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SCHATTEN–VON NEUMANN PROPERTIES FOR FOURIER INTEGRAL OPERATORS WITH NON-SMOOTH SYMBOLS II

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Abstract

We establish continuity and Schatten–von Neumann properties for Fourier integral operators with amplitudes in weighted modulation spaces, when acting on modulation spaces themselves. The phase functions are non smooth and admit second order derivatives again in suitable classes of modulation spaces.

0. Introduction

The aim of this paper is to investigate continuity and compactness properties for Fourier integral operators with non-smooth amplitudes (or symbols), when acting on (general, or weighted) modulation spaces. Especially we concern with detailed compactness investigations of such operators in background of Schatten–von Neumann theory, when acting on Hilbert modulation spaces. Here we recall that the spaces of trace-class or Hilbert–Schmidt operators are particular classes of Schatten–von Neumann type. More precisely, we establish sufficient conditions on the amplitudes and phase functions in order to allow the corresponding Fourier integral operators to be Schatten–von Neumann of certain degree. Since Sobolev spaces of Hilbert type are special cases of these Hilbert modulation spaces, it follows that our results can be applied to certain problems involving them.

The phase functions are assumed to be continuous, with second orders of derivatives belonging to appropriate modulation spaces (i.e. weighted "Sjöstrand classes") and satisfying appropriate non-degeneracy conditions. The amplitudes are assumed to belong to appropriate (weighted) modulation spaces, or more generally, appropriate (weighted) coorbit spaces of modulation type, where each such space is defined by imposing a mixed weighted Lebesgue norm on the short-time Fourier transform of distributions. These coorbit spaces contain various types of classical smooth amplitudes. For example, for any set of the smooth functions which belong to a fixed mixed Lebesgue spaces, together with all their derivatives, we may find a "small" such coorbit space which contains this set. Conse-

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quently, our main results apply on Fourier integral operators with such smooth amplitudes.

Furthermore, by letting the involved weight functions be trivially equal to one, these Sobolev spaces are equal to L^2. In this case, our results generalize those in [7, 8], where similar questions are discussed when the amplitudes and second order of derivatives belong to classical or non-weighted modulation spaces.

On the other hand, following some ideas of A. Boulkhemair in [5], the framework of the investigations in the present paper, as well as in [7, 8], is to localize the Fourier integral operators in terms of short-time Fourier transforms, and then making appropriate Taylor expansions and estimates. In fact, in [5], Boulkhemair considers a certain class of Fourier integral operators whose symbols are defined without any explicit regularity assumptions and with only small regularity assumptions on the phase functions. The symbol class considered by Boulkhemair, in the present paper denoted by $M^{\infty,1}$, is sometimes called the "Sjöstrand class", and contains $S^0_{0,0}$, the set of smooth functions which are bounded together with all their derivatives. In time-frequency community, $M^{p,q}$ is known as a (classical or non-weighted) modulation space with exponents $p \in [1, \infty]$ and $q \in [1, \infty]$. The strict definition may be found below or e.g. in [14, 17, 22]. Boulkhemair then proves that such operators extend uniquely to continuous operators on L^2. In particular it follows that pseudo-differential operators with symbols in $M^{\infty,1}$ are L^2-continuous, as proved by J. Sjöstrand in [35], where it seems that $M^{\infty,1}$ was used for the first time in this context.

Boulkhemair's result was extended in [7, 8], where it is proved that if the amplitude belongs to the classical modulation space $M^{p,1}$, then the corresponding Fourier integral operator is Schatten–von Neumann of order $p \in [1, \infty]$ on L^2. In [8] it is also proved that if the amplitude only depends on the phase space variables and belongs to $M^{p,p}$, then the corresponding Fourier integral operator is continuous from $M^{p',p'}$ to $M^{p,p}$, where $1/p + 1/p' = 1$. If in addition $1 \le p \le 2$, then it is also proved that the operator is Schatten–von Neumann of order p on L^2.

We remark that the assumptions on the phase functions imply that they are two times continuously differentiable. This property is usually violated by "classical" Fourier integral operators (see e.g. [24, 29, 30, 31, 32]). For example, this condition is not fulfilled in general when the phase function is homogeneous of degree one in the frequency variable. We refer to [9, 29, 30, 31, 32] for recent contributions to the theory of Fourier integral operators with non-smooth symbols, and in certain domains with few regularity assumptions of the phase functions.

In order to be more specific we recall some definitions. Assume that $p, q \in [1, \infty]$, $\chi \in \mathscr{S}(\mathbf{R}^n) \setminus 0$ and that $\omega \in \mathscr{P}(\mathbf{R}^{2n})$ are fixed. (See Section 1 for strict definition of $\mathscr{P}$.) Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^n)$ is the set of all $f \in \mathscr{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |V_\chi f(x, \xi)\omega(x, \xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q} < \infty \tag{0.1}$$

(with obvious modification when $p = \infty$ or $q = \infty$). Here $V_\chi f$ is the *short time Fourier*

transform of f with respect to the *window function* χ, i.e. $V_\chi f(x,\xi) \equiv \mathscr{F}(f\tau_x\overline{\chi})(\xi)$, where τ_x is the translation operator $\tau_x\chi(y) = \chi(y-x)$, $\mathscr{F}$ is the Fourier transform on $\mathscr{S}'(\mathbf{R}^n)$ which is given by

$$\mathscr{F}f(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-n/2}\int_{\mathbf{R}^n} f(x)e^{-i\langle x,\xi\rangle}\,dx$$

when $f \in \mathscr{S}(\mathbf{R}^n)$. For simplicity we set $M^{p,q} = M^{p,q}_{(\omega)}$ when $\omega = 1$.

Modulation spaces were introduced by H. Feichtinger in [14]. The basic theory of such spaces were thereafter extended by Feichtinger and Gröchenig in [17, 18], where the coorbit space theory was established. Here we note that the amplitude classes in the present paper consist of coorbit spaces, defined in such way that their norms are given by (0.1), after replacing the L^p and L^q norms by *mixed* Lebesgue norms and interchanging the order of integration (see Subsection 1.2 and Section 2). During the last twenty years, modulation spaces have been an active fields of research (see e.g. [14, 15, 22, 27, 39, 42]). They are rather similar to Besov spaces (see [46, 37, 42] for sharp embeddings) and it has appeared that they are useful in time-frequency analysis, signal processing, and to some extent also in pseudo-differential calculus.

Next we discuss the definition of Fourier integral operators. For simplicity we restrict ourself to operators which belong to $\mathscr{L}(\mathscr{S}(\mathbf{R}^{n_1}),\mathscr{S}'(\mathbf{R}^{n_2}))$. Here we let $\mathscr{L}(V_1,V_2)$ denote the set of all linear and continuous operators from V_1 to V_2, when V_1 and V_2 are topological vector spaces. For any appropriate $a \in \mathscr{S}'(\mathbf{R}^{N+m})$ (the *symbol* or *amplitude*) for $N = n_1 + n_2$, and real-valued $\varphi \in C(\mathbf{R}^{N+m})$ (the *phase function*), the Fourier integral operator $\operatorname{Op}_\varphi(a)$ is defined by the formula

$$\operatorname{Op}_\varphi(a)f(x) = (2\pi)^{-N/2}\iint_{\mathbf{R}^{n_1+m}} a(x,y,\xi)f(y)e^{i\varphi(x,y,\xi)}\,dy\,d\xi, \tag{0.2}$$

when $f \in \mathscr{S}(\mathbf{R}^{n_1})$. Here the integrals should be interpreted in distribution sense if necessary. By letting $m = n_1 = n_2 = n$, and choosing symbols and phase functions in appropriate ways, it follows that the pseudo-differential operator

$$\operatorname{Op}(a)f(x) = (2\pi)^{-n}\iint_{\mathbf{R}^n} a(x,y,\xi)f(y)e^{i\langle x-y,\xi\rangle}\,dy\,d\xi \tag{0.3}$$

is a special case of Fourier integral operator. Furthermore, if $t \in \mathbf{R}$ is fixed, and a is an appropriate function or distribution on $\mathbf{R}^{2n}$ instead of $\mathbf{R}^{3n}$, then the definition of the latter pseudo-differential operators covers the definition of pseudo-differential operators of the form

$$a_t(x,D)f(x) = (2\pi)^{-n}\iint_{\mathbf{R}^{2n}} a((1-t)x+ty,\xi)f(y)e^{i\langle x-y,\xi\rangle}\,dy\,d\xi. \tag{0.4}$$

On the other hand, in the framework of harmonic analysis it follows that the map $a \mapsto a_t(x, D)$ from $\mathscr{S}(\mathbf{R}^{2n})$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$ extends uniquely to a bijection from $\mathscr{S}'(\mathbf{R}^{2n})$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$.

In the literature it is usually assumed that a and φ in (0.2) are smooth functions. For example, if $n_1 = n_2 = n$, $a \in \mathscr{S}(\mathbf{R}^{2n+m})$ and $\varphi \in C^\infty(\mathbf{R}^{2n+m})$ satisfy $\varphi^{(\alpha)} \in S^0_{0,0}(\mathbf{R}^{2n+m})$ when $|\alpha| = N_1$ for some integer $N_1 \geq 0$, then it is easily seen that $\operatorname{Op}_\varphi(a)$ is continuous on $\mathscr{S}(\mathbf{R}^n)$ and extends to a continuous map from $\mathscr{S}'(\mathbf{R}^n)$ to $\mathscr{S}(\mathbf{R}^n)$. Here recall $S^0_{0,0}(\mathbf{R}^N)$ denotes the Hörmander symbol class which consists of all smooth functions on $\mathbf{R}^N$ which are bounded together with all their derivatives. In [1] it is proved that if $\varphi^{(\alpha)} \in S^0_{0,0}(\mathbf{R}^{2n+m})$ for all multi-indices α such that $|\alpha| = 2$ and satisfies

$$\left|\det\begin{pmatrix} \varphi''_{x,y} & \varphi''_{x,\xi} \\ \varphi''_{y,\xi} & \varphi''_{\xi,\xi} \end{pmatrix}\right| \geq \mathsf{d} \tag{0.5}$$

for some $\mathsf{d} > 0$, then the definition of $\operatorname{Op}_\varphi(a)$ extends uniquely to any $a \in S^0_{0,0}(\mathbf{R}^{2n+m})$, and then $\operatorname{Op}_\varphi(a)$ is continuous on $L^2(\mathbf{R}^n)$.

Next assume that φ instead satisfies $\varphi^{(\alpha)} \in M^{\infty,1}(\mathbf{R}^{3n})$ for all multi-indices α such that $|\alpha| = 2$ and that (0.5) holds for some $\mathsf{d} > 0$. This implies that the condition on φ is relaxed since $S^0_{0,0} \subseteq M^{\infty,1}$. Then Boulkhemair improves the result in [1] by proving that the definition of $\operatorname{Op}_\varphi(a)$ extends uniquely to any $a \in M^{\infty,1}(\mathbf{R}^{2n+m})$, and that $\operatorname{Op}_\varphi(a)$ is still continuous on $L^2(\mathbf{R}^n)$.

In Section 2 we discuss continuity and Schatten–von Neumann properties for Fourier integral operators which are related to those which were considered by Boulkhemair. More precisely, assume that ω, ω_j for $j = 1, 2$ and v are appropriate weight functions, $\varphi \in M^{\infty,1}_{(v)}$ and $1 < p < \infty$. Then we prove in Subsection 2.4 that the definition of $a \mapsto \operatorname{Op}_\varphi(a)$ from $\mathscr{S}$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$ extends uniquely to any $a \in M^{\infty,1}_{(\omega)}$, and that $\operatorname{Op}_\varphi(a)$ is continuous from $M^p_{(\omega_1)}$ to $M^p_{(\omega_2)}$. In particular we recover Boulkhemair's result by letting $\omega = \omega_j = v = 1$ and $p = 2$.

In Subsection 2.5 we consider more general Fourier integral operators, where we assume that the amplitudes belong to coorbit spaces which, roughly speaking, are like $M^{p,q}_{(\omega)}$ for $p, q \in [1, \infty]$ in certain variables and like $M^{\infty,1}_{(\omega)}$ in the other variables. (Note here that $M^{\infty,1}_{(\omega)}$ is contained in $M^{\infty,q}_{(\omega)}$ in view of Proposition 1.1 in Section 1.) If $q \leq p$, then we prove that such Fourier integral operators are continuous from $M^{p',p'}_{(\omega_1)}$ to $M^{p,p}_{(\omega_2)}$. Furthermore, by interpolation between the latter result and our extension of Boulkhemair's result we prove that if $q \leq \min(p, p')$, then these Fourier integral operators belong to $\mathscr{I}_p(M^{2,2}_{(\omega_1)}, M^{2,2}_{(\omega_2)})$. Here $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$ denotes the set of Schatten–von Neumann operators from the separable Hilbert space $\mathscr{H}_1$ to the separable Hilbert space $\mathscr{H}_2$ of order p. This means that $T \in \mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$ if and only if T is a linear

and continuous operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ which satisfy

$$\|T\|_{\mathscr{I}_p} \equiv \sup\Big(\sum |(Tf_j, g_j)_{\mathscr{H}_2}|^p\Big)^{1/p} < \infty,$$

where the supremum should be taken over all orthonormal sequences (f_j) in $\mathscr{H}_1$ and (g_j) in $\mathscr{H}_2$.

In Section 3 we list some consequences of our general results in Section 2. For example, assume that $p, q \in [1, \infty]$, $a(x, y, \xi) = b(x, \xi)$, for some $b \in M^{p,q}_{(\omega)}(\mathbf{R}^{2n})$, and that

$$|\det(\varphi''_{y,\xi})| \geq \mathsf{d} \tag{0.6}$$

holds for some constant $\mathsf{d} > 0$. Then it follows from the results in Section 2 that if $q = p$, then $\operatorname{Op}_\varphi(a)$ is continuous from $M^{p',p'}_{(\omega_1)}$ to $M^{p,p}_{(\omega_2)}$. Furthermore, if in addition (0.5) and $q \leq \min(p, p')$ hold, then $\operatorname{Op}_\varphi(a) \in \mathscr{I}_p$.

In the last part of Subsection 3.2 we present some consequences for Fourier integral operators with smooth symbols, and finally, in Subsection 3.3 we show how the results in Section 2 can be used to extend some results in [41, 43] on pseudo-differential operators of the form (0.3).

1. Preliminaries

In this section we discuss basic properties for modulation spaces. The proofs are in many cases omitted since they can be found in [12, 13, 14, 15, 17, 18, 19, 22, 40, 41, 42].

We start by discussing some notations. The duality between a topological vector space and its dual is denoted by $\langle \cdot\,, \cdot \rangle$. For admissible a and b in $\mathscr{S}'(\mathbf{R}^n)$, we set $(a, b) = \langle a, \overline{b} \rangle$, and it is obvious that $(\cdot\,, \cdot)$ on L^2 is the usual scalar product.

Assume that $\mathscr{B}_1$ and $\mathscr{B}_2$ are topological spaces. Then $\mathscr{B}_1 \hookrightarrow \mathscr{B}_2$ means that $\mathscr{B}_1$ is continuously embedded in $\mathscr{B}_2$. In the case that $\mathscr{B}_1$ and $\mathscr{B}_2$ are Banach spaces, $\mathscr{B}_1 \hookrightarrow \mathscr{B}_2$ is equivalent to $\mathscr{B}_1 \subseteq \mathscr{B}_2$ and $\|x\|_{\mathscr{B}_2} \leq C\|x\|_{\mathscr{B}_1}$, for some constant $C > 0$ which is independent of $x \in \mathscr{B}_1$.

Let $\omega, v \in L^\infty_{\mathrm{loc}}(\mathbf{R}^n)$ be positive functions. Then ω is called *v-moderate* if

$$\omega(x + y) \leq C\omega(x)v(y), \quad x, y \in \mathbf{R}^n, \tag{1.1}$$

for some constant $C > 0$, and if v in (1.1) can be chosen as a polynomial, then ω is called polynomially moderated. Furthermore, v is called *submultiplicative* if (1.1) holds for $\omega = v$ and v is even. In the sequel we always let v and v_j for $j \in \mathbf{N}$ stand for submultiplicative functions, if nothing else is stated. We denote by $\mathscr{P}(\mathbf{R}^n)$ the set of all polynomially moderated functions on $\mathbf{R}^n$.

Assume that $\omega(x_1, x_2) \in \mathscr{P}(\mathbf{R}^{n_1+n_2})$, where $x_j \in \mathbf{R}^{n_j}$ for $j = 1, 2$. If $\omega(x_1, x_2) = \omega_1(x_1)$ for some $\omega_1 \in \mathscr{P}(\mathbf{R}^{n_1})$, then we identify ω with ω_1 and write $\omega(x_1)$ instead of $\omega_1(x_1)$, i.e. $\omega(x_1, x_2) = \omega(x_1)$. In such situations we sometimes consider ω as an element in $\mathscr{P}(\mathbf{R}^{n_1})$.

1.1. Modulation spaces. Next we recall some properties on modulation spaces. We remark that the definition of modulation spaces $M^{p,q}_{(\omega)}(\mathbf{R}^n)$, given in (0.1) for $p, q \in [1, \infty]$, is independent of the choice of the window $\chi \in \mathscr{S}(\mathbf{R}^n) \setminus 0$. (See Proposition 1.1 below). For the short-time Fourier transform in (0.1) we note that the map $(f, \chi) \mapsto V_\chi f$ is continuous from $\mathscr{S}(\mathbf{R}^n) \times \mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}(\mathbf{R}^{2n})$ which extends uniquely to a continuous map from $\mathscr{S}'(\mathbf{R}^n) \times \mathscr{S}'(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^{2n})$.

For convenience we set $M^p_{(\omega)} = M^{p,p}_{(\omega)}$. Furthermore we set $M^{p,q} = M^{p,q}_{(\omega)}$ if $\omega \equiv 1$.

The proof of the following proposition is omitted, since the results can be found in [12, 13, 14, 15, 17, 18, 19, 22, 40, 41, 42]. Here and in what follows, $p' \in [1, \infty]$ denotes the conjugate exponent of $p \in [1, \infty]$, i.e. $1/p + 1/p' = 1$ should be fulfilled.

Proposition 1.1. *Assume that $p, q, p_j, q_j \in [1, \infty]$ for $j = 1, 2$, and $\omega, \omega_1, \omega_2, v \in \mathscr{P}(\mathbf{R}^{2n})$ are such that ω is v-moderate and $\omega_2 \le C\omega_1$ for some constant $C > 0$. Then the following are true:*
(1) *$f \in \mathscr{S}'(\mathbf{R}^n)$ belongs to $M^{p,q}_{(\omega)}(\mathbf{R}^n)$ if and only if (0.1) holds for $\chi \in M^1_{(v)}(\mathbf{R}^n) \setminus 0$. Moreover, $M^{p,q}_{(\omega)}(\mathbf{R}^n)$ is a Banach space under the norm in (0.1) and different choices of χ give rise to equivalent norms;*
(2) *if $p_1 \le p_2$ and $q_1 \le q_2$ then*

$$\mathscr{S}(\mathbf{R}^n) \hookrightarrow M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^n) \hookrightarrow M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^n) \hookrightarrow \mathscr{S}'(\mathbf{R}^n);$$

(3) *the L^2 product $(\,\cdot\,,\,\cdot\,)$ on $\mathscr{S}(\mathbf{R}^n)$ extends to a continuous map from $M^{p,q}_{(\omega)}(\mathbf{R}^n) \times M^{p',q'}_{(1/\omega)}(\mathbf{R}^n)$ to $\mathbf{C}$. On the other hand, if $\|a\| = \sup|(a, b)|$, where the supremum is taken over all $b \in \mathscr{S}(\mathbf{R}^n)$ such that $\|b\|_{M^{p',q'}_{(1/\omega)}} \le 1$, then $\|\cdot\|$ and $\|\cdot\|_{M^{p,q}_{(\omega)}}$ are equivalent norms;*
(4) *if $p, q < \infty$, then $\mathscr{S}(\mathbf{R}^n)$ is dense in $M^{p,q}_{(\omega)}(\mathbf{R}^n)$ and the dual space of $M^{p,q}_{(\omega)}(\mathbf{R}^n)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbf{R}^n)$, through the form $(\cdot, \cdot)_{L^2}$. Moreover, $\mathscr{S}(\mathbf{R}^n)$ is weakly dense in $M^\infty_{(\omega)}(\mathbf{R}^n)$.*

Proposition 1.1 (1) allows us be rather vague concerning the choice of $\chi \in M^1_{(v)} \setminus 0$ in (0.1). For example, if $C > 0$ is a constant and $\mathscr{A}$ is a subset of $\mathscr{S}'$, then $\|a\|_{M^{p,q}_{(\omega)}} \le C$ for every $a \in \mathscr{A}$, means that the inequality holds for some choice of $\chi \in M^1_{(v)} \setminus 0$ and every $a \in \mathscr{A}$. Evidently, a similar inequality is true for any other choice of $\chi \in M^1_{(v)} \setminus 0$, with a suitable constant, larger than C if necessary.

It is also convenient to let $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^n)$ be the completion of $\mathscr{S}(\mathbf{R}^n)$ under the norm $\|\cdot\|_{M^{p,q}_{(\omega)}}$. Then $\mathcal{M}^{p,q}_{(\omega)} \subseteq M^{p,q}_{(\omega)}$ with equality if and only if $p < \infty$ and $q < \infty$. It follows that most of the properties which are valid for $M^{p,q}_{(\omega)}(\mathbf{R}^n)$, also hold for $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^n)$.

We also need to use multiplication properties of modulation spaces. The proof of the following proposition is omitted since the result can be found in [14, 17, 41, 42].

Proposition 1.2. *Assume that* $p, p_j, q_j \in [1, \infty]$ *and* $\omega_j, v \in \mathscr{P}(\mathbf{R}^{2n})$, *for* $j = 0, \dots, N$, *satisfy*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_N} = \frac{1}{p_0}, \quad \frac{1}{q_1} + \cdots + \frac{1}{q_N} = N - 1 + \frac{1}{q_0},$$

and

$$\omega_0(x, \xi_1 + \cdots + \xi_N) \le C\omega_1(x, \xi_1) \cdots \omega_N(x, \xi_N), \quad x, \xi_1, \dots, \xi_N \in \mathbf{R}^n,$$

for some constant C. *Then* $(f_1, \dots, f_N) \mapsto f_1 \cdots f_N$ *from* $\mathscr{S}(\mathbf{R}^n) \times \cdots \times \mathscr{S}(\mathbf{R}^n)$ *to* $\mathscr{S}(\mathbf{R}^n)$ *extends uniquely to a continuous map from* $M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^n) \times \cdots \times M^{p_N,q_N}_{(\omega_N)}(\mathbf{R}^n)$ *to* $M^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^n)$, *and*

$$\|f_1 \cdots f_N\|_{M^{p_0,q_0}_{(\omega_0)}} \le C\|f_1\|_{M^{p_1,q_1}_{(\omega_1)}} \cdots \|f_N\|_{M^{p_N,q_N}_{(\omega_N)}}$$

for some constant C *which is independent of* $f_j \in M^{p_j,q_j}_{(\omega_j)}(\mathbf{R}^n)$ *for* $i = 1, \dots, N$.

Furthermore, if $u_0 = 0$ *when* $p < \infty$, $v(x, \xi) = v(\xi) \in \mathscr{P}(\mathbf{R}^n)$ *is submultiplicative,* $f \in M^{p,1}_{(v)}(\mathbf{R}^n)$, *and* ϕ, ψ *are entire functions on* $\mathbf{C}$ *with expansions*

$$\phi(z) = \sum_{k=0}^{\infty} u_k z^k, \quad \psi(z) = \sum_{k=0}^{\infty} |u_k| z^k,$$

then $\phi(f) \in M^{p,1}_{(v)}(\mathbf{R}^n)$, *and*

$$\|\phi(f)\|_{M^{p,1}_{(v)}} \le C\psi\big(C\|f\|_{M^{p,1}_{(v)}}\big),$$

for some constant C *which is independent of* $f \in M^{p,1}_{(v)}(\mathbf{R}^n)$.

In the following remark we list some other properties for modulation spaces. Here and in what follows we let $\langle x \rangle = (1 + |x|^2)^{1/2}$, when $x \in \mathbf{R}^n$.

REMARK 1.3. Assume that $p, q, q_1, q_2 \in [1, \infty]$ and that $\omega, v \in \mathscr{P}(\mathbf{R}^{2n})$ are such that ω is v-moderate. Then the following properties for modulation spaces hold:
(1) if $q_1 \le \min(p, p')$, $q_2 \ge \max(p, p')$ and $\omega(x, \xi) = \omega(x)$, then $M^{p,q_1}_{(\omega)} \subseteq L^p_{(\omega)} \subseteq M^{p,q_2}_{(\omega)}$. In particular, $M^2_{(\omega)} = L^2_{(\omega)}$;
(2) if $\omega(x, \xi) = \omega(x)$, then $M^{p,q}_{(\omega)}(\mathbf{R}^n) \hookrightarrow C(\mathbf{R}^n)$ if and only if $q = 1$;
(3) $M^{1,\infty}$ is a convolution algebra which contains all measures on $\mathbf{R}^n$ with bounded mass;
(4) if $x_0 \in \mathbf{R}^n$ is fixed and $\omega_0(\xi) = \omega(x_0, \xi)$, then $M^{p,q}_{(\omega)} \cap \mathscr{E}' = \mathscr{F}L^q_{(\omega_0)} \cap \mathscr{E}'$. Here $\mathscr{F}L^q_{(\omega_0)}(\mathbf{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbf{R}^n)$ such that

$$\|\hat{f}\, \omega_0\|_{L^q} < \infty.$$

Furthermore, if B is a ball with radius r and center at x_0, then

$$C^{-1}\|\hat{f}\|_{L^q_{(\omega_0)}} \le \|f\|_{M^{p,q}_{(\omega)}} \le C\|\hat{f}\|_{L^q_{(\omega_0)}}, \quad f \in \mathscr{E}'(B)$$

for some constant C which only depends on r, n, ω and the chosen window functions;
(5) if $\omega(x,\xi) = \omega_0(-\xi, x)$, then the Fourier transform on $\mathscr{S}'(\mathbf{R}^n)$ restricts to a homeomorphism from $M^p_{(\omega_0)}(\mathbf{R}^n)$ to $M^p_{(\omega)}(\mathbf{R}^n)$. In particular, if $\omega = \omega_0$, then $M^p_{(\omega)}$ is invariant under the Fourier transform. Similar facts hold for partial Fourier transforms;
(6) for each $x, \xi \in \mathbf{R}^n$ we have

$$\|e^{i\langle\cdot,\xi\rangle} f(\cdot - x)\|_{M^{p,q}_{(\omega)}} \le Cv(x,\xi)\|f\|_{M^{p,q}_{(\omega)}},$$

for some constant C;
(7) if $\tilde{\omega}(x,\xi) = \omega(x,-\xi)$ then $f \in M^{p,q}_{(\omega)}$ if and only if $\bar{f} \in M^{p,q}_{(\tilde{\omega})}$;
(8) if $s \in \mathbf{R}$ and $\omega(x,\xi) = \langle\xi\rangle^s$, then $M^2_{(\omega)}$ agrees with H^2_s, the Sobolev space of distributions with s derivatives in L^2. That is, H^2_s consists of all $f \in \mathscr{S}'$ such that $\mathscr{F}^{-1}(\langle\cdot\rangle^s \hat{f}) \in L^2$. If instead $\omega(x,\xi) = \langle x\rangle^s$, then $M^2_{(\omega)}$ agrees with L^2_s, the set of all $f \in L^1_{\mathrm{loc}}$ such that $\langle\cdot\rangle^s f \in L^2$.
See e.g. [12, 13, 14, 17, 18, 19, 22, 33, 42].

For future references we note that the constant C in Remark 1.3 (4) is independent of the center of the ball B.

In our investigations we need the following characterization of modulation spaces.

Proposition 1.4. *Let $\{x_\alpha\}_{\alpha\in I}$ be a lattice in $\mathbf{R}^n$, $B_\alpha = x_\alpha + B$ where $B \subseteq \mathbf{R}^n$ is an open ball, and assume that $f_\alpha \in \mathscr{E}'(B_\alpha)$ for every $\alpha \in I$. Also assume that $p, q \in [1,\infty]$. Then the following is true:*
(1) *if*

$$f = \sum_{\alpha\in I} f_\alpha \quad \text{and} \quad F(\xi) \equiv \left(\sum_{\alpha\in I} |\hat{f}_\alpha(\xi)\omega(x_\alpha,\xi)|^p\right)^{1/p} \in L^q(\mathbf{R}^n), \tag{1.2}$$

then $f \in M^{p,q}_{(\omega)}$, and $f \mapsto \|F\|_{L^q}$ defines a norm on $M^{p,q}_{(\omega)}$ which is equivalent to $\|\cdot\|_{M^{p,q}_{(\omega)}}$ in (0.1);
(2) *if in addition $\bigcup_\alpha B_\alpha = \mathbf{R}^n$, $\chi \in C_0^\infty(B)$ satisfies $\sum_\alpha \chi(\cdot - x_\alpha) = 1$, $f \in M^{p,q}_{(\omega)}(\mathbf{R}^n)$, and $f_\alpha = f\chi(\cdot - x_\alpha)$, then $f_\alpha \in \mathscr{E}'(B_\alpha)$ and* (1.2) *is fulfilled.*

Proof. (1) Assume that $\chi \in C_0^\infty(\mathbf{R}^n) \setminus 0$ is real-valued and fixed. Since there is a bound of overlapping supports of f_α, we obtain

$$\begin{aligned}|V_\chi f(x,\xi)\omega(x,\xi)| &= |\mathscr{F}(f\chi(\cdot - x))(\xi)\omega(x,\xi)| \\ &\le \sum |\mathscr{F}(f_\alpha\chi(\cdot - x))(\xi)\omega(x,\xi)| \\ &\le C\Big(\sum |\mathscr{F}(f_\alpha\chi(\cdot - x))(\xi)\omega(x,\xi)|^p\Big)^{1/p},\end{aligned}$$

for some constant C. From the support properties of χ, and the fact that ω is v-moderate for some $v \in \mathscr{P}(\mathbf{R}^{2n})$, it follows that

$$|\mathscr{F}(f_\alpha\chi(\cdot - x))(\xi)\omega(x,\xi)| \le C|\mathscr{F}(f_\alpha\chi(\cdot - x))(\xi)\omega(x_\alpha,\xi)|,$$

for some constant C independent of α. Hence, for some balls B' and $B'_\alpha = x_\alpha + B'$, we get

$$\begin{aligned}&\Big(\int_{\mathbf{R}^n} |\mathscr{F}(f\chi(\cdot - x))(\xi)\omega(x,\xi)|^p\,dx\Big)^{1/p} \\ &\le C\Big(\sum_\alpha \int_{B'_\alpha} |\mathscr{F}(f_\alpha\chi(\cdot - x))(\xi)\omega(x_\alpha,\xi)|^p\,dx\Big)^{1/p} \\ &\le C'\Big(\sum_\alpha \int_{B'_\alpha} (|\hat{f}_\alpha\omega(x_\alpha,\cdot)| * |\hat{\chi}v(0,\cdot)|(\xi))^p\,dx\Big)^{1/p} \\ &\le C''\Big(\sum_\alpha (|\hat{f}_\alpha\omega(x_\alpha,\cdot)| * |\hat{\chi}v(0,\cdot)|(\xi))^p\Big)^{1/p} \le C''F * |\hat{\chi}v(0,\cdot)|(\xi),\end{aligned}$$

for some constants C, C' and C''. Here the last estimate follows from Minkowski's inequality. By applying the L^q-norm and using Young's inequality we get

$$\|f\|_{M^{p,q}_{(\omega)}} \le C''\|F * |\hat{\chi}v(0,\cdot)|\|_{L^q} \le C''\|F\|_{L^q}\|\hat{\chi}v(0,\cdot)\|_{L^1}.$$

Since we have assumed that $F \in L^q$, it follows that $\|f\|_{M^{p,q}_{(\omega)}}$ is finite. By similar arguments we get $\|F\|_{L^q} \le C\|f\|_{M^{p,q}_{(\omega)}}$ for some constant C. This proves (1).

The assertion (2) follows immediately from the general theory of modulation spaces. (See e.g. Chapter 12 in [22].) The proof is complete. □

Next we discuss (complex) interpolation properties for modulation spaces. Such properties were carefully investigated in [14] for classical modulation spaces, and thereafter extended in several directions in [18], where interpolation properties for coorbit spaces were established, see also Subsection 1.2. The following proposition is an immediate consequence of Theorem 4.7 in [17].

Proposition 1.5. *Assume that* $0 < \theta < 1$, $p_j, q_j \in [1, \infty]$ *and that* $\omega_j \in \mathscr{P}(\mathbf{R}^{2n})$ *for* $j = 0, 1, 2$ *satisfy*

$$\frac{1}{p_0} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q_0} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2} \quad \text{and} \quad \omega_0 = \omega_1^{1-\theta}\omega_2^{\theta}.$$

Then

$$\left(\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^n), \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^n)\right)_{[\theta]} = \mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^n),$$

and

$$\left(M^{p_1,q_1}_{(\omega_0)}(\mathbf{R}^n), M^{p_2,q_2}_{(\omega_0)}(\mathbf{R}^n)\right)_{[\theta]} = M^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^n).$$

Next we recall some facts in Section 2 in [44] on narrow convergence. For any $f \in \mathscr{S}'(\mathbf{R}^n)$, $\omega \in \mathscr{P}(\mathbf{R}^{2n})$, $\chi \in \mathscr{S}(\mathbf{R}^n)$ and $p \in [1, \infty]$, we set

$$H_{f,\omega,p}(\xi) = \left(\int_{\mathbf{R}^n} |V_\chi f(x, \xi)\omega(x, \xi)|^p \, dx\right)^{1/p}.$$

DEFINITION 1.6. Assume that $f, f_j \in M^{p,q}_{(\omega)}(\mathbf{R}^n)$, $j = 1, 2, \ldots$. Then f_j is said to converge *narrowly* to f (with respect to $p, q \in [1, \infty]$, $\chi \in \mathscr{S}(\mathbf{R}^n)\backslash 0$ and $\omega \in \mathscr{P}(\mathbf{R}^{2n})$), if the following conditions are satisfied:
(1) $f_j \to f$ in $\mathscr{S}'(\mathbf{R}^n)$ as j tends to ∞;
(2) $H_{f_j,\omega,p}(\xi) \to H_{f,\omega,p}(\xi)$ in $L^q(\mathbf{R}^n)$ as j tends to ∞.

REMARK 1.7. Assume that $f, f_1, f_2, \ldots \in \mathscr{S}'(\mathbf{R}^n)$ satisfies (1) in Definition 1.6, and assume that $\xi \in \mathbf{R}^n$. Then it follows from Fatou's lemma that

$$\liminf_{j\to\infty} H_{f_j,\omega,p}(\xi) \geq H_{f,\omega,p}(\xi)$$

and

$$\liminf_{j\to\infty} \|f_j\|_{M^{p,q}_{(\omega)}} \geq \|f\|_{M^{p,q}_{(\omega)}}.$$

The following proposition is important to us later on. We omit the proof since the result is a restatement of Proposition 2.3 in [44].

Proposition 1.8. *Assume that* $p, q \in [1, \infty]$ *with* $q < \infty$ *and that* $\omega \in \mathscr{P}(\mathbf{R}^{2n})$. *Then* $C_0^\infty(\mathbf{R}^n)$ *is dense in* $M^{p,q}_{(\omega)}(\mathbf{R}^n)$ *with respect to the narrow convergence.*

1.2. Coorbit spaces of modulation space types. Next we discuss a family of Banach spaces of time-frequency type which contains the modulation spaces. Certain types of these Banach spaces are used as symbol classes for Fourier integral operators which are considered in Subsection 2.5. (Cf. the introduction.) After submitting the paper, we got knowledge that our coorbit spaces may be, in a broader context, considered as modulation spaces (cf. [16]).

Assume that V_j and W_j for $j = 1, \dots, 4$ are vector spaces of dimensions n_j and m_j respectively such that

$$V_1 \oplus V_2 = V_3 \oplus V_4 = \mathbf{R}^n, \quad W_1 \oplus W_2 = W_3 \oplus W_4 = \mathbf{R}^m. \tag{1.3}$$

We let the euclidean structure in V_j and W_j be inherited from $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. For convenience we use the notations

$$\overline{V} = (V_1, \dots, V_4), \quad \overline{W} = (W_1, \dots, W_4), \quad \text{and} \quad \mathsf{p} = (p, q, r, s),$$

for quadruples of vector spaces and the numbers $p, q, r, s \in [1, \infty]$, and we set

$$L^{\mathsf{p}}(\overline{V}) \equiv L^s(V_4; L^r(V_3; L^q(V_2; L^p(V_1)))).$$

Finally, if $\omega \in \mathscr{P}(\mathbf{R}^{2n})$, then we let $L^{\mathsf{p}}_{(\omega)}(\overline{V})$ be the Banach space which consists of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2n})$ such that $F\omega \in L^{\mathsf{p}}(\overline{V})$. This means that $L^{\mathsf{p}}_{(\omega)}(\overline{V})$ is the set of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2n})$ such that

$$\|F\|_{L^{\mathsf{p}}_{(\omega)}(\overline{V})} \equiv \left(\int_{V_4} \left(\int_{V_3} \left(\int_{V_2} \left(\int_{V_1} |F(x, \xi)\omega(x, \xi)|^p \, dx_1 \right)^{q/p} dx_2 \right)^{r/q} d\xi_1 \right)^{s/r} d\xi_2 \right)^{1/s}$$

is finite (with obvious modifications when one or more of p, q, r, s are equal to infinity). Here dx_1, dx_2, $d\xi_1$ and $d\xi_2$ denote the Lebesgue measure in V_1, V_2, V_3 and V_4 respectively.

Next, for $\chi \in \mathscr{S}(\mathbf{R}^n) \setminus 0$, we let $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ be the *coorbit space* which consists of all $f \in \mathscr{S}'(\mathbf{R}^n)$ such that $V_\chi f \in L^{\mathsf{p}}_{(\omega)}(\overline{V})$, i.e.

$$\|f\|_{\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})} \equiv \|V_\chi f\|_{L^{\mathsf{p}}_{(\omega)}(\overline{V})} < \infty. \tag{1.4}$$

We note that if $p = q$ and $r = s$, then $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ agrees with the modulation space $M^{p,r}_{(\omega)}$. On the other hand, if $p \neq q$ or $r \neq s$, then $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ is *not* a modulation space on such form. A more general definition of coorbit spaces can be found in [17, 18], where such spaces were introduced and briefly investigated.

The most of the properties for modulation spaces stated in Proposition 1.1 and Remark 1.3 carry over to $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}$ spaces. For example the analysis in [22] shows that the

following result holds. Here we use the convention

$$\mathsf{p}_1 \le \mathsf{p}_2 \quad \text{when} \quad \mathsf{p}_j = (p_j, q_j, r_j, s_j) \quad \text{and} \quad p_1 \le p_2,\ q_1 \le q_2,\ r_1 \le r_2,\ s_1 \le s_2,$$

and

$$t_1 \le \mathsf{p} \le t_2 \quad \text{when} \quad \mathsf{p} = (p, q, r, s), \quad t_1, t_2 \in [1, \infty] \quad \text{and} \quad t_1 \le p, q, r, s \le t_2.$$

Proposition 1.9. *Assume that* $\mathsf{p}, \mathsf{p}_j \in [1, \infty]$ *for* $j = 1, 2$, *and* $\omega, \omega_1, \omega_2, v \in \mathscr{P}(\mathbf{R}^{2n})$ *are such that* ω *is* v*-moderate and* $\omega_2 \le C\omega_1$ *for some constant* $C > 0$. *Then the following are true*:
(1) *if* $\chi \in M^1_{(v)}(\mathbf{R}^n) \setminus 0$, *then* $f \in \mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ *if and only if* (1.4) *holds, i.e.* $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ *is independent of the choice of* χ. *Moreover,* $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ *is a Banach space under the norm in* (1.4), *and different choices of* χ *give rise to equivalent norms*;
(2) *if* $\mathsf{p}_1 \le \mathsf{p}_2$ *then*

$$\mathscr{S}(\mathbf{R}^n) \hookrightarrow \mathbf{\Theta}^{\mathsf{p}_1}_{(\omega_1)}(\overline{V}) \hookrightarrow \mathbf{\Theta}^{\mathsf{p}_2}_{(\omega_2)}(\overline{V}) \hookrightarrow \mathscr{S}'(\mathbf{R}^n).$$

Later on we also need the following observation.

Proposition 1.10. *Assume that* $(x, y) \in V_1 \oplus V_2 = \mathbf{R}^{n_0+n}$ *with dual variables* $(\xi, \eta) \in V_4 \oplus V_3$, *where* $V_1 = V_4 = \mathbf{R}^{n_0}$ *and* $V_2 = V_3 = \mathbf{R}^n$. *Also assume that* $f \in \mathscr{S}'(\mathbf{R}^n)$, $f_0 \in \mathscr{S}'(\mathbf{R}^{n_0+n})$, $\omega \in \mathscr{P}(\mathbf{R}^{2n})$ *and* $\omega_0 \in \mathscr{P}(\mathbf{R}^{2(n_0+n)})$ *satisfy*

$$f_0(x, y) = f(y) \quad (\text{in } \mathscr{S}'(\mathbf{R}^{n_0+n}))$$

and

$$\omega_0(x, y, \xi, \eta) = \omega(y, \eta)\langle \xi \rangle^t$$

for some $t \in \mathbf{R}$, *and that* $p, q \in [1, \infty]$. *Then* $f \in M^{p,q}_{(\omega)}(\mathbf{R}^n)$ *if and only if* $f_0 \in \mathbf{\Theta}^{\mathsf{p}}_{(\omega_0)}(\mathbf{R}^{n_0+n})$ *and* $\mathsf{p} = (\infty, p, q, 1)$, *with* $\overline{V} = (V_1, V_2, V_3, V_4)$.

Proof. Let $\chi_0 = \chi_1 \otimes \chi$, where $\chi_1 \in \mathscr{S}(\mathbf{R}^{n_0})$ and $\chi \in \mathscr{S}(\mathbf{R}^n)$. By straightforward computations it follows that

$$|V_{\chi_0} f_0(x, y, \xi, \eta)\omega_0(x, y, \xi, \eta)| = |V_\chi f(y, \eta)\omega(y, \eta)|\, |\hat{\chi}_1(\xi)\langle \xi \rangle^t|. \tag{1.5}$$

Since $|\hat{\chi}_1(\xi)|\langle \xi \rangle^t$ turns rapidly to zero at infinity, the result follows by applying the $L^{\mathsf{p}}(\overline{V})$-norm on (1.5). □

Since interpolation properties for coorbit spaces are important to us, we next recall some of these properties. By Corollary 4.6 in [17] it follows that $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ is homeomorphic to a retract of $L^{\mathsf{p}}_{(\omega)}(\overline{V})$. This implies that the interpolation properties of $L^{\mathsf{p}}_{(\omega)}(\overline{V})$

spaces carry over to $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ spaces. (Cf. Theorem 4.7 in [17].) Furthermore, since the map $f \mapsto \omega \cdot f$ defines a homeomorphism from $L^{\mathsf{p}}_{(\omega)}(\overline{V})$ to $L^{\mathsf{p}}(\overline{V})$, it follows that $L^{\mathsf{p}}_{(\omega)}(\overline{V})$ has the same interpolation properties as $L^{\mathsf{p}}(\overline{V})$. From these observations together with the fact that the proof of Theorem 5.6.3 in [2] shows that

$$(L^{p_1}(\mathbf{R}^n; \mathscr{B}_1), L^{p_2}(\mathbf{R}^n; \mathscr{B}_2))_{[\theta]} = L^p(\mathbf{R}^n; \mathscr{B}),$$

$$\text{when} \quad p, p_1, p_2 \in [1, \infty], \quad \mathscr{B} = (\mathscr{B}_1, \mathscr{B}_2)_{[\theta]} \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$

it follows that the following result is an immediate consequence of Theorems 4.4.1 and 5.1.1 [2]. The second part is also a consequence of Corollary 4.6 in [17] and certain results in [26]. Here we use the convention

$$\frac{1}{\mathsf{p}} = \left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}\right) \quad \text{when} \quad \mathsf{p} = (p, q, r, s).$$

Proposition 1.11. *Assume that $V_j \subseteq \mathbf{R}^n$ and $W_j \subseteq \mathbf{R}^m$ for $j = 1, \dots, 4$ are vector spaces such that* (1.3) *holds, $\mathsf{p}_j, \mathsf{q}_j \in [1, \infty]^4$ for $j = 0, 1, 2$ satisfy*

$$\frac{1}{\mathsf{p}_0} = \frac{1-\theta}{\mathsf{p}_1} + \frac{\theta}{\mathsf{p}_2}, \quad \frac{1}{\mathsf{q}_0} = \frac{1-\theta}{\mathsf{q}_1} + \frac{\theta}{\mathsf{q}_2},$$

for some $\theta \in [0, 1]$. Also assume that $\omega, \omega_j \in \mathscr{P}(\mathbf{R}^{2n_j})$ for $j = 1, 2$. Then the following is true:

(1) *the complex interpolation space $(\mathbf{\Theta}^{\mathsf{p}_1}_{(\omega)}(\overline{V}), \mathbf{\Theta}^{\mathsf{p}_2}_{(\omega)}(\overline{V}))_{[\theta]}$ is equal to $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$;*

(2) *if T is a linear and continuous operator from $\mathbf{\Theta}^{\mathsf{p}_1}_{(\omega_1)}(\overline{V}) + \mathbf{\Theta}^{\mathsf{p}_2}_{(\omega_1)}(\overline{V})$ to $\mathbf{\Theta}^{\mathsf{q}_1}_{(\omega_2)}(\overline{W}) + \mathbf{\Theta}^{\mathsf{q}_2}_{(\omega_2)}(\overline{W})$, which restricts to a continuous map from $\mathbf{\Theta}^{\mathsf{p}_j}_{(\omega_1)}(\overline{V})$ to $\mathbf{\Theta}^{\mathsf{q}_j}_{(\omega_2)}(\overline{W})$ for $j = 1, 2$, then T restricts to a continuous mapping from $\mathbf{\Theta}^{\mathsf{p}_0}_{(\omega_1)}(\overline{V})$ to $\mathbf{\Theta}^{\mathsf{q}_0}_{(\omega_2)}(\overline{W})$.*

1.3. Schatten–von Neumann classes and pseudo-differential operators. Next we recall some facts in Chapter XVIII in [24] concerning pseudo-differential operators. Assume that $a \in \mathscr{S}(\mathbf{R}^{2n})$, and that $t \in \mathbf{R}$ is fixed. Then the pseudo-differential operator $a_t(x, D)$ in (0.4) is a linear and continuous operator on $\mathscr{S}(\mathbf{R}^n)$, as remarked in the introduction. For general $a \in \mathscr{S}'(\mathbf{R}^{2n})$, the pseudo-differential operator $a_t(x, D)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$ with distribution kernel

$$K_{t,a}(x, y) = (2\pi)^{-n/2}(\mathscr{F}_2^{-1}a)((1-t)x + ty, y - x), \tag{1.6}$$

where $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathscr{S}'(\mathbf{R}^{2n})$ with respect to the y-variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and $F(x, y) \mapsto F((1-t)x + ty, y - x)$ are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2n})$. Moreover, it agrees with the operator in (0.4) when $a \in \mathscr{S}(\mathbf{R}^{2n})$.

We recall that for any $t \in \mathbf{R}$ fixed, it follows from the kernel theorem by Schwartz that the map $a \mapsto a_t(x, D)$ is bijective from $\mathscr{S}'(\mathbf{R}^{2n})$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$ (see e.g. [24]). In particular, if $a \in \mathscr{S}'(\mathbf{R}^{2n})$ and $s, t \in \mathbf{R}$, then there is a unique $b \in \mathscr{S}'(\mathbf{R}^{2n})$ such that $a_s(x, D) = b_t(x, D)$. By straightforward applications of Fourier's inversion formula, it follows that

$$a_s(x, D) = b_t(x, D) \Longleftrightarrow b(x, \xi) = e^{i(t-s)\langle D_x, D_\xi\rangle} a(x, \xi). \tag{1.7}$$

(Cf. Section 18.5 in [24].)

Next we recall some facts on Schatten–von Neumann operators and pseudo-differential operators (cf. the introduction).

For each pairs of separable Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, the set $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$ is a Banach space which increases with $p \in [1, \infty]$, and if $p < \infty$, then $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$ is contained in the set of compact operators. Furthermore, $\mathscr{I}_1(\mathscr{H}_1, \mathscr{H}_2)$, $\mathscr{I}_2(\mathscr{H}_1, \mathscr{H}_2)$ and $\mathscr{I}_\infty(\mathscr{H}_1, \mathscr{H}_2)$ agree with the set of trace-class operators, Hilbert–Schmidt operators and continuous operators respectively, with the same norms.

Next we discuss complex interpolation properties of Schatten–von Neumann classes. Let $p, p_1, p_2 \in [1, \infty]$ and let $0 \le \theta \le 1$. Then similar *complex* interpolation properties hold for Schatten–von Neumann classes as for Lebesgue spaces, i.e. it holds

$$\mathscr{I}_p = (\mathscr{I}_{p_1}, \mathscr{I}_{p_2})_{[\theta]}, \quad \text{when} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}. \tag{1.8}$$

(Cf. [34].) Furthermore, by Theorem 2.c.6 in [25] and its proof, together with the remark which followed that theorem, it follows that the *real* interpolation property

$$\mathscr{I}_p = (\mathscr{I}_2, \mathscr{I}_\infty)_{\theta, p}, \quad \text{when} \quad \theta = 1 - \frac{2}{p} \tag{1.9}$$

holds. We refer to [34, 43] for a brief discussion of Schatten–von Neumann operators.

For any $t \in \mathbf{R}$ and $p \in [1, \infty]$, let $s_{t,p}(\omega_1, \omega_2)$ be the set of all $a \in \mathscr{S}'(\mathbf{R}^{2n})$ such that $a_t(x, D) \in \mathscr{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$. Also set

$$\|a\|_{s_{t,p}} = \|a\|_{s_{t,p}(\omega_1,\omega_2)} \equiv \|a_t(x, D)\|_{\mathscr{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})}$$

when $a_t(x, D)$ is continuous from $M^2_{(\omega_1)}$ to $M^2_{(\omega_2)}$. Since $a \mapsto a_t(x, D)$ is a bijective map from $\mathscr{S}'(\mathbf{R}^{2n})$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$, it follows that the map $a \mapsto a_t(x, D)$ restricts to an isometric bijection from $s_{t,p}(\omega_1, \omega_2)$ to $\mathscr{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$.

Proposition 1.12. *Assume that $p, q_1, q_2 \in [1, \infty]$ are such that $q_1 \le \min(p, p')$ and $q_2 \ge \max(p, p')$. Also assume that $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2n})$ and $\omega, \omega_0 \in \mathscr{P}(\mathbf{R}^{4n})$ satisfy*

$$\frac{\omega_2(x - ty, \xi + (1-t)\eta)}{\omega_1(x + (1-t)y, \xi - t\eta)} = \omega(x, \xi, \eta, y) \tag{1.10}$$

and

$$\omega_0(x, y, \xi, \eta) = \omega((1-t)x + ty, t\xi - (1-t)\eta, \xi + \eta, y - x). \tag{1.11}$$

Then the following is true:
(1) $M^{p,q_1}_{(\omega)}(\mathbf{R}^{2n}) \subseteq s_{t,p}(\omega_1, \omega_2) \subseteq M^{p,q_2}_{(\omega)}(\mathbf{R}^{2n})$;
(2) *the operator kernel K of $a_t(x, D)$ belongs to $M^p_{(\omega_0)}(\mathbf{R}^{2n})$ if and only if $a \in M^p_{(\omega)}(\mathbf{R}^{2n})$ and for some constant C, which only depends on t and the involved weight functions, it holds $\|K\|_{M^p_{(\omega_0)}} = C\|a\|_{M^p_{(\omega)}}$*

We note that (1.10) and (1.11) are equivalent to

$$\begin{aligned} &\omega(x, \xi, \eta, y) = \omega_0(x - ty, x + (1-t)y, \xi + (1-t)\eta, t\eta - \xi) \quad \text{and} \\ &\frac{\omega_2(x, \xi)}{\omega_1(y, -\eta)} \le C\omega_0(x, y, \xi, \eta). \end{aligned} \tag{1.12}$$

Proof. The assertion (1) is a restatement of Theorem 4.13 in [45]. The assertion (2) follows by similar arguments as in the proof of Proposition 4.8 in [45], which we recall here. Let $\chi, \psi \in \mathscr{S}(\mathbf{R}^{2n})$ be such that

$$\psi(x, y) = \int_{\mathbf{R}^n} \chi((1-t)x + ty, \xi) e^{i\langle y-x, \xi\rangle}\, d\xi.$$

By applying the Fourier inversion formula it follows by straightforward computations that

$$|\mathscr{F}(K\tau_{(x-ty, x+(1-t)y)}\psi)(\xi + (1-t)\eta, -\xi + t\eta)| = |\mathscr{F}(a\tau_{(x,\xi)}\chi)(\eta, y)|.$$

The result now follows by applying the $L^p_{(\omega)}$ norm on these expressions, and using (1.12). □

We also need the following proposition on continuity of linear operators with kernels in modulation spaces.

Proposition 1.13. *Assume that $p \in [1, \infty]$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2n_j})$, for $j = 1, 2$, and $\omega_0 \in \mathscr{P}(\mathbf{R}^{2n_1+2n_2})$ fulfill for some positive constant C*

$$\frac{\omega_2(x, \xi)}{\omega_1(y, -\eta)} \le C\omega_0(x, y, \xi, \eta). \tag{1.13}$$

Assume moreover that $K \in M^p_{(\omega_0)}(\mathbf{R}^{n_1+n_2})$ *and* T *is the linear and continuous map from* $\mathscr{S}(\mathbf{R}^{n_1})$ *to* $\mathscr{S}'(\mathbf{R}^{n_2})$ *defined by*

$$(Tf)(x) = \langle K(x, \cdot), f\rangle \tag{1.14}$$

when $f \in \mathscr{S}(\mathbf{R}^{n_1})$. *Then* T *extends uniquely to a continuous map from* $M^{p'}_{(\omega_1)}(\mathbf{R}^{n_1})$ *to* $M^p_{(\omega_2)}(\mathbf{R}^{n_2})$.

On the other hand, assume that T *is a linear continuous map from* $M^1_{(\omega_1)}(\mathbf{R}^{n_1})$ *to* $M^\infty_{(\omega_2)}(\mathbf{R}^{n_2})$, *and that equality is attained in* (1.13). *Then there is a unique kernel* $K \in M^\infty_{(\omega_0)}(\mathbf{R}^{n_1+n_2})$ *such that* (1.14) *holds for every* $M^1_{(\omega_1)}(\mathbf{R}^{n_1})$.

Proof. By Proposition 1.1 (3) and duality, it suffices to prove that for some constant C independent of $f \in \mathscr{S}(\mathbf{R}^{n_1})$ and $g \in \mathscr{S}(\mathbf{R}^{n_2})$, it holds:

$$|(K, g \otimes \bar{f})| \le C\|K\|_{M^p_{(\omega_0)}}\|g\|_{M^{p'}_{(1/\omega_2)}}\|f\|_{M^{p'}_{(\omega_1)}}.$$

Let $\omega_3(x, \xi) = \omega_1(x, -\xi)$. Then by straightforward calculation and using Remark 1.3 (7) we get

$$\begin{aligned}|(K, g \otimes \bar{f})| &\le C_1\|K\|_{M^p_{(\omega_0)}}\|g \otimes \bar{f}\|_{M^{p'}_{(1/\omega)}} \le C_2\|K\|_{M^p_{(\omega_0)}}\|g\|_{M^{p'}_{(1/\omega_2)}}\|\bar{f}\|_{M^{p'}_{(\omega_3)}}\\ &\le C\|K\|_{M^p_{(\omega_0)}}\|g\|_{M^{p'}_{(1/\omega_2)}}\|f\|_{M^{p'}_{(\omega_1)}}.\end{aligned}$$

The last part of the proposition concerning the converse property in the case $p = \infty$ is a restatement of Proposition 4.7 in [45] on generalization of Feichtinger–Gröchenig's kernel theorem. □

2. Continuity properties of Fourier integral operators

In this section we discuss Fourier integral operators with amplitudes in modulation spaces, or more generally in certain types of coorbit spaces. In Subsection 2.3 we extend Theorem 3.2 in [4] to more general modulation spaces.

2.1. Notation and general assumptions. In the most general situation, we assume that the phase function φ and the amplitude a depend on $x \in \mathbf{R}^{n_2}$, $y \in \mathbf{R}^{n_1}$ and $\zeta \in \mathbf{R}^m$, with dual variables respectively $\xi \in \mathbf{R}^{n_2}$, $\eta \in \mathbf{R}^{n_1}$ and $z \in \mathbf{R}^m$. For convenience we use the notation:

$$N = n_1 + n_2 \quad \text{and} \quad X = (x, y, \zeta) \in \mathbf{R}^{n_2} \oplus \mathbf{R}^{n_1} \oplus \mathbf{R}^m \simeq \mathbf{R}^{N+m}. \tag{2.1}$$

In order to state the results in Subsection 2.5 we let V_1 be a linear subspace of $\mathbf{R}^{N+m}$ of dimension N, $V_2 = V_1^\perp$, and let $V_j' \simeq V_j$ be the dual of V_j for $j = 1, 2$. Also let

any element $X = (x, y, \zeta) \in \mathbf{R}^{N+m}$ and $(\xi, \eta, z) \in \mathbf{R}^{N+m}$ be written as

$$\begin{aligned}(x, y, \zeta) &= \boldsymbol{t}_1 e_1 + \cdots + \boldsymbol{t}_N e_N + \boldsymbol{\varrho}_1 e_{N+1} + \cdots + \boldsymbol{\varrho}_m e_{N+m} \\ &\equiv (\boldsymbol{t}, \boldsymbol{\varrho}) = (\boldsymbol{t}, \boldsymbol{\varrho})_{V_1 \oplus V_2},\end{aligned}$$

and

$$\begin{aligned}(\xi, \eta, z) &= \boldsymbol{\tau}_1 e_1 + \cdots + \boldsymbol{\tau}_N e_N + \boldsymbol{u}_1 e_{N+1} \cdots + \boldsymbol{u}_m e_{N+m} \\ &\equiv (\boldsymbol{\tau}, \boldsymbol{u}) = (\boldsymbol{\tau}, \boldsymbol{u})_{V_1' \oplus V_2'},\end{aligned}$$

for some orthonormal basis $e_1, \dots, e_{N+m}$ in $\mathbf{R}^{N+m}$. We also let F_ζ' denote the gradient of $F \in C^1(\mathbf{R}^{N+m})$ with respect to the basis $e_{N+1}, \dots, e_{N+m}$.

In general we assume that the involved weight functions $\omega, v \in \mathscr{P}(\mathbf{R}^{N+m} \times \mathbf{R}^{N+m})$, $\omega_0 \in \mathscr{P}(\mathbf{R}^{2N})$ and $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2n_j})$ and the phase function $\varphi \in C(\mathbf{R}^{N+m})$ fulfill the following conditions:

(1) v is submultiplicative and satisfies

$$\begin{aligned}&v(X, \xi, \eta, z) = v(\xi, \eta, z) \quad \text{and} \quad v(t\,\cdot) \le Cv, \\ &X \in \mathbf{R}^{N+m},\ \xi \in \mathbf{R}^{n_2},\ \eta \in \mathbf{R}^{n_1},\ z \in \mathbf{R}^m,\end{aligned} \tag{2.2}$$

for some constant C which is independent of $t \in [0, 1]$. In particular, $v(X, \xi, \eta, z)$ is constant with respect to $X \in \mathbf{R}^{N+m}$;

(2) $\varphi \in C(\mathbf{R}^{N+m})$ and $\varphi^{(\alpha)} \in M_{(v)}^{\infty,1}(\mathbf{R}^{N+m})$ for all indices α such that $|\alpha| = 2$;

(3) there exist some constants C, C_1 and C_2 which are independent of

$$X = (x, y, \zeta) \in \mathbf{R}^{N+m}, \quad \xi, \xi_1, \xi_2 \in \mathbf{R}^{n_2},\ \eta, \eta_1, \eta_2 \in \mathbf{R}^{n_1},\ z_1, z_2 \in \mathbf{R}^m$$

such that

$$\begin{aligned}&\frac{\omega_2(x, \xi)}{\omega_1(y, -\eta)} \le C_1 \omega_0(x, y, \xi, \eta) \le C_2 \omega(X, \xi - \varphi_x'(X), \eta - \varphi_y'(X), -\varphi_\zeta'(X)), \\ &\omega(X, \xi_1 + \xi_2, \eta_1 + \eta_2, z_1 + z_2) \le C\omega(X, \xi_1, \eta_1, z_1) v(\xi_2, \eta_2, z_2).\end{aligned} \tag{2.3}$$

We note that the assumptions in (2) imply that the phase function φ belongs to $C^2(\mathbf{R}^{N+m})$ and is bounded by second order polynomials, since the condition that v is submultiplicative implies that $\varphi'' \in M_{(v)}^{\infty,1} \subseteq M^{\infty,1} \subseteq C \cap L^\infty$.

It is also convenient to set

$$\begin{aligned}\mathbf{E}_{a,\omega}(\boldsymbol{t}, \boldsymbol{\varrho}, \boldsymbol{\tau}, \boldsymbol{u}) &= |V_\chi a(X, \xi, \eta, z)\omega(X, \xi, \eta, z)|, \\ \mathcal{E}_{a,\omega}(x, y, \boldsymbol{u}) &= \sup_{\zeta, \boldsymbol{\tau}} \mathbf{E}_{a,\omega}(\boldsymbol{t}, \boldsymbol{\varrho}, \boldsymbol{\tau}, \boldsymbol{u}),\end{aligned}$$

when $a \in \mathscr{S}'(\mathbf{R}^{N+m})$. Here $\boldsymbol{u} \in V_2'$ and the supremum should be taken over $\zeta \in \mathbf{R}^m$ and $\boldsymbol{\tau} \in V_1'$.

2.2. The continuity assertions. In most of our investigations we consider Fourier integral operators $\operatorname{Op}_\varphi(a)$ where the amplitudes a belong to appropriate Banach spaces which are defined in similar way as certain types of coorbit spaces in Subsection 1.3, and that the phase function φ should satisfy the conditions in Subsection 2.1. In this context we list now the statements which will be proved in the following under appropriate assumptions on a, φ. Here the definition of admissible pairs (a, φ) is presented in Subsection 2.5 below.

(i) *the pair (a, φ) is admissible, the kernel $K_{a,\varphi}$ of $\operatorname{Op}_\varphi(a)$ belongs to $M^p_{(\omega_0)}$, and*

$$\|K_{a,\varphi}\|_{M^p_{(\omega_0)}} \le C\mathsf{d}^{-1}\exp\bigl(\|\varphi''\|_{M^{\infty,1}_{(v)}}\bigr)\|a\|,$$

for some constant C which is independent of $a \in \mathscr{S}'(\mathbf{R}^{N+m})$ and $\varphi \in C(\mathbf{R}^{N+m})$;

(ii) *the definition of $\operatorname{Op}_\varphi(a)$ extends uniquely to a continuous operator from $M^{p'}_{(\omega_1)}(\mathbf{R}^{n_1})$ to $M^p_{(\omega_2)}(\mathbf{R}^{n_2})$. Furthermore, for some constant C it holds*

$$\|\operatorname{Op}_\varphi(a)\|_{M^{p'}_{(\omega_1)}\to M^p_{(\omega_2)}} \le C\mathsf{d}^{-1}\exp\bigl(\|\varphi''\|_{M^{\infty,1}_{(v)}}\bigr)\|a\|;$$

(iii) *if in addition $1 \le p \le 2$, then $\operatorname{Op}_\varphi(a) \in \mathscr{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$.*

2.3. Reformulation of Fourier integral operators in terms of short time Fourier transforms. For each real-valued $\varphi \in C(\mathbf{R}^{N+m})$ which satisfies (2) in Subsection 2.1, and $a \in \mathscr{S}(\mathbf{R}^{N+m})$, it follows that the Fourier integral operator $f \mapsto \operatorname{Op}_\varphi(a)f$ in (0.2) is well-defined and makes sense as a continuous operator from $\mathscr{S}(\mathbf{R}^{n_1})$ to $\mathscr{S}'(\mathbf{R}^{n_2})$, that is

$$(\operatorname{Op}_\varphi(a)f, g) = (2\pi)^{-N/2}\int_{\mathbf{R}^{N+m}} a(X)e^{i\varphi(X)}f(y)\overline{g(x)}\,dX,$$

is well-defined when $f \in \mathscr{S}(\mathbf{R}^{n_1})$ and $g \in \mathscr{S}(\mathbf{R}^{n_2})$. In order to extend the definition we reformulate the latter relation in terms of short-time Fourier transforms.

Assume that $0 \le \chi, \psi \in C_0^\infty(\mathbf{R}^{N+m})$ and $0 \le \chi_j \in C_0^\infty(\mathbf{R}^{n_j})$ for $j = 1, 2$ are such that $\psi \equiv 1$ in the support of χ,

$$\iiint \chi(x, y, \zeta)\chi_1(x)\chi_2(x)\,dx\,dy\,d\zeta = 1,$$

and let $X_1 = (x_1, y_1, \zeta_1) \in \mathbf{R}^{N+m}$. By straightforward computations we get

$$\begin{aligned}
&(2\pi)^{N/2}(\operatorname{Op}_\varphi(a)f, g)\\
&= \int_{\mathbf{R}^{N+m}} a(X)f(y)\overline{g(x)}e^{i\varphi(X)}\,dX\\
&= \iint_{\mathbf{R}^{2(N+m)}} a(X+X_1)\chi(X_1)^2 f(y+y_1)\chi_1(y_1)\overline{g(x+x_1)\chi_2(x_1)}e^{i\psi(X_1)\varphi(X+X_1)}\,dX\,dX_1.
\end{aligned}$$

Then Parseval's formula gives

$$
\begin{aligned}
&(2\pi)^{N/2}(\mathrm{Op}_\varphi(a)f, g)\\
&= \iiiint_{\mathbf{R}^{2(N+m)}} F(X, \xi, \eta, \zeta_1)\mathscr{F}(f(y+\cdot)\chi_1)(-\eta)\overline{\mathscr{F}(g(x+\cdot)\chi_2)(\xi)}\, dX\, d\xi\, d\eta\, d\zeta_1\\
&= \iiiint_{\mathbf{R}^{2(N+m)}} F(X, \xi, \eta, \zeta_1)V_{\chi_1} f(y, -\eta)\overline{V_{\chi_2} g(x, \xi)}e^{-i(\langle x,\xi\rangle+\langle y,\eta\rangle)}\, dX\, d\xi\, d\eta\, d\zeta_1\\
&= \iiint_{\mathbf{R}^{2N+m}} \left(\int_{\mathbf{R}^m} F(X, \xi, \eta, \zeta_1)\, d\zeta_1\right) V_{\chi_1} f(y, -\eta)\overline{V_{\chi_2} g(x, \xi)}e^{-i(\langle x,\xi\rangle+\langle y,\eta\rangle)}\, dX\, d\xi\, d\eta,
\end{aligned}
$$

where

$$
F(X, \xi, \eta, \zeta_1) = \mathscr{F}_{1,2}(e^{i\psi(\cdot,\zeta_1)\varphi(X+(\cdot,\zeta_1))}a(X+(\cdot, \zeta_1))\chi(\cdot, \zeta_1)^2)(\xi, \eta).
$$

Here $\mathscr{F}_{1,2}a$ denotes the partial Fourier transform of $a(x, y, \zeta)$ with respect to the x and y variables.

By Taylor's formula it follows that

$$
\psi(X_1)\varphi(X+X_1) = \psi(X_1)\psi_{1,X}(X_1) + \psi_{2,X}(X_1),
$$

where

$$
\begin{aligned}
&\psi_{1,X}(X_1) = \varphi(X) + \langle \varphi'(X), X_1\rangle \quad \text{and}\\
&\psi_{2,X}(X_1) = \psi(X_1)\int_0^1 (1-t)\langle \varphi''(X+tX_1)X_1, X_1\rangle\, dt.
\end{aligned}
\tag{2.4}
$$

Inserting these expressions into the definition of $F(X, \xi, \eta, \zeta_1)$, and integrating with respect to the ζ_1-variable we obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^m} F(X, \xi, \eta, \zeta_1)\, d\zeta_1\\
&= (2\pi)^{m/2}\mathscr{F}((e^{i\psi_{2,X}}\chi)a(\cdot + X)\chi)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X))\\
&= (2\pi)^{N/2}\mathcal{H}_{a,\varphi}(X, \xi, \eta),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{H}_{a,\varphi}(X, \xi, \eta) = h_X * (\mathscr{F}(a(\cdot + X)\chi))(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X))\\
&\text{and} \quad h_X = (2\pi)^{-(N-m/2)}(\mathscr{F}(e^{i\psi_{2,X}}\chi)).
\end{aligned}
\tag{2.5}
$$

Note here that the convolution of the function

$$(X, \xi, \eta, z) \mapsto \mathscr{F}(a(\cdot + X)\chi))(\xi, \eta, z)$$

should be taken with respect to the variables ξ, η and z only.

Summing up we have proved that

$$\begin{aligned} &(\operatorname{Op}_\varphi(a)f, g) \\ &= T_{a,\varphi}(f, g) \\ &\equiv \iiint \mathcal{H}_{a,\varphi}(X, \xi, \eta)(V_{\chi_0} f)(y, -\eta)\overline{(V_{\chi_0} g)(x, \xi)} e^{-i(\langle x,\xi\rangle + \langle y,\eta\rangle)}\, dX\, d\xi\, d\eta. \end{aligned} \tag{2.6}$$

2.4. An extension of a result by Boulkhemair. Next we consider Fourier integral operators with amplitudes in the modulation space $M^{\infty,1}_{(\omega)}(\mathbf{R}^{2n+m})$, where we are able to state and prove the announced generalization of Theorem 3.2 in [5]. Here we assume that $n_1 = n_2 = n$ which implies that $N = 2n$.

Theorem 2.1. *Assume that* $1 < p < \infty$, $\varphi \in C(\mathbf{R}^{2n+m})$, $\omega, v \in \mathscr{P}(\mathbf{R}^{2(N+m)})$ *and* $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2n})$ *fulfill the conditions in* Subsection 2.1. *Also assume that* (0.5) *holds for some* $\mathsf{d} > 0$. *Then the following is true*:
(1) *the map* $a \mapsto \operatorname{Op}_\varphi(a)$ *from* $\mathscr{S}(\mathbf{R}^{2n+m})$ *to* $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$ *extends uniquely to a continuous map from* $M^{\infty,1}_{(\omega)}(\mathbf{R}^{2n+m})$ *to* $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$;
(2) *if* $a \in M^{\infty,1}_{(\omega)}(\mathbf{R}^{2n+m})$, *then the map* $\operatorname{Op}_\varphi(a)$ *from* $\mathscr{S}(\mathbf{R}^n)$ *to* $\mathscr{S}'(\mathbf{R}^n)$ *extends uniquely to a continuous operator from* $M^p_{(\omega_1)}(\mathbf{R}^n)$ *to* $M^p_{(\omega_2)}(\mathbf{R}^n)$. *Moreover, for some constant* C *it holds*

$$\|\operatorname{Op}_\varphi(a)\|_{M^p_{(\omega_1)} \to M^p_{(\omega_2)}} \le C\mathsf{d}^{-1}\|a\|_{M^{\infty,1}_{(\omega)}} \exp\bigl(C\|\varphi''\|_{M^{\infty,1}_{(v)}}\bigr). \tag{2.7}$$

The proof needs some preparing lemmas.

Lemma 2.2. *Assume that* $v(x, \xi) = v(\xi) \in \mathscr{P}(\mathbf{R}^n)$ *is submultiplicative and satisfies* $v(t\xi) \le Cv(\xi)$ *for some constant* C *which is independent of* $t \in [0, 1]$ *and* $\xi \in \mathbf{R}^n$. *Also assume that* $f \in M^{\infty,1}_{(v)}(\mathbf{R}^n)$, $\chi \in C_0^\infty(\mathbf{R}^n)$ *and that* $x \in \mathbf{R}^n$, *and let*

$$\varphi_{x,j,k}(y) = \chi(y) \int_0^1 (1-t) f(x + ty) y_j y_k\, dt.$$

Then there is a constant C *and a function* $g \in M^1_{(v)}(\mathbf{R}^n)$ *such that* $\|g\|_{M^1_{(v)}} \le C\|f\|_{M^{\infty,1}_{(v)}}$ *and* $|\mathscr{F}(\varphi_{x,j,k})(\xi)| \le \hat{g}(\xi)$.

Proof. We first prove the assertion when χ is replaced by $\psi_0(y) = e^{-2|y|^2}$. For convenience we let

$$\psi_1(y) = e^{-|y|^2}$$

and

$$\psi_2(y) = e^{|y|^2}$$

and

$$H_{\infty,f}(\xi) \equiv \sup_x |\mathscr{F}(f\psi_1(\cdot - x))(\xi)|.$$

We claim that g, defined by

$$\hat{g}(\xi) = \int_0^1 \int_{\mathbf{R}^n} (1-t) H_{\infty,f}(\eta) e^{-|\xi - t\eta|^2/16}\, d\eta\, dt, \tag{2.8}$$

fulfills the required properties.

In fact, if $v_1(\xi, x) = v(x, \xi) = v(\xi)$, then by applying $M^1_{(v)}$ norm on g, and using Remark 1.3 (6), (7) and Minkowski's inequality, we obtain

$$\begin{aligned}
\|g\|_{M^1_{(v)}} = \|\hat{g}\|_{M^1_{(v_1)}} &= \left\| \int_0^1 \int_{\mathbf{R}^n} (1-t) H_{\infty,f}(\eta) e^{-|\cdot - t\eta|^2/16}\, d\eta\, dt \right\|_{M^1_{(v_1)}} \\
&\le \int_0^1 \int_{\mathbf{R}^n} (1-t) H_{\infty,f}(\eta) \|e^{-|\cdot - t\eta|^2/16}\|_{M^1_{(v_1)}}\, d\eta\, dt \\
&\le C_1 \int_0^1 \int_{\mathbf{R}^n} (1-t) H_{\infty,f}(\eta) \|e^{-|\cdot|^2/16}\|_{M^1_{(v_1)}} v(t\eta)\, d\eta\, dt \\
&\le C_2 \int_0^1 \int_{\mathbf{R}^n} (1-t) H_{\infty,f}(\eta) v(\eta) \|e^{-|\cdot|^2/16}\|_{M^1_{(v_1)}}\, d\eta\, dt \\
&= C_3 \|H_{\infty,f} v\|_{L^1} = C_3 \|f\|_{M^{\infty,1}_{(v)}},
\end{aligned}$$

for some constants C_1, C_2 and C_3.

In order to prove that $|\mathscr{F}(\varphi_{x,j,k})(\xi)| \le \hat{g}(\xi)$, we let $\psi(y) = \psi_{j,k}(y) = y_j y_k \psi_0(y)$. Then

$$\varphi_{x,j,k}(y) = \psi(y) \int_0^1 (1-t) f(x + ty)\, dt.$$

By a change of variables we obtain

$$
\begin{aligned}
|\mathscr{F}(\varphi_{x,j,k})(\xi)| &= \left|(2\pi^{-n/2})\int_0^1 (1-t)\left(\int_{\mathbf{R}^n} f(x+ty)\psi(y)e^{-i\langle y,\xi\rangle}\,dy\right)dt\right| \\
&= \left|\int_0^1 t^{-n}(1-t)\mathscr{F}(f\psi((\cdot-x)/t))(\xi/t)e^{i\langle x,\xi\rangle/t}\,dt\right| \qquad (2.9)\\
&\le \int_0^1 t^{-n}(1-t)\sup_{x\in\mathbf{R}^n}|\mathscr{F}(f\psi((\cdot-x)/t))(\xi/t)|\,dt.
\end{aligned}
$$

We need to estimate the right-hand side. By straightforward computations we get

$$
\begin{aligned}
&|\mathscr{F}(f\psi((\cdot-x)/t))(\xi)| \\
&\le (2\pi)^{-n/2}(|\mathscr{F}(f\psi_1(\cdot-x))| * |\mathscr{F}(\psi((\cdot-x)/t)\psi_2(\cdot-x))|)(\xi) \\
&= (2\pi)^{-n/2}(|\mathscr{F}(f\psi_1(\cdot-x))| * |\mathscr{F}(\psi(\cdot/t)\psi_2)|)(\xi)
\end{aligned}
$$

where the convolutions should be taken with respect to the ξ-variable only. Then

$$
|\mathscr{F}(f\psi((\cdot-x)/t))(\xi)| \le (2\pi)^{-n/2}(H_{\infty,f} * |\mathscr{F}(\psi(\cdot/t)\psi_2)|)(\xi). \tag{2.10}
$$

For the estimate of the latter Fourier transform we observe that

$$
|\mathscr{F}(\psi(\cdot/t)\psi_2)| = |\partial_j\partial_k\mathscr{F}(\psi_0(\cdot/t)\,\psi_2)|. \tag{2.11}
$$

Since ψ_0 and ψ_2 are Gauss functions and $0 \le t \le 1$, a straightforward computation gives

$$
\mathscr{F}(\psi_0(\cdot/t)\psi_2)(\xi) = \pi^{n/2}t^n(2-t^2)^{-n/2}e^{-t^2|\xi|^2/(4(2-t^2))}. \tag{2.12}
$$

Thus a combination of (2.11) and (2.12) therefore give

$$
|\mathscr{F}(\psi(\cdot/t)\,\psi_2)(\xi)| \le Ct^n e^{-t^2|\xi|^2/16}, \tag{2.13}
$$

for some constant C which is independent of $t \in [0, 1]$. The assertion now follows by combining (2.8)–(2.10) and (2.13).

In order to prove the result for general $\chi \in C_0^\infty(\mathbf{R}^n)$ we set

$$
h_{x,j,h}(y) = \psi_0(y)\int_0^1 (1-t)f(x+ty)y_jy_k\,dt,
$$

and we observe that the result is already proved when $\varphi_{x,j,k}$ is replaced by $h_{x,j,h}$. Moreover $\varphi_{x,j,k} = \chi_1 h_{x,j,k}$, for some $\chi_1 \in C_0^\infty(\mathbf{R}^n)$. Hence if $\hat{g}_0$ is equal to the right-hand side of (2.8), the first part of the proof shows that

$$
|\mathscr{F}(\varphi_{x,j,k})(\xi)| = |\mathscr{F}(\chi_1 h_{x,j,k}(\xi))| \le (2\pi)^{-n/2}|\widehat{\chi_1}| * \hat{g}_0(\xi) \equiv \hat{g}(\xi).
$$

Since $\|g_0\|_{M^1_{(v)}} \le C\|f\|_{M^{\infty,1}_{(v)}}$ and $M^1_{(v_1)} * L^1_{(v)} \subseteq M^1_{(v_1)}$, we get for some positive constants C_1, C_2 and C_3 that

$$\|g\|_{M^1_{(v)}} \le C_1\|\hat{g}\|_{M^1_{(v_1)}} \le C_2\|\widehat{\chi}_1\|_{L^1_{(v)}}\|\hat{g}_0\|_{M^1_{(v_1)}} \le C_3\|f\|_{M^{\infty,1}_{(v)}},$$

which proves the result. □

As a consequence of Lemma 2.2 we have the following result.

Lemma 2.3. *Assume that $v(x,\xi) = v(\xi) \in \mathscr{P}(\mathbf{R}^n)$ is submultiplicative and satisfies $v(t\xi) \le Cv(\xi)$ for some constant C which is independent of $t \in [0,1]$ and $\xi \in \mathbf{R}^n$. Also assume that $f_{j,k} \in M^{\infty,1}_{(v)}(\mathbf{R}^n)$ for $j,k = 1,\dots,n$, $\chi \in C_0^\infty(\mathbf{R}^n)$ and that $x \in \mathbf{R}^n$, and let*

$$\varphi_x(y) = \sum_{j,k=1,\dots,n} \varphi_{x,j,k}(y), \quad \text{where} \quad \varphi_{x,j,k}(y) = \chi(y)\int_0^1 (1-t)f_{j,k}(x+ty)y_jy_k\,dt.$$

Then there is a constant C and a function $\Psi \in M^1_{(v)}(\mathbf{R}^n)$ such that

$$\|\Psi\|_{M^1_{(v)}} \le \exp\Big(C\sup_{j,k}\|f_{j,k}\|_{M^{\infty,1}_{(v)}}\Big)$$

and

$$|\mathscr{F}(\exp(i\varphi_x)(\xi))| \le (2\pi)^{n/2}\delta_0 + \hat{\Psi}(\xi). \tag{2.14}$$

Proof. By Lemma 2.2, we may find a function $g \in M^1_{(v)}$ and a constant $C > 0$ such that

$$|\mathscr{F}(\varphi_x)(\xi)| \le \hat{g}(\xi), \quad \|g\|_{M^1_{(v)}} \le C\sup_{j,k}\big(\|f_{j,k}\|_{M^{\infty,1}_{(v)}}\big).$$

Set

$$\Phi_{0,x} = (2\pi)^{n/2}\delta_0, \quad \Phi_{l,x} = |\mathscr{F}(\varphi_x)| * \cdots * |\mathscr{F}(\varphi_x)|, \quad l \ge 1,$$
$$\Upsilon_0 = (2\pi)^{n/2}\delta_0, \quad \hat{\Upsilon}_l = \hat{g} * \cdots * \hat{g}, \quad l \ge 1,$$

with l factors in the convolutions. Then by Taylor expansion, there are positive con-

stants C_1 and C_2 such that

$$|\mathscr{F}(\exp(i\varphi_x(\cdot))(\xi))| \le \sum_{l=0}^{\infty} \frac{C_1^l \Phi_{l,x}}{l!} \le \sum_{l=0}^{\infty} \frac{C_2^l \hat{\Upsilon}_l}{l!}.$$

Hence, if

$$\Psi \equiv \sum_{l=1}^{\infty} \frac{C_2^l \Upsilon_l}{l!},$$

then (2.14) follows with $C = C_2$. Furthermore, since v is submultiplicative, it follows from Proposition 1.2 that

$$\|\Upsilon_l\|_{M^1_{(v)}} = (2\pi)^{(l-1)n/2} \|g \cdots g\|_{M^1_{(v)}} \le \left(C_1 \|g\|_{M^1_{(v)}}\right)^l, \quad l \ge 1,$$

for some positive constant C_1. This gives

$$\begin{aligned} \|\Psi\|_{M^1_{(v)}} &\le \sum_{l=1}^{\infty} \frac{\|\Upsilon_l\|_{M^1_{(v)}}}{l!} \le \sum_{l=1}^{\infty} \frac{(C_1 \|g\|_{M^1_{(v)}})^l}{l!} \\ &\le \sum_{l=1}^{\infty} \frac{\left(C_2 \sup_{j,k}\left(\|f_{j,k}\|_{M^{\infty,1}_{(v)}}\right)\right)^l}{l!} \le \exp\left(C_2 \sup_{j,k} \|f_{j,k}\|_{M^{\infty,1}_{(v)}}\right), \end{aligned}$$

for some constants C_1 and C_2, and the result follows. □

Proof of Theorem 2.1. We shall mainly follow the proof of Theorem 3.2 in [5]. First assume that $a \in C_0^\infty(\mathbf{R}^{2n+m})$ and $f, g \in \mathscr{S}(\mathbf{R}^n)$. Then it follows that $\operatorname{Op}_\varphi(a)$ makes sense as a continuous operator from $\mathscr{S}$ to $\mathscr{S}'$. Since

$$|\mathscr{F}(e^{i\Psi_{2,X}}\chi)| \le (2\pi)^{-n+m/2} |\mathscr{F}(e^{i\Psi_{2,X}})| * |\hat{\chi}|, \quad |\mathscr{F}(a(\cdot + X)\chi)| = |V_\chi a(X, \cdot)|$$

and $M^1_{(v_1)} \subseteq L^1_{(v)}$, it follows from Remark 1.3 (5) and Lemma 2.3 that

$$|\mathcal{H}_{a,\varphi}(X, \xi, \eta)| \le C(G * |V_\chi a(X, \cdot)|)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)), \tag{2.15}$$

where $G \in L^1_{(v)}$ satisfies $\|G\|_{L^1_{(v)}} \le C \exp(C\|\varphi''\|_{M^{\infty,1}_{(v)}})$, since $|\hat{\chi}|$ turns rapidly to zero at infinity. Here $\mathcal{H}_{a,\varphi}$ is the same as in (2.5), $v_1(\xi, x) = v(\xi)$, and the convolution for $V_\chi a(X, \xi, \eta, z)$ should be taken with respect to the variables ξ, η and z only.

Next we set

$$
\begin{aligned}
\mathbf{E}_{a,\omega}(X, \xi, \eta, z) &= |V_\chi a(X, \xi, \eta, z)\omega(X, \xi, \eta, z)|, \\
\tilde{\mathrm{E}}_{a,\omega}(\xi, \eta, z) &= \sup_X \mathbf{E}_{a,\omega}(X, \xi, \eta, z), \\
F_1(y, \eta) &= |V_{\chi_1} f(y, \eta)\omega_1(y, \eta)|, \\
F_2(x, \xi) &= |V_{\chi_2} g(x, \xi)/\omega_2(x, \xi)|,
\end{aligned}
\tag{2.16}
$$

and

$$
\mathcal{Q}_{a,\omega}(\mathsf{X}, \zeta) = \mathbf{E}_{a,\omega}(X, \xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)),
\tag{2.17}
$$

and

$$
\mathcal{R}_{a,\omega,\varphi}(\mathsf{X}, \zeta) = ((Gv) * \mathbf{E}_{a,\omega}(X, \cdot))(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)),
\tag{2.18}
$$

where

$$
X = (x, y, \zeta) \quad \text{and} \quad \mathsf{X} = (x, y, \xi, \eta).
$$

Note here the difference between X and X. By combining (2.3) with (2.15) we get

$$
\begin{aligned}
&\iiint |\mathcal{H}_{a,\varphi}(X, \xi, \eta)(V_{\chi_1} f)(y, -\eta)(V_{\chi_2} g)(x, \xi)|\, dX\, d\xi\, d\eta \\
&\le C_1 \iiint (G * |V_\chi a(X, \cdot)|)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) \\
&\quad \times |(V_{\chi_1} f)(y, -\eta)(V_{\chi_2} g)(x, \xi)|\, dX\, d\xi\, d\eta \\
&\le C_2 \iiint \mathcal{R}_{a,\omega,\varphi}(\mathsf{X}, \zeta) F_1(y, -\eta) F_2(x, \xi)\, dX\, d\xi\, d\eta.
\end{aligned}
$$

Summing up we have proved that

$$
|(\mathrm{Op}_\varphi(a) f, g)| \le C \iiint \mathcal{R}_{a,\omega,\varphi}(\mathsf{X}, \zeta) F_1(y, -\eta) F_2(x, \xi)\, dX\, d\xi\, d\eta.
\tag{2.19}
$$

It follows from (2.16), (2.18), (2.19) and Hölder's inequality that

$$
|(\mathrm{Op}_\varphi(a) f, g)| \le C J_1 \cdot J_2,
\tag{2.20}
$$

where

$$
\begin{aligned}
J_1 &= \left(\iiint (Gv) * \tilde{\mathrm{E}}_{a,\omega}(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) F_1(y, -\eta)^p\, dX\, d\xi\, d\eta \right)^{1/p}, \\
J_2 &= \left(\iiint (Gv) * \tilde{\mathrm{E}}_{a,\omega}(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) F_2(x, \xi)^{p'}\, dX\, d\xi\, d\eta \right)^{1/p'}.
\end{aligned}
$$

We have to estimate J_1 and J_2. By taking $z = \varphi_\zeta'(X)$, $\zeta_0 = \varphi_y'(X)$, y, ξ and η as new variables of integrations, and using (0.5), it follows that

$$\begin{aligned} J_1 &\le \left(\mathsf{d}^{-1} \iiint (Gv) * \tilde{\mathrm{E}}_{a,\omega}(\xi - \kappa_1(y, z, \zeta_0), \eta - \zeta_0, z) F_1(y, -\eta)^p \, dy \, dz \, d\xi \, d\eta \, d\zeta_0 \right)^{1/p} \\ &= \left(\mathsf{d}^{-1} \iiint (Gv) * \tilde{\mathrm{E}}_{a,\omega}(\xi, \zeta_0, z) F_1(y, -\eta)^p \, dy \, dz \, d\xi \, d\eta \, d\zeta_0 \right)^{1/p} \\ &= \mathsf{d}^{-1/p} \|(Gv) * \tilde{\mathrm{E}}_{a,\omega}\|_{L^1}^{1/p} \|F_1\|_{L^p}, \end{aligned}$$

for some continuous function κ_1. It follows from Young's inequality and (2.3) that

$$\|(Gv) * \tilde{\mathrm{E}}_{a,\omega}\|_{L^1} \le \|G\|_{L^1_{(v)}} \|\tilde{\mathrm{E}}_{a,\omega}\|_{L^1}.$$

Hence

$$J_1 \le \mathsf{d}^{-1/p} \big(C \exp\big(C \|\varphi''\|_{M^{\infty,1}_{(v)}} \big) \|a\|_{M^{\infty,1}_{(\omega)}} \big)^{1/p} \|f\|_{M^p_{(\omega_1)}}. \tag{2.21}$$

If we instead take x, $y_0 = \varphi_\zeta'(X)$, ξ, η and $\zeta_0 = \varphi_x'(X)$ as new variables of integrations, it follows by similar arguments that

$$J_2 \le \mathsf{d}^{-1/p'} \big(C \exp\big(C \|\varphi''\|_{M^{\infty,1}_{(v)}} \big) \|a\|_{M^{\infty,1}_{(\omega)}} \big)^{1/p'} \|g\|_{M^{p'}_{(1/\omega_2)}}. \tag{2.21$'$}$$

A combination of (2.20), (2.21) and (2.21)$'$ now gives

$$|(\mathrm{Op}_\varphi(a) f, g)| \le C \mathsf{d}^{-1} \|a\|_{M^{\infty,1}_{(\omega)}} \|f\|_{M^p_{(\omega_1)}} \|g\|_{M^{p'}_{(1/\omega_2)}} \exp\big(C \|\varphi''\|_{M^{\infty,1}_{(v)}} \big),$$

which proves (2.7), and the result follows when $a \in C_0^\infty(\mathbf{R}^{2n+m})$ and $f, g \in \mathscr{S}(\mathbf{R}^n)$.

Since $\mathscr{S}$ is dense in $M^p_{(\omega_1)}$ and $M^{p'}_{(1/\omega_2)}$, the result also holds for $a \in C_0^\infty$ and $f \in M^p_{(\omega_1)}$. Hence it follows by Hahn–Banach's theorem that the asserted extension of the map $a \mapsto \mathrm{Op}_\varphi(a)$ exists.

It remains to prove that this extension is unique. Therefore assume that $a \in M^{\infty,1}_{(\omega)}$ is arbitrary, and take a sequence $a_j \in C_0^\infty$ for $j = 1, 2, \dots$ which converges to a with respect to the narrow convergence (see Definition 1.6). Then $\tilde{\mathrm{E}}_{a_j,\omega}$ converges to $\tilde{\mathrm{E}}_{a,\omega}$ in L^1 as j turns to infinity. By (2.4)–(2.6) and the arguments at the above, it follows from Lebesgue's theorem that

$$(\mathrm{Op}_\varphi(a_j) f, g) \to (\mathrm{Op}_\varphi(a) f, g)$$

as j tends to infinity. This proves the uniqueness, and the result follows. □

2.5. Fourier integral operators with amplitudes in coorbit spaces. A crucial point concerning the uniqueness when extending the definition of Op_φ to amplitudes

in $M^{\infty,1}_{(\omega)}$ in Theorem 2.1 is that C_0^∞ is dense in $M^{\infty,1}_{(\omega)}$ with respect to the narrow convergence. On the other hand, the uniqueness of the extension might be violated when spaces of amplitudes are considered where such density or duality properties are missing. In the present paper we use the reformulation (2.6) to extend the definition of the Fourier integral operator in (0.2) to certain amplitudes which are not contained in $M^{\infty,1}_{(\omega)}$.

More precisely, assume that $a \in \mathscr{S}'(\mathbf{R}^{N+m})$, $f \in \mathscr{S}(\mathbf{R}^{n_1})$, $g \in \mathscr{S}(\mathbf{R}^{n_2})$ and that the mapping

$$(X, \xi, \eta) \mapsto \mathcal{H}_{a,\varphi}(X, \xi, \eta)(V_{\chi_0} f)(y, -\eta)\overline{(V_{\chi_0} g)(x, \xi)}$$

belongs to $L^1(\mathbf{R}^{N+m} \times \mathbf{R}^N)$, where $\mathcal{H}_{a,\varphi}$ is given by (2.5). (Here recall that $N = n_1 + n_2$, where, from now on, n_1 and n_2 might be different.) Then we let $T_{a,\varphi}(f, g)$ be defined as the right-hand side of (2.6).

DEFINITION 2.4. Assume that $N = n_1 + n_2$, $v \in \mathscr{P}(\mathbf{R}^{N+m} \times \mathbf{R}^{N+m})$ is submultiplicative and satisfies (2.2), $\varphi \in C(\mathbf{R}^{N+m})$ fulfills the condition (2) in Subsection 2.1, and that $a \in \mathscr{S}'(\mathbf{R}^{N+m})$ is such that $f \mapsto T_{a,\varphi}(f, g_0)$ and $g \mapsto T_{a,\varphi}(f_0, g)$ are well-defined and continuous from $\mathscr{S}(\mathbf{R}^{n_1})$ and from $\mathscr{S}(\mathbf{R}^{n_2})$ respectively to $\mathbf{C}$, for each fixed $f_0 \in \mathscr{S}(\mathbf{R}^{n_1})$ and $g_0 \in \mathscr{S}(\mathbf{R}^{n_2})$. Then the pair (a, φ) is called *admissible*, and the Fourier integral operator $\operatorname{Op}_\varphi(a)$ is the linear continuous mapping from $\mathscr{S}(\mathbf{R}^{n_1})$ to $\mathscr{S}'(\mathbf{R}^{n_2})$ which is defined by the formulas (2.4), (2.5) and (2.6).

Here recall that if for each fixed $f_0 \in \mathscr{S}(\mathbf{R}^{n_1})$ and $g_0 \in \mathscr{S}(\mathbf{R}^{n_2})$, the mappings $f \mapsto T(f, g_0)$ and $g \mapsto T(f_0, g)$ are continuous from $\mathscr{S}(\mathbf{R}^{n_1})$ and from $\mathscr{S}(\mathbf{R}^{n_2})$ respectively to $\mathbf{C}$, then it follows by Banach–Steinhaus theorem that $(f, g) \mapsto T(f, g)$ is continuous from $\mathscr{S}(\mathbf{R}^{n_1}) \times \mathscr{S}(\mathbf{R}^{n_2})$ to $\mathbf{C}$.

The following theorem involves Fourier integral operators with amplitudes which are not contained in $M^{\infty,1}_{(\omega)}$.

Theorem 2.5. *Assume that* N, χ, ω, ω_j, v, φ, V_j, V_j', $\boldsymbol{t}$, $\boldsymbol{\tau}$, $\boldsymbol{\rho}$ *and* $\boldsymbol{u}$ *for* $j = 0, 1, 2$ *are the same as in* Subsection 2.1. *Also assume that* $a \in \mathscr{S}'(\mathbf{R}^{N+m})$ *fulfills* $\|a\| < \infty$, *where*

$$\|a\| \equiv \operatorname*{ess\,sup}_{x,y}\left(\int_{V_2'}\left(\sup_{\zeta\in\mathbf{R}^m,\boldsymbol{\tau}\in V_1'} |V_\chi a(X, \xi, \eta, z)\omega(X, \xi, \eta, z)|\right) d\boldsymbol{u}\right),$$

and that $|\det(\varphi''_{\boldsymbol{\rho},\zeta})| \geq \mathsf{d}$ *for some* $\mathsf{d} > 0$. *Then* (i)–(ii) *in* Subsection 2.2 *hold for* $p = \infty$.

We note that the conditions on a in Theorem 2.5 means that a should belong to a subspace of $M^\infty_{(\omega)}$ which is a superspace of $M^{\infty,1}_{(\omega)}$. Roughly speaking it follows that a should belong to $M^\infty_{(\omega)}$ in some variables and to $M^{\infty,1}_{(\omega)}$ in the other variables. In

fact, it follows that the amplitudes in Theorem 2.5 form a space of distributions which is equal to $\mathscr{F}_1\boldsymbol{\Theta}^{\mathsf{p}}_{(\tilde{\omega})}$, where $\mathsf{p} = (\infty, \infty, 1, \infty)$, $\tilde{\omega} \in \mathscr{P}(\mathbf{R}^{2(N+m)})$ is appropriate, and $\mathscr{F}_1$ is an appropriate partial Fourier transform on $\mathscr{S}'(\mathbf{R}^{N+m})$. Therefore, this space of distributions is not a coorbit space of that particular type which is considered in Subsection 1.2. On the other hand, it is a coorbit spaces in a more general context, considered in [17, 18].

Proof. It suffices to prove (i) in Subsection 2.2.

We use similar notations as in the proof of Theorem 2.1. Furthermore we let

$$\mathcal{E}_{a,\omega}(x, y, \boldsymbol{u}) = \sup_{\zeta,\boldsymbol{\tau}} \mathbf{E}_{a,\omega}(X, \xi, \eta, z)$$

and

$$G_{1,v}(\boldsymbol{u}) = \int_{V_1'} G(\xi, \eta, z)\, d\boldsymbol{\tau},$$

where $\mathbf{E}_{a,\omega}$ is given by (2.16). By taking $x, y, -\varphi_{\boldsymbol{\varrho}}'(X), \xi, \eta$ as new variables of integration in (2.19), and using the fact that $|\det(\varphi''_{\boldsymbol{\varrho},\zeta})| \geq \mathsf{d}$ we get

$$\begin{aligned} |(\mathrm{Op}_\varphi(a)f, g)| &\leq C\mathsf{d}^{-1} \int_{\mathbf{R}^{2N}} \mathcal{K}_{a,\omega,Gv}(\mathsf{X}) F_1(y, -\eta) F_2(x, \xi)\, d\mathsf{X} \\ &\leq C\mathsf{d}^{-1} \|\mathcal{K}_{a,\omega,Gv}\|_{L^\infty} \|F_1\|_{L^1} \|F_2\|_{L^1}, \end{aligned} \tag{2.22}$$

where $\mathsf{X} = (x, y, \xi, \eta)$ and

$$\mathcal{K}_{a,\omega,Gv}(\mathsf{X}) = \int_{V_2'} ((Gv) * (\mathbf{E}_{a,\omega}(x, y, \kappa_1, \cdot)))((\xi, \eta, 0)_{\mathbf{R}^{N+m}} - (\kappa_2, \boldsymbol{u})_{V_1'\oplus V_2'})\, d\boldsymbol{u},$$

for some continuous functions $\kappa_1 = \kappa_1(x, y, \boldsymbol{u})$ and $\kappa_2 = \kappa_2(x, y, \boldsymbol{u})$.

We need to estimate $\|\mathcal{K}_{a,\omega,Gv}\|_{L^\infty}$. By Young's inequality and simple change of variables it follows that

$$\|\mathcal{K}_{a,\omega,Gv}\|_{L^\infty} \leq \|Gv\|_{L^1} \cdot J_{a,\omega},$$

where

$$\begin{aligned} J_{a,\omega} &= \operatorname*{ess\,sup}_{\mathsf{X}} \left(\int_{V_2'} \mathbf{E}_{a,\omega}(x, y, \kappa_1(x, y, \boldsymbol{u}), (\kappa_2(x, y, \boldsymbol{u}), \boldsymbol{u})_{V_1'\oplus V_2'})\, d\boldsymbol{u} \right) \\ &\leq \operatorname*{ess\,sup}_{\mathsf{X}} \left(\int_{V_2'} \sup_{\zeta,\boldsymbol{\tau}} \mathbf{E}_{a,\omega}(x, y, \zeta, (\boldsymbol{\tau}, \boldsymbol{u})_{V_1'\oplus V_2'})\, d\boldsymbol{u} \right) = \|a\|. \end{aligned}$$

Hence

$$\|\mathcal{K}_{a,\omega,Gv}\|_{L^\infty} \leq \|Gv\|_{L^1} \|a\| \leq C \exp(C\|\varphi''\|_{M^{\infty,1}_{(\omega)}}) \|a\|. \tag{2.23}$$

A combination of (2.22), (2.23), and the facts that $\|F_1\|_{L^1} = \|f\|_{M^1_{(\omega_1)}}$ and $\|F_2\|_{L^1} = \|g\|_{M^1_{(1/\omega_2)}}$ now gives that the pair (a, φ) is admissible, and that (i) in Subsection 2.2 holds. The proof is complete. □

Corollary 2.6. *Assume that N, χ, ω_0, ω_j, v and φ for $j = 1, 2$ are the same as in Subsection 2.1. Also assume that $a \in \mathscr{S}'(\mathbf{R}^{N+m})$, and that one of the following conditions holds*:

(1) $|\det(\varphi''_{\zeta,\zeta})| \geq \mathsf{d}$ *and* $\|a\| < \infty$, *where*

$$\|a\| = \sup_{x,y}\left(\int_{\mathbf{R}^m} \sup_{\zeta,\xi,\eta} |V_\chi a(X, \xi, \eta, z)\omega(X, \xi, \eta, z)|\, dz\right); \tag{2.24}$$

(2) $m = n_1$, $|\det(\varphi''_{x,\zeta})| \geq \mathsf{d}$ *and* $\|a\| < \infty$, *where*

$$\|a\| = \sup_{x,y}\left(\int_{\mathbf{R}^{n_1}} \sup_{\zeta,\eta,z} |V_\chi a(X, \xi, \eta, z)\omega(X, \xi, \eta, z)|\, d\xi\right); \tag{2.25}$$

(3) $m = n_2$, $|\det(\varphi''_{y,\zeta})| \geq \mathsf{d}$ *and* $\|a\| < \infty$, *where*

$$\|a\| = \sup_{x,y}\left(\int_{\mathbf{R}^{n_2}} \sup_{\zeta,\xi,z} |V_\chi a(X, \xi, \eta, z)\omega(X, \xi, \eta, z)|\, d\eta\right). \tag{2.26}$$

Then the (i)–(ii) *in* Subsection 2.2 *hold for $p = \infty$.*

Proof. If (1) is fulfilled, then the result follows by choosing

$$V_1 = V_1' = \{(\xi, \eta, 0) \in \mathbf{R}^{N+m};\ \xi \in \mathbf{R}^{n_2},\ \ \eta \in \mathbf{R}^{n_1}\},$$
$$V_2 = V_2' = \{(0, 0, \zeta) \in \mathbf{R}^{N+m};\ \zeta \in \mathbf{R}^m\},$$

$\boldsymbol{\varrho} = \zeta$, $\boldsymbol{\tau} = (\xi, \eta)$ and $\boldsymbol{u} = z$ in Theorem 2.5. If instead (2) is fulfilled, then the result follows by choosing

$$V_1 = V_1' = \{(0, \eta, z) \in \mathbf{R}^{N+m};\ \eta \in \mathbf{R}^{n_1},\ \ z \in \mathbf{R}^m\},$$
$$V_2 = V_2' = \{(\xi, 0, 0) \in \mathbf{R}^{N+m};\ \xi \in \mathbf{R}^{n_2}\},$$

$\boldsymbol{\varrho} = x$, $\boldsymbol{\tau} = (\eta, z)$ and $\boldsymbol{u} = \xi$ in Theorem 2.5. The result follows by similar arguments if instead (3) is fulfilled. The details are left for the reader. □

Next we discuss continuity and Schatten–von Neumann properties for Fourier integral operators with related conditions on the amplitudes belong to coorbit spaces which are related to the amplitude space in Theorem 2.5. These computations are based on

estimates of the short-time Fourier transform of the distribution kernels of these operators.

Assume that $a \in \mathscr{S}'(\mathbf{R}^{N+m})$, $\omega, v \in \mathscr{P}(\mathbf{R}^{2(N+m)})$, $\omega_j(\mathbf{R}^{2n_j})$ and $\varphi \in C(\mathbf{R}^{N+m})$ satisfy $\varphi^{(\alpha)} \in M^{\infty,1}_{(v)}(\mathbf{R}^{N+m})$ for each multi-indices α such that $|\alpha| = 2$, (2.2) and (2.3), as before. Formally, the kernel can be written as

$$K_{a,\varphi}(x, y) = (2\pi)^{-N/2} \int_{\mathbf{R}^m} a(X) e^{i\varphi(X)}\, d\zeta.$$

(Cf. Theorem 3.1.) Hence, if $0 \le \chi_j \in C_0^\infty(\mathbf{R}^{n_j})$ for $j = 1, 2$ are the same as in Subsection 2.3, then it follows by straightforward computations that the short-time Fourier transform of K can be expressed in terms of the formula

$$(V_{\chi_1 \otimes \chi_2} K_{a,\varphi})(x, y, \xi, \eta) = (\mathrm{Op}_\varphi(a)(\chi_1(\cdot - y)e^{-i\langle \cdot, \eta\rangle}), \chi_2(\cdot - x)e^{i\langle \cdot, \xi\rangle})_{L^2}. \tag{2.27}$$

By letting $f = \chi_1(\cdot - y)e^{-i\langle \cdot, \eta\rangle}$ and $g = \chi_2(\cdot - x)e^{i\langle \cdot, \xi\rangle}$, it follows that

$$|V_{\chi_1} f(y_1, \eta_1)| = |(V_{\chi_1}\chi_1)(y_1 - y, \eta_1 + \eta)| \tag{2.28}$$

and

$$|V_{\chi_2} g(x_1, \xi_1)| = |(V_{\chi_2}\chi_2)(x_1 - x, \xi_1 - \xi)|. \tag{2.29}$$

Now we choose N_0 large enough such that ω_0 is moderate with respect to $\langle \cdot \rangle^{N_0}$, and we set

$$F(\mathsf{X}) = |(V_{\chi_1}\chi_1)(y, -\eta)(V_{\chi_2}\chi_2)(x, \xi)\langle \mathsf{X}\rangle^{N_0}|, \quad \text{where} \quad \mathsf{X} = (x, y, \xi, \eta).$$

Then F is a continuous function which turns rapidly to zero at infinity. Furthermore, it follows from (2.28) and (2.29) that

$$|V_{\chi_1} f(y_1, -\eta_1) V_{\chi_2} g(x_1, \xi_1)\omega_0(\mathsf{X})| \le C F(\mathsf{X}_1 - \mathsf{X})\omega_0(\mathsf{X}_1), \tag{2.30}$$

where the inequality follows from the fact that

$$\omega_0(\mathsf{X}) \le C\omega_0(\mathsf{X}_1)\langle \mathsf{X} - \mathsf{X}_1\rangle^{N_0}.$$

By combining (2.2), (2.3), (2.19) and (2.27)–(2.30) we obtain

$$|(V_{\chi_1 \otimes \chi_2} K_{a,\varphi})(\mathsf{X})\omega_0(\mathsf{X})| \le C \iint_{\mathbf{R}^{2N+m}} \mathcal{R}_{a,\omega,\varphi}(\mathsf{X}_1, \zeta_1) F(\mathsf{X}_1 - \mathsf{X})\, d\zeta_1\, d\mathsf{X}_1, \tag{2.31}$$

for some constant C, with $\mathcal{R}_{a,\omega,\varphi}$ defined in (2.18).

We have now the following result related to Theorem 2.5.

Theorem 2.7. *Assume that N, χ, ω, ω_j for $j = 0, 1, 2$, v and φ are the same as in* Subsection 2.1. *Also assume that $p \in [1, \infty]$, and that one of the following conditions hold*:

(1) $a \in \mathscr{S}'(\mathbf{R}^{N+m})$ *and* $\|a\| < \infty$, *where*

$$\|a\| = \left(\iint_{\mathbf{R}^N}\left(\int_{\mathbf{R}^m}\sup_z\left(\iint_{\mathbf{R}^N}|V_\chi a(X,\xi,\eta,z)\omega(X,\xi,\eta,z)|^p d\xi\, d\eta\right)^{1/p} d\zeta\right)^p dx\, dy\right)^{1/p};$$

(2) $|\det(\varphi''_{\zeta,\zeta})| \ge \mathsf{d}$ *for some* $\mathsf{d} > 0$, $a \in \mathscr{S}'(\mathbf{R}^{N+m})$, *and* $\|a\| < \infty$, *where*

$$\|a\| = \left(\iint_{\mathbf{R}^N}\left(\int_{\mathbf{R}^m}\sup_\zeta\left(\iint_{\mathbf{R}^N}|V_\chi a(X,\xi,\eta,z)\omega(X,\xi,\eta,z)|^p d\xi\, d\eta\right)^{1/p} dz\right)^p dx\, dy\right)^{1/p}.$$

Then the (i)–(iii) *in* Subsection 2.2 *hold.*

We note that if $a \in M^1$, then the hypothesis in Theorem 2.7 (1) is fulfilled for $\omega = \omega_j = 1$. Hence Theorem 2.7 generalizes Proposition 2.3 in [7] or [8].

Proof. It suffices to prove (i). We only consider the case when (2) and $p < \infty$ are fulfilled. The other cases follow by similar arguments and are left for the reader.

Let G be the same as in the proof of Theorem 2.1, and let $\mathbf{E}_{a,\omega}$, $\mathcal{Q}_{a,\omega}$ and $\mathcal{R}_{a,\omega}$ be as in (2.16)–(2.18). It follows from (2.31) and Hölder's inequality that

$$\begin{aligned}
&|(V_{\chi_1\otimes\chi_2}K_{a,\varphi})(\mathsf{X})\omega_0(\mathsf{X})| \\
&\le C\iint_{\mathbf{R}^{m+2N}}(\mathcal{R}_{a,\omega,\varphi}(\mathsf{X}_1,\zeta_1)F(\mathsf{X}_1-\mathsf{X})^{1/p})F(\mathsf{X}_1-\mathsf{X})^{1/p'}\,d\zeta_1\,d\mathsf{X}_1 \\
&\le C\|F\|_{L^1}^{1/p'}\left(\int_{\mathbf{R}^{2N}}\left(\int_{\mathbf{R}^m}\mathcal{R}_{a,\omega,\varphi}(\mathsf{X}_1,\zeta_1)\,d\zeta_1\right)^p F(\mathsf{X}_1-\mathsf{X})\,d\mathsf{X}_1\right)^{1/p},
\end{aligned}$$

where $\|F\|_{L^1}$ is finite, since F turns rapidly to zero at infinity. By letting $C_\varphi = C\exp\big(C\|\varphi''\|_{M^{\infty,1}_{(v)}}\big)$ for some large constant C, and applying the L^p norm and Young's inequality, we get

$$\begin{aligned}
\|K\|^p_{M^p_{(\omega_0)}} &\le C_1\int_{\mathbf{R}^{2N}}\left(\int_{\mathbf{R}^m}\mathcal{R}_{a,\omega,\varphi}(\mathsf{X},\zeta)\,d\zeta\right)^p d\mathsf{X} \\
&\le C_1\|G\|^p_{L^1_{(v)}}\int_{\mathbf{R}^{2N}}\left(\int_{\mathbf{R}^m}\mathcal{Q}_{a,\omega,\varphi}(\mathsf{X},\zeta)\,d\zeta\right)^p d\mathsf{X} \qquad (2.32)\\
&\le C_\varphi\int_{\mathbf{R}^{2N}}\left(\int_{\mathbf{R}^m}\mathbf{E}_{a,\omega}(X,\xi-\varphi'_x(X),\eta-\varphi'_y(X),-\varphi'_\zeta(X))\,d\zeta\right)^p d\mathsf{X},
\end{aligned}$$

for $\mathcal{Q}_{a,\omega,\varphi}$ as in (2.17) and some constant C_1. It follows now from Minkowski's in-

equality that the latter integral can be estimated by

$$\iint_{\mathbf{R}^N}\left(\int_{\mathbf{R}^m}\left(\iint_{\mathbf{R}^N}\mathbf{E}_{a,\omega}(X,\xi-\varphi'_x(X),\eta-\varphi'_y(X),-\varphi'_\zeta(X))^p\,d\xi\,d\eta\right)^{1/p}d\zeta\right)^p dx\,dy$$
$$=\iint_{\mathbf{R}^N}\left(\int_{\mathbf{R}^m}\left(\iint_{\mathbf{R}^N}\mathbf{E}_{a,\omega}(X,\xi,\eta,-\varphi'_\zeta(X))^p\,d\xi\,d\eta\right)^{1/p}d\zeta\right)^p dx\,dy.$$

By letting $C_\varphi = C_2\exp\bigl(C_2\|\varphi''\|_{M^{\infty,1}_{(v)}}\bigr)$, taking $\xi,\eta,-\varphi'_\zeta(X),x,y$ as new variables of integration, and using the fact that $|\det(\varphi''_{\zeta,\zeta})|\ge \mathsf{d}$, we get for some function κ that

$$\begin{aligned}\|K\|^p_{M^p_{(\omega_0)}} &\le \frac{C_\varphi}{\mathsf{d}}\iint_{\mathbf{R}^N}\left(\int_{\mathbf{R}^m}\left(\iint_{\mathbf{R}^N}\mathbf{E}_{a,\omega}(x,y,\kappa(x,y,z),\xi,\eta,z)^p\,d\xi\,d\eta\right)^{1/p}dz\right)^p dx\,dy\\ &\le \frac{C_\varphi}{\mathsf{d}}\iint_{\mathbf{R}^N}\left(\int_{\mathbf{R}^m}\sup_\zeta\left(\iint_{\mathbf{R}^N}\mathbf{E}_{a,\omega}(x,y,\zeta,\xi,\eta,z)^p\,d\xi\,d\eta\right)^{1/p}dz\right)^p dx\,dy\\ &= C_\varphi\|a\|^p.\end{aligned}$$

This proves the assertion □

Next we have the following result, parallel to Theorem 2.7.

Theorem 2.8. *Assume that* N, χ, ω, ω_j, v, φ, V_j, V'_j, $\boldsymbol{t}$, $\boldsymbol{\tau}$, $\boldsymbol{\rho}$ *and* $\boldsymbol{u}$ *for* $j=0,1,2$ *are the same as in Subsection* 2.1. *Also assume that* $p\in[1,\infty]$, $a\in\mathscr{S}'(\mathbf{R}^{N+m})$ *fulfills* $\|a\|<\infty$, *where*

$$\|a\| = \int_{V'_2}\left(\iint_{V_1\times V'_1}\left(\operatorname*{ess\,sup}_{\boldsymbol{\varrho}\in V_2}|V_\chi a(X,\xi,\eta,z)\omega(X,\xi,\eta,z)|\right)^p d\boldsymbol{t}\,d\boldsymbol{\tau}\right)^{1/p}d\boldsymbol{u},$$

and that $|\det(\varphi''_{\boldsymbol{\varrho},\zeta})|\ge\mathsf{d}$. *Then* (i)–(iii) *in* Subsection 2.2 *hold.*

We note that the norm estimate on a in Theorem 2.8 means that $a\in\boldsymbol{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ with $\mathsf{p}=(\infty,p,p,1)$ and $\overline{V}=(V_2,V_1,V'_1,V'_2)$. The proof of Theorem 2.8 is based on Theorem 2.5 and the following result which generalizes Theorem 2.7 in the case $p=1$.

Proposition 2.9. *Assume that* N, χ, ω, ω_j, v, φ, V_j, V'_j, $\boldsymbol{t}$, $\boldsymbol{\tau}$, $\boldsymbol{\rho}$ *and* $\boldsymbol{u}$ *for* $j=0,1,2$ *are the same as in* Subsection 2.1. *Also assume that* $a\in\mathscr{S}'(\mathbf{R}^{N+m})$ *satisfies* $\|a\|<\infty$, *where*

$$\|a\| = \iint_{\mathbf{R}^m\times V_1}\operatorname*{ess\,sup}_{\boldsymbol{\varrho}\in V_2}\left(\iint_{\mathbf{R}^N}|V_\chi a(X,\xi,\eta,z)\omega(X,\xi,\eta,z)|\,d\xi\,d\eta\right)d\boldsymbol{t}\,dz,$$

and that $|\det(\varphi''_{\boldsymbol{\varrho},\zeta})|\ge\mathsf{d}$. *Then* (i)–(iii) *in* Subsection 2.2 *hold for* $p=1$.

Proof. We use the same notations as in Subsection 2.1 and the proof of Theorem 2.1. It follows from (2.32) that

$$\begin{aligned}\|K\|_{M^1_{(\omega_0)}} &\le C_\varphi \iint_{\mathbf{R}^{2N+m}} \mathbf{E}_{a,\omega}(X, \xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X))\, d\zeta\, d\mathsf{X} \\ &= C_\varphi \iint_{\mathbf{R}^{2N+m}} \mathbf{E}_{a,\omega}(X, \xi, \eta, -\varphi'_\zeta(X))\, d\zeta\, d\mathsf{X} \\ &= C_1 C_\varphi \iint_{V_1\times V_2} \left(\iint_{\mathbf{R}^N} \mathbf{E}_{a,\omega}(X, \xi, \eta, -\varphi'_\zeta(X))\, d\xi\, d\eta \right) dt\, d\varrho.\end{aligned}$$

By taking t and $-\varphi'_\zeta$ as new variables of integration in the outer double integral, and using the fact that $|\det(\varphi''_{\varrho,\zeta})| \ge \mathsf{d}$, we get

$$\begin{aligned}\|K\|_{M^1_{(\omega_0)}} &\le C_1 C_\varphi \mathsf{d}^{-1} \int_{\mathbf{R}^m} \left(\int_{V_1} \left(\iint_{\mathbf{R}^N} \mathbf{E}_{a,\omega}(X, \xi, \eta, z)\, d\xi\, d\eta \right) dt \right) dz \\ &\le C_1 C_\varphi \mathsf{d}^{-1} \int_{\mathbf{R}^m} \left(\int_{V_1} \sup_{\varrho\in V_2} \left(\iint_{\mathbf{R}^N} \mathbf{E}_{a,\omega}(X, \xi, \eta, z)\, d\xi\, d\eta \right) dt \right) dz \\ &= C_1 C_\varphi \mathsf{d}^{-1} \|a\|.\end{aligned}$$

This proves the result. □

Proof of Theorem 2.8. We start to consider the case $p = 1$. By Proposition 2.9 (i), Minkowski's inequality and substitution of variables we obtain

$$\begin{aligned}\|K\|_{M^1_{(\omega_0)}} &\le C_\varphi \mathsf{d}^{-1} \int_{\mathbf{R}^m} \left(\int_{V_1} \operatorname*{ess\,sup}_{\varrho\in V_2} \left(\iint_{\mathbf{R}^N} \mathbf{E}_{a,\omega}(X, \xi, \eta, z)\, d\xi\, d\eta \right) dt \right) dz \\ &\le C_\varphi \mathsf{d}^{-1} \iiiint_{V_1\times\mathbf{R}^{N+m}} \sup_{\varrho\in V_2} (\mathbf{E}_{a,\omega}(X, \xi, \eta, z))\, dt\, d\xi\, d\eta\, dz \\ &= C_1 C_\varphi \int_{V_2'} \left(\iint_{V_1\times V_1'} \sup_{\varrho\in V_2} (\mathbf{E}_{a,\omega}(X, \xi, \eta, z))\, dt\, d\tau \right) du,\end{aligned}$$

for some constant C_1, and the result follows in this case.

Next we consider the case $p = \infty$. By Theorem 2.5 (i) we get

$$\begin{aligned}\|K\|_{M^\infty_{(\omega_0)}} &\le C_\varphi \sup_{x,y} \left(\int_{V_2'} \sup_{\zeta,\tau} (\mathbf{E}_{a,\omega}(X, \xi, \eta, z))\, du \right) \\ &\le C_\varphi \int_{V_2'} \left(\operatorname*{ess\,sup}_{(t,\tau)\in V_1\times V_1'} \left(\sup_{\varrho\in V_2} (\mathbf{E}_{a,\omega}(X, \xi, \eta, z)) \right) \right) du,\end{aligned}$$

and the result follows in this case as well.

The theorem now follows for general p by interpolation, using Proposition 1.11. The proof is complete. □

By interpolating Theorem 2.1 and Theorem 2.8 we get the following result.

Theorem 2.10. *Assume that* N, χ, ω, ω_j, v, φ, V_j, V_j', $\boldsymbol{t}$, $\boldsymbol{\tau}$, $\boldsymbol{\rho}$ *and* $\boldsymbol{u}$ *for* $j = 0, 1, 2$ *are the same as in* Subsection 2.1. *Also assume that* $p, q \in [1, \infty]$, $a \in \mathscr{S}'(\mathbf{R}^{N+m})$ *fulfills* $\|a\| < \infty$, *where*

$$\|a\| = \int_{V_2'}\left(\int_{V_1'}\left(\int_{V_1}\left(\sup_{\varrho\in V_2}|V_\chi a(X, \xi, \eta, z)\omega(X, \xi, \eta, z)|\right)^p d\boldsymbol{t}\right)^{q/p} d\boldsymbol{\tau}\right)^{1/q} d\boldsymbol{u},$$

and that in addition $n_1 = n_2$ *and* (0.5) *and* $|\det(\varphi''_{\varrho,\xi})| \geq \mathsf{d}$ *hold for some* $\mathsf{d} > 0$. *Then the following is true*:
(1) *if in addition* $p' \leq q \leq p$, *and* $p_1, p_2 \in [1, \infty]$ *satisfy*

$$\frac{1}{p} + \frac{1}{q'} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p'} + \frac{1}{q} \quad \text{and} \quad \frac{1}{p_1'} + \frac{1}{p_2} = \frac{1}{p} + \frac{1}{q}, \tag{2.33}$$

with strict inequality in (2.33) *when* $q < p$, *then the definition of* $\operatorname{Op}_\varphi(a)$ *extends uniquely to a continuous map from* $M^{p_1}_{(\omega_1)}$ *to* $M^{p_2}_{(\omega_2)}$;
(2) *if* $q \leq \min(p, p')$, *then* $\operatorname{Op}_\varphi(a) \in \mathscr{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$.

We note that the norm estimate on a in Theorem 2.10 means that $a \in \boldsymbol{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ with $\mathsf{p} = (\infty, p, q, 1)$ and $\overline{V} = (V_2, V_1, V_1', V_2')$.

Proof. In order to prove (1) we note that the result holds when $(p, q) = (\infty, 1)$ or $q = p$, in view of Theorems 2.1 and 2.8. Next assume that $q = p'$ for $p \geq 2$, and set $\mathsf{p}_1 = (\infty, \infty, 1, 1)$ and $\mathsf{p} = (\infty, 2, 2, 1)$. Then it follows from Theorems 2.1 and 2.8 that the bilinear form

$$T(a, f) \equiv \operatorname{Op}_\varphi(a) f$$

is continuous from

$$\boldsymbol{\Theta}^{\mathsf{p}_1}_{(\omega)} \times M^p_{(\omega_1)} \quad \text{to} \quad M^p_{(\omega_2)}, \quad 1 < p < \infty,$$

and from

$$\boldsymbol{\Theta}^{\mathsf{p}_2}_{(\omega)} \times M^2_{(\omega_1)} \quad \text{to} \quad M^2_{(\omega_2)}.$$

By interpolation, using Theorem 4.4.1 in [2], Proposition 1.5 and Proposition 1.11, it follows that if $q = p' < 2$, then T extends uniquely to a continuous map from

$$\boldsymbol{\Theta}^{\mathsf{p}}_{(\omega)} \times M^{p_1}_{(\omega_1)} \quad \text{to} \quad M^{p_2}_{(\omega_2)},$$

when $p' < p_1 = p_2 < p$. This proves (1) when $q = p$ or $q = p'$.

For $q \in (p', p)$, the result now follows by interpolation between the case $q = p'$ and $p_1 = p_2 = p_0$ where $p' < p_0 < p$, and the case $q = p$ and $p_1' = p_2 = p$. In fact, by interpolation it follows that T extends to a continuous map from

$$\mathbf{\Theta}^{\mathsf{p}}_{(\omega)} \times M^{p_1}_{(\omega_1)} \quad \text{to} \quad M^{p_2}_{(\omega_2)}$$

when

$$\frac{1}{q} = \frac{1-\theta}{p'} + \frac{\theta}{p}, \quad \frac{1}{p_1} = \frac{1-\theta}{p_0} + \frac{\theta}{p'}, \quad \frac{1}{p_2} = \frac{1-\theta}{p_0} + \frac{\theta}{p}.$$

It is now straightforward to control that these conditions are equivalent with those conditions in (1), and the assertion follows for $p' \le q \le p$.

In a similar way, the case $p \in [q, q']$ follows by interpolation between the cases $\mathsf{p}_1 = (\infty, q, q, 1)$ and $\mathsf{p}_2 = (\infty, q', q, 1)$. The details are left for the reader.

In order to prove (2), it is no restriction to assume that $q = \min(p, p')$. If $p = \infty$ and $q = 1$, then the result is a consequence of Theorem 2.1. If instead $1 \le q = p \le 2$, then the result follows from Theorem 2.8. The remaining case $2 \le p = q' \le \infty$ now follows by interpolation between the cases $(p, q) = (2, 2)$ and $(p, q) = (\infty, 1)$, using (1.8) or (1.9), and the interpolation properties in Section 1.2. The proof is complete. □

3. Consequences

In this section we list some consequences of the results in Section 2. In Subsection 3.1 we consider Fourier integral operators where the amplitudes depend on two variables only. In Subsection 3.2 we consider Fourier integral operators with smooth amplitudes.

3.1. Fourier integral operators with amplitudes depending on two variables. We start to discuss Schatten–von Neumann operators for Fourier integral operators with symbols in $M^{p,q}_{(\omega)}(\mathbf{R}^{2n})$ and phase functions which admit second order derivatives in $M^{\infty,1}_{(v)}(\mathbf{R}^{3n})$, for appropriate weight functions ω and v. We assume here that the phase functions depend on $x, y, \zeta \in \mathbf{R}^n$ and that the amplitudes only depend on the x and ζ variables and are independent of the y variable. Note that here we have assumed that the numbers n_1, n_2 and m in Section 2 are equal to n. As in the previous section, we use the notation $X, Y, Z, \dots$ for triples of the form $(x, y, \zeta) \in \mathbf{R}^{3n}$.

The first aim is to establish a weighted version of Theorem 2.5 in [8]. To this purpose, we need to transfer the conditions for the weight and phase functions from Section 2. Namely here and in the following we assume that $\varphi \in C(\mathbf{R}^{3n})$, $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{4n})$, $v_1 \in \mathscr{P}(\mathbf{R}^n)$, $v_2 \in \mathscr{P}(\mathbf{R}^{2n})$ and $v \in \mathscr{P}(\mathbf{R}^{6n})$. A condition on the phase function is

$$|\det(\varphi''_{y,\zeta}(X))| \ge \mathsf{d}, \quad X = (x, y, \zeta) \in \mathbf{R}^{2n} \tag{3.1}$$

for some constant $\mathsf{d} > 0$, and the conditions in (2.3) in Subsection 2.1 are modified into:

$$
(2.3)' \qquad \begin{aligned}
\omega_0(x, y, \xi, \varphi_y'(X)) &\le C\omega(x, \zeta, \xi - \varphi_x'(X), -\varphi_\zeta'(X)),\\
\frac{\omega_2(x, \xi)}{\omega_1(y, -\eta)} &\le C\omega_0(x, y, \xi, \eta),\\
\omega_0(x, y, \xi, \eta_1 + \eta_2) &\le C\omega_0(x, y, \xi, \eta_1)v_1(\eta_2),\\
\omega(x, \zeta, \xi_1 + \xi_2, z_1 + z_2) &\le \omega(x, \zeta, \xi_1, z_1)v_2(\xi_2, z_2),\\
v(X, \xi, \eta, z) &= v_1(\eta)v_2(\xi, z), \quad x, y, z, z_j, \xi, \xi_j, \eta, \zeta \in \mathbf{R}^n.
\end{aligned}
$$

For convenience we also set $\operatorname{Op}_{1,0,\varphi}(a) = \operatorname{Op}_\varphi(a_1)$ when $a_1(x, y, \zeta) = a(x, \zeta)$.

Proposition 3.1. *Assume that $p \in [1, \infty]$, $\mathsf{d} > 0$, $v \in \mathscr{P}(\mathbf{R}^{6n})$ is submultiplicative and satisfies $v(t\,\cdot) \le Cv$ when $t \in [0, 1]$, $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{4n})$, $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2n})$ and that $\varphi \in C(\mathbf{R}^{3n})$ are such that φ is real-valued, $\varphi^{(\alpha)} \in M^{\infty,1}_{(v)}$ for all multi-indices α such that $|\alpha| = 2$, and (3.1) and (2.3)$'$ are fulfilled for some constant C. Then the following is true:*

(1) *the map*

$$
a \mapsto K_{a,\varphi}(x, y) \equiv \int a(x, \zeta)e^{i\varphi(x,y,\zeta)}\, d\zeta,
$$

from $\mathscr{S}(\mathbf{R}^{2n})$ to $\mathscr{S}'(\mathbf{R}^{2n})$ extends uniquely to a continuous map from $M^p_{(\omega)}(\mathbf{R}^{2n})$ to $M^p_{(\omega_0)}(\mathbf{R}^{2n})$;

(2) *the map $a \mapsto \operatorname{Op}_{1,0,\varphi}(a)$ from $\mathscr{S}(\mathbf{R}^{2n})$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$ extends uniquely to a continuous map from $M^p_{(\omega)}(\mathbf{R}^{2n})$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$;*

(3) *if $a \in M^p_{(\omega)}(\mathbf{R}^{2n})$, then the definition of $\operatorname{Op}_{1,0,\varphi}(a)$ extends uniquely to a continuous operator from $M^{p'}_{(\omega_1)}(\mathbf{R}^n)$ to $M^p_{(\omega_2)}(\mathbf{R}^n)$. Furthermore, for some constant C it holds*

$$
\|\operatorname{Op}_{1,0,\varphi}(a)\|_{M^{p'}_{(\omega_1)} \to M^p_{(\omega_2)}} \le C\mathsf{d}^{-1} \exp\bigl(\|\varphi''\|_{M^{\infty,1}_{(v)}}\bigr)\|a\|_{M^p_{(\omega)}};
$$

(4) *if $a \in M^{\infty,1}_{(\omega)}(\mathbf{R}^{2n})$ and $1 < p < \infty$, then the definition of $\operatorname{Op}_{1,0,\varphi}(a)$ from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$ extends uniquely to a continuous operator from $M^p_{(\omega_1)}(\mathbf{R}^n)$ to $M^p_{(\omega_2)}(\mathbf{R}^n)$;*

(5) *if $q \le \min(p, p')$, $a \in M^{p,q}_{(\omega)}(\mathbf{R}^{2n})$, and in addition condition (0.5) holds, then $\operatorname{Op}_{1,0,\varphi}(a) \in \mathscr{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$.*

Proof. We start to prove the continuity assertions. Let $a_1(x, y, \zeta) = a(x, \zeta)$, and let

$$
\tilde{\omega}(x, y, \zeta, \xi, \eta, z) = \omega(x, \zeta, \xi, z)v_1(\eta).
$$

By Proposition 1.10 it follows that $a_1 \in \mathbf{\Theta}^{\mathsf{p}}_{(\tilde{\omega})}(\overline{V})$ with $\mathsf{p} = (\infty, p, p, 1)$ and $\overline{V} = (V_2, V_1, V_1', V_2')$. Hence Theorem 2.8 shows that it suffices to prove that (2.3) holds after ω has been replaced by $\tilde{\omega}$.

By (2.3)′ we have

$$\begin{aligned}\omega_0(x, y, \xi, \eta) &\le C\omega_0(x, y, \xi, \varphi'_y(X))v_1(\eta - \varphi'_y(X))\\ &\le C^2\omega(x, \zeta, \xi - \varphi'(X), -\varphi'_\zeta(X))v_1(\eta - \varphi'_y(X))\\ &= C^2\tilde{\omega}(x, y, \zeta, \xi - \varphi'(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)).\end{aligned}$$

This proves that the first two inequalities in (2.3) hold. Furthermore, since v_1 is submultiplicative we have

$$\begin{aligned}\tilde{\omega}(X, \xi_1 + \xi_2, \eta_1 + \eta_2, z_1 + z_2) &= \omega(x, \zeta, \xi_1 + \xi_2, z_1 + z_2)v_1(\eta_1 + \eta_2)\\ &\le C\omega(x, \zeta, \xi_1, z_1)v_2(\xi_2, z_2)v_1(\eta_1)v(\eta_2)\\ &= C\tilde{\omega}(X, \xi_1, \eta_1, z_1)v(\xi_2, \eta_2, z_2),\end{aligned}$$

for some constant C. This proves the last inequality in (2.3), and the continuity assertions follow.

It remains to prove the uniqueness. If $p < \infty$, then the uniqueness follows from the fact that $\mathscr{S}$ is dense in $M^p_{(\omega)}$.

Next we consider the case $p = \infty$. Assume that $a \in M^1_{(\omega)}(\mathbf{R}^{2n})$ and $b \in M^1_{(1/\omega_0)}(\mathbf{R}^{2n})$, and let $\tilde{\varphi}(x, y, \xi) = -\varphi(x, \xi, y)$. Since (3.1) also holds when φ is replaced by $\tilde{\varphi}$, the first part of the proof shows that $K_{b,\tilde{\varphi}} \in M^1_{(1/\omega)}$. Furthermore, by straightforward computations we have

$$(K_{a,\varphi}, b) = (a, K_{b,\tilde{\varphi}}). \tag{3.2}$$

In view of Proposition 1.1 (3), it follows that the right-hand side in (3.2) makes sense if, more generally, a is an arbitrary element in $M^\infty_{(\omega)}(\mathbf{R}^{2n})$, and then

$$|(a, K_{b,\tilde{\varphi}})| \le C\mathsf{d}^{-1}\|a\|_{M^\infty_{(\omega)}}\|b\|_{M^1_{(1/\omega_0)}}\exp\bigl(C\|\varphi''\|_{M^{\infty,1}_{(v)}}\bigr),$$

for some constant C which is independent of d, $a \in M^\infty_{(\omega)}$ and $b \in M^1_{(1/\omega_0)}$.

Hence, by letting $K_{a,\varphi}$ be defined as (3.2) when $a \in M^\infty$, it follows that $a \mapsto K_{a,\varphi}$ on M^1 extends to a continuous map on M^∞. Furthermore, since $\mathscr{S}$ is dense in M^∞ with respect to the weak* topology, it follows that this extension is unique. The proof is complete. □

Finally we remark that the arguments above also give Theorem 3.1′ below, which concerns Fourier integral operators of the form

$$\mathrm{Op}_{t_1,t_2,\varphi}(a)f(x) \equiv \iint a(t_1x + t_2y, \xi)f(y)e^{i\varphi(t_1x+t_2y, -t_2x+t_1y, \xi)}\,dy\,d\xi.$$

It is then natural to assume that the conditions (3.1) and (2.3)′ are replaced by

$$t_1^2 + t_2^2 = 1, \quad |\det(\varphi''_{y,\xi}(X))| \geq \mathsf{d}, \tag{3.1$'$}$$

and

$$\begin{aligned}
&\omega_0(t_1x + t_2y, -t_2x + t_1y, t_1\xi + t_2\varphi'_y(X), -t_2\xi + t_1\varphi'_y(X)) \\
&\leq C\omega(x, \zeta, \xi - \varphi'_x(X), -\varphi'_\zeta(X)) \\
&\frac{\omega_2(x, \xi)}{\omega_1(y, -\eta)} \leq C\omega_0(x, y, \xi, \eta), \\
&\omega_0(x, y, \xi + t_2\eta_2, \eta_1 + t_1\eta_2) \leq \omega_0(x, y, \xi, \eta_1)v_1(\eta_2) \\
&\omega(x, \zeta, \xi_1 + \xi_2, z_1 + z_2) \leq \omega(x, \zeta, \xi_1, z_1)v_2(\xi_2, z_2), \\
&v(X, \xi, \eta, z) = v_1(\eta)v_2(\xi, z), \quad x, y, z, z_j, \xi, \xi_j, \eta, \zeta \in \mathbf{R}^n.
\end{aligned} \tag{2.3$''$}$$

Proposition 3.1′. *Assume that $p \in [1, \infty]$, $\mathsf{d} > 0$, $v \in \mathscr{P}(\mathbf{R}^{6n})$ is submultiplicative and satisfies $v(t\,\cdot) \leq Cv$ when $t \in [0, 1]$, $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{4n})$, $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2n})$ and that $\varphi \in C(\mathbf{R}^{3n})$ are such that φ is real-valued, $\varphi^{(\alpha)} \in M^{\infty,1}_{(v)}$ for all multi-indices α such that $|\alpha| = 2$, and (3.1)′ and (2.3)″ are fulfilled for some constants t_1, t_2 and C. Then the following is true:*

(1) *the map*

$$a \mapsto K_{a,\varphi}(x, y) \equiv \int a(t_1x + t_2y, \zeta)e^{i\varphi(t_1x+t_2y, -t_2x+t_1y, \zeta)}\, d\zeta,$$

from $\mathscr{S}(\mathbf{R}^{2n})$ to $\mathscr{S}'(\mathbf{R}^{2n})$ extends uniquely to a continuous map from $M^p_{(\omega)}(\mathbf{R}^{2n})$ to $M^p_{(\omega_0)}(\mathbf{R}^{2n})$;

(2) *the map $a \mapsto \mathrm{Op}_{t_1,t_2,\varphi}(a)$ from $\mathscr{S}(\mathbf{R}^{2n})$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$ extends uniquely to a continuous map from $M^p_{(\omega)}(\mathbf{R}^{2n})$ to $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n))$;*

(3) *if $a \in M^p_{(\omega)}(\mathbf{R}^{2n})$, then the definition of $\mathrm{Op}_{t_1,t_2,\varphi}(a)$ extends uniquely to a continuous operator from $M^{p'}_{(\omega_1)}(\mathbf{R}^n)$ to $M^p_{(\omega_2)}(\mathbf{R}^n)$. Furthermore, for some constant C it holds*

$$\|\mathrm{Op}_{t_1,t_2,\varphi}(a)\|_{M^{p'}_{(\omega_1)} \to M^p_{(\omega_2)}} \leq C\mathsf{d}^{-1} \exp\big(\|\varphi''\|_{M^{\infty,1}_{(v)}}\big)\|a\|_{M^p_{(\omega)}};$$

(4) *if $a \in M^{\infty,1}_{(\omega)}(\mathbf{R}^{2n})$ and $1 < p < \infty$, then the definition of $\mathrm{Op}_{t_1,t_2,\varphi}(a)$ from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$ extends uniquely to a continuous operator from $M^p_{(\omega_1)}$ to $M^p_{(\omega_2)}$;*

(5) *if $q \leq \min(p, p')$, $a \in M^{p,q}_{(\omega)}(\mathbf{R}^{2n})$, and in addition condition (0.5) holds, then $\mathrm{Op}_{t_1,t_2,\varphi}(a) \in \mathscr{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$.*

Proof. By letting

$$x_1 = t_1x + t_2y, \quad y_1 = -t_2x + t_1y$$

as new coordinates, it follows that we may assume that $t_1 = 1$ and $t_2 = 0$, and then the result agrees with Theorem 3.1. The proof is complete. □

3.2. Fourier integral operators with smooth amplitudes. Next we apply Theorem 2.10 to Fourier integral operators with smooth amplitudes. We recall that the condition on a in Theorem 2.10 means exactly that $a \in \mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ with $\mathsf{p} = (\infty, p, q, 1)$ and $\overline{V} = (V_2, V_1, V_1', V_2')$. In what follows we consider the case when $n_1 = n_2 = m = n$ and

$$\begin{aligned} V_1 &= V_1' = \{(x, 0, \zeta) \in \mathbf{R}^{3n};\ x, \zeta \in \mathbf{R}^n\} \quad \text{and} \\ V_2 &= V_2' = \{(0, y, 0) \in \mathbf{R}^{3n};\ y \in \mathbf{R}^n\}. \end{aligned} \tag{3.3}$$

However, the analysis presented here also holds without these restrictions. The details are left for the reader. We are especially concerned with spaces of amplitudes of the form

$$C^{N,p}_{(\omega)}(\mathbf{R}^{3n}) = \left\{a \in C^N(\mathbf{R}^{3n});\ \|a\|_{C^{N,p}_{(\omega)}} < \infty\right\},$$

where $N \geq$ is an integer, $\omega \in \mathscr{P}(\mathbf{R}^{3n})$ and

$$\|a\|_{C^{N,p}_{(\omega)}} \equiv \sum_{|\alpha| \leq N} \left(\iint_{\mathbf{R}^{2n}} \|a^{(\alpha)}(x, \cdot, \zeta)\omega(x, \cdot, \zeta)\|^p_{L^\infty} dx\, d\zeta \right)^{1/p}.$$

We also set

$$\begin{aligned} C^{\infty,p}_{(\omega)}(\mathbf{R}^{3n}) &= \bigcap_{N \geq 0} C^{N,p}_{(\omega)}(\mathbf{R}^{3n}), \\ S_{(\omega_0)}(\mathbf{R}^n) &= \{f \in C^\infty(\mathbf{R}^n);\ \omega_0 \partial^\alpha f \in L^\infty,\ \forall \alpha\}, \end{aligned}$$

when $\omega_0 \in \mathscr{P}(\mathbf{R}^n)$.

The following proposition links $C^{N,p}_{(\omega)}$ with $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$.

Proposition 3.2. *Assume that* (3.3) *is fulfilled for* $\overline{V} = (V_2, V_1, V_1', V_2)$, $N \geq 0$ *is an integer,* $p, q, r \in [1, \infty]$, $\mathsf{p} = (\infty, p, q, 1)$, $\mathsf{p}_1 = (\infty, p, 1, 1)$ *and that* $\mathsf{p}_2 = (\infty, p, \infty, r)$. *Also assume that* $\omega \in \mathscr{P}(\mathbf{R}^{3n})$, *and let*

$$\omega_s(X, \xi, \eta, z) = \omega(X)\langle \xi, \eta, z\rangle^s, \quad s \in \mathbf{R}.$$

If $s_1 < -2n/q'$ *when* $q > 1$ *and* $s_1 \leq 0$ *when* $q = 1$, *and* $s_2 > n(q+2)/q$, *then the*

following embedding holds:

$$\Theta^{\mathsf{p}_1}_{(\omega_N)} \hookrightarrow \Theta^{\mathsf{p}}_{(\omega_N)} \hookrightarrow \Theta^{\mathsf{p}_2}_{(\omega_N)}, \tag{3.4}$$

$$\Theta^{\mathsf{p}_2}_{(\omega_{N+s_2})} \hookrightarrow \Theta^{\mathsf{p}}_{(\omega_N)} \hookrightarrow \Theta^{\mathsf{p}_1}_{(\omega_{N+s_1})} \tag{3.5}$$

and

$$C^{N+3n+1,p}_{(\omega)} \hookrightarrow \Theta^{\mathsf{p}_1}_{(\omega_N)} \hookrightarrow C^{N,p}_{(\omega)}. \tag{3.6}$$

For the proof we consider the set $\mathscr{P}_0(\mathbf{R}^n)$ of all $\omega \in \mathscr{P}(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$ such that $\omega^{(\alpha)}/\omega$ is bounded for all multi-indices α. (Cf. [43, 45].)

Lemma 3.3. *Assume that* $\mathsf{p} = (p, q, r, s) \in [1, \infty]^4$, *and that* $N \geq 0$ *is an integer. Then the following is true*:
(1) *if* $\omega \in \mathscr{P}(\mathbf{R}^n)$, *then it exists an element* $\omega_0 \in \mathscr{P}_0(\mathbf{R}^n)$ *such that*

$$C^{-1}\omega_0 \leq \omega \leq C\omega_0, \tag{3.7}$$

for some constant C;
(2) *if* $\omega \in \mathscr{P}(\mathbf{R}^{2n})$, $\tilde{\omega}_j \in \mathscr{P}_0(\mathbf{R}^{2n})$ *for* $j = 1, 2$ *are such that* $\tilde{\omega}_1(x, \xi) = \tilde{\omega}_1(x)$ *and* $\tilde{\omega}_2(x, \xi) = \tilde{\omega}_2(\xi)$, *and that* $a_j \in S_{(1/\tilde{\omega}_j)}(\mathbf{R}^n)$, *then the mappings*

$$f \mapsto \tilde{\omega}_1 \cdot f, \quad \text{and} \quad f \mapsto \tilde{\omega}_2(D)f$$
$$(f \mapsto \tilde{a}_1 \cdot f, \quad \text{and} \quad f \mapsto \tilde{a}_2(D)f)$$

are homeomorphisms (continuous) from $\Theta^{\mathsf{p}}_{(\tilde{\omega}_1\omega)}(\overline{V})$ *and from* $\Theta^{\mathsf{p}}_{(\tilde{\omega}_2\omega)}(\overline{V})$ *respectively to* $\Theta^{\mathsf{p}}_{(\omega)}(\overline{V})$. *Furthermore, if*

$$\omega_{N_1,N_2}(x, \xi) = \omega(x, \xi)\langle x\rangle^{N_2}\langle \xi\rangle^{N_1},$$

then
(3.8)

$$\begin{aligned}\Theta^{\mathsf{p}}_{(\omega_{N_1,N_2})}(\overline{V}) &= \{f \in \mathscr{S}'(\mathbf{R}^n);\ x^\alpha \partial^\beta f \in \Theta^{\mathsf{p}}_{(\omega)}(\overline{V}),\ |\alpha| \leq N_2,\ |\beta| \leq N_1\} \\ &= \{f \in \mathscr{S}'(\mathbf{R}^n);\ f, x_j^{N_2} f, D_k^{N_1} f, x_j^{N_2} D_k^{N_1} f \in \Theta^{\mathsf{p}}_{(\omega)}(\overline{V}),\ 1 \leq j,\ k \leq n\};\end{aligned}$$

(3) *if* $\omega \in \mathscr{P}(\mathbf{R}^{6n})$ *and* $\omega_0 \in \mathscr{P}_0(\mathbf{R}^{6n})$ *are such that* $\omega(X, \xi, \eta, z) = \omega(X)$ *and* $\omega_0(X, \xi, \eta, z) = \omega_0(X)$, *then the map* $a \mapsto \omega_0 \cdot a$ *is a bijection from* $C^{N,p}_{(\omega_0\omega)}(\mathbf{R}^{3n})$ *to* $C^{N,p}_{(\omega)}(\mathbf{R}^{3n})$.

Proof. The assertion (1) follows from Lemma 1.2 in [44], and (3) is a straightforward consequence of the definitions. The continuity assertions on $\tilde{\omega}_j$ in (2) follows

from Theorem 2.2 in [43] when $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ is a modulation space. The general case follows by similar arguments as in the proof of that theorem. We omit the details. (Cf. Remark 2.8 in [43].)

Next we prove the continuity for the map $f \mapsto a_1 \cdot f$. Let

$$\omega_{a_1} = C\tilde{\omega}_1 + a_1,$$

where the constant C is chosen such that $|a_1| \le C\tilde{\omega}_1/2$. Then it follows from the definitions that

$$C_1^{-1}\tilde{\omega}_1 \le \omega_{a_1} \le C_1\tilde{\omega}_1,$$

for some constant C_1. This proves that $\omega_{a_1} \in \mathscr{P}_0(\mathbf{R}^n)$, and the first part of (2) now shows that the mappings

$$f \mapsto \tilde{\omega}_1 \cdot f \quad \text{and} \quad f \mapsto a_1 \cdot f$$

are continuous from $\mathbf{\Theta}^{\mathsf{p}}_{(\tilde{\omega}_1\omega)}(\overline{V})$ to $\mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$. Since $f \mapsto a_1 \cdot f$ is a linear combination of these mappings, the result follows.

The continuity assertions for the map $f \mapsto a_2(D)f$ follows by similar arguments. The details are left for the reader.

It remains to prove (3.8). It is convenient to set

$$\sigma_{N_1,N_2}(x,\xi) = \langle x\rangle^{N_2}\langle\xi\rangle^{N_1}.$$

Furthermore, let M_0 be the set of all $f \in \mathbf{\Theta}^{\mathsf{p}}_{(\omega)}$ such that $x^\beta\partial^\alpha f \in \mathbf{\Theta}^{\mathsf{p}}_{(\omega)}$ when $|\alpha| \le N_1$ and $|\beta| \le N_2$, and let $\tilde{M}_0$ be the set of all $\in \mathbf{\Theta}^{\mathsf{p}}_{(\omega)}$ such that $x_j^{N_2}\partial_k^{N_1} f \in \mathbf{\Theta}^{\mathsf{p}}_{(\omega)}$ for $j,k = 1,\dots,N$. We shall prove that $M_0 = \tilde{M}_0 = \mathbf{\Theta}^{\mathsf{p}}_{(\sigma_{N_1,N_2}\omega)}$. Obviously, $M_0 \subseteq \tilde{M}_0$. By the first part of (2) it follows that $\mathbf{\Theta}^{\mathsf{p}}_{(\sigma_{N_1,N_2}\omega)} \subseteq M_0$. The result therefore follows if we prove that $\tilde{M}_0 \subseteq \mathbf{\Theta}^{\mathsf{p}}_{(\sigma_{N_1,N_2}\omega)}$.

In order to prove this, assume first that $N_1 = N$, $N_2 = 0$, $f \in \tilde{M}_0$, and choose open sets

$$\Omega_0 = \{\xi \in \mathbf{R}^n;\ |\xi| < 2\}$$

and

$$\Omega_j = \{\xi \in \mathbf{R}^n;\ 1 < |\xi| < n|\xi_j|\}.$$

Then $\bigcup_{j=0}^n \Omega_j = \mathbf{R}^n$, and there are non-negative functions $\varphi_0,\dots,\varphi_n$ in S_0^0 such that $\operatorname{supp}\varphi_j \subseteq \Omega_j$ and $\sum_{j=0}^n \varphi_j = 1$. In particular, $f = \sum_{j=0}^n f_j$ when $f_j = \varphi_j(D)f$. The result follows if we prove that $f_j \in \mathbf{\Theta}^{\mathsf{p}}_{(\sigma_{N,0}\omega)}$ for every j.

Now set $\psi_0(\xi) = \sigma_N(\xi)\varphi_0(\xi)$ and $\psi_j(\xi) = \xi_j^{-N}\sigma_N(\xi)\varphi_j(\xi)$ when $j = 1, \dots, n$. Then $\psi_j \in S_0^0$ for every j. Hence the first part of (2) gives

$$\|f_j\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\sigma_N,0^\omega)}} \le C_1\|\sigma_N(D)f_j\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}} = C_1\|\psi_j(D)\partial_j^N f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}} \le C_2\|\partial_j^N f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}} < \infty$$

and

$$\|f_0\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\sigma_N,0^\omega)}} \le C_1\|\sigma_N(D)f_0\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}} = C_1\|\psi_0(D)f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}} \le C_2\|f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}} < \infty,$$

for some constants C_1 and C_2. This proves that

$$\|f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\sigma_N,0^\omega)}} \le C\left(\|f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}} + \sum_{j=1}^{N}\|\partial_j^N f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}}\right), \tag{3.9}$$

for some constant C, and the result follows in this case.

If we instead split up f into $\sum \varphi_j f$, then similar arguments show that

$$\|f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\sigma_{0,N},\omega)}} \le C\left(\|f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}} + \sum_{k=1}^{N}\|x_k^N f\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega)}}\right), \tag{3.10}$$

and the result follows in the case $N_1 = 0$ and $N_2 = N$ from this estimate.

The general case now follows if combine (3.9) with (3.10), which proves (2). The proof is complete. □

Proof of Proposition 3.2. The first embeddings in (3.4) follows immediately from Proposition 1.9. Next we prove (3.5). Let $\varepsilon > 0$ be chosen such that $s_2 - 2\varepsilon > n(q+1)/q$, $\mathbf{E}_{a,\omega_N}$ be as in Section 2, and set

$$F_{a,\omega_N}(\xi, \eta, z) = \left(\iint_{\mathbf{R}^{2n}} \sup_{y\in\mathbf{R}^n} \mathbf{E}_{a,\omega_N}(X, \xi, \eta, z)^p\, dx\, d\zeta\right)^{1/p}.$$

Then Hölder's inequality gives

$$\begin{aligned}
\|a\|_{\boldsymbol{\Theta}^{\mathrm{p}}_{(\omega_N)}} &= \int_{\mathbf{R}^n}\left(\iint_{\mathbf{R}^{2n}} F_{a,\omega_N}(\xi, \eta, z)^q\, d\xi\, dz\right)^{1/q} d\eta \\
&= \int_{\mathbf{R}^n}\left(\iint_{\mathbf{R}^{2n}} F_{a,\omega_{N+s_2}}(\xi, \eta, z)^q \langle \xi, \eta, z\rangle^{-s_2 q}\, d\xi\, dz\right)^{1/q} d\eta \\
&\le \int_{\mathbf{R}^n}\left(\iint_{\mathbf{R}^{2n}} F_{a,\omega_{N+s_2}}(\xi, \eta, z)^q \langle \xi, z\rangle^{-(2n+\varepsilon)}\, d\xi\, dz\right)^{1/q} \langle \eta\rangle^{-(n+\varepsilon)}\, d\eta
\end{aligned}$$

$$\leq C\|F_{a,\omega_{N+s_2}}\|_{L^\infty} = C\|a\|_{\Theta^{p_2}_{(\omega_{N+s_2})}},$$

where

$$C = \left(\iint_{\mathbf{R}^{2n}} \langle \xi, z\rangle^{-(2n+\varepsilon)}\, d\xi\, dz\right)^{1/q} \int_{\mathbf{R}^n} \langle \eta\rangle^{-(n+\varepsilon)}\, d\eta < \infty.$$

This proves the first inclusion in (3.5). The second inclusion follows by similar arguments. The details are left for the reader.

Next we prove (3.6). By Lemma 3.3 it follows that we may assume that $\omega = 1$ and $N = 0$. By Remark 1.3 (2) we have

$$\Theta^{p_1} \subseteq M^{\infty,1} \subseteq C \cap L^\infty.$$

Furthermore, if $\chi \in \mathscr{S}(\mathbf{R}^{3n})$ is such that $\chi(0) = (2\pi)^{-3n/2}$, then it follows by Fourier's inversion formula that

$$a(X) = \iiint_{\mathbf{R}^{3n}} V_\chi a(X, \xi, \eta, z) e^{i(\langle x,\xi\rangle + \langle y,\eta\rangle + \langle \zeta, z\rangle)}\, d\xi\, d\eta\, dz.$$

Hence Minkowski's inequality gives

$$\begin{aligned}
\|a\|_{C^{0,p}} &= \left(\iint_{\mathbf{R}^{2n}} \left(\sup_{y\in\mathbf{R}^n} |a(X)|\right)^p dx\, d\zeta\right)^{1/p} \\
&\leq \left(\iint_{\mathbf{R}^{2n}} \sup_{y\in\mathbf{R}^n} \left(\iiint_{\mathbf{R}^{3n}} |V_\chi a(X, \xi, \eta, z)|\, d\xi\, d\eta\, dz\right)^p dx\, d\zeta\right)^{1/p} \\
&\leq \iiint_{\mathbf{R}^{3n}} \left(\iint_{\mathbf{R}^{2n}} \sup_{y\in\mathbf{R}^n} |V_\chi a(X, \xi, \eta, z)|^p dx\, d\zeta\right)^{1/p} d\xi\, d\eta\, dz = \|a\|_{\Theta^{p_1}}.
\end{aligned}$$

This proves the right embedding in (3.6).

In order to prove the left embedding in (3.6) we observe that

$$|V_\chi a(X, \xi, \eta, z)| \leq (2\pi)^{-3n/2} \int |\chi(X_1 - X) a(X_1)|\, dX_1 = (2\pi)^{-3n/2}(|a| * |\check{\chi}|)(X),$$

which together with Young's inequality give

$$\begin{aligned}
\|a\|_{\Theta^{p_2}} &= \sup_{\xi,\eta,z} \left(\iint_{\mathbf{R}^{2n}} \sup_{y\in\mathbf{R}^n} |V_\chi a(X, \xi, \eta, z)|^p dx\, d\zeta\right)^{1/p} \\
&\leq C\left(\iint_{\mathbf{R}^{2n}} \sup_{y\in\mathbf{R}^n} (|a| * |\check{\chi}|)(X)^p dx\, d\zeta\right)^{1/p} \\
&\leq C\|\chi\|_{L^1} \left(\iint_{\mathbf{R}^{2n}} \sup_{y\in\mathbf{R}^n} |a(X)|^p dx\, d\zeta\right)^{1/p} = C\|\chi\|_{L^1}\|a\|_{C^{0,p}},
\end{aligned}$$

for some constant C. Hence if $\omega(X, \xi, \eta, z) = \langle \xi, \eta, z\rangle^{-3n-1}$, then it follows from Lemma 3.3 that

$$\begin{aligned}\|a\|_{\Theta^{\mathsf{p}_1}} &\le C_1\|\omega^{-1}(D)a\|_{\Theta^{\mathsf{p}_1}_{(\omega)}} \le C_2 \sum_{|\alpha|\le 3n+1} \|a^{(\alpha)}\|_{\Theta^{\mathsf{p}_1}_{(\omega)}} \\ &\le C_2 \sum_{|\alpha|\le 3n+1} \|a^{(\alpha)}\|_{\Theta^{\mathsf{p}_2}} \le C_3 \sum_{|\alpha|\le 3n+1} \|a^{(\alpha)}\|_{C^{0,p}} = C_3\|a\|_{C^{N,p}},\end{aligned}$$

for some constants $C_1, \dots, C_3$. This proves (3.6) and the result follows. □

Corollary 3.4. *Let N, ω_s and p be as in* Proposition 3.2. *Then*

$$C^{\infty,p}_{(\omega)} = \bigcap_{N\ge 0} \Theta^{\mathsf{p}}_{(\omega_N)}.$$

REMARK 3.5. Similar properties with similar motivations as those in Proposition 3.2, Lemma 3.3 and Corollary 3.4, and their proofs, also holds when the $\Theta^{\mathsf{p}}_{(\omega_N)}$ spaces and $C^{N,p}_{(\omega)}$ spaces are replaced by the modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^n)$ for $\omega \in \mathscr{P}(\mathbf{R}^{2n})$ and

$$\{f \in \mathscr{S}'(\mathbf{R}^n);\ f^{(\alpha)} \in M^{p,q}_{(\omega)}(\mathbf{R}^n),\ |\alpha| \le N\}$$

respectively. (Cf. [44].)

Now we may combine Proposition 3.2 with the results in Section 2 to obtain continuity properties for certain type of Fourier integral operator when acting on modulation spaces. For example, the following result is a consequence of Theorem 2.10 and Proposition 3.2.

Theorem 3.6. *Assume that $n_1 = n_2 = m = n$, $\omega \in \mathscr{P}(\mathbf{R}^{6n})$ and $\tilde\omega \in \mathscr{P}(\mathbf{R}^{3n})$ satisfy*

$$\omega(X, \xi, \eta, z) = \tilde\omega(X)\langle \xi, \eta, z\rangle^N$$

for some constant N, and that χ, ω_j, v and φ for $j = 0, 1, 2$ are the same as in Subsection 2.1. *Also assume that $p \in [1, \infty]$, $a \in C^{\infty,p}_{(\tilde\omega)}(\mathbf{R}^{3n})$, and that $|\det(\varphi''_{y,\zeta})| \ge \mathsf{d}$ and* (0.5) *hold for some $\mathsf{d} > 0$. Then the following is true*:
(1) (i)–(ii) *in* Subsection 2.2 *holds*;
(2) $\operatorname{Op}_\varphi(a) \in \mathscr{I}_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$.

3.3. Some consequences in the theory of pseudo-differential operators. The results in Secton 2 also allow us to extend some properties in [41, 43] for pseudo-differential operators of the form (0.3). In this case we have that $\varphi(x, y, \zeta) = \langle x - y, \zeta\rangle$,

where $x, y, \zeta \in \mathbf{R}^n$, and the conditions in (2.3) imply that

$$\frac{\omega_2(x, \xi + \zeta)}{\omega_1(y, \eta + \zeta)} \leq C\omega(x, y, \zeta, \xi, -\eta, y - x) \tag{3.11}$$

for some constant C which is independent of $x, y, \xi, \eta, \zeta \in \mathbf{R}^n$. Hence the following result is an immediate consequence of Theorem 2.10.

Proposition 3.7. *Assume that $\omega_j \in \mathscr{P}(\mathbf{R}^{2n})$ and $\omega \in \mathscr{P}(\mathbf{R}^{6n})$ satisfy* (3.11), *V_j and V_j' are the same as in* (3.3) *for $j = 1, 2$, and assume that $a \in \mathbf{\Theta}^{\mathsf{p}}_{(\omega)}(\overline{V})$ for some $\mathsf{p} = (\infty, p, q, 1)$ with $p, q \in [1, \infty]$. Then* (1) *and* (2) *in* Theorem 2.10 *hold for $\varphi(x, y, \zeta) = \langle x - y, \zeta \rangle$.*

We may now prove the following result.

Proposition 3.8. *Assume that $\omega_j \in \mathscr{P}(\mathbf{R}^{2n})$ for $j = 1, 2$ and $\omega \in \mathscr{P}(\mathbf{R}^{3n})$ satisfy*

$$\frac{\omega_2(x, \xi)}{\omega_1(x, \xi)} \leq C\omega(x, x, \xi), \quad x, \xi \in \mathbf{R}^n, \tag{3.12}$$

and that $a \in C^{p,\infty}_{(\omega)}(\mathbf{R}^{3n})$ for $p \in [1, \infty]$. Then (1) *and* (2) *in* Theorem 2.10 *hold for $\varphi(x, y, \zeta) = \langle x - y, \zeta \rangle$.*

Proof. Since ω_j and ω are moderated by $\langle \cdot \rangle^N$ for some $N \geq 0$, it follows from (3.12) that

$$\frac{\omega_2(x, \xi + \zeta)}{\omega_1(y, \eta + \zeta)} \leq C\omega_{N_0}(x, y, \zeta, \xi, -\eta, y - x),$$

for some constants C and N_0, where

$$\omega_{N_0}(x, y, \zeta, \xi, \eta, z) = \omega(x, y, \zeta)\langle \xi, \eta, z \rangle^{N_0}.$$

Hence (3.11) is fulfilled after replacing ω by ω_{N_0}. The result follows now by combining Proposition 3.2 with Proposition 3.7. The proof is complete. □

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