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RESEARCH ARTICLE

L^p-microlocal regularity for pseudodifferential operators of quasi-homogeneous type

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The authors consider pseudodifferential operators whose symbols have decay at **infinity** of quasi-homogeneous type and study their behavior on the wave front set of distributions in weighted Zygmund-Hölder spaces and weighted Sobolev spaces in L^p framework. Then microlocal properties for solutions to linear partial differential equations with coefficients in weighted Zygmund-Hölder spaces are obtained.

Keywords: non regular pseudodifferential operators, microlocal properties, weighted Zygmund-Hölder and Sobolev spaces.

AMS Subject Classification: 35S05, 35A17.

1. Introduction

Pseudodifferential operators whose smooth symbols have a quasi-homogeneous decay at infinity were firstly introduced in 1977 in Lascar [4], where their microlocal properties in the L^2-framework were also studied.
Symbol classes of quasi-homogeneous type and several related problems have been developed in the mean time, see e.g. Segàla [5] for the local solvability, Garello [1] for symbols with decay of type $(1, 1)$, Yamazaki [10] where non-smooth symbols in the L^p-framework are introduced and studied under suitable restrictive conditions on the Fourier transform of the symbols themselves.
In [2], [3] the authors prove the L^p-boundedness of a class of pseudodifferential operators with non-smooth symbols of quasi-homogeneous type, taking their values in Zygmund-Hölder spaces with respect to the first variable. Precisely, in [3] continuity for pseudodifferential operators of $(1, \delta)$ quasi-homogeneous type, for $0 \le \delta \le 1$, is considered and applied to obtain local regularity results.
In the present paper, the arguments in [2], [3] are suitably adapted to studying microlocal Sobolev and Zygmund-Hölder regularity of pseudodifferential operators with both smooth and non-smooth symbols of quasi-homogeneous type.
In §2, quasi-homogeneous weight functions are introduced together with their main properties. Moreover quasi-homogeneous smooth symbols of type $(1, \delta)$ are considered and their symbolic calculus is developed.
Quasi-homogeneous function spaces of Sobolev type, $H^{s,p}_M$, and Zygmund-Hölder

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type, $B^{s,M}_{\infty,\infty}$, are considered in §3. Here symbols $a(x,\xi)$ belonging to $B^{r,M}_{\infty,\infty}$, with respect to the x variable, are introduced and their boundedness in $H^{s,p}_M$ and $B^{s,M}_{\infty,\infty}$ is studied.
A microlocal version of quasi-homogeneous smooth symbols is considered in §4, where the property of ellipticity is stated in microlocal terms and a related microlocal parametrix is constructed.
The quasi-homogeneous $H^{s,p}_M$ and $B^{s,M}_{\infty,\infty}$ wave front sets of distributions are defined in §5. Here, the so-called microlocal properties of pseudodifferential operators in $\mathrm{Op}S^m_{M,\delta}$ are proved. Then these properties are applied to obtain microregularity results for operators with smooth symbols, in §6.
Finally, in §7 the previous results are used for studying the microlocal regularity of solutions to linear PDEs with quasi-homogeneous Zygmund-Hölder coefficients and pseudodifferential equations with symbols in $B^{s,M}_{\infty,\infty}$, with respect to x.

2. Quasi-homogeneous pseudodifferential operators

In the following $M=(m_1,\dots,m_n)$ will be a vector with positive integer components such that $\min\limits_{1\le j\le n} m_j = 1$. The *quasi-homogeneous weight function* in $\mathbb{R}^n$ is defined by

$$|\xi|_M := \left(\sum_{j=1}^n \xi_j^{2m_j}\right)^{\frac12}, \quad \xi\in\mathbb{R}^n\,. \tag{1}$$

We set $1/M := (1/m_1,\dots,1/m_n)$, $\alpha\cdot 1/M := \sum\limits_{j=1}^n \alpha_j/m_j$, $m^* := \max\limits_{1\le j\le n} m_j$ and $\langle\xi\rangle_M^2 := 1+|\xi|_M^2$. Clearly the usual euclidean norm $|\xi|$ corresponds to the quasi-homogeneous weight function (1) in the case $M=(1,\dots,1)$.
The following properties can be easily proved.

Proposition 2.1: *for any vector $M=(m_1,\dots,m_n)\in\mathbb{N}^n$ satisfying the previous assumptions, there exists a suitable positive constant C such that*

i) $\frac{1}{C}(1+|\xi|)\le\langle\xi\rangle_M\le C(1+|\xi|)^{m^*}, \quad \forall\,\xi\in\mathbb{R}^n\,;$
ii) $|\xi+\eta|_M\le C(|\xi|_M+|\eta|_M), \quad \forall\,\xi,\eta\in\mathbb{R}^n\,;$
iii) (quasi-homogeneity) $|t^{1/M}\xi|_M = t|\xi|_M\,, \quad \forall\,t>0\,,\forall\,\xi\in\mathbb{R}^n\,,$ *where* $t^{1/M}\xi=(t^{1/m_1}\xi_1,\dots,t^{1/m_n}\xi_n)$;
iv) $\xi^\gamma\partial^{\alpha+\gamma}|\xi|_M\le C_{\alpha,\gamma}\langle\xi\rangle_M^{1-\alpha\cdot1/M}, \ \forall\,\alpha,\gamma\in\mathbb{Z}^n_+\,,\forall\,\xi\ne0.$

The detailed proof of Proposition 2.1 can be found in [2] and the references given there.

Definition 2.2: given $m\in\mathbb{R}$ and $\delta\in[0,1]$, $S^m_{M,\delta}$ will be the class of functions $a(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ such that for all multi-indices $\alpha,\beta\in\mathbb{Z}^n_+$ there exists $C_{\alpha,\beta}>0$ such that:

$$|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)|\le C_{\alpha,\beta}\langle\xi\rangle_M^{m-\alpha\cdot1/M+\delta\beta\cdot1/M}, \quad \forall x,\,\xi\in\mathbb{R}^n\,. \tag{2}$$

Notice that, in principle, the previous definition could be stated also for symbols displaying a (ρ,δ)-type decay at infinity. Actually, for every $\rho,\delta\in[0,1]$, $\delta\le\rho$, the quasi-homogeneous class $S^m_{M,\rho,\delta}$ could be defined, in natural way, to be the class of all the smooth functions $a(x,\xi)$, whose

derivatives obey the estimates:

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle_M^{m-\rho\alpha\cdot 1/M+\delta\beta\cdot 1/M}, \quad \forall x,\, \xi \in \mathbb{R}^n\,,$$

for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ and suitable positive constants $C_{\alpha,\beta}$.
We here only deal with the case $\rho = 1$ for the main reason that symbols in the classes $S^m_{M,\delta} := S^m_{M,1,\delta}$ plainly satisfy the Lizorkin-Marcinkiewicz Theorem for L^p-Fourier multipliers [6, Ch. IV, §6], which allows to develop the L^p-theory of the pseudodiffererential operators for $1 < p < \infty$, [3].
The estimates in Proposition 2.1.i yield the inclusion

$$S^m_{M,\delta} \subset S^{\max\{mm^*,m\}}_{1/m^*,\delta m^*} \tag{3}$$

which establishes a suitable relation between the *quasi-homogeneous classes* $S^m_{M,\delta}$ and the Hörmander symbol classes $S^m_{\rho,\delta}$.
Henceforth, we set $S^m_M := S^m_{M,0}$, $S^\infty_{M,\delta} := \bigcup_{m\in\mathbb{R}} S^m_{M,\delta}$, $S^{-\infty} := \bigcap_{m\in\mathbb{R}} S^m_{1,0}$.
Again Proposition 2.1.i yields that for all vectors $M \in \mathbb{N}^n$ and $\delta \in [0,1]$ there holds $S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m_{M,\delta}$.
The derivatives and pointwise product of symbols in $S^m_{M,\delta}$ enjoy all the expected rules: they are summarized by the next Propositions 2.3 and 2.4, see [3] for the proofs.

Proposition 2.3: *for $m, m' \in \mathbb{R}$, $\delta, \delta' \in [0,1]$, $\mu, \nu \in \mathbb{Z}_+^n$ we have*

$$\sigma \in S^m_{M,\delta} \;\Rightarrow\; \partial_\xi^\mu \partial_x^\nu \sigma \in S^{m-\mu\cdot 1/M+\delta\,\nu\cdot 1/M}_{M,\delta}; \tag{4}$$

$$\sigma \in S^m_{M,\delta}\,,\;\; \tau \in S^{m'}_{M,\delta'} \;\Rightarrow\; \sigma\,\tau \in S^{m+m'}_{M,\max\{\delta,\delta'\}}. \tag{5}$$

Proposition 2.4: *let $\{m_j\}_{j\ge 0}$ be a real decreasing sequence with $\lim\limits_{j\to+\infty} m_j = -\infty$. For every $j \ge 0$, let σ_j be a symbol in $S^{m_j}_{M,\delta}$. Then there exists a symbol $\sigma \in S^{m_0}_{M,\delta}$ such that*

$$\sigma - \sum_{j<N} \sigma_j \in S^{m_N}_{M,\delta}.$$

We write $\sigma \sim \sum\limits_j \sigma_j$ and we call $\{\sigma_j\}_{j\ge 0}$ asymptotic expansion of σ.
Moreover, if τ is any other symbol for which the sequence $\{\sigma_j\}_{j\ge0}$ is an asymptotic expansion, then $\sigma - \tau \in S^{-\infty}$.

For each symbol $a \in S^m_{M,\delta}$ ($m \in \mathbb{R}$, $\delta \in [0,1]$), the pseudodifferential operator $a(x,D) = \mathrm{Op}(a)$ is defined on $\mathcal{S}(\mathbb{R}^n)$ by the usual quantization

$$a(x,D)u = (2\pi)^{-n} \int e^{i\,x\cdot\xi} a(x,\xi)\hat{u}(\xi)\,d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{6}$$

where $x\cdot\xi := \sum_{j=1}^n x_j\,\xi_j$ and $\hat{u} = \mathcal{F}u$ is the Fourier transform of u. It is well-known that (6) defines a linear bounded operator from $\mathcal{S}(\mathbb{R}^n)$ to itself. In the following, we will denote by $\mathrm{Op}\, S^m_{M,\delta}$ the set of pseudodifferential operators with symbol in the class $S^m_{M,\delta}$.

For the adjoint and the product of pseudodifferential operators in Op $S^m_{M,\delta}$, a suitable symbolic calculus is developed in [3] with some restrictions on δ; we quote here the result

Proposition 2.5: symbolic calculus.

1. *If* $a(x, D) \in \mathrm{Op}S^m_{M,\delta}$ *with* $0 \le \delta < 1/m^*$, *the adjoint operator* $a(x, D)^*$ *still belongs to* $\mathrm{Op}S^m_{M,\delta}$ *and its symbol* $a^*(x, \xi)$ *satisfies for any integer* $k > 0$

$$a^*(x,\xi) - \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi \partial^\alpha_x \overline{a(x,\xi)} \in S^{m-(1/m^*-\delta)k}_{M,\delta};$$

according to Proposition 2.4, we write: $a^* \sim \sum_{\alpha \ge 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi \partial^\alpha_x \overline{a}$.

2. *If* $a(x, D) \in \mathrm{Op}S^{m_1}_{M,\delta_1}$, $b(x, D) \in \mathrm{Op}S^{m_2}_{M,\delta_2}$ *with* $0 \le \delta_2 < 1/m^*$, *then*

$$a(x, D)b(x, D) \in \mathrm{Op}S^{m_1+m_2}_{M,\delta},$$

with $\delta := \max\{\delta_1, \delta_2\}$, *and the symbol* $a\sharp b$ *of the product satisfies for any integer* $k > 0$

$$a\sharp b(x,\xi) - \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi a(x,\xi) \partial^\alpha_x b(x,\xi) \in S^{m_1+m_2-(1/m^*-\delta_2)k}_{M,\delta}; \qquad (7)$$

we write: $a\sharp b \sim \sum_{\alpha \ge 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi a \partial^\alpha_x b$.

3. Function spaces

Definition 3.1: for $s \in \mathbb{R}$ and $p \in]1, +\infty[$, we say that a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the *quasi-homogeneous Sobolev space* $H^{s,p}_M$, if

$$\langle D \rangle^s_M u := \mathcal{F}^{-1}(\langle \cdot \rangle^s_M \hat{u}) \in L^p(\mathbb{R}^n) \,. \qquad (8)$$

$H^{s,p}_M$ becomes a Banach space, when provided with the norm

$$||u||_{H^{s,p}_M} := ||\langle D \rangle^s_M u||_{L^p} \,. \qquad (9)$$

The case $p = 1$ is not considered since, in order to prove the continuity result in Theorem 3.3, $H^s_{M,p}$ must be stated in terms of dyadic decompositions, which characterize $H^{s,p}_M$ only for $1 < p < \infty$, see [9, §2.3.5], [3].

The following continuous embeddings can be easily established

$$\mathcal{S}(\mathbb{R}^n) \subset H^{s,p}_M \subset H^{r,p}_M \,, \qquad (10)$$

whenever $r < s$ and $p \in]1, +\infty[$; $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $H^{s,p}_M$, for all $s \in \mathbb{R}$ and $p \in]1, +\infty[$.

In the sequel, we will also use a quasi-homogeneous version of Zygmund-Hölder's classes, which can be introduced by means of a quasi-homogeneous partition of

unity.
For a given $K > 1$, let ϕ be a function in $C^\infty([0,+\infty))$ such that $0 \le \phi(t) \le 1$, $\phi(t) = 1$ for $0 \le t \le \frac{1}{2K}$, $\phi(t) = 0$, when $t > K$. Set

$$\varphi_{-1}(\xi) = \phi\left(|\xi|_M\right), \quad \varphi_h(\xi) = \phi\left(\frac{|\xi|_M}{2^{h+1}}\right) - \phi\left(\frac{|\xi|_M}{2^h}\right), \ h = 0, 1, \dots . \tag{11}$$

The sequence $\Phi := \{\varphi_h\}_{h=-1}^\infty$, defined in (11), is a *quasi-homogeneous partition of unity.* We set $u_h = \varphi_h(D)u$ for $u \in \mathcal{S}'(\mathbb{R}^n)$ and $h = -1, 0, \dots$.
For any $s \in \mathbb{R}$, the *quasi-homogeneous Zygmund-Hölder class* $B^{s,M}_{\infty,\infty}$ is defined to be the set of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|^\Phi_{B^{s,M}_{\infty,\infty}} := \sup_{h=-1,\dots} 2^{sh}\|u_h\|_{L^\infty} < \infty \tag{12}$$

is satisfied for some quasi-homogeneous partition of unity Φ. Moreover (12) is a norm in $B^{s,M}_{\infty,\infty}$, and turns it into a Banach space. Different choices of Φ lead to equivalent norms of $B^{s,M}_{\infty,\infty}$.
The analysis of linear partial differential equations with rough coefficients, performed in §7, needs the use of certain classes of non-smooth symbols studied in [3]. We recall here the main definition.

Definition 3.2: for $r > 0$, $m \in \mathbb{R}$ and $\delta \in [0,1]$, $B^{r,M}_{\infty,\infty}S^m_{M,\delta}$ is defined to be the set of measurable functions $a(x,\xi)$ such that for every $\alpha \in \mathbb{Z}^n_+$, the following inequalities hold true

$$|\partial^\alpha_\xi a(x,\xi)| \le C_\alpha \langle\xi\rangle_M^{m-\alpha\cdot 1/M}, \quad \forall x,\, \xi \in \mathbb{R}^n; \tag{13}$$

$$\|\partial^\alpha_\xi a(\cdot,\xi)\|_{B^{r,M}_{\infty,\infty}} \le C_\alpha \langle\xi\rangle_M^{m-\alpha\cdot 1/M+\delta r}, \quad \forall \xi \in \mathbb{R}^n. \tag{14}$$

As in the case of smooth symbols, we set for brevity $B^{r,M}_{\infty,\infty}S^m_M := B^{r,M}_{\infty,\infty}S^m_{M,0}$. In [3], the following continuity result is proved.

Theorem 3.3: *if $r > 0$, $m \in \mathbb{R}$, $\delta \in [0,1]$ and $a(x,\xi) \in B^{r,M}_{\infty,\infty}S^m_{M,\delta}$, then for all $s \in](\delta-1)r, r[$ and $p \in]1,+\infty[$ the following*

$$a(x,D) : H^{s+m,p}_M \to H^{s,p}_M \tag{15}$$

$$a(x,D) : B^{s+m,M}_{\infty,\infty} \to B^{s,M}_{\infty,\infty} \tag{16}$$

are linear continuous operators.
If in addition $\delta < 1$, then the mapping property (16) *is still true for $s = r$.*

Since the inclusion $S^m_{M,\delta} \subset B^{r,M}_{\infty,\infty}S^m_{M,\delta}$ is true for all $r > 0$, a straightforward consequence of Theorem 3.3 is the following

Corollary 3.4: *if $a \in S^m_{M,\delta}$, for $m \in \mathbb{R}$ and $\delta \in [0,1[$, then* (15), (16) *are true for all $s \in \mathbb{R}$. If $\delta = 1$,* (15) *and* (16) *are true for all $s > 0$.*

Recall that pseudodifferential operators with symbol $a \in S^{-\infty}$ are *regularizing* operators, namely they can be extended to linear bounded operators from $\mathcal{S}'(\mathbb{R}^n)$ into the space of *polynomially bounded* C^∞ functions in $\mathbb{R}^n$, with polynomially bounded derivatives, and from the space $\mathcal{E}'(\mathbb{R}^n)$ of the compactly supported distributions in $\mathbb{R}^n$ into the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

4. Microlocal properties

In the sequel, we will set $T^\circ\mathbb{R}^n := \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Let $M = (m_1, \dots, m_n) \in \mathbb{N}^n$ be a vector satisfying the assumptions of §2.

Definition 4.1: we say that a set $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ is $M-$conic, if

$$\xi \in \Gamma_M \quad \Rightarrow \quad t^{1/M}\xi \in \Gamma_M\ ,\ \forall\, t > 0\,.$$

Definition 4.2: let U be an open subset of $\mathbb{R}^n$ and $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ an open $M-$conic set. For $m \in \mathbb{R}$ and $\delta \in [0,1]$, we say that $a \in S'(\mathbb{R}^{2n})$ belongs microlocally to $S^m_{M,\delta}$ on $U \times \Gamma_M$ if $a_{|\,U\times\Gamma_M} \in C^\infty(U \times \Gamma_M)$ and for all $\alpha, \beta \in \mathbb{Z}^n_+$ there exists $C_{\alpha,\beta} > 0$ such that:

$$|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)| \le C_{\alpha,\beta} \langle\xi\rangle_M^{m-\alpha\cdot 1/M+\delta\,\beta\cdot 1/M}\,, \quad \forall\,(x,\xi) \in U \times \Gamma_M\,; \tag{17}$$

we will write in this case $a \in mclS^m_{M,\delta}(U \times \Gamma_M)$.
For $(x_0, \xi_0) \in T^\circ\mathbb{R}^n$, we set

$$mclS^m_{M,\delta}(x_0,\xi_0) := \bigcup_{U,\,\Gamma_M} mclS^m_{M,\delta}(U \times \Gamma_M)\,, \tag{18}$$

where the union in the right-hand side is taken over all of the open neighborhoods $U \subset \mathbb{R}^n$ of x_0 and the open $M-$conic neighborhoods $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ of ξ_0.

With the above stated notations, we say that $a \in \mathcal{S}'(\mathbb{R}^n)$ is *microlocally regularizing* on $U \times \Gamma_M$ if $a_{|\,U\times\Gamma_M} \in C^\infty(U \times \Gamma_M)$ and for every $m > 0$ and all $\alpha, \beta \in \mathbb{Z}^n_+$ a positive constant $C_{m,\alpha,\beta} > 0$ is found in such a way that:

$$|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)| \le C_{m,\alpha,\beta}(1+|\xi|)^{-m}\,, \quad \forall\,(x,\xi) \in U \times \Gamma_M\,. \tag{19}$$

Let us denote by $mclS^{-\infty}(U \times \Gamma_M)$ the set of all microlocally regularizing symbols on $U \times \Gamma_M$. For $(x_0, \xi_0) \in T^\circ\mathbb{R}^n$, we set:

$$mclS^{-\infty}(x_0,\xi_0) := \bigcup_{U,\,\Gamma_M} mclS^{-\infty}(U \times \Gamma_M)\,. \tag{20}$$

In view of Proposition 2.1, it is easily seen that for all $\delta \in [0,1]$ and $M \in \mathbb{N}^n$

$$mclS^{-\infty}(U \times \Gamma_M) = \bigcap_{m>0} mclS^{-m}_{M,\delta}(U \times \Gamma_M)\,; \tag{21}$$

a similar identity holds for $mclS^{-\infty}(x_0, \xi_0)$.
It is immediate to check that symbols in $mclS^m_{M,\delta}(U \times \Gamma_M)$, $mclS^m_{M,\delta}(x_0,\xi_0)$ behave according to the same rules of "global" symbols, collected in Proposition 2.3. Moreover $S^m_{M,\delta} \subset mclS^m_{M,\delta}(U \times \Gamma_M) \subset mclS^m_{M,\delta}(x_0,\xi_0)$ hold true, whenever $(x_0,\xi_0) \in T^\circ\mathbb{R}^n$, U is an open neighborhood of x_0 and Γ_M is an open $M-$conic neighborhood of ξ_0.

Definition 4.3: we say that a symbol $a \in mclS^m_{M,\delta}(x_0,\xi_0)$ is microlocally $M-$elliptic at $(x_0,\xi_0) \in T^\circ\mathbb{R}^n$ if there exist an open neighborhood U of x_0 and an $M-$conic open neighborhood Γ_M of ξ_0 such that for $c_0 > 0$, $\rho_0 > 0$:

$$|a(x,\xi)| \geq c_0 \langle\xi\rangle_M^m\,, \quad \forall\,(x,\xi) \in U \times \Gamma_M\,, \quad |\xi|_M > \rho_0\,. \tag{22}$$

Moreover the characteristic set of $a \in S^m_{M,\delta}$ is $\mathrm{Char}(a) \subset T^\circ\mathbb{R}^n$ defined by

$$(x_0,\xi_0) \in T^\circ\mathbb{R}^n \setminus \mathrm{Char}(a) \;\Leftrightarrow\; a \text{ is microlocally M-elliptic at } (x_0,\xi_0)\,. \tag{23}$$

Proposition 4.4: microlocal parametrix. *Assume that* $0 \leq \delta < 1/m^*$. *Then* $a \in S^m_{M,\delta}$ *is microlocally* $M-$*elliptic at* $(x_0,\xi_0) \in T^\circ\mathbb{R}^n$ **if and only if** *there exist symbols* $b, c \in S^{-m}_{M,\delta}$ *such that*

$$a \,\sharp\, b - 1 \quad \textit{and} \quad c \,\sharp\, a - 1 \tag{24}$$

are microlocally regularizing at (x_0,ξ_0).

Proof: Firstly assume that a satisfies (22) for $(x,\xi) \in U \times \Gamma_M$ and large ξ. The construction of a symbol $b \in S^{-m}_{M,\delta}$, satisfying the left condition in (24), follows as in the case of a (global) $M-$elliptic symbol a. For the sake of completeness, let us summarize the main steps. Let $F(z) = \frac{1}{z}$ for $|z| \geq c_0$ be a smooth function of complex variable and set: $b_0(x,\xi) = \langle\xi\rangle_M^{-m} F(\langle\xi\rangle_M^{-m} a(x,\xi))$. Then $b_0 \in S^{-m}_{M,\delta}$ and $a(x,\xi)b_0(x,\xi) = 1$ in $U \times \Gamma_M$ for $|\xi|_M > \rho_0$.
It follows that the symbol $\sigma := ab_0 - 1$ is microlocally in $S^{-(1/m^*-\delta)}_{M,\delta}$ on $U \times \Gamma_M$. Using the asymptotic expansion (7) we also obtain that $a\sharp b_0 - 1 \in mclS^{-(1/m^*-\delta)}_{M,\delta}(U \times \Gamma_M)$. We define recursively the symbols r_j, b_j, for $j \geq 1$, by

$$\begin{aligned} &r_1 := 1 - a\sharp b_0\,, \quad r_j := r_1\sharp r_{j-1}\,,\; j = 2,3,\dots \\ &b_j := b_0\sharp r_j\,,\; j = 1,2,\dots\,. \end{aligned} \tag{25}$$

It follows that

$$r_j \in mclS^{-(1/m^*-\delta)j}_{M,\delta}(U \times \Gamma_M) \cap S^0_{M,\delta}\,, \quad b_j \in mclS^{-m-(1/m^*-\delta)j}_{M,\delta}(U \times \Gamma_M) \cap S^{-m}_{M,\delta}\,.$$

Let now $\phi \in C_0^\infty(U)$ be identically one on some open neighborhood $\mathcal{U}$ of x_0, contained in U. Moreover, let $\widetilde{\Gamma}_M$ be an open $M-$conic neighborhood of ξ_0, such that $\widetilde{\Gamma}_M \cap \{|\xi|_M = 1\}$ has compact closure in $\Gamma_M \cap \{|\xi|_M = 1\}$, and choose $\widetilde{\psi}(\xi) \in S^0_M$ such that for some $0 < \varepsilon_0 < |\xi_0|_M$:

$$\mathrm{supp}\,\widetilde{\psi} \subset \Gamma_M \quad \text{and} \quad \widetilde{\psi} = 1 \text{ on } \widetilde{\Gamma}_M \cap \{|\xi|_M > \varepsilon_0\}. \tag{26}$$

For all $j \geq 0$, define the symbol $\widetilde{b}_j(x,\xi) := \phi(x)\widetilde{\psi}(\xi)b_j(x,\xi) \in S^{-m-(1/m^*-\delta)j}_{M,\delta}$.
By Proposition 2.4 there exists a symbol $b \in S^{-m}_{M,\delta}$ such that

$$b \sim \sum_{j\geq 0} \widetilde{b}_j\,.$$

To showing that $a\sharp b - 1$ is microlocally regularizing at (x_0,ξ_0) split $a\sharp b$ as:

$$a\sharp b = a\sharp \sum_{j<k} \widetilde{b}_j + a\sharp\mathcal{R}_k, \quad \text{where} \quad \mathcal{R}_k \in S^{-m-(1/m^*-\delta)k}_{M,\delta}. \tag{27}$$

For each $j \geq 0$, $a\sharp\widetilde{b}_j - a\sharp b_j$ is microlocally regularizing at (x_0, ξ_0); indeed, for any integer $k > 0$ we obtain in $\mathcal{U} \times \widetilde{\Gamma}_M$:

$$\begin{aligned} a\sharp\widetilde{b}_j &= \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha (\phi\widetilde{\psi} b_j) + \mathcal{S}_{k,j} = \widetilde{\psi} \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b_j + \mathcal{S}_{k,j} \\ &= \widetilde{\psi}(a\sharp b_j + \mathcal{T}_{k,j}) + \mathcal{S}_{k,j} = a\sharp b_j + (\widetilde{\psi} - 1) a\sharp b_j + \widetilde{\psi}\mathcal{T}_{k,j} + \mathcal{S}_{k,j}\,, \end{aligned}$$

with $\mathcal{S}_{k,j} \in S_{M,\delta}^{-(1/m^*-\delta)(j+k)}$, $\mathcal{T}_{k,j} \in S_{M,\delta}^{-(1/m^*-\delta)k}$. We can also prove that $\sigma := (\widetilde{\psi}-1)a\sharp b_j$ is microlocally regularizing on $\mathcal{U}\times\widetilde{\Gamma}_M$; indeed, in view of (26), σ vanishes identically on $\mathcal{U}\times(\widetilde{\Gamma}_M \cap \{|\xi|_M > \varepsilon_0\})$; on the other hand in $\mathcal{U}\times(\widetilde{\Gamma}_M \cap \{|\xi|_M \leq \varepsilon_0\})$, we obtain for all $m > 0$ and $\alpha, \beta \in \mathbb{Z}_+^n$:

$$\begin{aligned} &|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \leq C \sum_{\nu\leq\alpha} |\partial_\xi^\nu (\widetilde{\psi} - 1)(\xi)| |\partial_\xi^{\alpha-\nu} \partial_x^\beta (a\sharp b_j)(x,\xi)| \\ &\leq C\langle\xi\rangle_M^{-\alpha\cdot 1/M + \delta\beta\cdot 1/M} \leq C\langle\xi\rangle_M^{-m-\alpha\cdot 1/M+\delta\beta\cdot 1/M} \langle\xi\rangle_M^m \\ &\leq C(1+\varepsilon_0^2)^{m/2} \langle\xi\rangle_M^{-m-\alpha\cdot 1/M+\delta\beta\cdot 1/M}\,. \end{aligned}$$

Hence we conclude that $a\sharp\widetilde{b}_j - a\sharp b_j \in \bigcap_{k>0} mclS_{M,\delta}^{-(1/m^*-\delta)k}(\mathcal{U}\times\widetilde{\Gamma}_M)$, that means $a\sharp\widetilde{b}_j - a\sharp b_j \in mclS^{-\infty}(\mathcal{U}\times\widetilde{\Gamma}_M)$, in view of (21).
Set $q_j := a\sharp\widetilde{b}_j - a\sharp b_j$, $j \geq 0$; then for any integer $k > 0$ from (27) we obtain:

$$\begin{aligned} a\sharp b &= a\sharp \sum_{j<k} \widetilde{b}_j + a\sharp\mathcal{R}_k = \sum_{j<k} a\sharp b_j + \sum_{j<k} q_j + a\sharp\mathcal{R}_k \\ &= a\sharp b_0 + \sum_{1\leq j<k} (a\sharp b_0)\sharp r_j + \sum_{j<k} q_j + a\sharp\mathcal{R}_k \\ &= 1 - r_1 + \sum_{1\leq j<k} (1-r_1)\sharp r_j + \sum_{j<k} q_j + a\sharp\mathcal{R}_k \\ &= 1 - r_k + \sum_{j<k} q_j + a\sharp\mathcal{R}_k = 1 - \tau_k\,, \end{aligned}$$

where $\tau_k := r_k - \sum_{j<k} q_j - a\sharp\mathcal{R}_k \in mclS_{M,\delta}^{-(1/m^*-\delta)k}(\mathcal{U}\times\widetilde{\Gamma}_M)$.
Since k is arbitrary, we conclude that $a\sharp b - 1$ is microlocally regularizing at $(x_0,\xi_0) \in \mathcal{U}\times\widetilde{\Gamma}_M$.
In a similar way we can construct a symbol $c \in S_{M,\delta}^{-m}$, such that $c\sharp a - 1$ is microlocally regularizing at (x_0, ξ_0) .
Conversely if we assume (24) satisfied by some $b, c \in S_{M,\delta}^{-m}$, then (22) follows by standard arguments. □

5. Microlocal Sobolev and Hölder spaces

Throughout this section, for $s \in \mathbb{R}$ and M satisfying the assumptions in §2, E_M^s denotes either the Sobolev space $H_M^{s,p}$, with an arbitrarily fixed summability exponent $p \in]1, +\infty[$, or the Zygmund-Hölder space $B_{\infty,\infty}^{s,M}$.

Definition 5.1: for $(x_0, \xi_0) \in T^\circ\mathbb{R}^n$, $s \in \mathbb{R}$, we define $mclE_M^s(x_0,\xi_0)$ as the set of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that:

$$\psi(D)(\phi u) \in E_M^s\,, \tag{28}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$ is identically one in a neighborhood of x_0, $\psi(\xi) \in S_M^0$ is a symbol identically one on $\Gamma_M \cap \{|\xi|_M > \varepsilon_0\}$, for $0 < \varepsilon_0 < |\xi_0|_M$, and finally $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ is a $M-$conic neighborhood of ξ_0.
Under the same assumptions, we also write

$$(x_0, \xi_0) \notin WF_{E_M^s}(u)\,.$$

The set $WF_{E_M^s}(u) \subset T^\circ\mathbb{R}^n$ is called the E_M^s-wave front set of u.
Finally we say that $x_0 \notin E_M^s - \text{singsupp}\,(u)$ if and only if there exists a function $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi \equiv 1$ in some open neighborhood of x_0, such that $\phi u \in E_M^s$.

We say that a distribution satisfying the previous definition is *microlocally in* E_M^s *at* (x_0, ξ_0). Moreover the closed set $WF_{E_M^s}(u)$ is $M-$ *conic* in the ξ variable.
Assume that $u \in mclE_M^s(x_0, \xi_0)$ and consider Γ_M, ϕ, ψ as in Definition 5.1. Let $\widetilde{\psi}(\xi)$ be a symbol in S_M^0 satisfying $\text{supp}\,\widetilde{\psi} \subset \Gamma_M$. Take also a function $\widetilde{\phi} \in C_0^\infty(\mathbb{R}^n)$ such that $\widetilde{\phi}\,\phi = \widetilde{\phi}$, namely $\phi = 1$ in supp $\tilde{\phi}$. Then

$$\begin{aligned}&\widetilde{\psi}(D)(\widetilde{\phi}u) = \widetilde{\psi}(D)(\widetilde{\phi}\phi u) = \widetilde{\psi}(D)\widetilde{\phi}\,[\psi(D)(\phi u) + (I - \psi(D))(\phi u)]\\ &= \widetilde{\psi}(D)\widetilde{\phi}\,[\psi(D)(\phi u)] + \widetilde{\psi}(D)\widetilde{\phi}(I - \psi(D))(\phi u) = T(\psi(D)(\phi u)) + R(\phi u)\,,\end{aligned}$$

where $Tw := \widetilde{\psi}(D)(\widetilde{\phi}w)$ and $Rw := \widetilde{\psi}(D)(\widetilde{\phi}(I - \psi(D))w)$. Since $T \in \text{Op}\,S_M^0$, from $\psi(D)(\phi u) \in E_M^s$ and Corollary 3.4 it follows that $T(\psi(D)(\phi u)) \in E_M^s$. On the other hand, the operator R is regularizing; indeed, R has symbol $(1 - \psi)\,(\widetilde{\psi}\sharp\widetilde{\phi})$ such that for any integer $k > 0$:

$$(1 - \psi)\,(\widetilde{\psi}\sharp\widetilde{\phi})(x, \xi) = \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!}(1 - \psi(\xi))\partial_\xi^\alpha\widetilde{\psi}(\xi)\partial_x^\alpha\widetilde{\phi}(x) + \tau_k(x, \xi)\,, \tag{29}$$

with $\tau_k \in S_M^{-k/m^*}$. Since $1 - \psi(\xi)$ vanishes identically on $\text{supp}\,\widetilde{\psi} \cap \{|\xi|_M > \varepsilon_0\}$, one has in such set $(1 - \psi(\xi))\,(\widetilde{\psi}\sharp\widetilde{\phi})(x, \xi) = \tau_k(x, \xi)$. On the other hand, in $\text{supp}\,\widetilde{\psi} \cap \{|\xi|_M \le \varepsilon_0\}$ one has for all $m > 0$:

$$\begin{aligned}&|\partial_\xi^\nu\partial_x^\beta((1 - \psi(\xi))\partial_\xi^\alpha\widetilde{\psi}(\xi)\partial_x^\alpha\widetilde{\phi}(x))| = |\partial_\xi^\nu((1 - \psi(\xi))\partial_\xi^\alpha\widetilde{\psi}(\xi))\partial_x^{\alpha+\beta}\widetilde{\phi}(x)|\\ &\le C_\nu \sum_{\mu\le\nu} |\partial_\xi^{\mu+\alpha}\widetilde{\psi}(\xi)||\partial_\xi^{\nu-\mu}(1 - \psi)(\xi)||\partial^{\alpha+\beta}\widetilde{\phi}(x)|\\ &\le C_{\nu,\alpha,\beta} \sum_{\mu\le\nu} \langle\xi\rangle_M^{-(\mu+\alpha)\cdot 1/M}\langle\xi\rangle_M^{-(\nu-\mu)\cdot 1/M} \le C_{\nu,\alpha,\beta}\langle\xi\rangle_M^{-m-\alpha\cdot 1/M-\nu\cdot 1/M}(1 + \varepsilon_0^2)^{\frac{m}{2}}\,.\end{aligned}$$

This means that $(1 - \psi)\,(\widetilde{\psi}\sharp\widetilde{\phi})$ belongs to S_M^{-k/m^*} for all $k > 0$, hence it belongs to $S^{-\infty}$. Thus $R(\phi u) \in \mathcal{S}(\mathbb{R}^n)$. We can then conclude that for $u \in mclE_M^s(x_0, \xi_0)$, one has $\widetilde{\psi}(D)(\widetilde{\phi}u) \in E_M^s$, provided that $\widetilde{\psi}$ and $\widetilde{\phi}$ are taken to satisfy the assumptions before. We obtain then the following property:

Proposition 5.2: *if* $u \in mclE_M^s(x_0, \xi_0)$, *with* $(x_0, \xi_0) \in T^\circ\mathbb{R}^n$, *then* $\varphi u \in mclE_M^s(x_0, \xi_0)$ *for any* $\varphi \in C_0^\infty(\mathbb{R}^n)$, *such that* $\varphi(x_0) \neq 0$.

Let now π_1 be the canonical projection of $T^\circ\mathbb{R}^n$ onto $\mathbb{R}^n$, $\pi_1(x, \xi) = x$.

Proposition 5.3: *for every* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $s \in \mathbb{R}$ *we have:*

$$E_M^s - \text{singsupp}(u) = \pi_1(WF_{E_M^s}(u))\,.$$

Proof: for proving that $\pi_1(WF_{E_M^s}(u)) \subseteq E_M^s - \text{singsupp}(u)$, assume that $x_0 \notin$

E^s_M-singsupp(u). Then there exists a function $\phi\in C_0^\infty(\mathbb{R}^n)$, $\phi\equiv 1$ in some open neighborhood of x_0, such that $\phi u\in E^s_M$. Let ξ_0 be any nonzero vector in $\mathbb{R}^n$ and Γ_M an arbitrary $M-$conic open neighborhood of ξ_0. For every symbol $\psi=\psi(\xi)\in S^0_M$ identically equal to 1 on $\Gamma_M\cap\{|\xi|_M>\varepsilon_0\}$, for some $0<\varepsilon_0<|\xi_0|_M$, Corollary 3.4 gives $\psi(D)(\phi u)\in E^s_M$, that is $(x_0,\xi_0)\notin WF_{E^s_M}(u)$.
In order to prove the converse inclusion assume that x_0 does not belong to $\pi_1(WF_{E^s_M}(u))$. This implies that $(x_0,\eta)\notin WF_{E^s_M}(u)$ for any vector η, such that $|\eta|_M=1$; thus, for such a vector η, there exist corresponding $\phi_\eta\in C_0^\infty(\mathbb{R}^n)$, satisfying $\phi_\eta\equiv 1$ in an open neighborhood of x_0, $\Gamma_{M,\eta}$, $M-$conic open neighborhood of η, and $\psi_\eta(\xi)\in S^0_M$, satisfying $\psi_\eta\equiv 1$ in $\Gamma_{M,\eta}\cap\{|\xi|_M>\varepsilon_\eta\}$ for suitable $0<\varepsilon_\eta<1$, such that:

$$\psi_\eta(D)(\phi_\eta u)\in E^s_M\,. \tag{30}$$

Because of the compactness of the *quasi-homogeneous unit sphere* $\{|\eta|_M=1\}$ and the observation above, following the arguments of [8, Proposition 6.3], we can find finitely many open $M-$conic sets $\Gamma^1_M,\dots,\Gamma^k_M$ and corresponding symbols $\psi_1=\psi_1(\xi),\dots,\psi_k=\psi_k(\xi)$ in S^0_M such that

$$\begin{aligned}&\operatorname{supp}\psi_l\subset\Gamma^l_M\cap\{|\xi|_M>\varepsilon_l\},\ l=1,\dots,k\,,\\ &\sum_{l=1}^k\psi_l(\xi)=1\,,\quad \text{as } |\xi|_M>\varepsilon^*\,,\end{aligned} \tag{31}$$

where for all $l=1,\dots,k$, $0<\varepsilon_l<1$ are suitably fixed and $\varepsilon^*:=\max\{\varepsilon_l\,,\ l=1,\dots,k\}$.
Take also a symbol $\chi=\chi(\xi)\in S^0_M$ such that

$$\chi(\xi)=0\,,\ \text{for } |\xi|_M\le\varepsilon^*\quad\text{and}\quad \chi(\xi)=1\,,\ \text{for } |\xi|_M>2\varepsilon^*\,. \tag{32}$$

At last, take a function $\phi\in C_0^\infty(\mathbb{R}^n)$, $\phi\equiv 1$ in a sufficiently small open neighborhood of x_0. In view of (31) and (32), we can write

$$\phi u=(1-\chi(D))(\phi u)+\sum_{l=1}^k\chi(D)\psi_l(D)(\phi u)\,.$$

It follows that $\phi u\in E^s_M$, since $1-\chi(\xi)\in S^{-\infty}$ and for each $l=1,\dots,k$, $\chi(D)\psi_l(D)(\phi u)\in E^s_M$ (because of (30) and Corollary 3.4). This proves that $x_0\notin E^s_M-$singsupp(u). □

The following microlocal counterpart of the boundedness properties given in Corollary 3.4 can be proved.

Theorem 5.4: *for $\delta\in[0,1/m^*[$, and $(x_0,\xi_0)\in T^\circ\mathbb{R}^n$, assume that $a\in S^\infty_{M,\delta}\cap mclS^m_{M,\delta}(x_0,\xi_0)$, $m\in\mathbb{R}$. Then for all $s\in\mathbb{R}$*

$$u\in mclE^{s+m}_M(x_0,\xi_0)\quad\Rightarrow\quad a(x,D)u\in mclE^s_M(x_0,\xi_0)\,. \tag{33}$$

Proof: let U^0 be an open neighborhood of x_0 and $\Gamma^0_M\subset\mathbb{R}^n\setminus\{0\}$ an open neighborhood of ξ_0 for which (17) is satisfied by the symbol $a(x,\xi)$. In view of the previous argument, we may find an open neighborhood U of x_0, with compact closure contained in U_0, and an open $M-$conic neighborhood Γ_M of ξ_0, such that

$\Gamma_M \cap \{|\xi|_M = 1\}$ has compact closure in $\Gamma^0_M \cap \{|\xi|_M = 1\}$, in such a way that

$$\psi(D)(\phi u) \in E^{s+m}_M\,, \tag{34}$$

where $\phi = \phi(x) \in C^\infty_0(U^0)$ is identically one on U, and $\psi = \psi(\xi) \in S^0_M$, satisfies

$$\operatorname{supp}\psi \subset \Gamma^0_M \quad \text{and} \quad \psi = 1 \text{ on } \Gamma_M \cap \{|\xi|_M > \varepsilon_0\}\,, \quad 0 < \varepsilon_0 < |\xi_0|_M.$$

We take now $\widetilde{\phi} \in C^\infty_0(\mathbb{R}^n)$ and $\widetilde{\psi} = \widetilde{\psi}(\xi) \in S^0_M$ exactly as in the argument following Definition 5.1. Setting $Au := a(x,D)u$, let us consider $\widetilde{\psi}(D)(\widetilde{\phi} Au)$. Then $\widetilde{\psi}(D)(\widetilde{\phi}\, Au) = \widetilde{\psi}(D)\left[\widetilde{\phi}\, A(\phi u)\right] + \widetilde{\psi}(D)\left[\widetilde{\phi}\, A(1-\phi)u\right]$.

The operator $\widetilde{\phi}A(1-\phi)$ is regularizing, since for an arbitrary integer $k > 0$ one finds, by the use of the asymptotic expansion (7),

$$\widetilde{\phi}(x)(a\sharp(1-\phi))(x,\xi) = \widetilde{\phi}(x) \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi a(x,\xi) \partial^\alpha_x (1-\phi(x)) + \sigma_k(x,\xi)\,,$$

where $\sigma_k \in S^{l-(1/m^*-\delta)k}_{M,\delta}$ for some $l \geq m$ and $\widetilde{\phi}(x)\partial^\alpha_\xi a(x,\xi)\partial^\alpha_x(1-\phi(x))$ are identically zero for all α, since $1-\phi(x) = 0$ for $x \in \operatorname{supp}\widetilde{\phi}$. Hence $\widetilde{\phi}\, a\sharp(1-\phi) \in S^{-\infty}$ and we conclude $\widetilde{\psi}(D)\left[\widetilde{\phi}A(1-\phi)u\right] \in \mathcal{S}(\mathbb{R}^n)$.

Concerning $\widetilde{\psi}(D)\left[\widetilde{\phi}\, A(\phi u)\right]$, observe that :

$$\begin{aligned}
&\widetilde{\psi}(D)\left[\widetilde{\phi}\, A(\phi u)\right] = \widetilde{\psi}(D)\widetilde{\phi}\, A\,[\psi(D)(\phi u)] + \widetilde{\psi}(D)\,\widetilde{\phi}\, A\,(I - \psi(D))(\phi u) \\
&= T[\psi(D)(\phi u)] + R(\phi u)\,,
\end{aligned}$$

where $T := \widetilde{\psi}(D)\widetilde{\phi}\, A$ and $R := \widetilde{\psi}(D)\,\widetilde{\phi}\, A\,(I - \psi(D))$. Again, the operator R is regularizing, since in view of (7) its symbol $(1-\psi)[\widetilde{\psi}\,\sharp\,(\widetilde{\phi}\, a)]$ expands in a similar way as (29) and $1-\psi$ vanishes identically on $\operatorname{supp}\widetilde{\psi} \cap \{|\xi|_M > \varepsilon_0\}$.

Concerning the operator T, using the assumptions on $\widetilde{\phi}$ and $\widetilde{\psi}$, it can be considered as an operator with symbol in $S^m_{M,\delta}$, modulo a regularizing reminder. In order to prove it, we apply again the asymptotic expansion (7) to the symbol $\widetilde{\psi}\sharp(\widetilde{\phi}a)$:

$$\widetilde{\psi}\,\sharp\,(\widetilde{\phi}a)(x,\xi) = \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi \widetilde{\psi}(\xi) \partial^\alpha_x (\widetilde{\phi}(x) a(x,\xi)) + \tau_k(x,\xi)\,, \tag{35}$$

where $\tau_k \in S^{l-(1/m^*-\delta)k}_{M,\delta}$ and $l \geq m$. Now by means of $\widetilde{\widetilde{\psi}}(\xi) \in S^0_M$ such that $\operatorname{supp}\widetilde{\widetilde{\psi}} \subset \Gamma_M$ and $\widetilde{\widetilde{\psi}} = 1$ on $\operatorname{supp}\widetilde{\psi}$, let us set $\widetilde{a}(x,\xi) := \widetilde{\widetilde{\psi}}(\xi)\widetilde{\phi}(x)a(x,\xi) \in S^m_{M,\delta}$. Then (35) implies

$$(\widetilde{\psi}\,\sharp\,(\widetilde{\phi}\, a))(x,\xi) = (\widetilde{\psi}\,\sharp\,\widetilde{a})(x,\xi) + \tau_k(x,\xi) - \theta_k(x,\xi)\,, \tag{36}$$

for a suitable $\theta_k \in S^{m-(1/m^*-\delta)k}_{M,\delta}$. Since k is arbitrary, we get $T = \widetilde{\psi}(D)\widetilde{A} + \mathcal{R}$, where $\widetilde{A} := \widetilde{a}(x,D)$ and $\mathcal{R}$ is a regularizing operator.

From the last equality, we conclude that

$$T(\psi(D)(\phi u)) = \widetilde{\psi}(D)\widetilde{A}(\psi(D)(\phi u)) + \mathcal{R}(\psi(D)(\phi u))$$

belongs to E^s_M, by Corollary 3.4. Then $\widetilde{\psi}(D)(\widetilde{\phi}Au) \in E^s_M$ and the proof is concluded. □

6. Microlocal Sobolev and Hölder regularity

This section is devoted to prove the following microregularity result.

Theorem 6.1 : *let $a \in S^m_{M,\delta}$, for $m \in \mathbb{R}$, $\delta \in [0, 1/m^*[$, be microlocally $M-$elliptic at $(x_0, \xi_0) \in T^\circ\mathbb{R}^n$. For $s \in \mathbb{R}$ and $p \in]1, +\infty[$ assume that $u \in \mathcal{S}'(\mathbb{R}^n)$ fulfills $a(x, D)u \in mclE^s_M(x_0, \xi_0)$. Then $u \in mclE^{s+m}_M(x_0, \xi_0)$.*

Proof: consider Γ_M, $\phi \in C^\infty_0(\mathbb{R}^n)$ and $\psi = \psi(\xi) \in S^0_M$ satisfying the assumptions of Definition 5.1. Then for $Au := a(x, D)u$:

$$\psi(D)(\phi Au) \in E^s_M\,. \tag{37}$$

Let us take an open neighborhood $\mathcal{U}$ of x_0, such that $\phi(x) = 1$ in $\overline{\mathcal{U}}$, and $\widetilde{\phi} \in C^\infty_0(\mathcal{U})$; moreover, arguing as in the proof of Proposition 4.4, let $\widetilde{\Gamma}_M$ be an open $M-$conic neighborhood of ξ_0 such that $\widetilde{\Gamma}_M \cap \{|\xi|_M = 1\}$ has compact closure in $\Gamma_M \cap \{|\xi|_M = 1\}$ and $\widetilde{\psi} = \widetilde{\psi}(\xi)$ be a symbol in S^0_M satisfying (26). We can then find a symbol $b \in S^{-m}_{M,\delta}$ such that for $B := b(x, D)$

$$BAu = u + \mathcal{R}u\,, \tag{38}$$

where $\mathcal{R}$ is microlocally regularizing at $\mathcal{U} \times \widetilde{\Gamma}_M$. Using (38), we can write

$$\widetilde{\psi}(D)(\widetilde{\phi}u) = \widetilde{\psi}(D)(\widetilde{\phi}BAu) - \widetilde{\psi}(D)(\widetilde{\phi}\mathcal{R}u)\,.$$

Arguing now as in the proof of Theorem 5.4, one can see that the operator $\widetilde{\psi}(D)(\widetilde{\phi}\mathcal{R})$ is regularizing, hence $\widetilde{\psi}(D)(\widetilde{\phi}\mathcal{R}u) \in E^{s+m}_M$.
Moreover by means of a suitable regularizing operator $\mathcal{S}$ we have

$$\widetilde{\psi}(D)(\widetilde{\phi}BAu) = \widetilde{\psi}(D)(\widetilde{\phi}B(\phi Au)) + \mathcal{S}u. \tag{39}$$

Indeed, applied to the symbol $\widetilde{\phi}\,(b\,\sharp\,(1-\phi)) \in S^{-m}_{M,\delta}$ of $\widetilde{\phi}B(1-\phi)$ the asymptotic formula (7) gives for any integer $k > 0$:

$$\widetilde{\phi}(x)(b\,\sharp\,(1-\phi))(x,\xi) = \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!}\widetilde{\phi}(x)\partial^\alpha_\xi b(x,\xi)\partial^\alpha_x(1-\phi(x)) + s_k(x,\xi)\,,$$

where $s_k \in S^{m-(1/m^*-\delta)k}_{M,\delta}$ and, for all α, the symbol $\widetilde{\phi}(x)\partial^\alpha_\xi b(x,\xi)\partial^\alpha_x(1-\phi(x))$ vanishes identically, since ϕ is identically one on supp $\widetilde{\phi}$. Then the operator $\widetilde{\phi}B(1-\phi)$ is regularizing and (39) follows, applying the identity $w = \phi w + (1-\phi)w$ to $w = Au$.
Now, we decompose $\widetilde{\psi}(D)(\widetilde{\phi}B(\phi Au))$ as

$$\widetilde{\psi}(D)(\widetilde{\phi}B(\phi Au)) = \widetilde{\psi}(D)\widetilde{\phi}B\left[\psi(D)(\phi Au)\right] + \widetilde{\psi}(D)\widetilde{\phi}B(I-\psi(D))\left[\phi Au\right]\,. \tag{40}$$

Again, by the help of (7), we find that the operator $\widetilde{\psi}(D)\widetilde{\phi}B(I-\psi(D))$ is regularizing, hence

$$\widetilde{\psi}(D)\widetilde{\phi}B(I-\psi(D))\,[\phi Au]\in E^{s+m}_M\,.$$

On the other hand, $\widetilde{\psi}(D)\widetilde{\phi}B$ belongs to $\mathrm{Op}\,S^{-m}_{M,\delta}$. Thus from (37) we get

$$\widetilde{\psi}(D)\widetilde{\phi}B\,[\psi(D)(\phi Au)]\in E^{s+m}_M\,,$$

in view of Corollary 3.4. This completes the proof of Theorem 6.1. □

As a consequence of Theorems 5.4, 6.1, there holds the following

Corollary 6.2: *for $a\in S^m_{M,\delta}$, $m\in\mathbb{R}$, $\delta\in[0,1/m^*[$ and $u\in\mathcal{S}'(\mathbb{R}^n)$, the following inclusions:*

$$WF_{E^s_M}(a(x,D)u)\subset WF_{E^{s+m}_M}(u)\subset WF_{E^s_M}(a(x,D)u)\cup\mathrm{Char}(a)\,, \tag{41}$$

hold true for every $s\in\mathbb{R}$.

Proof: the first inclusion directly follows from Definition 5.1 and Theorem 5.4. To prove the second inclusion in (41), assume that $(x_0,\xi_0)\notin\mathrm{Char}(a)$; that is $a(x,\xi)$ is microlocally $M-$elliptic at (x_0,ξ_0). If in addition $(x_0,\xi_0)\notin WF_{E^s_M}(a(x,D)u)$, that is $a(x,D)u\in mclE^s_M(x_0,\xi_0)$, Theorem 6.1 implies $u\in mclE^{s+m}_M(x_0,\xi_0)$, namely that $(x_0,\xi_0)\notin WF_{E^{s+m}_M}(u)$. □

7. Non regular symbols

In this section, the microlocal regularity results discussed in §5 and §6 are applied to obtain microlocal regularity results for a linear partial differential equation of quasi-homogeneous order $m\in\mathbb{N}$ of the form

$$A(x,D)u:=\sum_{\alpha\cdot 1/M\le m}a_\alpha(x)D^\alpha u=f(x)\,, \tag{42}$$

where $D^\alpha:=(-i)^{|\alpha|}\partial^\alpha$ and the coefficients a_α are assumed in a Zygmund-Hölder class $B^{r,M}_{\infty,\infty}$ of positive order r.
Concerning the operator $A(x,D)$, we assume it is *microlocally* $M-$elliptic at a given point $(x_0,\xi_0)\in T^\circ\mathbb{R}^n$; according to Definition 4.3 and the quasi-homogeneity of the norm $|\xi|_M$, this means that there exist an open neighborhood U of x_0 and an open $M-$conic neighborhood Γ_M of ξ_0 such that the $M-$*principal symbol* of $A(x,D)$ satisfies

$$A_m(x,\xi)=\sum_{\alpha\cdot 1/M=m}a_\alpha(x)\xi^\alpha\neq 0,\quad \text{for}\,(x,\xi)\in U\times\Gamma_M,\quad \xi\neq 0. \tag{43}$$

The forcing term f is assumed to be in some space E^s_M, with a suitable order of smoothness s, *microlocally* at (x_0,ξ_0) (cf. Definition 5.1).

Theorem 7.1: *let $A(x,D)u=f$ be a linear partial differential equation, as in* (42)*, with coefficients in the space $B^{r,M}_{\infty,\infty}$ of positive order r. Assume that $A(x,D)$ is microlocally M-elliptic at $(x_0,\xi_0)\in T^\circ\mathbb{R}^n$. Moreover, for $p\in]1,+\infty[$, $0<\delta<1/m^*$*

and $(\delta-1)r+m<s<r+m$*, assume that* $f\in mclH^{s-m,p}_M(x_0,\xi_0)$ *and* $u\in H^{s-\delta r,p}_M$*. Then we have* $u\in mclH^{s,p}_M(x_0,\xi_0)$*.*
Under the same assumptions, if $f\in mclB^{s-m,M}_{\infty,\infty}(x_0,\xi_0)$ *and* $u\in B^{s-\delta r,M}_{\infty,\infty}$*, for given* $0<\delta<1/m^*$ *and* $(\delta-1)r+m<s\le r+m$*, then* $u\in mclB^{s,M}_{\infty,\infty}(x_0,\xi_0)$*.*

Remark 1: assuming in (42) $A(x,D)$ with coefficients in $B^{r,M}_{\infty,\infty}$, $r>0$, u a priori in $H^{s-\delta r,p}_M$ (resp. $B^{s-\delta r,M}_{\infty,\infty}$) for $1<p<\infty$, $(\delta-1)r+m<s<r+m$ (resp. $(\delta-1)r+m<s\le r+m$), $\delta\in]0,1/m^*[$, we obtain

$$WF_{H^{s,p}_M}(u)\subset WF_{H^{s-m,p}_M}(A(x,D)u)\cup \mathrm{Char}(A)$$
$$\left(\text{resp. } WF_{B^{s,M}_{\infty,\infty}}(u)\subset WF_{B^{s-m,M}_{\infty,\infty}}(A(x,D)u)\cup \mathrm{Char}(A)\right).$$

Following [7], [3], non-smooth symbols in $B^{r,M}_{\infty,\infty}S^m_M$ can be decomposed, for a given $\delta\in]0,1]$, into the sum of a smooth symbol in $S^m_{M,\delta}$ and a non-smooth symbol of lower order. Namely, let ϕ be a fixed C^∞ function such that $\phi(\xi)=1$ for $\langle\xi\rangle_M\le 1$ and $\phi(\xi)=0$ for $\langle\xi\rangle_M>2$. For given $\varepsilon>0$ we set $\phi(\varepsilon^{1/M}\xi):=\phi(\varepsilon^{1/m_1}\xi_1,\dots,\varepsilon^{1/m_n}\xi_n)$.
Any symbol $a(x,\xi)\in B^{r,M}_{\infty,\infty}S^m_M$ may be split in

$$a(x,\xi)=a^\sharp(x,\xi)+a^\natural(x,\xi), \tag{44}$$

where for some $\delta\in]0,1]$

$$a^\#(x,\xi):=\sum_{h=-1}^{\infty}\phi(2^{-h\delta/M}D_x)a(x,\xi)\varphi_h(\xi).$$

One can prove the following proposition (see [3, Proposition 3.9] and [7]).

Proposition 7.2: *if* $a(x,\xi)\in B^{r,M}_{\infty,\infty}S^m_M$*, with* $r>0$*,* $m\in\mathbb{R}$*, and* $\delta\in]0,1]$*, then* $a^\#(x,\xi)\in S^m_{M,\delta}$ *and* $a^\natural(x,\xi)\in B^{r,M}_{\infty,\infty}S^{m-r\delta}_{M,\delta}$*.*

The following microlocal version of [3, Proposition 3.10] can be also proved.

Proposition 7.3: *assume that* $a(x,\xi)\in B^{r,M}_{\infty,\infty}S^m_M$*,* $m\in\mathbb{R}$*, is microlocally* M*-elliptic at* $(x_0,\xi_0)\in T^\circ\mathbb{R}^n$*, then for any* $\delta\in]0,1]$*,* $a^\#(x,\xi)\in S^m_{M,\delta}$ *is still microlocally* $M-$*elliptic at* (x_0,ξ_0)*.*

Proof: the microlocal $M-$ellipticity of a yields the existence of positive constants c_1, ρ_1 such that

$$|a(x,\xi)|\ge c_1\langle\xi\rangle^m_M\,,\ \text{when } (x,\xi)\in U\times\Gamma_M \text{ and } |\xi|_M>\rho_1\,, \tag{45}$$

where U is a suitable open neighborhood of x_0 and Γ_M an open $M-$conic neighborhood of ξ_0. On the other hand, for any $\rho_0>0$ we can find a positive integer h_0, which increases together with ρ_0, such that $\varphi_h(\xi)=0$ as long as $|\xi|_M>\rho_0$ and $h=-1,\dots,h_0-1$. We can then write:

$$a^\#(x,\xi)=\sum_{h=h_0}^{\infty}\phi\left(2^{-h\delta/M}D_x\right)a(x,\xi)\varphi_h(\xi),\quad |\xi|_M>\rho_0\,. \tag{46}$$

Set for brevity $\phi\left(2^{-h\delta/M}\cdot\right)=\phi_h(\cdot)$
By means of (46), the Cauchy-Schwarz inequality and [3, Lemma 3.8], when $|\xi|_M>\rho_0$ we can estimate

$$\begin{aligned}
&|a^{\#}(x,\xi)-a(x,\xi)|^2\\
&=\left|\sum_{h=h_0}^{\infty}\left(\phi_h(D_x)-I\right)a(x,\xi)\varphi_h(\xi)\right|^2\\
&=\sum_{h=h_0}^{\infty}\sum_{k=h-N_0}^{h+N_0}\left\langle\left(\left(\phi_h(D_x)\right)-I\right)a(x,\xi)\varphi_h(\xi),\left(\phi_k(D_x)-I\right)a(x,\xi)\varphi_k(\xi)\right\rangle\\
&=\sum_{t=-N_0}^{N_0}\sum_{h=h_0}^{\infty}\left\langle\left(\left(\phi_h(D_x)-I\right)a(x,\xi)\varphi_h(\xi),\left(\phi_{h+t}(D_x)-I\right)a(x,\xi)\varphi_{h+t}(\xi)\right)\right\rangle\\
&\le\sum_{t=-N_0}^{N_0}\sum_{h=h_0}^{\infty}\|\left(\left(\phi_h(D_x)\right)-I\right)a(\cdot,\xi)\|_{L^\infty}|\varphi_h(\xi)|\\
&\times\|\left(\phi_{h+t}(D_x)-I\right)a(\cdot,\xi)\|_{L^\infty}|\varphi_{h+t}(\xi)|\\
&\le C^2\sum_{t=-N_0}^{N_0}\sum_{h=h_0}^{\infty}2^{-h\delta r}2^{-(h+t)\delta r}\|a(\cdot,\xi)\|^2_{B^{r,M}_{\infty,\infty}}\\
&\le C^2\sum_{h=h_0}^{\infty}2^{-2h\delta r}\|a(\cdot,\xi)\|^2_{B^{r,M}_{\infty,\infty}}\le C^2 2^{-2h_0\delta r}\|a(\cdot,\xi)\|^2_{B^{r,M}_{\infty,\infty}},
\end{aligned}$$

where C denotes different positive constants depending only on δ, N_0 and r. Since $\|a(\cdot,\xi)\|_{B^{r,M}_{\infty,\infty}}\le c^*\langle\xi\rangle^m_M$, let us fix ρ_0 large enough to have $C2^{-h_0\delta r}<\frac{c_1}{2c^*}$ (with c_1 from (45)). Then for $(x,\xi)\in U\times\Gamma_M$ and $|\xi|_M>\max\{\rho_0,\rho_1\}$

$$|a^{\#}(x,\xi)|\ge|a(x,\xi)|-|a^{\#}(x,\xi)-a(x,\xi)|\ge\tfrac{c_1}{2}\langle\xi\rangle^m_M \tag{47}$$

follows and the proof is concluded. □

Consider now the linear partial differential equation (42), with $A(x,D)$ microlocally $M-$elliptic at (x_0,ξ_0). For an arbitrarily fixed $\delta\in]0,1/m^*[$, we split the symbol $A(x,\xi)$ as $A(x,\xi)=A^{\#}(x,\xi)+A^{\natural}(x,\xi)$, according to Proposition 7.2. In view of Propositions 7.3, 4.4 there exists a smooth symbol $B(x,\xi)\in S^{-m}_{M,\delta}$ such that

$$B(x,D)A^{\#}(x,D)=I+R(x,D)\,,$$

where $R(x,D)$ is microlocally regularizing at (x_0,ξ_0).
Applying now $B(x,D)$ to both sides of (42), on the left, we obtain:

$$u=B(x,D)f-R(x,D)u-B(x,D)A^{\natural}(x,D)u\,. \tag{48}$$

Assume that $f\in mclE^{s-m}_M(x_0,\xi_0)$ and $u\in E^{s-\delta r}_M$ for $(\delta-1)r+m<s<r+m$ (also $s=r+m$ in the case of Zygmund-Hölder spaces). Since $A^{\natural}(x,\xi)\in B^{r,M}_{\infty,\infty}S^{m-r\delta}_{M,\delta}$, one can apply Theorem 3.3 and Corollary 3.4 to find that $B(x,D)A^{\natural}(x,D)u\in E^s_M$; moreover Theorem 5.4 and Corollary 3.4 give $B(x,D)f\in mclE^s_M(x_0,\xi_0)$ and $R(x,D)u\in E^s_M$. This shows the result of Theorem 7.1.
By means of the argument above stated, we obtain the following general result for non regular pseudodifferential operators.

Corollary 7.4: *for $a(x,\xi)\in B^{r,M}_{\infty,\infty}S^m_M$, $r>0$, u belonging a priori to $H^{s-\delta r,p}_M$ (resp. $B^{s-\delta r,M}_{\infty,\infty}$) for $1<p<\infty$, $(\delta-1)r+m<s<r+m$ (resp. $(\delta-1)r+m<$*

$s \le r+m$), $\delta \in]0,1/m^*[$, we have

$$WF_{H_M^{s,p}}(u) \subset WF_{H_M^{s-m,p}}(a(x,D)u) \cup \mathrm{Char}(a)$$
$$(\text{resp. } WF_{B^{s,M}_{\infty,\infty}}(u) \subset WF_{B^{s-m,M}_{\infty,\infty}}(a(x,D)u) \cup \mathrm{Char}(a))\,.$$

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