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The semi-classical approach to the exclusive electron scattering

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Abstract

The semi-classical approach, successfully applied in the past to the inelastic, inclusive electron scattering off nuclei, is extended to the treatment of exclusive processes. The associated formalism goes beyond the plane wave impulse approximation, since it can account also for the final states interaction in the mean field approximation, respecting the Pauli principle. We then explore, in the frame of two admittedly crude approximations, the impact on the exclusive cross section of the distortion of the outgoing nucleon wave. In addition also the influence, on the same observable, of the shape of the potential binding the nucleons into the nucleus is investigated. In accord with the findings of fully quantum mechanical calculations, the exclusive scattering is found to be quite sensitive to the mean field final states interaction, unlike the inclusive one. Indeed we verify that the latter is not affected, as implied by unitarity, by the distortion of the outgoing nucleon wave except for the effect of relativity, which is modest in the range of momenta up to about 500 MeV/c. Furthermore, depending upon the correlations between the directions of the outgoing and of the initial nucleon, the exclusive cross-section turns out to be remarkably sensitive to the shape of the potential binding the nucleons. These correlations also critically affect the domain in the missing energy - missing momentum plane where the exclusive process occurs. Finally the issue of the validity of the semi-classical framework, in leading order of the $\hbar$ expansion, in providing a description of the exclusive processes is shortly addressed. © 1998 Elsevier Science B.V.

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1. Introduction

The plane wave impulse approximation (PWIA) has been a framework extensively employed in analysing the exclusive (or semi-inclusive) processes of inelastic scattering of electrons off nuclei, like, e.g., the $(e,e'p)$ one [1]. The advantage of such an approach lies of course in its simplicity : indeed, in the PWIA, the final nucleon is described by a plane wave and is not antisymmetrized with the daughter $(A-1)$ nucleus. Accordingly in PWIA one deals with the diagonal component of the spectral function $S(\boldsymbol{p},E_p)$ only, which is the easiest to calculate.

The chief flaw of PWIA is of course the neglect of final state interactions (FSI). These should be calculated on the basis of a realistic model. The simplest available model, namely the Fermi gas (FG), appears hardly suitable for describing exclusive processes, being an uniform distribution of unbound hadrons in an infinite volume. Yet finite size (surface) and binding effects can be properly inserted into the FG model within a semi-classical formalism, which has been developed in Ref. [3,4]. It has been satisfactorily tested in the "inclusive" nuclear response functions, but for their low energy side, where the semi-classical approach interpolates the quantum mechanical cross sections without reproducing their rapid variation with the energy.

The aim of this paper is to extend the semi-classical method to deal with exclusive processes, going beyond the PWIA by accounting for the FSI. Of course the problem of the FSI in the exclusive processes has already been widely addressed [5]: not, however, in the semi-classical framework. In this paper we shall show that the latter allows not only for an adequate treatment of the inclusive processes, but of the exclusive (or semi-inclusive) ones as well, at least in a mean field framework, the scheme we shall adhere to, for simplicity, in the present work.

Indeed when the momenta transferred to the nucleus responding to an external electromagnetic field are not too small, the semi-classical spectral function $S(\boldsymbol{p},\boldsymbol{p}',E)$ and distortion operator $\widehat{\rho}_N$ (to be later defined) are expected to satisfactorily account for the dynamics of a nucleon inside the nucleus (the first) and for the distortion the outgoing nucleon wave suffers in crossing the nuclear surface (the second). We actually conjecture the semi-classical method to be applicable to exclusive processes when the variation of the nuclear density and mean field over distances of, say, half a wavelength of the impinging photon is modest. Accordingly momentum transfers down to 300–500 MeV/c, with associated half-wavelengths of 1–2 fm, should fulfill the above requirement except, possibly, in the surface of the nucleus. The latter is known to play a minor role in the physics of the quasi-elastic "inclusive" scattering (witnessing the success of the Fermi gas in accounting for the quasi-elastic peak). This paper suggests that this occurrence remains true for "exclusive" processes also, in the kinematical domain above referred to and in the mean field framework.

Whether the extension of the semi-classical method to encompass nucleon–nucleon (NN) dynamical correlations is justified, both for the spectral function and for the distortion of the final nucleon wave, is a subject yet to be explored. We intend to carry out this task since the data unambiguously point to the existence of a substantial exclusive cross-section in kinematical regions hardly compatible with a pure mean field description of the $(e, e'p)$ process [7].

As it is well-known the semi-classical method expresses the physical observables in powers of the Planck's constant $\hbar$. It turns out that the leading term of the expansion already accounts, in the mean field approximation, for the basic elements of the exclusive physics, namely the nuclear confinement and the FSI. It becomes then possible to address more directly several questions related to the physics of the exclusive (and inclusive) processes, whose answer would otherwise be much harder to get in a fully quantum mechanical many-body scheme.

As a first point one would like to appreciate the impact on the exclusive process of the FSI. For this purpose we describe the distortion of the outgoing nucleon wave in two opposite, schematic models, the so-called eikonal and uniform approximations: in the first the outgoing nucleon is not deflected from the direction of the initial momentum, while in the latter the final nucleon is isotropically emitted from the nucleus. We compare the results we obtain with the exclusive cross-sections in the semi-classical PWIA.

As a second point the sensitivity of the semi-classical exclusive cross-section to the shape of the shell model potential binding the nucleons in the nucleus should be tested: accordingly we employ in our calculations both an harmonic oscillator and a Woods–Saxon potential well. Here, for simplicity, the field felt by the particles and the holes is taken to be real and non-relativistic. Actually an approach with a *complex* shell model potential which extrapolates the mean field from positive toward negative energies has been developed in Ref. [8,9], while a relativistic approach, both for the shell model [10] and for the optical potential (which accounts for the distortion of the ejected nucleon) has been recently used in exclusive processes [11].

We have found, as expected, that indeed in the exclusive $(e, e'p)$ cross-section the outgoing nucleon wave can keep track of the original bound state, again depending upon the distortion mechanism, in contradistinction to the inclusive cross-section, which is almost unaffected by the shape of the potential. In other words the exclusive cross-section is sensitive to the structure of the spectral function.

Accordingly, although the focus of this work is largely centered on the effects of FSI, we have also paid special attention to the spectral function, a key ingredient of the exclusive cross-section. Direct calculations of the hole spectral density have been performed long ago with variational methods in few-nucleon systems [12,13] and later on in nuclear matter [14,15] in the frame of the correlated basis theory. More recently spectroscopic factors have been evaluated in the relativistic shell model approach of Ref. [10], both in medium and heavy nuclei. While the semi-classical approach admittedly fails in reproducing specific quantum mechanical aspects of the spectral function, it keeps the basic average features of the latter (which instead are missed by the Fermi gas description), still retaining a great deal of simplicity.

We want to point out that the semi-classical formalism developed in detail in this work looks somewhat intricate, but in fact its application leads to relatively simple calculations.

We summarize now the organization of the paper: in Section 2 the general expressions

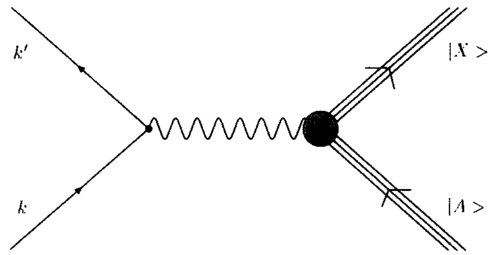

Fig. 1. The Feynman diagram representing the inclusive (e,e') process.

for the inclusive and exclusive cross-section in the one-photon exchange approximation are shortly revisited. Furthermore the off-diagonal spectral function and the distortion operator are introduced. In Section 3 we derive and discuss the semi-classical expression for the exclusive cross-section, both in PWIA and DWIA. In Section 4 the diagonal part of the semi-classical spectral function is obtained in the mean field framework. Analytic expressions of this quantity for a few one-body potential wells are provided in Appendix A. In Section 5 we deduce the mean field expression for the distortion operator and discuss it in the context of the two rather extreme models (eikonal and uniform approximation) referred to above. In Section 6 we calculate, for both models, the exclusive cross-sections and, in Section 7, the inclusive ones, by integrating the former over the appropriate regions of the missing energy-missing momentum plane. Finally in Sections 8 and 9 we present and discuss our numerical results and examine the validity of the semi-classical approach together with its possible extensions.

2. The cross-sections

The inclusive cross-section for the scattering of an electron, with initial and final four-momenta k and k' respectively, out of a nucleus, initially in its ground state $|A\rangle$ and then excited into "any" final state $|X\rangle$, reads

$$\frac{d^2\sigma}{d\Omega_e\, d\epsilon'} = 2\alpha^2 \frac{1}{(Q^2)^2}\frac{k'}{k}\eta^{\mu\nu}\sum_X \langle A|\widehat{J}^{\dagger}_{\mu}(\boldsymbol{q})|\, X\rangle\langle X|\widehat{J}_{\nu}(\boldsymbol{q})|A\rangle\delta(E_X - E_A - \omega) \quad (2.1)$$

(all the states are normalized to one in a large box of volume V). In the above $Q^2 = -q^2 \equiv \boldsymbol{q}^2 - \omega^2$ is the space-like four momentum transferred from the electron to the nucleus and

$$\eta^{\mu\nu} = k^{\mu}k'^{\nu} + k^{\nu}k'^{\mu} - g^{\mu\nu}k\cdot k' \quad (2.2)$$

is the well-known symmetric leptonic tensor of rank two. The diagram describing the inclusive process is displayed in Fig. 1.

In the present work we confine ourselves, for sake both of simplicity and of illustration, to consider one-body current only, disregarding meson exchange currents (MEC). Likewise correlations among nucleons beyond the mean field will be dealt with in future research. In any case the matrix elements of the nucleon's one-body current $\widehat{J}_\nu$ entering into (2.1) read (the symbols are self-explanatory)

$$\langle X|\widehat{J}_\nu(\boldsymbol{q})|A\rangle = \sum_s \int \frac{d\boldsymbol{p}}{(2\pi)^3} \int \frac{d\boldsymbol{p}'}{(2\pi)^3} \langle X|\widehat{a}^\dagger(\boldsymbol{p}', s')\widehat{a}(\boldsymbol{p}, s)|A\rangle\langle \boldsymbol{p}', s'|j_\nu(\boldsymbol{q})|\boldsymbol{p}, s\rangle, \tag{2.3}$$

the standard annihilation and creation operators $\widehat{a}$ and $\widehat{a}^\dagger$ being normalized according to the anticommutation rule

$$\{\widehat{a}\,(\boldsymbol{p}, s), \widehat{a}^\dagger(\boldsymbol{p}', s')\} = (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}')\delta_{ss'}. \tag{2.4}$$

In the above

$$\langle \boldsymbol{p}', s'|j_\nu(\boldsymbol{q})|\boldsymbol{p}, s\rangle = (2\pi)^3\delta(\boldsymbol{p}'-\boldsymbol{p}-\boldsymbol{q})j_\nu^{s's}(\boldsymbol{p}+\boldsymbol{q}, \boldsymbol{p}) \tag{2.5}$$

and

$$j_\nu^{s's}(\boldsymbol{p}+\boldsymbol{q}, \boldsymbol{p}) = \bar{u}(\boldsymbol{p}+\boldsymbol{q}, s')\left[F_1(Q^2)\gamma_\nu + F_2(Q^2)i\frac{\kappa}{2M_N}\sigma_{\mu\nu}q^\mu\right]u(\boldsymbol{p}, s), \tag{2.6}$$

where F_1 and F_2 are the Dirac and Pauli nucleon's form factors, $\kappa = 1.79$ for protons and $\kappa = -1.91$ for neutrons. Finally the spinor normalization is $u^\dagger u = 1$.

Clearly the insertion of (2.5) into (2.3) leads to

$$\langle X|\widehat{J}_\nu(\boldsymbol{q})|A\rangle = \sum_s \int \frac{d\boldsymbol{p}}{(2\pi)^3}\langle X|\widehat{a}^\dagger(\boldsymbol{p}+\boldsymbol{q}, s')\widehat{a}\,(\boldsymbol{p}, s)|A\rangle j_\nu^{s's}(\boldsymbol{p}+\boldsymbol{q}, \boldsymbol{p}). \tag{2.7}$$

Now in the PWIA framework the final nuclear state is factorized as follows (V is the normalization volume)

$$|X\rangle = |n\rangle \otimes \frac{1}{\sqrt{V}}|\widetilde{\boldsymbol{p}}_N, s_N\rangle, \tag{2.8}$$

where $|n\rangle$ represents an excited state, normalized to one, of the residual $(A-1)$ nucleus and $|\widetilde{\boldsymbol{p}}_N, s_N\rangle$ is just a plane wave, normalized according with the anticommutators (2.4). In such a scheme, to the matrix element (2.7) only a single nucleon with a given momentum $\boldsymbol{p}$ will contribute, the rest of the nucleus behaving just as a spectator.

In the present approach instead the wave of the outgoing nucleon, $|\widetilde{\boldsymbol{p}}_N, s_N\rangle$, is "distorted" by the interaction with the residual nucleus and no longer is a momentum eigenstate; as a consequence the nuclear matrix element (2.7) will be expressed through an integral (sum) over all possible initial nucleons' momenta (spin) and will read:

$$\langle X|\widehat{J}_\nu(\boldsymbol{q})|A\rangle = \frac{1}{\sqrt{V}}\sum_s \int \frac{d\boldsymbol{p}}{(2\pi)^3}\langle \widetilde{\boldsymbol{p}}_N|\boldsymbol{p}+\boldsymbol{q}\rangle j_\nu^{s_N s}(\boldsymbol{p}+\boldsymbol{q}, \boldsymbol{p})\langle n\,|\widehat{a}\,(\boldsymbol{p}, s)|A\rangle \tag{2.9}$$

providing the spin of the outgoing nucleon, which will no longer be explicitly indicated in the state vectors, is unaffected by the distortion.

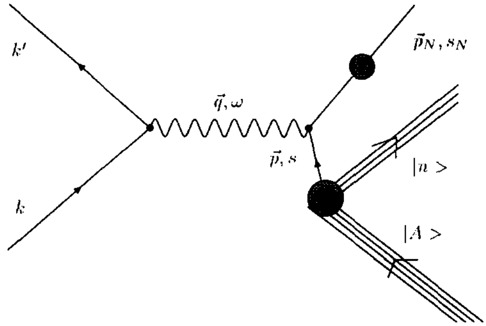

Fig. 2. Diagrammatic representation of an exclusive $(e, e'N)$ process.

In Fig. 2 the exclusive process is diagrammatically displayed : the bubble on the final nucleon leg should be ignored in the PWIA scheme, whereas in the framework of the present paper it is meant to embody the FSI.

Using (2.9) we can now compute the cross-section for the exclusive process. In the laboratory frame, where the nucleus is at rest (hence $E_A = M_A$), we obtain

$$\frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, d\boldsymbol{p}_N} = \frac{2\alpha^2}{(2\pi)^3}\frac{1}{(Q^2)^2}\frac{k'}{k}\eta_{\mu\nu}\int\frac{d\boldsymbol{p}}{(2\pi)^3}\int\frac{d\boldsymbol{p}'}{(2\pi)^3}$$
$$\times\sum_{s_N,s',s} j^{\mu}_{s's_N}(\boldsymbol{p}',\boldsymbol{p}'+\boldsymbol{q})\langle\boldsymbol{p}'+\boldsymbol{q}|\widetilde{\boldsymbol{p}}_N\rangle\langle\widetilde{\boldsymbol{p}}_N|\boldsymbol{p}+\boldsymbol{q}\rangle j^{\nu}_{s_N s}(\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p})$$
$$\times\sum_n\langle A|\widehat{a}^\dagger(\boldsymbol{p}',s')|n\rangle\delta[E^n_{A-1}-(\omega+M_A-E_N)]\langle n|\widehat{a}(\boldsymbol{p},s)|A\rangle \tag{2.10}$$

the sum extending over the whole spectrum of eigenfunctions of the residual $(A-1)$ nucleus, with eigenvalues E^n_{A-1}, while E_N is the energy of the outgoing nucleon.

Concerning the electromagnetic vertices they will be taken, according to (2.6), with the nucleon on the mass-shell (which is the case for the FG, but clearly not for a finite nucleus). Of course this assumption can be corrected by employing, e.g., the CC1 prescription of De Forest [6] to move the nucleon off the energy shell preserving gauge invariance.

It is customary to introduce the positive "nuclear separation energy"

$$E_S = M_N + M_{A-1} - M_A = M_N - \mu, \tag{2.11}$$

μ being the nuclear chemical potential, and the positive "excitation energy"

$$\mathcal{E}_n = E^n_{A-1} - E^0_{A-1} \cong E^n_{A-1} - M_{A-1} - E_{\rm rec} \tag{2.12}$$

of the residual $(A-1)$ nucleus recoiling with momentum $\boldsymbol{q}-\boldsymbol{p}_N$ (the "missing momentum") and energy (in the non-relativistic approximation)

$$E_{\rm rec} \cong \frac{(\boldsymbol{q}-\boldsymbol{p}_N)^2}{2\,M_{A-1}}. \tag{2.13}$$

Accordingly we can write

$$\delta[E^n_{A-1} - (\omega + M_A - E_N)] = \delta[\mathcal{E}_n - (\omega - T_N - E_S - E_{\rm rec})] = \delta(\mathcal{E}_n - \mathcal{E}), \quad (2.14)$$

where

$$\mathcal{E} = \omega - T_N - E_S - E_{\rm rec} \quad (2.15)$$

is the so-called "missing energy", fixed by the external kinematics, and T_N the kinetic energy of the outgoing nucleon.

To proceed further we introduce now three quantities, related to different one-body operators, which allow to express the exclusive cross-section in a compact form.

The first of these is the general (off-diagonal) *spectral function* defined as follows

$$\begin{aligned}\langle \boldsymbol{p}, s|S(\mathcal{E})|\boldsymbol{p}', s'\rangle &= \sum_n \langle A|\widehat{a}^\dagger(\boldsymbol{p}', s')|n\rangle\delta[\mathcal{E} - (E^n_{A-1} - E^0_{A-1})]\langle n|\widehat{a}(\boldsymbol{p}, s)|A\rangle \\ &= \langle A|\widehat{a}^\dagger(\boldsymbol{p}', s)\delta(\widehat{\mathcal{H}} - \mathcal{E})\widehat{a}(\boldsymbol{p}, s)|A\rangle. \end{aligned} \quad (2.16)$$

In the above closure has been applied and $\widehat{\mathcal{H}}$ is the Hamiltonian whose eigenvalues are the excitation energies of the residual nucleus.

Upon integration over the excitation energy, the spectral function, diagonal in the spin indices for parity conserving interactions, yields

$$\int d\mathcal{E}\langle \boldsymbol{p}, s|S(\mathcal{E})|\boldsymbol{p}', s\rangle = \langle A|\widehat{a}^\dagger(\boldsymbol{p}', s)\widehat{a}(\boldsymbol{p}, s)|A\rangle, \quad (2.17)$$

which, for $\boldsymbol{p} = \boldsymbol{p}'$, is just the momentum distribution of the nucleons inside the nucleus. The latter becomes experimentally accessible if the data span a range of missing energy large enough.

Obviously for the diagonal part of the spectral function the well-known sum rule

$$\int \frac{d\boldsymbol{p}}{(2\pi)^3} \int d\mathcal{E}\langle \boldsymbol{p}, s|S(\mathcal{E})\,|\boldsymbol{p}, s\rangle = A, \quad (2.18)$$

holds, A being the number of nucleons in the nucleus and a spin-isospin summation being implicitly assumed.

Next the matrix elements in momentum space

$$\langle \boldsymbol{p}'|\widehat{\rho}_N|\boldsymbol{p}\rangle = \langle \boldsymbol{p}'|\widetilde{\boldsymbol{p}}_N\rangle\langle\widetilde{\boldsymbol{p}}_N|\boldsymbol{p}\rangle \quad (2.19)$$

of the *one-body projection operator* $\widehat{\rho}_N$ should be introduced. The operator $\widehat{\rho}_N$ embodies the distortion of the outgoing nucleon's wave and it is clearly of central relevance for the present treatment. In PWIA the outgoing nucleon state is a plane wave and thus (2.19) becomes:

$$\langle \boldsymbol{p}'|\widehat{\rho}_N|\boldsymbol{p}\rangle = (2\pi)^6\delta\left(\boldsymbol{p}' - \boldsymbol{p}_N\right)\delta\left(\boldsymbol{p}_N - \boldsymbol{p}\right). \quad (2.20)$$

In the next three sections we shall study in detail both S and $\widehat{\rho}_N$ in the semi-classical framework.

Finally the one-body operator associated with the electromagnetic vertices, namely the *single nucleon tensor* (which we shall assume it to be on the mass-shell)

$$\langle \boldsymbol{p}', s'|W_{sn}^{\mu\nu}(\boldsymbol{q})|\boldsymbol{p}, s\rangle = j_{s_N s'}^{\mu *}(\boldsymbol{p}' + \boldsymbol{q}, \boldsymbol{p}')j_{s_N s}^{\nu}(\boldsymbol{p} + \boldsymbol{q}, \boldsymbol{p}), \tag{2.21}$$

is the third element which enters into the physics of the electromagnetic exclusive processes.

On the basis of the above definitions the exclusive cross-section (2.10) can be rewritten as follows

$$\frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, d\boldsymbol{p}_N} = \frac{2\alpha^2}{(2\pi)^3}\frac{1}{(Q^2)^2}\frac{k'}{k}\eta_{\mu\nu}\int\frac{d\boldsymbol{p}}{(2\pi)^3}\int\frac{d\boldsymbol{p}'}{(2\pi)^3}\sum_{s_N,s',s}\langle \boldsymbol{p}' + \boldsymbol{q}|\widehat{\rho}_N|\boldsymbol{p} + \boldsymbol{q}\rangle$$
$$\times\langle \boldsymbol{p}, s|S(\mathcal{E})|\boldsymbol{p}', s'\rangle\langle \boldsymbol{p}', s'|W_{sn}^{\mu\nu}(\boldsymbol{q})|\boldsymbol{p}, s\rangle. \tag{2.22}$$

We shall discuss in the following section the condition under which the off-diagonal matrix elements of $W_{sn}^{\mu\nu}$ can be disregarded. The diagonal $W_{sn}^{\mu\nu}$ directly relates to the physical "on shell" electron-nucleon cross-section according to

$$\left(\frac{d\sigma}{d\Omega_e}\right)_{sn} = 2\alpha^2\frac{1}{(Q^2)^2}\frac{k'}{k}\eta_{\mu\nu}\langle \boldsymbol{p}, s|W_{sn}^{\mu\nu}(\boldsymbol{q})|\boldsymbol{p}, s\rangle. \tag{2.23}$$

3. The semi-classical approach

For the details of the method we refer the reader to Ref. [4]. Here it suffices to remind the definition of the Wigner transform (WT) of a one-body operator O, namely

$$O_W(\boldsymbol{R}, \boldsymbol{p}) = \int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\left\langle \boldsymbol{p} + \frac{\boldsymbol{k}}{2}\middle|\widehat{O}\middle|\boldsymbol{p} - \frac{\boldsymbol{k}}{2}\right\rangle \tag{3.1}$$

and, inversely,

$$\langle \boldsymbol{p}|\widehat{O}|\boldsymbol{p}'\rangle = \int d\boldsymbol{R}\, e^{i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}}O_W\left(\boldsymbol{R}, \frac{\boldsymbol{p}+\boldsymbol{p}'}{2}\right). \tag{3.2}$$

The core of the semi-classical approach lies in the systematic expansion in $\hbar$ (or in the gradient with respect to $\boldsymbol{R}$ or $\boldsymbol{p}$) of the Wigner transform of operators; in our case only the leading order of the expansion will be kept: accordingly the Wigner transform of the product of operators reduces to the product of the Wigner transforms of each factor. We shall repeatedly exploit this rule in the following.

Introducing the Wigner transforms and performing the change of variables

$$\boldsymbol{p} = \boldsymbol{K} + \frac{\boldsymbol{u}}{2} \qquad \text{and} \qquad \boldsymbol{p}' = \boldsymbol{K} - \frac{\boldsymbol{u}}{2} \tag{3.3}$$

we can recast the expression (2.22) into the form

$$\frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, d\boldsymbol{p}_N} = \frac{2\alpha^2}{(2\pi)^3}\frac{1}{(Q^2)^2}\frac{k'}{k}\eta_{\mu\nu}\int\frac{d\boldsymbol{K}}{(2\pi)^3}\int\frac{d\boldsymbol{u}}{(2\pi)^3}$$

$$
\begin{aligned}
&\times \sum_{s_N s' s} \int d\boldsymbol{R} e^{-i\boldsymbol{u}\cdot\boldsymbol{R}} [\widehat{\rho}_N]_{\mathrm{W}}(\boldsymbol{R}, \boldsymbol{K}+\boldsymbol{q}) \int d\boldsymbol{S}\, e^{i\boldsymbol{u}\cdot\boldsymbol{S}} [S_{ss'}(\mathcal{E})]_{\mathrm{W}}(\boldsymbol{S}, \boldsymbol{K}) \\
&\times \left\langle \boldsymbol{K} - \frac{\boldsymbol{u}}{2}, s' \right| W^{\mu\nu}_{sn}(\boldsymbol{q}) \left| \boldsymbol{K} + \frac{\boldsymbol{u}}{2}, s \right\rangle \\
&= \frac{2\alpha^2}{(2\pi)^3} \frac{1}{(Q^2)^2} \frac{k'}{k} \eta_{\mu\nu} \sum_{s's} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \int d\boldsymbol{R}\, d\boldsymbol{S} [S_{s's}(\mathcal{E})]_{\mathrm{W}}(\boldsymbol{S}, \boldsymbol{K}) \\
&\times [\widehat{\rho}_N]_{\mathrm{W}}(\boldsymbol{R}, \boldsymbol{K}+\boldsymbol{q}) [W^{\mu\nu}_{s',s}(\boldsymbol{q})]_{\mathrm{W}}(\boldsymbol{S}-\boldsymbol{R}, \boldsymbol{K}),
\end{aligned} \tag{3.4}
$$

where

$$
\begin{aligned}
\left[W^{\mu\nu}_{s',s}(\boldsymbol{q})\right]_{\mathrm{W}}(\boldsymbol{S}-\boldsymbol{R}, \boldsymbol{K}) &= \int \frac{d\boldsymbol{t}}{(2\pi)^3} e^{i\boldsymbol{t}\cdot(\boldsymbol{S}-\boldsymbol{R})} \\
&\times \sum_{s_N} \left[j^{\mu}_{s_N s'} \left(\boldsymbol{K} + \frac{\boldsymbol{t}}{2} + \boldsymbol{q}, \boldsymbol{K} + \frac{\boldsymbol{t}}{2} \right) \right]^* \\
&\times j^{\nu}_{s_N s} \left(\boldsymbol{K} - \frac{\boldsymbol{t}}{2} + \boldsymbol{q}, \boldsymbol{K} - \frac{\boldsymbol{t}}{2} \right) \\
&\simeq \delta(\boldsymbol{R}-\boldsymbol{S}) \langle \boldsymbol{K}, s | W^{\mu\nu}_{sn}(\boldsymbol{q}) | \boldsymbol{K}, s \rangle .
\end{aligned} \tag{3.5}
$$

To leading order in $\hbar$ the t-dependence in the electromagnetic current can be disregarded. Hence the Wigner transform in Eq. (3.5) depends on the diagonal nucleonic matrix elements only. This approximation leads to a local expression for the exclusive cross-section. The interference between the elementary contributions arising from different regions of the nucleus is neglected in first order and would come into play through higher order terms in the $\hbar$ expansion. The latter involve derivatives of the mean binding field, thus they are mainly relevant for surface effects, namely for the removal of peripheral nucleons.

By inserting (3.5) into (3.4) one finally obtains for the exclusive cross-section the semi-classical expression

$$
\begin{aligned}
\frac{d^4\sigma}{d\Omega_e d\epsilon' d\boldsymbol{p}_N} &= \frac{2\alpha^2}{(2\pi)^3} \frac{1}{(Q^2)^2} \frac{k'}{k} \eta_{\mu\nu} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \langle \boldsymbol{K}, s | W^{\mu\nu}_{sn}(\boldsymbol{q}) | \boldsymbol{K}, s \rangle \\
&\times \int d\boldsymbol{R} [S_{ss}(\mathcal{E})]_{\mathrm{W}}(\boldsymbol{R}, \boldsymbol{K}) \left[\widehat{\rho}_N\right]_{\mathrm{W}}(\boldsymbol{R}, \boldsymbol{K}+\boldsymbol{q}) \\
&= \frac{1}{(2\pi)^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \left(\frac{d\sigma}{d\Omega_e} \right)_{sn} (\boldsymbol{K}, \boldsymbol{q}) \\
&\times \int d\boldsymbol{R} [S_{ss}(\mathcal{E})]_{\mathrm{W}}(\boldsymbol{R}, \boldsymbol{K}) \left[\widehat{\rho}_N\right]_{\mathrm{W}}(\boldsymbol{R}, \boldsymbol{K}+\boldsymbol{q})
\end{aligned} \tag{3.6}
$$

(repeated indices are meant to be summed) where, considering, for sake of illustration, the longitudinal channel only,

$$
\left[\frac{d\sigma}{d\Omega_e} \right]_{sn} (\boldsymbol{p}, \boldsymbol{q}) = \sigma_{\mathrm{Mott}} \frac{M_N}{E(p)} \frac{M_N}{E(|\boldsymbol{p}+\boldsymbol{q}|)} \left(\frac{Q^2}{q^2} \right)^2 R_L, \tag{3.7}
$$

σ_{Mott} being the Mott cross-section, $E(p) = \sqrt{p^2 + M_N^2}$,

$$R_L = \frac{q^2}{Q^2} \left\{ G_{\mathrm{E}}^2(Q^2) + W_2(Q^2)\chi^2 \right\} \tag{3.8}$$

and

$$W_2(Q^2) = \frac{1}{1 + Q^2/4M_N^2} \left[G_{\mathrm{E}}^2(Q^2) + \tau G_{\mathrm{M}}^2(Q^2) \right]. \tag{3.9}$$

Furthermore the dimensionless variable $\chi = p \sin\theta / M_N$ (θ being the angle between $\boldsymbol{p}$ and $\boldsymbol{q}$) is utilized, and G_{E} and G_{M} are the Sach's electric and magnetic nucleon's form factors. In the following it will be understood, without explicit mention, that all our calculations and results refer to the longitudinal channel.

Eq. (3.6) clearly shows the modification with respect to the PWIA scheme, where only one nucleon with momentum $\boldsymbol{K} = \boldsymbol{p}_N - \boldsymbol{q}$ takes part in the exclusive process. Here instead because of the FSI embodied in $\widehat{\rho}_N$, the momentum of the nucleon actually involved into the process might take any value: hence the integration over the variable $\boldsymbol{K}$.

The above formulae also transparently show how the Wigner transform operates: it replaces the off-diagonal momentum matrix elements of the one-body operators entering into the expression for the exclusive cross-section with diagonal "space-dependent" matrix elements.

Before ending this section we write the expression for the semi-classical PWIA cross-section; introducing into (2.22) the explicit form (2.20) for the matrix element of the one-body projection operator and considering the inverse WT of the spectral function and the single nucleon tensor, one easily ends up with:

$$\left(\frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, d\boldsymbol{p}_N} \right)_{\mathrm{PWIA}} = \frac{1}{(2\pi)^3} \left(\frac{d\sigma}{d\Omega_e} \right)_{sn} (\boldsymbol{p}_m, \boldsymbol{q}) \int d\boldsymbol{R} \left[S_{ss}(\mathcal{E}) \right]_{\mathrm{W}}(\boldsymbol{R}, \boldsymbol{p}_m), \tag{3.10}$$

where the so-called missing momentum $\boldsymbol{p}_m = \boldsymbol{q} - \boldsymbol{p}_N$ has been introduced. The above formula clearly shows that in PWIA the cross-section is directly proportional to the spectral function.

In the next two sections we shall calculate the WT of the spectral function and of the distortion operator in leading order.

4. The diagonal semi-classical spectral function

Let us consider a closed shell nucleus having A nucleons sitting in the lowest orbits of a potential well $V(R)$ (in principle the Hartree-Fock (HF) mean field) and an $(A-1)$ daughter nucleus obtained by creating a hole in a generic occupied level of the former. The $(A-1)$ nucleus thus obtained will generally be in an excited state with an energy given by (in the non-relativistic HF approximation and neglecting the small recoil energy)

$$E_{A-1} \equiv E_{A-1,h} = M_A - M_N - \epsilon_h \tag{4.1}$$

$$= \sum_{\beta<\epsilon_F} t_\beta + \tfrac{1}{2}\sum_{\beta,\beta'} \langle \beta\beta'|v|\beta\beta'\rangle_a + AM_N - t_h - \sum_{\beta}\langle h\beta|v|h\beta\rangle_a - M_N, \tag{4.2}$$

where the Eq. (4.1) defines the hole energy ϵ_h. In (4.2) the first three terms on the RHS correspond to the energy of the nucleus with mass number A, v is a suitable two-body interaction and the subtracted terms represent the energy of a particle in the h orbit. Of course in the HF scheme the ground state energy of the $A-1$ daughter nucleus is obtained by removing a particle at the Fermi energy, namely

$$E^0_{A-1} = M_A - M_N - \epsilon_F \equiv M_{A-1}, \tag{4.3}$$

from where, according to (2.11), the relation $\epsilon_F = -E_S$ follows. We then get for the (positive) excitation energy

$$\mathcal{E} = E_{A-1,h} - M_{A-1} = \epsilon_F - \epsilon_h. \tag{4.4}$$

In the semi-classical approximation the "negative" Fermi energy is

$$\epsilon_F = \frac{k_F^2(R)}{2M_N^*} + V(R), \tag{4.5}$$

R being the radial variable; a nucleon effective mass M_N^* has been introduced to account, together with the potential well V(R), for the Hartree–Fock mean field. Eq. (4.5) locally defines a Fermi momentum $k_F(R)$ for $R \leqslant R_c$, R_c being the classical turning point fixed by the equation

$$\epsilon_F = V(R_c). \tag{4.6}$$

We now express the spectral function in the basis of the eigenfunctions of the single particle hamiltonian

$$h = \frac{p^2}{2M_N^*} + V(R), \tag{4.7}$$

rather than in the basis of the momentum eigenfunctions. The former of course obey the equation

$$h|\alpha\rangle = \epsilon_\alpha|\alpha\rangle, \tag{4.8}$$

ϵ_α being the associated eigenvalues.

We thus get

$$\begin{aligned}\langle \boldsymbol{p}|S(\mathcal{E})|\boldsymbol{p}'\rangle &= \sum_{\alpha\alpha'}\langle \boldsymbol{p}'|\alpha'\rangle\langle A|\widehat{a}^\dagger_{\alpha'}\delta(\widehat{\mathcal{H}} - \mathcal{E})\widehat{a}_\alpha|A\rangle\langle\alpha|\boldsymbol{p}\rangle \\ &= \sum_{\alpha\alpha'}\langle \boldsymbol{p}'|\alpha'\rangle\langle\alpha'|\delta\left[\mathcal{E} - (\epsilon_F - \epsilon_\alpha)\right]\theta(\epsilon_F - \epsilon_\alpha)|\alpha\rangle\langle\alpha|\boldsymbol{p}\rangle \\ &= \langle \boldsymbol{p}'|\theta(\epsilon_F - h)\delta\left[\mathcal{E} - (\epsilon_F - h)\right]|\boldsymbol{p}\rangle.\end{aligned} \tag{4.9}$$

By applying then the definition (3.1) of the Wigner transform it is an easy matter to verify that, to leading order of the $\hbar$ expansion, one has

$$[\theta(\epsilon_F - h)]_W(\boldsymbol{R},\boldsymbol{p}) = \theta\left[\epsilon_F - \left(\frac{p^2}{2M_N^*} + V(R)\right)\right] + O(\hbar^2) \tag{4.10}$$

and

$$\{\delta[\mathcal{E} - (\epsilon_F - h)]\}_W(\boldsymbol{R},\boldsymbol{p}) = \delta\left[\mathcal{E} - \left(\epsilon_F - \frac{p^2}{2M_N^*} - V(R)\right)\right] + O(\hbar^2). \tag{4.11}$$

Hence it follows that, in leading order, the WT of the spectral function is just the product of the WT (4.10) and (4.11), namely

$$[S(\mathcal{E})]_W(\boldsymbol{R},\boldsymbol{p}) = \theta\left(\epsilon_F - \frac{p^2}{2M_N^*} - V(R)\right)\delta\left[\mathcal{E} - \left(\epsilon_F - \frac{p^2}{2M_N^*} - V(R)\right)\right]. \tag{4.12}$$

An integration over the whole nucleus yields then for the diagonal spectral function in the semi-classical approximation the expression

$$S(\boldsymbol{p},\mathcal{E}) = \int d\boldsymbol{R}\,\theta\left(\frac{k_F^2(R)}{2M_N^*} - \frac{p^2}{2M_N^*}\right)\delta\left[\mathcal{E} - \left(\frac{k_F^2(R)}{2M_N^*} - \frac{p^2}{2M_N^*}\right)\right], \tag{4.13}$$

which displays a striking similarity with the one of a Fermi gas. In (4.13) the definition (4.5) of the local Fermi momentum has been used. Actually, in leading order, the HF semi-classical approximation of the spectral function for a finite nucleus might be viewed as arising from a superposition of a large set of FG each one characterized by a different k_F, the latter being defined according to the prescription (4.5), or, equivalently,

$$k_F(R) = \sqrt{2M_N^*(\epsilon_F - V(R))}. \tag{4.14}$$

To illustrate the method we report in Appendix A the calculation of the diagonal semi-classical spectral function for a few specific single-particle potentials.

In a square well potential the spectral function is different from zero only along the curve (V_0 is the depth of the well)

$$\mathcal{E} = \epsilon_F + V_0 - \frac{p^2}{2M_N^*}, \tag{4.15}$$

in analogy with the FG.

For the harmonic oscillator potential

$$V(R) = \tfrac{1}{2}M_N^*\omega_0^2R^2 - V_0, \tag{4.16}$$

V_0 being a positive constant and $\hbar\omega_0 = 41/A^{1/3}$ MeV the harmonic oscillator frequency, the semi-classical spectral function is displayed in Fig. 3: here we consider the nucleus of ^{40}Ca ($\hbar\omega_0 \simeq 12$ MeV and $V_0 = 55$ MeV). It can be compared with the quantum mechanical situation, illustrated in Fig. 8 of Ref. [2], where $\mathcal{E}$ is quantized and p is a continuous variable. In the semi-classical case instead both $\mathcal{E}$ and p are continuous. Furthermore the semi-classical $S_{ho}(p,\mathcal{E})$ vanishes abruptly along the line (4.15) whereas

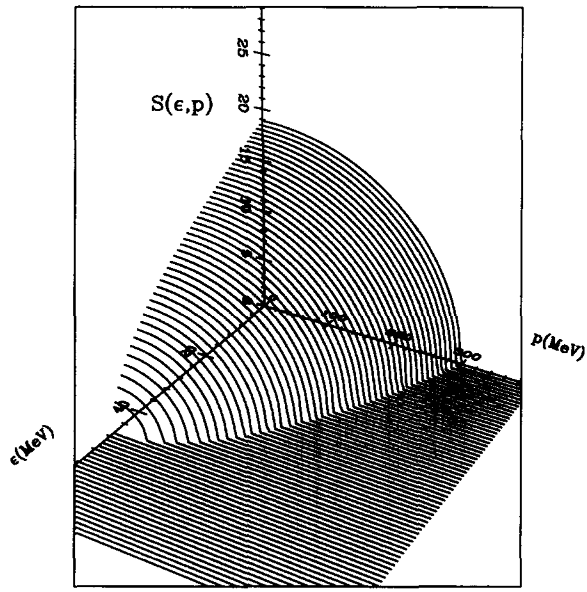

Fig. 3. Semi-classical spectral function obtained with the harmonic oscillator potential, as a function of missing energy and missing momentum.

in the quantum case there is a small leakage across this line. Finally the oscillations characterizing the quantum momentum distribution of the nucleons in a given shell are now replaced by a smooth behaviour with p. The semi-classical approach averages the quantum mechanical spectral function and fills up the domain of the $(\mathcal{E}, p_m)$ plane where the strength of the latter is concentrated, keeping nevertheless to a large extent the simplicity of the quantum Fermi gas.

In Fig. 4 the semi-classical spectral function is displayed for the Woods–Saxon well

$$V(R) = -\frac{V_1}{1 + e^{(R-R_0)/a}}, \tag{4.17}$$

with $V_1 = 50$ MeV, $R_0 = 1.2A^{1/3}$ fm and $a = 0.65$ fm. The strength distribution is somewhat different than in the previous case: it also vanishes abruptly along the line

$$\mathcal{E} + \frac{p^2}{2M_N^*} - \epsilon_F - V_1 = \left(\epsilon_F - \mathcal{E} - \frac{p^2}{2M_N^*}\right) e^{-R_0/a}, \tag{4.18}$$

which is in practice quite close to the one of the square potential well (if $V_0 = V_1$) defined in Eq. (4.15). The two lines coincide, as expected, in the limit $a \to 0$. Indeed the Woods–Saxon semi-classical spectral function becomes sizable close to the line (4.18), in resemblance with the square well case.

This can be better understood from the momentum distribution

$$n(p) = \int d\mathcal{E}\, S(\boldsymbol{p}, \mathcal{E}), \tag{4.19}$$

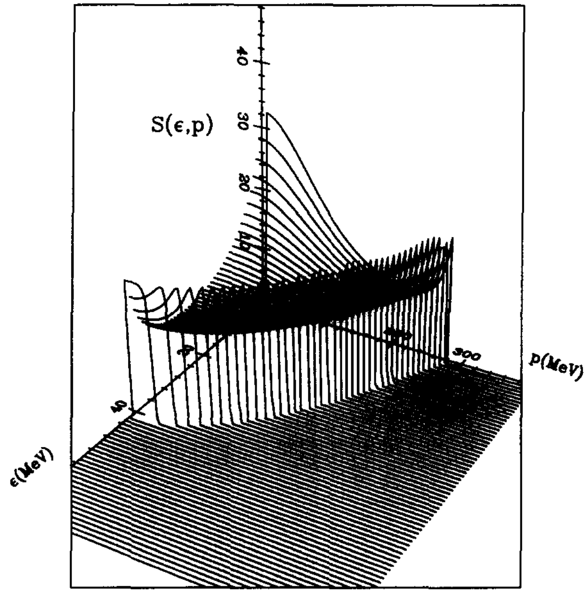

Fig. 4. Semi-classical spectral function obtained with the Woods-Saxon potential, as a function of missing energy and missing momentum.

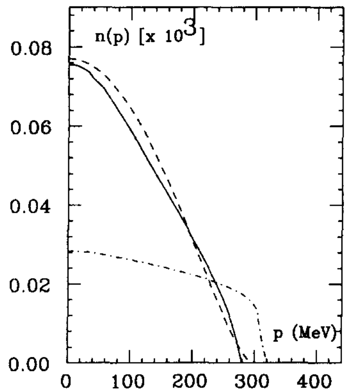

Fig. 5. Semi-classical momentum distribution (multiplied by 10^3), Eq. (4.19), as a function of p, for the harmonic oscillator well (dashed line) and the "customary" Woods-Saxon well (solid line). The dot-dashed line corresponds to the same quantity for a Woods-Saxon potential with a sharp surface ($a = 0.1$ fm).

which is illustrated in Fig. 5 for three cases: (a) the harmonic oscillator oscillator potential, (b) the Woods-Saxon well with the "standard" parameters referred to above and (c) with a sharper surface ($a = 0.1$ fm, $V_1 = 60$ MeV and $R_0 = 1.2A^{1/3}$ fm, to preserve the correct normalization). In spite of the marked differences between the corresponding spectral functions, the cases (a) and (b) are quite similar. On the contrary a very thin surface in the Woods-Saxon well [case (c)] leads to a momentum distribution which is reminiscent of the FG one. This confirms the above interpretation of the behaviour of $S_{WS}(p, \mathcal{E})$ in approaching the boundary (4.18).

5. The distortion operator

The distortion operator

$$\widehat{\rho}_N = |\widetilde{\boldsymbol{p}}_N\rangle\langle\widetilde{\boldsymbol{p}}_N| \tag{5.1}$$

obeys, in the HF scheme, the equation

$$h\widehat{\rho}_N = \frac{p_N^2}{2M_N}\widehat{\rho}_N, \tag{5.2}$$

where, on the RHS, the free nucleon mass appears.

Taking the WT of the above one gets

$$(h\widehat{\rho}_N)_{\mathrm{W}} \cong (h)_{\mathrm{W}}(\widehat{\rho}_N)_{\mathrm{W}} = \frac{p_N^2}{2M_N}(\widehat{\rho}_N)_{\mathrm{W}}, \tag{5.3}$$

or, focussing on the dependence upon the variables $\boldsymbol{R}$ and $\boldsymbol{p}$,

$$\left[(h)_{\mathrm{W}}(\boldsymbol{R},\boldsymbol{p}) - \frac{p_N^2}{2M_N}\right](\widehat{\rho}_N)_{\mathrm{W}}(\boldsymbol{R},\boldsymbol{p}) = 0. \tag{5.4}$$

Now since

$$(h)_{\mathrm{W}}(\boldsymbol{R},\boldsymbol{p}) = \frac{p^2}{2M_N^*} + V(R), \tag{5.5}$$

$V(R)$ being the chosen potential well, one has

$$(\widehat{\rho}_N)_{\mathrm{W}}(\boldsymbol{R},\boldsymbol{p}) = Z(\boldsymbol{R},\boldsymbol{p};\boldsymbol{p}_N)\delta\left(\frac{p_N^2}{2M_N} - V(R) - \frac{p^2}{2M_N^*}\right), \tag{5.6}$$

where the factor $Z(\boldsymbol{R},\boldsymbol{p};\boldsymbol{p}_N)$ can be partially fixed by the closure condition:

$$\int \frac{d\boldsymbol{p}_N}{(2\pi)^3}|\widetilde{\boldsymbol{p}}_N\rangle\langle\widetilde{\boldsymbol{p}}_N| + \sum_\alpha |\alpha\rangle\langle\alpha|\theta(-h) = 1, \tag{5.7}$$

the sum being extended over the bound HF orbits. In Wigner transform (5.7) becomes

$$\int \frac{d\boldsymbol{p}_N}{(2\pi)^3}(\widehat{\rho}_N)_{\mathrm{W}}(\boldsymbol{R},\boldsymbol{p}) = \int \frac{d\boldsymbol{p}_N}{(2\pi)^3}Z(\boldsymbol{R},\boldsymbol{p};\boldsymbol{p}_N)\delta\left(\frac{p_N^2}{2M_N} - V(R) - \frac{p^2}{2M_N^*}\right)$$
$$= \theta\left(\frac{p^2}{2M_N^*} + V(R)\right), \tag{5.8}$$

or, after performing the integration over the modulus of $\boldsymbol{p}_N$,

$$M_N p_N(R,p)\int \frac{d\widehat{p}_N}{(2\pi)^3}Z\left(\boldsymbol{R},\boldsymbol{p};\boldsymbol{p}_N(R,p)\right) = \theta\left(\frac{p^2}{2M_N^*} + V(R)\right), \tag{5.9}$$

the integral being over the direction of $\boldsymbol{p}_N(R,p) \equiv p_N(R,p)\widehat{p}_N$, with

$$p_N(R,p) = \sqrt{\frac{M_N}{M_N^*}p^2 + 2M_N V(R)}. \tag{5.10}$$

In a quantum framework one would obtain the states $|\widetilde{\boldsymbol{p}}_N\rangle$ from the positive energy solutions of an appropriate optical potential: the distortion operator, as well as its Wigner transform (5.6), would then be fixed. Here we heuristically set

$$Z\left(\boldsymbol{R},\boldsymbol{p};\boldsymbol{p}_N(R,p)\right) = \frac{(2\pi)^3}{M_N p_N(R,p)} F(\widehat{p}_N,\widehat{p})\theta\left(\frac{p^2}{2M_N^*} + V(R)\right), \tag{5.11}$$

with the additional condition

$$\int d\widehat{p}_N\, F(\widehat{p}_N,\widehat{p}) = 1. \tag{5.12}$$

As an illustration we shall consider in the following two extreme assumptions for F, namely:

$$F(\widehat{p}_N,\widehat{p}) = \delta(\widehat{p}_N - \widehat{p}), \tag{5.13}$$

which corresponds to the *eikonal approximation* and therefore is expected to be valid only for large enough energies of the outgoing nucleon. In this case in fact the final nucleon should be little deflected from the direction of the initial one, which has absorbed the photon inside the nucleus;

$$F(\widehat{p}_N,\widehat{p}) = \frac{1}{4\pi}, \tag{5.14}$$

which obviously corresponds to a final nucleon escaping the nucleus with the same probability in all directions (*uniform approximation*).

One clearly expects the true physics to lie in between the predictions of (5.13) and (5.14), respectively.

6. The semi-classical exclusive cross-section

In this section we derive explicit expressions for the exclusive cross-section in the semi-classical approximation, treating the distortion of the outgoing nucleon wave in the HF approximation as discussed in Section 5. For sake of comparison we use both the harmonic oscillator (cut at the classical turning point) and the Woods–Saxon wells.

To settle the basis for this scope, we insert into the expression of the exclusive cross-section (3.6) the distortion operator as given by (5.6) and (5.11). Then the δ-function appearing into the latter allows us to perform the integration over the modulus of the momentum variable. We thus obtain:

$$\frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, dp_N} = \frac{1}{(2\pi)^3}\left(\frac{M_N^*}{M_N}\right)\int d\widehat{p}\int d\boldsymbol{R}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left(\mathcal{P}(R)\widehat{p}-\boldsymbol{q},\boldsymbol{q}\right)$$
$$\times\left[S_{ss}(\mathcal{E})\right]_{\mathrm{W}}\left(\boldsymbol{R},\mathcal{P}(R)\widehat{p}-\boldsymbol{q}\right)\frac{\mathcal{P}(R)}{p_N}F(\widehat{p}_N,\widehat{p}), \tag{6.1}$$

with

$$\mathcal{P}(R) = \sqrt{\frac{M_N^*}{M_N}}\sqrt{p_N^2 - 2M_N V(R)}. \tag{6.2}$$

To proceed further we exploit the general expression (4.12) for the semi-classical spectral function in Wigner transform. Then (6.1) can be recast as follows

$$\frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, dp_N} = \frac{1}{2\pi^2}\left(\frac{M_N^*}{M_N}\right)\int R^2\, dR\frac{\mathcal{P}(R)}{|\boldsymbol{q}-\boldsymbol{p}_m|}$$
$$\times\int d\widehat{p}\, F\left(\widehat{\boldsymbol{q}-\boldsymbol{p}_m},\widehat{p}\right)\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left(\mathcal{P}(R)\widehat{p}-\boldsymbol{q},\boldsymbol{q}\right)$$
$$\times\delta\left[\mathcal{E}-\left(\epsilon_{\mathrm{F}}-\frac{(\boldsymbol{q}-\boldsymbol{p}_m)^2}{2M_N}-\frac{q^2}{2M_N^*}+\frac{\mathcal{P}(R)}{M_N^*}\widehat{p}\cdot\boldsymbol{q}\right)\right]\theta(\mathcal{E}), \tag{6.3}$$

where the trivial integration over the angles of the vector $\boldsymbol{R}$ has been performed and the missing momentum variable $\boldsymbol{p}_m$ has been used.

We now separately investigate the exclusive cross section in the two limiting approximations for the "distortion function" F discussed at the end of Section 5 (for simplicity here and in the following we shall ignore the effective mass, setting $M_N^* = M_N$).

To start with we consider the *eikonal approximation* [formula (5.13)]. In this case it is straightforward to obtain the following one-dimensional integral expression for the exclusive cross-section

$$\frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, dp_N} = \frac{1}{2\pi^2}\int_0^{R_c} R^2 dR\frac{\mathcal{P}(R)}{|\boldsymbol{q}-\boldsymbol{p}_m|}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left(\frac{\mathcal{P}(R)(\boldsymbol{q}-\boldsymbol{p}_m)}{|\boldsymbol{q}-\boldsymbol{p}_m|}-\boldsymbol{q},\boldsymbol{q}\right)$$
$$\times\delta\left\{\mathcal{E}-\left[\epsilon_{\mathrm{F}}-\frac{p_m^2}{2M_N}+\left(\frac{q^2-\boldsymbol{q}\cdot\boldsymbol{p}_m}{M_N}\right)\left(\frac{\mathcal{P}(R)}{|\boldsymbol{q}-\boldsymbol{p}_m|}-1\right)\right]\right\}\theta(\mathcal{E}). \tag{6.4}$$

Note that the "exclusive variables" $\mathcal{E}$ and p_m appear explicitly in (6.4); among the "inclusive" ones only q does, whereas the transferred energy ω is hidden in the scalar product $\boldsymbol{q}\cdot\boldsymbol{p}_m$. Moreover the upper limit of the R-integration in (6.4) is set by the Eq. (4.6), which clearly entails $V(R_c) < 0$ and $\sqrt{1-2M_N V(R)/(\boldsymbol{q}-\boldsymbol{p}_m)^2} > 1$. Hence, since the hole energy

$$\epsilon_h = \frac{p_m^2}{2M_N}-\frac{q^2-\boldsymbol{q}\cdot\boldsymbol{p}_m}{M_N}\left(\frac{\mathcal{P}(R)}{|\boldsymbol{q}-\boldsymbol{p}_m|}-1\right) \tag{6.5}$$

is obviously negative, it follows that in the semi-classical eikonal approximation the projection of the missing momentum $\boldsymbol{p}_m$ on the momentum transfer $\boldsymbol{q}$ has to be less than q.

It remains now to exploit the energy conserving δ-function in (6.4) to perform the R-integration. This is easily achieved by taking advantage of the identity

$$\delta\left[\mathcal{E}-(\epsilon_{\mathrm{F}}-\epsilon_h)\right]=\frac{\left(\boldsymbol{q}-\boldsymbol{p}_m\right)^2}{\left(q^2-\boldsymbol{q}\cdot\boldsymbol{p}_m\right)}\frac{\delta(R-\widetilde{R})}{|dV/dR|}\frac{\mathcal{P}(R)}{|\boldsymbol{q}-\boldsymbol{p}_m|}, \tag{6.6}$$

$\widetilde{R}$ being the root of the equation

$$\mathcal{A}(\widetilde{R})=\mathcal{A}\equiv\frac{\mathcal{E}-\epsilon_{\mathrm{F}}+p_m^2/2M_N}{\left(q^2-\boldsymbol{q}\cdot\boldsymbol{p}_m\right)/M_N}+1, \tag{6.7}$$

where

$$\mathcal{A}(R)\equiv\frac{\mathcal{P}(R)}{|\boldsymbol{q}-\boldsymbol{p}_m|}=\sqrt{1-\frac{V(R)}{(\boldsymbol{q}-\boldsymbol{p}_m)^2/2M_N}}. \tag{6.8}$$

One then obtains the following "analytic" expression for the semi-classical exclusive cross-section in the eikonal approximation for the mean field distortion operator:

$$\begin{aligned}\frac{d^4\sigma}{d\Omega_e\,d\epsilon'\,d\boldsymbol{p}_N}=&\frac{1}{2\pi^2}\widetilde{R}^2\mathcal{A}^2(\widetilde{R})\frac{1}{|dV/dR|_{R=\widetilde{R}}}\frac{\left(\boldsymbol{q}-\boldsymbol{p}_m\right)^2}{\left(q^2-\boldsymbol{q}\cdot\boldsymbol{p}_m\right)}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\\&\times\left[\mathcal{A}(\boldsymbol{q}-\boldsymbol{p}_m)-\boldsymbol{q},\boldsymbol{q}\right].\end{aligned} \tag{6.9}$$

For a full exploitation of formula (6.9) an expression for the cosine of the angle between $\boldsymbol{p}_m$ and $\boldsymbol{q}$ is still needed: it is most easily obtained by applying the energy conservation to the right sector of the diagram displayed in Fig. 2. One gets:

$$\cos\theta_{p_m q}=\frac{M_N}{qp_m}\left(\mathcal{E}+E_S-\omega\right)+\frac{1}{2}\left[\frac{p_m}{q}\left(1+\frac{M_N}{M_{A-1}}\right)+\frac{q}{p_m}\right]. \tag{6.10}$$

Choosing now for $V(R)$ the harmonic oscillator potential, as given by formula (4.16) with the bare nucleon's mass, we easily get the following cross-section

$$\begin{aligned}\frac{d^4\sigma}{d\Omega_e\,d\epsilon'\,d\boldsymbol{p}_N}=&\frac{1}{\sqrt{2}\pi^2}\frac{1}{M_N^{3/2}\omega_0^3}\left\{\frac{(\boldsymbol{q}-\boldsymbol{p}_m)^2}{2M_N}\left(1-\mathcal{A}^2\right)+V_0\right\}^{1/2}\\&\times\mathcal{A}^2\frac{\left(\boldsymbol{q}-\boldsymbol{p}_m\right)^2}{\left(q^2-\boldsymbol{q}\cdot\boldsymbol{p}_m\right)}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left[\mathcal{A}(\boldsymbol{q}-\boldsymbol{p}_m)-\boldsymbol{q},\boldsymbol{q}\right],\end{aligned} \tag{6.11}$$

where the solution of (6.7) is

$$\widetilde{R}=\frac{1}{\omega_0}\sqrt{\frac{2}{M_N}\left[\frac{(\boldsymbol{q}-\boldsymbol{p}_m)^2}{2M_N}\left(1-\mathcal{A}^2\right)+V_0\right]}. \tag{6.12}$$

For a Woods-Saxon well [Eq. (4.17)] formula (6.9) holds with

$$\widetilde{R} = R_0 + a \ln \left\{ \frac{2M_N V_1}{(\boldsymbol{q} - \boldsymbol{p}_m)^2 (\mathcal{A}^2 - 1)} - 1 \right\}. \tag{6.13}$$

Next we turn to the *uniform approximation* for the distortion function F [formula (5.14)]. In this case a fully analytic expression for the exclusive cross-section cannot be achieved. Indeed by exploiting the δ-function appearing in the definition of the distortion operator and the azimuthal angle independence of the elementary single nucleon cross-section, one gets

$$\frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, d\boldsymbol{p}_N} = \frac{1}{(2\pi)^2} \int_0^{R_c} R^2 dR \frac{\mathcal{P}(R)}{|\boldsymbol{q} - \boldsymbol{p}_m|} \int d\cos\theta \left[\frac{d\sigma}{d\Omega_e}\right]_{sn} (\mathcal{P}(R)\widehat{p} - \boldsymbol{q}, \boldsymbol{q})$$
$$\times \delta \left\{ \mathcal{E} - \left[\epsilon_F - \frac{(\boldsymbol{q} - \boldsymbol{p}_m)^2}{2M_N} - \frac{q^2}{2M_N} + \frac{\mathcal{P}(R)}{M_N} q \cos\theta \right] \right\}, \tag{6.14}$$

where $\mathcal{P}(R)$ and R_c are again fixed by Eq. (6.2) and (4.6), and θ is the angle between $\boldsymbol{q}$ and $\boldsymbol{p}$. The integration over the latter variable is trivial and one finally obtains for the semi-classical exclusive cross-section, in the uniform approximation for the distortion operator, the following one-dimensional integral expression

$$\frac{d^4\sigma}{d\Omega_e d\epsilon' d\boldsymbol{p}_N} = \frac{1}{(2\pi)^2} \frac{M_N}{q} \frac{1}{|\boldsymbol{q} - \boldsymbol{p}_m|} \int_0^{R_c} R^2 dR$$
$$\times \left[\frac{d\sigma}{d\Omega_e}\right]_{sn} (\mathcal{P}(R)\widehat{p} - \boldsymbol{q}, \boldsymbol{q}) \Big|_{\cos\theta = y_0(R)} \theta(1 - |y_0(R)|), \tag{6.15}$$

where

$$y_0(R) = \frac{M_N}{\mathcal{P}(R) q} \left[\mathcal{E} - \left(\epsilon_F - \frac{(\boldsymbol{q} - \boldsymbol{p}_m)^2}{2M_N} - \frac{q^2}{2M_N} \right) \right]. \tag{6.16}$$

Notably the one-body potential confining the nucleons into the nucleus does not explicitly appear in (6.15): it is however hidden in the equations fixing R_c, ϵ_F and $\mathcal{P}(R)$.

7. The semi-classical t-inclusive cross-section

To get the inclusive cross-section in the t-channel one should integrate the exclusive one, obtained in the previous section, over the outgoing nucleon's momentum. Indeed in the t-inclusive scattering only the final electron is detected: accordingly the momentum $\boldsymbol{q}$ transferred to the nucleus is kept fixed in the process. By contrast in the u-channel, where the outgoing nucleon only is detected, the vector $\boldsymbol{\xi} = \boldsymbol{p}_N - \boldsymbol{k}$ is kept fixed, whereas $\boldsymbol{q}$ varies.

We intend to apply the semi-classical formalism to the u-inclusive scattering in a forthcoming paper: here we focus instead on the t-inclusive channel, where the vast majority of the electron scattering experiments have been performed.

To start with we first show that by integrating the exclusive cross-section over the momentum of the emitted nucleon we recover the correct inclusive cross-section if and only if the distortion of the outgoing particle is properly accounted for. For this purpose the integral of (3.6) over $\boldsymbol{p}_N$ is carried out by exploiting the semi-classical expressions (4.12) for the spectral function and (4.5) for the Fermi energy. We get

$$\frac{d^2\sigma}{d\Omega_e\, d\epsilon'} = \int \frac{d\boldsymbol{p}_N}{(2\pi)^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \left(\frac{d\sigma}{d\Omega_e}\right)_{sn} (\boldsymbol{K},\boldsymbol{q}) \int d\boldsymbol{R} (\widehat{\rho}_N)_{\mathrm{W}} (\boldsymbol{R},\boldsymbol{K}+\boldsymbol{q})$$
$$\times \theta\left[\epsilon_{\mathrm{F}} - \frac{K^2}{2M_N^*} - V(R)\right] \delta\left\{\mathcal{E} - \left[\epsilon_{\mathrm{F}} - \frac{K^2}{2M_N^*} - V(R)\right]\right\}. \tag{7.1}$$

Now, using the δ-function explicitly embodied in the distortion operator [see (5.6)], we transform the argument of the δ-function in Eq. (7.1) (neglecting the recoil energy) as follows:

$$\mathcal{E} - \left[\epsilon_{\mathrm{F}} - \frac{K^2}{2M_N^*} - V(R)\right] = \omega - \left(\frac{q^2}{2M_N^*} + \frac{\boldsymbol{q}\cdot\boldsymbol{K}}{M_N^*}\right). \tag{7.2}$$

Then the integration over $\boldsymbol{p}_N$ is immediately done with the help of (5.8) and we end up with

$$\frac{d^2\sigma}{d\Omega_e\, d\epsilon'} = \int \frac{d\boldsymbol{K}}{(2\pi)^3} \int d\boldsymbol{R} \left(\frac{d\sigma}{d\Omega_e}\right)_{sn} (\boldsymbol{K},\boldsymbol{q})\, \theta\left[\frac{|\boldsymbol{K}+\boldsymbol{q}|^2}{2M_N^*} + V(R)\right]$$
$$\times \theta\left[\epsilon_{\mathrm{F}} - \frac{K^2}{2M_N^*} - V(R)\right] \delta\left\{\omega - \left(\frac{q^2}{2M_N^*} + \frac{\boldsymbol{q}\cdot\boldsymbol{K}}{M_N^*}\right)\right\}, \tag{7.3}$$

which is the well-known semi-classical expression for the inclusive cross-section, limited however to the continuum spectrum for the emitted nucleon. The internal consistency of the semi-classical approach is thus proved. In connection with this result we note that it extends the PWIA of Ref. [2] where it is shown that for the fully quantum mechanical relativistic Fermi gas the integral of the exclusive cross-section leads to the inclusive one only in the non-Pauli blocked domain. [1] The expression (7.3) instead fully respects the Pauli principle: of course it does not account for the contribution to the inclusive cross section arising from the unoccupied bound states lying in between the Fermi energy and the continuum.

For the actual evaluation of (7.3), however, we choose here to follow the approach of integrating over the "exclusive" variables $\mathcal{E}$ and p_m. For this purpose we observe that, owing to the independence of the exclusive cross-section upon the azimuthal angle of $\boldsymbol{p}_N$, the integration over the latter can be converted into an integration over an ap-

[1] The same result is obtained in the present framework, when the PWIA for $\widehat{\rho}_N$, expression (2.20), is employed.

propriate domain of the missing-energy–missing-momentum plane. Indeed the following relationship holds:

$$\frac{d^2\sigma}{d\Omega_e\, d\epsilon'} = 2\pi \frac{M_N}{q} \int_{\Gamma} p_m\, dp_m\, d\mathcal{E} \frac{d^4\sigma}{d\Omega_e\, d\epsilon'\, dp_N}, \tag{7.4}$$

the boundaries of the integration domain Γ being given (in the positive quadrant of the $(\mathcal{E}, p_m)$ plane) by the curves [2]

$$\mathcal{E}^- = \omega - E_S - \frac{(q-p_m)^2}{2M_N} - \frac{p_m^2}{2M_{A-1}} \tag{7.5}$$

$$\mathcal{E}^+ = \omega - E_S - \frac{(q+p_m)^2}{2M_N} - \frac{p_m^2}{2M_{A-1}}, \tag{7.6}$$

which are obtained by setting $\cos\theta_{p_m q} = -1$ and $\cos\theta_{p_m q} = +1$, respectively, in Eq. (6.10). They represent the largest ($\mathcal{E}^-$) and the lowest ($\mathcal{E}^+$) excitation energy of the residual nucleus compatible with the kinematical constraints in an exclusive process.

We remind that, while $\mathcal{E}^-$ always extends to the first quadrant of the $(\mathcal{E}, p_m)$ plane, this might not be the case for $\mathcal{E}^+$. Indeed, for $\omega \leqslant q^2/2M_N + E_S \equiv \widetilde{\omega}$, $\mathcal{E}^+$ lives entirely outside the first quadrant: in this case the lower limit of integration over the variable $\mathcal{E}$ should simply be set equal to zero. On the other hand, for $\omega \geqslant \widetilde{\omega}$, $\mathcal{E}^+$ extends to the first quadrant as well.

8. Results

In this section the predictions of our theory are numerically appraised. We first consider the exclusive cross-sections. They are displayed in Figs. 6a–d as a function of the missing momentum for various missing energies at $q = 300$ MeV/c (Figs. 6a and 6b) and at $q = 500$ MeV/c (Figs. 6c and 6d); results both in the eikonal (Figs. 6a and 6c) and in the uniform (Figs. 6b and 6d) approximation for the distortion of the outgoing nucleon are shown. The energy transfer ω has been chosen here to be close to the quasi-elastic peak ($\omega \simeq q^2/2M_N + E_S$). All the figures displayed in this first set refer to the harmonic oscillator well, with $V_0 = 55$ MeV and $\hbar\omega_0 = 41/A^{1/3} \simeq 12$ MeV for the ^{40}Ca nucleus.

The corresponding results for the Woods–Saxon well (in the same kinematical conditions and in the two approximations employed for the distortion operator) are shown in Figs. 7a–d; we use $V_1 = 49.8$ MeV, $R_0 = 1.2A^{1/3}$ fm and $a = 0.65$ fm.

A few features are worth commenting: one is related to the remarkable difference between the exclusive cross-sections evaluated in the uniform and in the eikonal approximations. In the first case the cross-sections are seen to be fairly constant over the whole range of the kinematically allowed values of the missing momentum p_m, *independently* from the potential well binding the nucleons into the nucleus. In the second

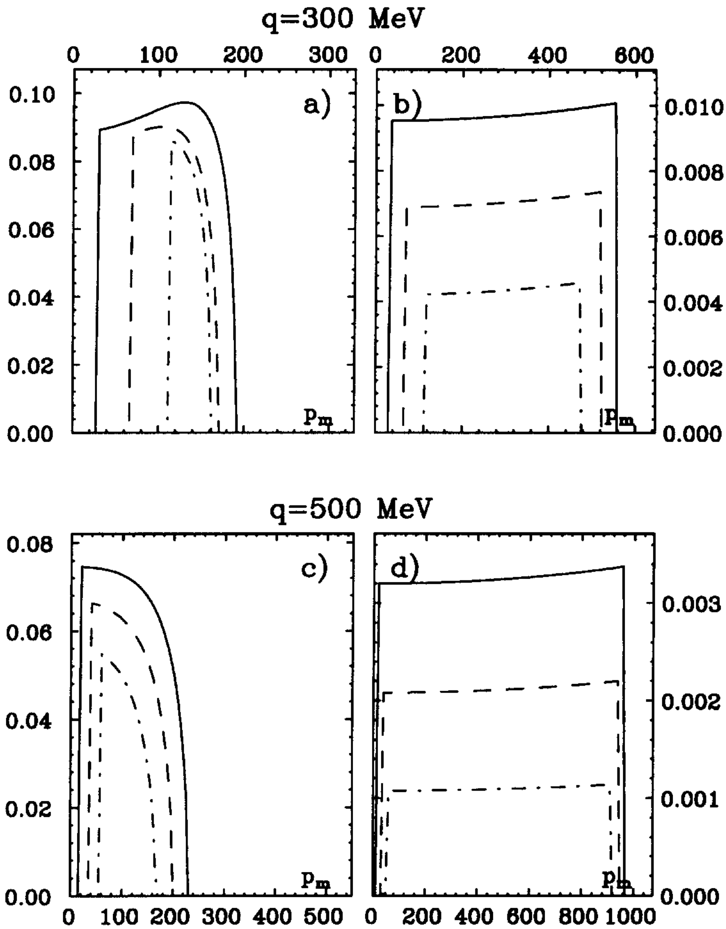

Fig. 6. (a)-(d) Exclusive longitudinal cross sections divided by $\sigma_{\rm Mott}$ as a function of the missing momentum p_m (in MeV/c) at q = 300 MeV/c (a and b) and q = 500 MeV/c (c and d); the vertical scale is in fm^{3}MeV^{-1}; $\omega = q^2/2M_N + E_S$. The eikonal (on the left) and the uniform (on the right) approximations for the distortion operator are used; the harmonic oscillator potential is employed. In all figures curves corresponding to three different values for the missing energy are displayed: $\mathcal{E}$ = 10 MeV (continuous line), $\mathcal{E}$ = 20 MeV (dashed line) and $\mathcal{E}$ = 30 MeV (dot-dashed line).

case the cross-sections tend to peak at low missing momenta; moreover they turn out to be restricted to a rather limited range of p_m, much smaller than the one allowed by pure kinematics.

These outcomes clearly reflect the drastic difference between the functions F characterizing our two models for the distortion operator. However, while the cross-section in the eikonal approximation is quite sensitive to the specific form of the spectral function and, in turn, to the potential well, this out not to be the case for the uniform distortion. The question then arises whether the semi-classical method can be satisfactorily employed in both situations. In the light of the previous discussion the $\hbar$ expansion seems to be more adequate in dealing with physical situations described by the eikonal

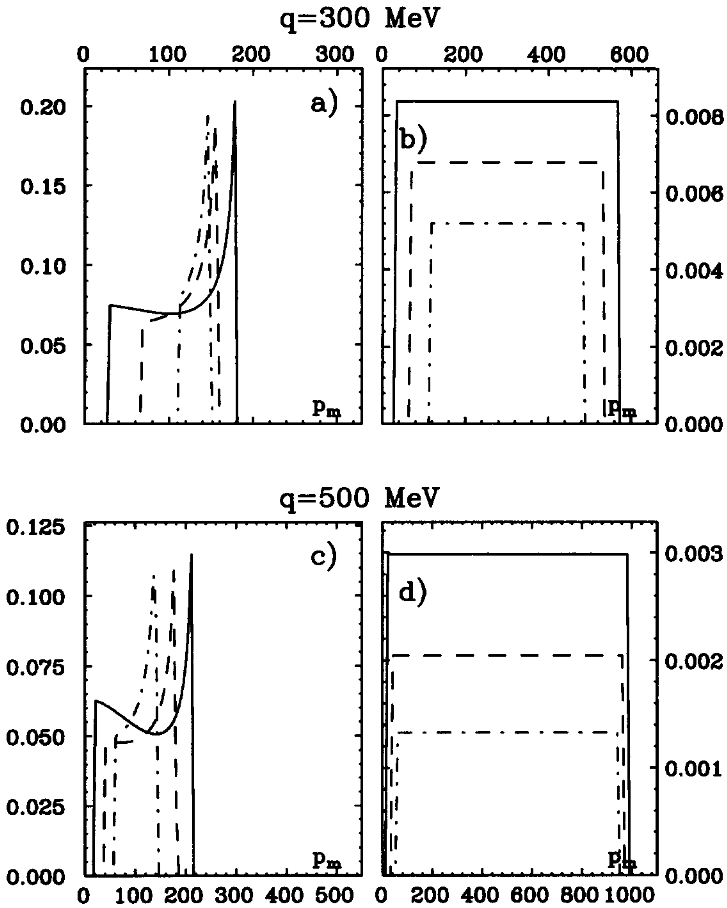

Fig. 7. (a)-(d) The same as in Fig. 6a-d, but employing the Woods-Saxon potential well.

approximation (high energy, small scattering angles) rather than for the ones described by the uniform one, the latter stemming from complex quantum interactions inside the nucleus. Yet in both cases the present approach provides a valuable orientation on the physics of the exclusive processes.

We also notice that the exclusive cross-section in eikonal approximation is quite sensitive to the shape of the potential well: indeed in this case the outgoing particle keeps memory of the initial momentum and hence of the spectral function inside the nucleus. The eikonal cross-sections turn out in fact to reflect quite closely the structure of $S(p,\mathcal{E})$. This is clearly shown in Figs. 8a and 8b (for the harmonic oscillator and the Woods-Saxon well, respectively), where the "distorted" exclusive cross-sections are compared with the corresponding ones evaluated in PWIA, which, according to Eq. (3.10), are directly proportional to the spectral function. In both cases $q = 300$ MeV/c, ω is close to the quasi-elastic peak and $\mathcal{E} = 10$ MeV/c. The PWIA turns out to be similar to the distorted cross-section evaluated in eikonal approximation, but, at least for the kinematics chosen here, the range of missing momenta (p_m) spanned by (3.10) is somewhat larger

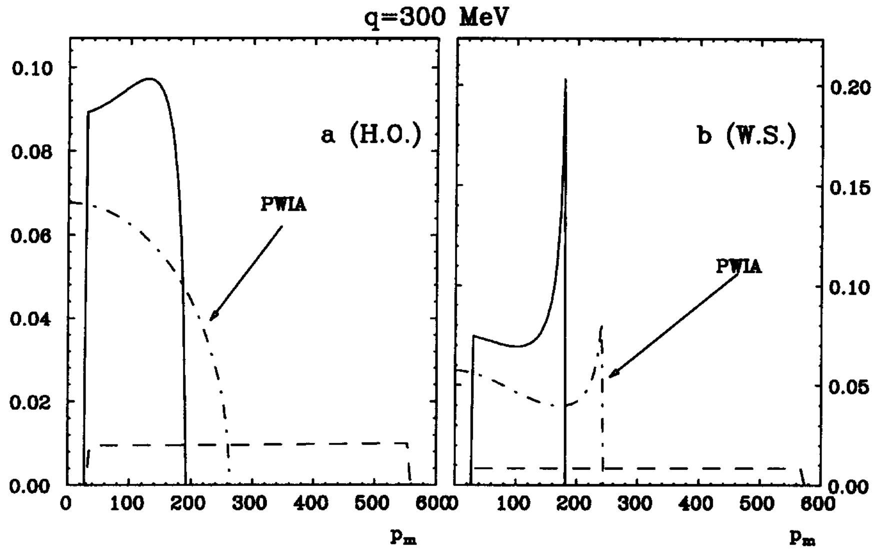

Fig. 8. (a),(b) Exclusive longitudinal cross-sections divided by σ_{Mott} as a function of the missing momentum p_m (in MeV/c), for $q = 300$ MeV/c, $\omega = q^2/2M_N + E_S$, $\mathcal{E} = 10$ MeV, in the eikonal (continuous line) and uniform (dashed line) DWIA, as well as in PWIA (dot-dashed line); the vertical scale is in fm^{3}MeV^{-1}. In (a) the harmonic oscillator potential is employed, in (b) the Woods–Saxon well.

than in the eikonal approximation. Indeed in both cases the outgoing nucleon keeps the direction of the one which has absorbed the virtual photon but, according to (5.10), the modulus of the final nucleon momentum is reduced by the mean field acting through the distortion operator: this, in turn, can lower the maximum allowed missing momentum with respect to the PWIA.

We consider it premature to test our semi-classical approach against either the experiment or quantum calculations before working out a realistic distortion function F, in particular before correcting the "extreme" (eikonal or uniform) assumptions of Section 5. In this connection we have already pointed out the inability of the semi-classical framework to account for specific quantum effects, like the shell structure, which show up in the spectral function. In fact, as already discussed in Section 4, the semi-classical spectral function averages the fluctuations of the quantum one.

Concerning the inclusive cross-sections, they are displayed in Figs. 9a and 9b at $q =$ 200 MeV/c and $q = 500$ MeV/c, respectively, for both the harmonic oscillator and the Woods–Saxon well; they are obtained by integrating in the $(\mathcal{E}, p_m)$ plane the exclusive cross-section, using for the single nucleon cross-section σ_{Mott} only. The calculation has been performed both in the uniform and in the eikonal approximations, which however cannot be distinguished in the figures, owing to their identity. The expected numerical coincidence of the inclusive cross-sections obtained by integrating the two, markedly different, exclusive cross-sections is thus seen to be realized, providing that higher order relativistic effects in the single nucleon cross-section are ignored (this amounts to keep only the Mott cross-section). When however the fully relativistic expression for the single nucleon cross-section, Eq. (3.7), is employed, then the above

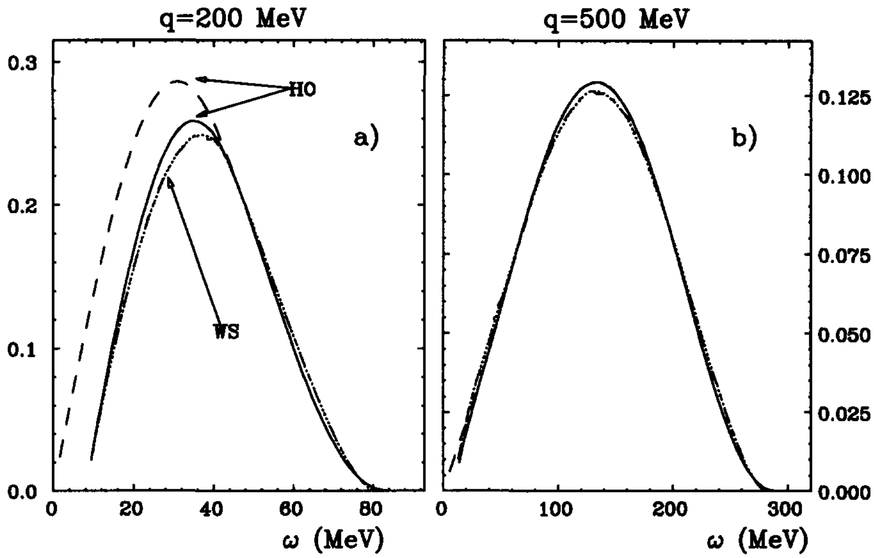

Fig. 9. (a),(b) Inclusive longitudinal cross-sections divided by $\sigma_{\rm Mott}$ at q = 200 MeV/c (a) and q = 500 MeV/c (b), as a function of the energy transfer ω (in MeV): the single nucleon cross section coincides with $\sigma_{\rm Mott}$; the vertical scale is in MeV^{-1}. The continuous and dashed (coincident) lines refer to the integral of the exclusive cross-section in eikonal and uniform approximation, respectively, using the harmonic oscillator potential. The dot-dashed (eikonal) and dotted (uniform) lines are obtained with the Woods-Saxon well. The long-dashed line is the direct evaluation (with the harmonic oscillator potential) of the inclusive cross-section, through the polarization propagator.

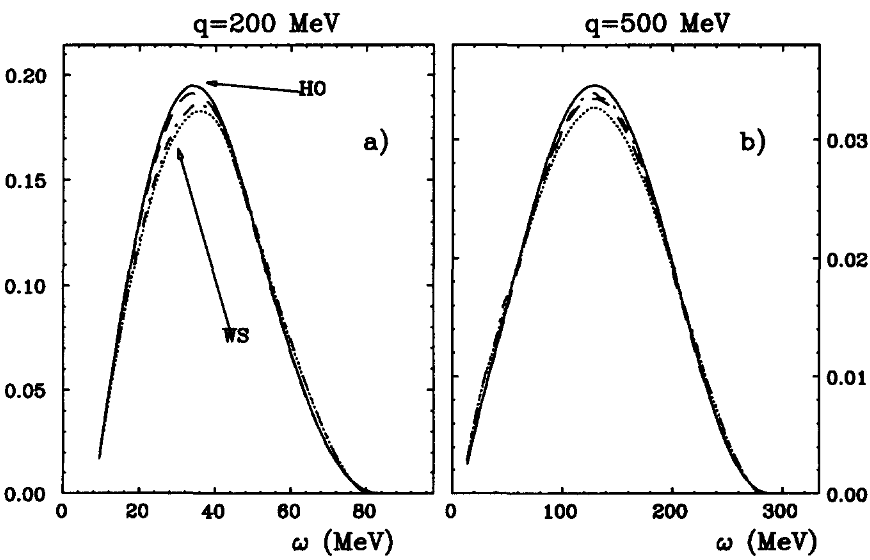

Fig. 10. (a),(b) The same as in Fig. 9a,b but using the fully relativistic single nucleon cross section (and omitting the direct calculation).

mentioned coincidence between inclusive cross-sections no longer holds, especially near the maximum, but the discrepancy remains mild up to $q = 500$ MeV/c, as illustrated in Figs. 10a and 10b, which exhibit the same cases as in Fig. 9a and 9b.

Also displayed in Figs. 9a and 9b are the inclusive cross-sections obtained through the semi-classical expression for the polarization propagator. These differ in the low energy side from the cross sections produced by the integration of the exclusive processes in the missing energy-missing momentum plane: at low q the difference is sizeable, but it becomes negligible as q increases. In general the inclusive cross sections obtained through the polarization propagator turn out to be larger. This difference arises from the bound unoccupied orbits, which are embodied in the polarization propagator, but are not allowed to the outgoing nucleon in exclusive processes. The effect disappears at large q, as it should.

As a consequence of these findings, we confirm that little can be learned from the inclusive electron scattering, treated within the mean field approximation, about the mechanism responsible for the distortion of the outgoing nucleon wave [18]. Furthermore only a very weak dependence of the inclusive cross-sections on the shape of the potential well is observed. Accordingly one is lead to conclude that while the nucleon-nucleon correlations (of short and long range) in the initial state affect both the inclusive and exclusive inelastic electron scattering, other details of the dynamics of the emitted nucleon, mainly reflecting the shape of the mean field and how it affects the way of the nucleon out of the nucleus, are more conveniently studied with exclusive processes.

9. Conclusions

In this paper we have applied the semi-classical approach to the exclusive electron scattering, by addressing two basic questions. First we wished to explore whether the semi-classical method can also be applied to processes connected with more detailed aspects of nuclear structure than those entering into the inclusive scattering. We did not expect the semi-classical method to be able of encompassing finer features of an exclusive process (this, after all, does not happen even for the inclusive scattering at low frequencies).

Having however recognized that the semi-classical approach provides a valid average description at least for the nuclear quasi-elastic scattering, we wished to go beyond the PWIA accounting for the FSI in the mean field approximation, including antisymmetrization. In this connection we succeeded in setting up the formalism, in leading order of the $\hbar$ expansion, in a way suitable for the treatment of this physics. In order to provide a preliminary illustration of how the formalism works, we have resorted to two admittedly approximate, actually extreme, models to account for the mechanism of the distortion of the outgoing nucleon wave.

A few outcomes of the present study are worth to be recalled. The first one is the strong sensitivity of the exclusive process to the distortion of the outgoing nucleon wave:

the stronger is the latter, the weakest are the remnants of the nuclear mean field in the exclusive cross-section, as it happens for the rather extreme situation described by the uniform approximation. On the other hand a "weaker" distortion, like the one entailed by the eikonal approximation, yields cross-sections more strictly related to the spectral function and hence closer to the PWIA, as it should: in this framework we have shown that the exclusive cross-sections are in fact quite different in shape, whether we adopt the harmonic oscillator or the Woods–Saxon well for the nuclear mean field.

A second result relates to the strong correlation between the distortion mechanism and the domain, in the missing energy - missing momentum plane, where the exclusive cross-section exists. We have found that in the eikonal approximation this domain is quite restricted, whereas in the uniform one it essentially covers the whole kinematically allowed region.

Concerning the inclusive cross-section, obtained by integrating the exclusive ones in the appropriate domain of the $(\mathcal{E}, p_m)$ plane, we have shown that they are quite insensitive both to the distortion of the outgoing nucleon, as we prove also with an analytic calculation, and to the specific shape of the mean binding field, in agreement with previous direct evaluations and measurements of the quasi-elastic inclusive process.

Finally we have observed, in the inclusive cross-section, a mild dependence upon the distortion mechanism when higher order relativistic effects are taken into account in the expression of the single nucleon cross-section: this effect, which is modest up to the largest momenta (500–600 MeV/c) considered here, signals the need of a fully consistent relativistic approach, both in the currents and in the mean field utilized to describe the nucleon dynamics inside the nucleus, especially when more extreme kinematical conditions, typical of the experiments planned at CEBAF, should be accounted for

Although we have found that the semi-classical method yields sensible results, the validity of the scheme can only be ultimately assessed by testing it against both the experiment and fully quantum mechanical calculations.

Yet we consider the semi-classical method attractive by itself on three counts, namely:
(i) for its simplicity,
(ii) for allowing to grasp the role of the FSI in a remarkably transparent way (this, as we have seen, follows either by comparing the eikonal with the uniform approximations for the distortion or by a comparison with the PWIA),
(iii) for retaining in the exclusive process the simplicity of the Fermi gas model but taking into account the local features of the nuclear mean field through the folding of Fermi gases at different densities. Quantum mechanical interferences between different points in the nucleus, not considered here, occur in the second and higher orders of the $\hbar$ expansion.

With reference to point 2 we note that in our formalism the mechanism of the distortion is embedded into the function $F(\widehat{p}_N, \widehat{p})$ of Section 5, whereas in the traditional (quantum mechanical) approaches it is included in the distorted wave function of the outgoing nucleon. Clearly, as previously stated, before attempting to test the semi-classical approach against the experimental data, a realistic expression for the function

F should be developed: indeed the ones discussed in this paper were just meant as an illustration.

Concerning point 3 it is worth pointing out the correspondence with the quantum mechanical approach of Ref. [2], where a different Fermi gas is associated with each individual orbit of the nucleons inside the nucleus rather than with elementary volumes into which the nucleus is ideally split.

In the present work we have left out important dynamical effects arising from

(i) the nucleon–nucleon correlations
(ii) the meson exchange currents
(iii) the two-step processes, which occur when the nucleon which has absorbed the photon scatters, in its way out, with another one in the nucleus or when a second nucleon is emitted from the excited daughter nucleus and comes out almost simultaneously to the one directly hit by the impinging photon. These mechanisms can of course contribute to both the $(e,e'p)$ and to the $(e,e'NN)$ exclusive cross-sections.

Undoubtedly these processes can be, and in fact have partly been, treated in a fully quantum mechanical framework [19,20]. Yet, as in the case of the FSI in the mean field approximation, we believe that the semi-classical method can be advantageously employed in dealing also with this far from simple physics to gain at least an orientation on its role in shaping the remarkably complex structure of the exclusive response in the missing energy-missing momentum plane.

Appendix A

We calculate here explicitly the diagonal semi-classical spectral function for a few typical potential wells. We use natural units, $\hbar = c = 1$.

A.1. Square potential well

Let us call $V_0(>0)$ the depth of the well and $\bar{R}$ its range:

$$V(R) = \begin{cases} -V_0 & \text{for} \quad 0 \leqslant R \leqslant \bar{R}, \\ 0 & \text{for} \quad R > \bar{R}. \end{cases} \tag{A.1}$$

Then one gets for the Fermi momentum, only defined for $R < \bar{R}$,

$$k_F = \sqrt{2M_N^*(\epsilon_F + V_0)}. \tag{A.2}$$

An easy calculation yields then for the spectral function the following expression

$$S_{sw}(p,\mathcal{E}) = \tfrac{4}{3}\pi\bar{R}^3\, \theta\left(\sqrt{2M_N^*(\epsilon_F + V_0)} - p\right) \delta\left[\mathcal{E} - \left(\epsilon_F + V_0 - \frac{p^2}{2M_N^*}\right)\right], \tag{A.3}$$

which looks indeed like the FG one.

The support of (A.3) in the $(\mathcal{E}, p)$ plane is a line. Note that

$$4\int_0^\infty d\mathcal{E}\, S_{\rm sw}(p,\mathcal{E}) = 4\tfrac{4}{3}\pi\bar{R}^3\theta\left(\sqrt{2M_N^*(\epsilon_{\rm F}+V_0)}-p\right) \tag{A.4}$$

provides the momentum distribution of the nucleons inside the nucleus, while

$$4\int\frac{dp}{(2\pi)^3}\int d\mathcal{E}\, S_{\rm sw}(p,\mathcal{E}) = \tfrac{4}{3}\pi\bar{R}^3\frac{2k_{\rm F}^3}{3\pi^2} = A, \tag{A.5}$$

[with $k_{\rm F}$ given by (A.2)] is the number of nucleons, the factor of 4 accounting for the spin-isospin degeneracy. Of course the normalisation (A.5) holds valid if the Fermi energy fulfills the condition

$$\epsilon_{\rm F} = \frac{1}{8}\frac{(9\pi A)^{2/3}}{M_N^*\bar{R}^2} - V_0. \tag{A.6}$$

A.2. Harmonic oscillator potential well

With the potential (4.16) one easily derives the spectral function

$$S_{\rm ho}(p,\mathcal{E}) = \sqrt{2}\frac{4\pi}{(M_N^*\omega_0^2)^{3/2}}\theta(\mathcal{E})\sqrt{\epsilon_{\rm F}+V_0-p^2/2M_N^*-\mathcal{E}}, \tag{A.7}$$

whose support is no longer a line in the $(\mathcal{E}, p)$ plane, but rather a two-dimensional domain bound by the curve

$$\mathcal{E} = \epsilon_{\rm F} + V_0 - p^2/2M_N^*, \tag{A.8}$$

the range of the allowed momenta being

$$0 \leqslant p \leqslant p_{\rm ho}^{\rm max}, \tag{A.9}$$

with

$$p_{\rm ho}^{\rm max} = \sqrt{2M_N^*(\epsilon_{\rm F}+V_0)}. \tag{A.10}$$

Although (A.8) is reminiscent of the Fermi gas, the actual support of $S_{\rm ho}$ is more closely related to the one of a quantum mechanical harmonic oscillator. Indeed in the latter case the spectral function is substantially different from zero only in the region where the semi-classical $S_{\rm ho}(p,\mathcal{E})$ does not vanish, although it lives only on a set of parallel lines corresponding to the discrete harmonic oscillator eigenvalues.

The corresponding momentum distribution reads

$$\begin{aligned}4\int_0^\infty d\mathcal{E}\, S_{\rm ho}(p,\mathcal{E}) &= \frac{16\pi}{3}\frac{1}{M_N^*\omega_0^3}\left(2M_N^*(\epsilon_{\rm F}+V_0)-p^2\right)^{3/2}\\ &= \frac{16\pi}{3}\frac{1}{M_N^{*3}\omega_0^3}\left[\left(p_{\rm ho}^{\rm max}\right)^2-p^2\right]^{3/2},\end{aligned} \tag{A.11}$$

with the normalization

$$4\int \frac{dp}{(2\pi)^3} d\mathcal{E} S_{\rm ho}(p,\mathcal{E}) = \frac{2}{3}\left(\frac{\epsilon_{\rm F}+V_0}{\omega_0}\right)^3 = A, \tag{A.12}$$

which fixes $\epsilon_{\rm F}$.

A.3. Woods–Saxon potential well

The semi-classical spectral function associated with the Woods–Saxon well (4.17) reads:

$$S_{\rm WS}(p,\mathcal{E}) = 4\pi R_\delta^2 \theta\left[\varepsilon_{\rm F} - \frac{p^2}{2M_N} - V(R_\delta)\right]\frac{\theta(R_\delta)}{|dV/dR|_{R=R_\delta}}, \tag{A.13}$$

where

$$R_\delta = R_0 + a\ln\left(\frac{V_1}{\mathcal{E}+p^2/2M_N - \varepsilon_{\rm F}} - 1\right) \tag{A.14}$$

and dV/dR is the derivative of the Woods–Saxon well (4.17).

The corresponding momentum distribution and normalization condition cannot be expressed analytically, but must be numerically evaluated.

A.4. Power-law potential well

For the potential

$$V(R) = -V_0 + (\epsilon_{\rm F}+V_0)\left(\frac{R}{R_{\rm c}}\right)^n, \tag{A.15}$$

V_0 and $R_{\rm c}$ being positive constants and $n = 1,2,3,\ldots$, one finds

$$S_{\rm plw}(p,\mathcal{E}) = \tfrac{4}{3}\pi R_{\rm c}^3 \frac{3}{n(\epsilon_{\rm F}+V_0)}\left(1 - \frac{\mathcal{E}+p^2/2M_N^*}{\epsilon_{\rm F}+V_0}\right)^{3/n-1} \times\theta\left[\epsilon_{\rm F}+V_0 - \left(\mathcal{E}+\frac{p^2}{2M_N^*}\right)\right], \tag{A.16}$$

the Fermi energy being linked to the potential through the relation (4.6).

As in the previous instances $S_{\rm plw}(p,\mathcal{E})$ lives in a domain of the plane $(\mathcal{E},p)$ bound by the curve

$$\mathcal{E} = \epsilon_{\rm F} + V_0 - \frac{p^2}{2M_N^*} \tag{A.17}$$

on which it vanishes, and within the range of momenta

$$0 \leqslant p \leqslant p_{\rm plw}^{\rm max}, \tag{A.18}$$

with, as before,

$$p_{\rm plw}^{\rm max} = \sqrt{2M_N^*(\epsilon_{\rm F} + V_0)}. \tag{A.19}$$

Note that for $n = 3$ the spectral function is constant. Furthermore it becomes closer and closer to the one of the square well (and, hence, of the FG) as n becomes large.

The momentum distribution is given by

$$\int_0^\infty d\mathcal{E}\, S_{\rm plw}(p,\mathcal{E}) = \frac{16\pi R_c^3}{3}\left(\frac{\epsilon_{\rm F} + V_0 - p^2/2M_N^*}{\epsilon_{\rm F} + V_0}\right)^{3/n}, \tag{A.20}$$

with normalization

$$\int \frac{d\boldsymbol{p}}{(2\pi)^3}\int_0^\infty d\mathcal{E} S_{\rm plw}(p,\mathcal{E}) = \tfrac{4}{3}(p_{\rm plw}^{\rm max}R_c)^3 B\left(\frac{3}{2},\frac{3}{n}+1\right), \tag{A.21}$$

B being the Euler function of second kind, and yields the nucleons' number A when the Fermi energy is fixed according to

$$\epsilon_{\rm F} = \frac{1}{2M_N^* R_c^2}\left(\frac{3\pi A}{4B(3/2,3/n+1)}\right)^{2/3} - V_0. \tag{A.22}$$

We notice that (A.15) interpolates between various forms of the mean field.

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