On the boundedness of solutions to a nonlinear singular oscillator

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On the boundedness of solutions to a nonlinear singular oscillator *

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1 Introduction

In this paper we are concerned with a second order scalar equation of the form

$$x'' + V'(x) = p(t), \tag{1.1}$$

where p is a π-periodic function and, for $x > -1$,

$$V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\gamma} - 1, \tag{1.2}$$

being γ is a positive integer.

Our main result consists of proving that all solutions to (1.1) are bounded; moreover, we deal, in the more general case when $\gamma > 2$ is any real number, with the existence of Aubry-Mather sets for (1.1).

These two problems have been considered by various authors in the last years.

The question of the boundedness of all solutions is the famous Littlewood problem. It has been studied, among others, by D. Bonheure-C. Fabry [2], M. Levi [6], R. Ortega [12] and the third author [7] in the case when, instead of (1.2), a regular potential is treated. In particular, attention has been devoted to the asymmetric resonant potential $W(x) = \frac{a}{2}x_+^2 + \frac{b}{2}x_-^2$, being $1/\sqrt{a} + 1/\sqrt{b} = 2/n$ for some $n \in \mathbf{N}$. In this framework, we refer to the papers by R. Ortega [11] and the third author [8],[9].

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In the above quoted results, the main tool for the proofs is Moser twist theorem, or its variant given by R. Ortega in [12].

The question whether an equation of the form (1.1) has solutions of Mather type is interesting as well, since Mather sets (and the knowledge of their rotation number) provide a rather complete qualitative description of the dynamics of (1.1). In this framework, together with the pioneering work of J. Moser [10], we refer in particular to the paper [13] by M.L. Pei, who gave a sufficient condition for the existence of Aubry-Mather sets for some planar maps.

Our results may be considered of some interest since, to our knowledge, it is the first time that the question of the boundedness of all solutions (as well as the existence of Aubry-Mather sets) is treated in case of a singular potential. As far as the existence of periodic solutions (of fixed period) is concerned, for the case of singular potentials we refer to the the work by P.J. Torres [14]. It has to be pointed out that the singularites considered in [14] (and references therein) are different from those we consider in this paper. Finally, it is worth mentioning that in the framework of Moser twist theorem it is announced in the Introduction of the paper [6] the possibility of treating singular potentials, which are different from (1.2).
Concerning the choice of the potential V, we point out that the crucial estimates in Lemma 2.1 and Lemma 2.2 are valid for a class of functions satisfying some regularity and growth conditions that we state at the end of Section 4. In particular, we deal with the case when $\lim_{x\to+\infty} V(x)/x^2 = n^2/2$, for some $n \in \mathbf{N}$; this means we consider potentials which are "asymptotically resonant" at $+\infty$. We decided to treat a potential of the form (1.2) since it represents a typical example where these conditions are easily obtained.

We point out that our main results (Theorem 4.1 and Theorem 4.2) are proved under the assumption that

$$1 + \frac{1}{2}\int_0^\pi p\,(t_0+\theta)\sin\theta\,d\theta > 0, \quad \forall\ t_0 \in \mathbf{R}. \tag{1.3}$$

This condition obviously holds in case $p \equiv 0$, p "small" and also in case the function $(1/2)\int_0^\pi p\,(t_0+\theta)\sin\theta\,d\theta$ vanishes at some point. For comments on (1.3) in relation with the Lazer-Leach condition, we refer to Remark 4.3.

In case hypothesis (1.3) fails, then the existence of unbounded solutions can be proved on the lines of the work [1] by J.M. Alonso and R. Ortega. We refer to Remark 4.4 for more details on this subject.

The proofs of our results consist of two main steps. First, equation (1.1) is written as an equivalent planar hamiltonian system, using

$$H(x,y,t) = \frac{1}{2}y^2 + \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\gamma} - 1 - xp(t).$$

Then, new variables of "area-angle" type are introduced. The main task consists of developing careful estimates on the Poincaré map in order to apply Ortega's variant of Moser twist theorem and Pei's theorem on the existence of Aubry-Mather sets. It is clear that the singularity of the potential V represents a very serious difficulty.

More precisely, we need to estimate (among others) the n-th derivative ($n = 0, \dots, 5$) of the time-map

$$T_-(h) = 2\int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(s))}}\, ds,$$

where, for a fixed h, the number $\alpha_h \in (-1, 0)$ is such that $V(-\alpha_h) = h$ (cf. Lemma 2.1). Moreover, the map

$$I_-(h) = 2\int_0^{\alpha_h} \sqrt{2(h - V(s))}\, ds,$$

together with its derivatives, has to be estimated as well. For these estimates, we have borrowed some techniques from the Appendix of the paper by M. Levi [6].

2 Preliminary lemmata

Let us consider the second order equation

$$x'' + V'(x) = p(t), \tag{2.1}$$

where p is a π-periodic function, γ is a positive integer and

$$V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1 - x_-^2)^\gamma} - 1, \quad \forall\, x > -1.$$

2.1. Definition of the time and area maps.

For every $h > 0$ we denote by $I_0(h)$ the area enclosed by the (closed) curve

$$\frac{1}{2}y^2 + V(x) = h.$$

Let $-1 < -\alpha_h < 0 < \beta_h$ be such that

$$V(-\alpha_h) = V(\beta_h) = h.$$

It is easy to see that

$$I_0(h) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2(h - V(s))}\, ds, \quad \forall\, h > 0. \tag{2.2}$$

By means of a simple computation we get

$$I_0(h) = 2\int_0^{\beta_h} \sqrt{2(h - V(s))}\, ds + 2\int_{-\alpha_h}^0 \sqrt{2(h - V(s))}\, ds = \pi h + 2\int_0^{\alpha_h} \sqrt{2(h - V(-s))}\, ds.$$

For every $h > 0$ let

$$I_-(h) = 2\int_0^{\alpha_h} \sqrt{2(h - V(-s))}\, ds. \tag{2.3}$$

We then have

$$I_0(h) = \pi h + I_-(h), \quad \forall\ h > 0. \tag{2.4}$$

Moreover, let

$$T_0(h) = I_0'(h) = \pi + 2\int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}\, ds, \tag{2.5}$$

and

$$T_-(h) = 2\int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}\, ds.$$

Note also that $T^{(n)}(h) = I^{(n+1)}(h)$, for all $n \geq 0$.

2.2. Estimates on the maps I_- and T_- and their derivatives.

The following lemma (whose proof is given in the Appendix) is crucial for the sequel.

Lemma 2.1 *For every $n = 0, 1, 2, \ldots, 5$ we have*

$$\frac{d^n T_-}{dh^n}(h) = (-1)^n\ \frac{1}{2^n}\ (2n-1)!!\ \frac{\sqrt{2}}{h^{(2n+1)/2}} + o\left(\frac{1}{h^{(2n+1)/2}}\right), \quad h \to +\infty. \tag{2.6}$$

As a consequence of Lemma 2.1, we can obtain that

$$\left| h^k \frac{d^k I_-(h)}{dh^k} \right| \leq C_0 I_-(h), \quad \text{for } k \geq 1. \tag{2.7}$$

Moreover, for $h = h_0(I_0)$ (the inverse function of I_0), it follows that

$$\left| I^k \frac{d^k h(I)}{dI^k} \right| \leq C_0 h(I), \quad \text{for } k \geq 1, \tag{2.8}$$

$$\left| h^k \frac{d^k T_0(h)}{dh^k} \right| \leq C_0 T_0(h), \quad \left| h^k \frac{d^k T_-(h)}{dh^k} \right| \leq C_0 T_-(h), \quad \text{for } k \geq 1. \tag{2.9}$$

$$\left| I^k \frac{d^k T_0(h(I))}{dI^k} \right| \leq C_0 T_0(h(I)), \quad \left| I^k \frac{d^k T_-(h(I))}{dI^k} \right| \leq C_0 T_-(h(I)), \quad \text{for } k \geq 1. \tag{2.10}$$

We observe that, in particular, there exist positive constants $c_0, C_0, c_0', C_0', c_0'', C_0''$ such that

$$c_0\sqrt{h} \leq I_-(h) \leq C_0\sqrt{h}, \tag{2.11}$$

$$\frac{c_0'}{\sqrt{h}} \leq T_-(h) \leq \frac{C_0'}{\sqrt{h}}, \tag{2.12}$$

$$\frac{c_0''}{h\sqrt{h}} \leq T_-'(h) \leq \frac{C_0''}{h\sqrt{h}}, \tag{2.13}$$

for h sufficiently large.

2.3. Some useful changes of variables.

Let us observe that (2.1) can be written as a Hamiltonian system of the form

$$\begin{cases} x' = \dfrac{\partial H}{\partial y} \\ \\ y' = -\dfrac{\partial H}{\partial x}, \end{cases} \tag{2.14}$$

where

$$H(x,y,t) = \frac{1}{2}y^2 + \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\gamma} - 1 - xp(t).$$

For every $(x,y) \in (-1,+\infty) \times \mathbf{R}$, let us define (θ, I) by

$$I = I_0(h(x,y)) \tag{2.15}$$

and

$$\theta(x,y) = \begin{cases} \dfrac{\pi}{T_0(h(x,y))}\left(\frac{T_-(h)}{2} + \arcsin\dfrac{x}{\sqrt{2h(x,y)}}\right) & \text{if } x>0,\ y>0 \\ \\ \dfrac{\pi}{T_0(h(x,y))}\left(\frac{T_-(h)}{2} + \pi - \arcsin\dfrac{x}{\sqrt{2h(x,y)}}\right) & \text{if } x>0,\ y<0 \\ \\ \dfrac{\pi}{T_0(h(x,y))}\left(\displaystyle\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y)+1-(1-s^2)^{-\gamma})}}\,ds\right) & \text{if } x<0,\ y>0 \\ \\ \dfrac{\pi}{T_0(h(x,y))}\left(T_0(h(x,y)) - \displaystyle\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y)+1-(1-s^2)^{-\gamma})}}\,ds\right) & \text{if } x<0,\ y<0, \end{cases} \tag{2.16}$$

where

$$h(x,y) = \frac{1}{2}y^2 + V(x).$$

In the new variables (θ, I) system (2.14) becomes

$$\begin{cases} \theta' = \dfrac{\partial H}{\partial I} \\ \\ I' = -\dfrac{\partial H}{\partial \theta}, \end{cases} \tag{2.17}$$

where

$$H(\theta, I, t) = \pi h_0(I) - \pi x(I, \theta) p(t). \tag{2.18}$$

Before stating the next lemma, we observe (and refer to the Appendix for details) that we can write

$$\Big(\frac{V(x)}{V'(x)}\Big)' = 1 - \phi(x), \quad \phi(x) := \frac{1}{2\gamma} \; \frac{(1-(1-x^2)^\gamma)(1+(2\gamma+1)x^2)}{x^2}.$$

Lemma 2.2 *For I sufficient large, the following estimates hold:*

$$\left| I^k \frac{\partial^k x(I,\theta)}{\partial I^k} \right| \le c\sqrt{I} \quad \text{for} \quad 0 \le k \le 6.$$

Proof. For $x > 0$, by the definition of θ, we have

$$x = \sqrt{2h} \sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_0(h)}{2} \right).$$

It is a direct computation to prove that, for $x > 0$, the Lemma holds true. Hence, it is sufficient to prove that, for $x < 0$,

$$\left| I^k \frac{\partial^k x}{\partial I^k} \right| \le C(1 + x), \quad \text{for} \quad 1 \le k \le 6. \tag{2.19}$$

- $k = 1$.

From the definition of θ, we have

$$T_0(h)\frac{\theta}{2\pi} = \int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h - V(s))}} ds, \quad \text{for } x < 0, y > 0.$$

Take the derivative with respect to the action variable I in both sides of the above equality (the angle variable θ is independent on I)

$$T_0'(h) h_I \frac{\theta}{2\pi} = \frac{\partial}{\partial I} \int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h - V(s))}} ds.$$

As what was done in the paper of Levi [6], we may get

$$\frac{\partial}{\partial I}\int_{-\alpha_h}^{x}\frac{ds}{\sqrt{2(h-V)}}=\frac{1}{\sqrt{2(h-V(x))}}\left(x_I-\frac{h_I}{h}\frac{V(x)}{V'(x)}\right)+\frac{h_I}{h}\int_{-\alpha_h}^{x}\left(\frac{1}{2}-\phi(s)\right)\frac{ds}{\sqrt{2(h-V)}},$$

which yields that

$$x_I=\sqrt{2(h-V(x))}\left[T_0'(h)h_I\frac{\theta}{2\pi}-\frac{h_I}{h}\int_{-\alpha_h}^{x}\left(\frac{1}{2}-\phi(s)\right)\frac{1}{\sqrt{2(h-V(s))}}ds\right]+\frac{h_I}{h}\frac{V(x)}{V'(x)}. \quad (2.20)$$

By the definition of θ, we have

$$T_0'(h)h_I\frac{\theta}{2\pi}-\frac{h_I}{h}\int_{-\alpha_h}^{x}\left(\frac{1}{2}-\phi(s)\right)\frac{1}{\sqrt{2(h-V(s))}}ds=\frac{h_I}{h}\left[\int_{-\alpha_h}^{x}\left(\frac{hT_0'(h)}{T_0(h)}-\frac{1}{2}+\phi(s)\right)\frac{1}{\sqrt{2(h-V(s))}}ds\right).$$

Let

$$F(x,I)=\int_{-\alpha_h}^{x}\left(\frac{hT_0'(h)}{T_0(h)}-\frac{1}{2}+\phi(s)\right)\frac{1}{\sqrt{2(h-V(s))}}ds. \quad (2.21)$$

Then

$$x_I=\sqrt{2(h-V(x))}\frac{h_I}{h}F(x,I)+\frac{h_I}{h}\frac{V(x)}{V'(x)}. \quad (2.22)$$

Now, it is useful to observe that since

$$\frac{V(x)}{V'(x)}=\frac{1-(1-x^2)^\gamma)}{2\gamma x}\cdot(1-x)(1+x),$$

there is a constant c_0 such that

$$\left|\frac{V(x)}{V'(x)}\right|\le c_0(1+x),\quad \text{for } x\in(-1,0).$$

Because $|h_I/h|\le c'I^{-1}$ and $\alpha_h\le 1$, it is enough to prove that there is a constant c_2 such that

$$\sqrt{2(h-V(x))}|F(x,I)|\le c_2(\alpha_h+x). \quad (2.23)$$

Let

$$G(x,I)=\frac{\alpha_h+x}{\sqrt{2(h-V(x))}}.$$

Then there is a large constant c_2 such that

$$-c_2G(x,I)\le F(x,I)\le c_2G(x,I).$$

Indeed, note that since

$$F(-\alpha_h,I)=G(-\alpha_h,I)=0,$$

it is enough to prove that

$$-c_2\frac{\partial G}{\partial x}(x,I) \le \frac{\partial F}{\partial x}(x,I) \le c_2\frac{\partial G}{\partial x}(x,I).$$

However, from (2.9) it follows that there is a constant c_2' such that

$$\left|\frac{\partial F}{\partial x}(x,I)\right| = \left|\left(\frac{hT_0'(h)}{T_0(h)} - \frac{1}{2} + \phi(x)\right)\frac{1}{\sqrt{2(h-V(x))}}\right| \le c_2'\frac{1}{\sqrt{2(h-V(x))}}.$$

By a direct computation, we obtain

$$\frac{\partial G}{\partial x}(x,I) = \left(1 + \frac{V'(x)(\alpha_h + x)}{2(h-V(x))}\right)\cdot\frac{1}{\sqrt{2(h-V(x))}}.$$

¿From $V''(x) > 0$ and $h = V(-\alpha_h)$, it follows that

$$\left|\frac{V'(x)(\alpha_h + x)}{2(h-V(x))}\right| \le \frac{1}{2},$$

so for $c_2 > 2c_2'$, we may get

$$-c_2\frac{\partial G}{\partial x}(x,I) \le \frac{\partial F}{\partial x}(x,I) \le c_2\frac{\partial G}{\partial x}(x,I).$$

Note that $\alpha_h < 1$ and if we choose

$$c_1 = (c_2 + c_0)c'$$

then we get the estimate (2.19) for $k = 1$.

- $k = 2$.

First we introduce an operator $\mathcal{L}$ as follows. For a function $f(x,I)$, define

$$\mathcal{L}(f) = \frac{h_I}{h}\left[\left(f\frac{V}{V'}\right)_x - \frac{1}{2}f\right] + f_I. \tag{2.24}$$

Then

$$x_I = \sqrt{2(h-V(x))}\int_{-\alpha_h}^{x}\left(\frac{T_0'(h)h_I}{T_0(h)} - \mathcal{L}(1)\right)\frac{1}{\sqrt{2(h-V(s))}}ds + \frac{h_I}{h}\cdot\frac{V(x)}{V'(x)}.$$

The following equality (whose proof can be found in [6]) is crucial for the sequel:

$$\frac{d}{dI}\int_{-\alpha_h}^{x} g(s,I)\frac{1}{\sqrt{2(h(I)-V(s))}}ds = \int_{-\alpha_h}^{x}\mathcal{L}(g)\frac{1}{\sqrt{2(h(I)-V(s))}}ds + g(x,I)\frac{h_I}{h}F(x,I). \tag{2.25}$$

We now state the important

CLAIM 1. Suppose that the function $g(x,I)$ is continuous and there is a constant c_0 such that $|g(x,I)| \le c_0 I^{-k}$, for some $k \in \mathbf{N}$. Then we can find a constant c_1 such that, for $-\alpha_h \le x \le 0$,

$$\sqrt{2(h-V(x))}\left|\int_{-\alpha_h}^{x} g(s,I)\frac{1}{\sqrt{2(h-V(s))}}ds\right| \le c_1 I^{-k}(\alpha_h + x). \tag{2.26}$$

In particular,

$$|\sqrt{2(h-V(x))}\cdot F(x,I)| \le c(\alpha_h + x). \tag{2.27}$$

The proof of this claim is just like what we do in the proof of the estimate on x_I.

We prove now (2.19) for $k=2$. Let $g_1(x,I) = I^k g(x,I)$. Then the above inequality (2.26) is equivalent to the following

$$\sqrt{2(h-V(x))}\left|\int_{-\alpha_h}^{x} g_1(s,I)\frac{1}{\sqrt{2(h-V(s))}}ds\right| \le c_1(\alpha_h + x).$$

From (2.22) it follows that

$$\frac{d}{dI}\sqrt{2(h-V(x))} = \frac{h_I}{h}\left(\frac{1}{2}\sqrt{2(h-V(x))} - V'(x)F\right). \tag{2.28}$$

Hence, differentiation in both sides of (2.22) with respect to I gives

$$\begin{aligned} x_{II} \;&=\; \frac{h_I^2}{h^2}\left(\frac{1}{2}\sqrt{2(h-V(x))} - V'(x)F(x,I)\right)F(x,I) + \frac{hh_{II}-h_I^2}{h^2}\left(\sqrt{2(h-V(x))}F(x,I) + \frac{V(x)}{V'(x)}\right) \\ &\quad + \frac{h_I}{h}\left[\sqrt{2(h-V(x))}\cdot\frac{d}{dI}F(x,I) + \left(1 - \frac{V(x)V''(x)}{(V'(x))^2}\right)x_I\right]. \end{aligned}$$

¿From (2.21), (2.25) and (2.26), it suffices to prove the following estimates for obtaining (2.19) for $k=2$:

$$|V'(x)F(x,I)| \le c\cdot\sqrt{2(h-V(x))}, \tag{2.29}$$

$$\left|\sqrt{2(h-V(x))}\cdot\frac{d}{dI}F(x,I)\right| \le c\cdot I^{-1}(\alpha_h + x). \tag{2.30}$$

Indeed, if the above inequalities hold, then we have

$$\begin{aligned} |x_{II}| \;&\le\; cI^{-2}|\sqrt{2(h-V(x))}F(x,I)| + cI^{-2}\left(|\sqrt{2(h-V(x))}F(x,I)| + (1+x)\right) \\ &\quad + cI^{-1}\left[|\sqrt{2(h-V(x))}\cdot\frac{d}{dI}F(x,I)| + I^{-1}(1+x)\right] \le cI^{-2}(1+x). \end{aligned}$$

Note that here the constant c in the different lines has different quantities. However, these constants are independent of I and x.

The proof of (2.29).

Let

$$f(x,I)=\frac{T_0'(h)h}{T_0(h)}-\frac{1}{2}+\phi(x). \tag{2.31}$$

It is enough to prove that there is a constant $c>0$ such that

$$c\cdot\frac{\sqrt{2(h-V(x))}}{V'(x)}\le F(x,I)\le -c\cdot\frac{\sqrt{2(h-V(x))}}{V'(x)}.$$

Arguing as in the proof of the estimate of x_I, it suffices to prove that

$$-c\cdot\frac{\frac{(V'(x))^2}{\sqrt{2(h-V(x))}}+\sqrt{2(h-V(x))}\cdot V''(x)}{(V'(x))^2}\le\frac{f(x,I)}{\sqrt{2(h-V(x))}}\le c\cdot\frac{\frac{(V'(x))^2}{\sqrt{2(h-V(x))}}+\sqrt{2(h-V(x))}\cdot V''(x)}{(V'(x))^2}. \tag{2.32}$$

Since $T_0'(h)h/T_0(h)$ is bounded, ϕ is continuous in the interval $[-1,0]$ (cf. Appendix), there is a constant $c'>0$ such that $|f(x,I)|\le c'$. Choose $c=2c'$. Then (2.32) holds, which yields (2.29).

Indeed, we can prove the following fact (which contains (2.29) as a particular case) that will be used later.

CLAIM 2. Suppose that there is a constant c_0 such that $|g(x,I)|\le c_0I^{-k}$, then one may find a constant $c>0$ such that, for $-\alpha_h\le x\le 0$,

$$\left|V'(x)\int_{-\alpha_h}^{x}g(s,I)\frac{1}{\sqrt{2(h-V(s))}}ds\right|\le cI^{-k}\sqrt{2(h-V(x))}. \tag{2.33}$$

The proof of (2.30).
By (2.25) and the definition of $f(x,I)$, we have

$$\frac{d}{dI}F(x,I)=\frac{d}{dI}\int_{-\alpha_h}^{x}f(s,I)\frac{1}{\sqrt{2(h-V)}}ds=\int_{-\alpha_h}^{x}\mathcal{L}(f)\frac{1}{\sqrt{2(h-V)}}ds+f(x,I)\frac{h_I}{h}F(x,I). \tag{2.34}$$

Note that

$$\frac{\partial f(x,I)}{\partial x}=\phi_x$$

and

$$\frac{\partial f(x,I)}{\partial I}=\frac{h_I}{h}\cdot\frac{T_0T_0''h^2+T_0T_0'h-(T_0'h)^2}{T_0^2};$$

by (2.8), (2.9), the definitions of $\mathcal{L}$ and $f(t,x)$, there is a constant $c' > 0$ such that

$$|\mathcal{L}(f)| \leq c' I^{-1}.$$

An application of Claim 1 shows that

$$\Big|\sqrt{2(h-V(x))}\int_{-\alpha_h}^{x} \mathcal{L}(f)\frac{1}{\sqrt{2(h-V)}}ds\Big| \leq c\, I^{-1}(\alpha_h + x).$$

Moreover, using the fact that f is bounded, inequality $|h_I/h| \leq c\, I^{-1}$, and (2.27), we get

$$\Big|\sqrt{2(h-V(x))}\int_{-\alpha_h}^{x} f(x,I)\frac{h_I}{h}F(x,I)ds\Big| \leq c\, I^{-1}(\alpha_h + x).$$

This proves (2.30).

- $k = 3$.

Let

$$h_1(I) = \frac{h_I^2}{h^2}, \quad h_2(I) = \frac{hh_{II} - h_I^2}{h^2}, \quad h_3(I) = \frac{h_I}{h}$$

and

$$\begin{aligned} f_1(x,I) &= \left(\frac{1}{2}\sqrt{2(h-V(x))} - V'(x)F(x,I)\right)F(x,I) \\ f_2(x,I) &= \left(\sqrt{2(h-V(x))}F(x,I) + \frac{V(x)}{V'(x)}\right) \\ f_3(x,I) &= \sqrt{2(h-V(x))}\cdot\frac{d}{dI}F(x,I) + \left(1 - \frac{V(x)V''(x)}{(V'(x))^2}\right)x_I. \end{aligned}$$

Then

$$x_{II} = h_1(I)f_1(x,I) + h_2(I)f_2(x,I) + h_3(I)f_3(x,I).$$

Moreover, from the discussions in the proof for the case $k = 2$, we have

$$|f_1(x,I)| \leq c(1+x), \quad |f_2(x,I)| \leq c(1+x), \quad |f_3(x,I)| \leq cI^{-1}(1+x). \tag{2.35}$$

By the estimates (2.8) and (2.9), it is easy to verify that

$$|h_1|,\ |h_2| \leq cI^{-2}, \quad |h_3| \leq cI^{-1}, \quad \left|\frac{dh_1(I)}{dI}\right|,\ \left|\frac{dh_2(I)}{dI}\right| \leq cI^{-3}, \quad \left|\frac{dh_3(I)}{dI}\right| \leq cI^{-2}. \tag{2.36}$$

Because of (2.35) and (2.36) and

$$\frac{\partial^3 x}{\partial I^3} = \sum_{j=1}^{3}\left(\frac{dh_j(I)}{dI}f_j(x,I) + h_j(I)\cdot\frac{d}{dI}f_j(x,I)\right),$$

it suffices to prove the following results for obtaining the estimate on x_{III}:

$$\left|\frac{d}{dI}f_1(x,I)\right| \le cI^{-1}(1+x), \tag{2.37}$$

$$\left|\frac{d}{dI}f_2(x,I)\right| \le cI^{-1}(1+x), \tag{2.38}$$

$$\left|\frac{d}{dI}f_3(x,I)\right| \le cI^{-2}(1+x). \tag{2.39}$$

The proof of (2.37).

From the definition of f_1 and (2.28), it follows that

$$\frac{d}{dI}f_1(x,I) = \left(\frac{h_I}{2h}\left(\frac{1}{2}\sqrt{2(h-V)} - V'F\right) - V''x_IF - V'\frac{d}{dI}F\right)F + \left(\frac{1}{2}\sqrt{2(h-V)} - V'F\right)\frac{d}{dI}F.$$

¿From (2.8), (2.23), (2.29), (2.30), and the estimate on x_I, it follows that

$$\begin{aligned} |\frac{d}{dI}f_1(x,I)| &\le c\left(I^{-1}\sqrt{2(h-V(x))}|F| + I^{-1}|V''(x)(1+x)F(x,I)|F(x,I) + \sqrt{2(h-V(x))}|\frac{d}{dI}F(x,I)|\right) \\ &\le cI^{-1}(1+x) + I^{-1}|V''(x)(1+x)F(x,I)|F(x,I). \end{aligned}$$

Now we estimate the last term in the above inequality.

From the definition of V, there is a constant $c>0$ such that

$$|V''(x)(1+x)| \le c\left(|V'(x)| + (1+x)\right).$$

Indeed, if $x\in[-1/2,0]$, we know that $|V''(x)|\le c$, while for $x\in[-1,-1/2]$, $|V''(x)(1+x)| \le c|V'(x)|$.

From this inequality, it follows that

$$\begin{aligned} I^{-1}|V''(x)(1+x)F(x,I)|F(x,I) &\le cI^{-1}|V'(x)F(x,I)|F(x,I) + cI^{-1}(1+x)(F(x,I))^2 \\ &\le cI^{-1}\sqrt{h-V(x)}F(x,I) + cI^{-1}(1+x) \\ &\le cI^{-1}(1+x), \end{aligned}$$

where we have used (2.29), (2.26) and $|F(x,I)|\le c$. The proof of (2.37) is completed.

The proof of (2.38) follows from (2.26), (2.28), (2.29), (2.30) and a direct computation.

The proof of (2.39).

First note that

$$\begin{aligned} \frac{d}{dI}f_3(x,I) &= \frac{h_I}{2h}\left(\frac{1}{2}\sqrt{2(h-V)} - V'F\right)\cdot\frac{d}{dI}F(x,I) \\ &\quad + \sqrt{2(h-V(x))}\cdot\frac{d^2}{dI^2}F(x,I) + (1-\phi(x))x_{II} - \phi'(x)(x_I)^2. \end{aligned}$$

In order to get (2.39), it suffices to show the following estimate:

$$\left|\sqrt{2(h-V(x))}\cdot\frac{d^2}{dI^2}F(x,I)\right|\leq cI^{-2}(1+x).$$

¿From (2.34) and (2.25), we have

$$\frac{d^2}{dI^2}F(x,I)=\int_{\alpha_h}^{x}\mathcal{L}(\mathcal{L}(f))\frac{1}{\sqrt{2(h-V(s))}}ds+\mathcal{L}(f)\frac{h_I}{h}F(x,I)+\frac{d}{dI}\left(f(x,I)\frac{h_I}{h}F(x,I)\right). \quad (2.40)$$

By the definitions of $\mathcal{L}$ and $f(x,I)$, it is easy to verify that

$$|\mathcal{L}(\mathcal{L}(f(x,I)))|\leq cI^{-2},\quad |\mathcal{L}(f(x,I))|\leq cI^{-1}.$$

Inequality (2.26) implies the following estimates

$$\left|\sqrt{2(h-V(x))}\left(\int_{\alpha_h}^{x}\mathcal{L}(\mathcal{L}(f))\frac{1}{\sqrt{2(h-V(s))}}ds+\mathcal{L}(f)\frac{h_I}{h}F(x,I)\right)\right|\leq cI^{-2}(1+x),$$

for $-1\leq x\leq 0$. Now we estimate the last term in (2.40). ¿From the definition of $f(x,I)$, we have

$$\begin{aligned}\frac{d}{dI}\left(f(x,I)\frac{h_I}{h}F(x,I)\right) &= \left(\phi'(x)x_I+\left[\frac{h^2T_0''}{T_0}+\frac{hT_0'}{T_0}-\left(\frac{hT_0'}{T_0}\right)^2\right]\frac{h_I}{h}\right)\frac{h_I}{h}F(x,I)\\ &\quad +f(x,I)\frac{h_{II}h-h_I^2}{h^2}F(x,I)+f(x,I)\frac{h_I}{h}\cdot\frac{d}{dI}F(x,I).\end{aligned}$$

¿From (2.23) and (2.30), it follows that, for $-1\leq x\leq 0$,

$$\left|\sqrt{2(h-V(x))}\cdot\frac{d}{dI}\left(f(x,I)\frac{h_I}{h}F(x,I)\right)\right|\leq cI^{-2}(1+x).$$

Then, (2.39) follows from the above discussions.

- $4\leq k\leq 6$.

For obtaining the estimates in the general case, we must prove the following:

$$\left|\frac{d^n}{dI^n}f_1(x,I)\right|\leq cI^{-n}(1+x),\quad \left|\frac{d^n}{dI^n}f_2(x,I)\right|\leq cI^{-n}(1+x),\quad \left|\frac{d^n}{dI^n}f_3(x,I)\right|\leq cI^{-n-1}(1+x), \quad (2.41)$$

for $-1\leq x\leq 0$.

In the proof of the above estimates, one may meet the terms $V'(x)\frac{d^k}{dI^k}F(x,I)$, $V''(x)x_I\frac{d}{dI}F$, $V_{xx}x_IF$ and so on. In order to estimate these terms, we have to use (2.33) and the fact:

$$(1+x)^{k-1}\left|\frac{d^k}{dx^k}V(x)\right|\leq c\left(|V'(x)|+(1+x)^{k-1}\right),\quad \text{for}\quad -1<x<0.$$

Moreover, we also need the following property of the operator $\mathcal{L}$:

CLAIM 3. Suppose the function $g : [-1, 0] \times \mathbb{R}^+ \to \mathbb{R}$, $(x, I) \mapsto g(x, I)$ is smooth in x and

$$\left|\frac{\partial^{k+j}}{\partial x^k \partial I^j} g(x, I)\right| \leq cI^{-j}|g(x, I)|,$$

then

$$\left|\frac{\partial^{k+j}}{\partial x^k \partial I^j} \mathcal{L}(g(x, I))\right| \leq cI^{-j-1}|g(x, I)|.$$

For the proof of Claim 3, it is sufficient to observe that from the definition of the operator $\mathcal{L}$, it follows that

$$\frac{\partial^{k+j}}{\partial x^k \partial I^j} \mathcal{L}(g(x, I)) = \frac{\partial^{k+j+1}}{\partial x^k \partial I^{j+1}} g(x, I) + \frac{\partial^j}{\partial I^j} \left[\frac{h_I}{h} \cdot \left(\frac{\partial^{k+1}}{\partial x^{k+1}}(g\frac{V}{V'}) - \frac{1}{2}\frac{\partial^k}{\partial x^k} g\right)\right].$$

Because the proof of (2.41) is quite cumbersome, and contains no new difficulties, we omit it here. The proof of Lemma 2.2 is complete. ■

From Lemma 2.2 and (2.8), we know that H is invertible with respect to I for I large. Moreover, $H/I \to 1$ as $I \to +\infty$. So we assume that I can be written, as a function of θ, H and t, as

$$I = I_0\left(\frac{H}{\pi} + R(H, t, \theta)\right), \tag{2.42}$$

where R satisfies $|R| < H/\pi$. Recalling that h_0 is the inverse function of I_0, from (2.18) we deduce that

$$\frac{H}{\pi} + R(H, t, \theta) = h_0(I) \quad \Rightarrow \quad H + \pi R(H, t, \theta) = \pi h_0(I) \quad \Rightarrow \quad R(H, t, \theta) = x(I, \theta)p(t).$$

As a consequence, R is implicitly defined by

$$R(H, t, \theta) = x\left(I_0\left(\frac{H}{\pi} + R(H, t, \theta)\right), \theta\right) p(t). \tag{2.43}$$

Lemma 2.3 *The function $R(H, t, \theta)$ satisfies the following estimates:*

$$\left|H^k \frac{\partial^k R(H, t, \theta)}{\partial H^k}\right| \leq c\sqrt{H}, \quad \text{for} \quad 0 \leq k \leq 6. \tag{2.44}$$

Proof.

- $k=0$. This follows from the expression of R in (2.43) and Lemma 2.2. Moreover, from now on, we assume that

$$\frac{1}{2\pi}H \leq \frac{H}{\pi} + R \leq \frac{2}{\pi}H. \tag{2.45}$$

- $k=1$. Taking the derivative with respect to H in both sides of (2.43), we obtain

$$\frac{\partial R}{\partial H} = \frac{1}{\pi} \cdot \frac{\frac{\partial x}{\partial I} \cdot I_0'(\frac{H}{\pi}+R)p(t)}{1 - \frac{\partial x}{\partial I} \cdot I_0'(\frac{H}{\pi}+R)p(t)}.$$

Hence, the result follows from Lemma 2.2, (2.7) and (2.45).

In the case of $k \geq 2$, one may get

$$\left(1 - \frac{\partial x}{\partial I} \cdot I_0'(\frac{H}{\pi}+R)p(t)\right)\frac{\partial^k R}{\partial H^k} = \sum c_{n,j_1\cdots j_n}\frac{\partial^n x}{\partial I^n} \cdot \frac{\partial^{j_1}}{\partial H^{j_1}} I_0(\frac{H}{\pi}+R) \cdot \cdots \cdot \frac{\partial^{j_n}}{\partial H^{j_n}} I_0(\frac{H}{\pi}+R),$$

where $1 \leq n \leq k$, $j_1 + \cdots + j_n = k$, $1 \leq j_1, \cdots, j_n < k$. It is easy to verify (2.44) for $k \geq 2$. ■

Analogously, one can prove the following more general

Lemma 2.4 *The function $R(H,t,\theta)$ satisfies the following estimates:*

$$\left| H^k \frac{\partial^{k+l} R(H,t,\theta)}{\partial H^k \partial t^l} \right| \leq c\sqrt{H}, \quad \text{for} \quad 0 \leq k+l \leq 6. \tag{2.46}$$

We are now ready to rewrite (2.17) with new variables. To this end, observe that from (2.43) and Lemma 2.3, it follows that

$$R(H,t,\theta) = x(H,\theta)p(t) + R_1(H,t,\theta), \tag{2.47}$$

where the function R_1 satisfies

$$\left| H^k \frac{\partial^k R_1(H,t,\theta)}{\partial H^k} \right| \leq c, \quad \text{for} \quad 0 \leq k \leq 6. \tag{2.48}$$

From the above discussions, we obtain

$$\begin{aligned} I &= H + \pi R(H,t,\theta) + I_-\left(\frac{H}{\pi} + R(H,t,\theta)\right) = \\ &= H + \pi x(H,\theta)p(t) + I_-\left(\frac{H}{\pi}\right) + \int_0^1 I_-'\left(\frac{H}{\pi} + sR(H,t,\theta)\right) R(H,t,\theta)\,ds + \pi R_1(H,t,\theta) \\ &= H + \pi x(H,\theta)p(t) + I_-\left(\frac{H}{\pi}\right) + \int_0^1 T_-\left(\frac{H}{\pi} + sR(H,t,\theta)\right) R(H,t,\theta)\,ds + \pi R_1(H,t,\theta). \end{aligned} \tag{2.49}$$

¿From this relation we infer that

$$\frac{\partial I}{\partial H} = 1+\frac{1}{\pi}T_-\left(\frac{H}{\pi}\right)+\pi\partial_H x(H,\theta)p(t)+\partial_H\left(\int_0^1 T_-\left(\frac{H}{\pi}+sR(H,t,\theta)\right)R(H,t,\theta)\,ds+\pi R_1(H,t,\theta)\right).$$

Assuming now θ as a time variable, (2.17) is transformed in the system

$$\begin{cases} \dfrac{dt}{d\theta} = 1+\dfrac{1}{\pi}T_-\left(\dfrac{H}{\pi}\right)+\pi x_H(H,\theta)p(t)+\partial_H\left(\displaystyle\int_0^1 T_-\left(\dfrac{H}{\pi}+sR(H,t,\theta)\right)R(H,t,\theta)\,ds+\pi R_1(H,t,\theta)\right) \\ \dfrac{dH}{d\theta} = -\pi x(H,\theta)p'(t)-\partial_t\left(\displaystyle\int_0^1 T_-\left(\dfrac{H}{\pi}+sR(H,t,\theta)\right)R(H,t,\theta)\,ds+\pi R_1(H,t,\theta)\right). \end{cases} \tag{2.50}$$

Suppose $(t(\theta,H_0,t_0),H(\theta,H_0,t_0))$ is the solution of (2.50) with the initial data (t_0,H_0). Then Lemma 2.2 implies that there is a constant c such that

$$|\sqrt{H(\theta)}-\sqrt{H_0}|\leq c, \quad \text{for} \quad 0\leq\theta\leq\pi. \tag{2.51}$$

A final change of variables is needed. Indeed, for every $H>0$ let us define

$$H=\frac{\rho}{\epsilon^2}, \tag{2.52}$$

with $\rho\in[1/\eta,\eta]$ and $\epsilon>0$. The constant $\eta>1$ will be conveniently chosen in Section 4. System (2.50) is then transformed into

$$\frac{dt}{d\theta}=1+\epsilon F(\rho,t,\theta;\epsilon), \quad \frac{d\rho}{d\theta}=\epsilon G(\rho,t,\theta;\epsilon), \tag{2.53}$$

where

$$F=\frac{1}{\epsilon}\left[\frac{1}{\pi}T_-\left(\frac{H}{\pi}\right)+\pi x_H(H,\theta)p(t)+\partial_H\left(\int_0^1 T_-\left(\frac{H}{\pi}+sR(H,t,\theta)\right)R(H,t,\theta)\,ds+\pi R_1(H,t,\theta)\right)\right],$$

$$G=\epsilon\left\{-\pi x(H,\theta)p'(t)-\partial_t\left(\int_0^1 T_-\left(\frac{H}{\pi}+sR(H,t,\theta)\right)R(H,t,\theta)\,ds+\pi R_1(H,t,\theta)\right)\right\}$$

with $H=\rho/\epsilon^2$. This system is a π-periodic Hamiltonian system in the new time variable θ.

3 Estimates on the Poincaré map

In this section we deduce the asymptotic development of the Poincaré map of (2.53), as $\epsilon\to 0$. This will be the crucial step in applying Moser twist theorem and Aubry-Mather theory to prove our main results for (2.1).

Definition 3.1 *We say a function $g(\rho,t,\theta;\epsilon)\in O_k(1)$ if g is smooth in (ρ,t) and*

$$\left|\frac{\partial^{k_1+k_2}}{\partial t^{k_1}\partial\rho^{k_2}}g(\rho,t,\theta;\epsilon)\right|\leq C,$$

for some constant $C>0$ which is independent of the arguments ρ,t,θ,ϵ, where $k_1+k_2\leq k$. Similarly, we say a function $g(\rho,t,\theta;\epsilon)\in o_k(1)$ if g is smooth in (ρ,t) and

$$\lim_{\epsilon\to0}\left|\frac{\partial^{k_1+k_2}}{\partial t^{k_1}\partial\rho^{k_2}}g(\rho,t,\theta;\epsilon)\right|=0,\quad \text{uniformly in } (\rho,t,\theta),$$

where $k_1+k_2\leq k$.

¿From the previous lemmas and the definitions of the functions F and G, we have

$$F,\ G\in O_6(1).$$

Now, let us denote by P the Poincaré map associated to (2.53), i.e. for every (t_0,ρ_0) let

$$(t_1,\rho_1)=P(t_0,\rho_0)=(t(\pi;t_0,\rho_0),\rho(\pi;t_0,\rho_0)),$$

where $(t(\cdot;t_0,\rho_0),\rho(\cdot;t_0,\rho_0))$ denotes the solution of (2.53) satisfying $(t(0;t_0,\rho_0),\rho(0;t_0,\rho_0))=(t_0,\rho_0)$.

We will prove the following result:

Proposition 3.2 *The map P satisfies*

$$\begin{cases} t_1=t_0+\pi+\epsilon\sqrt{2\pi\rho_0^{-1}}\left(1+\frac{1}{2}\int_0^\pi p\,(t_0+\theta)\sin\theta\,d\theta\right)+\epsilon o_5(1)\\ \\ \rho_1=\rho_0-\epsilon\sqrt{2\pi\rho_0}\int_0^\pi p'\,(t_0+\theta)\sin\theta\,d\theta+\epsilon o_5(1), \end{cases}\tag{3.1}$$

for $\epsilon\to0$.

We will prove Proposition 3.2 by means of the estimates that we are going to deduce below.

First of all, from (2.53) we may assume that the solution of (2.53) with the initial data (t_0,ρ_0) is of the form

$$t(\theta)=t_0+\theta+\epsilon\Gamma_1(\theta,t_0,\rho_0;\epsilon),\quad \rho(\theta)=\rho_0+\epsilon\Gamma_2(\theta,t_0,\rho_0;\epsilon).$$

Substitution into (2.53) yields that

$$\Gamma_1=\int_0^\theta F(\rho_0+\epsilon\Gamma_2,t_0+\theta+\epsilon\Gamma_1,\theta;\epsilon)d\theta,\quad \Gamma_2=\int_0^\theta G(\rho_0+\epsilon\Gamma_2,t_0+\theta+\epsilon\Gamma_1,\theta;\epsilon)d\theta.$$

¿From standard differential inequalities, it follows that

$$\left|\frac{\partial^{k+l}\Gamma_1(\theta,\rho_0,t_0;\epsilon)}{\partial\rho_0^k\partial t_0^l}\right|+\left|\frac{\partial^{k+l}\Gamma_2(\theta,\rho_0,t_0;\epsilon)}{\partial\rho_0^k\partial t_0^l}\right|\le C_0,\quad \text{for}\quad k+l\le 6,$$

that is, $\Gamma_1,\ \Gamma_2\in O_6(1)$. It is easy to show that

$$\Gamma_1=\int_0^\theta F(\rho_0,t_0+\theta,\theta;\epsilon)d\theta+\epsilon O_6(1),\quad \Gamma_2=\int_0^\theta G(\rho_0,t_0+\theta,\theta;\epsilon)d\theta+\epsilon O_6(1).$$

By Lemmas 2.3 and 2.1, the definitions of F and G, we have

$$F(\rho_0,t_0+\theta,\theta;\epsilon)=\frac{1}{\epsilon}\left[\frac{1}{\pi}T_-\left(\frac{\rho_0}{\pi\epsilon^2}\right)+\pi x_H(\frac{\rho_0}{\epsilon^2},\theta)p(t_0+\theta)\right]+\epsilon O_6(1),$$

$$G(\rho_0,t_0+\theta,\theta;\epsilon)=-\pi\epsilon x(\frac{\rho_0}{\epsilon^2},\theta)p'(t_0+\theta)+\epsilon O_6(1).$$

Now we obtain that the Poincaré map of (2.53) is of the form

$$t_1=t_0+\pi+\epsilon L_1(\rho_0,t_0,\epsilon)+\epsilon o_5(1),\quad \rho_1=\rho_0+\epsilon L_2(\rho_0,t_0,\epsilon)+\epsilon o_5(1),\tag{3.2}$$

where

$$L_1(\rho_0,t_0,\epsilon)=\frac{1}{\epsilon}T_-\left(\frac{\rho_0}{\pi\epsilon^2}\right)+\frac{\pi}{\epsilon}\int_0^\pi x_H(\frac{\rho_0}{\epsilon^2},\theta)p(t_0+\theta)d\theta,\quad L_2(\rho_0,t_0,\epsilon)=-\pi\epsilon\int_0^\pi x(\frac{\rho_0}{\epsilon^2},\theta)p'(t_0+\theta)d\theta.\tag{3.3}$$

In the following lemma, we give estimates of L_1 and L_2.

Lemma 3.3 *The following estimates hold true:*

$$\begin{aligned}L_1(\rho_0,t_0,\epsilon)&=\sqrt{\frac{2\pi}{\rho_0}}+\sqrt{\frac{\pi}{2\rho_0}}\int_0^\pi p(t_0+\theta)\sin\theta d\theta+\epsilon O_5(1)\\ L_2(\rho_0,t_0,\epsilon)&=-\sqrt{2\pi\rho_0}\int_0^\pi p'(t_0+\theta)\sin\theta d\theta+\epsilon O_5(1),\end{aligned}\tag{3.4}$$

for $\epsilon\to 0$.

Proof. The proof is based on the following estimates

$$\text{meas}\,\{\theta\in[0,\pi],x(H_0,\theta)>0\}=\pi+\epsilon O_6(1),\quad \text{meas}\,\{\theta\in[0,\pi],x(H_0,\theta)<0\}=\epsilon O_6(1),\tag{3.5}$$

where $H_0=\epsilon^{-2}\rho_0$. To prove them, let us first observe that from (2.16) it follows that

$$\text{meas}\,\{\theta\in[0,\pi],x(H_0,\theta)>0\}=\frac{\pi^2}{T_0(h(x,y))},\tag{3.6}$$

where

$$h(x,y) = \frac{1}{2}y^2(H_0,\theta) + V(x(H_0,\theta)).$$

Here H_0 plays the role as the action variable I, that is (cf. (2.4)),

$$H_0 = \pi h + I_-(h).$$

Inequalities (2.11) and (2.7) imply that there is a unique function $\mathcal{H}$ such that

$$h = \frac{H_0}{\pi} + \mathcal{H}(H_0).$$

Moreover, similar to the proof of Lemma 2.3, one may get, for $k \geq 0$,

$$\left| H_0^k \frac{d^k}{dH_0^k}\mathcal{H}(H_0)\right| \leq C_0\sqrt{H_0}.$$

In particular, we have

$$\frac{1}{2\pi}H_0 \leq h \leq \frac{3}{2\pi}H_0.$$

From these estimates, (2.12), (2.13) and (2.9), it follows that

$$\left| H_0^k \frac{d^k}{dH_0^k}\left(T_-(h) - T_-(\frac{H_0}{\pi})\right)\right| \leq C_0 \cdot H_0^{-1}.$$

Hence,

$$T_-(h) = T_-(\frac{H_0}{\pi}) + \epsilon^2 O_6(1) = \frac{\sqrt{2\pi}}{\sqrt{\rho_0}} \cdot \epsilon + \epsilon^2 O_6(1). \tag{3.7}$$

The first conclusion in (3.5) follows from (3.6), (2.5) and the above estimate. The second conclusion in (3.5) is obtained by

$$\operatorname{meas}\{\theta \in [0,\pi], x(H_0,\theta) < 0\} = T_0(h) - \operatorname{meas}\{\theta \in [0,\pi], x(H_0,\theta) > 0\}.$$

Now, let us complete the proof of the Lemma. We recall that, when $x < 0$, we have

$$|x(H_0,\theta)| = O_6(1), \quad |x_H(H_0,\theta)| = \epsilon^2\ O_5(1).$$

When $x > 0$, from the definition of θ, it follows that

$$\arcsin\frac{x(H_0,\theta)}{\sqrt{2h}} = \frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2} = \theta + \epsilon O_5(1),$$

which yields that

$$x(H_0,\theta) = \sqrt{2h}\sin(\theta + \epsilon O_5(1)) = \sqrt{\frac{2H_0}{\pi}}\sin\theta + O_5(1), \quad x_H(H_0,\theta) = \sqrt{\frac{1}{2\pi H_0}}\sin\theta + \epsilon^2 O_5(1).$$

Hence, we have

$$\begin{aligned} L_1(\rho_0,t_0,\epsilon) &= \frac{1}{\epsilon}T_-\left(\frac{\rho_0}{\pi\epsilon^2}\right)+\frac{\pi}{\epsilon}\int_0^\pi x_H(\frac{\rho_0}{\epsilon^2},\theta)p(t_0+\theta)d\theta \\ &= \sqrt{\frac{2\pi}{\rho_0}}+\frac{\pi}{\epsilon}\int_{\{\theta\in[0,\pi]:x(H_0,\theta)>0\}} x_H(H_0,\theta)p(t_0+\theta)d\theta+\epsilon O_5(1) \\ &\quad +\frac{\pi}{\epsilon}\int_{\{\theta\in[0,\pi]:x(H_0,\theta)<0\}} x_H(H_0,\theta)p(t_0+\theta)d\theta \\ &= \sqrt{\frac{2\pi}{\rho_0}}+\frac{\pi}{\epsilon}\int_{\{\theta\in[0,\pi]:x(H_0,\theta)>0\}}\sqrt{\frac{1}{2\pi H_0}}p(t_0+\theta)\sin\theta d\theta+\epsilon O_5(1) \\ &= \sqrt{\frac{2\pi}{\rho_0}}+\sqrt{\frac{\pi}{2\epsilon^2 H_0}}\int_0^\pi p(t_0+\theta)\sin\theta d\theta+\epsilon O_5(1). \end{aligned}$$

Recalling (2.52), we deduce that the first estimate in (3.4) holds true.
In an analogous way, we infer that

$$\begin{aligned} L_2(\rho_0,t_0,\epsilon) &= -\pi\epsilon\int_{\{\theta\in[0,\pi]:x(H_0,\theta)>0\}} x(H_0,\theta)p'(t_0+\theta)d\theta \\ &\quad -\pi\epsilon\int_{\{\theta\in[0,\pi]:x(H_0,\theta)<0\}} x(H_0,\theta)p'(t_0+\theta)d\theta \\ &= -\pi\epsilon\int_{\{\theta\in[0,\pi]:x(H_0,\theta)>0\}} x(H_0,\theta)p'(t_0+\theta)d\theta+\epsilon O_5(1) \\ &= -\pi\epsilon\cdot\sqrt{\frac{2H_0}{\pi}}\int_0^\pi p'(t_0+\theta)\sin\theta d\theta+\epsilon O_5(1). \end{aligned}$$

Recalling (2.52) again, this concludes the proof. ■

From Lemma 3.3 and the relations (3.2)-(3.3) we immediately deduce the validity of Proposition 3.2.

4 The main results

In the previous section we have proved that the Poincaré map associated to the initial equation has the asymptotic development

$$\begin{aligned} t_1 &= t_0+\pi+\epsilon\sqrt{2\pi\rho_0^{-1}}\left(1+\frac{1}{2}\int_0^\pi p\,(t_0+\theta)\sin\theta\,d\theta\right)+\epsilon o_5(1) \\ \rho_1 &= \rho_0-\epsilon\sqrt{2\pi\rho_0}\int_0^\pi p'\,(t_0+\theta)\sin\theta\,d\theta+\epsilon o_5(1), \end{aligned} \tag{4.1}$$

for $\epsilon \to 0$.

This expression can be written in the form

$$\begin{aligned} t_1 &= t_0 + \pi + \epsilon\ l_1(\rho_0, t_0) + \epsilon\phi_1(\rho_0, t_0) \\ \rho_1 &= \rho_0 + \epsilon\ l_2(\rho_0, t_0) + \epsilon\phi_2(\rho_0, t_0), \end{aligned} \tag{4.2}$$

with

$$l_1(\rho_0, t_0) = \sqrt{2\pi\rho_0^{-1}} \left(1 + \frac{1}{2}\int_0^{\pi} p\,(t_0 + \theta)\sin\theta\, d\theta\right)$$

$$l_2(\rho_0, t_0) = -\sqrt{2\pi\rho_0}\int_0^{\pi} p'\,(t_0 + \theta)\sin\theta\, d\theta$$

and

$$\phi_i = o_5(1), \quad \epsilon \to 0,\ i = 1, 2.$$

Moreover, it is easy to show that $l_1 \in C^6$ and $l_2, \phi_1, \phi_2 \in C^5$ when $p \in C^6(\mathbf{R})$.

We are now in position to state and prove our result on boundedness of solutions of the initial equation.

Theorem 4.1 *Assume that γ is a positive integer and that $p \in C^6(\mathbf{R})$ satisfies*

$$1 + \frac{1}{2}\int_0^{\pi} p\,(t_0 + \theta)\sin\theta\, d\theta > 0, \quad \forall\ t_0 \in \mathbf{R}. \tag{4.3}$$

Then, all the solutions of (2.1) *are bounded.*

Proof. We apply Ortega's variant of Moser theorem as it is stated in Theorem 3.1 in [12]. Assumption (3.2) follows from (4.3); the regularity hypothesis (3.3) holds as well. In order to check the validity of (3.5), it is sufficient to slightly modify the corresponding step in the proof of Theorem 1 in [7]; indeed, one can prove the existence of $\mathcal{I}(\rho_0, t_0)$ of the form $\rho_0\alpha(t_0)$, being α a suitable function satisfying $(3.4) - (3.5) - (3.6)$. This is the point where we suitably choose the constant η introduced at the end of Section 2. ∎

Using the estimates developed in the previous sections, we can also obtain a result of Aubry-Mather type.

Theorem 4.2 *Assume that γ is a positive integer and that $p \in C(\mathbf{R})$ satisfies* (4.3)*; then, there is an $\epsilon_0 > 0$ such that, for any $\omega \in (1/\pi, 1/\pi + \epsilon_0)$, equation (2.1) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number ω. More precisely,*

- *if $\omega = p/q$ is rational, the solutions $(x_\omega(t + 2i\pi), x'_\omega(t + 2i\pi)), 1 \le i \le q - 1$ are mutually unlinked periodic solutions of period $q\pi$; moreover, in this case,*

$$\lim_{q\to\infty} \min_{t\in\mathbf{R}}(|x_\omega(t)| + |x'_\omega(t)|) = +\infty;$$

• *if ω is irrational, the solution $(x_\omega(t), x'_\omega(t))$ is either a usual quasi-periodic solution or a generalized one.*

We recall that a solution is called generalized quasi-periodic if the closed set

$$\overline{\{(x_\omega(2i\pi), x'_\omega(2i\pi)), i \in \mathbf{Z}\}}$$

is a Denjoy's minimal set.

Proof. We apply Aubry-Mather theory. More precisely, according to [13], we first recall that $\lim_{h\to+\infty} T_0(h) = \pi$. For the applicability of Theorem B in [13], it is sufficient to check that the Poincaré map P has the monotone twist property. To this aim, we need to show (cf. (4.2)) that

$$\frac{\partial t_1}{\partial \rho_0} > 0,$$

for $\epsilon > 0$ sufficiently small. It is straightforward to check that this is a consequence of assumption (4.3).

■

The above results are valid for any continuous potential $V : (a, +\infty) \to \mathbf{R}^+$, $a \in \mathbf{R}$, which satisfies the following assumptions:

(V_1) there exists $b > a$ s.t. $V(b) = 0 = V'(b)$;

(V_2) $V \in \mathcal{C}^7((a,+\infty)\setminus\{b\})$, $\lim_{x\to a^-} V(x) = +\infty$;

(V_3) the function $W(x) = V(x)/V'(x)$ is of class $\mathcal{C}^6((a,+\infty)\setminus\{b\})$ and

$$\lim_{x\to a^-, x\to b^\pm} \left| W^{(j)}(x) \right| < +\infty, \quad j = 0, \ldots, 6; \tag{4.4}$$

(V_4) there exists $c > 0$ s.t.

$$(x-a)^{k-1} \left| \frac{d^k}{dx^k} V(x) \right| \leq c \left(|V'(x)| + (x-a)^{k-1} \right), \quad \text{for} \quad x > a \quad \text{and} \quad k = 0, \ldots 6;$$

(V_5) $V(x) = \frac{1}{2} n^2 x^2 + r(x)$, where r is e.g. of the form

$$r(x) = O(x), \;\; r'(x) = c + O\Big(\frac{1}{x^2}\Big), \;\; r^{(k)}(x) = O\Big(\frac{1}{x^{1+k}}\Big), \; k \geq 2, \;\; x \to +\infty.$$

It is easy to observe that the function $V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\gamma} - 1$ that we have treated is an example of a singular "asymptotically resonant" potential satisfying $(V_1) - (V_5)$. Moreover, we remark that, as far as Theorem 4.1 is concerned, the restriction $\gamma \in \mathbf{N}$ can be weakened; indeed, it is sufficient to require $\gamma \in \mathbf{R}$ with $\gamma > 5$.

As for Theorem 4.2, we observe that in the proof it is sufficient to perform a $\mathcal{C}^1$-development of the Poincaré map. This means we just have to estimate I_-, T_-, T'_-, T''_-; these estimates can be obtained once, besides (V_1) and (V_5), it is assumed
$(V_2)'$ $V \in \mathcal{C}^3((a,+\infty) \setminus \{b\})$;
$(V_3)'$ the function $W(x) = V(x)/V'(x)$ is of class $\mathcal{C}^2((a,+\infty) \setminus \{b\})$ and

$$\lim_{x\to a^-, x\to b^\pm} \left|W^{(j)}(x)\right| < +\infty, \quad j = 0, 1, 2; \tag{4.5}$$

$(V_4)'$ there exists $c > 0$ s.t.

$$(x-a)^{k-1}\left|\frac{d^k}{dx^k}V(x)\right| \le c\left(|V'(x)| + (x-a)^{k-1}\right), \quad \text{for} \quad x > a \quad \text{and} \quad k = 0, \dots 2.$$

In particular, if we take V as in (1.2), then Theorem 4.2 is valid in case $\gamma > 2$ is any real number.

Remark 4.3 *Assumption* (4.3) *can be compared with the well-known Lazer-Leach condition*

$$\left|\int_0^{2\pi} p(t)e^{int}\,dt\right| < 2(\underline{h}(+\infty) - \bar{h}(-\infty)) \tag{4.6}$$

being $\underline{h}(+\infty) = \liminf_{x\to+\infty} h(x)$ *and* $\bar{h}(-\infty) = \limsup_{x\to-\infty} h(x)$. *According to [5], condition* (4.6) *guarantees (and in some cases characterizes) the existence of* 2π*-periodic solutions for* $x'' + n^2x + h(x) = p(t)$. *For improvements and comments on Lazer-Leach type conditions in a more general context, and in relation with the question of coexistence of periodic and unbounded solutions, we refer to the paper by C. Fabry-J. Mawhin [3]. In case of resonant asymmetric (or isochronous) oscillators and* $h \equiv 0$*, it is now known (cf. [8]) that Lazer-Leach type conditions play a central role also when the question of boundedness is treated. On the other hand, boundedness of all solutions is guaranteed by [11] assuming that* p *is "sufficiently small" for an equation of the form* $x'' + ax^+ - bx^- = 1 + p(t)$. *Theorem 4.1 provides a condition for the boundedness of all solutions in case* p *is not necessarily small. Moreover, our condition* (1.3) *obviously holds in case* $p \equiv 0$*,* p *"small" and also in case the function* $(1/2)\int_0^\pi p(t_0 + \theta)\sin\theta\,d\theta$ *vanishes at some point. In other words, we are able to treat cases when a Lazer-Leach type condition might fail.*
In the framework of Lazer-Leach type conditions, an interesting contribution has been recently given by A. Fonda-J. Mawhin [4]. More precisely, in the case of planar first order systems in [4] it is examined the question of the coexistence of 2π*-periodic and unbounded solutions.*

Remark 4.4 *An asymptotic development of the Poincaré map of the form* (4.1) *is treated by J.M. Alonso and R. Ortega in [1]. From Proposition 2.1 in [1], it follows that if the function* $\theta \mapsto \left(1 + \frac{1}{2}\int_0^\pi p(t_0 + \theta)\sin\theta\,d\theta\right)$ *has simple zeros then there are unbounded solutions. In particular, we observe that we can study the existence of unbounded solutions in case our hypothesis* (4.3) *fails.*

5 Appendix

Proof of Lemma 2.1**.**

We first develop some preliminaries; the proof of Lemma 2.1 can be found at the end of this section.

Let

$$W(x) = \frac{V(x)}{V'(x)}, \quad \forall\ x \in (-1,0).$$

For every $x \in (-1,0)$, we have

$$W(x) = \frac{(1-x^2)(1-(1-x^2)^\gamma)}{2\gamma x},$$

$$W'(x) = 1-\phi(x), \quad \phi(x) = \frac{1}{2\gamma}\ \frac{(1-(1-x^2)^\gamma)(1+(2\gamma+1)x^2)}{x^2}.$$

The function W is even; from now on, we will denote by V the function $V(-|x|)$. In this way, we can consider the interval $(0,1)$ instead of $(-1,0)$. It is easy to see that

$$\begin{aligned} &\lim_{x\to 0^+} W(x) = 0, \quad \lim_{x\to 1^-} W(x) = 0; \\ &\lim_{x\to 0^+} \left|W^{(j)}(x)\right| < +\infty, \quad \lim_{x\to 1^-} \left|W^{(j)}(x)\right| < +\infty. \end{aligned} \tag{5.1}$$

For every $u = u(h,x)$, let $\mathcal{K}(u)$ be defined by

$$\mathcal{K}(u)(h,x) = u_h(h,x) + \frac{1}{2h}u(h,x) + \frac{1}{h}\ \left(u\frac{V}{V'}\right)'(h,x), \tag{5.2}$$

where $'$ means the derivative with respect to the x-variable.

For every positive integer n let $\mathcal{K}^n = \mathcal{K}\circ\ldots\circ\mathcal{K}$ (n times). We will be interested in a formula for $\mathcal{K}^n(1)$:

Proposition 5.1 *For every $n \geq 1$ we have*

$$\mathcal{K}^n(1)(h,x) = \frac{1}{h^n}P_n(x), \tag{5.3}$$

where P_n is recursively defined by

$$\begin{cases} P_{n+1}(x) = \left(\dfrac{1}{2}-n\right)P_n(x) + (W\ P_n)'(x), \quad n\geq 1 \\ \\ P_1(x) = \dfrac{1}{2} + W'(x). \end{cases} \tag{5.4}$$

Proof. 1. From (5.2) we immediately deduce

$$\mathcal{K}(1)(h,x) = \frac{1}{2h} + \frac{1}{h}\ W'(x) = \frac{1}{h}\ P_1(x),$$

which proves (5.3) for $n = 1$.

2. Assume now that (5.3) holds true for every integer from 1 to n; we show that it is fulfilled also for $n+1$. Indeed, we have

$$\mathcal{K}^{n+1}(1)(h,x) = \mathcal{K}(\mathcal{K}^n(1))(h,x) = \mathcal{K}\left(\frac{1}{h^n}P_n(x)\right)(h,x) =$$

$$= -\frac{n}{h^{n+1}}P_n(x) + \frac{1}{2h}\ \frac{1}{h^n}P_n(x) + \frac{1}{h^{n+1}}(WP_n)'(x) = \frac{1}{h^{n+1}}\ P_{n+1}(x),$$

where P_{n+1} is defined in (5.4). ■

Proposition 5.2 *For every positive integer $n = 0, 1, \ldots, 5$ we have*

$$\begin{aligned} &(W\ P_n)(0) = 0, \quad (W\ P_n)(1) = 0; \\ &\int_0^1 P_n(x)\, dx = (-1)^{n-1}\frac{(2n-3)!!}{2^n}, \end{aligned} \tag{5.5}$$

where

$$(2k-1)!! = \begin{cases} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) & k \geq 1 \\ 1 & k = 0. \end{cases}$$

Proof. The first formulas in (5.5) easily follow from (5.1). We prove the formula for the integral of P_n; let

$$\epsilon_n = \int_0^1 P_n(x)\, dx.$$

¿From (5.4) we deduce that

$$\epsilon_n = \int_0^1 \left\{\left(\frac{3}{2} - n\right) P_{n-1}(x) + (W\ P_{n-1})'(x)\right\} dx =$$

$$= \left(\frac{3}{2} - n\right)\ \epsilon_{n-1} + (W\ P_{n-1})(1) - (W\ P_{n-1})(0) = \left(\frac{3}{2} - n\right)\ \epsilon_{n-1}.$$

As a consequence, we get

$$\epsilon_n = \left(\frac{3}{2} - n\right) \cdot \left(\frac{3}{2} - (n-1)\right) \cdot \ldots \left(\frac{3}{2} - 3\right) \cdot \left(\frac{3}{2} - 2\right) \epsilon_1. \tag{5.6}$$

Now, let us observe that

$$\epsilon_1 = \int_0^1 P_1(x)\,dx = \int_0^1 \left(W'(x) + \frac{1}{2}\right)\,dx = W(1) - W(0) + \frac{1}{2} = \frac{1}{2};$$

from this relation and (5.6) we deduce that

$$\epsilon_n = (-1)^{n-1}\frac{(2n-3)\cdot(2n-1)\cdot\ldots\cdot 5\cdot 3}{2^n},$$

which completes the proof. ■

We recall now formula (A3.2) from [6]; for every function $u = u(h,x)$, let

$$\mathcal{I}(h) = \int_{-\alpha_h}^{0} u(h,s)\sqrt{h - V(s)}\,ds = \int_0^{\alpha_h} u(h,-s)\sqrt{h-V(s)}\,ds,$$

where α_h is defined by $V(-\alpha_h) = h$, for every $h > 0$. Moreover, let $\tilde{u}(h,x) = u(h,-x)$. Then we have

$$\frac{d}{dh}\mathcal{I}(h) = \int_0^{\alpha_h} \mathcal{K}(\tilde{u})(h,s)\sqrt{h-V(s)}\,ds. \tag{5.7}$$

By iterating this formula, we get

$$\frac{d^n}{dh^n}\mathcal{I}(h) = \int_0^{\alpha_h} \mathcal{K}^n(\tilde{u})(h,s)\sqrt{h-V(s)}\,ds, \tag{5.8}$$

for every integer $n \geq 1$.

We recall that

$$I_-(h) = 2\int_0^{\alpha_h} \sqrt{2(h-V(s))}\,ds = 2\sqrt{2}\int_0^{\alpha_h} \sqrt{h-V(s)}\,ds, \tag{5.9}$$

for every $h > 0$; moreover, we have

$$T_-(h) = I'_-(h), \quad \forall\ h > 0. \tag{5.10}$$

Proposition 5.3 *For every integer $n = 0, 1, \ldots, 5$ we have*

$$\frac{d^n}{dh^n}T_-(h) = 2\sqrt{2}\ \frac{1}{h^{n+1}}\ \int_0^{\alpha_h} P_{n+1}(s)\sqrt{h-V(s)}\,ds. \tag{5.11}$$

Proof. From (5.10) we infer that

$$\frac{d^n}{dh^n}T_-(h) = \frac{d^{n+1}}{dh^{n+1}}I_-(h);$$

we then apply (5.8) with $u = \tilde{u} \equiv 1$ and (5.3):

$$\frac{d^n}{dh^n}T_-(h) = 2\sqrt{2}\int_0^{\alpha_h} \mathcal{K}^{n+1}(1)(h,s)\sqrt{h - V(s)}\,ds = 2\sqrt{2}\,\frac{1}{h^{n+1}}\int_0^{\alpha_h} P_{n+1}(s)\sqrt{h - V(s)}\,ds.$$

This completes the proof. ■

We are now in position to prove Lemma 2.1; for the reader's convenience, we recall its statement below.

Lemma 2.1. *For every $n = 0, 1, 2, \dots, 5$ we have*

$$\frac{d^n T_-}{dh^n}(h) = (-1)^n\,\frac{1}{2^n}\,(2n-1)!!\,\frac{\sqrt{2}}{h^{(2n+1)/2}} + o\left(\frac{1}{h^{(2n+1)/2}}\right), \quad h \to +\infty. \tag{5.12}$$

PROOF. From (5.11) we deduce that

$$h^{(2n+1)/2}\,\frac{d^n}{dh^n}T_-(h) = 2\sqrt{2}\,\frac{1}{\sqrt{h}}\int_0^{\alpha_h} P_{n+1}(s)\sqrt{h - V(s)}\,ds$$

and then

$$h^{(2n+1)/2}\,\frac{d^n}{dh^n}T_-(h) = 2\sqrt{2}\int_0^{\alpha_h} P_{n+1}(s)\sqrt{1 - \frac{V(s)}{h}}\,ds.$$

Recalling that $\alpha_h \to 1$, for $h \to +\infty$, an application of Lebesgue dominated convergence Theorem gives

$$\lim_{h\to+\infty} h^{(2n+1)/2}\,\frac{d^n}{dh^n}T_-(h) = 2\sqrt{2}\int_0^1 P_{n+1}(s)\,ds.$$

Using (5.5) from this relation we get (5.12).

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