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On Unequally Smooth Bivariate Quadratic Spline Spaces

C. Dagnino, P. Lamberti and S. Remogna*

Abstract

In this paper we consider spaces of unequally smooth local bivariate quadratic splines, defined on criss-cross triangulations of a rectangular domain.
For such spaces we present some results on the dimension and on a local basis.
Finally an application to B-spline surface generation is provided.

Keywords: bivariate spline approximation, unequally smooth bivariate spline space, B-spline basis
Subject classification AMS (MOS): 65D07; 41A15

1 Introduction

Aim of this paper is the investigation of bivariate quadratic spline spaces with less than maximum C^1 smoothness on criss-cross triangulations of a rectangular domain, with particular reference to their dimension and to the construction of a local basis. Indeed, in many practical applications, piecewise polynomial surfaces need to be connected by using different smoothness degrees and, in literature, tensor product spline surfaces of such a kind have already been investigated (see e.g. [1, 5]). In [2] the dimension and a B-spline basis for the space of all quadratic C^1 splines on a criss-cross triangulation are obtained. Since some supports of such B-splines are not completely contained in the rectangular domain, in [7] a new B-spline basis for such space is proposed, with all supports included in the domain.

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The paper is organized as follows. In Section 2 we present some results on the dimension of the unequally smooth spline space and on the construction of a B-spline basis with different types of smoothness. In Section 3 an application to B-spline surface generation is presented.

2 Bases of unequally smooth bivariate quadratic spline spaces

Let $\Omega = [a, b] \times [c, d]$ be a rectangle decomposed into $(m+1)(n+1)$ subrectangles by two partitions

$$\bar{\xi} = \{\xi_i, \quad i = 0, \ldots, m+1\},$$
$$\bar{\eta} = \{\eta_j, \quad j = 0, \ldots, n+1\},$$

of the segments $[a, b] = [\xi_0, \xi_{m+1}]$ and $[c, d] = [\eta_0, \eta_{n+1}]$, respectively. Let $\mathcal{T}_{mn}$ be the criss-cross triangulation associated with the partition $\bar{\xi} \times \bar{\eta}$ of the domain Ω.

Given two sets $\bar{m}^\xi = \{m_i^\xi\}_{i=1}^m$, $\bar{m}^\eta = \{m_j^\eta\}_{j=1}^n$, with m_i^ξ, $m_j^\eta = 1, 2$ for all i, j, we set

$$M = 3 + \sum_{i=1}^{m} m_i^\xi, \quad N = 3 + \sum_{j=1}^{n} m_j^\eta \tag{1}$$

and let $\bar{u} = \{u_i\}_{i=-2}^M$, $\bar{v} = \{v_j\}_{j=-2}^N$ be the nondecreasing sequences of knots, obtained from $\bar{\xi}$ and $\bar{\eta}$ by the following two requirements:

(i) $u_{-2} = u_{-1} = u_0 = \xi_0 = a, \qquad b = \xi_{m+1} = u_{M-2} = u_{M-1} = u_M,$
$v_{-2} = v_{-1} = v_0 = \eta_0 = c, \qquad d = \eta_{n+1} = v_{N-2} = v_{N-1} = v_N;$

(ii) for $i = 1, \ldots, m$, the number ξ_i occurs exactly m_i^ξ times in $\bar{u}$ and for $j = 1, \ldots, n$, the number η_j occurs exactly m_j^η times in $\bar{v}$.

For $0 \le i \le M-1$ and $0 \le j \le N-1$, we set $h_i = u_i - u_{i-1}$, $k_j = v_j - v_{j-1}$ and $h_{-1} = h_M = k_{-1} = k_N = 0$. In the whole paper we use the following notations

$$\begin{aligned} \sigma_{i+1} &= \tfrac{h_{i+1}}{h_i + h_{i+1}}, \quad \sigma_i' = \tfrac{h_{i-1}}{h_{i-1} + h_i}, \\ \tau_{j+1} &= \tfrac{k_{j+1}}{k_j + k_{j+1}}, \quad \tau_j' = \tfrac{k_{j-1}}{k_{j-1} + k_j}. \end{aligned} \tag{2}$$

When in (2) we have $\frac{0}{0}$, we set the corresponding value equal to zero.

On the triangulation $\mathcal{T}_{mn}$ we can consider the spline space of all functions s, whose restriction to any triangular cell of $\mathcal{T}_{mn}$ is a polynomial in two variables of total degree two. The smoothness of s is related to the multiplicity of knots in $\bar{u}$ and $\bar{v}$ [4]. Indeed let m_i^ξ (m_j^η) be the multiplicity of ξ_i (η_j), then

$$m_i^\xi \quad (m_j^\eta) \quad + \quad \text{degree of smoothness for } s \text{ crossing the line } u=\xi_i \quad (v=\eta_j) \\ = 2.$$

We call such space $\mathcal{S}_2^{\bar{\mu}}(\mathcal{T}_{mn})$. We can prove [4] that

$$\dim \mathcal{S}_2^{\bar{\mu}}(\mathcal{T}_{mn}) = 8 - mn + m + n + (2+n)\sum_{i=1}^{m} m_i^\xi + (2+m)\sum_{j=1}^{n} m_j^\eta. \tag{3}$$

Now we denote by

$$\mathcal{B}_{MN} = \{B_{ij}(u,v)\}_{(i,j)\in\mathcal{K}_{MN}}, \quad \mathcal{K}_{MN} = \{(i,j) : 0 \le i \le M-1, 0 \le j \le N-1\}, \tag{4}$$

the collection of $M \cdot N$ quadratic B-splines defined in [4], that we know to span $\mathcal{S}_2^{\bar{\mu}}(\mathcal{T}_{mn})$. In $\mathcal{B}_{MN}$ we find different types of B-splines. There are $(M-2)(N-2)$ inner B-splines associated with the set of indices $\widehat{\mathcal{K}}_{MN} = \{(i,j) : 1 \le i \le M-2, 1 \le j \le N-2\}$, whose restrictions to the boundary $\partial\Omega$ of Ω are equal to zero.

To the latter, we add $2M+2N-4$ boundary B-splines, associated with

$$\widetilde{\mathcal{K}}_{MN} := \{(i,0),(i,N-1), 0 \le i \le M-1; (0,j),(M-1,j), 0 \le j \le N-1\},$$

whose restrictions to the boundary of Ω are univariate B-splines [7].

Any B_{ij} in $\mathcal{B}_{MN}$ is given in Bernstein-Bézier form. Its support is obtained from the one of the quadratic C^1 B-spline $\bar{B}_{ij}$, with octagonal support (Fig. 1) [2, 7], by conveniently setting h_i and/or k_j equal to zero in Fig. 1, when there are double (or triple) knots in its support. The B_{ij}'s BB-coefficients different from zero are computed by using Table 1, evaluating the corresponding ones related to the new support [3]. The symbol "O" denotes a zero BB-coefficient.

Since $\bar{u}$ and $\bar{v}$ can have multiple knots, then the B_{ij} smoothness changes and the B-spline support changes as well, because the number of triangular cells on which the function is nonzero is reduced. For example, in Fig. 2 we propose: (a) the graph of a B-spline B_{ij}, with the double knot $v_{j-1} = v_j$, (b) its support with its BB-coefficients different from zero, computed by setting $k_j = 0$ in Fig. 1 and Table 1. Analogously in Figs. 3÷6 we propose some other multiple knot B-splines. In Figs. 2(b)÷6(b) a thin line means that the B-spline is C^1 across it, while a thick line means that the function is continuous across it, but not C^1 and a dotted line means that the function has a jump across it.

All B_{ij}'s are non negative and form a partition of unity.

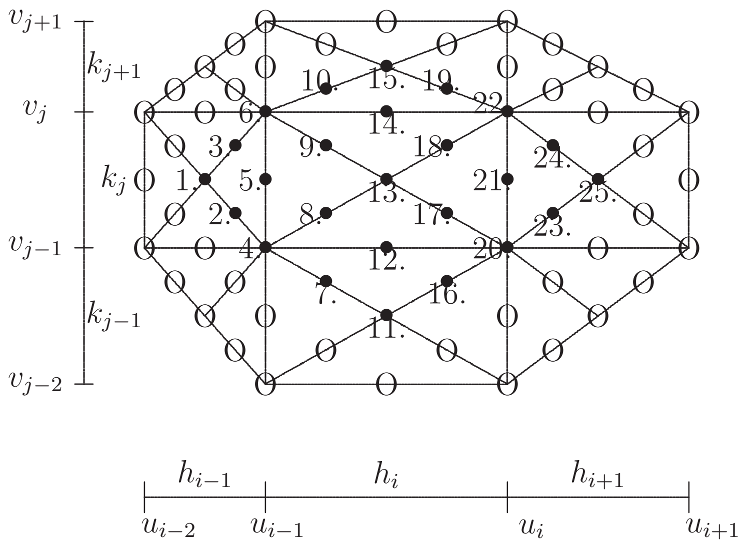


Figure 1: Support of the C^1 B-spline $\bar{B}_{ij}(u,v)$.

1. $\dfrac{\sigma_i'}{4}$,	2. $\dfrac{\sigma_i'}{2}$,	3. $\dfrac{\sigma_i'}{2}$,	4. $\sigma_i'\tau_j'$,	5. σ_i',
6. $\sigma_i'\tau_{j+1}$,	7. $\dfrac{\tau_j'}{2}$,	8. $\dfrac{\sigma_i'+\tau_j'}{2}$,	9. $\dfrac{\sigma_i'+\tau_{j+1}}{2}$,	10. $\dfrac{\tau_{j+1}}{2}$,
11. $\dfrac{\tau_j'}{4}$,	12. τ_j',	13. $\dfrac{\sigma_i'+\sigma_{i+1}+\tau_j'+\tau_{j+1}}{4}$,	14. τ_{j+1},	15. $\dfrac{\tau_{j+1}}{4}$,
16. $\dfrac{\tau_j'}{2}$,	17. $\dfrac{\sigma_{i+1}+\tau_j'}{2}$,	18. $\dfrac{\sigma_{i+1}+\tau_{j+1}}{2}$,	19. $\dfrac{\tau_{j+1}}{2}$,	20. $\sigma_{i+1}\tau_j'$,
21. σ_{i+1},	22. $\sigma_{i+1}\tau_{j+1}$,	23. $\dfrac{\sigma_{i+1}}{2}$,	24. $\dfrac{\sigma_{i+1}}{2}$,	25. $\dfrac{\sigma_{i+1}}{4}$,

Table 1: B-net of the C^1 B-spline $\bar{B}_{ij}(u,v)$.

Since $\sharp\mathcal{B}_{MN} = M\cdot N$, from (3) and (1) it results that $\sharp\mathcal{B}_{MN} > \dim \mathcal{S}_2^{\bar{\mu}}(\mathcal{T}_{mn})$. Therefore the set $\mathcal{B}_{MN}$ is linearly dependent and we can prove [4] that the number of linearly independent B-splines in $\mathcal{B}_{MN}$ coincides with $\dim \mathcal{S}_2^{\bar{\mu}}(\mathcal{T}_{mn})$. Then we can conclude that the algebraic span of $\mathcal{B}_{MN}$ is all $\mathcal{S}_2^{\bar{\mu}}(\mathcal{T}_{mn})$.

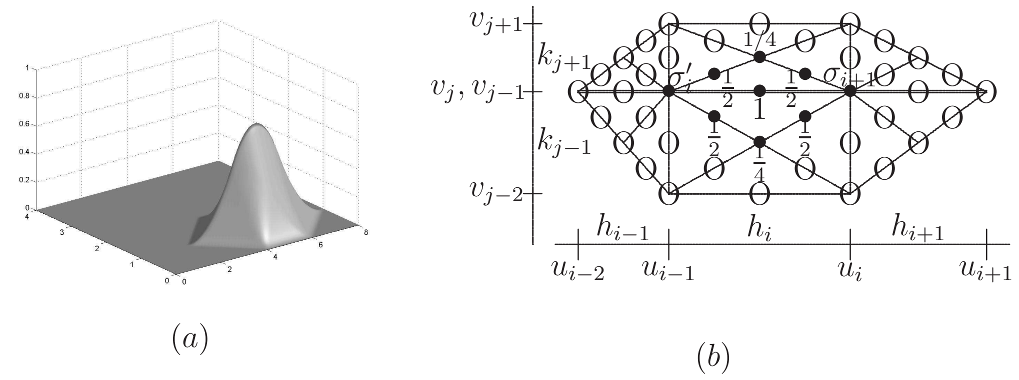


Figure 2: A double knot quadratic C^0 B-spline B_{ij} with $v_{j-1} = v_j$ and its support.

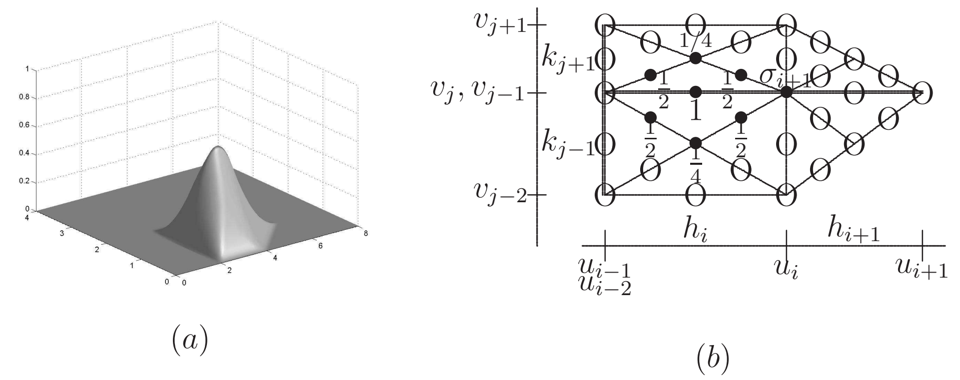


Figure 3: A double knot quadratic C^0 B-spline B_{ij} with $u_{i-2} = u_{i-1}$, $v_{j-1} = v_j$ and its support.

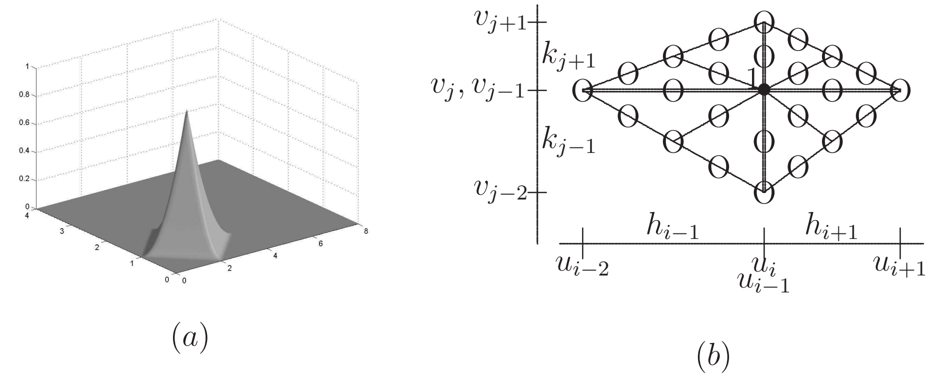


Figure 4: A double knot quadratic C^0 B-spline B_{ij} with $u_{i-1} = u_i$, $v_{j-1} = v_j$ and its support.

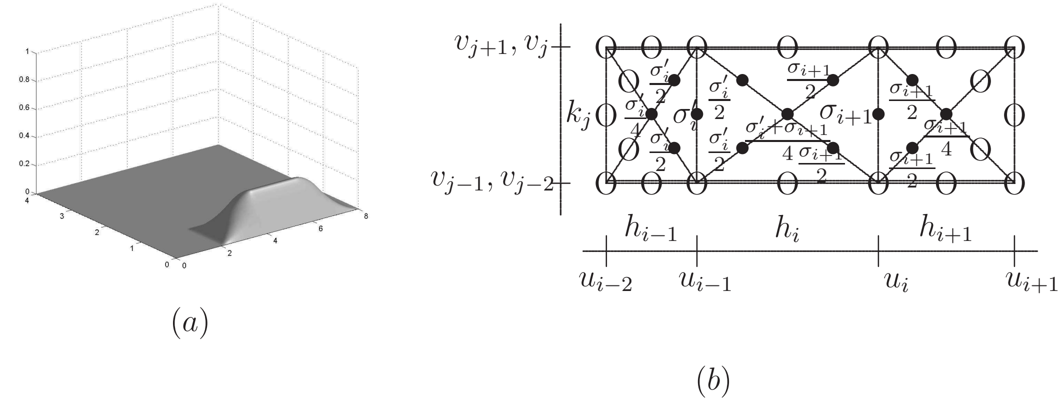


Figure 5: A double knot quadratic C^0 B-spline B_{ij} with $v_{j-2} = v_{j-1}$, $v_j = v_{j+1}$ and its support.

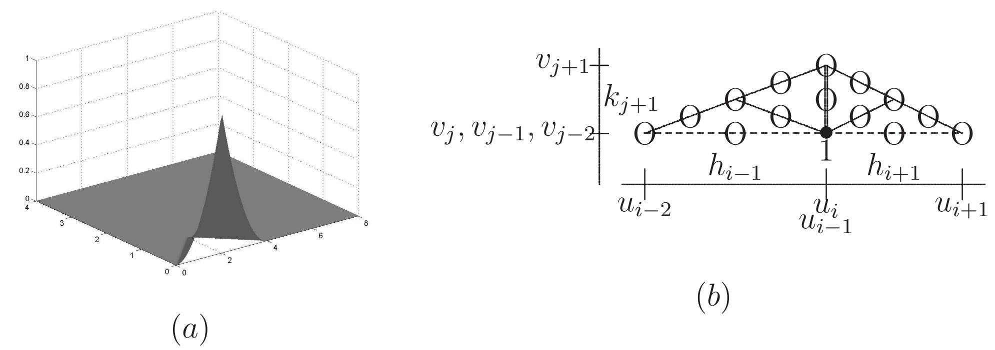


Figure 6: A triple knot quadratic B-spline B_{ij} with $u_{i-1} = u_i$, $v_{j-2} = v_{j-1} = v_j$ and its support.

3 An application to surface generation

In this section we propose an application of the above obtained results to the construction of unequally smooth quadratic B-spline surfaces.

An unequally smooth B-spline surface can be obtained by taking a bidirectional net of control points $\mathbf{P}_{ij}$, two knot vectors $\bar{u}$ and $\bar{v}$ in the parametric domain Ω, as in Section 2, and assuming the B_{ij}'s (4) as blending functions. It has the following form

$$\mathbf{S}(u,v) = \sum_{(i,j)\in\mathcal{K}_{MN}} \mathbf{P}_{ij}\ B_{ij}(u,v), \quad (u,v) \in \Omega. \tag{5}$$

Here we assume $(s_i, t_j) \in \Omega$ as the pre-image of $\mathbf{P}_{ij}$, with $s_i = \frac{u_{i-1}+u_i}{2}$ and $t_j = \frac{v_{j-1}+v_j}{2}$.

We remark that in case of functional parametrization, $\mathbf{S}(u,v)$ is the spline function defined by the well known bivariate Schoenberg-Marsden operator (see e.g. [6, 9]), which is "variation diminishing" and reproduces bilinear functions.

Since the B-splines in $\mathcal{B}_{MN}$ are non negative and satisfy the property of unity partition, the surface (5) has both the convex hull property and the affine transformation invariance one.

Moreover $\mathbf{S}(u,v)$ has C^1 smoothness when both parameters $\bar{u}$ and $\bar{v}$ have no double knots. When both/either $\bar{u}$ and/or $\bar{v}$ have/has double knots, then the surface is only continuous at such knots [8].

Finally, from the B-spline locality property, the surface interpolates both the four points $\mathbf{P}_{00}$, $\mathbf{P}_{M-1,0}$, $\mathbf{P}_{0,N-1}$, $\mathbf{P}_{M-1,N-1}$ and the control points $\mathbf{P}_{ij}$ if both u_i and v_j occur at least twice in $\bar{u}$ and $\bar{v}$, respectively.

Example 1.

We consider a test surface, given by the following functional parametrization:

$$\begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases},$$

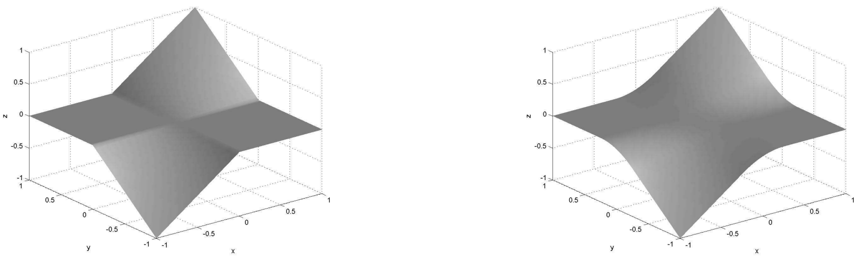
with

$$f(u,v) = \begin{cases} \mid u \mid v & \text{if } uv > 0 \\ 0 & \text{elsewhere} \end{cases}.$$

We assume $\Omega = [-1,1] \times [-1,1]$ as parameter domain and $m = n = 5$. Moreover we set $\bar{\xi} = \{-1, -0.5, -0.25, 0, 0.25, 0.5, 1\}$ and $\bar{\eta} = \bar{\xi}$. We choose $\bar{m}^{\xi} = \{1, 1, 2, 1, 1\}$ and $\bar{m}^{\eta} = \bar{m}^{\xi}$. Therefore we have $M = N = 9$ and

$$\bar{u} = \{-1, -1, -1, -0.5, -0.25, 0, 0, 0.25, 0.5, 1, 1, 1\}, \quad \bar{v} = \bar{u}.$$

In this case $\mathbf{P}_{ij} = f(s_i, t_j)$. The graph of the corresponding surface (5) is reported in Fig. 7(a). It is obtained by evaluating $\mathbf{S}$ on a 55×55 uniform rectangular grid of points in the domain Ω. In Fig. 7(b) we present the quadratic C^1 B-spline surface, obtained if all knots in $\bar{u}$ and $\bar{v}$, inside Ω, are assumed simple.



(a) (b)

Figure 7: $\mathbf{S}$ with double (a) and simple (b) knots at $\xi_3 = \eta_3 = 0$.

We remark how the presence of double knots allows to well simulate a discontinuity of the first partial derivatives across the lines $u = 0$ and $v = 0$.

Example 2.

We want to reconstruct the spinning top in Fig. 8 by a non uniform quadratic B-spline surface (5).

Figure 8: A spinning top.

In order to do it we consider the following control points

$$\mathbf{P}_{00} = \mathbf{P}_{10} = \mathbf{P}_{20} = \mathbf{P}_{30} = \mathbf{P}_{40} = \mathbf{P}_{50} = (0, 0, 0),$$

$$\begin{array}{lll} \mathbf{P}_{01} = (0, \frac{1}{2}, \frac{1}{2}), & \mathbf{P}_{11} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & \mathbf{P}_{21} = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ \mathbf{P}_{31} = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & \mathbf{P}_{41} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & \mathbf{P}_{51} = \mathbf{P}_{01}, \end{array}$$

$$
\begin{array}{lll}
\mathbf{P}_{02} = (0, \frac{3}{4}, \frac{7}{12}), & \mathbf{P}_{12} = (\frac{3}{4}, \frac{3}{4}, \frac{7}{12}), & \mathbf{P}_{22} = (\frac{3}{4}, -\frac{3}{4}, \frac{7}{12}), \\
\mathbf{P}_{32} = (-\frac{3}{4}, -\frac{3}{4}, \frac{7}{12}), & \mathbf{P}_{42} = (-\frac{3}{4}, \frac{3}{4}, \frac{7}{12}), & \mathbf{P}_{52} = \mathbf{P}_{02},
\end{array}
$$

$$
\begin{array}{lll}
\mathbf{P}_{03} = (0, \frac{13}{10}, \frac{5}{6}), & \mathbf{P}_{13} = (\frac{13}{10}, \frac{13}{10}, \frac{5}{6}), & \mathbf{P}_{23} = (\frac{13}{10}, -\frac{13}{10}, \frac{5}{6}), \\
\mathbf{P}_{33} = (-\frac{13}{10}, -\frac{13}{10}, \frac{5}{6}), & \mathbf{P}_{43} = (-\frac{13}{10}, \frac{13}{10}, \frac{5}{6}), & \mathbf{P}_{53} = \mathbf{P}_{03},
\end{array}
$$

$$
\begin{array}{lll}
\mathbf{P}_{04} = (0, 1, 1), & \mathbf{P}_{14} = (1, 1, 1), & \mathbf{P}_{24} = (1, -1, 1), \\
\mathbf{P}_{34} = (-1, -1, 1), & \mathbf{P}_{44} = (-1, 1, 1) & \mathbf{P}_{54} = \mathbf{P}_{04},
\end{array}
$$

$$
\begin{array}{lll}
\mathbf{P}_{05} = (0, \frac{1}{2}, 1), & \mathbf{P}_{15} = (\frac{1}{2}, \frac{1}{2}, 1), & \mathbf{P}_{25} = (\frac{1}{2}, -\frac{1}{2}, 1), \\
\mathbf{P}_{35} = (-\frac{1}{2}, -\frac{1}{2}, 1), & \mathbf{P}_{45} = (-\frac{1}{2}, \frac{1}{2}, 1) & \mathbf{P}_{55} = \mathbf{P}_{05},
\end{array}
$$

$$
\begin{array}{lll}
\mathbf{P}_{06} = (0, \frac{1}{8}, 1), & \mathbf{P}_{16} = (\frac{1}{8}, \frac{1}{8}, 1), & \mathbf{P}_{26} = (\frac{1}{8}, -\frac{1}{8}, 1), \\
\mathbf{P}_{36} = (-\frac{1}{8}, -\frac{1}{8}, 1), & \mathbf{P}_{46} = (-\frac{1}{8}, \frac{1}{8}, 1) & \mathbf{P}_{56} = \mathbf{P}_{06},
\end{array}
$$

$$
\begin{array}{lll}
\mathbf{P}_{07} = (0, \frac{1}{8}, \frac{3}{2}), & \mathbf{P}_{17} = (\frac{1}{8}, \frac{1}{8}, \frac{3}{2}), & \mathbf{P}_{27} = (\frac{1}{8}, -\frac{1}{8}, \frac{3}{2}), \\
\mathbf{P}_{37} = (-\frac{1}{8}, -\frac{1}{8}, \frac{3}{2}), & \mathbf{P}_{47} = (-\frac{1}{8}, \frac{1}{8}, \frac{3}{2}) & \mathbf{P}_{57} = \mathbf{P}_{07},
\end{array}
$$

$$
\begin{array}{lll}
\mathbf{P}_{08} = (0, \frac{1}{8}, 2), & \mathbf{P}_{18} = (\frac{1}{8}, \frac{1}{8}, 2), & \mathbf{P}_{28} = (\frac{1}{8}, -\frac{1}{8}, 2), \\
\mathbf{P}_{38} = (-\frac{1}{8}, -\frac{1}{8}, 2), & \mathbf{P}_{48} = (-\frac{1}{8}, \frac{1}{8}, 2) & \mathbf{P}_{58} = \mathbf{P}_{08},
\end{array}
$$

$$
\mathbf{P}_{09} = \mathbf{P}_{19} = \mathbf{P}_{29} = \mathbf{P}_{39} = \mathbf{P}_{49} = \mathbf{P}_{59} = (0, 0, 2),
$$

defining the control net in Fig. 9. Here $M = 6$ and $N = 10$.

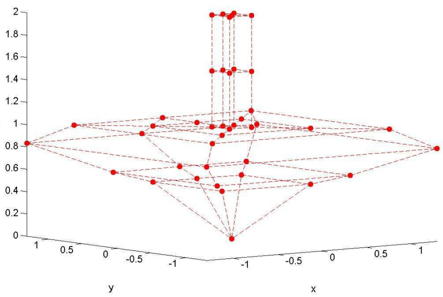
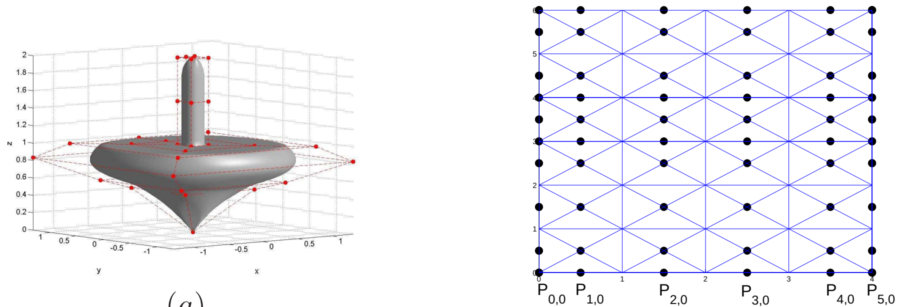


Figure 9: The control net corresponding to $\{\mathbf{P}_{ij}\}_{(i,j)\in\mathcal{K}_{6,10}}$.

Then, to well model our object, we assume $\bar{u} = \{0, 0, 0, 1, 2, 3, 4, 4, 4\}$ and $\bar{v} = \{0, 0, 0, 1, 2, 3, 3, 4, 4, 5, 6, 6, 6\}$. The graph of the B-spline surface of type (5) is reported in Fig. $10(a)$, while in Fig. $10(b)$ the corresponding criss-cross triangulation of the parameter domain is given.



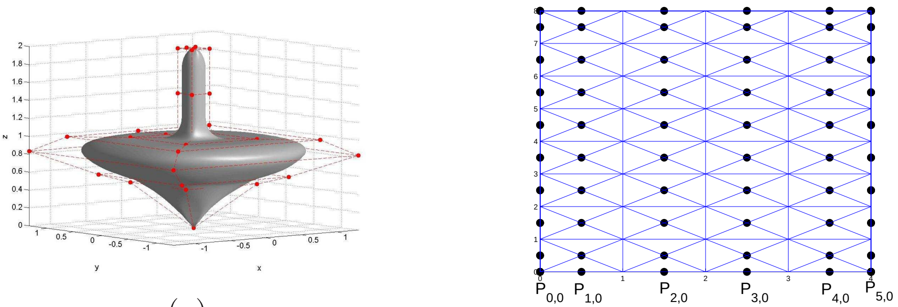
(a) (b)

Figure 10: The surface $\mathbf{S}(u, v)$ with double knots in $\bar{v}$ and its parameter domain.

In Fig. 11 we present the quadratic C^1 B-spline surface based on the same control points and obtained if all knots in $\bar{u}$ and $\bar{v}$, inside Ω, are assumed simple, i.e.

$$\bar{u} = \{0, 0, 0, 1, 2, 3, 4, 4, 4\}, \quad \bar{v} = \{0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 8, 8\}.$$

In Fig. 12(a) and (b) the effects of multiple knots are emphasized. We



(a) (b)

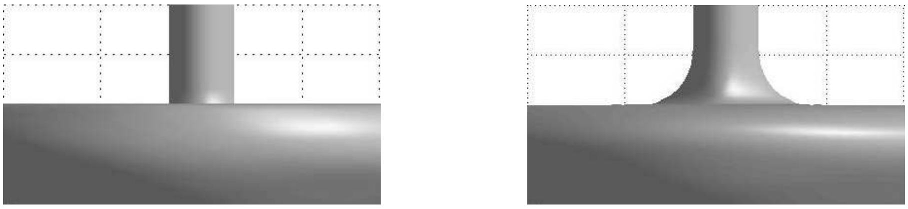
Figure 11: The surface $\mathbf{S}(u, v)$ with simple knots inside Ω and its parameter domain.

remark that in such a way we can better model the real object.

The construction of the B-spline basis and the B-spline surfaces has been realized by Matlab codes.

4 Conclusions

In this paper we have presented some results on the dimension of the unequally smooth spline space $\mathcal{S}_2^{\bar{\mu}}(\mathcal{T}_{mn})$ and on the construction of a B-spline



(a) (b)

Figure 12: In (a) zoom of Fig. 10(a) and in (b) zoom of Fig. 11(a).

basis with different types of smoothness.

We plan to use these results in the construction of blending functions for multiple knot NURBS surfaces with a criss-cross triangulation as parameter domain. Moreover such results could be also applied in reverse-engineering techniques, by using surfaces based on spline operators reproducing higher degree polynomial spaces [6, 9].

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