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Fast Growing Stochastic Dynamic Models*

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Abstract

A stochastic intertemporal optimization model with stationary discounted one-period utility and stationary dynamic constraints is studied. The main goal is to extend standard dynamic programming techniques, which require utilities to be bounded, to the case of unbounded utilities. This is possible by imposing a limit on the growth rate of state variables. A relationship between this growth rate and the discount factor is established. These results are then applied to the consumption-saving model with no borrowing.

1 Introduction

The goal of this paper is to extend to the stochastic setting an earlier result by Montrucchio and Uberti [11] on Dynamic Programming models with unbounded short-run rewards.

It is well known that the core of dynamic programming rests on the close link between the optimal value function and the solution of the stochastic Bellman functional equation. Following Blackwell [3], to guarantee the existence of a unique measurable function that satisfies the Bellman equation, the one-period utility is required to be bounded and continuous. Under these assumptions, the solution of the functional equation turns out to be the unique fixed-point of a contraction mapping in the Banach space of bounded and continuous functions.

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Having in mind economic growth models, it is immediate to realize that the restriction on utilities to be bounded is almost never satisfied: capital keeps growing in time and utility does the same (see e.g. [5], [8], [9] and [13]). A commonly way to get around this difficulty is to restrict the domain of the one-period utility to the (bounded) set of sustainable capital stocks (see [8]). Yet, there are many situations where this cannot be done. An important example is given by the consumption-saving model studied in Section 4 of the present work.

Since dynamic programming theory happens to be a powerful method for obtaining meaningful characterizations of the value function and related policy functions (see e.g. [5], [9], [10], [13] and [12]), it seems quite useful to find a way to preserve the principle of dynamic programming also in an environment where the state space and the utility are unbounded. This is what the present research is concerned about. Throughout this paper, we show that extending dynamic programming methods to the unbounded case is possible under sufficiently mild conditions: the only restriction required will be an upper bound on the asymptotic rate of growth of the economy.

The idea underlining our analysis is to construct the state space of the system as the limit of an increasing sequence of closed sets such that utility is bounded on each of these sets. This device permits the use of standard methods over each set of the sequence. Through the introduction of a suitable metric, the whole state space can be successively covered. Such a construction is compatible with some assumptions on the growth rate of the system: in particular it will be required that the asymptotic rate of growth must be smaller than the reciprocal of the discount factor.

Section 2 is devoted to notation and the setting of a general stochastic infinite-horizon optimization model. The main facts of dynamic programming known in the literature for the (standard) bounded case are mentioned. In Section 3 we develop the main results of our approach. Section 4 is dedicated to the study of a no-borrowing consumption-saving model with exogenous endowment (for an excellent overview on this kind of models see, e.g., [4], Chapt. 6.2), while Section 5 further specifies the result of Section 4 for utility functions that exhibit a smooth behavior at infinity.

2 A Stochastic Model

In this section, after introducing notation, we briefly recall some well known results of dynamic programming.

The uncertainty of the environment is described by an exogenous stochastic process $\{z_t\}_{t=1}^{\infty}$, where each random variable z_t takes values in some Borel space[1] $(Z, \mathcal{Z})$. Such a process is assumed to be Markovian with stationary transition function (stochastic kernel) $Q : Z \times \mathcal{Z} \to [0,1]$. For each $t \geq 1$, let $Z^t = \prod_{i=1}^t Z_i$, $\Omega = \prod_{i=1}^\infty Z_i$, where $Z_i = Z$ for $i = 1, 2, \ldots$, and let $\mathcal{F}_t$ be the σ-algebra generated by rectangle sets in Z^t. We denote by $\omega_t = (z_1, z_2, \ldots, z_t) \in Z^t$ a finite history of shocks, while by $\omega = (z_1, z_2, \ldots) \in \Omega$ a whole history. Given any initial shock $z_0 \in Z$, all finite probabilities on rectangles of $\mathcal{F}_t$ are given by $\mu_{z_0}^t(C_1 \times \cdots \times C_t) = \int_{C_1} Q(z_0, dz_1) \cdots \int_{C_t} Q(z_{t-1}, dz_t)$, where $C_i \in \mathcal{Z}$ for $1 \leq i \leq t$. Let $\mathcal{F}$ be the smallest σ-algebra containing $\bigcup_{t=1}^\infty \mathcal{F}_t$; we denote by μ_{z_0} the probability on $(\Omega, \mathcal{F})$ induced by the $\mu_{z_0}^t$'s.

A plan $\{x_t\}_{t=1}^\infty$ is represented by a vector $x_1 \in X$ and random variables $x_t : Z^{t-1} \to X$, for $t = 2, 3, \ldots$, where X is a Polish space. Let $\mathcal{X} \subseteq \mathcal{B}^n$ be the Borel σ-algebra on X. For each $t \geq 2$, x_t is $\mathcal{F}_{t-1}$-measurable. The dynamic constraint is represented by a $\mathcal{X} \otimes \mathcal{X} \otimes \mathcal{Z}$-measurable set $D \subset X \times X \times Z$. For each $(x, z) \in X \times Z$, let $\Gamma : X \times Z \to X$ defined as $\Gamma(x, z) = \{y \in X : (x, y, z) \in D\}$ be the correspondence representing the set of feasible actions when the current state of the system is (x, z). The one-period return function $u : D \to \mathbb{R}$ is $\mathcal{X} \otimes \mathcal{X} \otimes \mathcal{Z}$-measurable. The discount factor β is a constant parameter belonging to the interval $(0, 1)$.

With these ingredients at hand, the stochastic optimization problem under study can be stated directly in the reduced form as follows:

$$v(x_0, z_0) = \sup \left\{ u(x_0, x_1, z_0) + \mathbb{E}_{z_0} \left[\sum_{t=1}^{\infty} \beta^t u\left(x_t(\omega_{t-1}), x_{t+1}(\omega_t), z_t\right) \right] \right\} \quad (1)$$

$$\text{s.t. } (x_0, z_0) \in X \times Z \text{ is given,}$$

$$x_1 \in \Gamma(x_0, z_0) \text{ and}$$

$$x_{t+1}(\omega_t) \in \Gamma\left[x_t(\omega_{t-1}), z_t\right] \ \mu_{z_0}^t - a.s., \ t = 1, 2, \ldots,$$

where expectation $\mathbb{E}_{z_0}$ is defined by the probability μ_{z_0} and in the last line we agree to denote $\omega_0 = z_0$ and $x_1(\omega_0) \equiv x_1$.

Let $\mathcal{L}^+(X \times Z)$ denote the space of positive real-valued measurable functions on $X \times Z$. Then the Markov operator $Mf : \mathcal{L}^+(X \times Z) \to \mathcal{L}^+(X \times Z)$ is defined as $(Mf)(y, z) = \int_Z f(y, z')Q(z, dz')$. Markov assumption on the stochastic process of the exogenous shocks establishes an important relationship between the infinite-horizon problem (1) and the Bellman equation

$$w(x, z) = \sup_{y \in \Gamma(x,z)} \left[u(x, y, z) + \beta \left(Mw\right)(y, z)\right], \quad (2)$$

[1] That is, a Borel subset of a Polish space.

as the next result states. Define the associated policy correspondence $G: X \times Z \to X$ by

$$G(x,z) = \{y \in \Gamma(x,z) : w(x,z) = u(x,y,z) + \beta\,(Mw)\,(y,z)\}\,. \qquad (3)$$

If there exists a measurable selection $g\,(x,z) \in G\,(x,z)$, called *optimal policy*, then we say that a plan $\mathrm{x}^* = \{x_t^*\}_{t=1}^{\infty}$ is generated by g starting at (x_0, z_0) if $x_1^* = g(x_0, z_0)$ and $x_{t+1}^*(\omega_t) = g\,[x_t^*(\omega_{t-1}), z_t]\ \mu_{z_0}^t - a.s.$, $t = 1, 2, \ldots$.

Fact 1 *If w is a measurable function satisfying (2) such that, for each initial condition $(x_0, z_0) \in X \times Z$,*

$$\lim_{t\to\infty} \beta^t \mathbb{E}_{z_0}\{w\,[x_t(\omega_{t-1}), z_t]\} = 0, \qquad (4)$$

for all feasible sequence $\{x_t\}$ starting from (x_0, z_0), and G permits a measurable selection g, then w is the value function v of (1), and any plan x^* generated by g is optimal.

A good reference for a complete discussion on all the assumptions and the statement of the problem, as well as a proof for Fact 1, is [13].

The most important application of the principle of verification stated in Fact 1 is due to Blackwell [3] in the case where the function u is bounded and continuous[2]. The main idea behind this approach is to see the function w in (2) as a fixed point of a contraction mapping. Let $C(X \times Z)$ denote the Banach space of bounded continuous functions on $X \times Z$ with the sup norm: $\|f\| = \sup_{(x,z)\in X\times Z} |f(x,z)|$. Consider the Bellman operator T associated to (2):

$$(Tf)(x,z) = \sup_{y\in\Gamma(x,z)} [u(x,y,z) + \beta\,(Mf)\,(y,z)]\,. \qquad (5)$$

We say that the transition function Q satisfies the Feller property if the Markov operator M maps the space $C(X \times Z)$ into itself.

Fact 2 *If Z is a Borel space and X is a Polish space, Γ is nonempty, compact-valued and continuous, u is bounded and continuous and the transition function Q has the Feller property, then $T : C(X \times Z) \to C(X \times Z)$ is a contraction with modulus β, has a unique fixed point w in $C(X \times Z)$ and, for each $w_0 \in C(X \times Z)$,*

$$\|T^n w_0 - w\| \le \beta^n \|w_0 - w\|\,, \qquad n = 1, 2, \ldots.$$

Moreover, the policy correspondence G is nonempty, compact-valued and upper semicontinuous.

[2] A different proof of the same result can be found in [7]. There u is assumed to be upper semicontinuous and X is a subset of a compact metric space.

Thus, w in (2) is the only fixed point of T, i.e., is the only solution of (2) in $C(X \times Z)$. Since w is bounded, it satisfies (4), and the properties of G are such that there is a measurable selection in G. Hence, by Fact 1, w is the solution v of program (1). The proof of Fact 2 is a straightforward extension of the proof in [13] for the case of Polish spaces. About the existence of the measurable selection, see also [2].

3 Growth Assumptions

In this section we develop an approach that allows to cover also unbounded problems.

Assumption 1 *X is a Polish space and Z is a Borel space with their σ-algebras $\mathcal{X}$ and $\mathcal{Z}$. The correspondence $\Gamma : X \times Z \to X$ is nonempty, compact-valued and continuous.*

Assumption 2 *The transition function $Q : Z \times \mathcal{Z} \to [0, 1]$ has the Feller property.*

Assumption 3 *There exists an increasing sequence of closed sets $X_i \uparrow X$ such that, for each $i \geq 1$, $x \in X_i \Longrightarrow \Gamma(x, z) \subseteq X_{i+1}$, for all $z \in Z$.*

Assumption 4 *The one-period return u is continuous and, for each i, there exists a constant L_i such that $|u(x, y, z)| \leq L_i$ for all $x \in X_i$, $z \in Z$ and $y \in \Gamma(x, z)$.*

Since $X_i \subseteq X_{i+1}$, there is no loss of generality assuming $L_i \leq L_{i+1}$. More specifically, we shall always take

$$L_i = \sup \{ |u(x, y, z)| : x \in X_i, \ z \in Z, \ y \in \Gamma(x, z) \} .$$

Define

$$\limsup_{i \to \infty} \frac{L_{i+1}}{L_i} = \vartheta, \tag{6}$$

where $\vartheta \geq 1$. The main result of this section will establish an important relation between ϑ and the discount factor β.

Let us introduce the sequence $\alpha = \{\alpha_i\}$, where $\alpha_i = \sum_{t=0}^{\infty} \beta^t L_{i+t}$. Clearly these series are convergent provided that $\beta < 1/\vartheta$. Let $\mathcal{H}(X \times Z)$ be the space of continuous functions $f : X \times Z \to \mathbb{R}$ such that the restrictions $f_i = f \mid_{X_i \times Z}$ on subsets $X_i \times Z$ are bounded for each i. Define in $\mathcal{H}(X \times Z)$ the family of seminorms $\|f\|_i = \sup_{(x,z)\in X_i\times Z} |f(x,z)| = \|f_i\|$. These seminorms endows $\mathcal{H}(X\times Z)$ with the topology of the uniform convergence over each $X_i \times Z$. Now consider the subset $\mathcal{H}_\alpha(X\times Z)$ in $\mathcal{H}(X\times Z)$ defined as $f \in \mathcal{H}_\alpha(X\times Z)$ if and only if there exists a constant $L \geq 1$ such that $\|f\|_i \leq L\alpha_i$, for all $i = 1, 2, \ldots$.

Proposition 1 *The Bellman operator T defined in (5) maps $\mathcal{H}_\alpha(X\times Z)$ into itself.*

Proof. We first show that T preserves continuity. By Assumption 4, u is continuous, and, by Assumption 2, also Mf is continuous, provided that f is. Hence, as the correspondence Γ satisfies Assumption 1, the Berge Maximum Theorem applies and Tf turns out to be continuous.

To see that $Tf \in \mathcal{H}_\alpha(X\times Z)$, note that

$$\begin{aligned} |(Tf)(x,z)| &\leq \sup_{y\in\Gamma(x,z)} |u(x,y,z) + \beta\,(Mf)\,(y,z)| \\ &\leq \sup_{y\in\Gamma(x,z)} |u(x,y,z)| + \beta \sup_{y\in\Gamma(x,z)} |(Mf)\,(y,z)|. \end{aligned} \tag{7}$$

If $x \in X_i$, then $y \in \Gamma(x,z) \subseteq X_{i+1}$. Thus, for each $f \in \mathcal{H}_\alpha(X\times Z)$, $\|Tf\|_i \leq L_i + \beta L\alpha_{i+1} = (1-L)L_i + L\alpha_i \leq L\alpha_i$, and the proof is now complete. ■

Let us introduce the following metric in $\mathcal{H}_\alpha(X\times Z)$:

$$d(f,g) = \sup_{i=1,2,\ldots} \left(\frac{\|f-g\|_i}{\alpha_i}\right), \qquad \forall\, f,g \in \mathcal{H}_\alpha(X\times Z).$$

Endowed with this metric, $\mathcal{H}_\alpha(X\times Z)$ turns out to be a complete metric space. Completeness can be established by showing that $\mathcal{H}_\alpha(X\times Z)$ is a closed subset of the Fréchet space $\mathcal{H}(X\times Z)$. We are now ready to state the main result.

Theorem 1 *For each discount factor $\beta < 1/\vartheta$, the Bellman equation (2) has one and only one solution w in $\mathcal{H}_\alpha(X\times Z)$, and the sequence $w_{n+1} = Tw_n$ converges uniformly over $X_i \times Z$ for each initial condition $w_0 \in \mathcal{H}_\alpha(X\times Z)$. Moreover, the policy correspondence G defined in (3) is nonempty, compact-valued and upper semicontinuous.*

Proof. By using the standard technique due to Blackwell, it is easy to show that $\|Tf - Tg\|_i \leq \beta\,\|f-g\|_{i+1}$, for all f, $g \in \mathcal{H}_\alpha(X\times Z)$ and all $i = 1, 2, \ldots$. For each i, the following holds:

$$\frac{\|Tf-Tg\|_i}{\alpha_i} \leq \frac{\beta\,\|f-g\|_{i+1}}{\alpha_i} = \frac{\beta\alpha_{i+1}}{\alpha_i}\left(\frac{\|f-g\|_{i+1}}{\alpha_{i+1}}\right). \tag{8}$$

Since $\beta\alpha_{i+1}/\alpha_i = (\alpha_i - L_i)/\alpha_i = 1 - L_i/\alpha_i$ and $L_i < \alpha_i$, $\beta\alpha_{i+1}/\alpha_i < 1$ for all i. Moreover, since we assume $\beta\vartheta < 1$, there exists $\widehat{\vartheta} > \vartheta$ such that $\beta\widehat{\vartheta} < 1$ and, by (6), there is $N \geq 1$ such that for all $i \geq N$ $L_{i+1}/L_i < \widehat{\vartheta}$. Now, for all $i \geq N$,

$$\frac{\beta\alpha_{i+1}}{\alpha_i} = 1 - \frac{1}{\sum_{t=0}^{\infty} \beta^t \left(L_{i+t}/L_i\right)} < 1 - \frac{1}{\sum_{t=0}^{\infty} \left(\beta\widehat{\vartheta}\right)^t} = \beta\widehat{\vartheta} < 1,$$

that is, the $\beta\alpha_{i+1}/\alpha_i$'s are uniformly smaller than 1. Let

$$\lambda = \max\left\{\max_{1\leq i\leq N-1} \left(\beta\alpha_{i+1}/\alpha_i\right), \beta\widehat{\vartheta}\right\} < 1,$$

hence $\beta\alpha_{i+1}/\alpha_i \leq \lambda < 1$, for all i. From (8), $\|f-g\|_i/\alpha_i \leq \lambda\|f-g\|_{i+1}/\alpha_i$ for all i, which yields

$$\sup_i \frac{\|Tf - Tg\|_i}{\alpha_i} \leq \lambda \sup_i \frac{\|f-g\|_{i+1}}{\alpha_{i+1}} \leq \lambda \sup_i \frac{\|f-g\|_i}{\alpha_i}.$$

This leads to the statement because, for all $f, g \in \mathcal{H}_\alpha(X \times Z)$, we have found a real number $\lambda \in (0,1)$ such that $d(Tf, Tg) \leq \lambda d(f,g)$; this is enough to show that T is a contraction on $\mathcal{H}_\alpha(X \times Z)$.

Properties of the policy correspondence G follow from the Berge Maximum Theorem. ∎

Now it is immediate to prove the following corollary.

Corollary 1 *For each discount factor $\beta < 1/\vartheta$, the unique function $w \in \mathcal{H}_\alpha(X \times Z)$ which satisfies the Bellman equation (2) is the value function v of (1) and there exists at least one optimal plan.*

Proof. In order to apply Fact 1, it only remains to show that, for each initial condition $(x_0, z_0) \in X \times Z$, (4) holds and that there is a measurable selection. For the sake of simplicity, we shall drop the arguments of random vectors $x_t(\omega_{t-1})$.

If $\mathrm{x} = \{x_t\}$ is a feasible plan starting from $(x_0, z_0) \in X_i \times Z$, then $x_t \in X_{i+t}$ $\mu_{z_0}^t - a.s.$ Since w satisfies (2), by the same argument as in the proof of Proposition 1, $|w(x,z)| \leq L\alpha_i$ for each $(x,z) \in X_i \times Z$. Hence $|w(x_t, z)| \leq L\alpha_{i+t}$ $\mu_{z_0}^t - a.s.$ for all $z \in Z$, and, by taking expectations, $\mathbb{E}_{z_0}\{|w(x_t, z_t)|\} \leq L\alpha_{i+t}$ for all $t \geq 1$. Thus

$$\begin{aligned}\lim_{t\to\infty} \beta^t \mathbb{E}_{z_0}\{|w(x_t, z)|\} &\leq L \lim_{t\to\infty} \beta^t L\alpha_{i+t} \\ &= L \lim_{t\to\infty} \beta^t \sum_{\tau=0}^{\infty} \beta^\tau L_{i+t+\tau}.\end{aligned}$$

Since $\sum_{\tau=0}^{\infty} \beta^\tau L_{i+t+\tau}$ is convergent, the result follows.

As for the last part of the statement, note that any plan generated by a measurable selection from the policy correspondence G is optimal. ∎

4 The Consumption-Saving Model without Borrowing

Consider an infinitely lived consumer with utility function $u(c)$, who earns a stochastic endowment $w_t \geq 0$ at each time t and holds an initial financial asset $a_0 \geq 0$. We denote by c_t and a_t the consumption and saving variable respectively at time t. We assume that no borrowing is allowed, i.e., $a_t \geq 0$. The interest factor $R_t = 1 + r_t \in [1, \overline{R}]$, $\overline{R} > 1$, of the asset market at each time t is also random. The stochastic process (w_t, R_t) is governed by a transition function Q which, given each realization of (w_t, R_t) of current endowment and interest factor, represents the joint probability of endowment and interest factor (w_{t+1}, R_{t+1}) at time $t+1$. It is assumed that Q has the Feller property and is such that endowment and interest factor are independent, i.e., for each given pair (w, R), it takes the product form $Q((w, R), C \times D) = q(w, C)p(R, D)$, where C and D are measurable sets in $\mathbb{R}_+$ and $[1, R]$ respectively. Moreover, we shall assume that the endowment may grow through time, but not faster than a certain rate. Specifically, w_{t+1} at time $t+1$ will not be larger than a function φ of the endowment w_t at time t, which is bounded itself by a linear rate of growth: $w_{t+1} \leq \varphi(w_t) \leq A + Bw_t$ almost surely with respect to probability $q(w_t, \cdot)$, where $A \geq 0$ and $B > 1$. In other words, at each time t, the support of the stochastic kernel $q(w_t, \cdot)$ lies in $[0, \varphi(w_t)]$.

Given the initial endowment w_0, the initial asset a_0 and the interest factor $R_0 = 1$ at time $t = 0$, the consumer faces the following problem:

$$\begin{aligned}
v(a_0, w_0, R_0) &= \sup \left\{ u(c_0) + \mathbb{E}_{(w_0, R_0)} \left[\sum_{t=1}^{\infty} \beta^t u(c_t) \right] \right\} \\
\text{s.t. } c_t &= R_t a_t + w_t - a_{t+1}, \text{ for all } t \geq 0, \\
& \quad R_0 = 1 \text{ and } a_0, w_0, \text{ are given.}
\end{aligned}$$

The reduced model turns out:

$$
\begin{aligned}
v(a_0, w_0, R_0) &= . \sup \{u(a_0 + w_0 - a_1) \\
&\quad + \mathbb{E}_{(w_0,R_0)} \left[\sum_{t=1}^{\infty} \beta^t u \left(R_t a_t + w_t - a_{t+1}\right) \right] \bigg\} \\
\text{s.t. } 0 &\leq a_{t+1} \leq R_t a_t + w_t, \text{ for all } t \geq 0, \\
& R_0 = 1 \text{ and } a_0, w_0, \text{ are given.}
\end{aligned}
$$

The dynamic constraint satisfies Assumptions 1 and 3. The increasing sequence of sets is constructed by setting $X_i = [0, x_i]$, where $x_0 = a_0 + w_0$ and $x_{i+1} = \overline{R} x_i + \overline{w}_{i+1}$, where $\overline{w}_{i+1} = A + B\overline{w}_i$, for all $i \geq 0$. Clearly, since $\overline{R} > 1$ and $\overline{w}_i \geq 0$, $x_i \to +\infty$.

In this example it is easy to determine the upper bounds L_i and the range of discount factors β which assure that Theorem 1 and Corollary 1 are true. It is convenient to pick the $L_i = u(x_i)$, whenever utility u is increasing. Therefore the asymptotic rate of growth of the economy is given by

$$
\vartheta = \limsup_{i \to \infty} \frac{L_{i+1}}{L_i} = \limsup_{i \to \infty} \frac{u(x_{i+1})}{u(x_i)} = \limsup_{i \to \infty} \frac{u\left(\overline{R} x_i + \overline{w}_{i+1}\right)}{u(x_i)}.
$$

Next proposition relates the growth rate ϑ (or, which is the same, the discount factor β) to the maximum interest factor $\overline{R}$.

Proposition 2 *Let $u(c)$ be increasing and concave over $\mathbb{R}_+$.*

i) if $B < \overline{R}$, then $\vartheta \leq \overline{R}$, i.e. Theorem 1 and Corollary 1 hold for discount factors $\beta < \overline{R}^{-1}$;

ii) if $B > \overline{R}$, then Theorem 1 and Corollary 1 hold for discount factors $\beta < B^{-1}$.

In words, this result establishes that the largest between the maximum interest factor $\overline{R}$ and the maximum growth rate of the exogenous endowment B determines the range of discount factors such that our technique can be used. It is interesting to note that in (i) the exogenous endowment lies almost surely in a compact set, that is, it is equivalent to the case with constant maximum endowment (see Proposition 3 in [11] for the deterministic setting).

Proof. $i)$ Let $\lambda_i = \overline{R} + \overline{w}_{i+1}/x_i$. Therefore

$$
\vartheta = \limsup_{i \to \infty} \frac{u\left(\overline{R} x_i + \overline{w}_{i+1}\right)}{u(x_i)} = \limsup_{i \to \infty} \frac{u(\lambda_i x_i)}{u(x_i)}. \tag{9}
$$

It is immediate to see that $\lambda_i \to \overline{R}$ as i goes to infinity, as

$$\lambda_i = \overline{R} + \frac{A}{x_i} + B\frac{\overline{w}_i}{x_i} \tag{10}$$

and the last addend converges to zero if $B < \overline{R}$.

Take a supergradient p_i of u at x_i, then $u(\lambda_i x_i) \leq u(x_i) + p_i(\lambda_i x_i - x_i)$. Pick a point $\epsilon \geq 0$ such that $u(\epsilon) = 0$. Of course there is no loss in generality, because by shifting the utility function it is always possible to let it meet the axis. A similar argument provides $u(x_i) \geq p_i(x_i - \epsilon)$, from which follows that $p_i x_i / u(x_i) \leq x_i/(x_i - \epsilon)$. Thus

$$\frac{u(\lambda_i x_i)}{u(x_i)} \leq 1 + \frac{p_i x_i}{u(x_i)}(\lambda_i - 1) \leq 1 + \frac{x_i}{x_i - \epsilon}(\lambda_i - 1).$$

By taking the limit, we have $\limsup_{i\to\infty} u(\lambda_i x_i)/u(x_i) \leq \overline{R}$, and the proof of (i) is complete.

(ii) The proof when $B > \overline{R}$ reflects exactly the pattern above, with the difference that in this case $\lambda_i \to B$ as i goes to infinity. To prove this, note that in this case the sequence $\{\overline{w}_i/x_i\}$ in (10) converges to $1 - \overline{R}/B$, thus establishing the result. ■

5 Regular Utility Functions

The class of problems where the technique described in the previous section applies can be further specified if the utility behaves nicely toward infinity. In this vein, we provide the following definition, which is a slight modification of the one in [6], Chapt. VII.

Definition 1 *A concave function $f : \mathbb{R}_+ \to \mathbb{R}$ varies regularly at infinity with exponent at most $0 \leq \gamma \leq 1$ if*

$$\limsup_{n\to\infty} \frac{f(\lambda_n x_n)}{f(x_n)} \leq \lambda^\gamma,$$

for each sequence $\lambda_n \to \lambda \geq 1$ and each sequence $x_n \to +\infty$.

In view of (9) in the proof of Proposition 2, it is clear that $\gamma \leq 1$ for concave utility functions. In order to improve the estimation $\beta < \overline{R}^{-1}$, we need an exponent $\gamma < 1$. As an immediate consequence of Definition 1 applied to the sequences λ_i and x_i in (9), the following result is valid.

Proposition 3 *If the utility function u varies regularly at infinity with exponent at most* $0 \leq \gamma \leq 1$, *then Theorem 1 and Corollary 1 hold for each discount factor* $\beta < \overline{R}^{-\gamma}$..

In the extreme case $\gamma = 0$ we say that f has slow variation at infinity. This is especially interesting since Theorem 1 and Corollary 1 hold for each discount factor $0 < \beta < 1$. Most of the commonly used utility functions vary regularly at infinity. For example $u(c) = \log c$ exhibits slow variation at infinity, while the CRRA utility $u(c) = c^\gamma$ vary regularly at infinity with exponent γ.

6 Concluding Remarks

In Section 4 we have showed that our theory may find a useful application in growth models where it is natural to assume that the state space is not bounded. This example makes also clear that condition (6), which limits the rate of growth of the economy and is necessary to apply Theorem 1 and Corollary 1, is not restrictive whenever growth models are involved. It is also worth emphasizing that the restriction on the maximum growth rate given by φ in our consumption-saving model turns out to be the weakest condition which is compatible with convexity assumption of the constraints. There are several other cases in the economic literature where the state space is not bounded and our method could be successfully adopted. For example, the A-K model of growth (see [1]) belongs to this framework.

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