowledge starting from the previous one. Some students remembers from the physics the naive linear relation "k times x plus a number" and Lorenza uses this occasion for recalling when they have spoken about it, that is, with the treatment of functions. The interesting thing is that she involves also herself in the discourse ("let's remember.." # 496), but it is quite obvious that she is expecting that students answer to her question, because she stops, giving them time for answering (*communicative emotionality* of Lorenza). She justifies why they have used functions in physics, that is, to introduce the dependence among the physical quantities and the proportionality, that are, the link between two variables into play. It is interesting that she accompanies this knowledge at play by phrases that, at the beginning, should be questions, but then continue with the tone of voice of a state-

ment rather than of a question (# 498 "do yo remember we have seen..."). The knowledge (rational key) comes along with this particular tone of voice (emotional key) and they all together surface an *epistemic emotionality* of Lorenza. The tone of voice could show us her expectation about the fact that students are quite confident of what they already done in physics about functions. Her speech is quite insistent when she repeats many times that they have already seen functions, that they remember it, that they spoke with her about "physical quantities" and so on (#498). This way of communication is full of emotional elements that stress her hope that they should remember the previous knowledge (*communicative emotionality*). At the end of this brief excerpt a student says the relation $y = kx + y_0$.

502	T: both k and y_0 were numbers, real, that represented something. Do you remember (*posture as* Fig. 4.27)? It is just to make (*gesture to mime the "box" of the classroom culture*) the review of what we have already known, uhm (ok?)? k represents	 Figure 4.27
503	S1: proportionality	

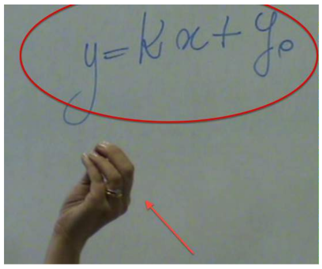
<table>
<tr><td>504</td><td>T: (gestures in Fig. 4.28, Fig. 4.29, Fig. 4.30) (insistent rhythm) But, how did we repre(sent it), dra(w), did we draw a sketch?</td><td>

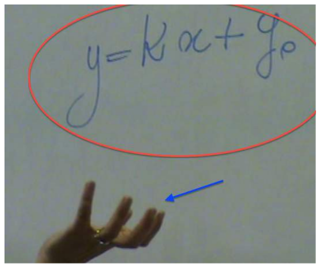
Figure 4.28


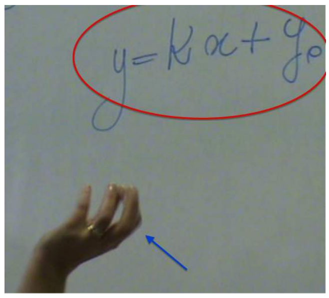
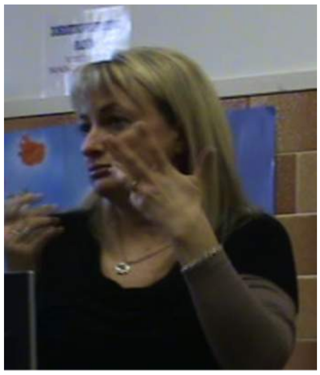
Figure 4.29


Figure 4.30</td></tr>
<tr><td>505</td><td>S3: a straight line that passes...</td><td></td></tr>
</table>

506	T: that is (*nodding*), it was a straight line, it was a random straight line of the plane and then k and y_0 gave us particular values of the straight line (*facial expression:* Fig. 4.31), so, Riccardo	 Figure 4.31
507	R: k is the gradient	
508	T: it is called gradient, but (*inquiring way of the request*) what does it mean in short?	
509	S2: the slope (Fig. 4.32: *S2 mimes the slope*)	 Figure 4.32
510	T: the slope with respect to the x-axis and y_0 gave us another information	
511	R: the intersection point between the x-axis	
512	T: y_0 was the point that corresponded to the intersection of the straight line and the y-axis.	

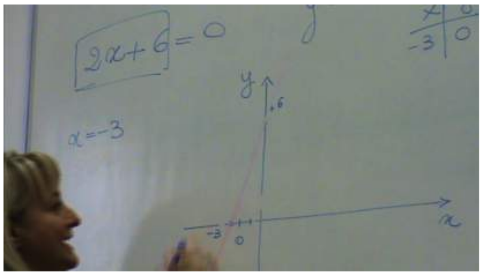
Lorenza recalls a specific knowledge (the concept of function and, in particular, the straight line) to show that the solution of an equation is the abscissa of the intersection point between the straight line associated to the equation and the x-axis. This action comes along with: the rhetorical question "It is just to make the review of what we have already known, uhm (ok?)?" (# 502); with the gesture of miming the "box of the classroom culture"; with the gestures in Fig. 4.28, Fig. 4.29, Fig. 4.30 (# 504). Moreover, when a student remembers the straight line (#505), she is nodding and she is satisfied for an answer ("that is (nodding)", #506). These emotional elements (gestures, rhetorical question, nodding) allow me to say something more rather than the pure rationality, through which, I could only describe well what actions Lorenza made to achieve her goal. Indeed, the emotional elements could give me informations concerning why she decides to act in that way. In fact, the emotions could reveal her expectation that the previous knowledge is valid for students and, then, that they are able to remember it. In particular, she draws upon the concept of function and the algebraic representation of the straight line ($y = kx + y_0$), because she wants to introduce the straight line associated to the equation. She recalls "the sketch" of that particular function and, then, she links the algebraic representation of a straight line to its graph in the cartesian plan. This knowledge is accompanied by a particular posture of Lorenza (Fig. 4.31, #506) that could testify that she is waiting for the answer from the students. The emotional aspect (e.g. the particular posture of the teacher) could reveal her expectation of the validity of what they have already done, but also her expectation that students are able to coordinate different registers of representation. With the pure rationality, I cannot say nothing about why she employs the concept of the straight line that they have already developed in physics, connecting the different registers of representation. Beyond what she is saying (few statements, many questions), it can be highlighted the insistent rhythm in which she asks questions to her students, without giving them time for answering. It seems that she is quite sure that they know the answer and, then, just because none answers, she takes time making many questions one after the other, expecting that her insistent questions remind her students about something of known. Moreover, she seems to have a pressing speech (#504), probably, because she hopes that students are able to reconstruct the coordination between the different register of representations. The discussion goes on and Lorenza underlines what each parameter in the algebraic formula of the straight line ($y = kx + y_0$) represent graphically. Lorenza recalls that, in physics, k and y_0 "gave" them particular values of the straight line. The former was the gradient and the latter the intersection point between the straight line and the y-axis. Actually, a student remembers that k was the

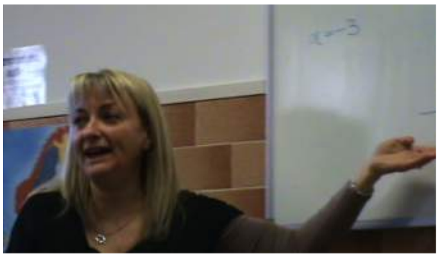
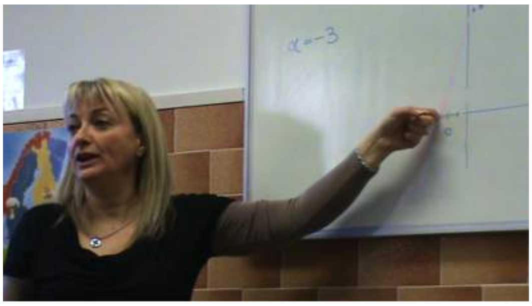
gradient, but Lorenza wants to go further and she asks, in a inquiring way (# 508), what it means the gradient "in short".
Hence, there is an *epistemic emotionality* because the epistemic is not only the knowledge concerning the straight line that includes also the coordination between different registers of representation (rational key), but also the fact that, having already explained it in physics and, probably, thinking she did it well, Lorenza is expecting that students remember it (emotional key, revealed, for example, by the posture in Fig. 4.31). Also in this second part we speak of epistemic, but at a meta-level. Then, from the teleological point of view she decides to recall that knowledge for introducing the geometrical interpretation of equations (rational key), but, simultaneously, she is expecting that students keep in mind what they have already seen, in order to construct this new mathematical meaning (emotional key, revealed by gestures in Fig. 4.28, Fig. 4.29, Fig. 4.30, by her nodding etc.), so, it emerges a *teleological emotionality* of Lorenza. At the same time, being emotionally involved (she has hopes and needs), Lorenza is not able to be neutral in her communication, because of she is, first of all, a human being. Indeed, in her speech, it can be highlighted the insistent rhythm in which she asks questions to her students, many rhetorical questions, the language of the body, the prosody. They, in turn, reflect her hopes and expectations on her class. Then, there are two intertwined aspects of communication: what she is saying through words that, generally, reflects her rationality and what she 'is saying' through the language of the body, the prosody etc. These two different sides coexist always in the discursive activity of Lorenza and, for this reason, we speak of *communicative emotionality* of the teacher.

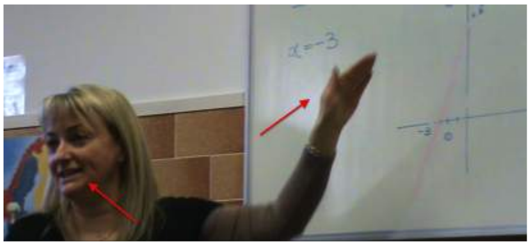
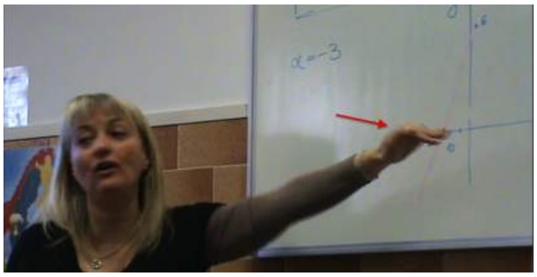
513	We know from physics, (*speeding up*) for the whole discourse we have done in physics that this one represents a straight line, we know also what these values say us, this one is the slope, the gradient and that one is the ordinate of the point with abscissa equal to 0, that is, the ordinate of point with abscissa equals to zero, the intersection between the straight line and the y-axis, right? Let's try to draw it and then we reason with the associated equation. Do you remember how is the sketch made? it's necessary to find two points, surely we have a point that is given from the ordinate (*pause*) at the origin, that is the point (0,6), and then we have to search for another point, it will be always two of them, right? (*tone of voice of a statement rather than of a question*) How do we find the other point, we take it randomly (Fig. 4.33), we choose it randomly, we make a little table, this time we put 0 for the y, we find that, look a little bit what we have to solve if we put 0 to the y (*pointing the table of values and the equation $2x + 6 = 0$*: Fig. 4.34), see it a little bit, (*tone of voice of a statement*) what we have to solve if we put 0 for y, we have to solve this one (*underlying with the finger the equation and with the tone of voice of a statement*) do you see? and then we find	 Figure 4.33  Figure 4.34
514	S2: -3	

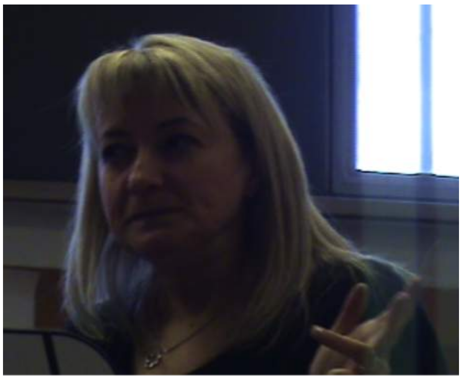
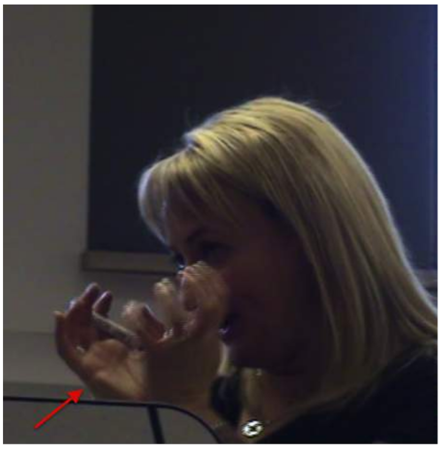
515	T: we find -3 for the y and, then, what have we discover? (*pause*) a point of the straight line that has (-3,0) as co-ordinates, that it will be more or less here.	
516	T: but, (*spelling and loudly*) where does -3 come from?	

Lorenza employs knowledge she has already explained in physics for drawing the straight line $y = 2x + 6$. In particular, she repeats that the value 2 is the slope of the straight line and the value 6 is the ordinate of the point with abscissa equals to 0. When she remembers this knowledge she speeds up, she repeats something that she has already said above and she concludes her statement with a "right?", as she is quite sure that they know what she is saying. Then, she puts into play the knowledge about how make the drawing of a straight line and she explained that it's necessary to have two points, because she has introduced in physics that for two points it passes just one single straight line. Recalling the fact that one point it is immediate, because it has 0 as abscissa and the value 6 as ordinate, she focuses on how to find the other point and she recalls the table of values in which she puts randomly a value for y, for example 0. This knowledge is accompanied by a rhetorical question ("Do you remember how is the sketch made?", #513), pauses for expecting an answer from her students, the tone of voice of a statement when she asks how to find the other point of the straight line. These emotional elements surface her hopes about the class culture and about the fact that examples are suitable for understanding better what she is doing. Then, *epistemic emotionality* of Lorenza involves both the knowledge of the straight line (rational key) and her expectation about the class culture and about the usefulness of examples (emotional key). It is not just a question of what knowledge she uses, but why she chooses that knowledge and not another one (e.g. she works on the example, because she believes that examples are more direct than the pure theory, she chooses as a first point of the straight line the value 6 because in physics they were used to consider it, because it is the most immediate point, she finds the other point constructing the table of values because they used to make it in physics and so on). The *teleological emotionality* of the teacher involves both her action to coordinate the algebraic register and the graphical one (rational key) and her hope that students are able to see this coordination (emotional key). This hope is disclosed by the tone of voice proper of a statement that accompanies the questions in #513. There is a mismatch between what she is saying by words and its tone of voice. Probably, she is quite sure that students are able to draw the graph of a straight line using the table of values. In particular, she hopes that students notices that finding the x associated to the y equal to 0 corresponds to solve the starting equation. In fact, at the end of the excerpt, she asks to students where -3 comes from spelling it and with a loud tone of voice, for drawing the attention of them on it (#516). Moreover, Lorenza's speech is full of many rhetorical questions. The insistent rhythm in which she speaks could show why she makes many questions and few statements. Indeed, it seems that she is expecting a reaction from her students. This could witness an *communicative emotionality*, because with the pure rationality I could describe what she is saying towards reaching understanding by the class, but the emotional overtones could give us reasons about her hopes and needs. For example, the teacher makes many questions in a rhythmic way, because she is expecting that she can construct new knowledge drawing on the previous one; because she hopes that examples are suitable and so on.

517	(*increasing the tone of voice*) then, the value $x = -3$, namely the solution of this equation, what does it represent (*spelling*) geometrically, on the cartesian plane? Does some of you see it? understand it?	 Figure 4.35
518	noise	

519	T: some of you have any idea? it represents (*smiling and pointing to the cartesian plane:* Fig. 4.36), (*pointing with the knuckle to* $x = -3$ *on the graph:* Fig. 4.37) the abscissa of the point [pauses] of intersection between the straight line (*smiling, miming the straight line, pausing*, Fig. 4.38) with the x-axis (*she mimes the x-axis:* Fig. 4.39)	 Figure 4.36  Figure 4.37

520		 Figure 4.38  Figure 4.39
521	S3: x	

522	T: it represents the point of the straight line that has 0 as ordinate, ok? (*she stops and she remains as in* Fig. 4.40) are you able to see it? (Fig. 4.41)	 Figure 4.40  Figure 4.41
523	s8: noise	
524	T: hence the solution of the associated equation is the value of the abscissa of the intersection point between the straight line and the x-axis, (*tone of voice of a statement*) do you understand?	
525	S7: yes	

The teacher wants, another time, to link the solution of the equation to the fact that it is the abscissa of the point of the straight line that has 0 as ordinate. In particular, she asks her students to make this coordination (#517). This action comes along with an increasing of the tone of voice of the questions, with spelling “geometrically” and with an insistent rhythm of these questions (#517). In addition, when none answers, she invites another time students to answer, but then she answers herself. She justifies on the cartesian plane that $x = -3$ represents the abscissa of the intersection point between the straight line and the x-axis. This justification is accompanied by several smiles of the teacher, gesture of miming the straight lines and with pauses waiting for feedback. Hence the *teleological emotionality* of the teacher involves both the coordination between the algebraic and the graphical registers (rational key) and her expectation that students are able to see this coordination (emotional key). This emotional key is revealed by her increasing of the tone of voice in #517; by the spelling “geometrically” in #517; by her insistent rhythm of the questions for asking them if they understood what she is explaining in #517, in #519, in #522 and in #524. In addition, in the last question in #524, she uses the tone of voice of a statement, probably because she hopes that, after coordinating the same concepts many times, students understand it. The *epistemic emotionality* of Lorenza involves both the geometrical interpretation of the solution (rational key) and her hope that the example is illuminating to understand it (emotional key). This emotional aspect is disclosed by her smiling when she explains what is -3 on the plane, probably, because she is self-confident that students have seen what she is saying (Fig. 4.36, Fig. 4.38); by her pausing, probably, because she is expecting that students continue the discourse; by gestures for animating straight lines, probably, because she hopes that students could be seen better what she is stating; by the eloquent facial expressions in Fig. 4.40 and in Fig. 4.41 in which she looks at the class with an inquiring way because she hopes that students have understood the discourse, but she is not completely sure. In addition, she repeats two times the fact that the solution of the equation, -3, is the abscissa of the intersection point between the straight line and the x-axis. Also these insistence is a hint of the fact she hopes that example is suitable to see the link between the solution of the equation and the graph. Having hopes and needs, for the teacher is very difficult to hide the emotional overtones. In fact, as I highlighted above, she increases the tone of voice, she looks at the class in eloquent ways, she makes emblematic gestures, she repeats with an insistent rhythm the key-concepts and so on. For this reason, we talk of the *communicative emotionality* of Lorenza.

4.2 The "fabric" of Rationality and Emotion in Carla

At the beginning of the first lesson I videotaped, Carla recalls what she did with her class the lesson before. In particular, she worked with her students on the activity 4.82,4.83 (attached in the subsection "Worksheets" at the end of this section) in which she introduced the concept of "open statement". When she introduces a new mathematical concept, she is used to give her class worksheets, to make them work in pairs and then to discuss all together what they have done. For the particular topic of equations, Carla starts with the definition of open statement as a logic proposition in which the truth value is determined by the value of one ore more independent variables. The open statement contains at least one variable, the value of which is selected within a referring universe set (e.g. U). The open statement is indicated by a capital letter of the alphabet (e.g. P) and the variable is indicated in brackets by a lower case letter of the alphabet (e.g. x). The open statement $P(x)$ can be true or false with respect to the value assumed by the variable x, chosen within the universe set U. For example, given the universe set $U = \{x \in \mathbb{N}/x \leq 20\}$, the open statement $A(x) : x$ is even, is true for $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ and false for the others. Carla explained what is an open statement in order to show linear equations as open statements in which the verb is "being equal to" and this open statement can be true for one value ascribed to the variable (determined equation), it can be false for each value of the variable (impossible equation), it can be true for all the values substituted to the variable (indeterminate equation). She uses another activity (4.84, 4.85) such that students construct themselves the concept of equation starting from the concept of open statement.

526	T: let's try to resume what was emerged working on that activity (referring to activity 4.82), to what conclusion we arrived	
527	S: I didn't write it	
528	T: I said you of reviewing these things for today. We arrived to see a very important definition, at the end of the first page (referring to activity 4.82)	
529	(*noise in the classroom*)	

530	T: we arrived to give the definition of "open statements" (*she pauses expecting that someone continues*)	
531	Ss: equivalent	
532	T: equivalent (*nodding*), let's remember this definition, when do we say that two open statements are equivalent, (*pronouncing*) within a given universe set, we underline it (Fig. 4.42)?	 Figure 4.42
533	S2: they are the same	
534	S3: when they have the same truth values set	
535	T: when they have the same truth value set and why, we have seen it in the second part of the activity (4.83), (*highest pitch*) why is it important to specify (*pronouncing*) "within a given universe set"?	
536	S4: the truth set can change with respect to the universe set	

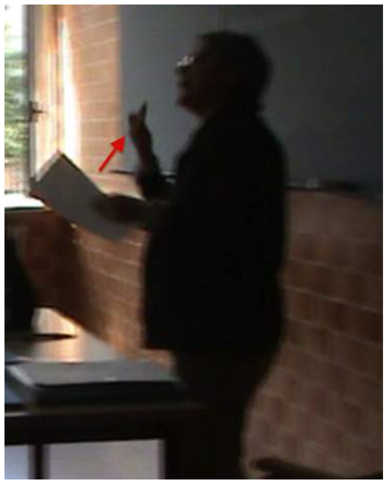
537	T: in the second part of the activity, we have verified it on some examples and that one was the conclusion that we have to write at the end that two open statements can be equivalent in a given universe set, but not in another one, this means that when we speak of (*pronouncing and highest pitch*) equivalent open statements we to specify within what set. Then, also today we work, you will work on worksheet that I have prepared and also here, as you will see, the worksheet has not title, and, at the end of the activity, you all together we will title the worksheet.	

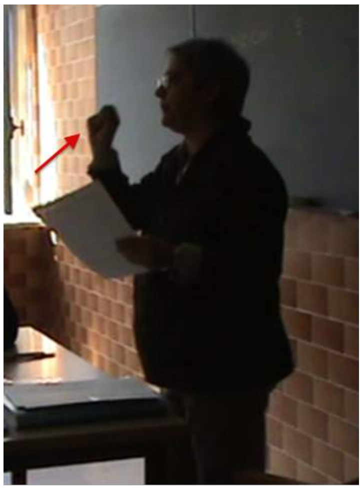
Carla recalls the worksheet of the lesson before (4.82 and 4.83), in which she introduced with her students the concept of "open statements" and of "equivalent open statements". In this brief excerpt, she focuses the attention of her students on why it is important to specify the universe set in which they consider "open statements". She accompanies this question with the increase of the tone of voice (# 535), the repetition for two times of the "why" (#535) and the pronouncing of "within a given universe set" (#535). The interesting change of the tone of voice, corresponding to the "why" (#535), could reveal her expectation on the fact that, for Carla, justifications give the sense of what they are doing. In particular, she reviews the definition of "equivalent open statements", because she is oriented towards clarifying the fact that two open statements can be equivalent in a certain universe set and not in another one, so it is important to specify it.
The *epistemic emotionality* is constituted both by the knowledge of what is an "open statement" and "equivalent open statements" (rational key), and by her expectation that students remember them: she has already explained them on the previous worksheet, and, probably, she thinks that her methodology of teaching through worksheets is effective (emotional key). This emotional key is revealed by her pause expecting feedback from students in #583; by her nodding; by pronouncing; by the gesture in Fig. 4.42 in#532 etcetera. Then, from a teleological point of view, she recalls the concept of open statement in order to stress why it is important to specify the universe set (rational key), but she is expecting that this justification comes from the students: she is very careful of justifying what she does and, then, she requires the same

attitude from her students (emotional key). This emotional counterpart is shown by the increasing the tone of voice; by the repetition of the "why"; by the pronouncing of "within a given universe set" in #535. So there is a *teleological emotionality* of Carla. Hence, at the same time, being emotionally involved, Carla cannot have a "plain" discourse, indeed she uses the language of the body, gestures, change of the tone of voice and so on. For this reason, it can be highlighted a *communicative emotionality* of Carla. With a pure rationality I could keep track just of what she is saying, but, looking at the emotional timbres, I could also say something about the fact that she hopes and needs that students react as she would like.

538	T: given that this worksheet (referring to activities 4.84, 4.85) isn't on your textbook you have to keep it and then, obviously, in the case of discussion we will correct it together.	
539	*the students work in pairs and the Carla corrects with the class the activity 4.84*	
540	*she begins to discuss with her class*	
541	T: follow on your worksheet, we have spoken generally of propositions, so then what is a proposition? let's try to remember it because it is (*she speaks loud referring to a student*) important	
542	S3: (*he speaks with a low voice*)	
543	T: then a proposition in mathematical sense of the term is a statement	
544	S4: that can be true or false	

545	T: that (*pronouncing*) must be true or false, then I have called propositions those that were in the first part of the activity then we have seen that the elementary propositions have a predicate and here the predicate is "to be equal to" and, every day, we have to do with propositions of this type and they are called (*gesture and facial expression:* Fig. 4.43)	 Figure 4.43
546	S3: equations	
547	T: no, there is also written here. how are they called?	
548	S8: open statements	

549	T: (*irritated*) we are still at the beginning (*miming the past:* Fig. 4.44), those propositions (Fig. 4.45: *irritated gesture for recalling just those propositions*)	 Figure 4.44  Figure 4.45
550	S7: equalities	

<table>
<tr><td>551</td><td>T: equalities. the equalities are (gesture: Fig. 4.46) particular propositions and as for all the propositions will can be true or false and within those I proposed what did you say?</td><td>
Figure 4.46</td></tr>
</table>

Carla does not use so much the textbook for introducing the new mathematical concepts. She refers to the mathematical textbook just at the end of having introduced a mathematical concept. At that moment she usually shows the topic is treated in the textbook and what are the differences and the analogies between what they did on the worksheets and what there is on the textbook. In fact, she prefers to employ several worksheets in which, through exercises and open questions – for justifying what they did in them – she introduces the theory. In the excerpt above, she comments the first activity of the 4.84 through which she wants to recall equalities (propositions with the predicate "to be equal to" that can be true or false) in order to introduce equations. Indeed, she decides to review the mathematical propositions anticipating, with a loud tone of voice (#541), that they are important. She remembers the definition of equalities and, then, she is expecting that someone recognizes that she speaks just of equalities. In fact, she is waiting for an answer from her students with the gesture and the emblematic facial expression in Fig. 4.43 (#545). A student responses equation instead equality. Then, with an irritated tone of voice and attitude, she recalls the past, miming it for helping students that those propositions are something that they have already seen in previous lessons. She accompanies the words "those propositions" with the gesture in Fig. 4.45 (#549), through which she seems to stress that students have already encountered them. Then, the *epistemic emotionality* of Carla concerns both the definition of an equality to construct that of equation (rational key) and her hope that students remember it because has already explained equalities to her class (emotional key). This emotional key is revealed, for example, by gesture in Fig. 4.43 (#545), by the fact that she seems irritated when a student gets wrong. Carla's *teleological emotionality* is constituted by the recalling equalities (rational key) and by the fact that they are important in order to introduce equations (emotional key). This emotional key is shown, for example, by the increasing the tone of voice when she says "important" (#541), by the gesture in Fig. 4.45 in which she seems that she wants to recall equalities with fervor (#549). At the same time, hence, being emotionally involved, Carla has not a neutral speech, but it is full of emotional aspects like the gestures, the change of the tone of voice, the facial expression and so on. Then, these several aspects in her speech highlighted the *communicative emotionality* of Carla.

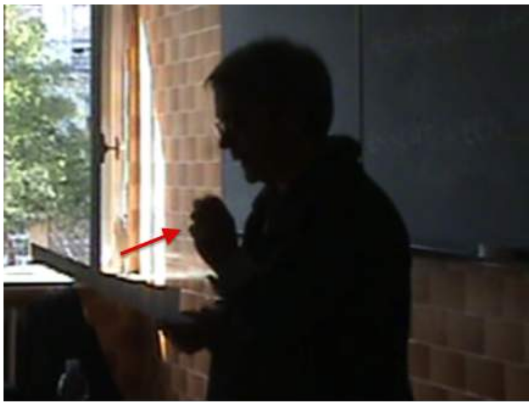
552	S8: false, true, false, false, true	
553	T: (*nodding*), well, Did all of you response that way?	
554	Ss: yes	
555	T: a false, b true, c false, d false, e true. Do you all agree?	
556	Ss: yes	
557	T: then, we considered open statements. In general what is an open statement? (*pause, gesture* Fig. 4.47)	 Figure 4.47
558	S10: a proposition that	

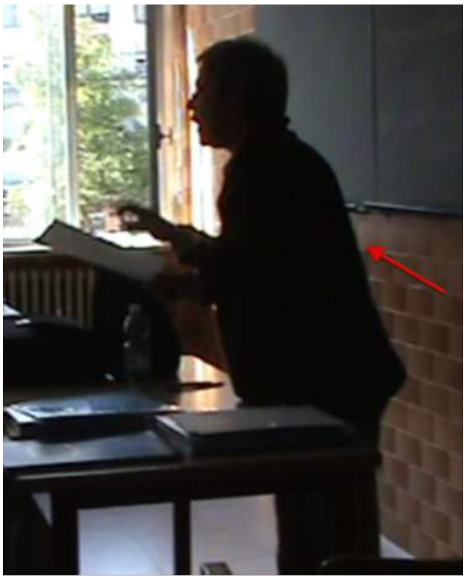
559	T: it is a phrase in which it appears a variable for which, first of all, we have to precise (*pause and waiting* Fig. 4.48) what is the universe set, right? try to answer to the questions I make in order to find of (Fig. 4.49) summarizing. Can we say for an open statement if it is true or false?	 Figure 4.48  Figure 4.49
560	Ss: no, we can say	
561	T: we can ask ourselves?	
562	S19: what is the truth set	
563	T: (*nodding*) in general, what do we call truth set?	
564	S3: solution	
565	S7: the set of solutions	

566	T: (*nodding*) in general, the truth set of an open statement is constituted by all the elements of the universe set (*pause, she is waiting with arms on her hip*, Fig. 4.50)	 Figure 4.50
567	S5: that make true the statement	
568	T: that make true it, then the equations, and this fact had to emerge here, are particular open statements in which the predicate is "to be equal to", then, as for all the open statements, of an equation (*raising her eyebrows*), given obviously the universe set, we ask ourselves what if the truth set, that we can continue to call this way, but for tradition it is called set of solutions, and an element of it, it is called solution.	

After correcting all together the first activity of the worksheet 4.84, Carla shifts the attention of students on how they have already defined an "open statement". Indeed, she wants to introduce the concept of equation as an open statement with the predicate "to be equal to". She asks to students what is an open statement. She accompanies this request with the gesture in Fig. 4.47, through which she seems to hope of having the culture classroom in the fist. Then, she wants to recall the importance of saying what is the referring universe set and, in fact, she is waiting for an answer from students as shown in Fig. 4.48 (#559). She hopes that someone says something, because, in the previous part of the lesson, she underlined very well the fact that, for an open statement, it is important to specify the universe set within it is considered: in fact its truth depends strictly on the set in which it is defined. At the end, she answers herself with a rhetorical question ("what is the universe set, right?" #559). Then, she goes on, because, probably, she feels confident that students are aware of this fact. Indeed, the discussion continues and Carla with her students consider what was the truth set of an open statement and, another time, she is expecting, probably a little irritated (Fig. 4.50: #566), that students remember the definition of it. The *epistemic emotionality* of Carla is constituted both by the knowledge of the concept of open statement, by the importance of the universe set and so on (rational key), and by the fact that she hopes that what is an open statement is already patrimony of the students (emotional key). In particular, she is waiting for a relevant feedback from her students because she has already explained open statements and, probably, she feels that she was effective in do that. This emotional key is disclosed, for example, by the gesture in Fig. 4.47, by Fig. 4.48, by the rhetorical question and so on. The *teleological emotionality* of Carla concerns both the recalling what they have already discussed about open statements in order to speak of equations (rational key) and her expectation that it is sufficient to just reviewing open statements, because she is expecting that students remember them (emotional key). This emotional key is disclosed by the verb "summarize" that comes along with the gesture in Fig. 4.49. In general, as human being, she cannot detach her body from her speech, then, her discourse is full of emotional elements, as particular postures, gestures, rhetorical intonation and so on and these facts all together contribute to emerge a *communicative emotionality* of the teacher.

569	T: In the central part of the worksheet there were equations in which you had to find the truth set applying, treating them as they are that is as open statements, without making particular calculation, trying to see what elements make true the statement. Then, in the case *a* what did you find? (Fig. 4.51: *she is waiting*)	 Figure 4.51
570	S3:4	
571	T: (*nodding*) in the case *b*?	
572	S5: 4	
573	T: because we work within $\mathbb{N}$, ok? then we will return on it, in the case *c*?	
574	S5: 8	
575	S9: -10	
576	Ss: -9	

577	T: the unique way for seeing who is right is trying to substitute. Try to substitute, then pay attention, I have said you and this fact has to be always keep in mind, you have to (*she inclines her body toward the class with open hand:* Fig. 4.52) give a (*pronouncing and highest pitch*) sense of what you read, then solving the equation $(x+1)^2 = 81$ means asking ourselves if it exist a value x such that doing $x+1$ and squaring it is 81, working in (*highest pitch and raising eyebrows*) $\mathbb{Z}$, this should make understand to you that (Fig. 4.53, *indicating* $x+1$) $x+1$, (*she inclines her body towards the class*) how it has to be?	 Figure 4.52  Figure 4.53
578	Ss: 9 o -9	

579	T: How $x+1$ has to be? if its square must be (*she inclines her body towards the class,* Fig. 4.54) 81? $x+1$ must be equal to 9 or -9 because of (*pronouncing and raising eyebrows*, Fig. 4.55) integers whose square is 81 there is not only 9 but also -9 and then what are the elements?	 Figure 4.54  Figure 4.55
580	Ss: 8 and -10	

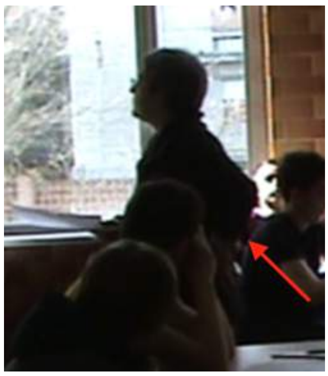
581	T: 8 and -10 (*nodding*), d, what have you found?	
582	Ss: 2 and -5	
583	T: and how have you found them? (Fig. 4.56: *biting her lips and waiting for an answer*)	 Figure 4.56
584	S6: putting one equals to zero	
585	T: we have applied an important property	
586	S3: "annullativo"	

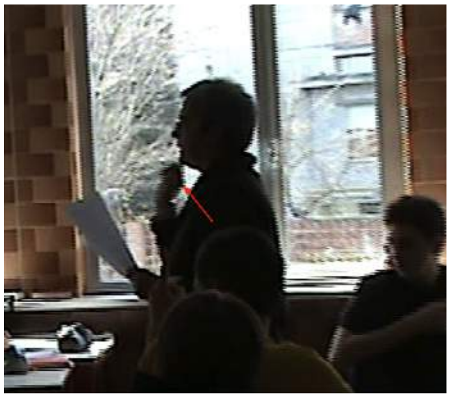
587	T: (*gesture of* Fig. 4.57, *for saying that what S3 said is wrong*) it is called property of	 Figure 4.57
588	S2: of equality	
589	T: (*annoyed*) property of	
590	S9: zero-product property	

591	T: Zero-product property, ok?try always to give a (*highest pitch and then she inclines her body towards the class:* Fig. 4.58) sense, to give (*raising eyebrows*) a meaning of what we read, solving the equation $(x+5)(x-2)=0$ means understanding when that product is (*raising eyebrows*) 0 and if we consider that the product of two factors is zero (Fig. 4.59) try to complete in order to review also the zero-product property, when the product of two factors is 0?	 Figure 4.58  Figure 4.59
592	Ss: when one of them is 0	
593	T: (*nodding and raising eyebrows*) when at least one of them is 0 and (*pause*) vice versa.	

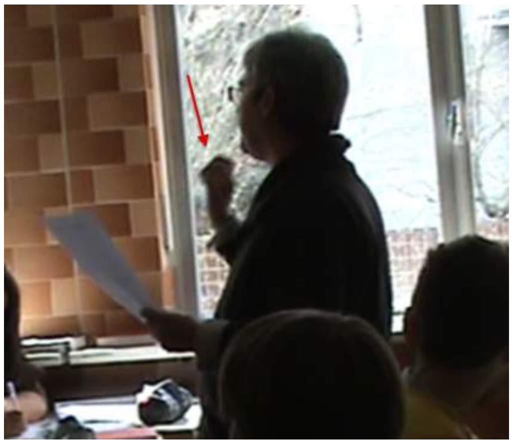
Carla discusses with her class the second activity of the worksheet 4.84 in which students had to find the solution set with respect to the assigned universe set. She is very careful in justifying what she does and she requires the same attitude from students. For example, in the case of the equation $(x+1)^2 = 81$ there were different opinions among students. Hence, Carla invites them to substitute the number for the unknown for verifying who is right. She explicitly explains what is the meaning of solving the equation $(x+1)^2 = 81$ (#577) and she stresses very well that students have to keep in mind always the sense to what they read or do (#577). This statement is accompanied by a particular facial expression of Carla and a gesture (Fig. 4.52) for marking what she is saying, by her inclining her body towards the class and the pronouncing, with highest pitch and eyebrows raised, the word "sense" (#577). Returning on the example, she repeats what students already said, that is, the solutions of the equation in $\mathbb{Z}$ are 9 or -9. She underlines why they are solutions: because 9 and -9 are the only integers whose square is 81. This justification comes along her inclining her body towards the class as in Fig. 4.54 and the pronouncing the word "integers", raising eyebrows (#632, biting her the lips (#583). In the justification of another equation, $(x+5)(x-2) = 0$ in $\mathbb{Z}$, she explicitly asks students how they have found that solution, expecting an answer from them, as in Fig. 4.56 (#583). Moreover, when she wants to justify the resolution of one equation with the zero-product property and students don't remember it, she seems very annoyed (#587, #589). Furthermore, she needs that students recall the zero-product property in order to understand the justification of the resolution of that equation (#591: In Fig. 4.58, Fig. 4.59 she inclines her body towards the class as to receive a feedback from students).
The *epistemic emotionality* is constituted both by the resolution of a linear equation (rational key) and by the fact that she is expecting that students feel the need of justification (emotional key). This emotional key is shown, for example, by her raising her eyebrows and her inclining her body towards the class corresponding to a request of justification from the students, her biting the lips and so on. She is expecting the same behaviour from students also when they don't justify correctly what they have done in an equation. In fact, she is seems very annoyed after the wrong answer of students (#589) and she is waiting that students recall the zero-product property as in Fig. 4.59 (#591). Her teleological is prompted by the fact that Carla wants to justify all the solutions of the equation (rational key), but she goes beyond, because she hopes that justifications are the means for giving a sense of what they are doing (emotional key, revealed by her facial expression and gesture in Fig. 4.52, by the pronouncing the term "sense" with an highest pitch #591). Then, this is a *teleological emotionality* of the teacher, because

there is also a teleological at a meta-level that goes beyond just clarifying of what students are doing. Her speech, being emotionally involved, cannot be neutral and for this reason there are two different aspects in her discourse: from one side there what she is saying, but, from the other one, there are all the different aspects (gestures, facial expression, prosody etcetera) that show her emotional engagement in the discussion. Hence, this is the *communicative emotionality* of the teacher.

594	T: the last one, working in $\mathbb{Q}$?	 Figure 4.60
595	Ss: -1/5	
596	T: it was sufficient to observe that 5 times $-\frac{1}{5}$ is equal to -1 and $-1+3$ is 2, ok? then in the next row it needed to arrive to a definition that, as we will see, is very important (*she is waiting feedback:* Fig. 4.61)	 Figure 4.61
597	S4: two equations defined in the same universe set U are equivalent in U if the two unknowns are equivalent	

598	S3: no, Prof.	
599	T: Can I speak? what does it mean that two unknowns are equivalent?	
600	S4: that are equal	
601	T: The premise was to remember (*irritated, gesture for specifying:* Fig. 4.62) what we did in the activity 1 (4.82, 4.83)	 Figure 4.62
602	S3: two equations defined in the same universe set U are equivalent in U if they have the same truth set or solution set	

603	T: that is (*nodding*), more or less, then remember (Fig. 4.63) the fundamental concept emerged in the activity 1 (4.82, 4.83) was that of equivalent open statements, that is, statements that within (*gesture with the hand*, Fig. 4.64) a given universe set have the (*highest pitch and she raises eyebrows*) same truth set. Now, given that equations are particular open statements we will speak of (*pauses and waiting for an answer:* Fig. 4.65)	 Figure 4.63  Figure 4.64  Figure 4.65

604	none answers	
605	T: of equivalent equation when they have the same truth set, in a given set. This was the (*pronouncing and gesture for specifying:* Fig. 4.66) very important definition that we wanted to construct, try to continue with the worksheet in the final part of the first page.	 Figure 4.66

Carla uses previous examples (solving $3x + 6 = 18$ in $\mathbb{N}$, $x(x + 1) = 20$ in $\mathbb{N}$ and so on) to construct the definition of equivalent equations. She explicitly declared that this definition is very important. She hopes that students are able to construct it, remembering that of equivalent open statements (#601), given that they have already shared that equations are open statements. In fact, she seems a little irritated when a student gives a wrong definition of equivalent equation and she has to explicitly recall what they did in the previous activity (#601). In particular, she focuses the attention of students (Fig. 4.64) on when open statements are equivalent and she is expecting that students see the analogy for equations (Fig. 4.65): also for equations, indeed, the important thing is that they must have the same truth set. She accompanies this property with an increasing the tone of voice and with raising eyebrows (#603).
Hence, there is the *teleological emotionality* of the teacher, because she doesn't limit herself to say that the previous examples were preparatory to introduce the definition of equivalent equations (rational key), but, at the same time, she is expecting that someone gives this definition. This because she has just now introduced the definition of equivalent open statements and, probably, she thinks that all of the students remember it and they are able, now, to apply it for the case of equations (emotional key). This emotional aspect is revealed by postures in Fig. 4.61, in Fig. 4.64, in Fig. 4.65, by her pronouncing the words "very important definition" in #605, by the fact that for

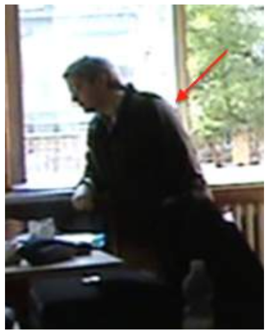
two times she explicitly repeats that this definition is important: #596 and #605. Moreover, the *epistemic emotionality* is constituted both by the definition of equivalent open statements and of equivalent equations (rational key) and by the fact that she is expecting that students construct themselves the definition of equivalent equations, seeing the analogies with that of equivalent open statements (emotional key). This emotional counterpart is revealed, for example, by the fact that she is a little bit irritated when a student gives the wrong definition (#601). In particular, she involves the fact that equivalent open statements have the same truth set. She stresses this fact with a highest pitch, raising eyebrows and with the gesture in Fig. 4.64. Probably, for drawing the attention of her students just on this fact, because it will be happened the same thing for equivalent equations. Then her speech cannot be extraneous to what she is feeling in that moment of the lesson. She has many hopes and needs because, probably, she believes that she taught to her students in an effective way. These emotional elements are shown by her many postures of waiting for feedback from students, by prosody (she changes her tone of voice when she wants to draw the attention of the students), by raising eyebrows that testify her underlying of what she is telling them and so on. For this reason, it emerges the *communicative emotionality* of Carla.
After working in pairs, students discuss with the teacher what they answered in the equations in the second page of the Activity 4.85.

606	*They are discussing the equation* $x^2 + 1 = 0$ *in* $U = \mathbb{Q}$	
607	T: then in the case f?	
608	Ss: empty set	
609	T: empty set, (*highest pitch*) this is very important to me! have all of you answered “empty set”?	
610	Ss: yes	

611	T: because x^2 cannot never be equal to -1, right? then (*highest pitch and she is specifying with finger*) pay attention! there are equations that have the empty set as the solution if we work in a (*pronouncing*) certain universe set, but changing the set universe they become (*inclining her body towards the class*) solvable. For example the equation a is not solvable in $\mathbb{N}$, but in $\mathbb{Z}$ it is. Now, h?	

Carla discusses with her class the different equations proposed in the worksheet 4.85. She continues the correction of the activity and she seems very careful to show that two equations with the same algebraic representation could have different solutions with respect to the universe set in which they are considered. The *epistemic emotionality* of Carla is based on the fact that she discusses with her class what happens for the equation $x^2 + 1 = 0$ in $\mathbb{Q}$ (rational key). Moreover, it also involves her expectations that students answer in a correct way and are able to justify their responses (emotional key). This, probably, because she thinks that in the past she was effective in giving a relevant role to the justification of what she did. This emotional counterpart is testified, for example, by the exclamations of Carla with a loud tone voice "this is very important to me!" (#609), by the rhetorical question and by the expression "pay attention!" (#611) specified with the finger, related to the solution of $x^2 + 1 = 0$ in $U = \mathbb{Q}$, namely the empty set. Furthermore, her posture is relevant: she is inclining her body towards her class when she stresses the fact that changing the universe set the equation $x^2 + 1 = 0$ can be solvable and, simultaneously, she pronounces the words "certain universe set" (#611). Hence, the *teleological emotionality* of Carla involved both the stating that changing the universe set that equation can admit solutions (rational key) and by her hope to be effective in stressing this observation (emotional key). This emotional counterpart is disclosed by the change of the tone of voice and the pronouncing of "certain universe set" and by her inclined her body towards the class. This attitude could prove that, for Carla, the justification of the answers related to the dependence of the solution on the universe set is so much relevant that she seems to have the will to involve students as much as possible. Then, being emotionally engaged during her discursive activity, her speech is full of emotional hues given, for example, by her prosody, by her attitude, by her posture, by her gesture. This shows the *communicative emotionality* of the teacher.

612	T: Now, h?	
613	Ss: empty set	
614	Ss: zero	
615	T: (*gesture in* Fig. 4.67, irritated) I hope none answered $\mathbb{Q}$	 Figure 4.67
616	Ss: no	
617	T: (*hands on hip*) Are we sure that none?	
618	Ss: yes, it is 0	
619	Ss: it is the empty set	
620	T: (*she nods and she looks at the class without speaking*) is there not any value that, substituted to x, makes it true?	
621	Ss: 0	
622	Ss: no	

623	T: (*increasing the volume of the tone of voice and she specifies with finger*) then, try to substitute to x the value 0 (*she is inclining her body towards the class as in* Fig. 4.68)	 Figure 4.68
624	Ss: *noise* with 0 it is valid.	
625	T: then (*highest pitch*) I have said (*specifying:* Fig. 4.69) to try to substitute the value 0 to x, what did you find?	 Figure 4.69
626	Ss: +1 and +1	
627	T: then you should verify that 0 satisfies that equation and there is only one, right? this equation has (*highest pitch*) as solution (*raising eyebrows and finger up to specify*) 0.	

In the excerpt above, Carla talks over with her students about the solution of the equation $(x+1)^2 = x^2+1$ in $U = \mathbb{Q}$. The class is divided between two different responses: one part of the class says that the solution is the empty set and the other one zero.
The *teleological emotionality* of Carla is constituted both by the several ac-

tions she makes (e.g. she asks many questions to her students) for being sure that none answers that the solution is $\mathbb{Q}$ (rational key) and, simultaneously, by her hope that none has written $\mathbb{Q}$ (gesture in # 615) (emotional key). She has this hope because she is expecting that students are used to give a sense to the algebraic expression they encounter and that they remember what she has explained about the development of the special product. This emotional aspect is revealed by her insistence in asking questions (#617, #620), by the facial expression in # 615 (Fig. 4.67); by the use of the verb "hope" in #615; by her posture with the hands on her hips (#617).
Given that some students answer that the solution set is $\mathbb{Q}$ and others say 0, the *epistemic emotionality* of the teacher concerns both the fact that she invites her class to try to substitute a value to the x, namely the 0 (rational key), and her expectation that students clarify their ideas through the example, constructed substituting a precise value to the unknown (emotional key). This emotional counterpart is revealed by her increasing the tone of voice, her pointing for specifying with her finger and her posture inclining her body towards the class as in Fig. 4.68, but, first of all, it is shown by the fact that she repeats for two times to substitute the value 0: #623 and #625. Moreover, her epistemic emotionality there is also when she says that the solution is 0 because it satisfies the equation (rational key), but also because she is expecting that the use of the example and the justification of why it's works (it makes true the equation) are illuminating for the students in giving the correct answer. Then, I points out that the speech of Carla cannot be neutral because she has hopes and needs, and, for this reason, the teacher has a *communicative emotionality* related to her expectations. For example, her prosody, her insistence in asking questions, her change of the tone of voice, her postures, her facial expressions are all together hints of the fact that she feels something when she explains to her class.

628	T: our goal is being able to (*raising eyebrows*) solve equations, then (*raising eyebrows*) to find the truth set (*pause*). For equalities the question can be if I have a true equality what can I do for obtaining again a true equality? speaking of (*highest pitch*) equations, instead, what should be the question? If I have an equation ok? what can I do for obtaining an equation (*raising eyebrows*)	

629	S12: equivalent	
630	T: (*she raises eyebrows and she nods, repeating what S12 said*) equivalent. Then, the next question you find on the worksheet, we read it together, is: let's suppose that we have an equation, then I have told you that you can help yourself with example (Fig. 4.70), ok? (*speeding up*) you can take any equations could you find (*highest pitch*) a procedure, a procedure means a series of operations, (*raising eyebrows*) a succession in a certain order, ok? that, given this equation, it allows you to obtain an equation equivalent to it? Have you understood what is required? ok? You have to translate the properties of equalities (*gesture in* Fig. 4.71 *and nodding*) (*speeding up*) that we have already done,ok? you have to see how we can use these (*raising eyebrows*) properties to operate with equations (*raising eyebrows, biting her lips and nodding*) (*pause and she looks at the class without speaking*). Then, here, the question has to be translated in terms in which it is posed in the worksheet (*she puts the hand on the worksheet and she nods looking at the class*) ok? given an equation how can I transform it in an equivalent equation to it? Have all of you understood the question?	 Figure 4.70  Figure 4.71
631	Ss: Yes	

632	T: then I have written also "help you with examples". (*increasing the tone of voice and nodding*) take an equation, ok? (*gesture with open hand that rotates to mime the randomness of the example*) try to write it on a sheet ok? (*she points the sheet of a student in the first row*) then you have to ask yourself what is it possible to do for obtaining an equivalent equation from this one (*she is referring to the same sheet of the student*), the equation is (*raising eyebrows*) an open statement.	

In the passage above, Carla wants to explain the principles of equivalence of equations drawing on the "leggi di monotonia" for equalities. The *teleological emotionality* of Carla is constituted, first, by the fact that she declares that the aim is solving equations, namely, finding the truth set of them (#628, rational key). Then, it involves her expectation that students themselves introduce the concept of equivalent equations, seeing the analogy with what they have already done in the case of equalities. She has this hope probably, because she thinks that she previously explained well how they can do for having again a true equality starting from a true equality. At the same time, she is needing that students remember what they have already studied with her about the concept of equality to construct the principles of equivalence of equations. These emotional elements constitute the emotional key shown by the fact that Carla raises eyebrows corresponding to the words "solve equations" and "to find the truth set", she increases her tone of voice when she says "equations" after remembering equalities, in order to stress that now they speak of equations and not more of equalities (#628). But, at the same time, she changes the tone of voice probably to highlight that there is an analogy between them. Finally, her hope that students are able to make the analogy is, also, proven by the fact that when a student answers "equivalent" (#629), she seems to be satisfied of the response and she nods, repeating what student has already said.

For constructing the principles of equivalence, Carla proposes to her class to choose an example on which identifying a procedure to obtain an equivalent equation of it. This choice comes along with a gesture of Carla in Fig. 4.70 in which she mimes the example, with the highest pitch when she says "a procedure" and with her raising eyebrows when she explains that a procedure is a series of operations in a certain order (# 630). She explicitly says that they have to translate the properties of equalities in terms of equations, miming the "transportation of the properties of equalities" in the world of equations (#630). After her speaking she nods, biting her lips and waiting for feedback from her students (#630). Moreover, she repeats a second time that students have to help ourselves with examples in order to find an equivalent equation to the chosen one. This time, she increases her tone of voice, always nodding as she is quite sure that for students are natural to construct examples. In addition, she mimes a random example pointing on the sheet of one student as if he has already written it and she refers always at that sheet within her discourse (#632). It could seem that Carla thinks that for student is so natural to construct example, that she is thinking as if students have already written it on the sheet. Hence, there is an *epistemic emotionality* because there is not only the knowledge of equalities and equations both considered as open statements (rational key), but also the fact that she is expecting that students are able to make an example (emotional key). This because examples could help them to construct the analogy between equalities and equations, namely to translate the properties of the equalities in the case of equations. This expectation is related, in turn, to the fact that students can remember and employ the previous knowledge. The emotional key is revealed by gestures, by the change of the tone of voice, by her facial expressions and so on. Hence, being emotionally involved, her discourse cannot be "plain", but is full of emotional aspects as, for the example, the repeating of the fact of taking example: it could show that for Carla it is important that students work on examples they constructed. In addition the changing of the tone of voice for stressing what are for Carla the crucial words or passages, the gestures, the raising eyebrows through which Carla seems to emphasize the relevance of what she is stating. For this reason, I talk of the *communicative emotionality* of the teacher.

633	(*they are working on the Worksheet 4.84*)	
634	T: then how are the "leggi di monotonia" translated (*pause*) in properties on equations (*she inclines her body towards the class, she raises eyebrows*)? (*nodding, she waits for an answer*) What can you say if you have an equation ok? and what do you do?	

635	S3: if we multiply or divide both sides of an equation for the same value we will obtain an equivalent equation.	
636	S8: also subtracting we obtain	
637	S3: yes	
638	T: then say, given an equation, for the first "legge di monotonia", if we add the same number to both sides or we subtract it, working in groups ok?, let's remember how the addition is defined, it means adding the opposite ok? then we can speak of sum. Then if we add to both sides of the equations the same number we obtain (*she looks at the class, she stops in speaking, biting her lips for waiting for an answer*)	
639	S6: an equivalent equation	
640	T: an equivalent equation (*repeating nodding*) to the given one. Instead for the second "legge di monotonia" (*nodding, she stops in speaking and she bites her lips:* Fig. 4.72)?	 Figure 4.72
641	S3: if we multiply or divide (*Carla nods, remaining with lips as in Fig. 4.72*)	
642	S5: by a number not equal to zero (*Carla nods, remaining with lips as in Fig. 4.72*)	

643	S7: both sides (*Carla nods, remaining with lips as in Fig. 4.72*)	
644	S3: we obtain an equivalent equation to the given one (*Carla nods, remaining with lips as in Fig. 4.72*)	
645	T: (*nodding*) do you all agree?	
646	Ss: Yes	
647	T: the question you find in the end of the page ok? (*highest pitch*) what does this serve us? (*increasing the tone of voice*) why is it helpful?	
648	S3: to being able to find the unknowns (*Carla nods*)	
649	T: (nodding) because we serve us to be able to transform an equation	
650	Ss: to simplify calculations	
651	T: then, let's remember what we have said, we have said that two equivalent equations are (*pause and gesture with the hand and she bites her lips:* Fig. 4.73)	 Figure 4.73
652	Ss: equivalent	
653	T: (*irritated and she beats her hand on the table*), equations that have (*pause*)	
654	Ss: the same	
655	T: the same set (*pause*)	

656	Ss: truth set	
657	T: (*nodding and speeding up*) the same truth set, then if my goal is solving an equation, will it change something if I solve an equation equivalent to a given one, from solving the starting equation?	

658	T: will it be the goal? The goal ok? (*pause, gesture and very significant face in* Fig. 4.74) the goal is arriving to an equation that is equivalent to the starting one, but that is so simple that it can be possible to read (*pause and significant gesture in* Fig. 4.75) which is the solution set.	 Figure 4.74  Figure 4.75

Going on in the lesson, after that students discuss in pairs on the example they chose, Carla decides to institutionalize the knowledge they have constructed and applied for that specific example in which they find an equivalent equation. In particular, Carla asks to her class how they have translated the "leggi di monotonia" in the case of equations and, then, she inclines her body towards the class and she raises eyebrows waiting for an answer from her students (#634). After two feedbacks of the students she summarizes how it happens if we apply the first "legge di monotonia" to an equation, but she doesn't want to say herself that doing that we obtain an equivalent equation, then, she stops her discourse, nodding and biting her lips expecting that someone finishes her statement (#638). The teacher has the same attitude for the second "legge di monotonia" (#640). Moreover, when students construct that knowledge, she remains for all the time in the same attitude that is constituted by her closed fist, her nodding and tightening her lips (#641, #642, #643,#644: Fig. 4.72). Hence, there is an *epistemic emotionality* of Carla that is not only the knowledge of the "leggi di monotonia" for equalities (rational key), but also the fact that Carla hopes that students remember what they have already done with her for equalities. This way they can see the analogy with the case of equations, given that she is expecting that students consider equations as open statement as equalities (emotional key). This emotional key is shown, for example, by her emblematic attitude when one student after the other construct all together the principles of equivalence in #641, #642, #643,#644 (Fig. 4.72), by her pauses, her nodding etcetera. The teacher is always very careful to give justification of what they do in the class. Indeed, after using the "leggi di monotonia" to find an equivalent equation to the starting one, Carla focuses the attention of her students on why they have employed that properties on equations (#647). This consideration comes along with a change of the tone of voice of the teacher when she stresses the last question of the worksheet related to why they have used that properties and with her nodding when a students answers "to being able to find the unknowns" (#648). Moreover, she recalls that their aim is to solve equation: the goal, then, is arriving to the most simple equivalent equation to the given one in order "to read" immediately what is the solution (#658). But, the interesting thing is that the development of the discourse does not contain only that words of Carla: there are also her insistence in asking what is the goal, the pauses and emblematic face waiting an answer in Fig. 4.73 her gesture in Fig. 4.74. In these figures she seems to bring out from her students the justification of the use of the "leggi di monotonia". Hence, the *teleological emotionality* of the teacher is involved, first, her explicitly aim of reducing an equation to the most simple form passing through equivalent equations (rational key). Then, it also involves the fact that she is expecting of bringing out that justification from her students, probably, because she hopes that students have learnt from her to justify all of what they do for giving it a meaning (emotional key, shown by her posture, she stopping in speaking, her gesture in Fig. 4.74 and so on). The teacher is engaged in the discursive activity with the class not only from a rational point of view, but also from an emotional point of view: she has hopes, expectations but also needs. Then in her speech are reflected both these sides because the teacher cannot be just pure rational in her discourse. For this reason, I speak of *communicative emotionality* of Carla. It is disclosed by her posture, her gestures, her stopping in speaking, the tightening of her lips and so on. These informations allow to say something about why she decides to act in a certain way. For example, very often, she stops in speaking, she tightens her lips, she pauses, she makes the gesture that mimes the "bringing out" of the knowledge from her students, probably, because she hopes that students continue the discourse, having learnt with her to construct new knowledge, to see analogy, to give justification of what they do and to work on example before institutionalizing the knowledge.

659	T: (*referring to equations*) they are (*raising eyebrows*) fundamental tool to solve problems (Fig. 4.76) ok? these are (*raising eyebrows*) very important activities because the resolution of problems is obviously, you should noticed it, one of (*raising eyebrows*) the fundamental application fields in mathematics. Then we consider problems that can be solved with first grade equations ok? I will give you a worksheet (in which there is the "Oil problem": 4.86), when you receive it (*pronouncing*) individually start to read it carefully. So, in this worksheet, it is proposed a problem, read it carefully and try to answer as it comes in your mind, ok? then arrive to the middle of the page where you have to verify the result you found. Hence, the first thing (Fig. 4.77) that we do when we read the text of a problem is understanding (*she nods while she speaks*) the text of the problem, ok? it is obvious but often the students don't do that. So, let's do the test: "is it clear for all of you the text of the problem?"'	 Figure 4.76  Figure 4.77
660	*they work individually on the problem and then they discuss all together*	

661	T: all this was to do (*pronouncing*) understand you (Fig. 4.78) why it is important to employ equations. Say that in this problem, if someone doesn't reflect on it and answers immediately (Fig. 4.79), often it happens that this answer is not the correct one. Some of you answered immediately (Fig. 4.80: *she is waiting that someone has actually done it*) and he has found that the content is (*she nods while she is speaking but always with the hand as in* Fig. 4.80, Fig. 4.81)	 Figure 4.78  Figure 4.79

<table>
<tr><td>662</td><td></td><td>
Figure 4.80

Figure 4.81</td></tr>
</table>

In the last lesson I videotaped, Carla explains that equations are fundamental tools for problem solving. She accompanies this important consideration with raising eyebrows for two times to stress the relevance of the issue (Fig. 4.76, Fig. 4.77: #659). In particular, Carla proposes the problem 4.86: “A drink costs 1.10 euro, included the empty; the content costs one euro more with respect to the empty. How much is the content?”. She is very careful to give importance to the text of the problem. Understanding it is the necessary condition to be able to solve it. After students work individually on the problem, she justifies why she has shifted from equations to problems. This justification comes along with pronouncing "understand" (#659) and by the open hands as in Fig. 4.78 in which, simultaneously, she seems to expect this justifications directly from students. Then, she wants to highlight that some of students have answered in a wrong way because they don't formalize the problem with an equation. She is sure that someone has answered in a wrong way because she has seen it passing through desks. Indeed she gestures as in Fig. 4.79, Fig. 4.80, Fig. 4.81 and she nods, expecting that who was wrong says what she did.

The *teleological emotionality* of Carla is constituted both by showing to students why it is important to study equations (rational key) and by her hope that students have already understand it (emotional key). The emotional key is revealed by her gestures with open hand in order "to receive" the explanation from students and by her raising eyebrows for expecting a feedback from them. The *epistemic emotionality* of the teacher is constituted by the resolution of the problem (rational key) and by her expectation that students who answered wrong, without passing through equations, understand the importance of construct an equation to solve the problem (emotional key). This emotional key is disclosed by her postures and gestures as in Fig. 4.80 and Fig. 4.81 in which she invites students who gave the wrong answers to intervene in the discussion in order to become aware of the importance of using equations in problem solving. Being emotionally involved her discourse cannot be plain, because Carla has hopes from her teaching and, in this case, she expects that students understand the meaning of why they have used five lessons for learning equations. It is very difficult to isolate her words from their emotional overtones. For this reason, I talk about the *communicative emotionality* of the teacher.

Worksheets

Nome e Cognome ……………………………………………… Classe ………. Data …………….

ATTIVITA' 1A – TITOLO: ……………………………………………..

ACTIVITY 1A - TITLE:

Considera le seguenti coppie di enunciati aperti, definiti nell'insieme universo U indicato, ed elenca tutti gli elementi dei rispettivi insiemi di verità A e B:

Consider the following couples of open statements, defined in the indicated universe set U and list the elements of the corresponding truth sets A and B:

a) $U = \{x \in N / x \leq 20\}$ $A(x)$: x è pari ; $B(x)$: x^2 è multiplo di 4

A(x): x is even ; B(x): x^2 is multiple of 4

A = ……………………………………………………………………………………

B = ……………………………………………………………………………………

b) $U = \{x \in N / x \leq 100\}$ $A(x)$: la cifra delle unità di x è 0 ; $B(x)$: x è multiplo di 10

A(x): the unit digit of x is 0 ; B(x): x is multiple of 10

A = ……………………………………………………………………………………

B = ……………………………………………………………………………………

c) $U = \{x \in N / x \leq 20\}$ $A(x)$: x è dispari ; $B(x)$: x^3 è dispari

A(x): x is odd ; B(x): x^3 is odd

A = ……………………………………………………………………………………

B = ……………………………………………………………………………………

d) $U = Z$ $A(x)$: $|x| \leq 5$; $B(x)$: $x^2 \leq 25$

A = ……………………………………………………………………………………

B = ……………………………………………………………………………………

e) $U = Z$ $A(x)$: $|x| < 5$; $B(x)$: $-4 < x + 1 < 6$

A = ……………………………………………………………………………………

B = ……………………………………………………………………………………

Che cosa puoi osservare? ……………………………………………………………………..

What can you observe?

……………………………………………………………………………………………

Come definiresti gli enunciati aperti di ciascuna delle coppie considerate? ……………………………

How would you define the open statements for each couple you considered?

Figure 4.82: Activity 1a

ATTIVITA' 1B

ACTIVITY 1B

a) Nell'insieme N sono definiti i seguenti enunciati aperti: $A(x):\ x < 6$; $B(x):\ x \leq 5$
In N are defined the following open statements:
Elenca tutti gli elementi dei rispettivi insiemi di verità:
List all of the elements of the corresponding truth sets:
A = ……………………………………………………………………

B = ……………………………………………………………………

Che cosa puoi concludere? ……………………………………………
What can you conclude?

b) Considera nell'insieme universo Q gli enunciati aperti definiti in a). Che cosa osservi? ……
In the universe set Q consider the open statements defined in a). What do you observe?
……………………………………………………………………

……………………………………………………………………

c) Considera ora i seguenti enunciati aperti: $A(x):\ x \leq 4$; $B(x):\ x^2 \leq 16$
Consider now the following open statements:
Nell'insieme N: ……………………………………………………
In N:
……………………………………………………………………

……………………………………………………………………

Nell'insieme Z: ……………………………………………………
In Z:
……………………………………………………………………

……………………………………………………………………

Quale importante osservazione ti suggeriscono gli esempi precedenti? ……………
Which important observation do the previous examples suggest to you?
……………………………………………………………………
……………………………………………………………………
……………………………………………………………………
……………………………………………………………………
……………………………………………………………………

Figure 4.83: Activity 1b

Nome e Cognome .... Classe .... Data ....

ATTIVITA' 2 – TITOLO ....

ACTIVITY 2 - TITLE

Considera le seguenti proposizioni e stabilisci quali sono vere e quali false:

Consider the following propositions and establish which are true and which are false:

a) $3\cdot 5+1=18$ ....

b) $3^5\cdot 3^2=3^7$ ....

c) $(2+1)^3=2^3+1$ ....

d) $30+10+5=5\cdot(6+2)$ ....

e) $(7+3)^2=7^2+6\cdot 7+3^2$ ....

Nelle precedenti proposizioni compare il predicato "essere uguale", esse sono esempi di <u>uguaglianze</u> e possono quindi essere vere o false.

In the previous propositions, it appears the predicate ``to be equal to", they are examples of equalities and they can be true or false.

Nei seguenti enunciati aperti compare il predicato "essere uguale", essi vengono detti <u>equazioni</u>.

Risolvere un'equazione significa determinare l'insieme di verità di un enunciato aperto in un insieme universo fissato. Sapresti risolvere le seguenti equazioni?

In the following open statements, it appears the predicate ``to be equal to", they are called equations. Solving an equation means determining the truth set of an open statement in a given universe set. Can you solve the following equations?

a) $3x+6=18$ $U=N$ ....

b) $x(x+1)=20$ $U=N$ ....

c) $(x+1)^2=81$ $U=Z$ ....

d) $(x+5)(x-2)=0$ $U=Z$ ....

e) $5x+3=2$ $U=Q$ ....

Nel caso delle equazioni, l'insieme di verità è detto <u>insieme delle soluzioni</u> ed i suoi elementi sono le <u>soluzioni</u> dell'equazione.

Ricordando gli esempi e la definizione incontrati nell'attività 1, completa la definizione:

Due equazioni definite, nello stesso insieme universo U, si dicono equivalenti in U se ....
....

Considera le seguenti equazioni: $2x+1=9$; $x^2-16=0$ e risolvile

nell'insieme N: ....

nell'insieme Z: ....

Che cosa è quindi <u>importante</u> osservare? ....
....

Figure 4.84: Activity 2a

L'insieme delle soluzioni S di un'equazione è un sottoinsieme dell'insieme universo U.

Risolvi le seguenti equazioni nell'insieme indicato:

a) $3x + 6 = 20 \quad U = N$ ....

b) $3x + 6 = 20 \quad U = Z$ ....

c) $3x + 6 = 20 \quad U = Q$ ....

d) $2x + 8 = 6 \quad U = N$ ....

e) $2x + 8 = 6 \quad U = Z$ ....

f) $x^2 + 1 = 0 \quad U = Q$ ....

g) $3(x - x) = 0 \quad U = Q$ ....

h) $(x + 1)^2 = x^2 + 1 \quad U = Q$ ....

i) $3(x + 2) = 3x + 6 \quad U = Q$ ....

j) $\frac{3}{x} = 0 \quad U = Q$ ....

Osservando i risultati ottenuti, quali osservazioni puoi fare riguardo all'insieme delle soluzioni di un'equazione? ....

....

....

....

....

....

Considera ora la seguente equazione, nell'incognita x, $ax + b = c$ con $a \neq 0$ e stabilisci quali condizioni devono soddisfare a, b e c affinché essa sia risolubile.........

In N ....

....

In Z ....

....

In Q ....

....

Figure 4.85: Activity 2b

Nome e Cognome ....Classe.............. Data............................

ATTIVITA' 1

Problema 1

Una bibita costa 1,10 euro, vuoto compreso; il contenuto costa un euro più del vuoto.
Quanto costa il contenuto?

....

Spiega come hai ottenuto il risultato che hai proposto:

....

....

VERIFICA SE IL TUO RISULTATO È COMPATIBILE CON I DATI DEL PROBLEMA:

La bibita senza vuoto costa........
Il vuoto costa......
La bibita, vuoto compreso costa....
La differenza tra il costo del contenuto e del vuoto è........

LA VERIFICA APPENA COMPIUTA TI PERMETTE DI CONFERMARE IL RISULTATO CHE HAI PROPOSTO ALL'INIZIO? **SI** **NO**

Se hai risposto **SI**, puoi passare alla fase successiva dell'attività.
Se hai risposto **NO**, allora proponi un nuovo risultato...

TRADUCI NEL LINGUAGGIO ALGEBRICO
Rileggi attentamente il testo del problema e rispondi alle domande che seguono, completando a parole la colonna che corrisponde al linguaggio naturale e, coi simboli dell'algebra, la colonna che si riferisce al linguaggio algebrico:
Quali sono i dati, ossia gli elementi noti desunti dal testo?
Quali sono le incognite?
Quali relazioni legano i dati alle incognite o le incognite tra di loro?

	Linguaggio naturale	Linguaggio algebrico
Richiesta o richieste		
Incognita o incognite		
Dati e relazioni fornite dal problema		
Equazione risolvente		
Risoluzione dell'equazione		
Controllo se la soluzione è accettabile		
Risposta		

Figure 4.86: "Oil problem"

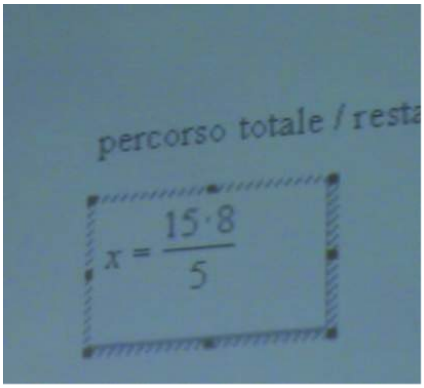
4.3 The "fabric" of Rationality and Emotion in Sara

In this lesson, Sara discusses with her class the following activity: *Luca has decided to participate in a foot race that takes place every year in his city; today he will train on the path of the race. From the starting line, he starts to walk in a regular way, namely with a constant velocity. He began walking at* $14:00$ *(2 p.m.), now it is* $14:15$ *(2:15 p.m.) and he covered* $3km$. *There still remains* $5km$*; at what time will he reach the arrival? How do you find the answer? Write the steps you did to arrive to the solution.*
At the beginning, students work in pairs to solve the activity proposed by the teacher. Then, they start with her to discuss what they have found. Students have suggested different ways of solving the problem: through numbers with proportions[8]; through segments; through their clocks. Sara is very careful to highlight that these different approaches to the problem produce the same result. They could appear different, but, actually, they are all equivalent methods to solve the problem ("All of them are (pronouncing) completely equivalent ways to respond to this question").
Sara uses this activity to introduce the concept of equation as a statement that contains the verb "to be equal to". The values that make it true constitute the solution set of the equation.

[8]the proportion $15 : 3 = x : 8$, where x is the total time to cover the entire path and and the proportion $15 : 3 = x : 5$, where x is the time to cover the remaining 5 kilometers.

663	T: If I wanted to (*pronouncing*) put in formula (*pause* Fig. 4.87) (*highest pitch*) what I'm saying, how could I do it (Fig. 4.88)? If I wanted to put it in formula, how could I do it (*she remains in the same manner for many seconds:* Fig. 4.89)? We have seen that a formula serves to speak in mathematics (*smiling*) a little bit, then if I wanted to put in a formula (*pronouncing*) the time used to cover the entire path, what can I do to put it in formula?	 Figure 4.87  Figure 4.88  Figure 4.89

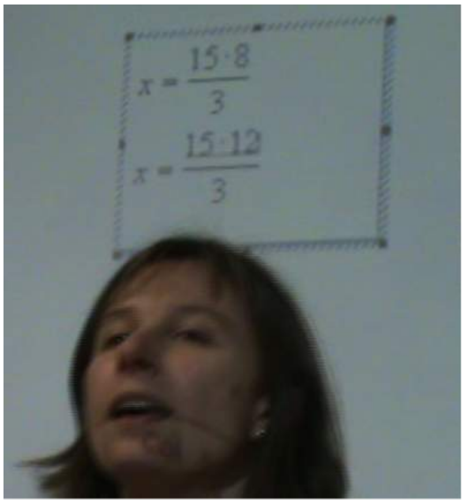
664	S1: but, in words?	
665	T: no, with a formula (*smiling* Fig. 4.90). We have already written it in words, you have told me that we have done the proportion and then you have solved the proportion or you have calculated the time necessary for doing one kilometer and then you have multiplied the time necessary for doing one kilometer times 8 (*she is waiting again on an answer as in* Fig. 4.91, *playing a little bit nervously with her ring*).	 Figure 4.90  Figure 4.91
666	S4: they are proportional	

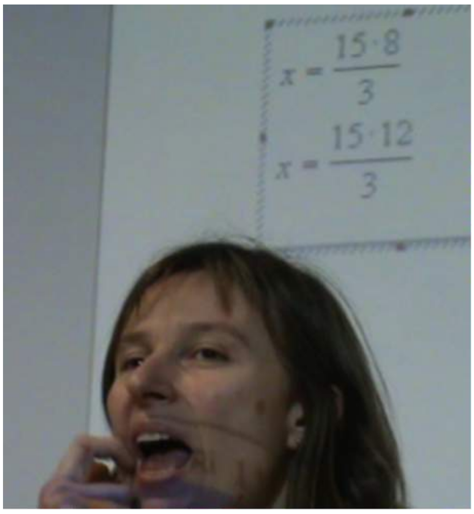
667	T: they are directly proportional yes, because we have made a proportion and then they are surely directly proportional	
668	S6: x is equal to the minutes already done times the total kilometers divided by the time for doing one kilometer	
669	T: Then if I had to write it, how can I do it?	
670	S6 dictates to the teacher	 Figure 4.92
671	T: (*she repeats what S6 says*) 15 times the total kilometers divided by the time used for one kilometer. Ok this one could be (*smiling*) right.	

Sara declares to the class that she wants to make a step forward, putting in the form of a formula what they have explored by words and proportions: namely to make a formula for the time used to cover the entire path[9]. This action comes along with pronouncing "to put in formula". There is a pause after these words, probably, to emphasize them. At the same time, there is a sudden increasing of the tone of voice to say what Sara wants to put in a formula (e.g. "what I'm saying") (#663). Moreover, this corresponds with the facial expression in Fig. 4.87. Furthermore, the declared goal is accompanied

[9]She introduces the verb "to put in a formula" to define what an equation is. This global aim will become visible going on in the lesson.

by her facial expression in Fig. 4.88, in which she seems to wait for feedback. The most interesting thing is that she repeats two times, one after the other, the same words "if I wanted to put it in a formula, how could I do it?" (#663). Then, she waits for an answer as she remains in the posture shown in Fig. 4.89. She recalls that they have seen formulas as ways of speaking of mathematics. Simultaneously, she ironical smiles, probably, because she uses the adverb "a little bit", even if she is perfectly aware that formulas are one of the fundamental mathematics tools. In particular, Sara wants to put in a formula what they have found with the proportion constructed to solve the starting problem. In that particular case, the distance was 8 kilometers and they considered the proportion $15 : 3 = x : 8$. In the excerpt above, there is an error that will be corrected by a student Arithmetic in the remaining of the lesson. Indeed, to calculate the time for covering the entire path, the operation is 15 minutes times the total distance divided by how many kilometers are done in 15 minutes, namely 3 instead of 5 (#671). The teacher does not notice the error and she smiles (#671), probably, because she is focused just the right structure of the formula suggested by a student. Hence the *teleological emotionality* of Sara is constituted both by her acting to put in a formula the verbal resolution of the problem (rational key) and by the need that students use Algebra to generalize what they have explored in Arithmetic. She hopes that students have learnt this attitude during the lessons with her. This emotional key is revealed, for example, by the pronouncing "to put in formula" as if to draw the attention of the class to them; by waiting feedback from the class as shown in Fig. 4.87, Fig. 4.88, Fig. 4.89. Moreover, her teleological emotionality involves also her hoping that students have learnt from her to see Algebra as a thinking tool. This emotional key could be marked by her ironically smile when she told her students that they already know that formulas serve to speak of mathematics *a little bit*. At the same time, her *epistemic emotionality* is based on the proportion they have already constructed in the resolution of the problem (rational key, #665). In fact, it consists in her hope that their previous work is illuminating for generalizing it through Algebra. This emotional element is revealed by her expecting an answer, playing a little bit nervously with her ring (Fig. 4.91) and by her satisfaction after the answer of S6 (#668), shown by her smiling (#671)). So, the teacher expresses this emotional engagement during her speech. For this reason, we speak of *communicative emotionality* of Sara: there are both the discourse constituted just by her words and the emotional counterpart of it. The latter is disclosed, for example, by her insistent repetition in #663; her prosody to focus the attention of the students on the key-words (e.g. "to put in a formula"); her facial expressions of wait and hoping that students are able to use Algebra, at least at the beginning, as a generalization of Arithmetic; her gestures and so on.

672	T: (*highest pitch*) but instead of calculating this way, (*very slowly and pronouncing all the words*) instead of calculating how much time Luca uses in 8 km, namely, how much time I use to make 8 km, I wanted to know how much time he uses to make 12 km, what should I do? (*gesturing, speeding up and smiling*) always thinking of the fact that Luca isn't ever tired and that he walks always at the same velocity?	
673	S10: 15 times 12 divided by 5	
674	T: 15 times 12 divided by 5	
675	T: where x is always the time, right? (*with a tone of voice indicating certainty*) let's write it somewhere (Fig. 4.93)	 Figure 4.93
676	S6: divided by 3	

677	T: good, why divided by 3 and not divided by 5 (*facial expression as in* Fig. 4.94)?	 Figure 4.94
678	noise	

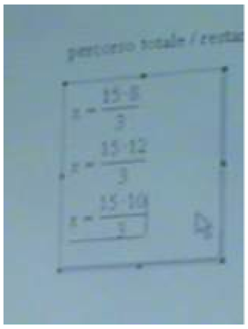
679	T: (*nodding*) not only, (*smiling:* Fig. 4.95) from a conceptual point of view, what does it mean 15 divided by 5? Make (gesture: Fig. 4.96)15 minutes divided by 5 kilometers, what does it mean (*she moves her fingers as if grasping for a justification, then she opens her hand towards the students* Fig. 4.97)	 Figure 4.95  Figure 4.96  Figure 4.97

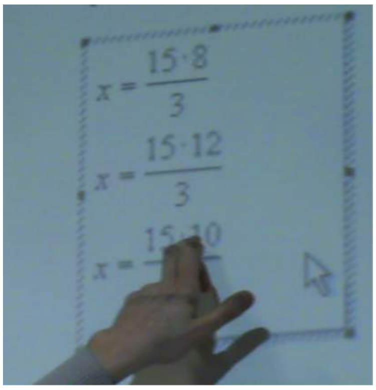
680	*noise*	
681	T: yes, it means that you have used 15 minutes to do 5 kilometers and not 15 minutes to do 3 kilometers, ok? And then Luca walks more slowly or more quickly?	
682	Ss: more quickly	
683	T: yes, he goes faster, then we would make Luca (*smiling*) a little bit too fast	

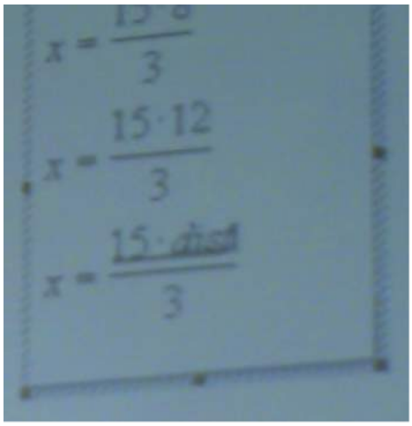
Sara accompanies gradually her class in finding an equation that expresses the time requested to cover any distance. This process comes along with an increasing of the tone of voice for the "but" (#672). Probably, she wants to contrast what they have just discussed for a given distance with its distance change. Moreover, she pronounces very slowly the sentences in #672. She likely wanted to turn the students' attention to the question of generalizing the problem. At the beginning, she decides just to change the number, 12 instead of 8, in order to arrive, through small steps, to introduce a variable for the distance. Then, she specifies, speeding up and smiling, that they can increase the distance supposing that Luca will not ever be tired while walking always at the same velocity. A student answers that the time requested to make $12km$ is 15 times 12 divided by 5. Sara writes it on the pc, when another student notices that it is divided by 3 and not by 5. The teacher seems satisfied by the observation and she asks why it is divided by 3 and not by 5. This request of justification comes along with a particular facial expression of Sara in Fig. 4.94. After an overlapping of answers of the class, Sara wants to focus the attention of her students on the meaning of the operation 15 divided by 5. This more specific request of justification is accompanied by the smirking of Sara in Fig. 4.95, probably, because someone realized the error; by her gesture for highlighting the two protagonists of the ratio; by her moving the fingers as grasping the justification from her students (Fig. 4.96, Fig. 4.97) (#679). At the end of this passage, the teacher, smiling, uses the error to make an important consideration: if Luca had employed 15 minutes to do $5km$ and not $3km$, he would have gone too fast.
Hence the *teleological emotionality* of the teacher is based on the change of the number for the distance, in order to introduce a variable for it (rational key) and the fact that she is expecting that this generalization comes naturally from her students (emotional key). This, probably, because she is quite sure that she has taught her students to see Algebra as a generalization of

Arithmetic. This emotional key is disclosed, for example, by pronouncing the "but" and by her slowly speaking explaining the change of the number #672. In addition, she hopes that students continue to link Algebra to the real context of the problem from which they started. Indeed, smiling, she underlines the fact that Luca is never tired to make this type of reasoning (#672). At the same time, there is an *epistemic emotionality* of Sara that is constituted by the arithmetical translation of the new problem (15 times 12 divided by 5) and by the justification of the error (rational key). Moreover, it involves the fact that she asks that justification from the students, because she hopes that they feel the need to justify in order to understand what they are considering (emotional key). This emotional counterpart is shown by her smirking when a student notices the wrong denominator; by her facial expression in Fig. 4.94; by her fingers movement that "grasps" the justification from them; by her open hand as she seems to want to receive that justification from the class #679. We can also speak of communicative emotionality of Sara, because she has certain hopes and needs in her teaching, which means that only with extreme difficulty would her discourse be stripped of emotional overtones.

684	T: good, then, if I went ahead this way, if I tried to continue this way (*gesture with hand to indicate the repetition and she is referring substituting* 8 *km with another number*) actually what am I doing? namely, if instead of having 12 *km* I would have uhm (*gesture for indicating the randomness*) 10 *km* what should I do?	
685	Ss: 15 times 10 divided by 3?	
686	T: x equal to	

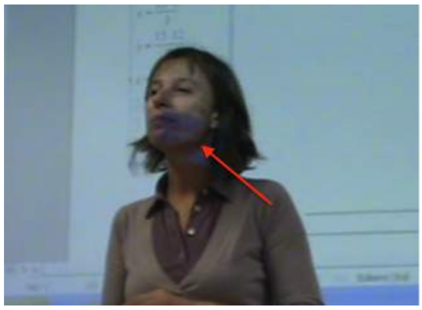
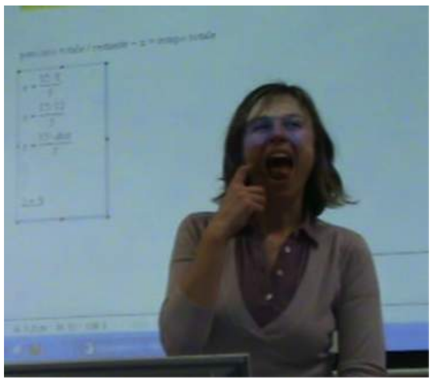
687	Ss: 15 times 10 divided by 3? (Fig. 4.98)	 Figure 4.98
688	T: ok, 15 times 10 divided by 3 and if we wanted to put somewhere some letters, what could we put? because she told me "lets' put letters, ok?" (*referring to a student*), ok?	
689	Ss: noise	
690	S3: instead of 10 I put	
691	T: instead of 10 I put (Fig. 4.99)?	 Figure 4.99
692	S3: a letter	
693	T: a letter	
694	S3: for every length that I covered	

695	T: for every length that I covered, I can put a number of any length of path and calculate what?	
696	S4: x, the time	
697	S5: the time	
698	T: the time (*nodding*)	
699	S3: the total time to cover that lenght	
700	T: oh (*as if to say "well done!"*) I could put here (*pointing the kilometers in the formula*) a measure, a letter that represents the distance and calculates the time used to cover that corresponding distance, right? (Fig. 4.100)	 Figure 4.100

701	T: instead of putting 10 that is the distance already covered, I could put a letter, I could put something and I'm able to calculate the distance (Fig. 4.100). In so doing you calculate the time that I use to cover a distance, obviously the time (.) depends on the distance. (*satisfied tone of voice*) Good job!In writing this (she returns to the screen and she points, Fig. 4.101), is there someone that notices something in mathematics, a mathematical object that you have already seen somewhere (Fig. 4.102: *she inclines her body towards the class waiting for an answer*)?	 Figure 4.101  Figure 4.102
702	Ss: noise	
703	T: when you write 15 times 8 divided by 3, 15 times 10 divided by 3, 15 times 12 divided by 3.	
704	S6: they are equations	
705	T: (*satisfied tone of voce*) they are equations	

Sara continues to push her students towards generalization, inviting them to consider, now, $10km$ as distance. This action comes along with a gesture of change the number and with that for miming the randomness of it (#684). A student answered: "x is equal to 15 times 10 divided by 3". But, Sara notices that they have suggested to put some letters in the formula. Hence, she is waiting for an answer (Fig. 4.99) and, actually, a student proposes to put the letter for the distance. The teacher exclaims "oh" to express her satisfaction for the answer (#700). In addition, justifying the introduction of the letter, she explicitly says "Good job" (#701). Then she shifts the attention of the student to the examples in #703. A student recognizes that they are equations and the teacher repeats what he said with a satisfied tone of voice (#705).

The *teleological emotionality* of Sara involves the changing of the value of the distance to arrive at a generalization (rational key). Moreover, it is also her expectation that the class is able to pass from the particular cases with numbers to a generalization with a letter. This emotional key is revealed by gestures that accompany the change of numbers and its randomness. The *epistemic emotionality* of the teacher is constituted both by the justification of putting the letter for the distance (rational key, #701) and by her need and hope that students are able to understand this justification. This emotional counterpart is shown by her satisfied tone of voice after feedbacks of the class (#700, #701). Furthermore, her epistemic emotionality includes her expectation that students remember that the equalities in #703 are equations, because they have already encountered them (emotional key, disclosed by her posture in Fig. 4.102). Her speech, then, is full of emotional elements, expressing the emotional side of her discursive activity. They are, for example, gestures for triggering the generalization, facial expressions for expecting that they remember previous knowledge and so on. For this reason, we talk of the *communicative emotionality* of the teacher.

706	T: Could you tell me examples of other equations? (*waiting, eloquent facial expression*: Fig. 4.103) (*she pauses for a while*) let's say an equation (*doing knee-bends*) an any one, simple (*smiling*).	 Figure 4.103
707	S7: 3 equal to 5	
708	T: (*in the meanwhile she writes*) 3 equal to 5? (Fig. 4.104, Fig. 4.105)	 Figure 4.104  Figure 4.105
709	S11: 5 minus 2	

710	T: 5 minus 2. Is it an equation?	
711	Ss: No	
712	T: No	
713	noise	
714	T: There must be an unknown	
715	S9: 8 equal to 2 times x	
716	S10: $x = 4$	
717	T: ok (*nodding*), If I write this way (*she writes* $x+45 = 4-3x$) is it an equation or not?	
718	S2: there isn't the equal sign	
719	Ss: yes there is the equal sign	
720	T: then, let's restart from the beginning. You have told me that the equation is an equality. Then when I have asked you to tell me an equation, you have said $3 = 5$ and $3 = 5 - 2$. Are these two equalities?	
721	S2: the second	
722	T: 3 equal to 5 is false, it is a false statement, but 3 equal to 5 is a statement, ok? I can perfectly say that 3 is equal to 5, but then it is not true, but I can say that 3 is equal to 5. Rightly, you have told me that for having an equation there must be an unknown, an unknown that I have called x, it could be called?	
723	Ss: any one	

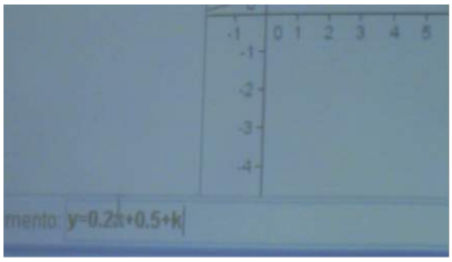
724	T: (*nodding*) any letter of the alphabet, but there must be an unknown because if there isn't an unknown then the things don't work, we aren't speaking of equations, then probably saying that an equation is an equality between two terms is not completely correct. Let's say that we define (*gesture with the hand with the thumb and the index joined moved up and down:* Fig. 4.106), let's say it immediately such that we are ok. We define (*again the same gesture,* Fig. 4.106) equation (*pronouncing*) as a statement in mathematics in which there is the verb, and that verb is (*pronouncing*) "to be equal to" (.) (*pause and she looks at the class*) ok? then we define equation as a statement in mathematics in which there is the verb "to be equal to". Obviously, in this statement there has to be at least one unknown, at least an unknown because there could be also more ok? (*she bites her lips, looking at the class*). Listen! If I defined the equation as a statement with the verb "to be equal to", (*speeding up*) then it doesn't change too much if, instead of saying that an equation is an equality, we say that is a statement that has the verb "to be equal to" and we say that there in an unknown when I search (*pronouncing*) for a solution of that equation.	 Figure 4.106

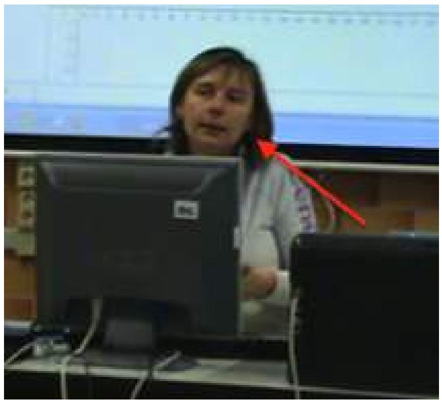
Sara decides to ask her class for other examples of equation with the aim of formalizing well what is an equation. This action is accompanied by a pause with the eloquent facial expression in Fig. 4.103, in which she seems to have a searching mood. Then, she changes her attitude: doing knee-bends and smiling, she declares that she is satisfied just by a simple example (#706).

She wants, probably, to put students at ease to answer. Actually, a student proposes $3 = 5$ and, first, she ironically smiles (Fig. 4.104). She is aware that this is not an equation but, probably, she doesn't want to say it. Then, she makes another interesting face (Fig. 4.105) in which she smiles again. She remains with the open mouth touching it with her finger. This posture could express her hope that someone asserts that $3 = 5$ is not an equation, even if, probably, it is hard, for her, to resist in saying it.
After the interventions of some students, the teacher states that there must be an unknown in an equation. She chooses the equation $x + 45 = 4 - 3x$ on which she discusses with her class. Then, she returns to the definition of equation as equality. At this point, she recalls equalities on the two examples made by the students ($3 = 5$, $3 = 5 - 2$) to notice the difference between them and an equation. In an equality there are just numbers, while in an equation there must be at least one unknown. She stresses this fact and she defines formally what it is an equation. She accompanies this definition with a gesture of moving up and down her hand joining her thumb and her index (Fig. 4.106). Probably, she wants to signal the fact that, now, she is giving a formal mathematical definition. Moreover, she pronounces the word "statement" and "to be equal to", repeating for two times, one after the other, the definition of equation (#724). This, probably, because, after the first time, while she is pausing and looking at the class, she is not sure that the whole class understands the definition. In addition, after clarification about how many unknowns there could be in an equation, she repeats for the third time its definition (#724).
The *teleological emotionality* of Sara involves both the use of examples of equations to give the formal definition of them (rational key) and her expectation and need that students are able to make examples (emotional key). This emotional key is revealed by her searching mood in Fig. 4.103; by her smiling to put students at ease to talk with her. In addition, it seems that she expects that students can construct right examples of equations when she, a little bit nervously, smiles after having heard $3 = 5$. She would like to correct immediately the student, but she hopes again that someone can do it in her place and she remains as in Fig. 4.105. The *epistemic emotionality* of the teacher is constituted by the definition of equation (rational key). Moreover, the epistemic is testified by her hope that they understand that formal definition they have previously discussed on examples (emotional key). This emotional counterpart is shown by her gesture moving up and down her hand while she says the definition as well as by her pronouncing of key-words as "statement" and "to be equal to". In addition, she repeats three times the definition of equation. Even if she hopes that examples were helped them in understanding the formal definition, she is also aware that they have given

wrong examples of equations. Then, it's better to stress the definition many times. In her speech she doesn't have to actually tell the students everything. For example, her pauses after questions are hints of this fact. She knows she has to tell the definition, but she does not want to have to tell the examples. This could be a secret constructivist kind of hope. Hence, we talk of *communicative emotionality* of the teacher.

725	T: Actually, we have already seen the first part of the activity. The activity asked us to prepare a slider, to call it k and to vary it from -15 to 15 and (*blaring tone of voice*) then it told us to take into account two equations: one we have already solved in the previous time and the other one was established, if you remember it. The equation already solved previously was $\frac{1}{5}x + \frac{1}{2} = 8$. To solve this equation we have already said that, actually, we could work on two different (*pronouncing*) functions, precisely on two (*pronouncing*) straight lines: one was this straight line (*she draws on GeoGebra the function* $y = \frac{1}{5}x + \frac{1}{2}$) (*pause and she looks at the screen*) and the other one was $y = 8$. Then, you have told me, if you remember, that the solution to the equation was the intersection point between these two straight lines. (*rhetorical question*) Do you remember it? There was Elena who said "(*highest pitch*) I go to see where I intersect and then I read the solution". Actually the solution we have to read is not on the $y - axis$, but it is on the $x - axis$, because it is the value of x that is of interest to us as solution and (*blaring tone of voice*) then the fact of asking to draw the perpendicular line to the $x - axis$ passing through A served simply to say that (*highest pitch*) I can go to read the solution.	

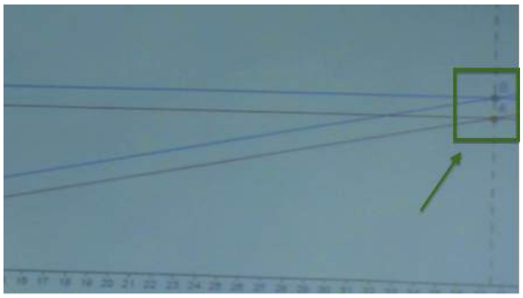
726	T: I can go to read the solution here (*she stands up and she goes on the screen, pointing to the abscissa of the intersection point and then she looks at the class as in* Fig. 4.107,). (*highest pitch*) Going to read this number or (*speeding up*) given that I cannot be sure of the value of this number because GeoGebra has limits (Fig. 4.108), I can read it here (*she points to the "Algebra view" of GeoGebra*). In the "Algebra view" the point A has coordinates 37.5 and 8 and, then, the solution of the equation is the number 37.5. If instead of x I put 37.5 the two straight lines intersect and they have the same value (Fig. 4.109), ok? (*she nods and she returns to the pc*). Then in the activity it was asked to have two different colours, namely to colour red this one (*she colors of red* $y = \frac{1}{5}x + \frac{1}{2}$, $y = 8$ *and A*). Then we were asked to draw another two straight lines. The other two straight lines are: one is $y = 0.2x + 0.5 + k$ (Fig. 4.110). In this case, k is equal to 1 and we see that GeoGebra writes (*pointing to the "Algebra view"*) $y = 0.2x + 1.5$. Why 1.5? (*pause, a student try to say something but he does not finish the sentence*) because k	 Figure 4.107  Figure 4.108  Figure 4.109  Figure 4.110

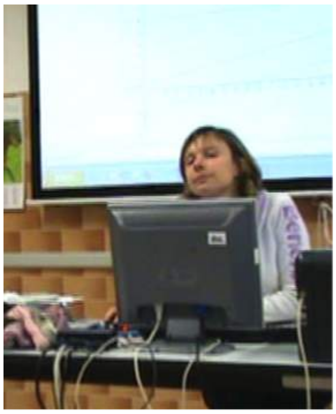
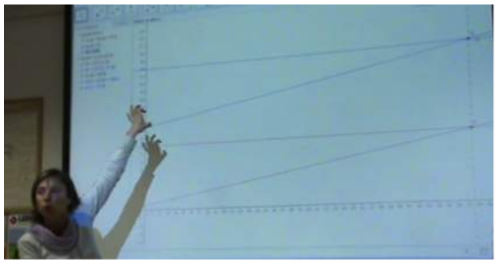
727	S1: it is 1	
728	T: it is 1and, then, 1 plus 0.5 is 1.5, it has already calculated (*referring to GeoGebra*). The other straight line is $y = 8+k$ (*blaring tone of voice and she looks at the class*) if I write $y = 8+k$, in the "Algebra view" it will write $8 + k$? (*facial expression in* Fig. 4.111, *long pause and she continues to look at the class*) (*smiling*) I don't hear answers (*she looks at the class smiling*).	 Figure 4.111
729	Ss: no	
730	T: No, What will it write? (*facial expression as in* Fig. 4.111)	
731	S1: It puts the value of k	
732	T: k is 1 now, then, will it be equal to?	
733	Ss: 9	

Sara works on GeoGebra in order to talk of the graphical solution of an equation. They have already introduced it in the previous lesson. She uses an activity of M@t.abel in which they have to solve the equation $\frac{1}{5}x+\frac{1}{2} = 8$. The solution of it is the time Luca uses to cover 8 km, starting from 500 m from the starting line. Hence, knowing that he uses 15 minutes to cover 3 km, namely his velocity is $\frac{1}{5}km/min$, the time he uses is the solution of $\frac{1}{5}x + \frac{1}{2} = 8$. The teacher reminds the students that they could work on two different straight lines: $y = \frac{1}{5}x + \frac{1}{2}$ and $y = 8$. She pronounces both "functions" and "straight lines" (#725). She recalls that the solution of the equation is related to the intersection point between them. Sara seems quite sure that the students remember this, indeed, she makes the rhetorical question "Do you remember it?". Then, she recalls what a student said in the previous lesson (#725). Moreover, she clarifies that they have constructed the perpendicular line to the $x - axis$ passing through A, because the solution of the equation can be "read" on the $x-axis$. Sara accompanies this justification with a blaring tone of voice. She repeats this fact looking at the class for feedback and pointing to the solution (Fig. 4.107). She uses many times the expressions "to read

the solution" and "to go to see" on the graph. Stressing that the software has limits, she invites students to read the solution not directly on the graph, but on the "Algebra view" of GeoGebra. This is interesting: she is saying that it has limits if we look visually, but not if we look numerically. She accompanies this statement with an eloquent facial expression in Fig. 4.108. After that, she continues with the activity in which they are requested to draw another couple of straight lines depending on k: $y = 0.2x + 0.5 + k$ and $y = 8 + k$. The teacher highlights that GeoGebra gives automatically the value of k to the first function ($y = 0.2x + 0.5 + k$) and then she seems to want from the students the response for the second one ($y = 8 + k$). In fact, after asking with a blaring tone of voice what happens for $y = 8 + k$, she pauses as in Fig. 4.111. Then, she smiles when she says that she isn't hearing any answers, probably, in order to keep the mood light (#728).
Hence the *teleological emotionality* of Sara is constituted by considering the two straight lines and their intersection point to find the solution (rational key). Moreover, the teleological involves the fact that she is expecting that students are used to "seeing" through the graphic register in order to find the solution. This emotional key is revealed, for example, by the fact that she often says, increasing the tone of voice, "to read the solution" and "to go to see" on the graph (#725, #726), probably, to draw the attention of the class to them. This emotional aspect is shown also by the rhetorical question "Do you remember it?" in #725. The *epistemic emotionality* of the teacher is, from one side, the geometrical interpretation the solution of an equation and how the software works (rational key). From the other side, it is related to her expectation that students know how to pass from one register of representation to another one (emotional key). This is strictly related to being able to "see" through the graphic register to reason about equations. For example, she hopes that students recognize the solution on the graph, pointing to it and maintaining a certain facial expression (Fig. 4.107) and waiting feedback from the class. Moreover, she hopes that students are able to link how the algebraic register and the graphic one of GeoGebra work together. In fact, after asking with blaring tone of voice what happens for $y = 8+k$, she pauses with a facial expression as in Fig. 4.111. It is quite clear that the teacher is expecting an intervention from the students. Moreover, her smiling, probably, aims to receive more participation from the class (#728). Actually, this attitude triggered several comments from the students. Her speech is full of emotional hues because she has certain hopes and needs in relation to her students. She changes her tone of voice to emphasize what she is saying, such that students understand the crucialness of it. Especially, in this part of the lesson she seems like a soloist, because she speaks almost all the time. Because of that, her pausing is meaningful: when she stops, she

has the need for students to speak. Being that her discourse is not neutral, we can speak of the *communicative emotionality* of Sara.

734	T: 9. In fact, it is what happens. We take the point B as the intersection point and then we colour blue these two new straight lines and B (Fig. 4.112). Now we could try to ask what happens if I vary the value of k, ok? Varying k something happens that is interesting. Then, let's start to see what happens on the graph varying *k* (*she moves the slider k*). On the graph, what happens while varying *k* (*she continues to move the slider k and she looks at the class*)?	 Figure 4.112
735	S2: (*perplexed*) uhm, it happens that the oblique straight lines remain parallel and the horizontal straight line remains parallel	
736	T: Ok, then the straight line are parallel (*pause*, Fig. 4.113: *she puts her lips in a strange way because she wants that students notice something more*)	 Figure 4.113
737	S3: They change position varying *k*	

738	T: How do they change position (*she continues to move the slider*)? What kind of movement they do (*she is referring to S3*, Fig. 4.114)	 Figure 4.114
739	S3: vertical	
740	T: (*nodding in marked way*) there is a vertical translation (*gesture for miming the vertical translation*) that can be up (*gesture for miming "up"*) or down (*gesture for miming "down"*), but they have simply a vertical translation (*miming the translation*) and they have not an horizontal translation (*miming it*). In terms of the time used by Luca in his foot race, the problem is "if I change of the same distance both the arrival and the starting point of Luca, what will be the time?". Namely, what the solution given by the blue straight lines (Fig. 4.115) will coincide with? (*pause and she remains as in* Fig. 4.116)	 Figure 4.115  Figure 4.116
741	S3: 37.5	

742	T: hence the time is again 37.5, (*highest pitch*) because the number of km he has to cover is the same.	

The teacher wants to let understand her students that the vertical translation of the straight lines due to varying k doesn't change the solution of the "new equations". Furthermore, she will stress that the new straight lines are related to the construction of equivalent equations. The *teleological emotionality* of Sara involves her expectation that students are able "to see" on the graph that, varying k, the solution of the equation doesn't change. She needs that they understand this fact because, after, she will link it to the concept of equivalent equations. Her hope is revealed by her continuously moving of the slider while she speaks, probably, to illuminate students "to see" that the abscissa of the intersection point between the straight lines is always the same (#734). Moreover, the emotional counterpart is shown by her posture in Fig. 4.113 and in Fig. 4.114, in which she seems to wait something more from the class. Her epistemic rationality is the justification of the fact that the solution of the equation doesn't change (rational key, #740, #742) and her expectation that Algebra is a thinking tool for students (emotional key). In fact, she returns to the problem of Luca in the real context and she explains, increasing the tone of voice, that the time (the solution of the equation) cannot be different because adding the same number both to the arrival and to the starting line, the distance is the same. This emotional key is revealed, for example, by her pause and her posture in Fig. 4.116, as she is expecting an answer. Furthermore, it is shown by the increasing of the tone of voice in #742, as drawing the attention of the class on the justification within the real problem. Having hopes and needs towards her class, she cannot be plain in her discourse. Hence, we talk of *communicative emotionality* of the teacher disclosed, for example, by her prosody, her gesture, her facial expression. All of them express that what she is saying is linked with her expectation.

743	T: (*pronouncing*) What are we doing (Fig. 4.117)?	 Figure 4.117
744	S3: equivalent equations	

745	(*repeating and nodding*) we are constructing many equivalent equations. You remember that in the previous lesson we have said that we have equivalent equations (Fig. 4.118), namely equations written (*pronouncing*) in a different way, but that they have (*pronouncing*) always (*pausing*) the same result. (*highest pitch*) Do we have equivalent equations just for $k = 7.5$, for $k = -3$ (*speeding up*) that are the equations we have just seen? or do we have equivalent equations for many values of k (*she returns on the pc and she moves k, looking at the class and smiling waiting for an answer,* Fig. 4.119)?	 Figure 4.118  Figure 4.119
746	Ss: Many	
747	T: For many or for each value of k (*she continues to move* k)?	
748	Ss: for all of them	
749	T: for each value of k. For each value of k I obtain however equivalent equations. The filling of the table was just to write equivalent equations. For example, when I write $0.2x + 1.5$, what value has k to have 1.5? (*pause and she lifts up her chin*)	

750	Ss: noise	
751	S2: 1	
752	T: 1. Then, If I give the value 1 (*she returns on GeoGebra to put k equal to 1*) I see that the equation is (*pointing*) $0.2x + 1.5 = 9$. (*pronouncing*) What happened to the sides of the equations? What did we do the sides of the equation (*she lifts up her chin,* Fig. 4.120)?	 Figure 4.120
753	S1: We have added 1	
754	T: we have added 1 (*pausing*)	
755	S1: to both sides	

756	T: (*smirking*) We have added 1 to both sides. In the previous lesson (*gesture for miming the past and she raises eyebrows*, Fig. 4.121), before easter holidays, we have said that the first principle of equivalence said us that we could add the same number to both sides and that the result of the equation continued to not change, ok? then I could add or subtract the same number to both terms and have (*pronouncing*) always equivalent equations. Then, what does it mean (*returning on the "Algebra view"*)? It means that I can add to both sides (*moving k*), see that the blue straight lines have the same movement, they have the same translations (*she mimes the translation:* Fig. 4.122), namely they have exactly the same movement, then we add or subtract to both sides exactly the same quantity, our result doesn't change. If I wanted to obtain the result of the equation, I would take k, I would do such that B coincide (*pronouncing*) exactly with the $x-axis$ (*she is doing it on GeoGebra*). To let coincide B exactly with the $x-axis$, what value I have to give to k?	 Figure 4.121  Figure 4.122
757	S11: -8	

<table>
<tr>
<td>758</td>
<td>T: -8. If I give -8 to k, what happens is that B belongs to the $x-axis$ (Fig. 4.123). The second side of the equation (pronouncing) takes the value 0. The first side of our equation has a certain expression and I, actually, go (pronouncing) to see where the blue equation intersects the $x-axis$ (Fig. 4.124). I go to find what it is called the (pronouncing) zero of function (gesture to accompany the pronouncing) because it is the point in which the straight line touches the $x-axis$, ok? More or less it was to try to remember the first principle of equivalence, so, if you pay attention to what we have done, we should have seen the first principle of the equivalence working with balance (she mimes the balance: Fig. 4.125), because we added (she mimes the balance also with the body) or subtracted small weights from the balance in equivalent manner to both sides and it remains in balance (pronouncing) or we can obtain the first principle starting from a situation of this type (she moves again the slider k and she looks at the class in an emblematic way: Fig. 4.126).</td>
<td>
Figure 4.123

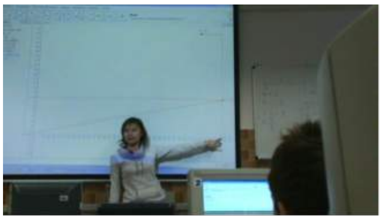



Figure 4.124

Figure 4.125

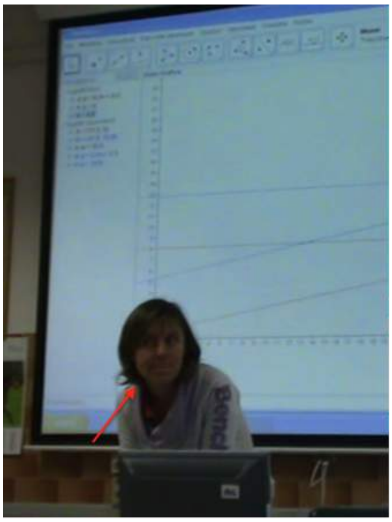



Figure 4.126</td>
</tr>
</table>

After working on the graph, Sara prompts the students to link the vertical translation of the straight lines to the concept of equivalent equations. In particular, she explicitly asks to her class what they are doing. She accompanies this action with pause as in Fig. 4.117, in which she is waiting for feedback from the class (#743). Then, satisfied, she repeats, nodding, what a student answers (#745). To accomplish her goal, Sara remembers what is the definition of equivalent equations. She pronounces “in a different way” and “always” (#745). Moreover, Sara shifts the attention of the students on for how many values of k they can have equivalent equations. This question comes along with an increasing of the tone of voice and her emblematic posture in Fig. 4.119, in which she seems quite relaxed that students are able to response. Actually, while Sara moves the slider k, students become aware that they can have equivalent equations for infinite values of k. She explains how the first principle works, showing that if k is 1, GeoGebra adds automatically 1 on both sides (#749, #752). She accompanies this discussion with many questions to her students, pauses and facial expressions with the chin up (#749, #752). It is quite clear that she is waiting answers from the class. Furthermore, this expectation is proven also by her smirk in #756 when a student says that they have added 1 to both sides. Then she repeats what the first principle says, with the facial expression in Fig. 4.121 and pronouncing another time “always” (#756). In terms of what happens on the graph, she highlights that the straight lines are translated of the same value, hence the result doesn’t change. To explain what happens she uses a specific example: adding 0 to both sides. In fact, she invites her students to move the intersection point of the straight lines on the $x-axis$. She stresses this fact pronouncing “exactly with the $x - axis$” (#756). At this moment, Sara introduces the concept of the zero of a function and she pronounces both “zero of function” and “because it is the point in which the straight line touches the $x - axis$” (#758). At the end, she explicitly links the first principle of equivalence using the balance and what they are doing now with GeoGebra. She accompanies this fact gesturing the balance and pronouncing the sentence of the geometrical interpretation of the principle of equivalence (#758). At the end, she finishes looking at the class remaining in posture as in Fig. 4.126.

Hence, her *teleological emotionality* involves her expectation that students are able to link the vertical translations of the straight lines to the concept of equivalent equations. She is always very careful to coordinate the different registers of representation. Then, now, she hopes that it is not so difficult for students seeing the translations of the straight lines as adding a quantity to both sides of an equations. This emotional counterpart is revealed, for example, by her pausing as in Fig. 4.117, her satisfaction after the answer of a student, her pronouncing key-words (#743, #745). She justifies how the first principle works how using GeoGebra works and she makes the specific example of adding 0 to both terms of the equation (rational key). At the same time, she is expecting that students are able to connect the algebraic register to the graphic one (emotional key). In particular, Sara introduces the zero of the function as the intersection point of the straight line with the $x - axis$. The emotional key is revealed by pronouncing few times "exactly with the $x - axis$" and by her gesture to recall what they have already done with the balance. These two intertwined aspects form the *epistemic emotionality* of the teacher.

Chapter 5

Conclusion

5.1 Ræmotionality

As shown in Chapter 4, the activity of the teachers is the combined result of both the rational component constituted by their decisions and the emotional one constituted by what they are feeling in that precise moment. The latter cannot be prepared a-priori. These emotions are triggered by the expectations teachers have for their teaching. I will use the metaphor of the actor in order to better capture this situation. The actor knows how the story he plays will finish. For example, he knows that Hamlet will be killed or that Violetta will die. Hence, the actor is prepared a-priori and, when he is playing the story, he has to communicate to the audience. In the case of the teacher, she also knows where she wants to go, but, unlike the actor, she doesn't know a-priori if she will obtain it, because it depends strictly on how students react. This fact produces expectations in the teacher and, then, she employs all of her means to arrive to her aim. The teacher establishes a communicative channel with students in order to have answers of a certain type from them.

Summarizing, the rationality and the emotionality of the teacher are not separable and coexist in the complex activity of the subject. I do not mean that emotion rules the rationality. The teacher does not base her teaching neither just on her sensibility nor just on her passion. It is just the combination of the rational and emotional sides that will allow her to obtain efficient results[1].

As shown in the analysis in Chapter 4, the three teachers have different emo-

[1] Referring another time to the theatre, the teacher is as the actor considered by Diderot in the "Paradoxe sur le comédien": the actor is an artist who bases her art neither just on her feeling nor just on her passion. In fact, it is necessary, in her emotions-filled interpretation, that she commits to her rationality.

tionalities and rationalities. This is not surprising if we think of the fact that each teacher has her own story and beliefs that determine what they actually are. Moreover, with a-posteriori reflection about their emotionalities, I can highlight differences in their way of acting within the classroom. In order to better appreciate these differences, I will now draw attention to the diversity of their teaching. This comparison does not aim to "judge" in a certain manner the work of the teachers. It just serves to reflect upon their decisions, trying to reinforce the following thesis. There are different emotional aspects that characterize each teacher and their decisions are often made visible through them.
Habermas speaks just of "rationality", but I have shown that, the whole activity of the subject is not only discursive, because there is an entire dimension of emotional aspects that intervene, and that cannot be captured by Habermas' notion of 'rationality'. For this reason, I have also accounted for the emotionality of the subject. Moreover, the rationality and the emotionality are strictly intertwined, constituting an *unicum*. Hence, we propose the term *rœmotionality* as a neologism to describe this idea.

5.2 Didactical phenomenologies

The view of Algebra

Lorenza, Carla and Sara have different views of Algebra. Lorenza, for the most part, wants that student see Algebra as an "authority" within the context in which they work. The teacher, often, uses Algebra as something that none could contradict. In fact, she personified Algebra as that which gives validity to justifications (see Section 3.5). This is the first role that Lorenza seems to give to Algebra. Then, just at the end of the explanation of equations, she adds that Algebra and, in this case, equations can solve mathematical problems. In particular, she does not have in mind real world problems. Instead, she has in mind "classical problems" such as the weight of half a brick[2] or the ages of people[3]. In constrast, Sara would like for students to see Algebra as a thinking tool. She declares explicitly in the a-priori interview that she wants students to learn to apply Algebra in every context of the real world. For example, Sara starts her explanation of equations with a real problem of a boy who has to cover a certain distance walking in a regular way. Also for Carla Algebra serves serves as a tool to solve problem.

[2] "If a brick weighs 3 pounds plus half a brick, how much does a brick and a half weigh?"

[3] "Marco, Luca and Andrea are cousins. Marco's age is one-third of Luca and Andrea is five years elder than Luca. If the sum of the age of the cousins is 40, find the ages of each."

She stresses this fact, for example, in the case of equations. Indeed, at the beginning, she treats equations from a technical point of view. But, then, she markedly says that the technical resolution of equations alone is not useful. Equations have to be seen as tool for solving problems. She introduces problems that are more similar to those of Lorenza, rather than to those of Sara.

Didactical material

All of the three teachers have adopted the same mathematical textbook (Bergamini et al., 2011). Lorenza is “faithful” to the mathematical textbook chosen by the school. Her teaching is in line with what the textbooks does. During her lessons, she dictates definitions to the students exactly as they are in the book. In the a-priori interview she declared that she likes it: “actually, it does what I would like to do”. Hence, the students are used to refer to the mathematical textbook even if Lorenza doesn't explicitly ask them to. The teacher loves the huge repertoire of exercises offered by the textbook. She hardly ever gives to her students other didactical materials. On the contrary, Carla does not use the mathematical textbook. She prefers to give worksheets prepared by herself in which students learn theory starting from mathematical activities. In the a-priori interview, she explicitly says that the definitions of the book are not so clear, so she prefers to reformulate them in another manner. Hence, in the class, she does not use the textbook with her students, except for when she finishes the explanation of a mathematical topic. Indeed, she asks her students to take the book, highlighting with them the analogies and the differences between what she has done and what the textbook does. She is very careful to stress that she wants students to learn the definition of equation as they have seen on the worksheets. Moreover, she explicitly affirms that the transportation rule, the cancellation rule and so on do not exist in mathematics, even if they are presented by their textbook. Similarly, Sara does not refer to the mathematical textbook. In contrast to Carla, Sara does not return to the book, not even at the end of the explanation of a mathematical topic. Sara uses the activities of M@t.abel in which every mathematical concept starts from a real problem. These mathematical activities allow students to use directly theory without already having a formalization of it. At the end or in the middle of these activities, Sara is careful to formalize the theory they introduced through the activities.

Classroom culture

[4] Lorenza recalls very often the classroom culture both of her current classroom and of her students' middle school classroom. She uses it as a sort of validation of what she and her students are doing. She expects that students will remember what they have already done, in order to prove the validity of what she is explaining in that moment (classroom culture as validation of the knowledge). In contrast, Carla does not refer to what students have seen in middle school. Rather, for example, when students speak of "the transportation rule" or "the cancellation rule" she responds nervously that they don't exist in mathematics and that they have to forget them. Carla considers just the classroom culture made with her. The aim of recalling it is again different from that of Lorenza. Indeed, Carla recalls it very often as a basis to construct new knowledge with her students (classroom culture as a basis to construct new knowledge). Similarly, Sara does not refer to what students have learnt in middle school. In general, she does not recall so much their previous knowledge. When it happens, it is just to remind her students that they have already seen the same thing with her, but from another point of view. For example, when she introduces the principles of equivalence with the translation of the straight lines, she recalls that they have already seen this principle with the metaphor of the balance.

Examples

Lorenza often uses and requires examples during her activity. She employs them because she expects that they are direct tools through which students have immediately the perception of where they want to arrive. For this reason, she makes and asks for examples both before formalizing the theory and after, when she wants to show students that what they have considered works from a theoretical point view too. Conversely, Carla mostly uses examples to construct theory. They constitute the first step towards the formalization of what they do. For this reason, she very often requires and employs them before formalizing the theory. Sometimes, she uses examples also because she expects that she can help students understand when they are wrong. Finally, Sara considers example in a bidirectional way. She expects that students understand better with the example, but, first of all, she requires examples from her class when she needs to know if students have understood.

[4] With "classroom culture", I mean just the previous contents.

Justifications

Lorenza often justifies what she does, referring always to what Algebra ensures and to what the mathematics textbook says. This way, she expects that justifications are automatically accepted by students because they come from the "authority". Hence, she requires the same behaviour of students: she needs that they justify using the algebraic rules they have learnt or the definition they have seen during the theoretical part of the explanation. On the contrary, Carla gives and requires justifications from students because she expects that just through justifications they can give sense to what they do. For example, she markedly stresses that equations alone are not so useful; they have learnt to work with them because they constitute one of the fundamental tools that are used in problem solving. Concerning the role of justification, Sara is quite in line with what Carla expects. Also, for Sara, justifications serve to go deep in the meaning of the mathematical content. She is not so concerned with students' correct use of formalism. She is more interested in their understanding of the mathematical meaning rather than their use of correct algebraic language.

In the Table 5.1, I summarized which elements characterize the teaching of the three teachers.

	Lorenza	Carla	Sara
Use of the mathematical textbook	✓	✓	✗
Use of other didactical material	✗	✓	✓
Recalling classroom culture (made only with her)	✗	✓	✓
Recalling classroom culture (made in middle school and with her)	✓	✗	✗
Classroom culture as validation	✓	✗	✗
Classroom culture as basis for constructing	✗	✓	✗
Classroom culture to link different registers	✗	✗	✓
Use of dynamic software (e.g. GeoGebra)	✗	✗	✓
Starting with a real problem	✗	✗	✓
Examples to construct theory	✗	✓	✗
Examples to justify theory	✓	✗	✗
Examples to check learning	✗	✗	✓
Algebra as thinking tool	✗	✗	✓
Algebra as an "authority"	✓	✗	✗

Table 5.1: Features of the didactical phenomenologies of Lorenza, Carla and Silvia

5.3 Explaining through Ræmotionality the didactical phenomenologies

The following table tries to make explicit the ræemotionality of the three teachers. For this reason, I will put both the description of what happens in the classroom, namely the decisions[5] of teachers, and pictures that should suggest something about the ræemotionality of the subject. In the table, the column of ræemotionality has made both by decisions and expectations of the teacher. In particular, I will recall the expectations of teachers I identified in Chapter 3.

[5]the decisions inform me about the rationality of the subject.

5.3.1 The case of Lorenza

Before summarizing the analyses made for Lorenza in Section 4.1, I recall her expectations I found in Section 3.5:

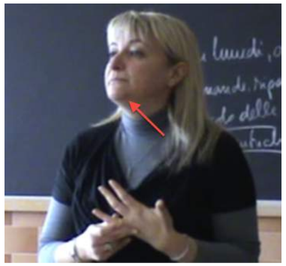
1. Expectation that the authority ensures acceptability, where the word "authority" is referred to the didactical material and to Algebra as discipline.

2. Expectation that the class culture is valid, where for "classroom culture" I mean both that constructed with her and that constructed in middle school.

3. Expectation that example are suitable for that precise context.

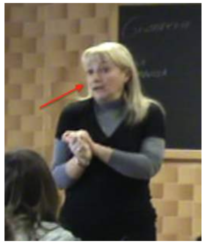
4. Expectation that justifications are made through what Algebra or the mathematical textbook state.

5. Expectation that students are able to coordinate different representation registers.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.1  Figure 5.2  Figure 5.3	✓ tone of voice proper of a statement ✓ pronouncing what she wants to recall ✓ insistent rhythm of the request of examples ✓ lifting up her chin ✓ smiling to put students at ease to answer	"I hope that someone remembers just something, we have spoken about (*pronouncing*) identities, then is there someone who wants to give, for now, the definition of identity and to do only an example (tone of voice of a statement and not of a question) of identity? Don't be shy! (she lifts up her chin, she smiles, she is waiting for an answer, biting her lips: Fig. 5.1, Fig. 5.2, Fig. 5.3)". "[...]One example, we have done an example within the classical ones (smiling)"	**Decisions**: recalling the concept of identity to introduce the definition of equation; requiring examples of identity to construct the definition of equation. **Expectations**: expectation that the classroom culture is valid; expectation that example are suitable for that precise context.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.4  Figure 5.5	✓ rhetorical questions ✓ “catching” knowledge with hands	“always?! (Fig. 5.4), do we always find it?(Fig. 5.5) for you this value or, let’s try to think a little bit”.	**Decisions**: clarifying the definition of the equation given by a student. **Expectations**: clarifying what students answer hoping that for students it’s result simpler to understand the different types of equations (determined, undetermined, impossible).
	✓ tone of voice proper of a statement ✓ insistent rhythm of the request of examples	“Let’s try to make an example of one thing of this type, we have given a definition that then we will write, so the fundamental difference is that an equation is satisfied, if it is possible, for a single value, let’s see to make an example of equation, who wants to make an example of equation? (tone of voice proper of a statement and not of a question)”	**Decisions**: requiring examples of equations. **Expectations**: expectation that example are suitable for that precise context.

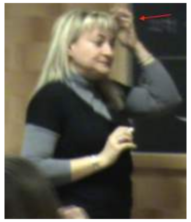
Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.6  Figure 5.7	✓ tone of voice proper of a statement ✓ insistent rhythm of the rhetorical questions ✓ miming the past ✓ smiling nervously	"Where does the concept of equivalence come from, eh? (she frowns, Fig. 5.20), we have already studied it, who remembers when we have spoken of equivalence, do you remember? (tone of voice proper of a statement not of a question) Do you remember (she miming the past, Fig. 5.7) the equivalence relation, never (scuote la testa), never (ride nervosa), we have done the relations, do you remember? We have defined the equivalence relations, those of admitted"	**Decisions**: recalling the concept of equivalence relation to see that the relation among equivalent equations is an equivalent one. **Expectations**: expectation that the classroom culture is valid.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.8	✓ rhetorical question ✓ raising up eyebrows and shoulders ✓ miming the past ✓ smiling nervously	" If we take into account two equations and we suppose that they are equivalent, the property, that is reflexivity, every equation is equivalent to itself, [rhetorical question]" *after an affirmative answer of a students* "yes (Fig. 5.8, she raising eyebrows and shoulders)"; "if an equation is equivalent to a second one, then the second one is equivalent to the first, because it admits the same solution, yes? [rhetorical question, another time she raises eyebrows and shoulders as in Fig. 5.8]"	**Decisions**: recalling "equivalence relations" to justify the terms "equivalent equations". **Expectations**: expectation that the classroom culture is valid.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.9 	✓ tone of voice proper of a statement ✓ speeding up ✓ rhetorical questions ✓ miming the past ✓ smiling nervously	"from the first principle will derive some calculation rules (speeding up and sure) that are those you apply (gesture for miming the "mechanically") mechanically, (speeding up and tone of voice proper of a statement, even if, at the beginning, she seems to be a question) you have already learnt them (She mimes the past, Fig. 5.9)"	**Decisions**: recalling the "calculation rules", asking them to students, to see them a consequence of the first principle of equivalence[a] of equations. [a] She explains that, from the first principle of equivalence, come from the transportation and the cancellation rules **Expectations**: expectation that the classroom culture is valid.

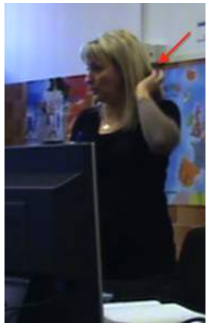
Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.10	✓ rhetorical questions	"What does the first principle say? it says (in the meanwhile she is reading from the textbooks and she is dictating to her students, Fig. 5.10): given an equation if we add..."	**Decisions**: dictating the definition of the first principle of equivalence from the mathematics textbook and personifying the principle.
			Expectations: expectation that the authority ensures acceptability, where the word "authority" is referred to the didactical material and to Algebra as discipline.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.11  Figure 5.12	✓ rhetorical questions ✓ smiling ✓ insistent rhythm in repeating the metaphor of balance	"This way, the balance returns to being in balance (she mimes the balance smiling, Fig. 5.11), otherwise it is unbalanced (she mimes the unbalance and she smiles, Fig. 5.12), right?"	**Decisions**: repeating the metaphor of the balance for the first principle of equivalence. **Expectations**: expectation that students are able to coordinate different representation registers.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.13  Figure 5.14	✓ rhetorical questions ✓ insistent rhythm in repeating that she has invented the example	“Let’s make an example (affirmative tone of voice). Let’s suppose that the starting equation is $x - 2$ equal to (she lifts up her chin as meaning that she doesn’t know what puts on the other side) $8x - 4$ (she scratches head, Fig. 5.13, and she makes a gesture as to mime that she is invented it, Fig. 5.14), ok? (she pauses and, then, she looks the equation for few seconds) ok? (in a whisper, but referring to the class: it seems a comment to herself as to mean that if it doesn’t work is because it is invented) I have invented it, ok? (she turns her hand in order to mime the “invention”)”; “Then (referring to the example just made) which quantity do we decide to add? [waiting for feedback, looking at the class)”	**Decisions**: making examples of equations in understanding how the first principle of equivalence works. **Expectations**: expectation that example are suitable for that precise context.

Phenomena	Observables	Speech	*Ræemotionality*
	✓ rhetorical questions ✓ tone of voice proper of a statement	"[self-confident tone of voice] this (pointing $8x$) is carried here (pointing the first side), then, it is added to that one (pointing x), it would result $-7x$, this (pointing -2) would be carry there (pointing the second side) and it would be added to this one (pointing -4) and we obtain"	**Decisions**: justification the resolution of equation through the "calculation rules" of equations.
			Expectations: expectation that the justifications are made through what Algebra states.

Phenomena	Observables	Speech	*Rœemotionality*
	✓ rhetorical questions ✓ insistent rhythm in repeating rhetorical questions	“Does it correspond, right? It corresponds with respect to the the things we have studied? (affirmative tone of voice) It corresponds to what we have already studied”	**Decisions**: recalling the transportation to see it as a consequence of the first principle of equivalence.
			Expectations: expectation that the classroom culture is valid.

Phenomena	Observables	Speech	*Ræmotionality*
 Figure 5.15	✓ rhetorical questions ✓ insistent rhythm in repeating that she has invented the example	“let’s think of an equation written in normal form, a random (she seems puzzled and, at the same, she scratches her head, Fig. 5.15) one . $2x + 6 = 0$, for example, right?”	**Decisions**: making examples of equations to introduce their geometrical interpretation. **Expectations**: expectation that example are suitable for that precise context.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.16	✓ rhetorical questions ✓ insistent rhythm in making questions ✓ tone of voice proper of a statement	"but the word (pronouncing) linear, (facial expression to catch her knowledge, Fig. 5.16) do you come into mind something that we have already seen?"; "(tone of voice of a statement rather than a question) do you remember we have seen the different proportionality, at a certain point we have met the dependence of the linear type and we have written an equation that represented that function, that link between two variables into play, which were the x variable and the y variable, do you remember? they represented two generic variables, in physics we spoke of physical quantities, who does remember that linear equation?"	**Decisions**: recalling the concept of function, asking it to students, to construct the geometrical interpretation of the solution of a linear equation. **Expectations**: expectation that the classroom culture is valid.

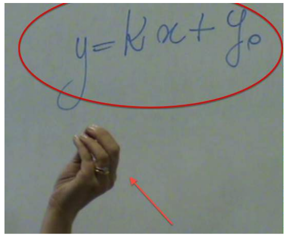
<table>
<tr><th>Phenomena</th><th>Observables</th><th>Speech</th><th>Ræemotionality</th></tr>
<tr>
<td rowspan="2">
Figure 5.17


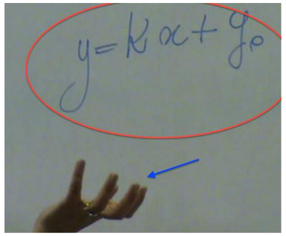
Figure 5.18


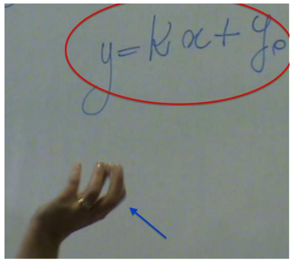
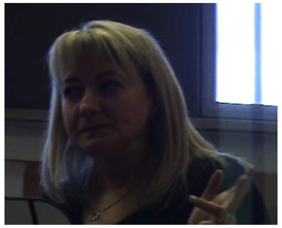
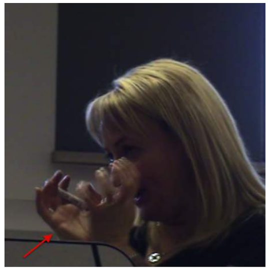
Figure 5.19


Figure 5.20</td>
<td rowspan="2">✓ rhetorical questions
✓ insistent rhythm in making questions (accompanied by many nervous gestures)
✓ nodding after a right answer of a student</td>
<td rowspan="2">“both k and y_0 were numbers, real, that represented something. Do you remember (posture as in picture)? It is just to make (gesture to mime the “box” of the class culture) the review of what we have already known, uhm (ok?)? k represents...”; “(gestures as in Figures) (insistent rhythm) But, how did we repre(sent it), dra(w), did we draw a sketch?”</td>
<td>Decisions: recalling the concept of a straight line, asking it to students in order to construct its graph.</td>
</tr>
<tr><td>Expectations: expectation that the classroom culture is valid.</td></tr>
</table>

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.21	✓ rhetorical questions ✓ insistent rhythm in making questions ✓ tone of voice proper of a statement when she makes questions about the coordination	"that is (nodding), it was a straight line, it was a random straight line of the plane and then k and y_0 gave us particular values of the straight line (Fig. 5.21)"; "it is called gradient, but what does it mean in short?"; "the slope with respect to the x-axis and y_0 gave us another information"	**Decisions**: recalling the coordination among the algebraic and the graphical registers of the straight line. **Expectations**: expectation that the classroom culture is valid.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.22	✓ rhetorical questions ✓ insistent rhythm in making questions ✓ tone of voice proper of a statement when she makes questions about the coordination	"Why this review? because if we think of the equation from which we started (pointing $2x + 6 = 0$) and, instead of writing this way, we write this way $y = 2x + 6$. This part (encircling the second side of $y = 2x + 6$), you see, is equal to (encircling $2x + 6$ in the starting equation), right? this one (pointing $y = 2x + 6$) has still the same expression of this one (pointing $y = kx + y_0$) and it represents, thus, in the cartesian plane (pause and the obvious facial expression: Fig. 5.22) a straight line. The only difference is that here we have "equal to 0" (pointing $2x + 6$) and there we have equal to y (pointing $y = 2x + 6$) and, then, what can be the interpretation of it from a geometrical point of view that can be help us to understand better the interpretation of an equation of the first grade with respect to our knowledge concerning the straight line of the plane (smiling and indicating the general straight line $y = kx + y_0$)? How can I compare them?"	**Decisions**: recalling another time the coordination between the algebraic representation of a straight line and its graph such that they see analogies between the equation of the straight line and the equation they have to solve. **Expectations**: expectation that the classroom culture is valid.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.23  Figure 5.24	✓ rhetorical questions ✓ insistent rhythm in making questions ✓ tone of voice proper of a statement when she makes questions about the coordination	"it represents the point of the straight line that has 0 as ordinate, ok? (she stops and she remains as in Fig. 5.23) are you able to see it? (Fig. 5.24)"; "hence the solution of the associated equation is the value of the abscissa of the intersection point between the straight line and the x-axis, [tone of voice of a statement] do you understand?"	**Decisions**: coordinating between the table of values of the straight line and its graph such that they see abscissa of the intersection point is the solution of the equation. **Expectations**: expectation that students are able to coordinate different representation registers.

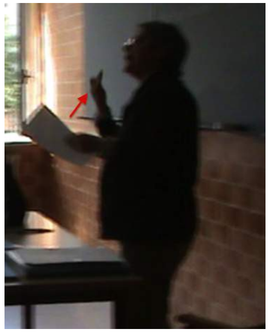
5.3.2 The case of Carla

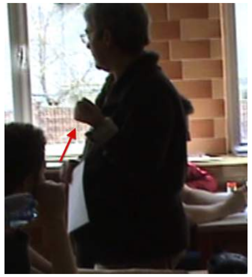
Before summarizing the analyses made for Carla in Section 4.2, I recall her expectations I found in Section 3.6:

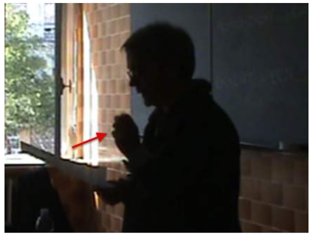
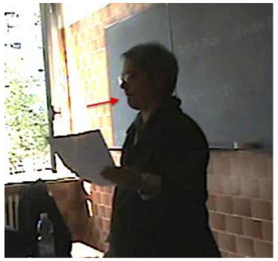
1. Expectation of constructing new knowledge from what has been already done in the classroom.
2. Expectation that the justifications are necessary to give sense to what she and her students do.
3. Expectation that also her students feel the need of justification to give sense to what they do.
4. Expectation that students see analogies.
5. Expectation that students learn to use the mathematical textbook in a critical way [6], referring to other didactical materials when it's necessary [7], underlying analogies and differences.
6. Expectation that students are able to make examples, because she thinks that examples are an useful tool to construct procedures or to review properties.

[6]to interpret correctly the definitions, to reflect upon its examples and so on
[7]e.g. the worksheets she prepares for the class

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.25	✓ increasing of the tone of voice ✓ pronouncing what she wants to justify ✓ insistent rhythm in questions ✓ raising eyebrows	"equivalent (nodding), let's remember this definition, when do we say that two open statements are equivalent, (pronouncing) within a given universe set, we underline it (Fig. 5.25)"; "when they have the same truth value set and why, we have seen it in the second part of the activity (4.83), (highest pitch) why is it important to specify (pronouncing) "within a given universe set"?"	**Decisions**: requiring justification of the importance of specifying the universe set of equations in which they work. **Expectations**: expectation that also her students feel the need of justification to give sense to what they do.

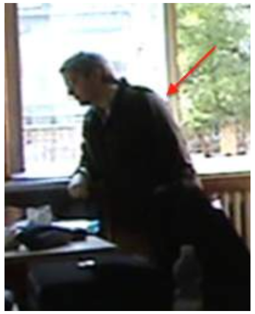
Phenomena	Observables	Speech	*Ræmotionality*
 Figure 5.26  Figure 5.27  Figure 5.28	✓ increasing of the tone of voice ✓ miming the classroom culture in her fist ✓ raising eyebrows ✓ irritated tone of voice if students don't remember it ✓ pause ✓ pronouncing ✓ gesture of specifying	"that (pronouncing) must be true or false, then I have called propositions those that were in the first part of the activity then we have seen that the elementary propositions have a predicate and here the predicate is "to be equal to" and, every day, we have to do with propositions of this type and they are called (gesture and facial expression, Fig. 5.26)"; "(irritated) we are still at the beginning (miming the past, Fig. 5.27), those propositions (Fig. 5.28: irritated gesture for recalling just those propositions)"	**Decisions**: recalling the definition of equalities in order to recall that of open statements **Expectations**: expectation of constructing new knowledge from what has been already done in the classroom.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.29  Figure 5.30	✓ increasing of the tone of voice ✓ miming the past ✓ raising eyebrows ✓ irritated tone of voice if students don't remember it ✓ pause ✓ rhetorical questions	"then, we considered open statements. In general what is an open statement? (pause, gesture Fig. 5.29)"; " it is a phrase in which it appears a variable for which, first of all, we have to precise (pause and waiting, Fig. 5.30) what is the universe set, right?"	**Decisions**: recalling the definition of open statement in order to construct the concept of equation. **Expectations**: expectation of constructing new knowledge from what has been already done in the classroom.

Phenomena	Observables	Speech	*Ræmotionality*
 Figure 5.31  Figure 5.32	✓ increasing of the tone of voice ✓ pronouncing ✓ raising eyebrows ✓ inclining her body towards the class ✓ waiting for an answer ✓ gesture of specifying	"T: the unique way for seeing who is right is trying to substitute. Try to substitute, then pay attention, I have said you and this fact has to be always keep in mind, you have to (she inclines her body towards the class with open hand, Fig. 5.31) give a (pronouncing and highest pitch) sense of what you read, then solving the equation $(x+1)^2 = 81$ means asking ourselves if it exist a value x such that doing $x+1$ and squaring it is 81, working in (highest pitch and she raises eyebrows)"; "and how have you found them?(Fig. 5.32: biting her lips and waiting for an answer)"	**Decisions**: justifying herself and asking justification of the meaning of the equation $(x+1)^2 = 81$. **Expectations**: expectation that the justifications are necessary to give sense to what she and her students do; expectation that also her students feel the need of justification to give sense to what they do.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.33  Figure 5.34	✓ increasing of the tone of voice ✓ raising eyebrows ✓ nodding ✓ irritated tone of voice if students don't remember it ✓ pause ✓ gesture for specifying	"that is (nodding), more or less, then remember (Fig. 5.33) the fundamental concept emerged in the activity 1 (4.82, 4.83) was that of equivalent open statements, that is, statements that within (gesture with the hand) a given universe set have the (highest pitch and she raises eyebrows) same truth set. Now, given that equations are particular open statements we will speak of (pauses and waiting for an answer: Fig. 5.34)"	**Decisions**: recalling the definition of equivalent open statements in order to construct the definition of equivalent equations. **Expectations**: expectation of constructing new knowledge from what has been already done in the classroom.

Phenomena	Observables	Speech	*Ræemotionality*
	✓ pronouncing ✓ raising eyebrows ✓ inclining her body towards the class ✓ increasing the tone of voice ✓ gesture of specifying	"because x^2 cannot never be equal to -1, right? then (highest pitch and she is specifying with finger) pay attention! there are equations that have the empty set as the solution if we work in a (pronouncing) certain universe set, but changing the set universe they become (inclining her body towards the class) solvable. For example the equation a is not solvable in $\mathbb{N}$, but in $\mathbb{Z}$ it is."	**Decisions**: justifying the importance of the universe set of an equation. **Expectations**: expectation that the justifications are necessary to give sense to what she and her students do.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.35	✓ pronouncing ✓ raising eyebrows ✓ inclining her body towards the class ✓ increasing the tone of voice ✓ gesture of specifying ✓ irritated tone of voice if students don't justify ✓ insistent rhythm in questions	"(increasing the volume of the tone of voice and she specifies with finger) then, try to substitute to x the value 0 (she inclines her body towards the class as in Fig. 5.35)."	**Decisions**: requiring justification from students. **Expectations**: expectation that also her students feel the need of justification to give sense to what they do.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.36	✓ increasing of the tone of voice ✓ pronouncing ✓ raising eyebrows ✓ repetition ✓ biting her lips (posture of waiting) ✓ nodding ✓ inclining her body towards the class	"Then, the next question you find on the worksheet, we read it together, is: let's suppose that we have an equation, then I have told you that you can help yourself with example (Fig. 5.36), ok?"	**Decisions**: requiring examples of equation to construct how the first of principle works. **Expectations**: expectation that students are able to do examples.

Phenomena	Observables	Speech	*Ræemotionality*
 	✓ repetition ✓ increasing the tone of voice ✓ raising eyebrows ✓ nodding ✓ irritated tone of voice if students don't remember it ✓ pause ✓ classroom culture in her fist ✓ gesture for specifying	"then how are the "leggi di monotonia" translated (pause) in properties on equations (she inclines her body towards the class, she raises eyebrows)? (nodding, she waits for an answer) What can you say if you have an equation ok? and what do you do?"; "let's remember what we have said, we have said that two equivalent equations are (pause and gesture with the hand and she bites her lips)"; " (irritated and she beats her hand on the table), equations that have (pause)"	**Decisions**: recalling the "leggi di monotonia" in order to construct from them the principle of equivalence for equations. **Expectations**: expectation of constructing new knowledge from what has been already done in the classroom.

5.3.3 The case of Sara

Before summarizing the analyses made for Lorenza in Section 4.3, I recall her expectations I found in Section 3.7:

1. Expectation that students learn "to see" through the graphic register in order to reason (think of) on equations.

2. Expectation that Algebra becomes for students a thinking tool.

3. Expectation that students learn to pass from one representation register to the other one.

4. Expectation that students learn to use the algebraic language as an extension of the arithmetical one.

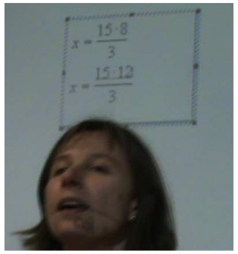
5. Expectation that examples are useful for students to understand the meaning of what they are doing. [8]

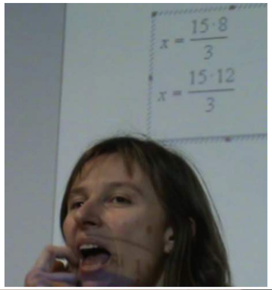
6. Expectation that justifications serve to go deep in the meaning of what they are doing.

[8]with examples she can also understand if students actually understand what they are doing.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.37	✓ smiling (to puts students at ease to answer) ✓ posture of waiting feedback ✓ pronouncing ✓ increasing of the tone of voice ✓ insistence in repetition of the aim	"no, with a formula (smiling Fig. 5.37). We have already written it in words, you have told me that we have done the proportion and then you have solved the proportion or you have calculated the time necessary for doing one kilometer and then you have multiplied the time necessary for doing one kilometer times 8 (she is waiting again on an answer, playing a little bit nervously with her ring)"	**Decisions**: recalling previous arithmetical knowledge to put into a formula what they have explored with numbers . **Expectations**: expectation that students learn to use the algebraic language as an extension of the arithmetical one.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.38	✓ smiling (after correct answers) ✓ posture of waiting feedback ✓ pronouncing ✓ increasing of the tone of voice ✓ insistence in repetition of the aim	"We have already written it in words, you have told me that we have done the proportion and then you have solved the proportion or you have calculated the time necessary for doing one kilometer and then you have multiplied the time necessary for doing one kilometer times 8 (she is waiting again on an answer as in Fig. 5.38, playing a little bit nervously with her ring)"	**Decisions**: recalling what they have explored with numbers.
			Expectations: expectation that students learn to use the algebraic language as an extension of the arithmetical one.

Phenomena	Observables	Speech	*Ræemotionality*
	✓ smiling (to puts students at ease to answer and when she uses Algebra as a thinking tool) ✓ posture of waiting feedback ✓ pronouncing ✓ increasing of the tone of voice	"(highest pitch) but instead of calculating this way, (very slowly and pronouncing all the words) instead of calculating how much time Luca uses in 8 km, namely, how much time I use to make 8 km, I want to know how much time he uses to make 12 km, what should I do?"; "(gesturing, speeding up and smiling) always thinking of the fact that Luca isn't ever tired and that he walks always at the same velocity"	**Decisions**: recalling the first number for the distance they use and then, she changes it, such that that students put a variable instead of numbers **Expectations**: expectation that Algebra becomes for students a thinking tool; expectation that students learn to use the algebraic language as an extension of the arithmetical one.

Phenomena	Observables	Speech	*Ræemotionality*
  Figure 5.39	✓ smiling (to puts students at ease to answer and when she uses Algebra as a thinking tool) ✓ facial expression of waiting ✓ open hand to receive the justification from them ✓ her moving finger to grasp the justification	"(nodding) not only, (smiling) from a conceptual point of view, what does it mean 15 divided by 5? Make (gesture Fig. 5.39)15 minutes divided by 5 kilometers, what does it mean (she moves her fingers as if grasping for a justification) (then opens her hand towards the students)"; "yes, he goes faster, then we would make Luca (smiling) a little bit too fast"	**Decisions**: justifying the sense of the mathematical operations. **Expectations**: expectation that justifications serve to go deep in the meaning of what they are doing.

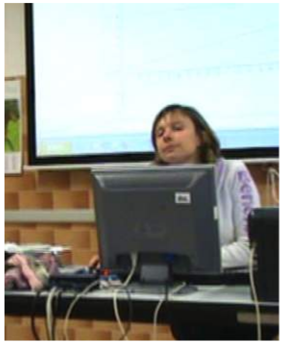
Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.40	✓ smiling (to puts students at ease to answer and when she uses Algebra as a thinking tool) ✓ posture of waiting feedback ✓ increasing of the tone of voice	"instead of 10 I put (Fig. 5.40)?"	**Decisions**: changing numbers to put the letter. **Expectations**: expectation that Algebra becomes for students a thinking tool; expectation that students learn to use the algebraic language as an extension of the arithmetical one.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.41	✓ smiling (when they recognize equations) ✓ facial expression of waiting ✓ open hand to receive the justification from them ✓ satisfied tone of voice (when they recognize equations)	"In writing this (she returns to the screen and she points), is there someone that notices something in mathematics, a mathematical object that you have already seen somewhere (Fig. 5.41: she inclines her body towards the class waiting for an answer)?"	**Decisions**: requiring justification of what they are constructing, namely, of equations. **Expectations**: expectation that justifications serve to go deep in the meaning of what they are doing.

Phenomena	Observables	Speech	*Ræemotionality*
 Figure 5.42	✓ smiling (to put students at ease to make examples) ✓ facial expression of waiting ✓ satisfied tone of voice (when they recognize equations)	"Could you tell me examples of other equations? (waiting, eloquent facial expression, Fig. 5.42) (she pauses for a while) let's say an equation (doing knee-bends) an any one, simple (smiling)"	**Decisions**: she requires examples of equations to construct the definition of equation. **Expectations**: expectation that examples can help students in understanding.

Phenomena	Observables	Speech	*Ræemotionality*
	✓ repetition ✓ gesture of specifying ✓ pronouncing	" (nodding) any letter of the alphabet, but there must be an unknown because if there isn't an unknown then the things don't work, we aren't speaking of equations, then probably saying that an equation is an equality between two terms is not completely correct. Let's say that we define (gesture with the hand with index and thumb joined moved up and down), let's say it immediately such that we are ok."	**Decisions**: she explains the definition of equations starting from the examples of them. **Expectations**: expectation that Algebra becomes for students a thinking tool; expectation that students learn to use the algebraic language as an extension of the arithmetical one.

Phenomena	Observables	Speech	*Ræemotionality*
	✓ pronouncing ✓ rhetorical question ✓ increasing the tone of voice	"Actually, we have already seen the first part of the activity. The activity asked us to prepare a slider, to call it k and to vary it from -15 to 15 and (blaring tone of voice) then it told us to take into account two equations: one we have already solved in the previous time and the other one was established, if you remember it. The equation already solved previously was $\frac{1}{5}x + \frac{1}{2} = 8$. To solve this equation we have already said that, actually, we could work on two different (pronouncing) functions, precisely on two (pronouncing) straight lines: one was this straight line (she draws on GeoGebra the function $y = \frac{1}{5}x + \frac{1}{2}$, foto 1) (pause and she looks at the screen) and the other one was $y = 8$ (Fig. 2). Then, you have told me, if you remember, that the solution to the equation was the intersection point between these two straight lines. (rhetorical question) Do you remember it? There was Elena who said "(highest pitch)I go to see where I intersect and then I read the solution". Actually the solution we have to read is not on the $y - axis$, but it is on the $x - axis$, because it is the value of x that is of interest to us as solution and (blaring tone of voice) then the fact of asking to draw the perpendicular line to the $x - axis$ passing through A served simply to say that (highest pitch) I can go to read the solution."	**Decisions**: recalling the geometrical interpretation of an equation to reason on equations on the graph. **Expectations**: expectation that students learn "to see" through the graphic register in order to reason (think of) on equations.

Phenomena	Observables	Speech	*Ræemotionality*
	✓ pronouncing ✓ rhetorical question ✓ increasing the tone of voice	"In fact, it is what happens. We take the point B as the intersection point and then we colour blue these two new straight lines and B. Now we could try to ask what happens if I vary the value of k, ok? Varying k something happens that is interesting. Then, let's start to see what happens on the graph varying k (she moves the slider k). On the graph, what happens while varying k (she continues to move the slider k and she looks at the class)?"	**Decisions**: exploring what happens moving the slider on GeoGebra such that students recognize the principles of equivalence on the graph **Expectations**: expectation that students learn "to see" through the graphic register in order to reason (think of) on equations.

5.4 Three models of teachers

Lorenza

As I summarized in section 5.3.1, when Lorenza introduces a new mathematical topic, she often recalls previous knowledge of students both from earlier in the year and from middle school. Moreover, in all the pieces of lessons I analysed, I observed almost the same indicators that accompany the decision to recall previous knowledge. For example, the teacher seems quite insistent in the rhythm of the rhetorical questions about previous knowledge, often also miming the past; there is often a mismatch between the affirmative tone of voice of asking students about previous facts and her facial expression, which does indeed look inquisitive. These observable can allow me to say something about the reasons for which she decides to recall previous knowledge when she is going to treat a new topic. In particular, the insistent rhythm of rhetorical questions about previous facts seems to suggest that Lorenza needs and, simultaneously, hopes that students remember what they have already done, because she considers it a valid start for the new knowledge. Also the interesting mismatch between the prosody and facial expression of the teacher conduct to the same conclusion: probably, the teacher does not want to markedly stress the request, because she is expecting that previous knowledge comes naturally from students.
Concerning the use of examples, I can conclude that the teacher often uses and requires examples after she introduces theory. In the segments of her lessons I studied previously, when she makes examples, she repeats in an insistent way that those examples are invented, scratching often her head. In addition, she accompanies the invention of the example with many rhetorical questions so as to be sure of the example she is constructing[9]. These observables can allow me to say that she would reassure herself about that the example properly fits with the topic. In addition, in turn, she requires examples from students and, often, this request comes along with an insistent rhythm in asking them, with the tone of voice proper of a statement. Probably, the teacher hopes that making examples becomes an habits for students, being a way to perceive immediately where they want to arrive.
Regarding the role the teacher gives to the justification, I noticed that she inclines her body to justify what she does with what Algebra "ensures", with

[9]"Let's suppose that the starting equation is $x-2$ equal to (she lifts up her chin as meaning that she doesn't know what puts on the other side) $8x-4$ (she scratches head, Fig. XX, and she makes a gesture as to mime that she is invented it, Fig. XX), ok? (she pauses and, then, she looks the equation for few seconds) ok? (in a whisper, but referring to the class: it seems a comment to herself as to mean that if it doesn't work is because it is invented) I have invented it, ok?"

what the mathematical textbook "says" or with what they have already seen in the past. The justifications almost always come along with rhetorical questions, with her raising eyebrows and shoulders, her smiling nervously and her tone of voice of a statement even when she asks questions. It seems that she expects that these justifications are naturally accepted by the students because they are based on what the "authorities" (Algebra, the textbook and previous knowledge) state.
Lorenza is quite careful to coordinate different representations of the same concept. Very often, she repeats the coordination among them and she asks many rhetorical questions with an insistent rhythm. Probably, Lorenza needs to test, in the ongoing activity, if students are understanding this coordination. For this reason, she appears quite insistent on capturing any possible reactions the students might have to what she is explaining.

Carla

As I briefly resumed in the section 5.3.2, Carla gives a very relevant role to justification in her classroom. Moreover, she requires of students the same orientation towards justification. In all the segments of lesson that referred to justification, she shows almost the same behaviour. For example, she pronounces and she increases the tone of voice for what she wants to justify: she raises eyebrows and she inclines her body towards the class. Probably, she hopes that students understand that she highlighting the need of justifying in order to give sense to what they do. In addition, it appears quite clear that she expects the same approach from students, in fact she inclines always towards them as "to transfer" this need of justification. In particular, she expresses her insistence that students give a meaning to what they do, waiting for justifications and being quite irritated when they avoid them.
Concerning the previous knowledge, the teacher often recalls it when she introduces the definition of a new mathematical object. In all the excerpts in which she recalls previous knowledge, she maintains the same "schema". In particular, she increases the tone of voice, she pronounces and she specifies with gesture what she wants to recall. Hence, it seems quite clear that she expects that students understand the importance of what they have already seen. In addition, she invites students to remember it shortly before the introduction of a new topic, raising eyebrows and gesturing to incite them in reviewing previous knowledge. In particular, she appears quite irritated when students don't react to her suggestions. Then, her hope that students remember previous knowledge in order to construct new knowledge drawing on the previous one becomes visible. In short, she seems "to pass on" the need of justifying and, simultaneously, she hopes that it was effective in doing it.

Regarding the role of example, the teacher often requires them before formalizing the mathematical concept. She requires examples from students, increasing the tone of voice, pronouncing, nodding and waiting for feedback from the class. It seems that she sees examples as an useful way for students to understand how things work and then to formalize. In fact, she nods before they have actually constructed examples because, probably, she is quite confident that the example will be helpful to the students.

Sara

As shortly summarized in section 5.3.3, the teacher recalls previous knowledge when she wants to generalize it or to link it to other knowledge. For example, she recalls previous knowledge to generalize it through Algebra or to coordinate it with a different register of representation. For example, in all the excerpts connected to the arithmetical previous knowledge, Sara pronounces and increases the tone of voice for what they have already done with numbers, then she smiles to put students at ease. It seems that she wants to stress what they have already seen with numbers now that they have to work with letters, hoping that students react, seeing Algebra as a natural extension of Arithmetic. The same emotional indicators appear when she recalls previous knowledge to coordinate that register with another register of representation [10]. It becomes quite clear that Sara hopes that students learn to pass from one register of representation to another one, considering them equivalent ways to say the same thing.
Sara gives a very relevant role to justification. She always "hammers" students in requiring justification of what they are doing. Sara always waits for justifications with open hands, often moving fingers, as to "grasp" them from students. In addition, she smiles to put students at ease to answer about the meaning of what they are doing without fear. These emotional observables could seem to signal her hope that justifications serve to go deep in the meaning of what they are doing.
Concerning the role of the examples, Sara often requires them after formalizing definitions. When she asks students to make example she waits for feedback, smiling and speaking with a satisfied tone of voice after hearing effective examples. It seems that she hopes that examples are a tool for checking if students have understood what they are doing. In fact, she appears very satisfied when they make good examples, because she interprets them as signals of students' understanding.

[10] when she recalls the algebraic treatment of the principles of equivalence in order to link it to the geometrical one.

Final remark

In Table 5.2, there are the emotional indicators that teachers reveal during their teaching concerning previous knowledge, examples and justifications. In the last column of the table, the complete set of their observables can be seen. Hence, the other columns show subsets of the observables. Different subsets can also admit overlaps. Indeed, for example, Lorenza uses rhetorical questions both when she recalls previous knowledge and when she justifies something.

Table 5.2 can be read horizontally or vertically. Reading it horizontally provides information about the emotional indicators of each teacher that are observed during her teaching. In other words, this reading provides a "sketch" of how she is emotionally involved in her teaching. As shown in the previous section, the three teachers are different in their teaching. This diversity does not completely depend on their decisions. In fact, at a first glance, they seem to take similar decisions: all of them recall previous knowledge, use examples, justify what they do and so on. What radically changes are the reasons why they make those decisions and not others. These reasons are strictly related to their expectations that, in turn, depend on their own beliefs and background. Their decisions are made visible through their emotionalities. In fact, reading vertically the Table 5.2, one can say something about the different reasons for their decisions.

For example, regarding the previous knowledge, Lorenza uses rhetorical questions. Probably, she thinks that students know already the answer because they are drawing upon the valid and already accepted previous knowledge. In addition, she nervously smiles when they don't remember it. Moreover, when she asks questions about previous knowledge she is expecting an answer as we can see from her facial expressions, but, simultaneously she uses an affirmative tone of voice. Probably, she would like that the previous facts are already valid for students such that they can naturally surface them. This discussion is in line with the expectation about the "classroom culture" I found from the a-priori interview.

On the contrary, Carla increases the tone of voice when she wants to recall previous facts, raises her eyebrows and mimes class culture in her "fist". In addition she makes the gesture of specifying, she pronounces, she pauses and she nods. The first difference from Lorenza is that Carla mimes the classroom culture having it in her fist, while Lorenza mimes just the past. This could be seen as a hint of the fact that Carla wants to hold in her fist the previous knowledge. Possibly, it is important for her not just to remind students that they have already seen it, rather she wants to employ it to construct the new one. Furthermore, if Lorenza never makes pauses in order to push students to make a step forward from the recalled knowledge (except for her atypical questions), it seems clear that Carla, instead, wants students to speak in order to develop previous facts. In fact she increases the tone of voice to draw the attention of students first on what they have recalled. Then, she raises her eyebrows and she pauses in order to expect a reaction from the class and she nods before they answer. Probably, she is quite sure that it is actually possible for students to construct new knowledge. This is also proved by the fact that she is disappointed when students don't react as she is expecting and, indeed, she uses an irritated tone of voice. As summarized in the Table 5.2, Sara smiles to put students at ease to answer, she has postures of waiting, she pronounces, she increases the tone of voice, she is insistent in the repetition of the aim. I can say that she is more in line with Carla than with Lorenza, even if not totally. She recalls previous facts they have done shortly before (for example, what they have found in certain problems with numbers), because she is interested in the fact that students shift from numbers to the generalization with letters. Then, it is a sort of previous knowledge different from the other two. Lorenza and Carla recall definitions or theory in general, while Sara recalls previous results they found with numbers. She knows very well that the passage to the letter is very delicate; in fact, she smiles to put students at ease to answer. I recall that, for example, Lorenza smiles nervously. Even just this difference suggests that Sara is expecting that they generalize Arithmetic, but she is aware that this is quite difficult for students and then she smiles, as if to say "try to say something even if you are not sure". This behaviour is different from Lorenza, who in a sense "demands" that students are sure in recalling previous knowledge. In this perspective, Sara is also different from Carla. In fact, while Carla uses an irritated tone of voice when students don't remember; Sara often recalls herself the previous knowledge. This because she is just interested in the fact that students are able to generalize from it, indeed she smiles to incite students into doing. One of the analogies between Sara and Carla is that both of them increase the tone of voice to focus the attention of students on what they want to recall and they pronounce it.

Regarding justification, Lorenza uses rhetorical questions and the tone of voice proper of a statement. It seems quite clear that she wants students to accept justifications because she often bases it on what Algebra "states" or on what the mathematical textbook "says". The rhetorical questions and her affirmative tone of voice are hints of the fact that she is sure that students will accept those justifications because they cannot contradict what Algebra or the textbook say. On the contrary, Sara never employs rhetorical questions, but she is expecting that students give justification because she sees justification as a tool to go deep into the meaning of what they are doing. In fact, she waits for a feedback, smiling to put students at ease to answer and, simultaneously, she stays with open hand towards the class to receive the justification from them. Otherwise, she moves fingers as to "grasp" the meaning that she expects from students. Hence, the role of justifications between Lorenza and Sara is quite different: for Lorenza, it is a sort of validation and acceptability of what she is explaining; for Sara, it is a means to dig deep in the sense of what she and the class are doing. In this perspective, Carla is more similar to Sara than to Lorenza. Also Carla waits for an answer, inclining her body towards the class in order to receive the justification from them. Also Carla wants students to feel the need for justifying and to give sense to what they are doing, but she expresses it in a different way from Sara. If Sara smiles to put students at ease to answer, Carla increases the tone of voice, raising her eyebrows and she appears irritated when they are not able to justify.

Concerning the role of examples, Lorenza is very insistent in asking examples. She always uses the tone of voice proper of a statement even when she is asking for examples. In addition, for justifying when they do not work she is insisted in repeating that examples are invented by herself. From the one side, she sees examples as a tool for showing to students the correctness of what she is doing and she is worried when it does not happen. From the other side, she requires them in an insistent affirmative way because she thinks that students understand immediately where she wants to go. She is quite sure that examples will ensure the acceptability of what they are doing. On the contrary, Carla increases the tone of voice when she requires examples, unlike Lorenza who uses an affirmative tone of voice. Indeed, Carla does not see examples as a "reliable" tool for the acceptability of what they are doing, rather a way through which students can construct the theory by themselves. In fact she pronounces and she raises eyebrows, biting her lips while waiting for an answer. In addition, it seems that the teacher invites students to make examples inclining her body towards them also when they are wrong: in fact she hopes that they become aware of their error through the example. Lastly, Sara smiles to put students at ease to make examples. For her, they constitute a way for knowing if students have understood. In fact, she thinks that for making an example they must have understood the meaning of the mathematical concept of the example. Obviously, it is important for her to have a feedback of this type. Hence, she is waiting for students' reaction. and she uses a satisfied tone of voice after they make correct examples.

If I had considered just their speech, without accounting for the audio and the video, I would have found three very similar teachers. Inserting also the emotional elements that constitute a significant part in their teaching, as prosody, gestures, facial expressions and so on, I found three really different and complex ræmotionalies. Actually, this corresponds to the global impression that one has when looking at the video.

	Observables			
	Knowledge	**Example**	**Justification**	*Set of observables*
Lorenza	rhetorical questions, tone of voice proper of a statement, facial expression of questioning, pronouncing, smiling nervously, miming the past	tone of voice proper of a statement, insistent rhythm in the request of examples, rhetorical questions, insistent rhythm that she has invented example, scratching head	rhetorical questions, tone of voice proper of a statement	*rhetorical questions, tone of voice proper of a statement, facial expression of questioning, pronouncing, smiling nervously, miming the past, insistent rhythm in the request of examples, scratching head*
Carla	increasing the tone of voice, raising eyebrows, miming class culture in her "fist", irritated tone of voice, pause, pronouncing, gesture of specifying, nodding	increasing the tone of voice, pronouncing, raising eyebrows, repetition, biting her lips (posture of waiting), nodding, inclining her body towards the class	increasing the tone of voice, pronouncing, insistent rhythm of questions, raising eyebrows, gesture of specifying, inclining her body towards the class, waiting for an answer, irritated tone of voice if students don't remember it	*increasing the tone of voice, raising eyebrows, miming class culture in her "fist", irritated tone of voice, pause, pronouncing, gesture of specifying, nodding, repetition, biting her lips, inclining her body towards the class, insistent rhythm of questions, waiting for an answer, irritated tone of voice if students don't remember it*
Sara	smiling to put students at ease to answer, posture of waiting, pronouncing, increasing the tone of voice, insistence in the repetition of the aim	smiling to put students at ease to make example, satisfied tone of voice after example from students, facial expression of waiting	smiling to put students at ease to answer, facial expression of waiting, open hand to receive justification, moving fingers to grasp justification, satisfied tone of voice after justification from students	*smiling to put students at ease to answer, posture of waiting, pronouncing, increasing the tone of voice, insistence in the repetition of the aim, satisfied tone of voice after example from students, open hand to receive justification, moving fingers to grasp justification, satisfied tone of voice after justification from students*

Table 5.2: Observables of the three different types: knowledge, example, justification

5.5 Future research

As I already stressed, my research shows the relevance of emotional aspects for analysing teachers' behaviour. This implies also that the constructivist lens is too strict for such an analysis. Indeed, the teacher actually "suggests" something to students through her emotionality. In other words, the teacher gives signals to students who, in turn, respond to them. This dynamics is unavoidable because the teacher is a human being and, as such, she cannot be "plain" in her activity. Possibly, there exists a form of communication with students through the emotional sphere.
Hence, for a future development of the research, on the one hand, it could be interesting to analyse what happens for students. It could be that they answer at the level of the rational key, but probably the emotional counterpart could trigger their answers. From the other hand, it could be intriguing to investigate the ræmotionalities of the same students involved in solving the activities I prepared for the research project. Surely, they also have a rational component and an emotional one that are intertwined. And, probably, one could find fresh results, because students could be more spontaneous and surely less "prepared" or "constructed" than their teachers. Lastly, I could also investigate how the ræmotionalities change depending on the topic. In Chapter 3, I said that I chose the topic of linear equations because it requires a shift. In fact, it is a crucial mathematical topic in which teachers have to accompany students from the arithmetical world to the algebraic one. Hence, perhaps, the ræmotionalities would be different for another mathematical topic.

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