On the Stability Criterion in a Saturated Atmosphere

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ABSTRACT

The expression of the moist buoyancy frequency indicates that it is not completely correct to use the moist adiabatic lapse rate as a static stability parameter of a saturated atmosphere, because the contribution of both the water vapor and condensed water gradients is ignored. This paper shows that the vapor gradient effect is usually larger than the condensed water one, and can be taken into account by introducing a new temperature lapse rate as a stability parameter. The new lapse rate is always greater than the moist adiabatic one, implying an increase of the atmospheric stability estimate at any temperature and for both liquid water and ice saturation conditions.

1. Introduction

It has been observed in the past (Lalas and Einaudi 1974; Durran and Klemp 1982) that the replacement of the dry adiabatic by the saturated adiabatic lapse rate in studying the static stability of a saturated atmosphere is inadequate, expecially when the stability is small. The reason comes from the expression of the moist buoyancy frequency (Durran and Klemp 1982), that by splitting the total water mixing ratio gradient dr_{tw}/dz into its vapor and liquid components gives

$$N_w^2 = \frac{g}{T}\left(1 + \frac{L_v r_w}{R_d T}\right)\left(\frac{dT}{dz} + \Gamma_w\right) - \frac{g}{1 + r_{tw}}\frac{dr_w}{dz} - \frac{g}{1 + r_{tw}}\frac{dr_l}{dz}. \quad (1)$$

The purpose of this note is to show that the gradient of saturated vapor profile in the second term on the rhs can be written in terms of the lapse rate, and (1) can then be expressed as

$$N_w^2 = \frac{g}{T}\left[1 + \frac{L_v r_w}{R_d T}\left(1 - \frac{\epsilon + r_w}{1 + r_{tw}}\right)\right]\left(\frac{dT}{dz} + \Gamma_w^*\right) - \frac{g}{1 + r_{tw}}\frac{dr_l}{dz}. \quad (2)$$

Thus Γ_w^* is the correct lapse to assess the stability of a saturated atmosphere for which the liquid water gradient is insignificant (a reasonable approximation in many realistic situations). The new stability parameter is derived in section 2 and is compared with Γ_w in section 3. Section 4 focuses then on the effect of the liquid water gradient.

2. Derivation of the stability parameter Γ_w^*

The following notation has been used in this work: subscripts d, v, w, l, tw indicate that the concerned quantities refer, respectively, to dry air, vapor, saturated vapor, condensed phase (liquid or solid), total water (saturated vapor and condensed phase); ρ denotes the density; r the mixing ratio; T the temperature; g the gravity acceleration; z the height; p_d the dry air partial pressure; e_w the saturation pressure of vapor over liquid water (replaced by e_i if saturation is over ice); and $p = p_d + e_w$ the total pressure. Moreover $\varepsilon = R_d/R_v$ is the ratio between the gas constants of dry air and water vapor; L_v is the specific latent heat of vaporization (replaced by the sublimation one if saturation is over ice); c_p is the specific heat at constant pressure; and Γ_d and Γ_w have the usual meaning of the dry and saturated adiabatic lapse rates, respectively.

In (1), with $r_w = \varepsilon e_w/p_d$, the gradient of the saturation mixing ratio of water vapor can be derived as

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$$\frac{dr_w}{dz} = r_w\left(\frac{1}{e_w}\frac{de_w}{dz} - \frac{1}{p_d}\frac{dp_d}{dz}\right). \qquad (3)$$

By multiplying now by $1/p_d = r_w/(\varepsilon e_w)$ the hydrostatic equation $d(p_d + e_w)/dz = -g\rho_d(1 + r_{tw})$ and by taking into account $p_d = R_d\rho_d T$, the last term of (3) can be expressed as

$$\frac{1}{p_d}\frac{dp_d}{dz} = -\frac{r_w}{\epsilon e_w}\frac{de_w}{dz} - \frac{g}{R_d T}(1 + r_{tw}). \qquad (4)$$

By substituting it into (3) and using the Clapeyron equation $(1/e_w)de_w/dz = L_v R_v^{-1}T^{-2}dT/dz$ we obtain

$$\frac{dr_w}{dz} = \frac{r_w}{R_d T}\left[\frac{L_v}{T}(\epsilon + r_w)\frac{dT}{dz} + g(1 + r_{tw})\right]. \qquad (5)$$

If (5) is inserted in (1), the expression for the square of the moist buoyancy frequency in a saturated atmosphere becomes

$$N_w^2 = \frac{g}{T}\left\{\left[1 + \frac{L_v r_w}{R_d T}\left(1 - \frac{\epsilon + r_w}{1 + r_{tw}}\right)\right]\frac{dT}{dz} + \left(1 + \frac{L_v r_w}{R_d T}\right)\Gamma_w - \frac{g r_w}{R_d}\right\} - \frac{g}{1 + r_{tw}}\frac{dr_l}{dz}. \qquad (6)$$

Then (2) is obtained by defining

$$\Gamma_w^* \equiv \frac{\left(1 + \frac{L_v r_w}{R_d T}\right)\Gamma_w - \frac{g r_w}{R_d}}{1 + \frac{L_v r_w}{R_d T}\left(1 - \frac{\epsilon + r_w}{1 + r_{tw}}\right)}. \qquad (7)$$

In this way (2) is equivalent to (1) and correctly reproduces the results of Durran and Klemp (1982) when $r_l = 0$. In this form the atmospheric lapse rate appears only explicitly, making it easier to evaluate the moist buoyancy frequency.

3. Lapse rates intercomparison

The saturated adiabatic lapse rate is given (Lalas and Einaudi 1974; Durran and Klemp 1982) by

$$\Gamma_w = \Gamma_d\frac{\left(1 + \frac{L_v r_w}{R_d T}\right)(1 + r_w + r_l)}{1 + \frac{c_{pv} r_w + c_l r_l}{c_{pd}} + \frac{L_v^2(\epsilon + r_w)r_w}{c_{pd} R_d T^2}}. \qquad (8)$$

By neglecting the contribution of the liquid phase, and with $c_{pv} \ll L_v^2\varepsilon/(R_d T^2)$, it can be simplified to the first order in r_w as

$$\Gamma_w \approx \Gamma_{w1} \equiv \Gamma_d\frac{1 + \frac{L_v r_w}{R_d T}}{1 + \frac{L_v^2 \epsilon r_w}{c_{pd} R_d T^2}}, \qquad (9)$$

a formula that is often found (sometimes using $r_w \approx \varepsilon e_w/p$) in meteorological books (Tverskoi 1965; Bluestein 1992).

By substituting (8) into (7) the new stability parameter can be written as

$$\Gamma_w^* = \Gamma_w\frac{\left(1 + \frac{L_v r_w}{R_d T}\right)(1 + r_w + r_l) - \dfrac{r_w\left[c_{pd} + c_{pv} r_w + c_l r_l + \frac{L_v^2 r_w}{R_d T^2}(\epsilon + r_w)\right]}{R_d\left(1 + \frac{L_v r_w}{R_d T}\right)}}{\left(1 + \frac{L_v r_w}{R_d T}\right)(1 + r_w + r_l) - \frac{L_v r_w}{R_d T}(\epsilon + r_w)}. \qquad (10)$$

This expression is of the fifth order in r_w. It can be truncated at the first order giving a formula similar to (9) but with the term $(L_v r_w)/(R_d T)$ at the numerator multiplied by 2. This is, however, a bad approximation, because the error that is introduced is higher than using Γ_w instead of Γ_w^*. A better, empirical, approximation is given by

$$\Gamma_w^* \approx \Gamma_{w1}^* \equiv \Gamma_d\frac{1 + \alpha\frac{L_v r_w}{R_d T}}{1 + \beta\frac{L_v^2 \epsilon r_w}{c_{pd} R_d T^2}}, \qquad (11)$$

where the values $\alpha = 1.18$ and $\beta = 0.93$ have been chosen with an optimization procedure in order to minimize the errors in the range of pressures and temperatures typical of the standard atmosphere.

From (10) it can be observed that $\Gamma_w^* > \Gamma_w$ when

$$\frac{c_{pd} + c_{pv} r_w + c_l r_l + \frac{L_v^2 r_w}{R_d T^2}(\epsilon + r_w)}{1 + \frac{L_v r_w}{R_d T}} < \frac{L_v}{T}(\epsilon + r_w),$$

TABLE 1. Values of $a = \Gamma^*_w$, $b = \Gamma_w - \Gamma^*_w$, $c = \Gamma^*_{w1} - \Gamma^*_w$ as functions of pressure and temperature in a saturated (over pure liquid water) atmosphere assuming $r_l = 0$. (Units are °C km^{-1}.)

Pressure (hPa)		Temperature (°C)										
		−50	−40	−30	−20	−10	0	10	20	30	40	50
100.	a	8.94	7.92	6.48	5.01	3.90	3.15	2.65	2.29	1.97	1.63	1.20
	b	−0.08	−0.18	−0.33	−0.47	−0.54	−0.53	−0.44	−0.33	−0.22	−0.13	−0.07
	c	0.02	0.02	0.00	−0.04	−0.06	0.01	0.16	0.37	0.65	1.00	1.50
200.	a	9.32	8.72	7.67	6.32	5.01	4.01	3.31	2.83	2.47	2.17	1.86
	b	−0.04	−0.10	−0.21	−0.36	−0.49	−0.56	−0.55	−0.47	−0.37	−0.26	−0.17
	c	0.01	0.02	0.02	−0.01	−0.05	−0.06	0.01	0.15	0.35	0.60	0.92
400.	a	9.53	9.20	8.55	7.53	6.30	5.12	4.20	3.54	3.06	2.71	2.41
	b	−0.02	−0.05	−0.12	−0.24	−0.38	−0.50	−0.57	−0.57	−0.51	−0.42	−0.32
	c	0.01	0.01	0.02	0.02	−0.01	−0.05	−0.05	0.00	0.12	0.30	0.53
600.	a	9.61	9.38	8.91	8.11	7.03	5.86	4.84	4.05	3.48	3.07	2.74
	b	−0.01	−0.04	−0.09	−0.18	−0.30	−0.44	−0.54	−0.59	−0.57	−0.51	−0.42
	c	0.00	0.01	0.02	0.02	0.00	−0.03	−0.05	−0.04	0.03	0.17	0.36
800.	a	9.64	9.47	9.10	8.45	7.51	6.40	5.34	4.47	3.83	3.36	2.99
	b	−0.01	−0.03	−0.07	−0.14	−0.25	−0.39	−0.51	−0.58	−0.59	−0.56	−0.48
	c	0.00	0.01	0.01	0.02	0.01	−0.01	−0.05	−0.05	−0.01	0.09	0.25
1000.	a	9.67	9.52	9.22	8.68	7.84	6.80	5.74	4.83	4.12	3.60	3.21
	b	−0.01	−0.02	−0.06	−0.12	−0.22	−0.34	−0.47	−0.56	−0.60	−0.59	−0.53
	c	0.00	0.01	0.01	0.02	0.01	0.00	−0.04	−0.05	−0.03	0.04	0.17

that is to say when $c_{pd} + c_{pv} r_w + c_l r_l < L_v(\varepsilon + r_w)/T$. This inequality is always satisfied, and so a stability criterion based on Γ^*_w always gives a greater stability than by using Γ_w. This is the consequence of the stabilizing effect of the vapor profile, as it will be showed now by means of the parcel theory.

Following Durran and Klemp (1982), N^2_w can be expressed as $gd(\ln\rho)/dz\,|^p_e$, where ρ is the total density and the subscripts p and e denote values for the parcel and the environment. By writing ρ as $\rho = \rho_d(1 + r_{tw})$, and with

$$\rho_d = \frac{p}{R_d T\left(1 + \dfrac{r_w}{\epsilon}\right)}, \tag{12}$$

N^2_w becomes

$$N^2_w = \frac{g}{\rho_d}\frac{d\rho_d}{dz}\bigg|^p_e + \frac{g}{1 + r_{tw}}\frac{dr_{tw}}{dz}\bigg|^p_e. \tag{13}$$

By now taking into account that the parcel is subject to a saturated adiabatic process in pressure equilibrium with the environment, and by using (3) and the Clapeyron equation, the first term in (13) can be transformed as

$$\frac{g}{\rho_d}\frac{d\rho_d}{dz}\bigg|^p_e = \frac{g}{T}\left(1 + \frac{L_v r_w}{R_d T}\right)\frac{dT}{dz}\bigg|^e_p, \tag{14}$$

that is to say the first term of (1). The total water content being conserved in the parcel, the second term in (13) takes the form of the second and third terms in (1).

The use of Γ_w as a stability parameter of a saturated atmosphere is therefore incorrect because it relies uniquely on the density variations that happen both in the parcel and in the environment and are proportional to $(dT/dz)_p = -\Gamma_w$ and $(dT/dz)_e = dT/dz$, respectively. The terms that are neglected rely, on the contrary, on the density variations that occur only in the environment, being proportional to dr_{tw}/dz, which is null in the parcel if there is no precipitation.

In a saturated atmosphere the water vapor mixing ratio r_w usually decreases with the height, due to the dominant influence of temperature on the saturation vapor pressure. The water vapor profile has therefore a stabilizing effect, implying $\Gamma^*_w > \Gamma_w$. The liquid water profile, on the contrary, can either increase or decrease the stability, depending on whether dr_l/dz is negative or positive.

The values of Γ^*_w and the errors that one makes by using Γ_w or the approximation Γ^*_{w1} are listed in Tables 1 and 2 for various values of pressure and temperature, neglecting the contribution of the condensed water mixing ratio. The tables of Iribarne and Godson (1981, appendix) have been employed for the thermodynamic properties of air and vapor, and the standard value $g = 9.80665$ m s^{-2} has been used.

Numerical values of the differences $(\Gamma_w - \Gamma^*_w)$ in Tables 1 and 2 confirm that a stability criterion based on Γ^*_w always gives a greater stability than the one obtained by using Γ_w. At a given pressure the upper values of $|\Gamma_w - \Gamma^*_w|$ are of about 0.6°C km^{-1} over water and 0.5°C km^{-1} over ice. Over water the temperature at which this upper value occurs decreases with pressure; over ice it occurs almost always at 0°C. Also Γ^*_w decreases with the pressure, and therefore the maximum value of the relative error that one makes by using Γ_w instead of Γ^*_w remains almost constant, around 17%. From a practical point of view an error of 0.6°C km^{-1} is negligible if compared with the uncertainties of measurements, but from a theoretical point of view and for modeling purposes it is more correct to use Γ^*_w.

TABLE 2. Values of $a = \Gamma^*_w$, $b = \Gamma_w - \Gamma^*_w$, $c = \Gamma^*_{w1} - \Gamma^*_w$ as functions of pressure and temperature in a saturated (over pure ice) atmosphere assuming $r_l = 0$. (Units are °C km^{-1}.)

Pressure (hPa)		Temperature (°C) −80	−70	−60	−50	−40	−30	−20	−10	0
100.	a	9.75	9.70	9.56	9.14	8.17	6.58	4.85	3.54	2.73
	b	0.00	0.00	−0.02	−0.05	−0.14	−0.29	−0.44	−0.51	−0.47
	c	0.00	0.00	0.01	0.02	0.03	0.01	−0.04	−0.06	0.01
200.	a	9.75	9.73	9.66	9.44	8.88	7.77	6.18	4.62	3.48
	b	0.00	0.00	−0.01	−0.03	−0.08	−0.18	−0.33	−0.47	−0.52
	c	0.00	0.00	0.00	0.01	0.02	0.03	0.00	−0.05	−0.06
400.	a	9.75	9.74	9.71	9.59	9.29	8.62	7.43	5.92	4.52
	b	0.00	0.00	0.00	−0.01	−0.04	−0.11	−0.22	−0.37	−0.49
	c	0.00	0.00	0.00	0.01	0.01	0.02	0.02	−0.01	−0.06
600.	a	9.76	9.75	9.72	9.65	9.44	8.96	8.04	6.70	5.25
	b	0.00	0.00	0.00	−0.01	−0.03	−0.07	−0.17	−0.30	−0.44
	c	0.00	0.00	0.00	0.00	0.01	0.02	0.02	0.01	−0.03
800.	a	9.76	9.75	9.73	9.67	9.52	9.14	8.39	7.21	5.80
	b	0.00	0.00	0.00	−0.01	−0.02	−0.06	−0.13	−0.25	−0.39
	c	0.00	0.00	0.00	0.00	0.01	0.02	0.02	0.02	−0.02
1000.	a	9.76	9.75	9.74	9.69	9.56	9.26	8.62	7.57	6.22
	b	0.00	0.00	0.00	−0.01	−0.02	−0.05	−0.11	−0.22	−0.36
	c	0.00	0.00	0.00	0.00	0.01	0.01	0.02	0.02	−0.01

Tables 1 and 2 show also that in the range of pressures and temperatures of meteorological interest the first-order empirical approximation (11) is a good approximation of Γ^*_w. It produces an error that is always lower than the one obtained by using Γ_w, and its similarity with the commonly used (9) makes it useful to quickly improve the accuracy of the stability estimates by simply turning the expression (9) into (11).

4. Influence of the liquid water profile

As mentioned above, the liquid water profile can either increase or decrease the stability, depending on whether dr_l/dz is negative or positive.

In clouds at any given level the water content varies considerably over short distances in a manner that is closely related to the variation of the vertical air velocity; typically, on the macroscale, the cloud water content increases with height above the cloud base, has a maximum somewhere in the upper half of the cloud, and then decreases again toward the cloud top (Pruppacher and Klett 1980). On the local scale the effect of the dr_l/dz term can therefore be as large as the effect of the dr_w/dz term retained here by introducing Γ^*_w, or even larger, and is quite unpredictable. On a wider scale, however, a rough estimate of their mutual importance can be given.

Figure 1 shows the vertical profiles of dr_w/dz and dr_l/dz calculated in a standard atmosphere assuming that the liquid content ρ_l increases linearly from 0 to 1 g

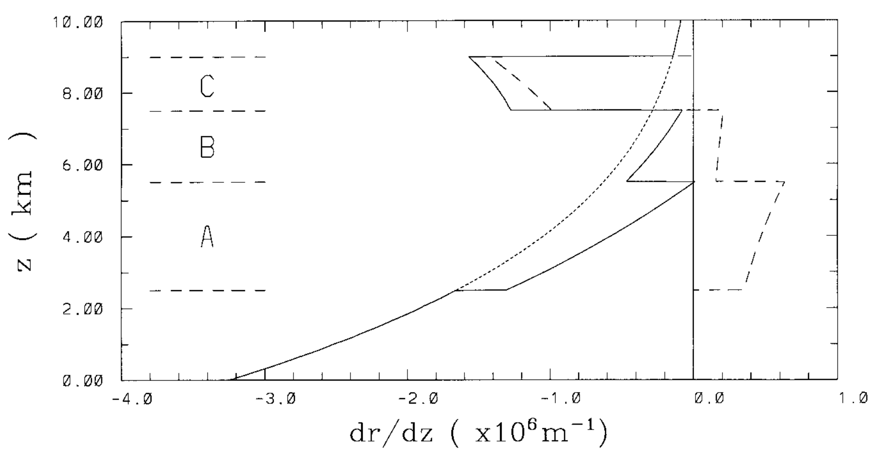

FIG. 1. Vertical profiles of dr_w/dz (dotted line), dr_l/dz (dashed line), and $dr_{tw}/dz = dr_w/dz + dr_l/dz$ (continuous line) in an idealized situation [standard atmosphere with a cumulus, assuming that the liquid content increases linearly from 0 to 1 g m^{-3} from 2.5 to 5.5 km (zone A), keeps constant up to 7.5 km (zone B), and then decreases linearly to zero between 7.5 and 9 km (zone C)]. The line $dr/dz = 0$ has been plotted as a reference.

m^{-3} from 2.5 to 5.5 km (zone A), keeps constant up to 7.5 km (zone B), and then decreases linearly to zero between 7.5 and 9 km (zone C); this schematic profile has been derived from an example concerning the cumulus clouds (Pruppacher and Klett 1980, their Fig. 2-13b). In Fig. 1 the dr_w/dz curve is almost always closer to the dr_{tw}/dz one than $dr/dz = 0$; therefore, in this example, the estimate of the atmospheric stability by using Γ_w^* (i.e., implicitely assuming $dr_{tw}/dz = dr_w/dz$) would be almost always more accurate than by using Γ_w (i.e., assuming $dr_{tw}/dz = 0$). Even if $d\rho_l/dz$ were doubled and the cloud extended to the ground this result would not change dramatically. Only at higher altitudes would Γ_w give a better prediction, but if one considers that when the liquid water gradient is negligible Γ_w^* gives the correct prediction and that the differences between Γ_w^* and Γ_w decrease with the height, the use of Γ_w^* instead of Γ_w can in general be suggested.

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