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AN APPROXIMATE FORMULA FOR THE FIRST-CROSSING-TIME DENSITY OF A WIENER PROCESS PERTURBED BY RANDOM JUMPS

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ABSTRACT. An approximate solution to an integral equation for the first-crossing-time density of a Wiener process with constant amplitude jumps separated by exponential random times is shown to hold under suitable conditions and to explain some multimodal behaviors.
Key words: jump-diffusion process, fixed point, multimodality.

INTRODUCTION

In many applications the necessity arises to introduce models where a jump component is superposed on an underlying diffusion process. The focus on jump-diffusion processes has then been constantly growing in literature. Jump-diffusion processes are largely employed in Mathematical Finance where they can be useful, among other instances, (cf. for example Merton (1976), Mercurio and Runggaldier (1993), Stoica (2001)) to study the pricing of some bonds (cf. Vaugirard (2004)) or the firm default probabilities (Zhang and Melnik (2007)). Processes, where jumps are superposed on diffusions, have been succesfully introduced in neuronal modeling (cf. Musila and Lánský (1991), Giraudo et al. (2002)) to allow separate descriptions for the contributions to the membrane potential from different types of synapses. In all such contexts the interest often lays in the characterization of the time at which the process first crosses a given level, the first-crossing-time (FCT). However the FCT problem is difficult even for simple diffusion processes (cf. Ricciardi et al. (1999)). Here we are concerned with the study of the FCT density for simple jump-diffusion processes composed by a Wiener process and jumps of constant amplitude separated by exponential random times. The amplitude of the jumps is supposed to be constant since in many applications one can consider their contributions indistinguishable in size or also as occurring in fixed amount releases. In this paper we prove an approximate formula for the FCT probability density function of such processes and we give an estimate of the related error, thus allowing to rely the different parameters involved with some observed multimodal behaviors of the FCT

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density (cf. Sacerdote and Sirovich(2003)). In Section 2 the main features of the considered processes are introduced while Section 3 contains the theoretical results. In Section 4 we exemplify our results by means of an application in the neuronal modeling context and we conclude in Section 5 with a brief discussion.

PRELIMINARIES

We consider the jump-diffusion process Y_t solution to the following stochastic differential equation:

$$dY_t = \mu dt + \sigma dW_t + a dN_t^u + i dN_t^d;\ Y_0 = 0 \tag{1}$$

where W_t is a Wiener process with drift $\mu \in \Re$ and variance per unit time $\sigma^2 \in (0,\infty)$, N_t^u and N_t^d are two homogeneous Poisson processes, independent from each other and from W_t, with intensities λ_u and λ_d respectively, and $a, i \in \Re, a > 0, i < 0$ are the corresponding constant jump amplitudes. It can be shown that the process Y_t has the following mean and variance:

$$E[Y_t] = \mu t + \mu_J t;\ Var[Y_t] = \sigma^2 t + \sigma_J^2 t \tag{2}$$

where we have set $\mu_J = a\lambda_u + i\lambda_d$; $\quad \sigma_J^2 = a^2\lambda_u + i^2\lambda_d$. Furthermore it can be shown (cf. Gihman and Skorohod (1975)) that the process Y_t possesses a conditional probability density function (p.d.f.). In analogy with the approach employed for pure diffusion models (cf. for example Ricciardi et al. (1999)) we introduce the random variable FCT from below, defined $\forall y < S$ as

$$\widetilde{T}_{S;y,\tau} = \inf\{t \geq \tau : Y_t \geq S; Y_\tau = y\} \tag{3}$$

We assume that after the occurrence of each crossing the process is reset to its initial value and hence the sequence of FCT's gives rise to a renewal process. We note that $\widetilde{T}_{S;y,\tau}$ is an absolutely continuous random variable (cf. Gihman and Skorohod (1975)). If $\mu \geq 0$ its p.d.f. $\widetilde{g}(S,t|\,y,\tau)$ is defined $\forall \tau < t$ as

$$\widetilde{g}(S,t|\,y,\tau) = \frac{\partial}{\partial t}\mathbb{P}\left\{\widetilde{T}_{S;y,\tau} < t\right\}. \tag{4}$$

Very few analytical results exist on FCT p.d.f. for jump-diffusion processes even in the case of very simple boundaries and they are mainly focused on the FCT moments (cf. Tuckwell (1976), Abundo (2000), Giraudo et al. (2002)) or on bounds or concern the special case of a double-exponential random jump amplitude (Kou and Wang (2003)).

Remark 1 *When $\lambda_u = 0$, $\lambda_d = 0$ the FCT becomes the first-passage-time random variable $T_{S;y,\tau}$ of the process W_t through the boundary S, whose p.d.f. is*

$$g(S,t|\,y,\tau) = \frac{S-y}{\sqrt{2\pi}\sigma(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{(S-y-\mu(t-\tau))^2}{2\sigma^2(t-\tau)}\right\}. \tag{5}$$

Remark 2 *The transition p.d.f. of the process W_t in the presence of an absorbing boundary in $S > 0$ is defined for $\tau < t$, $x, y \leq S$ (cf. Ricciardi (1990)) as*

$$f^a\left(x, t \mid y, \tau\right) = \frac{\partial}{\partial x} P\left(W_t < x; W_\theta < S, \forall\, \theta < t \mid W_\tau = y\right) =$$
$$\frac{1}{\sqrt{2\pi(t-\tau)}\sigma}\{\exp\left[-\frac{(x-y-\mu(t-\tau))^2}{2\sigma^2(t-\tau)}\right] - \exp\left[\frac{2\mu(S-y)}{\sigma^2} - \frac{(x+y-2S-\mu(t-\tau))^2}{2\sigma^2(t-\tau)}\right]\}. \tag{6}$$

The following relationship holds $\forall \tau < t$, $\forall\, x, y \leq S$ between f^a and the transition p.d.f. when no boundary is imposed, f:

$$f^a\left(x, t \mid y, \tau\right) \leq f\left(x, t \mid y, \tau\right) = \frac{1}{\sqrt{2\pi(t-\tau)\sigma^2}} \exp\left[-\frac{\left(x-y-\mu\left(t-\tau\right)\right)^2}{2\sigma^2\left(t-\tau\right)}\right]\}. \tag{7}$$

THE FCT DENSITY AND ITS APPROXIMATION

In this Section we show that the FCT p.d.f. $\widetilde{g}\left(S, t \mid y, \tau\right)$ defined in (4) is solution as a function of y and τ to an integral equation to which the Banach fixed point theorem can be applied. By employing the Corollary to such theorem (cf. for example Zeidler (1986)) the validity of the use of a suitable approximation to $\widetilde{g}\left(S, t \mid y, \tau\right)$ is then established.

Let us consider the jump-diffusion process Y_t defined by (1) with jump amplitudes $a = -i$ and denote as $\lambda = \lambda_u + \lambda_d$ the overall intensity of the jump process. It holds:

Lemma 3 *The FCT p.d.f. of the process Y_t solution to (1) is solution to the equation:*

$$\widetilde{g}\left(S, t \mid y, \tau\right) = e^{-\lambda(t-\tau)} g\left(S, t \mid y, \tau\right) + \int_\tau^t du \int_{-\infty}^S dz e^{-\lambda(u-\tau)}\{\lambda_u f^a\left(z-a, u \mid y, \tau\right)$$
$$+\lambda_d f^a\left(z+a, u \mid y, \tau\right) I_{(-\infty, S-a)}\left(z\right)\}\widetilde{g}\left(S, t \mid z, u\right) + \lambda_u e^{-\lambda(t-\tau)} \int_{S-a}^S dz f^a\left(z, t \mid y, \tau\right) \tag{8}$$

where $I_A(\cdot)$ is the indicator function over the set A, f^a is given in (6) and g is given in (5).

Proof By partitioning the sample paths of Y_t that enter into $[S, \infty)$ in the following three disjoint sets:

$A_1 = \{S$ is crossed the first time for diffusion before the occurrence of the first jump$\}$;

$A_2 = \{$at least one upward or downward jump occurs before entering $[S, \infty)$ for the first time for diffusion$\}$;

$A_3 = \{$the first upward jump occurs when the process is in $[S-a, S)$ without having crossed S before$\}$;

we can write the distribution function of $\widetilde{T}_{S;y,\tau}$ as

$$P\left(\widetilde{T}_{S;y,\tau} < t\right) = \int_\tau^t du e^{-\lambda(u-\tau)} g\left(S, u\left|y,\tau\right.\right)$$
$$+\lambda_u \int_\tau^t du \int_{-\infty}^{S-a} dz e^{-\lambda(u-\tau)} f^a\left(z, u\left|y,\tau\right.\right) P\left(\widetilde{T}_{S;z+a,u} < t\right)$$
$$+\lambda_d \int_\tau^t du \int_{-\infty}^{S} dz e^{-\lambda(u-\tau)} f^a\left(z, u\left|y,\tau\right.\right) P\left(\widetilde{T}_{S;z-a,u} < t\right) \qquad (9)$$
$$+\lambda_u \int_\tau^t du \int_{S-a}^{S} dz e^{-\lambda(u-\tau)} f^a\left(z, u\left|y,\tau\right.\right)$$

Note that the second and third term are obtained via a recursion formula discriminating on the occurrence of a first jump due to the memoryless property of the exponential interarrival time distribution for the processes N_t^u and N_t^d. By differentiating with respect to t one obtains (8). $\diamond$

For fixed $t \in [0, T]$, $T < \infty$, S and t in eq. (8) can be considered as parameters and eq. (8) can be written in operational form:

$$\widetilde{g}_t = F_t + \Psi \widetilde{g}_t \equiv \Phi \widetilde{g}_t \qquad (10)$$

where

$$F_t = \lambda_u e^{-\lambda(t-\tau)} \int_{S-a}^{S} dz f^a\left(z, t\left|y,\tau\right.\right) + e^{-\lambda(t-\tau)} g_t^S\left(y,\tau\right) \equiv F_t^1 + F_t^2 \qquad (11)$$

with $\widetilde{g}_t \equiv \widetilde{g}_t\left(S, t\left|y,\tau\right.\right)$, $g_t^{\cdot}\left(y,\tau\right) = g\left(\cdot, t\left|y,\tau\right.\right)$ and Ψ is the integral operator with kernel

$$k(z,u;y,\tau) = e^{-\lambda(u-\tau)}\{\lambda_u f^a\left(z-a, u\left|y,\tau\right.\right) + \lambda_d f^a\left(z+a, u\left|y,\tau\right.\right) I_{(-\infty,S-a)}\left(z\right)\}. \qquad (12)$$

Note that the integral equation in two dimensions introduced with (10) is of Volterra type with respect to the time variable and of Fredholm type on the half line with respect to the space variable. Let us now consider for each $t \in [0,T]$ the Banach space $\mathcal{X}$ of bounded and continuous real functions on $(-\infty, S] \times [0,t)$ equipped with the norm $\|x\| = \sup_{y\in(-\infty,S]} \sup_{\tau\in[0,t)} |x(y,\tau)|$. The operator Φ is a bounded continuous operator on $\mathcal{X}$. Indeed it can be shown by employing $(a.)$ the properties of the transition p.d.f.'s and $(b.)$ the bounded convergence Theorem that:

a. $\sup_y \sup_\tau \int_0^t \int_{-\infty}^S |k(z,u;y,\tau)| \, dz du < \infty$;

b. $\lim_{\tau'\to\tau, y'\to y} \int_0^t \int_{-\infty}^S \left|k(z,u;y',\tau') - k(z,u;y,\tau)\right| dz du = 0.$

From now on, whenever they remain the same as above, we will drop the specification of the domains over which *sup* are taken. We have the following

Theorem 4 *The operator Φ defined through (10)-(12) is a contraction.*

Proof Let $x_1(y,\tau)$ and $x_2(y,\tau)$ be two functions belonging to $\mathcal{X}$. When $t \in [0,T]$ it holds

$$
\begin{aligned}
&|\Phi x_1(y,\tau) - \Phi x_2(y,\tau)| = \left| \int_\tau^t du \int_{-\infty}^S dz e^{-\lambda(u-\tau)} \{\lambda_u f^a\left(z-a,u\,|y,\tau\right) \right.\\
&\left. +\lambda_d f^a\left(z+a,u\,|y,\tau\right) I_{(-\infty,S-a)}\left(z\right)\} \left[x_1(z,u) - x_2(z,u)\right] \right|\\
&\le \int_\tau^t du \int_{-\infty}^S dz \left| e^{-\lambda(u-\tau)} \ \{\lambda_u f^a\left(z-a,u\,|y,\tau\right) + \lambda_d f^a\left(z+a,u\,|y,\tau\right) I_{(-\infty,S-a)}\left(z\right)\}\right| \times\\
&|x_1(z,u) - x_2(z,u)| \le \lambda_u \int_\tau^t du \int_{-\infty}^S dz e^{-\lambda(u-\tau)} f^a\left(z-a,u\,|y,\tau\right) |x_1(z,u)-x_2(z,u)|\\
&+\lambda_d \int_\tau^t du \int_{-\infty}^{S-a} dz e^{-\lambda(u-\tau)} f^a\left(z+a,u\,|y,\tau\right) |x_1(z,u)-x_2(z,u)|
\end{aligned}
\tag{13}
$$

where the triangular inequality and obvious properties of the functions *exp* and f^a have been employed. By taking the sup of $|x_1(z,u) - x_2(z,u)|$ first over the space variable and then also over the time variable from (13) we get:

$$
\begin{aligned}
&|\Phi x_1(y,\tau) - \Phi x_2(y,\tau)| \le \int_\tau^t du \sup_z |x_1(z,u) - x_2(z,u)| \{\lambda_u \int_{-\infty}^S dz e^{-\lambda(u-\tau)} f^a\left(z-a,u\,|y,\tau\right)\\
&+\lambda_d \int_{-\infty}^{S-a} dz e^{-\lambda(u-\tau)} f^a\left(z+a,u\,|y,\tau\right)\}\\
&\le \sup_u \sup_z |x_1(z,u) - x_2(z,u)| \{\lambda_u \int_\tau^t du \int_{-\infty}^S dz e^{-\lambda(u-\tau)} f^a\left(z-a,u\,|y,\tau\right)\\
&+\lambda_d \int_\tau^t du \int_{-\infty}^{S-a} dz e^{-\lambda(u-\tau)} f^a\left(z+a,u\,|y,\tau\right)\}.
\end{aligned}
\tag{14}
$$

Let us now call the two terms inside curl brackets at the r.h.s. of the last inequality in (14) as I_1 and I_2 respectively. By exploiting the time homogeneity of the process W_t it holds:

$$
\begin{aligned}
&I_1 = \lambda_u \int_\tau^t du e^{-\lambda(u-\tau)} \int_{-\infty}^S dz f^a\left(z-a,u\,|y,\tau\right) = \lambda_u \int_0^{t-\tau} dv e^{-\lambda v} \int_{-\infty}^{S-a} dz f^a\left(z,v\,|y,0\right)\\
&\le \lambda_u \int_0^t dv e^{-\lambda v} \int_{-\infty}^{S-a} dz f^a\left(z,v\,|y,0\right) \le \lambda_u \sup_y \int_0^t dv e^{-\lambda v} \int_{-\infty}^{S-a} dz f^a\left(z,v\,|y,0\right) = \widetilde{I}_1
\end{aligned}
\tag{15}
$$

Proceeding in the same way for the term I_2 we obtain

$$
\begin{aligned}
&I_2 = \lambda_d \int_\tau^t du e^{-\lambda(u-\tau)} \int_{-\infty}^{S-a} dz f^a\left(z+a,u\,|y,\tau\right) = \lambda_d \int_0^{t-\tau} dv e^{-\lambda v} \int_{-\infty}^{S} dz f^a\left(z,v\,|y,0\right)\\
&\le \lambda_d \int_0^t dv e^{-\lambda v} \int_{-\infty}^{S} dz f^a\left(z,v\,|y,0\right) \le \lambda_d \sup_y \int_0^t dv e^{-\lambda v} \int_{-\infty}^{S} dz f^a\left(z,v\,|y,0\right) = \widetilde{I}_2
\end{aligned}
\tag{16}
$$

Calling now $\widetilde{I}_1 + \widetilde{I}_2 = K$ we have

$$
K < \lambda \sup_y \int_0^t dv e^{-\lambda v} \int_{-\infty}^S dz f^a\left(z,v\,|y,0\right) \le \lambda \sup_y \int_0^t dv e^{-\lambda v} \le 1. \tag{17}
$$

Note that this last passage stems from the property according to which, for $a, b, c \in \Re$, $a < b < c < S$:

$$\int_a^b dz f^a\left(z,\theta\left|y,\varsigma\right.\right) < \int_a^c dz f^a\left(z,\theta\left|y,\varsigma\right.\right) \forall\theta > \varsigma, \forall y < S. \tag{18}$$

Recalling (15) and (16) from (14) we get

$$|\Phi x_1(y,\tau) - \Phi x_2(y,\tau)| \leq K\left\|x_1 - x_2\right\| \tag{19}$$

Since (19) holds for any pair (y, τ) it holds for the supremum over the two variables, whence

$$\left\|\Phi x_1 - \Phi x_2\right\| \leq K\left\|x_1 - x_2\right\| \tag{20}$$

Hence the operator Φ is a contraction. $\diamond$

Let us now recall the definition of K:

$$K = \lambda_u \sup_y \int_0^t dv e^{-\lambda v} \int_{-\infty}^{S-a} dz f^a\left(z, v\left|y, 0\right.\right) + \lambda_d \sup_y \int_0^t dv e^{-\lambda v} \int_{-\infty}^{S} dz f^a\left(z, v\left|y, 0\right.\right). \tag{21}$$

From the result of the previous theorem it follows that the unique solution to (10) in $\mathcal{X}$ is

$$\widetilde{g}_t\left(y,\tau\right) = \lim_{n\to\infty} \Phi^n f^0\left(y,\tau\right) \tag{22}$$

where

$$\Phi^n f^0\left(y,\tau\right) \equiv \widetilde{g}_t^{(n)}\left(y,\tau\right) = \Phi\left(\Phi^{n-1} f^0\left(y,\tau\right)\right),\ n > 0;\ \Phi^0 f^0\left(y,\tau\right) \equiv f^0\left(y,\tau\right) \tag{23}$$

and $f^0\left(y,\tau\right)$ is an arbitrary function in $\mathcal{X}$. Furthermore the inequality

$$\left\|\Phi\widetilde{g}_t - \Phi^n f^0\right\| \leq \frac{K^n}{1-K}\left\|\Phi f^0 - f^0\right\| \tag{24}$$

holds for any integer $n \geq 0$ and any $f^0 \in \mathcal{X}$. If we make in particular the choice $f^0\left(y,\tau\right) \equiv \widetilde{g}_t^{(0)}\left(y,\tau\right) = e^{-\lambda(t-\tau)} g_t^S\left(y,\tau\right) \equiv F_t^2$, where F_t^2 has been defined in (11), the first iterate becomes

$$\begin{aligned}
&\widetilde{g}_t^{(1)}\left(y,\tau\right) = \Phi f^0\left(y,\tau\right) = e^{-\lambda(t-\tau)} g_t^S\left(y,\tau\right) + \lambda_u e^{-\lambda(t-\tau)} \textstyle\int_{S-a}^{S} dz f^a\left(z,t\left|y,\tau\right.\right) \\
&+\lambda_u e^{-\lambda(t-\tau)} \textstyle\int_\tau^t du \int_{-\infty}^{S-a} dz f_a\left(z,u\left|y,\tau\right.\right) g_t^{S-a}\left(z,u\right) \\
&+\lambda_d e^{-\lambda(t-\tau)} \textstyle\int_\tau^t du \int_{-\infty}^{S} dz f^a\left(z,u\left|y,\tau\right.\right) g_t^{S+a}\left(z,u\right).
\end{aligned} \tag{25}$$

Remark 5 *Note that (25) has an immediate interpretation in terms of sample path behavior for the process Y_t solution to (1) if suitable conditions are verified. It corresponds to considering the case of jumps of low frequency, but relevant amplitude with respect to the threshold, when most of the sample* *paths cross the boundary S either for pure diffusion without the occurrence of jumps or due to an upward jump when $Y_t \in [S-a, S)$ or for diffusion after the occurrence of at most a single (upward of downward) jump. The possible occurrence of a higher number of jumps is disregarded.*

Let us now consider the case for which

$$\lambda_u > \lambda_d; \lambda_u \ll 1. \tag{26}$$

We can now prove the following

Theorem 6 *Under the conditions (26) it holds for arbitrary $n > 1$:*

$$\left\| \widetilde{g}_t - \widetilde{g}_t^{(1)} \right\| \leq \frac{K^n}{1-K} \left(\lambda + \lambda_u \right) + \mathcal{O}\left(\lambda_u \right). \tag{27}$$

Proof From the triangular inequality for norms it follows that $\forall n \geq 2$

$$\left\| \widetilde{g}_t - \widetilde{g}_t^{(1)} \right\| \leq \left\| \widetilde{g}_t - \widetilde{g}_t^{(n)} \right\| + \left\| \widetilde{g}_t^{(n)} - \widetilde{g}_t^{(1)} \right\|. \tag{28}$$

As far as the first term on the r.h.s. of (28) is concerned, (24) gives

$$\left\| \widetilde{g}_t - \widetilde{g}_t^{(n)} \right\| \leq \frac{K^n}{1-K} \left\| \widetilde{g}_t^{(1)} - \widetilde{g}_t^{(0)} \right\|. \tag{29}$$

Let us now consider the norm $\left\| \widetilde{g}_t^{(1)} - \widetilde{g}_t^{(0)} \right\|$ in (29). From (25) one has

$$\begin{aligned} &\sup_y \sup_\tau \left| \widetilde{g}_t^{(1)} - \widetilde{g}_t^{(0)} \right| \leq \lambda_u \sup_y \sup_\tau \left| e^{-\lambda(t-\tau)} \int_{S-a}^{S} dz f^a\left(z, t \left| y, \tau\right.\right) \right| \\ &+ \lambda_u \sup_y \sup_\tau \left| e^{-\lambda(t-\tau)} \int_0^t du \sup_z \left[g_t^{S-a}\left(z, u\right) \right] \right| + \lambda_d \sup_y \sup_\tau \left| e^{-\lambda(t-\tau)} \int_0^t du \sup_z \left[g_t^{S+a}\left(z, u\right) \right] \right| \\ &\leq \lambda_u + \lambda_u + \lambda_d = \lambda_u + \lambda. \end{aligned} \tag{30}$$

Let us now turn to the second term in (28). By repeatedly employing the triangular inequality for norms we get

$$\left\| \widetilde{g}_t^{(n)} - \widetilde{g}_t^{(1)} \right\| \leq \sum_{k=1}^{n-1} \left\| \widetilde{g}_t^{(k+1)} - \widetilde{g}_t^{(k)} \right\|. \tag{31}$$

The first term in the sum at the r.h.s. of (31) is $\left\| \widetilde{g}_t^{(2)} - \widetilde{g}_t^{(1)} \right\|$. By applying the operator Φ defined in

(10) over the first iterate (25) and employing the relationship (7) we obtain that

$$\begin{aligned}
&\sup_y \sup_\tau \left|\widetilde{g}_t^{(2)} - \widetilde{g}_t^{(1)}\right| \le \lambda_u^2 \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_\tau^t du \int_{-\infty}^{S-a} dz f\left(z, u\,|y, \tau\right) \int_{S-a}^{S} dx f\left(x, t\,|z+a, u\right)\right] \\
&+\lambda_d \lambda_u \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_\tau^t du \int_{-\infty}^{S} dz f\left(z, u\,|y, \tau\right) \int_{S-a}^{S} dx f\left(x, t\,|z-a, u\right)\right] \\
&+\lambda_u^2 \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_\tau^t du \int_{-\infty}^{S-a} dz f\left(z, u\,|y, \tau\right) \int_u^t d\theta \int_{-\infty}^{S-a} dx f\left(x, \theta\,|z+a, u\right) g_t^{S-a}\left(x, \theta\right)\right] \\
&+\lambda_d \lambda_u \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_\tau^t du \int_{-\infty}^{S-a} dz f\left(z, u\,|y, \tau\right) \int_u^t d\theta \int_{-\infty}^{S} dx f\left(x, \theta\,|z+a, u\right) g_t^{S+a}\left(x, \theta\right)\right] \qquad (32) \\
&+\lambda_d \lambda_u \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_\tau^t du \int_{-\infty}^{S} dz f\left(z, u\,|y, \tau\right) \int_u^t d\theta \int_{-\infty}^{S-a} dx f\left(x, \theta\,|z-a, u\right) g_t^{S-a}\left(x, \theta\right)\right] \\
&+\lambda_d^2 \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_\tau^t du \int_{-\infty}^{S} dz f\left(z, u\,|y, \tau\right) \int_u^t d\theta \int_{-\infty}^{S} dx f\left(x, \theta\,|z-a, u\right) g_t^{S+a}\left(x, \theta\right)\right] \\
&\equiv D_1 + D_2 + D_3 + D_4 + D_5 + D_6.
\end{aligned}$$

Interchanging the order of integration term D_1 can be rewritten as

$$\begin{aligned}
D_1 &= \lambda_u^2 \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_\tau^t du \int_{S-a}^{S} dx \int_{-\infty}^{S-a} dz f\left(z, u\,|y, \tau\right) f\left(x, t\,|z+a, u\right)\right] \\
&\le \lambda_u^2 \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_0^t du \int_{S-a}^{S} dx f\left(x, t\,|y+a, \tau\right)\right] \le \lambda_u^2 t.
\end{aligned} \qquad (33)$$

where for the first inequality the spatial homogeneity of the process W_t and the Chapman-Kolmogorov equation for the transition p.d.f. of diffusion processes have been employed. Proceeding in an analogous way for terms D_2 and D_3 one has:

$$D_2 \le \lambda_d \lambda_u \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_0^t du \int_{S-a}^{S} dx f\left(x, t\,|y-a, \tau\right)\right] \le \lambda_d \lambda_u t; \qquad (34)$$

$$\begin{aligned}
D_3 &\le \lambda_u^2 \sup_y \sup_\tau \left[e^{-\lambda(t-\tau)} \int_0^t du \int_u^t d\theta \int_{-\infty}^{S-a} dx f\left(x, \theta\,|y+a, \tau\right) g_t^{S-a}\left(x, \theta\right)\right] \\
&\le \lambda_u^2 \sup_y \left[\int_0^t du \int_u^t d\theta \sup_x g_t^{S-a}\left(x, \theta\right)\right] \le \lambda_u^2 t.
\end{aligned} \qquad (35)$$

Terms D_4, D_5 and D_6 can be treated in a similar way. By iterating the previous passages for the terms with $k > 1$ in the sum at the r.h.s. of (31) and taking into account the conditions (26) it is possible to recognize that such terms are bounded from above for each k by $3 * 2^k \lambda_u^{k+1} \frac{t^k}{k!}$. From (31) we then get

$$\begin{aligned}
\sup_y \left|\widetilde{g}_t^{(n)} - \widetilde{g}_t^{(1)}\right| &\le \left[3 \sum_{k=1}^{n-1} 2^k \lambda_u^{k+1} \frac{t^k}{k!}\right] \le \left[3\lambda_u \sum_{k=1}^{\infty} 2^k \lambda_u^k \frac{t^k}{k!}\right] \\
&= \left[3\lambda_u \left(e^{2\lambda_u t} - 1\right)\right] \le 3\lambda_u e^{2\lambda_u t} = \mathcal{O}\left(\lambda_u\right).
\end{aligned} \qquad (36)$$

Inserting (36) into (28) one finally gets (27). $\diamond$

An example of the use of the approximate formula (25) is shown in Fig. 1. To validate the use of the approximation we compare the curve resulting from a suitable implementation of (25) with the corresponding sample frequency histogram obtained by adapting to jump-diffusion processes the FCT simulation technique proposed in Giraudo et al. (2001) for diffusion processes.

Remark 7 *In (25) the first-passage-time p.d.f.'s of the Wiener process through the different boundaries S, $S-a$ and $S+a$ are involved. Their contributions correspond to not overlapping terms in the FCT p.d.f., hence determine several peaks corresponding to the maxima of g_t^{S-a}, g_t^S and g_t^{S+a} respectively, if the jump amplitude is sufficiently high with respect to S and furthermore the variance per unit time σ^2 of the process W_t is sufficiently small to avoid the excessive superposition of the three shapes.*

Remark 8 *A different approximation for the FCT p.d.f. of the process solution to (1) can be employed if one chooses a parameter range corresponding to rather frequent but small amplitude jumps. In such instance the jump terms under suitable conditions can be summed up in the diffusion limit (cf. Kallianpur (1993)); the resulting process is another Wiener diffusion with first-passage-time p.d.f.*

$$g^{\widetilde{W}}\left(S,t|\,0,0\right)=\frac{S}{\sqrt{2\pi\left(\sigma^2+\sigma_J^2\right)}t^{\frac{3}{2}}}\exp\left\{-\frac{\left(S-\left(\mu+\mu_J\right)t\right)^2}{2\left(\sigma^2+\sigma_J^2\right)t}\right\}. \tag{37}$$

where μ_J and σ_J^2 are as in (2). The cases where (37) holds give rise to unimodal distributions.

An example of such type of FCT p.d.f. is shown in Fig. 2, where again the comparison is made with the corresponding sample frequency histogram.

EXAMPLES

In Sacerdote et al. (2003) a jump-diffusion model analoguous to (1) has been employed to describe a model neuron. Here we make use of the results of Section 3 to explain the multimodality in the FCT density shape that has been reported there to appear under suitable conditions and to relate the values of the parameters involved. In this application the diffusion component of the process accounts for the input contributions coming to the neuron from distal synapses with very high frequency but low intensity, while the jump processes correspond to much less frequent but relevant excitatory (upward jumps) or inhibitory (downward jumps) inputs from proximal synapses (cf. Musila and Lánský (1991)). The model neuron is supposed to produce an action potential or spike when the membrane potential reaches a given constant threshold level $S>0$. The mathematical counterpart of the intertimes between successive spikes is the FCT of the jump-diffusion process through S. According to biologically reasonable values we choose $S=10\ mV$, $\mu=1.5\ mVms^{-1}$ and $a=-i=7.5\ mV$. The jump size is fixed since the contributions to post-synaptic potentials coming from proximal synapses can be considered constant. In Fig. 3 we plot the FCT distribution for $\lambda_u = 0.048\ ms^{-1}$ and $\lambda_d = 0.015\ ms^{-1}$ with $\sigma^2 = 9, 2, 0.75, 0.25\ mV^2ms^{-1}$. This choice implies that $\mu_J = 0.25\ mVms^{-1}$ and $\sigma_J^2 = 3.5\ mV^2ms^{-1}$. Note that for small values of σ^2 the FCT p.d.f exhibits a multimodal shape that becomes best established when values of the order of 10^{-1} are chosen. This hints to the presence of preferential or characteristic times in the firing behavior of the model neuron considered, corresponding to the modes of the interspike times distribution.

To study the role of the noise originating from the jump processes, in Fig. 4 we show the FCT p.d.f. when the same Wiener process as above is considered, but for $\mu_J = 0.25\ mVms^{-1}$ and $\sigma_J^2 = 3.5, 5.5, 8\ mV^2ms^{-1}$. The jump frequency values, obtained by solving the equations for μ_J and σ_J^2 with respect to λ_u and λ_d, are indicated in the following Table:

Table 1 *here*

For $\mu = 1.5\ mVms^{-1}$ and $\sigma^2 = 0.25\ mV^2ms^{-1}$ the shape of the FCT p.d.f. is significantly influenced by the amount of noise originating from the jump process. Note that the case of $\sigma_J^2 = 8\ mV^2ms^{-1}$ belongs to the range for which the validity of the approximate formula (25) is not assured and the corresponding FCT curve is only qualitatively reliable. Interpreting the modes of the FCT p.d.f. as the characteristic times of the overall process, they are always unchanged when σ_J^2 is varied. Furthermore the relative location of the probability mass is strongly influenced by the ratio between the excitatory and inhibitory jump frequencies.

CONCLUSIONS

An approximate formula for the FCT p.d.f. of a Wiener process with superposed constant amplitude jumps occurring at exponentially distributed random times has been introduced that can allow to study the features of such function in a series of cases of interest for applications.

In particular it can be justified how a particular tuning of the diffusion and of the jump parameters can produce a multimodal behavior of the FCT p.d.f. This phenomenon had been already observed in pure diffusion models with an added periodic term in the drift (cf. Shimokawa et al. (1999)). The results obtained suggest that the multimodality can arise also in systems with no periodicity but in the presence of two different sources of noise.

Analoguous behaviors can be expected for more complex models, but in such case it seems difficult to determine approximate formulae for the FCT density such as (25).

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FIGURE CAPTIONS

Fig. 1. Comparison between FCT p.d.f. obtained by means of (25) (continuous line) and FCT normalized sample frequency histogram obtained via the simulation technique quoted in Section 3. The parameters are $\mu = 4$ and $\sigma^2 = 0.25$ for the underlying Wiener process with drift, $a = 7.5$, $\lambda_u = 0.089$ and $\lambda_d = 0.034$ so that $\mu_J = 0.4125$ and $\sigma_J^2 = 6.91875$ for the jump process.

Fig. 2. Comparison between FCT normalized sample frequency histogram obtained by means of simulation and of formula (37) (continuous line). Here $\mu = 1.5$, $\sigma^2 = 0.25$, $a = 0.2$, $\lambda_u = 6$ and $\lambda_d = 5$.

Fig. 3. FCT p.d.f. curves corresponding to different values of the parameter σ^2. Here $\mu = 1.25$ $mVms^{-1}$, $a = 7.5$ mV, $\lambda_u = 0.048$ ms^{-1}, $\lambda_d = 0.015$ ms^{-1} and $\sigma^2 = 9$ (dots), 2 (dash-dots), 0.75 (continuous line), 0.25 (dashes) mV^2ms^{-1}.

Fig. 4. FCT p.d.f. curves for different values of σ_J^2. Same values of the parameters μ and σ^2 as in Fig. 2, $a = 7.5$ mV, $\mu_J = 0.25$ $mVms^{-1}$ and $\sigma_J^2 = 3.5$ (dashes), 5.5 (continuous line), 8 (dots) mV^2ms^{-1}.

Table 1: Jump frequency values for $a = -i = 7.5$ mV, $\mu_J = 0.25$ $mVms^{-1}$ and $s_J^2 = 3.5, 5.5, 8$ mV^2ms^{-1} respectively.

σ_J^2	3.5	5.5	8
λ_u	0.048	0.065	0.088
λ_d	0.014	0.032	0.054

Fig. 1

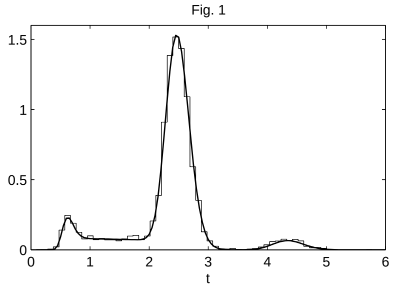

Fig. 2

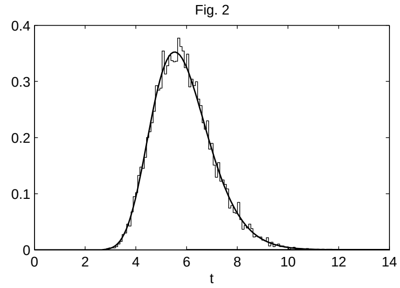

Fig. 3

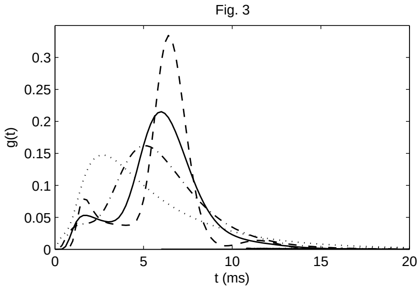

Fig. 4