Local Solvability for Partial Differential Equationswith Multiple Characteristics in Mixed Gevrey-C^∞ Spaces

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Local Solvability for Partial Differential Equations with Multiple Characteristics in Mixed Gevrey-C^∞ Spaces

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Abstract

In this paper we consider a class of linear partial differential equations with multiple characteristics, whose principal part is elliptic in a set of variables. We assume that the subprincipal symbol has real part different from zero and that its imaginary part does not change sign. We then prove the local solvability of such a class of operators in mixed Gevrey-C^∞ spaces, in the sense that the linear equation admits a local solution when the datum is Gevrey in some variables and only C^∞ in the other ones.

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1 Introduction

In this paper we study the local solvability of a class of linear partial differential equations with multiple characteristics. We say that an operator

$$A(w, D) = \sum_{|\alpha| \le m} c_\alpha(w) D^\alpha, \tag{1.1}$$

with $c_\alpha(w) \in C^\infty(\Omega)$, for an open set $\Omega \subset \mathbb{R}^n$, and $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$, is C^∞ locally solvable at $x_0 \in \Omega$ if there exists a neighborhood $V \subset \Omega$ of x_0 such that for every $f \in C_0^\infty(V)$ there exists $u \in \mathcal{D}'(V)$ satisfying $A(w, D)u = f$. It is well known that there exist non locally solvable operators, of a very simple form. The first example was given by Lewy [17], who proved that the equation

$$D_{w_1} u + i D_{w_2} u + i(w_1 + i w_2) D_{w_3} u = f$$

does not have any distribution solution u in any open non-void subset of $\mathbb{R}^3$ for a suitable $f \in C^\infty(\mathbb{R}^3)$. This example was generalized by Hörmander [11], who gave a necessary condition for the local solvability.

A natural question concerning non C^∞ locally solvable operators is the following: is it possible to restrict the set of right-hand sides f in such a way that for every f in such a smaller set the equation $A(w,D)u = f$ has a solution in some distribution space? A suitable functional setting for this purpose is given by Gevrey spaces. Let us fix $s \geq 1$ and an open set $\Omega \subset \mathbb{R}^n$; we say that f is Gevrey of order s in Ω, and we write $f \in G^s(\Omega)$, if $f \in C^\infty(\Omega)$ and for every compact set $K \subset \Omega$ there exists a positive constant C such that for every $\alpha \in \mathbb{Z}^n_+$ the following estimates hold:

$$\sup_{w \in K} |\partial^\alpha f(w)| \leq C^{|\alpha|+1}(\alpha!)^s.$$

For $s > 1$ we set $G^s_0(\Omega) = G^s(\Omega) \cap C^\infty_0(\Omega)$. The Gevrey spaces turn out to be a scale of spaces between the analytic and the C^∞ functions, as

$$\mathcal{A}(\Omega) = G^1(\Omega) \subset \cdots \subset G^s(\Omega) \subset G^t(\Omega) \subset \cdots \subset C^\infty(\Omega)$$

for every $1 \leq s \leq t$, $\mathcal{A}(\Omega)$ being the set of analytic functions in Ω. Similarly to the C^∞ case we say that a linear partial differential operator $A(w,D)$ with coefficients in $G^s(\Omega)$ is G^s locally solvable at $x_0 \in \Omega$, $s > 1$, if there exists a neighborhood $V \subset \Omega$ of x_0 such that for every $f \in G^s_0(V)$ there exists $u \in \mathcal{D}'_s(V)$ satisfying $A(w,D)u = f$, where $\mathcal{D}'_s(V)$ is the space of ultradistributions, dual of $G^s_0(V)$. Since C^∞ local solvability implies G^s local solvability for every $s > 1$ and G^t local solvability implies G^s local solvability for every $1 < s \leq t$, the set $\{s > 1 : A(w,D) \text{ is } G^s \text{ locally solvable at } x_0\}$ is either the empty set or an interval of the kind $(1, s_0)$ or $(1, s_0]$ for $s_0 > 1$. For an operator $A(w,D)$ that is not C^∞ locally solvable we can then look for which Gevrey indexes s it remains not G^s locally solvable and if, for s sufficiently small, it becomes G^s locally solvable.

Given an operator $A(w,D)$ as in (1.1) we consider its *principal symbol*, given by

$$a_m(w,\zeta) = \sum_{|\alpha|=m} c_\alpha(w)\zeta^\alpha,$$

and the corresponding *characteristic manifold*

$$\Sigma = \{(w,\zeta) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) : a_m(w,\zeta) = 0\}.$$

If $\nabla_\zeta a_m(w,\zeta) \neq 0$ for all $(w,\zeta) \in \Sigma$ we say that $A(w,D)$ is of principal type. For such operators a necessary and sufficient condition for the C^∞ local solvability has been provided, cf. Nirenberg-Trèves [22, 23], and such condition turns out to be necessary and sufficient also for the G^s local solvability, for every $s > 1$, cf. [30]. The attention has then been devoted in the last years to operators with multiple characteristics, i.e. operators with non-empty characteristic manifold Σ and for which the gradient of the principal symbol vanishes somewhere on Σ. In this case several results have been obtained, see for instance Mascarello-Rodino [21] and the references therein, but general necessary and sufficient conditions, comparable to the one of Nirenberg-Trèves for operators of principal type, are missing. In the case of operators with multiple characteristics the local solvability in Gevrey classes turns out to be an interesting problem, as there are operators that are not C^∞ locally solvable but are locally solvable in some Gevrey spaces, cf. for example Corli [4], Popivanov [28], [29], Oliaro-Rodino [25], Oliaro-Popivanov [24], Rodino [31], Gramchev-Popivanov [9]. Moreover, variants and generalizations of Gevrey spaces have been considered, such as e.g. ultradifferentiable classes, cf. for instance Braun-Meise-Taylor [3], Jornet-Oliaro [15], or anisotropic spaces, that permit to find more precise results by allowing different Gevrey orders in different variables, cf. for example Lorenz [19], Šananin [33], Liess-Rodino [18], Marcolongo-Oliaro [20], De Donno-Oliaro [6, 7].

In this paper we work with anisotropic spaces of functions that can be Gevrey of different order in different variables and just C^∞ in other variables. We introduce now the class of operators that we shall study and we illustrate the main result of the paper. Let us consider $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ with $p, q \in \mathbb{N}$, $p, q \geq 1$. Fix $\sigma \in \mathbb{R}^q$ with $\sigma_h \geq 1$ for every $h = 1, \dots, q$. We set

$$w = (x, y) \in \mathbb{R}^{p+q}, \quad \phi = (\mathbf{1}, \sigma) \in \mathbb{R}^{p+q}, \tag{1.2}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^p$; we study the following (anisotropic) partial differential operator:

$$P(w, D) = P_1(w, D_x) - P_2(w, D_y) + Q(w, D) \tag{1.3}$$

where

$$P_1(w, D_x) = \sum_{|\alpha|=m} a_\alpha(w) D_x^\alpha, \tag{1.4}$$

$$P_2(w, D_y) = \sum_{\langle \beta, \sigma \rangle = m} b_\beta(w) D_y^\beta \tag{1.5}$$

and

$$Q(w, D) = \sum_{\langle \gamma, \phi \rangle \leq m^*} c_\gamma(w) D_w^\gamma, \quad m^* < m; \tag{1.6}$$

the multi-indexes satisfy $\alpha \in \mathbb{Z}_+^p$, $\beta \in \mathbb{Z}_+^q$, $\gamma \in \mathbb{Z}_+^{p+q}$ and the notation $\langle \cdot, \cdot \rangle$ stands for the usual inner product in the respective space $\mathbb{R}^n$. The covariable associated to x, y, w, are denoted respectively by ξ, η, ζ; we then have $\zeta = (\xi, \eta)$.

We suppose that the following conditions are satisfied:

$$p_1(w, \xi) \text{ is } \xi\text{-elliptic and } \Re\, p_2(w, \eta) \neq 0 \text{ for } \eta \neq 0, \tag{1.7}$$

where $p_1(w, \xi)$ and $p_2(w, \eta)$ are the symbols of $P_1(w, D_x)$ and $P_2(w, D_y)$ respectively, and moreover the 'ξ-ellipticity' of $p_1(w, \xi)$ means that there exist positive constants c_1 and c_2, $c_1 < c_2$, such that

$$c_1 \langle \xi \rangle^m \leq |p_1(w, \xi)| \leq c_2 \langle \xi \rangle^m$$

for every $w \in \Omega$, $\xi \in \mathbb{R}^p$, $\langle \xi \rangle \gg 0$, where Ω is a neighborhood of the origin in $\mathbb{R}^{p+q}$ and

$$\langle \xi \rangle = (1 + |\xi|^2)^{1/2}. \tag{1.8}$$

We present now the main result, here in a simplified form. Let Ω be a neighborhood of the origin in $\mathbb{R}^{p+q}$. We shall write in the following $G_x^\lambda(\Omega)$, $\lambda \in \mathbb{R}^p$, $\lambda_j \geq 1$ for $j = 1, \dots, p$, for the space of all the functions that are, roughly speaking, Gevrey of order λ_j in the x_j-variable, $j = 1, \dots, p$, and C^∞ in the y-variables; analogously, $G_y^\mu(\Omega)$, $\mu \in \mathbb{R}^q$, $\mu_h \geq 1$ for $h = 1, \dots, q$, indicates the set of all the functions that are C^∞ in the x-variables and Gevrey of order μ_h in the y_h-variable, $h = 1, \dots, q$. For a precise definition of these spaces, see Definition 2.6.

Theorem 1.1. *Let the coefficients of $P(w, D)$ be analytic. We suppose that the operator $P(w, D)$ satisfies* (1.7), *and moreover that $\Im p_2(w, \eta)$ does not change sign for $(w, \eta) \in \Omega \times \mathbb{R}^q$ (we allow $\Im p_2(w, \eta)$ to vanish, even identically, on $\Omega \times \mathbb{R}^q$).*
Let us define $r_0 = \max\{\frac{1}{2}, 1 - (m - m^*)\}$ and fix $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^q$ such that $1 \leq \lambda_j < \frac{1}{r_0}$ for $j = 1, \dots, p$ and $1 \leq \mu_h < \frac{\sigma_h}{r_0}$ for $h = 1, \dots, q$. Then the equation

$$P(w, D)u = f(x, y)$$

admits a classical solution for $f \in G_x^\lambda(\Omega) \cap C_0^\infty(\Omega)$, or alternatively $f \in G_y^\mu(\Omega) \cap C_0^\infty(\Omega)$.

Theorem 1.1 states a sort of mixed Gevrey-C^∞ local solvability of the operator $P(w, D)$, in the sense that the datum is allowed to be C^∞ in a set of variables (x or y) but it is forced to be Gevrey in the other variables. For a more complete statement of Theorem 1.1 see Theorems 3.1 and 3.6, which establish the existence of a parametrix of P in the scale of Gevrey-Sobolev spaces $\mathbb{H}_{\phi,r}^{s,\psi}(\mathbb{R}^{p+q})$.
Basic examples and source for our work are operators whose principal part is the Laplace (or powers of the Laplace) operator. Consider for example

$$L_{m,d}(w, D) = \Delta_x^m + b(x, y) D_y^d, \tag{1.9}$$

where $\Delta_x = \sum_{j=1}^p D_{x_j}^2$, $x \in \mathbb{R}^p$, $y \in \mathbb{R}$, $m, d \in \mathbb{N}$ with $d < 2m$ and $b(x, y)$ is analytic with non vanishing real part. Interesting results on the operator $L_{m,d}(w, D)$, for $m = d = 1$ can be found in Boutet de Monvel [2] and in Popivanov [27], in which phenomena of hypoellipticity, local solvability and propagation of singularities for Schrödinger type equations are analyzed. Observe that if d is even and $\Re b(x, y) > 0$ then $L_{m,d}(w, D)$ is quasi elliptic, and then its C^∞ local solvability is known from general result, cf. for example Mascarello-Rodino [21]. Suppose now that

$$\Re b(x, y) < 0 \text{ and } \Im b(x, y) \text{ does not change sign on } \Omega, \tag{1.10}$$

Ω being a neighborhood of the origin in $\mathbb{R}^{p+1}$. In this case Popivanov [27] proved the C^∞ local solvability of $L_{1,1}(w, D)$, but under an additional condition, which actually implies that $\Im b(x, y)$ cannot vanish identically on Ω. Theorem 1.1 gives us the local solvability of $L_{1,1}(w, D)$ under the assumption (1.10), without any additional condition (in particular $\Im b(x, y)$ may vanish identically), but in mixed Gevrey-C^∞ spaces; more precisely, we obtain that the equation $L_{1,1}(w, D)u = f(x, y)$ is locally solvable when f is C^∞ in x and G^s, $s < 4$,

in y, or alternatively when f is G^s, $s < 2$, in x and C^∞ in y, G^s being the usual isotropic Gevrey space. More generally, for $m, d \in \mathbb{N}$, $d < 2m$, the equation $L_{m,d}(w, D)u = f(x, y)$ turns out to be locally solvable when f is C^∞ in x and G^s, $s < 4\frac{m}{d}$, in y, or alternatively when f is G^s, $s < 2$, in x and C^∞ in y.
Let us analyze another example, in the case $y \in \mathbb{R}^q$, $q > 1$:

$$\Lambda(w, D) = \Delta_x^{bp} - (1 + i|w|^2)(D_{y_1}^{2a} + D_{y_2}^{2b})^{p-1} + Q(w, D), \tag{1.11}$$

$x \in \mathbb{R}^p$, $y \in \mathbb{R}^2$, $p, a, b \in \mathbb{N}$, $1 \leq a \leq b$, where Δ_x^{bp} and $(1 + i|w|^2)(D_{y_1}^{2a} + D_{y_2}^{2b})^{p-1}$ in (1.11) play the role of $P_1(w, D_x)$ and $P_2(w, D_y)$, respectively, in (1.3); in particular we have $m = bp$, $\sigma = (\sigma_1, \sigma_2) = \left(\frac{bp}{a(p-1)}, \frac{p}{p-1}\right)$. De Donno [5] proved results of hypoellipticity for $\Lambda(w, D)$ under nonvanishing hypotheses on the imaginary part of $Q(w, D)$; for example the C^∞ hypoellipticity and local solvability at the origin of $\Lambda(w, D)$ is proved in [5] in the case $Q(w, D) = i(D_{y_1}^{2a} + D_{y_2}^{2b})^{p-2}\Delta_x^b$, $p \geq 4b+2$. Here we do not require non vanishing conditions on $Q(w, D)$, obtaining local solvability in mixed Gevrey-C^∞ classes; more precisely if in (1.11) we choose $Q(w, D) = 0$, $p, a, b \in \mathbb{N}$, $1 \leq a \leq b$, we have that the equation $\Lambda(w, D)u = f(x, y)$ admits a local solution when $f(x, y)$ is C^∞ in the x-variables, G^s, $s < \frac{2bp}{a(p-1)}$, in the y_1-variable and G^s, $s < \frac{2p}{p-1}$, in the y_2-variable, or alternatively when $f(x, y)$ is G^s, $s < 2$, in the x-variables and C^∞ in the y-variables.

In this paper we use the machinery of pseudo-differential operators (cf. for example Trèves [34]) and techniques from microlocal analysis. In particular the idea is to transform the operator $P(w, D)$ into another one, by means of a conjugation of the type

$$\widetilde{P}(w, D) = e^{\psi(w,D)} P(w, D) \left({}^t e^{-\psi(w,-D)}\right),$$

${}^t e^{-\psi(w,-D)}$ being the transposed of $e^{-\psi(w,-D)}$ (i.e. ${}^t e^{-\psi(w,-D)}$ is the pseudodifferential operator whose right symbol is $e^{-\psi(w,\zeta)}$). By a suitable choice of the phase $\psi(w, D)$ we can then apply to $\widetilde{P}(w, D)$ the results of De Donno [5], since we show that the operator $\widetilde{P}(w, D)$ contains a lower order term $\widetilde{Q}(w, D)$ whose imaginary part turns out to satisfy the non vanishing condition required in [5]. Then $\widetilde{P}(w, D)$ is C^∞ locally solvable. The conjugation allows us to transfer on $P(w, D)$ the local solvability in mixed Gevrey-C^∞ classes. This technique has already been used in some papers, starting from the work of Kajitani-Wakabayashi [16]; we refer also to Gramchev-Rodino [10] for the isotropic case, Marcolongo-Oliaro [20] in the anisotropic frame, and to De Donno-Oliaro [6, 7], in which the influence of the lower order terms is also taken into account. One of the novelties here is that we propose in the conjugation a microlocalisation near the anisotropic characteristic set $\{(w, \zeta) \in \Omega \times (\mathbb{R}^{p+q} \setminus \{0\}) : p_1(w, \xi) - p_2(w, \eta) = 0\}$, that allows us to obtain solvability in mixed Gevrey-C^∞ classes, whereas the results in the above quoted papers are only in Gevrey. Moreover, we consider in (1.4) a multidimensional variable x; this causes significant complications in the study of the conjugate operator and in the choice of the phase ψ. We finally note that some of the results contained in the present work were announced, in a very preliminary form and without proofs, in Oliaro [26].

2 Banach spaces of Gevrey-C^∞ functions

In this section we present a class of Gevrey-Sobolev spaces that we shall use as functional setting in the analysis of the local solvability of the operator (1.3). To start with, we say that $\Gamma \subset \mathbb{R}^{p+q}$ is a 'ϕ-conical set', ϕ as in (1.2), if it contains the points

$$(\varrho\xi_1, \dots, \varrho\xi_p, \varrho^{\sigma_1}\eta_1, \dots, \varrho^{\sigma_q}\eta_q)$$

for every $\varrho > 0$ whenever it contains the point (ξ, η).

Definition 2.1. *We say that an $L^2(\mathbb{R}^{p+q})$ function $f(w)$ belongs to the space $H^s_\phi(\mathbb{R}^{p+q})$, $s \geq 0$, $\phi = (\mathbf{1}, \sigma)$, if the following norm is finite:*

$$\|f\|_{H^s_\phi} = \left(\int_{\mathbb{R}^{p+q}} \langle\zeta\rangle_\phi^{2s} |\hat{f}(\zeta)|^2 \, d\zeta \right)^{1/2},$$

where

$$\langle\zeta\rangle_\phi = \langle\xi\rangle + \langle\eta\rangle_\sigma, \tag{2.1}$$

$\langle\eta\rangle_\sigma$ being given by

$$\langle\eta\rangle_\sigma = \sum_{h=1}^{q} (1 + |\eta_h|^{2/\sigma_h})^{\frac{1}{2}}. \tag{2.2}$$

Definition 2.2. *A function $\psi(w,\zeta) \in C^\infty(\mathbb{R}^{2(p+q)})$ is said to be a weight function of order (r, ϕ), $r \in (0,1)$, ϕ as in (1.2), if $\psi(w,\zeta)$ is non-negative and satisfies the following condition: for every $\gamma, \delta \in \mathbb{Z}^{p+q}_+$ there exists a constant C such that*

$$|\partial_w^\gamma \partial_\zeta^\delta \psi(w,\zeta)| \leq C^{1+|\gamma|+|\delta|} \gamma!^\epsilon \, \delta!^\epsilon \, \langle\zeta\rangle_\phi^{r-\langle\delta,\phi\rangle}, \tag{2.3}$$

for $\langle\zeta\rangle_\phi \gg 0$ and all $w \in \mathbb{R}^{p+q}$, where $\epsilon = (\epsilon_1, \dots, \epsilon_{p+q})$ with $\epsilon_j > 1$ for every $j = 1, \dots, p+q$.

Let us now introduce the (anisotropic) characteristic manifold of the operator P: it is denoted by Σ_ϕ and it is defined as follows:

$$\Sigma_\phi = \{(w,\zeta) \in \Omega \times (\mathbb{R}^{p+q} \setminus \{0\}) : p_1(w,\xi) - p_2(w,\eta) = 0\}, \tag{2.4}$$

where Ω is a neighborhood of the origin in $\mathbb{R}^{p+q}$, say, $\Omega = \{w \in \mathbb{R}^{p+q} : |w| < \delta_0\}$ for a fixed $\delta_0 > 0$.

Remark 2.3. *Since we have assumed that the condition (1.7) is satisfied, we have that Σ_ϕ is contained in a ϕ-conical set of the type $\Omega \times \Gamma_\phi$, where*

$$\Gamma_\phi = \{(\xi,\eta) \in \mathbb{R}^{p+q} \setminus \{0\} : c\langle\eta\rangle_\sigma \leq \langle\xi\rangle \leq C\langle\eta\rangle_\sigma\} \tag{2.5}$$

for suitable constants $c \leq C$.

This last remark gives rise to analyze the operator P only in a set of the type $\Omega \times \Gamma_\phi$, since outside such a set P is microlocally quasi-elliptic. We then suppose from now on that the weight function $\psi(w,\zeta)$ vanishes identically for $\zeta \notin \Gamma_\phi$. So we can assume (if necessary) that $\psi(w,\zeta)$ satisfies the following estimates for every $\gamma, \delta \in \mathbb{Z}_+^{p+q}$ in place of (2.3):

$$|\partial_w^\gamma \partial_\zeta^\delta \psi(w,\zeta)| \leq C^{1+|\gamma|+|\delta|}\, \gamma!^\epsilon\, \delta!^\epsilon\, \langle \xi \rangle^{r_1 - \langle \delta', \theta \rangle} \langle \eta \rangle_\sigma^{r_2 - \langle \delta'', \sigma \rangle}, \tag{2.6}$$

where $r_1, r_2 \in (0,1)$, $r_1 + r_2 = r$, and we have split $\mathbb{Z}_+^{p+q} \ni \delta = (\delta', \delta'')$ with $\delta' \in \mathbb{Z}_+^p$, $\delta'' \in \mathbb{Z}_+^q$. In particular it shall be useful to consider (2.6) in the cases $r_1 = 0$ or $r_2 = 0$, assuming then that the weight function satisfies estimates of the type

$$|\partial_w^\gamma \partial_\zeta^\delta \psi(w,\zeta)| \leq C^{1+|\gamma|+|\delta|}\, \gamma!^\epsilon\, \delta!^\epsilon \langle \xi \rangle^{- \langle \delta', \theta \rangle} \langle \eta \rangle_\sigma^{r - \langle \delta'', \sigma \rangle} \tag{2.7}$$

or

$$|\partial_w^\gamma \partial_\zeta^\delta \psi(w,\zeta)| \leq C^{1+|\gamma|+|\delta|}\, \gamma!^\epsilon\, \delta!^\epsilon \langle \xi \rangle^{r - \langle \delta', \theta \rangle} \langle \eta \rangle_\sigma^{- \langle \delta'', \sigma \rangle} \tag{2.8}$$

for every $(w,\zeta) \in \mathbb{R}^{2(p+q)}$, $\langle \zeta \rangle_\phi \gg 0$.

We now pass to define our class of Gevrey-Sobolev spaces.

Definition 2.4. *Let $\psi(w,\zeta)$ be a weight function of order (r,ϕ) and $s \geq 0$ a real number. Then $\mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q})$ is defined to be the set of all the L^2 functions $f(w)$ such that $\|f\|_{s,\psi,\phi,r} < \infty$, where*

$$\|f\|_{s,\psi,\phi,r} = \|e^{\psi(w,D)} f\|_{H^s_\phi};$$

the exponential operator $e^{\psi(w,D)}$ acts in the following way: $e^{\psi(w,D)} g(w) = \int e^{iw\zeta} e^{\psi(w,\zeta)} \hat{g}(\zeta)\, đ\zeta$, where $đ\zeta = (2\pi)^{-(p+q)}\, d\zeta$.

Spaces of the kind of $\mathbb{H}^{s,\psi}_{\phi,r}$ have been first introduced by Kajitani-Wakabayashi [16] and then used in various situations in the study of partial differential equations in Gevrey classes, cf. for example Gramchev-Rodino [10] for the isotropic case, Marcolongo-Oliaro [20], De Donno-Oliaro [6] in the anisotropic frame, Bourdaud-Reissig-Sickel [1] where the composition is investigated, Jornet-Oliaro [15] where spaces related to $\mathbb{H}^{s,\psi}_{\phi,r}$ are considered in the frame of ultradifferentiable functions.
We want now to prove that $\mathbb{H}^{s,\psi}_{\phi,r}$ contains suitable classes of Gevrey anisotropic functions. To this aim we need the following lemma.

Lemma 2.5. *Let $a(w,\zeta) \in C^\infty(\mathbb{R}^{2(p+q)})$ be a function such that for every $\gamma, \delta \in \mathbb{Z}_+^{p+q}$ and $N \in \mathbb{N}$ there exists a constant $C_{\gamma\delta}(N)$ satisfying*

$$|\partial_w^\gamma \partial_\zeta^\delta a(w,\zeta)| \leq C_{\gamma\delta}(N) |\zeta|^{-N},$$

for every $(w,\zeta) \in \mathbb{R}^{2(p+q)}$. Then

$$\mathcal{F}^{-1}_{\zeta \to w}\big(a(w,\zeta)\big) \in \mathcal{S}(\mathbb{R}^{p+q}), \tag{2.9}$$

where $\mathcal{F}^{-1}$ stands for the inverse Fourier transform.

Proof. Let us fix $\gamma, \delta \in \mathbb{Z}_+^{p+q}$; by a simple computation and an integration by parts, for every $N \in \mathbb{N}$ we have:

$$\begin{aligned}
\left| w^\gamma \partial_w^\delta \left(\mathcal{F}^{-1}_{\zeta \to w}(a(w,\zeta)) \right) \right| &= \left| w^\gamma \sum_{\nu \le \delta} \binom{\delta}{\nu} \int \zeta^\nu e^{iw\zeta} \partial_w^{\delta-\nu} a(w,\zeta)\, đ\zeta \right| \\
&\le \left| \sum_{\nu \le \delta} \binom{\delta}{\nu} \sum_{\substack{\mu \le \gamma \\ \mu \le \nu}} \binom{\gamma}{\mu} \nu! \int e^{iw\zeta} \zeta^{\nu-\mu} \partial_w^{\delta-\nu} \partial_\zeta^{\gamma-\mu} a(w,\zeta)\, đ\zeta \right| \\
&\le \sum_{\nu \le \delta} \binom{\delta}{\nu} \sum_{\substack{\mu \le \gamma \\ \mu \le \nu}} \binom{\gamma}{\mu} \int |\zeta|^{|\nu-\mu|} C_{\delta-\nu,\gamma-\mu}(N) |\zeta|^{-N}\, đ\zeta;
\end{aligned}$$

for N sufficiently large the last integral is convergent, and so (2.9) is true. □

Let us now define suitable classes of anisotropic Gevrey functions.

Definition 2.6. *Fix $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^q$ in such a way that $\lambda_j \ge 1$ for every $j = 1, \dots, p$ and $\mu_h \ge 1$ for every $h = 1, \dots, q$.*

(i) *We say that $g(w)$ belongs to the space $G^{(\lambda,\mu)}(\mathbb{R}^{p+q})$ if $g(w) \in C^\infty(\mathbb{R}^{p+q})$ and for every compact set $K \subset \mathbb{R}^{p+q}$ there exists a positive constant C_K such that for all $\alpha \in \mathbb{Z}_+^p$ and $\beta \in \mathbb{Z}_+^q$*

$$\sup_{w \in K} |\partial_x^\alpha \partial_y^\beta g(w)| \le C_K^{1+|\alpha|+|\beta|} \alpha!^\lambda \, \beta!^\mu, \tag{2.10}$$

where we have split $w = (x, y)$, cf. (1.2), and moreover $\alpha!^\lambda = \alpha_1!^{\lambda_1} \cdots \alpha_p!^{\lambda_p}$, $\beta!^\mu = \beta_1!^{\mu_1} \cdots \beta_q!^{\mu_q}$.

(ii) *The space $G_x^\lambda(\mathbb{R}^{p+q})$ is defined to be the set of all the functions $g(w) \in C^\infty(\mathbb{R}^{p+q})$ such that for every compact set $K \subset \mathbb{R}^{p+q}$ and for every $\beta \in \mathbb{Z}_+^q$ we can find $C_{\beta,K}$ such that for all $\alpha \in \mathbb{Z}_+^p$*

$$\sup_{w \in K} |\partial_x^\alpha \partial_y^\beta g(w)| \le C_{\beta,K}^{1+|\alpha|} \alpha!^\lambda. \tag{2.11}$$

(iii) *In an analogous way we set $G_y^\mu(\mathbb{R}^{p+q})$ to be the space of all $g(w) \in C^\infty(\mathbb{R}^{p+q})$ such that for every compact set $K \subset \mathbb{R}^{p+q}$ and for every $\alpha \in \mathbb{Z}_+^p$ there exists a positive constant $C_{\alpha,K}$ satisfying the following condition for all $\beta \in \mathbb{Z}_+^q$:*

$$\sup_{w \in K} |\partial_x^\alpha \partial_y^\beta g(w)| \le C_{\alpha,K}^{1+|\beta|} \beta!^\mu. \tag{2.12}$$

We then define

$$\begin{aligned}
G_0^{(\lambda,\mu)}(\mathbb{R}^{p+q}) &= G^{(\lambda,\mu)}(\mathbb{R}^{p+q}) \bigcap C_0^\infty(\mathbb{R}^{p+q}), \\
G_{x,0}^\lambda(\mathbb{R}^{p+q}) &= G_x^\lambda(\mathbb{R}^{p+q}) \bigcap C_0^\infty(\mathbb{R}^{p+q}), \\
G_{y,0}^\mu(\mathbb{R}^{p+q}) &= G_y^\mu(\mathbb{R}^{p+q}) \bigcap C_0^\infty(\mathbb{R}^{p+q}).
\end{aligned}$$

We have the following result.

Proposition 2.7. *Let us fix $r \in (0,1)$ and $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^q$ such that $1 < \lambda_j < \frac{1}{r}$ for every $j = 1, \ldots, p$ and $1 < \mu_h < \frac{\sigma_h}{r}$ for every $h = 1, \ldots, q$. Then*

$$G^{\lambda}_{x,0}(\mathbb{R}^{p+q}) \subset \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \quad \text{and} \quad G^{\mu}_{y,0}(\mathbb{R}^{p+q}) \subset \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}),$$

for all $s \geq 0$ and for every weight function $\psi(w,\zeta)$ of order (r,ϕ).

Proof. We observe at first that if $g(w) \in G^{\lambda}_{x,0}(\mathbb{R}^{p+q})$ (resp. $g(w) \in G^{\mu}_{y,0}(\mathbb{R}^{p+q})$) there exist positive constants C and ε such that:

$$|\hat{g}(\zeta)| \leq Ce^{-\varepsilon \sum_{j=1}^{p}(1+|\xi_j|)^{1/\lambda_j}} \quad (\text{resp. } |\hat{g}(\zeta)| \leq Ce^{-\varepsilon \sum_{h=1}^{q}(1+|\eta_h|)^{1/\mu_h}}); \tag{2.13}$$

the estimates (2.13) can be proved by a standard technique, cf. for example Theorem 2.1 in [20]. From now on we shall only study the case $g(w) \in G^{\lambda}_{x,0}(\mathbb{R}^{p+q})$, since the other one can be treated in the same way.

We have to prove that $e^{\psi(w,D)}g(w) \in H^s_{\phi}(\mathbb{R}^{p+q})$. Since $\mathcal{S}(\mathbb{R}^{p+q}) \subset H^s_{\phi}(\mathbb{R}^{p+q})$ for all s and ϕ, it is enough to show that $e^{\psi(w,D)}g(w) \in \mathcal{S}(\mathbb{R}^{p+q})$; by Lemma 2.5 we have only to prove that for every $\gamma, \delta \in \mathbb{Z}^{p+q}_{+}$ and $N \in \mathbb{N}$ there exists a constant $C_{\gamma\delta}(N)$ satisfying

$$\left|\partial^{\gamma}_{w}\partial^{\delta}_{\zeta}\left(e^{\psi(w,\zeta)}\hat{g}(\zeta)\right)\right| \leq C_{\gamma\delta}(N)|\zeta|^{-N}. \tag{2.14}$$

To estimate the left hand side ot (2.14) we can use Leibnitz rule and the following Faà di Bruno estimate, valid for $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$:

$$|\partial^{\alpha} f(g(z))| \leq C_{\alpha} \sum_{0<h\leq|\alpha|} |f^{(h)}(g(z))| \sum_{\alpha^{(1)}+\cdots+\alpha^{(h)}=\alpha} |g^{(\alpha^{(1)})}(z)| \cdots |g^{(\alpha^{(h)})}(z)|,$$

where the second sum in the right hand side is performed over all decomposition of α in a sum of h multi-indexes. We then obtain, using the well known relation $\partial^{\rho}_{\zeta}\hat{g}(\zeta) = (-i)^{|\rho|}\widehat{w^{\rho}g}(\zeta)$,

$$\begin{aligned}
\left|\partial^{\gamma}_{w}\partial^{\delta}_{\zeta}\left(e^{\psi(w,\zeta)}\hat{g}(\zeta)\right)\right| &\leq \sum_{\nu+\rho=\delta} \binom{\delta}{\nu} \left|\partial^{\gamma}_{w}\partial^{\nu}_{\zeta} e^{\psi(w,\zeta)}\partial^{\rho}_{\zeta}\hat{g}(\zeta)\right| \\
&\leq \sum_{\nu+\rho=\delta} \binom{\delta}{\nu} C_{\gamma+\nu} \sum_{0<h\leq|\gamma+\nu|} e^{\psi(w,\zeta)} \\
&\quad \times \sum_{\substack{\gamma^{(1)}+\cdots+\gamma^{(h)}=\gamma \\ \nu^{(1)}+\cdots+\nu^{(h)}=\nu}} |\partial^{\gamma^{(1)}}_{w}\partial^{\nu^{(1)}}_{\zeta}\psi(w,\zeta)| \ldots |\partial^{\gamma^{(h)}}_{w}\partial^{\nu^{(h)}}_{\zeta}\psi(w,\zeta)|\, |\widehat{w^{\rho}g}(\zeta)|;
\end{aligned}$$

since $g \in G^{\lambda}_{x,0}(\mathbb{R}^{p+q})$ we have $w^{\rho}g \in G^{\lambda}_{x,0}(\mathbb{R}^{p+q})$ for every $\rho \in \mathbb{Z}^{p+q}_{+}$; recall moreover that ψ is assumed to satisfy the estimates (2.6), in particular (2.7) and (2.8). Then we can apply (2.13) and (2.8), obtaining:

$$\left|\partial^{\gamma}_{w}\partial^{\delta}_{\zeta}\left(e^{\psi(w,\zeta)}\hat{g}(\zeta)\right)\right| \leq \tilde{C}_{\gamma\delta}\langle\xi\rangle^{r|\gamma+\nu|}e^{c\langle\xi\rangle^{r}}\, e^{-\varepsilon\sum_{j=1}^{p}(1+|\xi_j|)^{1/\lambda_j}}. \tag{2.15}$$

Recall that the weight function $\psi(w,\zeta)$ vanishes for $\zeta\notin\Gamma_\phi$, cf. (2.5); if $\zeta\in\Gamma_\phi$ by (2.15) we have that for every N we can find a constant $\overline{C}_{\gamma\delta}(N)$ satisfying

$$\left|\partial_w^\gamma\partial_\zeta^\delta\big(e^{\psi(w,\zeta)}\hat{g}(\zeta)\big)\right|\leq\overline{C}_{\gamma\delta}(N)\langle\xi\rangle^{-N}\leq C_{\gamma\delta}(N)\langle\zeta\rangle_\phi^{-N},$$

by definition of Γ_ϕ; this implies that (2.14) is satisfied for $\zeta\in\Gamma_\phi$. On the other hand for $\zeta\notin\Gamma_\phi$ we have $e^{\psi(w,\zeta)}\hat{g}(\zeta)\equiv\hat{g}(\zeta)$; since $g(w)\in C_0^\infty(\mathbb{R}^{p+q})\subset\mathcal{S}(\mathbb{R}^{p+q})$, then $\hat{g}(\zeta)\in\mathcal{S}(\mathbb{R}^{p+q})$, and so (2.14) is fulfilled for every (w,ζ). The proof is then complete. $\square$

3 Main result

We want to construct a parametrix of the operator (1.3) in the Gevrey-Sobolev spaces $\mathbb{H}^{s,\psi}_{\phi,r}$, and then we shall prove Theorem 1.1. To this aim, we start by fixing the weight function $\psi=\psi(w,\zeta)$ in the following way:

$$\psi(w,\zeta)=\left(1\pm\sum_{j=1}^p\frac{x_j}{2\delta}\,\varphi_j(\xi)\right)\langle\eta\rangle_\sigma^r\,\chi(\zeta),\tag{3.1}$$

with the usual splitting $w=(x,y)$, $\zeta=(\xi,\eta)$, and moreover:

(i) $\varphi_j(\xi)=c_j\frac{\partial_{\xi_j}p_1(0,\xi)}{\langle\xi\rangle^{m-1}}$ for every $j=1,\dots,p$, with $c_j>0$ sufficiently small;

(ii) $\chi(\zeta)\in G^\epsilon(\mathbb{R}^{p+q})$, $\epsilon=(\epsilon_1,\dots,\epsilon_{p+q})$ as in (2.3); moreover, if Γ_1 and Γ_2 are two (sufficiently small) ϕ-conical neighborhoods of the set $\{\zeta\in\mathbb{R}^{p+q}:p_1(0,\xi)=p_2(0,\eta)\}$, $\Gamma_1\subset\Gamma_2$, we suppose that:

- $\chi(\zeta)\equiv1$ for $\zeta\in\Gamma_1$, $\langle\zeta\rangle_\phi\geq M$
- $\chi(\zeta)\equiv0$ for $\zeta\notin\Gamma_2$, $\langle\zeta\rangle_\phi\geq M$

for a constant $M>0$.

We can now state the main result, concerning the existence of a parametrix.

Theorem 3.1. *Let λ and μ be fixed as in Proposition 2.7; we suppose that the coefficients of the operator $P(w,D)$ are in $G^\lambda_{x,0}(\Omega)$, or alternatively in $G^\mu_{y,0}(\Omega)$, Ω being a (sufficiently small) neighborhood of the origin. Let us assume that the condition (1.7) is satisfied, and that there exists a neighborhood $\Omega\times\Gamma_\phi$ of the anisotropic characteristic manifold Σ_ϕ, Γ_ϕ being a ϕ-conical set, such that*

$$\Im\,p_2(w,\eta)\text{ does not change sign for }(w,\zeta)=(w,\xi,\eta)\in\Omega\times\Gamma_\phi.\tag{3.2}$$

Moreover, let us fix $r\in(0,1)$ in such a way that

$$r>\max\Big\{\frac{1}{2},m^*-m+1\Big\},\tag{3.3}$$

m^* being the order of $Q(w,D)$, cf. (1.6). *We fix the weight function $\psi(w,\zeta)$ as in* (3.1), *with sign $-$ (respect. $+$) depending on the fact that $\Im p_2(w,\eta)\ge 0$ (resp. ≤ 0). Then there exists a linear map*

$$E:\mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q})\to\mathbb{H}^{s+m-(1-r),\psi}_{\phi,r}(\mathbb{R}^{p+q}) \tag{3.4}$$

such that

$$P(w,D)Eu=\chi(w)u+Ru \tag{3.5}$$

for every $u\in\mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q})$, where $\chi(w)\in G_0^{(\lambda,\mu)}(\Omega)$, $\chi(w)\equiv 1$ in a neighborhood of the origin, and

$$R:\mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q})\to\mathbb{H}^{s+(1-r),\psi}_{\phi,r}(\mathbb{R}^{p+q}). \tag{3.6}$$

In order to prove Theorem 3.1 we need some technical results. The idea is to transform (1.3) into another operator $\widetilde{P}$ ('conjugate operator', see below), prove the existence of a parametrix for $\widetilde{P}$ in H^s_ϕ spaces, and then transfer back to the original P the existence of a parametrix as stated in Theorem 3.1.

Let us fix now $\lambda\in\mathbb{R}^p$ and $\mu\in\mathbb{R}^q$ satisfying the assumptions of Proposition 2.7 and fix consequently ϵ in Definition 2.2 in such a way that

$$\epsilon\le(\lambda,\mu), \tag{3.7}$$

in the sense that $\epsilon_j\le\lambda_j$ for $j=1,\dots,p$ and $\epsilon_{p+h}\le\mu_h$ for $h=1,\dots,q$. We consider a symbol $a(w,\zeta)$ that is compactly supported in the w-variable and has the following property: there exists a positive constant C such that for every $\gamma,\delta\in\mathbb{Z}^{p+q}_+$ and $w,\zeta\in\mathbb{R}^{p+q}$

$$|D^\gamma_w D^\delta_\zeta a(w,\zeta)|\le C^{1+|\gamma|+|\delta|}\gamma!^{(\lambda,\mu)}\,\delta!^{(\lambda,\mu)}\langle\zeta\rangle_\phi^{m-\langle\delta,\phi\rangle}, \tag{3.8}$$

where we mean $\gamma!^{(\lambda,\mu)}=\gamma_1!^{\lambda_1}\dots\gamma_p!^{\lambda_p}\,\gamma_{p+1}!^{\mu_1}\dots\gamma_{p+q}!^{\mu_q}$, and similarly for $\delta!^{(\lambda,\mu)}$. We then write as usual $A(w,D)$ for the pseudodifferential operator with symbol $a(w,\zeta)$.
We fix now a weight function $\psi(w,\zeta)$ and consider the conjugate operator

$$\widetilde{A}(w,D)=e^{\psi(w,D)}A(w,D)\big({}^{\mathrm{t}}e^{-\psi(w,-D)}\big), \tag{3.9}$$

where ${}^{\mathrm{t}}e^{-\psi(w,-D)}$ denotes the transposed operator of $e^{-\psi(w,-D)}$.

Theorem 3.2. *The symbol $\tilde{a}(w,\zeta)$ admits the following asymptotic expansion:*

$$\tilde{a}(w,\zeta)\sim\sum_{\gamma,\delta}\frac{1}{\gamma!\,\delta!}\,\partial^\delta_\zeta\big[D^\gamma_w a(w,\zeta)\partial^\gamma_\zeta e^{\psi(w,\zeta)}D^\delta_w e^{-\psi(w,\zeta)}\big], \tag{3.10}$$

in the sense that

$$\tilde{a}(w,\zeta)-\sum_{\langle\gamma+\delta,\phi\rangle<N}\frac{1}{\gamma!\,\delta!}\,\partial^\delta_\zeta\big[D^\gamma_w a(w,\zeta)\partial^\gamma_\zeta e^{\psi(w,\zeta)}D^\delta_w e^{-\psi(w,\zeta)}\big]:=r_N(w,\zeta)$$

satisfies the following estimates:

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} r_N(w,\zeta)| \le C_N^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)}\langle\zeta\rangle_\phi^{m-(1-r)N-\langle\tilde\delta,\phi\rangle},$$

for every $N\in\mathbb{N}$. *More precisely we can write* $\widetilde{A}(w,D) = A_\psi(w,D) + R_\psi(w,D)$, *where (writing* $a_\psi(w,\zeta)$ *and* $r_\psi(w,\zeta)$ *for the symbols of* $A_\psi(w,D)$ *and* $R_\psi(w,D)$, *respectively):*

(i) *there exist positive constants* C *and* c *such that*

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} r_\psi(w,\zeta)| \le C^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)}e^{-c\langle\zeta\rangle_\phi^r}, \tag{3.11}$$

for every $\tilde\gamma,\tilde\delta\in\mathbb{Z}_+^{p+q}$;

(ii) *the remainder*

$$s_\psi^{(N)}(w,\zeta) := a_\psi(w,\zeta) - \sum_{\langle\gamma+\delta,\phi\rangle<N}\frac{1}{\gamma!\,\delta!}\,\partial_\zeta^\delta\big[D_w^\gamma a(w,\zeta)\partial_\zeta^\gamma e^{\psi(w,\zeta)}D_w^\delta e^{-\psi(w,\zeta)}\big] \tag{3.12}$$

satisfies the following estimates:

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} s_\psi^{(N)}(w,\zeta)| \le C^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)}\langle\zeta\rangle_\phi^{m-(1-r)N-\langle\tilde\delta,\phi\rangle},$$

for every $\tilde\gamma,\tilde\delta\in\mathbb{Z}_+^{p+q}$.

Corollary 3.3. *We have the following facts:*

$$e^{\psi(w,D)}\big({}^{\mathrm t}e^{-\psi(w,-D)}\big) = I_H + R_1 \tag{3.13}$$

$${}^{\mathrm t}e^{-\psi(w,-D)}e^{\psi(w,D)} = I_{\mathbb{H}} + R_2 \tag{3.14}$$

where I_H *means the identity in the space* $H_\phi^s(\mathbb{R}^{p+q})$, $I_{\mathbb{H}}$ *means the identity in the space* $\mathbb{H}_{\phi,r}^{s,\psi}(\mathbb{R}^{p+q})$, *and*

$$R_1 : H_\phi^s(\mathbb{R}^{p+q}) \to H_\phi^{s+(1-r)}(\mathbb{R}^{p+q}), \tag{3.15}$$

$$R_2 : \mathbb{H}_{\phi,r}^{s,\psi}(\mathbb{R}^{p+q}) \to \mathbb{H}_{\phi,r}^{s+(1-r),\psi}(\mathbb{R}^{p+q}). \tag{3.16}$$

The proofs of Theorem 3.2 and Corollary 3.3 are very technical, and shall be presented in Section 4. Now we want to apply the conjugation to the operator (1.3). Let us consider then

$$\widetilde{P}(w,D) = e^{\psi(w,D)}P(w,D)\big({}^{\mathrm t}e^{-\psi(w,-D)}\big), \tag{3.17}$$

where we suppose that the coefficients of $P(w,D)$ are in the space $G_0^{(\lambda,\mu)}(\mathbb{R}^{p+q})$, with λ and μ satisfying the hypotheses of Proposition 2.7; then the symbol $p(w,\zeta)$ satisfies the estimates (3.8), so we can apply Theorem 3.2 to $p(w,\zeta)$, obtaining:

$$\tilde p(w,\zeta) = p(w,\zeta) - ip_{\mathrm{add}}(w,\zeta) + p_{m-(1-r)-\nu}(w,\zeta) \tag{3.18}$$

with

$$
\begin{aligned}
p_{\mathrm{add}}(w,\zeta) = & \sum_{j=1}^{p+q} \partial_{w_j}\big(p_1(w,\xi)-p_2(w,\eta)\big)\partial_{\zeta_j}\psi(w,\zeta) \\
& -\sum_{j=1}^{p+q} \partial_{\zeta_j}\Big[\big(p_1(w,\xi)-p_2(w,\eta)\big)\partial_{w_j}\psi(w,\zeta)\Big],
\end{aligned} \tag{3.19}
$$

where (in the chosen splitting of variables $\zeta=(\xi,\eta)$) $p_1(w,\xi)$ and $p_2(w,\eta)$ are the symbols of (1.4) and (1.5) respectively, and moreover $p_{m-(1-r)-\nu}(w,\zeta)$ satisfies

$$
|\partial_w^\gamma \partial_\zeta^\delta p_{m-(1-r)-\nu}(w,\zeta)| \le C^{1+|\gamma|+|\delta|}\gamma!^{(\lambda,\mu)}\,\delta!^{(\lambda,\mu)}\,\langle\zeta\rangle_\phi^{m-(1-r)-\nu-\langle\delta,\phi\rangle}
$$

for every $\gamma,\delta\in\mathbb{Z}_+^{p+q}$ and for a suitable fixed $\nu>0$.

We want now to prove that, for the chosen weight function $\psi(w,\zeta)$, the symbol $p_{\mathrm{add}}(w,\zeta)$ is real and does not change sign.

Lemma 3.4. *Let us fix the weight function $\psi(w,\zeta)$ as in* (3.1)*, with sign $-$. Then the symbol* (3.19) *satisfies*

$$
p_{\mathrm{add}}(w,\zeta)\ge c\langle\zeta\rangle_\phi^{m-(1-r)}, \tag{3.20}
$$

for a positive constant c and for all ζ belonging to $\Omega_\delta\times\Gamma_\Sigma$, where Γ_Σ is a ϕ-conical neighborhood of the anisotropic characteristic manifold Σ_ϕ, and $\Omega_\delta:=\{w\in\mathbb{R}^{p+q}:|w|\le\delta\}$.
If we choose the sign $+$ in (3.1) *we have that*

$$
p_{\mathrm{add}}(w,\zeta)\le -c\langle\zeta\rangle_\phi^{m-(1-r)}, \tag{3.21}
$$

for $c>0$ and $\zeta\in\Omega_\delta\times\Gamma_\Sigma$.

Proof. We limit ourselves to the case when $\psi(w,\zeta)$ is chosen as in (3.1) with sign $-$, since the other case is similar. We observe that with our choice of $\psi(w,\zeta)$ the symbol $p_{\mathrm{add}}(w,\zeta)$ becomes:

$$
p_{\mathrm{add}}(w,\zeta)=p^{(1)}_{\mathrm{add}}(w,\zeta)+p^{(2)}_{\mathrm{add}}(w,\zeta)+p^{(3)}_{\mathrm{add}}(w,\zeta),
$$

where:

$$
\begin{aligned}
& p^{(1)}_{\mathrm{add}}(w,\zeta)= \\
& =\sum_{j=1}^{p}\partial_{x_j}\big(p_1(w,\xi)-p_2(w,\eta)\big)\left\{\Big(1-\sum_{k=1}^{p}\frac{x_k}{2\delta}\,\varphi_k(\xi)\Big)\langle\eta\rangle_\sigma^r\partial_{\xi_j}\chi(\zeta)-\sum_{k=1}^{p}\frac{x_k}{2\delta}\,\partial_{\xi_j}\varphi_k(\xi)\,\langle\eta\rangle_\sigma^r\,\chi(\zeta)\right\} \\
& \quad +\sum_{h=1}^{q}\partial_{y_h}\big(p_1(w,\xi)-p_2(w,\eta)\big)\Big(1-\sum_{k=1}^{p}\frac{x_k}{2\delta}\,\varphi_k(\xi)\Big)\partial_{\eta_h}\big(\langle\eta\rangle_\sigma^r\,\chi(\zeta)\big),
\end{aligned}
$$

$$
p^{(2)}_{\mathrm{add}}(w,\zeta)=\frac{1}{2\delta}\sum_{j=1}^{p}\partial_{\xi_j}p_1(w,\xi)\varphi_j(\xi)\langle\eta\rangle_\sigma^r\,\chi(\zeta)
$$

and

$$p_{\text{add}}^{(3)}(w,\zeta) = \frac{1}{2\delta}\sum_{j=1}^{p}\big(p_1(w,\xi) - p_2(w,\xi)\big)\partial_{\xi_j}\big(\varphi_j(\xi)\langle\eta\rangle_\sigma^r\,\chi(\zeta)\big).$$

The particular form of $\varphi_j(\xi)$ gives us $p_{\text{add}}^{(2)}(0,\zeta) = \frac{1}{2\delta}\sum\limits_{j=1}^{p} c_j \frac{(\partial_{\xi_j} p_1(0,\xi))^2}{\langle\xi\rangle^{m-1}}\langle\eta\rangle_\sigma^r\,\chi(\zeta)$ and so we obtain

$$p_{\text{add}}^{(2)}(w,\zeta) \geq \frac{c}{2\delta}\langle\zeta\rangle_\phi^{m-(1-r)}, \tag{3.22}$$

for $(w,\zeta) \in \Omega_\delta \times \Gamma_\Sigma$. Moreover, since $w \in \Omega_\delta$ we have $x_k \in (-\delta,\delta)$ for every $k = 1,\ldots,p$; then

$$|p_{\text{add}}^{(1)}(w,\zeta)| \leq C_1\langle\zeta\rangle_\phi^{m-(1-r)}, \tag{3.23}$$

for a constant C_1 independent of δ. Regarding $p_{\text{add}}^{(3)}(w,\zeta)$ we can obtain $|p_1(w,\xi) - p_2(w,\eta)| \leq \varepsilon\langle\zeta\rangle_\phi^m$ for every $\varepsilon > 0$, by taking δ and Γ_Σ sufficiently small; then we have that

$$|p_{\text{add}}^{(3)}(w,\zeta)| \leq \frac{c}{\delta}\langle\zeta\rangle_\phi^{m-(1-r)} \tag{3.24}$$

for $(w,\zeta) \in \Omega_\delta \times \Gamma_\Sigma$, where c is the constant of the estimate (3.22). Then (3.22), (3.23) and (3.24) give us (3.20). □

Now we are ready to prove the main Theorem 3.1. In the proof we shall use a result on the C^∞ hypoellipticity of operators of the kind (1.3), proved in De Donno [5, Theorem 2.10]. For the sake of completeness we recall here the statement.

Theorem 3.5. *Let us consider the operator*

$$L(w,D) = P_1(w,D_x) - P_2(w,D_y) + \Lambda(w,D),$$

where $P_1(w,D_x)$ and $P_2(w,D_y)$ are given by (1.4) and (1.5) respectively, and $\Lambda(w,D)$ is an anisotropic pseudodifferential operator, whose symbol $\lambda(w,\zeta)$ is C^∞ and satisfies

$$|D_w^\gamma D_\zeta^\delta \lambda(w,\zeta)| \leq C_{\gamma,\delta}\langle\zeta\rangle_\phi^{\overline{m}-\langle\delta,\phi\rangle}$$

for every $\gamma,\delta \in \mathbb{Z}_+^{p+q}$, where $\overline{m}$ is a real number such that $m - \frac{1}{2} < \overline{m} < m$.
We suppose that $P_1(w,D_x)$ and $P_2(w,D_y)$ have C^∞ coefficients and that they satisfy (1.7); in addition we assume that there exists a ϕ-conical neighborhood $\Gamma \subset \mathbb{R}^{p+q}$ of the anisotropic characteristic manifold (2.4), such that for every $(w,\zeta) \in \Omega \times \Gamma$, Ω being a neighborhood of the origin in $\mathbb{R}^{p+q}$, the following conditions hold:

(i) $|\Im\,\lambda(w,\zeta)| \geq c\langle\zeta\rangle_\phi^{\overline{m}}$;

(ii) $\Im\,\lambda(w,\zeta)\,\Im\,p_2(w,\eta) \leq 0$.

Then there exists a constant $d > 0$, and for every $\gamma, \delta \in \mathbb{Z}_+^{p+q}$ we can find a positive constant $M_{\gamma,\delta}$ such that

$$|\ell(w,\zeta)| \geq d\langle\zeta\rangle_\phi^{\overline{m}}; \tag{3.25}$$

$$|D_w^\gamma D_\zeta^\delta \ell(w,\zeta)| \leq M_{\gamma,\delta}|\ell(w,\zeta)|\langle\zeta\rangle_\phi^{(m-\overline{m})\langle\gamma,\phi\rangle-(\overline{m}-m+1)\langle\delta,\phi\rangle}, \tag{3.26}$$

for every $(w,\zeta) \in \Omega\times\Gamma$, where we have denoted by $\ell(w,\zeta)$ the symbol of the operator $L(w,D)$.

Proof of Theorem 3.1. We first observe that by Lemma 3.4 the conjugate operator (3.17) satisfies the hypotheses of Theorem 3.5, for $(w,\zeta) \in \Omega \times \Gamma_\phi$, with $\overline{m} = m - (1-r)$ (observe that $m-\frac{1}{2} < \overline{m} < m$ in view of (3.3)). Then by Theorem 3.5 we have that there exists a linear operator $\widetilde{E}_1 : H_\phi^s \to H_\phi^{s+m-(1-r)}$ satisfying $\widetilde{P}\widetilde{E} = \chi(w)\kappa(D) + \widetilde{R}$, where $\chi(w) \in G_0^{(\lambda,\mu)}(\Omega)$, the symbol $\kappa(\zeta)$ (anisotropic of order 0) is supported in a neighborhood of Σ_ϕ, and $\widetilde{R}$ is regularizing on H_ϕ^s, i.e. $\widetilde{R} : H_\phi^s \to H_\phi^t$ for every $t \geq 0$. Observe that we can choose $\kappa(\zeta)$ in such a way that the operator $\widetilde{P}$ is microlocally quasi elliptic on $\operatorname{supp}(1-\kappa(\zeta))$; we then have a microlocal parametrix of $\widetilde{P}$ also outside a neighborhood of the anisotropic characteristic manifold Σ_ϕ. Then a standard procedure of patching together there microlocal parametrices, cf. for example Gramchev-Rodino [10], Marcolongo-Oliaro [20], enables us to find a linear operator

$$\widetilde{E} : H_\phi^s(\mathbb{R}^{p+q}) \to H_\phi^{s+m-(1-r)}(\mathbb{R}^{p+q})$$

such that

$$\widetilde{P}\widetilde{E} = \chi(w) + \widetilde{R},$$

where $\chi(w) \in G_0^{(\lambda,\mu)}(\Omega)$ and

$$\widetilde{R} : H_\phi^s(\mathbb{R}^{p+q}) \to H_\phi^t(\mathbb{R}^{p+q})$$

for every $t \geq 0$. We then have

$${}^t e^{-\psi(w,-D)}\widetilde{P}\widetilde{E}e^{\psi(w,D)} = {}^t e^{-\psi(w,-D)}\chi(w)e^{\psi(w,D)} + {}^t e^{-\psi(w,-D)}\widetilde{R}e^{\psi(w,D)}. \tag{3.27}$$

Now, from Theorem 3.2 applied to the symbol $a(w,\zeta) = \chi(w)$ we have that

$$e^{\psi(w,D)}\chi(w){}^t e^{-\psi(w,-D)} - \chi(w) : H_\phi^s(\mathbb{R}^{p+q}) \to H_\phi^{s+(1-r)}(\mathbb{R}^{p+q});$$

then, from Corollary 3.3 it is easy to deduce that

$$R_\chi := {}^t e^{-\psi(w,-D)}\chi(w)e^{\psi(w,D)} - \chi(w) : \mathbb{H}_{\phi,r}^{s,\psi}(\mathbb{R}^{p+q}) \to \mathbb{H}_{\phi,r}^{s+(1-r),\psi}(\mathbb{R}^{p+q}). \tag{3.28}$$

By (3.28), (3.17) and (3.14), writing $E = {}^t e^{-\psi(w,-D)}\widetilde{E}e^{\psi(w,D)}$ and $S = {}^t e^{-\psi(w,-D)}\widetilde{R}e^{\psi(w,D)}$ we then have

$$(I_{\mathbb{H}} + R_2)PE = (\chi(w) + R_\chi) + S,$$

and so

$$PE = \chi(w) + R_\chi + S - R_2PE. \tag{3.29}$$

Now, for $\chi \in G_0^{(\lambda,\mu)}$ and $v \in \mathbb{H}^{s,\psi}_{\phi,r}$ with λ and μ satisfying the hypotheses of Proposition 2.7 we have that $\chi v \in \mathbb{H}^{s,\psi}_{\phi,r}$; this is a consequence of (2.13) and Lemma 2.5. Observe moreover that, by definition, E satisfies (3.4); since

$$S : \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \to \mathbb{H}^{t,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \tag{3.30}$$

for every $t \geq 0$,

$$R_2 : \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \to \mathbb{H}^{s+(1-r),\psi}_{\phi,r}(\mathbb{R}^{p+q}) \tag{3.31}$$

by (3.16), and

$$R_2 PE : \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \to \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \tag{3.32}$$

by the mapping properties of E, P, R_2 and R_χ, we have from (3.29) that

$$PE : \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \to \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}). \tag{3.33}$$

Then (3.33) and (3.16) imply that

$$R_2 PE : \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \to \mathbb{H}^{s+(1-r),\psi}_{\phi,r}(\mathbb{R}^{p+q}). \tag{3.34}$$

Now defining $R = R_\chi + S - R_2 PE$ we obtain (3.4) from (3.29) and we deduce (3.6) from (3.30)-(3.31)-(3.34). □

Theorem 3.6. *Let us suppose that the hypotheses of Theorem* 3.1 *are satisfied. Then the equation*

$$P(w, D)u = f \tag{3.35}$$

admits a local solution for every $f \in \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q})$; taking s sufficiently large the solution is classical.

Proof. We shall prove the local solvability of (3.35) by fixed point arguments. We want to find a solution u of the equation (3.35) of the form $u = Ev$; replacing in (3.35) such a function and using the fact that $PE = I + R$, cf. the previous Theorem 3.1 we obtain

$$v = f - Rv;$$

we then have to find a fixed point of the operator $\mathcal{Q}v = f - Rv$.
Now since $R : \mathbb{H}^{s,\psi}_{\phi,r} \to \mathbb{H}^{s+(1-r),\psi}_{\phi,r} \hookrightarrow \mathbb{H}^{s,\psi}_{\phi,r}$, the same technique as in Gramchev-Popivanov [8], cf. also Gramchev-Rodino [10] allows us to find a positive, continuous, non-decreasing function $\mathcal{R} : [0, \delta_0] \to [0, +\infty]$ such that $\mathcal{R}(0) = 0$ and, writing $\Omega_\delta = \{w \in \mathbb{R}^{p+q} : |w| \leq \delta\}$, $\delta \leq \delta_0$, we have

$$\|Rv\|_{s,\psi,\phi,r} \leq \mathcal{R}(\delta)\|v\|_{s,\psi,\phi,r}$$

for every $v \in \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q})$ with support contained in Ω_δ. Let us consider the complete metric space $\mathcal{B} = \{v \in \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \cap G_0^{(\lambda,\mu)}(\Omega_\delta) : \|v - f\|_{s,\psi,\phi,r} \leq 1\}$. We then have:

(i) $\mathcal{Q}: \mathcal{B} \to \mathcal{B}$: if $v \in \mathcal{B}$ we have

$$\|\mathcal{Q}v - f\|_{s,\psi,\phi,r} = \|Rv\|_{s,\psi,\phi,r} \leq \mathcal{R}(\delta)\|v\|_{s,\psi,\phi,r} \leq 1$$

for δ sufficiently small;

(ii) $\mathcal{Q}$ is a contraction on $\mathcal{B}$: if $u, v \in \mathcal{B}$ we obtain

$$\|\mathcal{Q}u - \mathcal{Q}v\|_{s,\psi,\phi,r} = \|R(u-v)\|_{s,\psi,\phi,r} \leq \mathcal{R}(\delta)\|u-v\|_{s,\psi,\phi,r},$$

and $\mathcal{R}(\delta) < 1$ for δ sufficiently small.

So if $\delta \ll 1$ we obtain the desired result by applying the Fixed Point Theorem in the space $\mathcal{B}$. □

4 Proof of the symbolic calculus for the conjugation

We give in this section the proof of Theorem 3.2 and Corollary 3.3.
In the following we shall need an anisotropic version of the Taylor formula.

Proposition 4.1. *Let ϕ be as in* (1.2)*. We consider a function $u(w)$ such that the derivatives $\partial^\alpha u$ exist and are continuous for all γ satisfying $\langle \gamma, \phi \rangle < k + \max_{j=1,\ldots,n} \phi_j$. Then for every $w, z \in \mathbb{R}^n$ we have:*

$$u(w+z) = \sum_{\langle \gamma,\phi \rangle < k} \frac{z^\gamma}{\gamma!}\partial^\gamma u(w) + \sum_{k \leq \langle \gamma,\phi \rangle < k + \max \phi_j} c_\gamma \frac{z^\gamma}{\gamma!} \int_0^1 (1-t)^{|\gamma|-1} \partial^\gamma u(w+tz)\, dt, \quad (4.1)$$

where

$$c_\gamma = \sum_{j \in I_\gamma} \gamma_j, \quad I_\gamma = \{j : \gamma_j \neq 0 \text{ and } \langle \gamma^{(j)}, \phi \rangle < k\}, \quad (4.2)$$

$\gamma^{(j)}$ *being the multi-index*

$$\gamma^{(j)} = (\gamma_1, \ldots, \gamma_{j-1}, \gamma_j - 1, \gamma_{j+1}, \ldots, \gamma_n). \quad (4.3)$$

We observe that the remainder in (4.1) contains derivatives of u whose anisotropic order $\langle \gamma, \phi \rangle$ is close to k; this is of crucial importance in the proof of the symbolic calculus stated in Theorem 3.2. Since we did not find, in the anisotropic case, a formula with such a feature in the literature, we give here the proof.

Proof of Proposition 4.1. Let us consider the function

$$v_k(w,z,t) = \sum_{\langle \gamma,\phi \rangle < k} (1-t)^{|\gamma|} \frac{z^\gamma}{\gamma!} (\partial^\gamma u)(w+tz);$$

we observe that

$$v_k(w,z,1) = u(w+z) \tag{4.4}$$

$$v_k(w,z,0) = \sum_{\langle\gamma,\phi\rangle<k} \frac{z^\gamma}{\gamma!}(\partial^\gamma u)(w). \tag{4.5}$$

Now we want to compute

$$\begin{aligned}\frac{\partial}{\partial t}v_k(w,z,t) = &\sum_{0<\langle\gamma,\phi\rangle<k} \Big\{-|\gamma|(1-t)^{|\gamma|-1}\frac{z^\gamma}{\gamma!}(\partial^\gamma u)(w+tz)\Big\} \\ &+ \sum_{\langle\gamma,\phi\rangle<k}\Big\{(1-t)^{|\gamma|}\frac{z^\gamma}{\gamma!}\sum_{j=1}^n(\partial^\gamma\partial_{w_j}u)(w+tz)z_j\Big\}.\end{aligned} \tag{4.6}$$

We can write the second sum in the right-hand side of (4.6) in the following way:

$$S := \sum_{\langle\gamma,\phi\rangle<k}\sum_{j=1}^n(\gamma_j+1)(1-t)^{|\gamma|}\frac{z^\gamma z_j}{\gamma!(\gamma_j+1)}(\partial^\gamma\partial_{w_j}u)(w+tz). \tag{4.7}$$

Let us put $\delta = (\gamma_1,\dots,\gamma_{j-1},\gamma_j+1,\gamma_{j+1},\dots,\gamma_n)$; observe that $\gamma = \delta^{(j)}$, cf. (4.3). Then S involves terms of the type $\delta_j(1-t)^{|\delta|-1}\frac{z^\delta}{\delta!}(\partial^\delta u)(w+tz)$, where δ satisfies $0 < \langle\delta,\phi\rangle < k + \max\limits_{j=1,\dots,n}\phi_j$. Let us consider two cases:

(i) if $0 < \langle\delta,\phi\rangle < k$, the addends on (4.7) involving $\partial^\delta u$ are exactly the terms with $\gamma = \delta^{(j)}$ for every j such that $\delta_j \neq 0$, so they are given by

$$\sum_{j:\delta_j\neq 0}\delta_j(1-t)^{|\delta|-1}\frac{z^\delta}{\delta!}(\partial^\delta u)(w+tz) = |\delta|(1-t)^{|\delta|-1}\frac{z^\delta}{\delta!}(\partial^\delta u)(w+tz);$$

(ii) let us consder now $k \le \langle\delta,\phi\rangle < k + \max\limits_{j=1,\dots,n}\phi_j$; the terms in (4.7) that give rise to such δ are the ones with $\gamma = \delta^{(j)}$ for all $j \in I_\delta$. Then in this case the terms of S involving $\partial^\delta u$, for fixed δ, are given by

$$\sum_{j\in I_\delta}\delta_j(1-t)^{|\delta|-1}\frac{z^\delta}{\delta!}(\partial^\delta u)(w+tz) = c_\delta(1-t)^{|\delta|-1}\frac{z^\delta}{\delta!}(\partial^\delta u)(w+tz).$$

Applying in (4.6) the arguments developed in (i)-(ii) we then have

$$\frac{\partial}{\partial t}v_k(w,z,t) = \sum_{k\le\langle\delta,\phi\rangle<k+\max\phi_j} c_\delta(1-t)^{|\delta|-1}\frac{z^\delta}{\delta!}(\partial^\delta u)(w+tz). \tag{4.8}$$

The conclusion then follows from (4.4), (4.5), (4.8) and the Fundamental Theorem of Calculus. □

The asymptotic expansion (3.10) has been proved, in the simpler case when the weight function $\psi(w,\zeta)$ does not depend at the same time on a variable and the corresponding covariable, in [20]; a proof in the more involved case $\psi=\psi(w,\zeta)$ was given (even for infinite order symbols $a(w,\zeta)$) in [16], Proposition 2.13, but only in the isotropic case and for analytic weight functions $\psi(w,\zeta)$. We give now the proof of Theorem 3.2, following the technique of [16]. Let us start by giving two technical lemmas.

Lemma 4.2. *Let $F(w,\zeta,\tilde\zeta)\in C^\infty(\mathbb{R}^{3(p+q)})$ be a function such that, for every fixed $w,\zeta\in\mathbb{R}^{p+q}$, $F(w,\zeta,\tilde\zeta)$ is bounded in $\tilde\zeta$ and all the derivatives $\partial_{\tilde\zeta_j}F(w,\zeta,\tilde\zeta)$, $j=1,\dots,p+q$, are compactly supported in $\tilde\zeta$. Suppose moreover that $F(w,\zeta,\tilde\zeta)\equiv 0$ for $|\tilde\zeta|<c_0$ for a positive constant c_0 (eventually depending on w and ζ). Then*

$$\int e^{-i\tilde w\tilde\zeta}F(w,\zeta,\tilde\zeta)\,d\tilde w\,d\tilde\zeta=0. \tag{4.9}$$

Proof. Let $[\tilde\zeta]$ be a C^∞ function such that $[\tilde\zeta]=|\zeta|$ for $|\zeta|\geq c_0$, $[\tilde\zeta]\neq 0$ for all $\tilde\zeta\in\mathbb{R}^{p+q}$. Then

$$\int e^{-i\tilde w\tilde\zeta}F(w,\zeta,\tilde\zeta)\,d\tilde w\,d\tilde\zeta=\int e^{-i\tilde w\tilde\zeta}F(w,\zeta,\tilde\zeta)\frac{|\tilde\zeta|^2}{[\tilde\zeta]^2}\,d\tilde w\,d\tilde\zeta;$$

since $|\tilde\zeta|^2e^{-i\tilde w\tilde\zeta}=\Delta_{\tilde w}\big(e^{-i\tilde w\tilde\zeta}\big)$, an integration by parts gives (4.9). □

Let us fix now a function $\Phi(\zeta)\in G^\epsilon(\mathbb{R}^{p+q})$, ϵ being the same as in (2.3), with the following properties:

(i) $\Phi(\zeta)\equiv 1$ for $|\zeta|_\phi\leq\frac14$;

(ii) $\Phi(\zeta)\equiv 0$ for $|\zeta|_\phi\geq\frac12$, where

$$|\zeta|_\phi:=\sum_{h=1}^{p+q}|\zeta_h|^{1/\phi_h}.$$

We consider now the function

$$\Xi(\zeta,\tilde\zeta)=\Phi\bigg(\frac{\tilde\zeta_1}{\langle\zeta\rangle_\phi^{\phi_1}},\dots,\frac{\tilde\zeta_{p+q}}{\langle\zeta\rangle_\phi^{\phi_{p+q}}}\bigg); \tag{4.10}$$

we have the following result.

Lemma 4.3. (i) $\Xi(\zeta,\tilde\zeta)\equiv 1$ *for* $\frac{|\tilde\zeta|_\phi}{\langle\zeta\rangle_\phi}\leq\frac14$;

(ii) $\Xi(\zeta,\tilde\zeta)\equiv 0$ *for* $\frac{|\tilde\zeta|_\phi}{\langle\zeta\rangle_\phi}\geq\frac12$;

(iii) *there exists a positive constant C such that for every $\delta,\tilde\delta\in\mathbb{Z}_+^{p+q}$*

$$|\partial_\zeta^\delta\partial_{\tilde\zeta}^{\tilde\delta}\,\Xi(\zeta,\tilde\zeta)|\leq C^{1+|\delta|+|\tilde\delta|}\delta!^\epsilon\,\tilde\delta!^\epsilon\,\langle\zeta\rangle_\phi^{-\langle\delta+\tilde\delta,\phi\rangle}.$$

The results of Lemma 4.3 can be easily deduced from the corresponding properties of Φ and from Faà di Bruno formula; the proof is omitted.

Now in order to prove Theorem 3.2 we start by considering the following operator:

$$T(w,D) = e^{\psi(w,D)}A(w,D); \tag{4.11}$$

moreover, we fix functions

$$\varphi_j^R(\zeta) \in G^{(\lambda,\mu)}(\mathbb{R}^{p+q}), \tag{4.12}$$

$R \in \mathbb{R}$, $R > 0$, satisfying the following conditions for every $j \in \mathbb{Z}_+$:

(i) $0 \leq \varphi_j^R(\zeta) \leq 1$ for all $\zeta \in \mathbb{R}^{p+q}$;

(ii) $\varphi_j^R(\zeta) \equiv 1$ for $\langle\zeta\rangle_\phi^r > 2Rj$;

(iii) $\varphi_j^R(\zeta) \equiv 0$ for $\langle\zeta\rangle_\phi^r < Rj$;

(iv) $|\partial_\zeta^\delta \varphi_j^R(\zeta)| \leq C^{1+|\delta|}\delta!^{(\lambda,\mu)}\langle\zeta\rangle_\phi^{-\langle\delta,\phi\rangle}$, for every $\zeta \in \mathbb{R}^{p+q}$ and $\delta \in \mathbb{Z}_+^{p+q}$.

Let us now define

$$q(w,\zeta) = \sum_{j=0}^{\infty} \varphi_j^R(\zeta) q_j(w,\zeta), \tag{4.13}$$

where

$$q_j(w,\zeta) = \sum_{j \leq \langle\gamma,\phi\rangle < j+1} \frac{1}{\gamma!}\partial_\zeta^\gamma e^{\psi(w,\zeta)} D_w^\gamma a(w,\zeta),$$

and set

$$r(w,\zeta) := t(w,\zeta) - q(w,\zeta), \tag{4.14}$$

$t(w,\zeta)$ being the symbol of the operator (4.11), The following result then holds.

Proposition 4.4. (i) *There exist positive constants C and c such that*

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} q(w,\zeta)| \leq C^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)}\langle\zeta\rangle_\phi^{m-\langle\tilde\delta,\phi\rangle} e^{c\langle\zeta\rangle_\phi^r} \tag{4.15}$$

for every $\tilde\gamma, \tilde\delta \in \mathbb{Z}_+^{p+q}$;

(ii) *For every $c_0 > 0$ we can find a constant $C_0 = C_0(c_0)$ such that*

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} r(w,\zeta)| \leq C_0 C^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)} e^{-c_0\langle\zeta\rangle_\phi^r}, \tag{4.16}$$

$\tilde\gamma, \tilde\delta \in \mathbb{Z}_+^{p+q}$.

Proof. (i) Let us start by proving that for every $\tilde{\gamma}, \tilde{\delta} \in \mathbb{Z}_+^{p+q}$ the following estimate holds:

$$|\partial_w^{\tilde{\gamma}} \partial_\zeta^{\tilde{\delta}} e^{\psi(w,\zeta)}| \leq C^{1+|\tilde{\gamma}|+|\tilde{\delta}|} \tilde{\gamma}!^\epsilon \, \tilde{\delta}!^\epsilon \langle\zeta\rangle_\phi^{-\langle\tilde{\delta},\phi\rangle} e^{c\langle\zeta\rangle_\phi^r}. \tag{4.17}$$

To this aim, we need the Faà di Bruno formula: if $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}$ are two C^h functions, for every $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq h$, we have:

$$\partial^\alpha F(G(x)) = \sum_{k=1}^{|\alpha|} F^{(k)}(G(x)) B_{\alpha,k}\big(\{\partial^\beta G(x)\}\big), \tag{4.18}$$

where $B_{\alpha,k}(\{x_\beta\})$, for a sequence $\{x_\beta\}$, are the so called 'Bell polynomials', defined as follows:

$$B_{\alpha,k}(\{x_\beta\}) = \alpha! \sum_{\substack{\sum_{|\beta|>0} c_\beta = k \\ \sum_{|\beta|>0} c_\beta \beta = \alpha}} \prod_{\beta>0} \frac{1}{c_\beta!} \left(\frac{x_\beta}{\beta!}\right)^{c_\beta}. \tag{4.19}$$

We then have the following well know identity:

$$\sum_{|\alpha|\geq k} B_{\alpha,k}(\{x_\beta\}) \frac{z^\alpha}{\alpha!} = \frac{1}{k!} \left(\sum_{|\beta|>0} x_\beta \frac{z^\beta}{\beta!}\right)^k. \tag{4.20}$$

So, by (4.18) with $\alpha = (\tilde{\gamma}, \tilde{\delta})$ we have:

$$|\partial_w^{\tilde{\gamma}} \partial_\zeta^{\tilde{\delta}} e^{\psi(w,\zeta)}| \leq \tilde{\gamma}! \, \tilde{\delta}! \sum_{k=1}^{|\tilde{\gamma}|+|\tilde{\delta}|} e^{\psi(w,\zeta)} \sum_{\substack{\sum_{|\beta|>0} c_\beta = k \\ \sum_{|\beta|>0} c_\beta \beta = (\tilde{\gamma},\tilde{\delta})}} \prod_{\beta>0} \frac{1}{c_\beta!} \left(\frac{|\partial_w^{\beta^{(1)}} \partial_\zeta^{\beta^{(2)}} \psi(w,\zeta)|}{\beta!}\right)^{c_\beta}, \tag{4.21}$$

where we have split $\mathbb{Z}_+^{2(p+q)} \ni \beta = (\beta^{(1)}, \beta^{(2)})$, with $\beta^{(1)}, \beta^{(2)} \in \mathbb{Z}_+^{p+q}$. Now, from (2.3) and since $\sum_{|\beta|>0} c_\beta \beta^{(2)} = \tilde{\delta}$ we easily deduce that

$$\begin{aligned}
\prod_{\beta>0} |\partial_w^{\beta^{(1)}} \partial_\zeta^{\beta^{(2)}} \psi(w,\zeta)|^{c_\beta} &\leq C^{1+|\tilde{\gamma}|+|\tilde{\delta}|} \langle\zeta\rangle_\phi^{rk} \langle\zeta\rangle_\phi^{-\langle\tilde{\delta},\phi\rangle} \prod_{\beta>0} \big[\beta^{(1)}!^\epsilon \, \beta^{(2)}!^\epsilon\big]^{c_\beta} \\
&\leq C^{1+|\tilde{\gamma}|+|\tilde{\delta}|} \langle\zeta\rangle_\phi^{rk} \langle\zeta\rangle_\phi^{-\langle\tilde{\delta},\phi\rangle} \prod_{\beta>0} \big(\beta^{(1)}! \, \beta^{(2)}!\big)^{c_\beta} \tilde{\gamma}!^{\tilde{\epsilon}} \, \tilde{\delta}!^{\tilde{\epsilon}}
\end{aligned}$$

where

$$\tilde{\epsilon} := (\epsilon_1 - 1, \ldots, \epsilon_{p+q} - 1). \tag{4.22}$$

Then from (4.21) we deduce that

$$|\partial_w^{\tilde{\gamma}}\partial_\zeta^{\tilde{\delta}}e^{\psi(w,\zeta)}| \le \tilde{\gamma}!^{\tilde{\epsilon}}\,\tilde{\delta}!^{\tilde{\epsilon}}\sum_{k=1}^{|\tilde{\gamma}|+|\tilde{\delta}|} e^{\psi(w,\zeta)}C^{1+|\tilde{\gamma}|+|\tilde{\delta}|}\langle\zeta\rangle_\phi^{rk}\langle\zeta\rangle_\phi^{-\langle\tilde{\delta},\phi\rangle}B_{(\tilde{\gamma},\tilde{\delta}),k}\big(\{\beta!\}\big). \tag{4.23}$$

For every $z\in\mathbb{R}^{2(p+q)}$, $z\neq 0$, we have $B_{(\tilde{\gamma},\tilde{\delta}),k}\big(\{\beta!\}\big)\le \tilde{\gamma}!\,\tilde{\delta}!\frac{1}{z^{(\tilde{\gamma},\tilde{\delta})}}\sum_{|\gamma|+|\delta|\ge k} B_{(\gamma,\delta),k}\big(\{\beta!\}\big)\frac{z^{(\gamma,\delta)}}{\gamma!\,\delta!} = \tilde{\gamma}!\,\tilde{\delta}!\frac{1}{z^{(\tilde{\gamma},\tilde{\delta})}}\frac{1}{k!}\bigg(\sum_{|\beta|>0} z^\beta\bigg)^k$, as we can deduce from (4.20); this implies that, taking $|z|$ sufficiently small,

$$B_{(\tilde{\gamma},\tilde{\delta}),k}\big(\{\beta!\}\big)\le C\frac{\tilde{\gamma}!\,\tilde{\delta}!}{k!}. \tag{4.24}$$

Moreover, we observe that

$$\langle\zeta\rangle_\phi^{rk}\le k!e^{\langle\zeta\rangle_\phi^r}, \tag{4.25}$$

for every $\zeta\in\mathbb{R}^{p+q}$ and $k\in\mathbb{Z}_+$. We then obtain (4.17) by applying (4.24), (4.25) and (2.3) in (4.23), since by definition of $\tilde{\epsilon}$ we have $\tilde{\gamma}!^{\tilde{\epsilon}}\,\tilde{\delta}!^{\tilde{\epsilon}}\,\tilde{\gamma}!\,\tilde{\delta}! = \tilde{\gamma}!^{\epsilon}\,\tilde{\delta}!^{\epsilon}$.

We can then simply estimate the derivatives of $q_j(w,\zeta)$ by applying Leibnitz formula, (3.8), (4.17) and (3.7):

$$|\partial_w^{\tilde{\gamma}}\partial_\zeta^{\tilde{\delta}}q_j(w,\zeta)|\le C^{1+|\tilde{\gamma}|+|\tilde{\delta}|}\tilde{\gamma}!^{(\lambda,\mu)}\,\tilde{\delta}!^{(\lambda,\mu)}\langle\zeta\rangle_\phi^{m-\langle\tilde{\delta},\phi\rangle}e^{c\langle\zeta\rangle_\phi^r}\sum_{j\le\langle\gamma,\phi\rangle<j+1}\gamma!^{\tilde{\epsilon}}\,\gamma!^{(\lambda,\mu)}C^{1+|\gamma|}\langle\zeta\rangle_\phi^{-\langle\gamma,\phi\rangle}, \tag{4.26}$$

$\tilde{\epsilon}$ being given by (4.22).
We now pass to analyze the symbol $q(w,\zeta)$. Let us fix $\overline{\zeta}\in\mathbb{R}^{p+q}$, and prove (4.15) for this $\overline{\zeta}$. We can find $k\in\mathbb{N}$ such that $Rk\le\langle\overline{\zeta}\rangle_\phi^r<R(k+1)$. Then, due to the properties of $\varphi_j^R(\zeta)$, cf. (4.12), we have:

$$q(w,\overline{\zeta})=\sum_{j=0}^{k}\varphi_j^R(\overline{\zeta})q_j(w,\overline{\zeta}).$$

By Leibnitz rule, (4.26) and the property (iv) of the functions $\varphi_j^R(\zeta)$ we obtain

$$|\partial_w^{\tilde{\gamma}}\partial_\zeta^{\tilde{\delta}}q(w,\overline{\zeta})|\le\sum_{j=0}^{k}C^{1+|\tilde{\gamma}|+|\tilde{\delta}|}\tilde{\gamma}!^{(\lambda,\mu)}\,\tilde{\delta}!^{(\lambda,\mu)}\langle\overline{\zeta}\rangle_\phi^{m-\langle\tilde{\delta},\phi\rangle}e^{c\langle\overline{\zeta}\rangle_\phi^r}\sum_{j\le\langle\gamma,\phi\rangle<j+1}\gamma!^{\tilde{\epsilon}}\,\gamma!^{(\lambda,\mu)}C^{1+|\gamma|}\langle\overline{\zeta}\rangle_\phi^{-\langle\gamma,\phi\rangle}. \tag{4.27}$$

Observe that $\langle\overline{\zeta}\rangle_\phi^{-\langle\gamma,\phi\rangle}\le\langle\overline{\zeta}\rangle_\phi^{-j}\le R^{-j/r}k^{-j/r}$; the Stirling formula gives us $n^n\ge Cn!\,e^n\frac{1}{\sqrt{n}}$, for every $n\in\mathbb{N}$, and so

$$\langle\overline{\zeta}\rangle_\phi^{-\langle\gamma,\phi\rangle}\le\frac{j^{r/2}}{R^{j/r}C^{1/r}j!^{1/r}e^{j/r}}. \tag{4.28}$$

Observe now that $\gamma!^{\tilde{\epsilon}}\le|\gamma|!^{\max\tilde{\epsilon}_h}\le(j+1)!^{\max\tilde{\epsilon}_h}$. Moreover, since λ and μ satisfy the hypotheses of Proposition 2.7 we can find $\kappa>0$ such that $\lambda_j\le\frac{1}{r}-\kappa$ for every $j=1,\dots,p$

and $\frac{\mu_h}{\sigma_h} \le \frac{1}{r} - \kappa$ for every $h = 1, \dots, q$; then $\gamma!^{(\lambda,\mu)} \le \big(\gamma_1! \dots \gamma_p!\, \gamma_{p+1}!^{\sigma_1} \dots \gamma_{p+q}!^{\sigma_{p+q}}\big)^{\frac{1}{r}-\kappa} \le \widetilde{C}(j+1)!^{\frac{1}{r}-\kappa}$. This last inequality, together with (4.28), enables us to deduce from (4.27) that

$$\begin{aligned} |\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} q(w,\overline{\zeta})| \le \sum_{j=0}^{k} & C^{1+|\tilde\gamma|+|\tilde\delta|} \tilde\gamma!^{(\lambda,\mu)}\, \tilde\delta!^{(\lambda,\mu)} \langle\overline{\zeta}\rangle_\phi^{m-\langle\tilde\delta,\phi\rangle} e^{c\langle\overline{\zeta}\rangle_\phi^r} \\ & \times \left\{ \sum_{j\le\langle\gamma,\phi\rangle<j+1} C^{2+j} \frac{j!^{\frac{1}{r}-\kappa+\max\tilde\epsilon_h}(j+1)^{\frac{1}{r}-\kappa+\max\tilde\epsilon_h} j^{r/2}}{R^{j/r}C^{1/r} j!^{1/r}\, e^{j/r}} \right\}; \end{aligned} \tag{4.29}$$

then, choosing $\tilde\epsilon_h$ in such a way that $\max\tilde\epsilon_h \le \kappa$ we have that for R sufficiently large the quantity in brackets $\{\dots\}$ in (4.29) is the j-th term of a convergent series, so (4.15) is proved for $\zeta = \overline{\zeta}$; since $\overline{\zeta}$ is arbitrary (4.15) is true for all $\zeta \in \mathbb{R}^{p+q}$.

(ii) We pass now to analyze the remainder $r(w,\zeta)$, cf. (4.14), and we prove (4.16). From standard computations we have that the symbol $t(w,\zeta)$ of the operator (4.11) is given by

$$t(w,\zeta) = \int e^{-i\tilde w\tilde\zeta} e^{\psi(w,\zeta+\tilde\zeta)} a(w+\tilde w,\zeta)\, d\tilde w\, đ\tilde\zeta;$$

then, remembering the definition of $q(w,\zeta)$, cf. (4.13), we can write $r(w,\zeta)$ in the following way:

$$r(w,\zeta) = \sum_{N=0}^{\infty} \big\{ r_{1N}(w,\zeta) + r_{2N}(w,\zeta) \big\}, \tag{4.30}$$

where

$$\begin{aligned} r_{1N}(w,\zeta) = & \big(\varphi_N^R(\zeta) - \varphi_{N+1}^R(\zeta)\big) \\ & \times \left\{ \int e^{-i\tilde w\tilde\zeta} e^{\psi(w,\zeta+\tilde\zeta)} a(w+\tilde w,\zeta)\, \Xi(\zeta,\tilde\zeta)\, d\tilde w\, đ\tilde\zeta - \sum_{j=0}^{N} q_j(w,\zeta) \right\} \end{aligned} \tag{4.31}$$

and

$$r_{2N}(w,\zeta) = \big(\varphi_N^R(\zeta) - \varphi_{N+1}^R(\zeta)\big) \int e^{-i\tilde w\tilde\zeta} e^{\psi(w,\zeta+\tilde\zeta)} a(w+\tilde w,\zeta) \big[1 - \Xi(\zeta,\tilde\zeta)\big]\, d\tilde w\, đ\tilde\zeta, \tag{4.32}$$

Ξ being the function (4.10). Consider at first $r_{1N}(w,\zeta)$: using the anisotropic Taylor formula, cf. Proposition 4.1, for $a(w+\tilde w,\zeta)$ we can split

$$r_{1N}(w,\zeta) = r_{1N}^{(1)}(w,\zeta) + r_{1N}^{(2)}(w,\zeta), \tag{4.33}$$

where

$$\begin{aligned} r_{1N}^{(1)}(w,\zeta) = & \big(\varphi_N^R(\zeta) - \varphi_{N+1}^R(\zeta)\big) \\ \times & \left\{ \sum_{\langle\gamma,\phi\rangle<N+1} \int e^{-i\tilde w\tilde\zeta} e^{\psi(w,\zeta+\tilde\zeta)} \frac{\tilde w^\gamma}{\gamma!} \partial_w^\gamma a(w,\zeta)\, \Xi(\zeta,\tilde\zeta)\, d\tilde w\, đ\tilde\zeta - \sum_{j=0}^{N} q_j(w,\zeta) \right\}, \end{aligned} \tag{4.34}$$

$$
\begin{aligned}
r_{1N}^{(2)}(w,\zeta) &= \big(\varphi_N^R(\zeta)-\varphi_{N+1}^R(\zeta)\big) \sum_{N+1\le\langle\gamma,\phi\rangle<N+1+\max\phi_j} c_\gamma \\
&\quad \times \int_0^1 \int_{\mathbb{R}^{2(p+q)}} e^{-i\tilde{w}\tilde{\zeta}}\, \frac{\tilde{w}^\gamma}{\gamma!} e^{\psi(w,\zeta+\tilde{\zeta})}(1-t)^{|\gamma|-1}\, \partial_w^\gamma a(w+t\tilde{w},\zeta)\, \Xi(\zeta,\tilde{\zeta})\, d\tilde{w}\, đ\tilde{\zeta}\, dt.
\end{aligned} \tag{4.35}
$$

Regarding $r_{1N}^{(1)}(w,\zeta)$, since $\tilde{w}^\gamma e^{-i\tilde{w}\tilde{\zeta}} = (-1)^{|\gamma|} D_{\tilde{\zeta}}^\gamma e^{-i\tilde{w}\tilde{\zeta}}$, we can integrate by parts, obtaining

$$
\begin{aligned}
r_{1N}^{(1)}(w,\zeta) &= \big(\varphi_N^R(\zeta)-\varphi_{N+1}^R(\zeta)\big) \\
&\times \sum_{\langle\gamma,\phi\rangle<N+1} \left\{ \int e^{-i\tilde{w}\tilde{\zeta}} \frac{1}{\gamma!} \partial_{\tilde{\zeta}}^\gamma e^{\psi(w,\zeta+\tilde{\zeta})} D_w^\gamma a(w,\zeta)\, \Xi(\zeta,\tilde{\zeta})\, d\tilde{w}\, đ\tilde{\zeta} - \frac{1}{\gamma!} \partial_\zeta^\gamma e^{\psi(w,\zeta)} D_w^\gamma a(w,\zeta) \right\} \\
&\quad + \big(\varphi_N^R(\zeta)-\varphi_{N+1}^R(\zeta)\big) \\
&\quad\quad \times \sum_{\langle\gamma,\phi\rangle<N+1} \sum_{\substack{\delta\le\gamma \\ \delta\neq 0}} \binom{\gamma}{\delta} \int e^{-i\tilde{w}\tilde{\zeta}} \frac{1}{\gamma!} \partial_{\tilde{\zeta}}^{\gamma-\delta} e^{\psi(w,\zeta+\tilde{\zeta})} D_w^\gamma a(w,\zeta)\, \partial_{\tilde{\zeta}}^\delta \Xi(\zeta,\tilde{\zeta})\, d\tilde{w}\, đ\tilde{\zeta};
\end{aligned}
$$

the condition $\delta \neq 0$ in the second addend implies that $\partial_{\tilde{\zeta}}^\delta \Xi(\zeta,\tilde{\zeta}) \equiv 0$ for $|\tilde{\zeta}| \ll 1$, cf. (i) in Lemma 4.3, and so by Lemma 4.2 the second addend of $r_{1N}^{(1)}(w,\zeta)$ vanishes identically. Then a taylor expansion of $\partial_{\tilde{\zeta}}^\gamma e^{\psi(w,\zeta+\tilde{\zeta})}$ stopped at the first order gives us

$$
\begin{aligned}
& r_{1N}^{(1)}(w,\zeta) = \\
&= \big(\varphi_N^R(\zeta)-\varphi_{N+1}^R(\zeta)\big) \sum_{\langle\gamma,\phi\rangle<N+1} \frac{1}{\gamma!} \partial_\zeta^\gamma e^{\psi(w,\zeta)} D_w^\gamma a(w,\zeta) \int e^{-i\tilde{w}\tilde{\zeta}} \big[\Xi(\zeta,\tilde{\zeta})-1\big]\, d\tilde{w}\, đ\tilde{\zeta} \\
&\quad + \big(\varphi_N^R(\zeta)-\varphi_{N+1}^R(\zeta)\big) \sum_{\langle\gamma,\phi\rangle<N+1} \frac{1}{\gamma!} D_w^\gamma a(w,\zeta) \\
&\quad\quad \times \sum_{j=1}^{p+q} \int_0^1 \int_{\mathbb{R}^{2(p+q)}} e^{-i\tilde{w}\tilde{\zeta}}\, \tilde{\zeta}_j\, \partial_{\zeta_j} \partial_\zeta^\gamma e^{\psi(w,\zeta+t\tilde{\zeta})}\, \Xi(\zeta,\tilde{\zeta})\, d\tilde{w}\, đ\tilde{\zeta}\, dt,
\end{aligned} \tag{4.36}
$$

since

$$
\int e^{-i\tilde{w}\tilde{\zeta}}\, d\tilde{w}\, đ\tilde{\zeta} = 1. \tag{4.37}
$$

By Lemma 4.2 the first addend in the right hand side of (4.36) vanishes, since $\Xi(\zeta,\tilde{\zeta})-1 \equiv 0$ for $|\tilde{\zeta}| \ll 1$; the second addend also vanishes, as we can deduce by an integration by parts. This implies that

$$
r_{1N}^{(1)}(w,\zeta) \equiv 0. \tag{4.38}
$$

We pass now to the analysis of (4.35); multiplying in the integral by $\frac{(1+|\tilde{w}|^2)^M}{(1+|\tilde{w}|^2)^M}$ and integrating by parts we can write $r_{1N}^{(2)}(w,\zeta)$ in the following form:

$$\begin{aligned} r_{1N}^{(2)}(w,\zeta) = \big(\varphi_N^R(\zeta)-\varphi_{N+1}^R(\zeta)\big) &\sum_{N+1\le\langle\gamma,\phi\rangle<N+1+\max\phi_j} c_\gamma \int_0^1 (1-t)^{|\gamma|-1} \\ &\times \int e^{-i\tilde{w}\tilde{\zeta}}(1+|\tilde{w}|^2)^{-M} r_{1N}^{\gamma}(w,\zeta,t\tilde{w},\tilde{\zeta})\, d\tilde{w}\, đ\tilde{\zeta}\, dt, \end{aligned} \tag{4.39}$$

where

$$r_{1N}^{\gamma}(w,\zeta,t\tilde{w},\tilde{\zeta}) = \frac{1}{\gamma!}\partial_w^\gamma a(w+t\tilde{w},\zeta)(1+\Delta_{\tilde{\zeta}})^M D_{\tilde{\zeta}}^\gamma\Big\{e^{\psi(w,\zeta+\tilde{\zeta})}\,\Xi(\zeta,\tilde{\zeta})\Big\}, \tag{4.40}$$

M being a fixed positive integer, sufficiently large. Since $\Xi(\zeta,\tilde{\zeta})=0$ for $\frac{|\tilde{\zeta}|_\phi}{\langle\zeta\rangle_\phi}\ge\frac{1}{2}$, cf. Lemma 4.3, in the analysis of $r_{1N}^{\gamma}(w,\zeta,t\tilde{w},\tilde{\zeta})$ we can suppose that $|\tilde{\zeta}|_\phi\le\frac{1}{2}\langle\zeta\rangle_\phi$. This implies that we may assume

$$\frac{1}{2}\langle\zeta\rangle_\phi \le \langle\zeta+\tilde{\zeta}\rangle_\phi \le \frac{3}{2}\langle\zeta\rangle_\phi, \tag{4.41}$$

since $\langle\zeta+\tilde{\zeta}\rangle_\phi\le\langle\zeta\rangle_\phi+|\tilde{\zeta}|_\phi$. Now (3.8), (4.17), (4.41), (3.7), condition (iii) of Lemma 4.3 and Leibnitz rule enable us to prove that for every $\tilde{\gamma},\tilde{\delta}\in\mathbb{Z}_+^{p+q}$ there exist positive constants C and c such that

$$|\partial_w^{\tilde{\gamma}}\partial_\zeta^{\tilde{\delta}} r_{1N}^{\gamma}(w,\zeta,t\tilde{w},\tilde{\zeta})| \le C^{1+|\tilde{\gamma}|+|\tilde{\delta}|}\tilde{\gamma}!^{(\lambda,\mu)}\,\tilde{\delta}!^{(\lambda,\mu)}C^{1+|\gamma|}\gamma!^{(\lambda,\mu)+\tilde{\epsilon}}\,\langle\zeta\rangle_\phi^{m-\langle\gamma+\tilde{\delta},\phi\rangle}e^{\bar{c}\langle\zeta\rangle_\phi^r}, \tag{4.42}$$

$\tilde{\epsilon}$ being given by (4.22). Let us analyze the constant c_γ in (4.39): by (4.2) we have

$$c_\gamma \le |\gamma| \le \langle\gamma,\phi\rangle \le N+1+\max\phi_j. \tag{4.43}$$

Now by Leibnitz rule, (4.42), the property (iv) of the functions (4.12) and the fact that $|\tilde{\zeta}|_\phi\le\frac{1}{2}\langle\zeta\rangle_\phi$, we then have:

$$\begin{aligned} |\partial_w^{\tilde{\gamma}}\partial_\zeta^{\tilde{\delta}} r_{1N}^{(2)}(w,\zeta)| \le (N+1+\max\phi_j)&C^{1+|\tilde{\gamma}|+|\tilde{\delta}|}\tilde{\gamma}!^{(\lambda,\mu)}\tilde{\delta}!^{(\lambda,\mu)}\langle\zeta\rangle_\phi^{m-\langle\tilde{\delta},\phi\rangle}e^{\bar{c}\langle\zeta\rangle_\phi^r}\langle\zeta\rangle_\phi^D \\ \times &\sum_{N+1\le\langle\gamma,\phi\rangle<N+1+\max\phi_j} C^{1+|\gamma|}\gamma!^{(\lambda,\mu)+\tilde{\epsilon}}\langle\zeta\rangle_\phi^{-\langle\gamma,\phi\rangle}\Upsilon_N(\zeta), \end{aligned} \tag{4.44}$$

where $\tilde{\epsilon}$ is given by (4.22), D satisfies

$$\int_{|\tilde{\zeta}|\le\frac{1}{2}\langle\zeta\rangle_\phi} d\tilde{\zeta} \le \mathrm{const}\langle\zeta\rangle_\phi^D,$$

and $\Upsilon_N(\zeta)$ is the characteristic function of the set

$$K_N = \{\zeta\in\mathbb{R}^{p+q} : RN \le \langle\zeta\rangle_\phi^r \le 2R(N+1)\} \tag{4.45}$$

(observe that $\operatorname{supp}(\varphi_N^R(\zeta) - \varphi_{N+1}^R(\zeta)) \subset K_N$). Since λ and μ satisfy the hypotheses of Proposition 2.7, we can find $\kappa > 0$ such that $\lambda_j \leq \frac{1}{r} - \kappa$, $j = 1, \dots, p$ and $\mu_h \leq \sigma_h(\frac{1}{r} - \kappa)$, $h = 1, \dots, q$. Then, since for every $\vartheta > 1$, $\vartheta \in \mathbb{R}$, and $n \in \mathbb{N}$

$$n! \leq \big([\vartheta n] + 1\big)!^{\frac{1}{\vartheta}}$$

($[\vartheta n]$ being the largest integer $\leq \vartheta n$) we have:

$$\begin{aligned}\gamma!^{(\lambda,\mu)} &\leq \gamma_1!^{\lambda_1} \dots \gamma_p!^{\lambda_p} \big([\sigma_1\gamma_{p+1}] + 1\big)!^{\frac{\mu_1}{\sigma_1}} \dots \big([\sigma_q\gamma_{p+q}] + 1\big)!^{\frac{\mu_q}{\sigma_q}} \\ &\leq \big\{\gamma_1! \dots \gamma_p! \big([\sigma_1\gamma_{p+1}] + 1\big)! \dots \big([\sigma_q\gamma_{p+q}] + 1\big)!\big\}^{\frac{1}{r}-\kappa} \\ &\leq \big(\gamma_1 + \dots + \gamma_p + [\sigma_1\gamma_{p+1}] + \dots + [\sigma_q\gamma_{p+q}] + q\big)!^{\frac{1}{r}-\kappa};\end{aligned}$$

now, since $\gamma_1 + \dots + \gamma_p + [\sigma_1\gamma_{p+1}] + \dots + [\sigma_q\gamma_{p+q}] \leq \langle\gamma, \phi\rangle < N + 1 + \max \phi_j$, using the well known estimate $(N + k)! \leq 2^k k! 2^N N!$ we can conclude that

$$\gamma!^{(\lambda,\mu)} \leq C 2^N N!^{\frac{1}{r}-\kappa}, \tag{4.46}$$

where C does not depend on N.
By (4.46) and since by Stirling formula $N^N \geq \frac{CN!e^N}{\sqrt{N}}$ for all $N \in \mathbb{N}$, we have:

$$\begin{aligned}(N + 1 + \max \phi_j) &\sum_{N+1\leq\langle\gamma,\phi\rangle<N+1+\max\phi_j} C^{1+|\gamma|}\gamma!^{(\lambda,\mu)+\tilde{\epsilon}}\langle\zeta\rangle_\phi^{-\langle\gamma,\phi\rangle}\Upsilon_N(\zeta) \\ &\leq \sum_{N+1\leq\langle\gamma,\phi\rangle<N+1+\max\phi_j} C_0C_1^N N!^{\frac{1}{r}-\kappa+\max\tilde{\epsilon}_j} R^{-N} N^{-\frac{N}{r}}\Upsilon_N(\zeta) \\ &\leq \frac{C_0C_1^N N^{\frac{1}{2r}}}{e^{\frac{N}{r}}R^N} N!^{\frac{1}{r}-\kappa+\max\tilde{\epsilon}_j-\frac{1}{r}}\Upsilon_N(\zeta);\end{aligned}$$

observe now that, choosing $\tilde{\epsilon}$ in such a way that $\max\tilde{\epsilon}_j < \kappa$ we have $N!^{\max\tilde{\epsilon}_j-\kappa} \leq C(\tilde{c})e^{-\tilde{c}N}$ for every $\tilde{c} > 0$; moreover, taking R sufficiently large we have that $A_N := \frac{C_0C_1^N N^{\frac{1}{2r}}}{e^{\frac{N}{r}}R^N}$ satisfies $\sum_{N=0}^{\infty} A_N, \infty$. We then have

$$\begin{aligned}(N + 1 + \max \phi_j) \sum_{N+1\leq\langle\gamma,\phi\rangle<N+1+\max\phi_j} C^{1+|\gamma|}\gamma!^{(\lambda,\mu)+\tilde{\epsilon}}\langle\zeta\rangle_\phi^{-\langle\gamma,\phi\rangle}\Upsilon_N(\zeta) &\leq C(\tilde{c})e^{-\tilde{c}N}A_N\Upsilon_N(\zeta) \\ &\leq C(\tilde{c})e^{-\frac{\tilde{c}}{2R}\langle\zeta\rangle_\phi^r}A_N,\end{aligned}$$

since $\langle\zeta\rangle_\phi^r \leq 2R(N + 1)$ on $\operatorname{supp}\Upsilon_N(\zeta)$. Taking into account the inequality $\langle\zeta\rangle_\phi^k \leq C_{k,r}e^{\langle\zeta\rangle_\phi^r}$ we then conclude by (4.44) that

$$|\partial_w^{\tilde{\gamma}}\partial_\zeta^{\tilde{\delta}} r_{1N}^{(2)}(w, \zeta)| \leq C(c_0)C^{1+|\tilde{\gamma}|+|\tilde{\delta}|}\tilde{\gamma}!^{(\lambda,\mu)}\,\tilde{\delta}^{(\lambda,\mu)}\,e^{-c_0\langle\zeta\rangle_\phi^r}A_N, \tag{4.47}$$

for every $c_0 > 0$, where A_N satisfies $\sum\limits_{N=0}^{\infty} A_N < \infty$.
So from (4.33), (4.38) and (4.47) we have:

$$|\partial_w^{\tilde{\gamma}} \partial_\zeta^{\tilde{\delta}} r_{1N}(w,\zeta)| \leq C(c_0) C^{1+|\tilde{\gamma}|+|\tilde{\delta}|} \tilde{\gamma}!^{(\lambda,\mu)}\, \tilde{\delta}!^{(\lambda,\mu)} e^{-c_0 \langle \zeta \rangle_\phi^r} A_N, \tag{4.48}$$

where $\sum\limits_{N=0}^{\infty} A_N < \infty$.

We have now to analyze $r_{2N}(w,\zeta)$, cf. (4.32); applying the change of variables $w + \tilde{w} = \overline{w}$ (and rewriting $\tilde{w}$ in place of $\overline{w}$) we have

$$r_{2N}(w,\zeta) = \big(\varphi_N^R(\zeta) - \varphi_{N+1}^R(\zeta)\big) \int e^{iw\tilde{\zeta}} e^{\psi(w,\zeta+\tilde{\zeta})} \big[1 - \Xi(\zeta,\tilde{\zeta})\big] \left(\int e^{-i\tilde{w}\tilde{\zeta}} a(\tilde{w},\zeta)\, d\tilde{w} \right) đ\tilde{\zeta}. \tag{4.49}$$

Recall that the symbol $a(w,\zeta)$ is compactly supported in the w-variables and satisfies the estimates (3.8); then, since λ and μ satisfy the hypotheses of Propostion 2.7, for every $\tilde{\gamma}, \tilde{\delta}, \tilde{\nu} \in \mathbb{Z}_+^{p+q}$ we have:

$$\left| \partial_{\tilde{\zeta}}^{\tilde{\gamma}} \partial_\zeta^{\tilde{\delta}} \int e^{-i\tilde{w}\tilde{\zeta}} \partial_{\tilde{w}}^{\tilde{\nu}} a(\tilde{w},\zeta)\, d\tilde{w} \right| \leq C^{1+|\tilde{\nu}|} \tilde{\nu}!^{(\lambda,\mu)}\, C^{1+|\tilde{\delta}|} \tilde{\delta}!^{(\lambda,\mu)}\, \langle \zeta \rangle_\phi^{m - \langle \tilde{\delta}, \phi \rangle} e^{-d \langle \tilde{\zeta} \rangle_\phi^{r+\tilde{\kappa}}}, \tag{4.50}$$

for a positive constant $\tilde{\kappa}$ depending on λ and μ. The estimate (4.50) can be proved by standard techniques, cf. for example [20], formula (2.2). Then the derivatives of $r_{2N}(w,\zeta)$ can be estimated by applying Leibnitz rule, by using (4.17), (3.7), the property (iv) of the functions (4.12), the property (iii) of Lemma 4.3 and (4.50), obtaining:

$$|\partial_w^{\tilde{\gamma}} \partial_\zeta^{\tilde{\delta}} r_{2N}(w,\zeta)| \leq C^{1+|\tilde{\gamma}|+|\tilde{\delta}|} \tilde{\gamma}!^{(\lambda,\mu)}\, \tilde{\delta}!^{(\lambda,\mu)}\, \Upsilon_N(\zeta) \int \langle \zeta \rangle_\phi^m e^{c\langle \zeta+\tilde{\zeta} \rangle_\phi^r} e^{-d\langle \tilde{\zeta} \rangle_\phi^{r+\tilde{\kappa}}} \Psi(\zeta,\tilde{\zeta})\, đ\tilde{\zeta},$$

where $\Psi(\cdot)$ is the characteristic function of $\mathrm{supp}\big(1 - \Xi(\cdot)\big)$ and $\Upsilon_N(\zeta)$ is the characteristic function of K_N, cf. (4.45). Then we can assume that $|\tilde{\zeta}|_\phi \geq \frac{1}{4}\langle \zeta \rangle_\phi$. Observe now that, since $|\tilde{\zeta}|_\phi \leq c\langle \tilde{\zeta} \rangle_\phi$, we have $\langle \zeta \rangle_\phi^m \leq C' \langle \tilde{\zeta} \rangle_\phi^m \leq C e^{\langle \tilde{\zeta} \rangle_\phi^r}$; moreover, since $\tilde{\kappa} > 0$ is fixed, we have that for every $M_1, M_2 > 0$ there exists a constant $C_{M_1 M_2}$ satisfying $e^{-d\langle \tilde{\zeta} \rangle_\phi^{r+\tilde{\kappa}}} \leq C_{M_1 M_2} e^{-M_1 \langle \tilde{\zeta} \rangle_\phi^r - 4M_2 |\tilde{\zeta}|_\phi^r} \leq C_{M_1 M_2} e^{-M_1 \langle \tilde{\zeta} \rangle_\phi^r} e^{-M_2 \langle \zeta \rangle_\phi^r}$; we then conclude that

$$|\partial_w^{\tilde{\gamma}} \partial_\zeta^{\tilde{\delta}} r_{2N}(w,\zeta)| \leq C(c_0) C^{1+|\tilde{\gamma}|+|\tilde{\delta}|} \tilde{\gamma}!^{(\lambda,\mu)}\, \tilde{\delta}!^{(\lambda,\mu)} e^{-c_0 \langle \zeta \rangle_\phi^r} e^{-N}, \tag{4.51}$$

by a suitable choice of M_1 and M_2, since on $\mathrm{supp}\,\Upsilon_N(\zeta)$ we have $\langle \zeta \rangle_\phi^r \geq RN$. Then (4.16) follows immediately from (4.30), (4.48) and (4.51). $\square$

We can now prove the asymptotic expansion (3.10) of the conjugate operator (3.9).

Proof of Theorem 3.2. Let us define the symbol $a_\psi(w,\zeta)$ in the following way:

$$a_\psi(w,\zeta)=\sum_{j=0}^{\infty}\varphi_j^R(\zeta)\sum_{j\le\langle\delta,\phi\rangle<j+1}\frac{1}{\delta!}\,\partial_\zeta^\delta\{q(w,\zeta)D_w^\delta e^{-\psi(w,\zeta)}\},\tag{4.52}$$

$q(w,\zeta)$ being the symbol (4.13). We then consider the operators

$$R'_\psi(w,D)=R(w,D)\big({}^{\mathrm{t}}e^{-\psi(w,-D)}\big)\tag{4.53}$$

and

$$T_1(w,D)=Q(w,D)\big({}^{\mathrm{t}}e^{-\psi(w,-D)}\big),$$

whose symbols are denoted by $r'_\psi(w,\zeta)$ and $t_1(w,\zeta)$ respectively; $Q(w,D)$ and $R(w,D)$ are the operators whose symbols are (4.13) and (4.14), respectively. By definition of $R(w,D)$ and $Q(w,D)$ we have that $\widetilde{A}(w,D)=T_1(w,D)+R'_\psi(w,D)$; then, setting

$$r''(w,\zeta)=t_1(w,\zeta)-a_\psi(w,\zeta)$$

and

$$r_\psi(w,\zeta)=r'_\psi(w,\zeta)+r''_\psi(w,\zeta)$$

we have

$$\tilde{a}(w,\zeta)=a_\psi(w,\zeta)+r_\psi(w,\zeta).$$

We have now to prove that $a_\psi(w,\zeta)$ and $r_\psi(w,\zeta)$ satisfy the requested properties (i)-(ii).

(i) Consider at first $R_\psi(w,D)$: to start with, we analyze $r'_\psi(w,\zeta)$: by (4.53) we have

$$r'_\psi(w,\zeta)=\int e^{-i\tilde{w}\tilde{\zeta}}r(w,\zeta+\tilde{\zeta})e^{-\psi(w+\tilde{w},\zeta+\tilde{\zeta})}\,d\tilde{w}\,đ\tilde{\zeta};$$

so, multiplying by $\frac{(1+|\tilde{w}|^2)^M}{(1+|\tilde{w}|^2)^M}$ in the integral and integrating by parts we obtain:

$$\partial_w^\gamma\partial_\zeta^\delta r'_\psi(w,\zeta)=\int f_{\gamma,\delta}(w,\zeta,\tilde{\zeta})\,đ\tilde{\zeta},$$

where

$$f_{\gamma,\delta}(w,\zeta,\tilde{\zeta})=\partial_w^\gamma\partial_\zeta^\delta\int e^{-i\tilde{w}\tilde{\zeta}}(1+|\tilde{w}|^2)^{-M}\big(1+\Delta_{\tilde{\zeta}}\big)^M\Big[r(w,\zeta+\tilde{\zeta})e^{-\psi(w+\tilde{w},\zeta+\tilde{\zeta})}\Big]\,d\tilde{w}.$$

We now observe that $e^{-\psi(w,\zeta)}$ satisfies similar estimates as in (4.17); by (4.16) we then have that $r(w,\zeta+\tilde{\zeta})e^{-\psi(w+\tilde{w},\zeta+\tilde{\zeta})}$ is Gevrey of anisotropic order $\epsilon=(\epsilon_1,\dots,\epsilon_{p+q})$ in the $\tilde{w}$ variables; so, since for M sufficiently large $(1+|\tilde{w}|^2)^{-M}$ ensures us the convergence of the integral, we can prove as in (4.50) that

$$|f_{\gamma,\delta}(w,\zeta,\tilde{\zeta})|\le e^{-d\sum\limits_{j=1}^{p+q}(1+|\tilde{\zeta}_j|^2)^{\frac{1}{2\epsilon_j}}}C(\overline{c})C^{1+|\gamma|+|\delta|}\gamma!^{(\lambda,\mu)}\,\delta!^{(\lambda,\mu)}\,e^{-\overline{c}\langle\zeta+\tilde{\zeta}\rangle_\phi^r},$$

where $\overline{c}$ is arbitrary and $C(\overline{c})$ is a positive constant depending on $\overline{c}$. We now have $\sum_{j=1}^{p+q}(1+|\tilde{\zeta}_j|^2)^{\frac{1}{2\epsilon_j}} \geq c\langle\tilde{\zeta}\rangle_\phi^r$, by choosing $\epsilon_j > 1$ sufficiently small; moreover $\langle\zeta+\tilde{\zeta}\rangle_\phi^r \geq c_0\big(\langle\zeta\rangle_\phi^r - \langle\tilde{\zeta}\rangle_\phi^r\big)$; so we obtain

$$|f_{\gamma,\delta}(w,\zeta,\tilde{\zeta})| \leq C(\overline{c})C^{1+|\gamma|+|\delta|}\gamma!^{(\lambda,\mu)}\,\delta!^{(\lambda,\mu)}\,e^{-\overline{c}\,c_0\langle\zeta\rangle_\phi^r}e^{(\overline{c}\,c_0-\tilde{d})\langle\tilde{\zeta}\rangle_\phi^r}.$$

It then follows that

$$|\partial_w^\gamma\partial_\zeta^\delta r'_\psi(w,\zeta)| \leq C(\overline{c})C^{1+|\gamma|+|\delta|}\gamma!^{(\lambda,\mu)}\,\delta!^{(\lambda,\mu)}\,e^{-\overline{c}c_0\langle\zeta\rangle_\phi^r}\int e^{(\overline{c}c_0-\tilde{d})\langle\tilde{\zeta}\rangle_\phi^r}\,đ\tilde{\zeta}.$$

Choosing $\overline{c} > 0$ in such a way that $\overline{c}c_0 - \tilde{d} < 0$ we then have

$$|\partial_w^\gamma\partial_\zeta^\delta r'_\psi(w,\zeta)| \leq C^{1+|\gamma|+|\delta|}\gamma!^{(\lambda,\mu)}\,\delta!^{(\lambda,\mu)}\,e^{-c\langle\zeta\rangle_\phi^r}. \tag{4.54}$$

Regarding $r''(w,\zeta)$ the same arguments of Propostion 4.4 enable us to prove that $r''(w,\zeta)$ satisfies estimates as in (4.16), and so (3.11) is proved.

(ii) Observe that, by (4.52) and (4.13) we can write

$$a_\psi(w,\zeta) = \sum_{\gamma,\delta} h_{\gamma,\delta}(w,\zeta),$$

where

$$h_{\gamma,\delta}(w,\zeta) = \frac{1}{\gamma!\,\delta!}\varphi^R_{[\delta,\phi]}(\zeta)\partial_\zeta^\delta\Big[\varphi^R_{[\gamma,\phi]}(\zeta)\partial_\zeta^\gamma e^{\psi(w,\zeta)}D_w^\gamma a(w,\zeta)D_w^\delta e^{-\psi(w,\zeta)}\Big],$$

where $[\gamma,\phi]$ (resp. $[\delta,\phi]$) stands for the largest integer $\leq \langle\gamma,\phi\rangle$ (resp. $\langle\delta,\phi\rangle$). Writing

$$g_{\gamma,\delta}(w,\zeta) = \frac{1}{\gamma!\,\delta!}\partial_\zeta^\delta\Big[\partial_\zeta^\gamma e^{\psi(w,\zeta)}D_w^\gamma a(w,\zeta)D_w^\delta e^{-\psi(w,\zeta)}\Big]$$

we want to analyze the remainder

$$\begin{aligned} s_\psi^{(N)}(w,\zeta) &= \sum_{\gamma,\delta} h_{\gamma,\delta}(w,\zeta) - \sum_{\langle\gamma+\delta,\phi\rangle<N} g_{\gamma,\delta}(w,\zeta) \\ &= s_{\psi,1}^{(N)}(w,\zeta) + s_{\psi,2}^{(N)}(w,\zeta), \end{aligned} \tag{4.55}$$

where

$$s_{\psi,1}^{(N)}(w,\zeta) = \sum_{\langle\gamma+\delta,\phi\rangle<N}\big\{h_{\gamma,\delta}(w,\zeta) - g_{\gamma,\delta}(w,\zeta)\big\}, \tag{4.56}$$

$$s_{\psi,2}^{(N)}(w,\zeta) = \sum_{\langle\gamma+\delta,\phi\rangle\geq N} h_{\gamma,\delta}(w,\zeta). \tag{4.57}$$

Regarding (4.56) we observe that for $\langle\zeta\rangle_\phi^r \geq 2R(N-1)$, $\varphi^R_{[\gamma,\phi]}(\zeta) = \varphi^R_{[\delta,\phi]}(\zeta) \equiv 1$ for $\langle\gamma+\delta,\phi\rangle < N$, and so $s^{(N)}_{\psi,1}(w,\zeta) \equiv 0$ for $\langle\zeta\rangle_\phi^r \geq 2R(N-1)$; then we can suppose that $\langle\zeta\rangle_\phi^r \leq 2R(N-1)$. Now writing explicitly $\partial_\zeta^\delta e^{\psi(w,\zeta)}$ and $D_w^\gamma e^{-\psi(w,\zeta)}$ by means of Faà di Bruno formula (4.18), using (2.3), (3.8), the property (iv) of the functions (4.12) and the same procedure used to obtain (4.17) we can prove that:

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} h_{\gamma,\delta}(w,\zeta)| \leq C_{\gamma,\delta}^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)}\,\langle\zeta\rangle_\phi^{m-\langle\tilde\delta,\phi\rangle}, \tag{4.58}$$

and the same holds for $g_{\gamma,\delta}(w,\zeta)$. Then we have:

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} s^{(N)}_{\psi,1}(w,\zeta)| \leq C_N^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)}\,\langle\zeta\rangle_\phi^{m-\langle\tilde\delta,\phi\rangle}\big[2R(N-1)\big]^{\frac{(1-r)N}{r}}\big[2R(N-1)\big]^{-\frac{(1-r)N}{r}};$$

since we assume $\langle\zeta\rangle_\phi^r \leq 2R(N-1)$ we have $\big[2R(N-1)\big]^{-\frac{(1-r)N}{r}} \leq \langle\zeta\rangle_\phi^{-(1-r)N}$; thus

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} s^{(N)}_{\psi,1}(w,\zeta)| \leq \widetilde{C}_N^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)}\,\langle\zeta\rangle_\phi^{m-(1-r)N-\langle\tilde\delta,\phi\rangle}. \tag{4.59}$$

Regarding $s^{(N)}_{\psi,2}(w,\zeta)$ we obtain that

$$|\partial_w^{\tilde\gamma}\partial_\zeta^{\tilde\delta} s^{(N)}_{\psi,2}(w,\zeta)| \leq \sum_{\langle\gamma+\delta,\phi\rangle\geq N} C^{1+|\tilde\gamma|+|\tilde\delta|}\tilde\gamma!^{(\lambda,\mu)}\,\tilde\delta!^{(\lambda,\mu)}\,\langle\zeta\rangle_\phi^{m-\langle\tilde\delta,\phi\rangle-(1-r)\langle\gamma+\delta,\phi\rangle}$$

by the same technique used to prove (4.58); since now $\langle\gamma+\delta,\phi\rangle \geq N$ we have that $s^{(N)}_{\psi,2}(w,\zeta)$ satisfies the same estimates as in (4.59). By (4.55) we then obtain (3.12). □

We want now to prove Corollary 3.3. We observe that Theorem 3.2 cannot be applied in general to operators $A(w,D)$ whose symbol is not compactly supported in the w-variable; in fact, the estimate (4.50) does not hold if $a(w,\zeta)$ does not have compact support in the w-variable. However, in some particular cases we can avoid this requirement; the proof of Corollary 3.3 consists in showing that for $a(w,\zeta) \equiv 1$ the asymptotic expansion (3.10) still holds.

Proof of Corollary 3.3. An inspection of the proof of Theorem 3.2 shows that for $a(w,\zeta) \equiv 1$ the formula (4.49) becomes

$$r_{2N}(w,\zeta) = (2\pi)^{p+q}\big(\varphi_N^R(\zeta) - \varphi_{N+1}^R(\zeta)\big)e^{\psi(w,\zeta)}\big[1 - \Xi(\zeta,0)\big],$$

and by Lemma 4.3, (i) we then have

$$r_{2N}(w,\zeta) \equiv 0;$$

so the estimate (4.51) trivially holds and then the asymptotic expansion of the conjugation in Theorem 3.2 is true also for $a(w,\zeta) \equiv 1$. Then, (3.13) follows immediately from (3.10) applied

to the symbol $a(w,\zeta) \equiv 1$ and stopped at $N=1$; the estimate of the remainder, together with known results about mapping properties in Sobolev spaces of anisotropic pseudo-differential operators, cf. for example [13], [14], give us (3.15).
Concerning (3.14) we proceed in the following way: we want to prove that

$$ {}^{\mathrm{t}}e^{-\psi(w,-D)}e^{\psi(w,D)}f - f \in \mathbb{H}^{s+(1-r),\psi}_{\phi,r}(\mathbb{R}^{p+q}), $$

for every $f \in \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q})$; by Definiton 2.4, this is equivalent to show that

$$ e^{\psi(w,D)}\big({}^{\mathrm{t}}e^{-\psi(w,-D)}\big)e^{\psi(w,D)}f - e^{\psi(w,D)}f \in H^{s+(1-r)}_{\phi}(\mathbb{R}^{p+q}); \tag{4.60} $$

writing $e^{\psi(w,D)}\big({}^{\mathrm{t}}e^{-\psi(w,-D)}\big)$ in (4.60) using the expression (3.13) it remains to prove

$$ R_1 e^{\psi(w,D)}f \in H^{s+(1-r)}_{\phi}(\mathbb{R}^{p+q}), $$

which is true since $e^{\psi(w,D)} : \mathbb{H}^{s,\psi}_{\phi,r}(\mathbb{R}^{p+q}) \to H^{s}_{\phi}(\mathbb{R}^{p+q})$ by definition, and R_1 satisfies (3.15). $\square$

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