Entire extensions and exponential decay for semilinear elliptic equations

Entire extensions and exponential decay for semilinear elliptic equations

Marco Cappiello [a], Todor Gramchev [b] and Luigi Rodino [c]

Abstract

We consider semilinear partial differential equations in $\mathbb{R}^n$ of the form

$$\sum_{\frac{|\alpha|}{m}+\frac{|\beta|}{k}\leq 1} c_{\alpha\beta}x^\beta D_x^\alpha u = F(u)$$

where k and m are given positive integers. Relevant examples are semilinear Schrödinger equations

$$-\Delta u + V(x)u = F(u)$$

where the potential $V(x)$ is given by an elliptic polynomial. We propose techniques, based on anisotropic generalizations of the global ellipticity condition of M. Shubin and multiparameter Picard type schemes in spaces of entire functions, which lead to new results for entire extensions and asymptotic behaviour of the solutions. Namely we study solutions (eigenfunctions and homoclinics) in the framework of the Gelfand-Shilov spaces $S_\nu^\mu(\mathbb{R}^n)$. Critical thresholds are identified for the indices μ and ν, corresponding to analytic regularity and asymptotic decay, respectively. In the one-dimensional case $-u''+V(x)u=F(u)$ our results for linear equations link up with those given by the classical asymptotic theory and by the theory of ODE in the complex domain, whereas for homoclinics new phenomena concerning analytic extensions are described.

2000 Mathematical Subject Classification: Primary 35B65, Secondary 35B40.

1 Introduction and main results

In this paper we study the exponential decay and the holomorphic extensions of the solutions to semilinear equations of the form $Pu = F(u)$ globally defined in $\mathbb{R}^n$, where the linear term P is an anisotropic globally elliptic partial differential operator with polynomial coefficients, cf. Shubin [23], Helffer [15], Boggiatto, Buzano and Rodino [3]. Such class of operators generalize the Schrödinger operators with elliptic polynomial potentials

$$H = -\Delta + V(x), \qquad x \in \mathbb{R}^n. \tag{1.1}$$

To introduce the reader to our results and to the function spaces used in the following, let us consider (1.1) in the one-dimensional case

$$Hu = -u''(x) + (a_0x^{2h} + a_1x^{2h-1} + \ldots + a_{2h})u(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where h is a positive integer, $a_j \in \mathbb{C}, j = 0, 1, \ldots, 2h$, and

$$a_0 \notin \mathbb{R}_- \cup \{0\}. \tag{1.3}$$

Note the study of the asymptotic behaviour of the solutions of

$$-u''(x) + (a_0 x^{2h} + a_1 x^{2h-1} + \ldots + a_{2h})u(x) = 0 \tag{1.4}$$

for $x \to \infty$ is a classical subject and interesting "per se" in the asymptotic theory of linear ODEs (we cite some fundamental works: Sibuya [24], [25], Szegö [26], Wasov [28], see also Mascarello and Rodino [17], Chapter 7). For the solutions $u \in L^2(\mathbb{R})$ of (1.4), the theory of the asymptotic integration implies that u decays like $\exp(-\varepsilon|x|^{h+1})$, $\varepsilon > 0$, for $x \to \infty$. Main issue in the following will be to combine this information on the decay with the one on the regularity. Namely, it will follow from our results that such solutions, extending to entire functions $u(z)$ in the complex domain, satisfy for some $A > 0, \varepsilon > 0$ an estimate of the form

$$|\partial_z^\alpha u(z)| \leq A^{|\alpha|+1}(\alpha!)^{h/(h+1)} e^{-\varepsilon|z|^{h+1}} \tag{1.5}$$

for z in a conic neighborhood of the real axis in $\mathbb{C}$. Such estimates, with term $(\alpha!)^{h/(h+1)}$ for α-derivatives, are optimal and, as far as we know, new in literature. They apply to a number of special functions appearing as solutions of (1.4), see Section 5. It is interesting to observe that our global ellipticity condition (1.3) for (1.2) corresponds to a dichotomy exponential growth/decay for the solutions of (1.4), see Section 5 for a more precise description in terms of asymptotic theory. By a rotation in the complex plane, this property transfers to straight lines in the complex plane, provided global ellipticity is preserved.

The estimates (1.5) lead in a natural way to the idea that the appropriate functional framework, to study the holomorphic extensions and the decay on infinity simultaneously, is given by the Gelfand-Shilov spaces of type S (cf. the classical book of Gelfand and Shilov [12], see also Mityagin [18], Pilipovic [19]). We recall that $f \in S_\nu^\mu(\mathbb{R}^n), \mu > 0, \nu > 0, \mu + \nu \geq 1$, iff $f \in C^\infty(\mathbb{R}^n)$ and there exist $A > 0$, $\varepsilon > 0$ such that

$$|\partial_x^\alpha f(x)| \leq A^{|\alpha|+1}(\alpha!)^\mu e^{-\varepsilon|x|^{1/\nu}} \tag{1.6}$$

for all $x \in \mathbb{R}^n, \alpha \in \mathbb{Z}_+^n$ or, equivalently, one can find $C > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial_x^\alpha f(x)| \leq C^{|\alpha|+|\beta|+1}(\alpha!)^\mu (\beta!)^\nu, \qquad \alpha, \beta \in \mathbb{Z}_+^n. \tag{1.7}$$

The bounds (1.6), (1.7) with $\mu < 1$ grant that f extends to $\mathbb{C}^n$ as an entire function with uniform estimates, see [12] for precise statements. So for example (1.5) reads as $u \in S_{1/(h+1)}^{h/(h+1)}(\mathbb{R})$.
Concerning recent applications of Gelfand-Shilov spaces, we mention that for traveling (solitary) wave solutions to dispersive and dissipative equations, the S_ν^μ-regularity with index $\mu = 1$, joint to exponential decay, i.e. $\nu = 1$, was recently studied by Bona and Li [4], Bondareva and Shubin [5], Biagioni and Gramchev [2], Gramchev [13], Cappiello, Gramchev and Rodino [8].
Let us now go back to the initial model, i.e. the Schrödinger operator (1.1) in $\mathbb{R}^n$. We assume that

$$V(x) = V_0(x) + R(x), \qquad x \in \mathbb{R}^n, \tag{1.8}$$

where $V_0(x)$ is a homogeneous elliptic polynomial with complex coefficients of degree $2h$. Generalizing the condition (1.3) of the one-dimensional case, we set

$$V_0(x) \notin \mathbb{R}_- \cup \{0\}, \qquad x \in \mathbb{R}^n \setminus 0, \tag{1.9}$$

while $R(x)$ is a polynomial of degree at most $2h-1$ (i.e. anisotropic generalizations of the multidimensional harmonic oscillator $-\Delta + |x|^2$ appearing in Quantum Mechanics). It is known that super-exponential decay estimates of type $\exp[-\varepsilon|x|^{h+1}]$, $\varepsilon > 0$, hold also for second order partial differential equations, under the assumptions (1.8), (1.9). The main interest here comes historically from Quantum Mechanics, where the exponential decay of eigenfunctions has been intensively studied, see for instance Agmon [1], Hislop and Sigal [16], Rabinovich [22], Buzano [6] and the references quoted therein. We also mention Davies [10], Davies and Simon [11] and the recent works of Rabier [20], Rabier and Stuart [21].
It is natural to discuss the validity of the bound (1.5), i.e. the information $u \in S^{h/(h+1)}_{1/(h+1)}$ in the n-dimensional case. To this end, further generalizing to higher order linear operators, we first study the S^μ_ν-regularity of eigenfunctions to anisotropic Shubin type partial differential operators in $\mathbb{R}^n$

$$P = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} x^\beta D^\alpha_x, \tag{1.10}$$

where k and m are positive integers. Here we use the standard notation $D^\alpha_x = (-i)^{|\alpha|}\partial^\alpha_x$. We assume that P is anisotropic (m,k)-globally elliptic, namely, there exist $C > 0$ and $R > 0$ such that

$$\left| \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} x^\beta \xi^\alpha \right| \geq C(|x|^{2k} + |\xi|^{2m})^{1/2}, \qquad |x| + |\xi| \geq R. \tag{1.11}$$

Note that the operator H in (1.1), (1.2) satisfies (1.11) for $m = 2, k = 2h$ under the assumptions (1.8), (1.9). Anisotropic global ellipticity in the previous sense implies both local regularity and asymptotic decay of the solutions, namely we have the following basic result (see [3]): $Pu = f \in \mathcal{S}(\mathbb{R}^n)$ for $u \in \mathcal{S}'(\mathbb{R}^n)$ implies actually $u \in \mathcal{S}(\mathbb{R}^n)$. In this paper we want to improve this result focusing on the regularity of P in the Gelfand-Shilov classes $S^\mu_\nu(\mathbb{R}^n)$. Namely we shall prove the following theorem.

Theorem 1.1. *Assume that P in (1.10) is (m,k)-globally elliptic, i.e. (1.11) is satisfied. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is a solution of $Pu = f \in S^\mu_\nu(\mathbb{R}^n)$, with*

$$\mu \geq \mu_{cr} := \frac{k}{k+m}, \qquad \nu \geq \nu_{cr} := \frac{m}{k+m}, \tag{1.12}$$

then also $u \in S^\mu_\nu(\mathbb{R}^n)$. In particular, $Pu = 0$, $u \in \mathcal{S}'(\mathbb{R}^n)$, implies $u \in S^{k/(k+m)}_{m/(k+m)}(\mathbb{R}^n)$.

The proof of Theorem 1.1 will be given in the next Section 3. We address to Section 5 for a simple alternative proof in the one-dimensional case by means of asymptotic theory, and some examples of solutions. In the ODE case see also Gramchev and Popivanov [14] for related results. From (1.6), cf. [12], one easily deduces the following result in the complex domain, which we refer to eigenfunctions of P (if P is (m,k)-globally elliptic, also $P - \lambda, \lambda \in \mathbb{C}$, is (m,k)-globally elliptic).

Proposition 1.2. *Under the previous assumptions on P, if $u \in \mathcal{S}'(\mathbb{R}^n)$ is a solution of $Pu = \lambda u$, for some $\lambda \in \mathbb{C}$, then u extends to an entire function on $\mathbb{C}^n$ and, for suitable constants $\varepsilon > 0$, $\gamma > 0$ and $C > 0$*

$$|\partial_z^\alpha u(z)| \leq C^{|\alpha|+1}(\alpha!)^{\mu_{cr}} e^{-\varepsilon|z|^{1/\nu_{cr}}}, \quad z \in \mathbb{C}^n,\ |Imz| < \gamma|Rez|,\ \alpha \in \mathbb{Z}_+^n. \tag{1.13}$$

Notice that for $m = 2, k = 2h$, (1.13) gives the estimates (1.5).
The proof of Theorem 1.1 will provide also precise bounds for the constant ε in (1.13), which does not depend on compact perturbations of P.

We pass now to semilinear equations. With respect to the linear case, we shall require in addition that the spectrum $\sigma(P)$ of P in $L^2(\mathbb{R}^n)$ does not coincide with the whole complex plane. This assumption is not necessary in the linear case as we can read in the proof of Theorem 1.1. Concerning the nonlinear term, we shall assume that $F(u)$ is of the form

$$F(u) = \sum_{\ell=2}^{d} F_\ell u^\ell, \qquad F_\ell \in \mathbb{C}, \tag{1.14}$$

Hence we shall consider the equation

$$Pu = \sum_{\frac{|\alpha|}{m}+\frac{|\beta|}{k}\leq 1} c_{\alpha\beta} x^\beta D_x^\alpha u = F(u) + f, \tag{1.15}$$

where f is given, $f = 0$ or $f \in S_\nu^\mu(\mathbb{R}^n), \mu \geq \mu_{cr}, \nu \geq \nu_{cr}$. In Section 4 we shall prove the following theorem.

Theorem 1.3. *Let P of the form* (1.10) *satisfy* (1.11) *and assume that $\sigma(P) \neq \mathbb{C}$; let $F(u)$ be as in* (1.14) *and let $f \in S_\nu^\mu(\mathbb{R}^n)$, $\mu \geq \mu_{cr} = \frac{k}{k+m}$, $\nu \geq \nu_{cr} = \frac{m}{k+m}$. Let $s > n/2$ and let $u \in H^s(\mathbb{R}^n)$ be a solution of (1.15). Then*

$$u \in S_\nu^{\max\{1,\mu\}}(\mathbb{R}^n). \tag{1.16}$$

In particular, if $f = 0$ we obtain that any solution $u \in H^s(\mathbb{R}^n)$ of (1.15) *belongs to $S_{\nu_{cr}}^1(\mathbb{R}^n)$, that is, we have for positive constants C and ε:*

$$|\partial_x^\beta u(x)| \leq C^{|\beta|+1}\beta! e^{-\varepsilon|x|^{1/\nu_{cr}}}, \qquad x \in \mathbb{R}^n. \tag{1.17}$$

The key point in Theorem 1.3, that we want to emphasize, is that in the semilinear case we still have super-exponential decay of order $1/\nu_{cr}$, however in view of (1.17) the extension to the complex domain $u(z)$ is analytic in a strip $\{z \in \mathbb{C}^n : |\mathrm{Im} z| < T\}$ for some $T > 0$, not entire in general.

As we shall see in Section 4, our method allows to treat, at least for particular models, more general nonlinear terms than (1.14). Namely, we give a generalization of Theorem 1.3 for Schrödinger operators H defined by (1.1), (1.8), with $V_0(x) > 0$ for $x \in \mathbb{R}^n \setminus 0$ and $R(x)$ polynomial of degree at most $2h-1$ with real coefficients. We shall allow for H a more general nonlinear term of the form

$$F(x, u, \nabla u) = \sum_{2\leq \ell+|\gamma|\leq d} F_{\ell,\gamma}(x) u^\ell (\nabla u)^\gamma, \tag{1.18}$$

with $F_{\ell,\gamma}(x)$ polynomials in x such that

$$F_{\ell,\gamma}(x) = F_{\ell,\gamma} \in \mathbb{C} \quad \textit{if} \quad \gamma \neq 0, \qquad \textit{and} \qquad \deg(F_{\ell,0}(x)) \leq h. \tag{1.19}$$

We will obtain the following result.

Theorem 1.4. *Let H be the operator defined by* (1.1), (1.8), *with $V_0(x) > 0$ for $x \in \mathbb{R}^n \setminus 0$ and $R(x)$ real-valued and let $f \in S^\mu_\nu(\mathbb{R}^n)$ for some $\mu \geq \mu_{cr} = \frac{h}{h+1}, \nu \geq \nu_{cr} = \frac{1}{h+1}$. Then, if $u \in H^{s+1}(\mathbb{R}^n), s > n/2$, is a solution of the equation*

$$Hu = f + F(x, u, \nabla u), \tag{1.20}$$

with F as in (1.18), (1.19), *then $u \in S^{\max\{1,\mu\}}_\nu(\mathbb{R}^n)$.*

Theorem 1.3 in the particular case $k = m$, i.e. $\mu = \nu$, and Theorem 1.4 in the case $V_0(x) = |x|^2$ were already in [7]. With respect to [7], we follow here a different approach in the proofs, taking advantage of the next Proposition 2.4, joined with inductive estimates.

It is worth, in conclusion, to return to the one-dimensional equation (1.4) in the semilinear version

$$-u'' + (a_0x^{2h} + a_1x^{2h-1} + \ldots + a_{2h})u = F(x, u, u')$$

under the preceding assumptions on the coefficients a_j and the nonlinearity F. We have from Theorem 1.4 that every solution $u \in H^{s+1}(\mathbb{R}), s > 1/2$, extends to a holomorphic function $u(z)$ in the strip $\{z \in \mathbb{C} : |\text{Im}z| < T\}$ satisfying there

$$|\partial^\alpha_z u(z)| \leq A^{|\alpha|+1}\alpha! e^{-\varepsilon|z|^{h+1}},$$

for suitable positive constants A, T, ε. With respect to (1.5), entire extension is lost in general. We shall test this on a simple example in Section 5. The same example exhibits a solution with algebraic growth. This contraddicts in the semilinear case the dichotomy exponential growth/decay from the asymptotic theory.

2 Preliminaries

In this section we illustrate some basic properties of anisotropic globally elliptic operators of the form (1.10) and recall some equivalent formulations of the ellipticity condition (1.11). Moreover we prove that the Fourier transformation preserves the global ellipticity. This property will be crucial in the next sections to derive decay estimates for the solutions of (1.15). Finally we recall some recent characterization of Gelfand-Shilov spaces $S^\mu_\nu(\mathbb{R}^n)$ that will be instrumental in the proofs of our results in the next sections.
To place the operator (1.10) in the general theory of anisotropic operators, cf. [3], we recall that the Newton polyhedron of P is defined as the convex hull of the set $\mathcal{A} \cup \{(0, 0)\}$, where

$$\mathcal{A} = \{(\alpha, \beta) \in \mathbb{Z}^{2n}_+ : \frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1 \quad \textit{and} \quad c_{\alpha\beta} \neq 0\}.$$

We can also define the principal part of P as follows.

Definition 2.1. *Let P be defined by* (1.10) *for some positive integers k, m. We define the principal symbol $p_{m,k}(x, \xi)$ of P as the function*

$$p_{m,k}(x, \xi) = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} = 1} c_{\alpha\beta} x^\beta \xi^\alpha. \tag{2.1}$$

The global ellipticity condition (1.11) can be easily reformulated as follows, cf. [3].

Proposition 2.2. *Let P be an operator of the form* (1.10)*. Then* (1.11) *holds if and only if*

$$p_{m,k}(x,\xi) \neq 0 \qquad \textit{for all} \quad (x,\xi) \neq (0,0). \tag{2.2}$$

We now describe the action of the Fourier transformation on the operator (1.10).

Proposition 2.3. *Let P be an operator of the form* (1.10) *and let $u \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\widehat{Pu} = Q\hat{u}$$

where Q is an operator of the form

$$Q = \sum_{\frac{|\rho|}{k} + \frac{|\sigma|}{m} \leq 1} a_{\rho\sigma} y^\sigma D_y^\rho. \tag{2.3}$$

Moreover, P is (m,k)-globally elliptic if and only if Q is (k,m)-globally elliptic, i.e. the following estimate holds true for some positive constants C', R':

$$\left| \sum_{\frac{|\rho|}{k} + \frac{|\sigma|}{m} \leq 1} a_{\rho\sigma} y^\sigma \eta^\rho \right| \geq C'(|y|^{2m} + |\eta|^{2k})^{1/2} \qquad \textit{for} \qquad |y| + |\eta| \geq R' > 0. \tag{2.4}$$

Proof. Applying the standard properties of the Fourier transform and Leibniz formula we can compute as follows

$$\begin{aligned}
\widehat{Pu}(\xi) &= \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} (\widehat{x^\beta D_x^\alpha u})(\xi) \\
&= \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} D_x^\beta (\xi^\alpha \widehat{u}(\xi)) \\
&= \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} \sum_{\substack{\gamma \leq \alpha \\ \gamma \leq \beta}} \binom{\beta}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} \xi^{\alpha-\gamma} D_\xi^{\beta-\gamma} \widehat{u}(\xi) \\
&= Q\hat{u}(\xi),
\end{aligned}$$

where

$$Q = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} \sum_{\substack{\gamma \leq \alpha \\ \gamma \leq \beta}} \binom{\beta}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} y^{\alpha-\gamma} D_y^{\beta-\gamma} \tag{2.5}$$

and we observe that $\frac{|\alpha-\gamma|}{m} + \frac{|\beta-\gamma|}{k} \leq 1$ in (2.5). The first part of the proposition is proved. Moreover we notice from (2.5) that the principal symbol of Q is given by

$$q_{k,m}(y,\eta) = \sum_{\frac{|\rho|}{k} + \frac{|\sigma|}{m} = 1} c_{\sigma\rho} y^\sigma \eta^\rho = p_{m,k}(\eta, y) \qquad \textit{for all} \quad (y,\eta) \in \mathbb{R}^{2n}.$$

Then we can conclude the proof applying Proposition 2.2. □

To derive our estimates in Gelfand-Shilov classes, in the sequel we shall take advantage of a nice characterization of the space $S^\mu_\nu(\mathbb{R}^n)$ given by Chung, Chung and Kim [9] showing that it is sufficient to check (1.7) for $\alpha = 0$ and, separately, for $\beta = 0$. Moreover the space $S^\mu_\nu(\mathbb{R}^n)$ is also characterized via the Fourier transform. We recall this result in detail since it will be largely used in the next sections.

Proposition 2.4. *Let $\mu > 0, \nu > 0$ with $\mu + \nu \geq 1$ and let $f \in C^\infty(\mathbb{R}^n)$. Then the following conditions are equivalent:*
i) $f \in S^\mu_\nu(\mathbb{R}^n)$;
ii) There exist positive constants A_o, B_o and C_o such that

$$\sup_{x\in\mathbb{R}^n} |\partial_x^\alpha f(x)| \leq C_o A_o^{|\alpha|}(\alpha!)^\mu \qquad \textit{and} \qquad \sup_{x\in\mathbb{R}^n} |x^\beta f(x)| \leq C_o B_o^{|\beta|}(\beta!)^\nu$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$;
iii) There exist positive constants A_1, B_1 and C_1 such that

$$\sup_{x\in\mathbb{R}^n} |x^\beta f(x)| \leq C_1 A_1^{|\beta|}(\beta!)^\nu \qquad \textit{and} \qquad \sup_{\xi\in\mathbb{R}^n} |\xi^\alpha \hat{f}(\xi)| \leq C_1 B_1^{|\alpha|}(\alpha!)^\mu$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$;
iv) There exist positive constants A_2, B_2 and C_2 such that

$$\sup_{x\in\mathbb{R}^n} |\partial_x^\alpha f(x)| \leq C_2 A_2^{|\alpha|}(\alpha!)^\mu \qquad \textit{and} \qquad \sup_{\xi\in\mathbb{R}^n} |\partial_\xi^\beta \hat{f}(\xi)| \leq C_2 B_2^{|\beta|}(\beta!)^\nu$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$.

3 Linear estimates

In this section we prove regularity and decay estimates for the solutions of the linear equation $Pu = f$. Although the approach will be essentially the same as for the general equation (1.15), we prefer to treat the linear case separately for two reasons. The first is that for $F = 0$ in (1.15) the results hold under weaker assumptions on P and on the a priori regularity of the solution. The second, more important, reason is that in the linear case we are able to prove a stronger regularity for the solution as we already claimed in the Introduction. Let us start from the study of the Gevrey-analytic regularity of the solutions. To this end we need to introduce suitable scales of Sobolev norms.

Let $\mu \geq \mu_{cr} = \frac{k}{k+m}$. For fixed $\varepsilon > 0, s \geq 0$ we define the norm

$$|\!|\!| u |\!|\!|_{\{s,\mu;\varepsilon\}} = \sum_{\alpha\in\mathbb{Z}^n_+} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \|\partial_x^\alpha u\|_s$$

and the corresponding partial sum

$$E_N^{s,\mu;\varepsilon}[u] = \sum_{|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \|\partial_x^\alpha u\|_s,$$

where $\|\cdot\|_s$ denotes the standard norm in the Sobolev space $H^s(\mathbb{R}^n)$. By Stirling formula and Sobolev embedding estimates it easily follows that if a function u in

$C^\infty(\mathbb{R}^n)$ is such that $|||u|||_{\{s,\mu;\varepsilon\}} < +\infty$ for some $\varepsilon > 0, s \geq 0$, then u satisfies the global estimate

$$\sup_{\alpha\in\mathbb{Z}^n_+} C^{-|\alpha|}(\alpha!)^{-\mu} \sup_{x\in\mathbb{R}^n} |\partial_x^\alpha u(x)| < +\infty. \tag{3.1}$$

for some positive constant C.

Let us now consider the equation $Pu = f$, where P is an operator of the form (1.10) satisfying (1.11). Assume that we can find $\lambda \in \mathbb{C} \setminus \sigma(P)$. Since also $P - \lambda$ satisfies (1.11), then by the results in [3], the linear operator

$$(P - \lambda)^{-1} \circ x^q\partial_x^p : H^s(\mathbb{R}^n) \mapsto H^s(\mathbb{R}^n) \tag{3.2}$$

is continuous for any $p, q \in \mathbb{Z}^n_+$ with $\frac{|p|}{m} + \frac{|q|}{k} \leq 1$ and for every $s \geq 0$. Differentiating and introducing commutators in the equation $Pu = f$, we get for every $\alpha \in \mathbb{Z}^n_+$:

$$P(\partial_x^\alpha u) = \partial_x^\alpha f - [\partial_x^\alpha, P]u.$$

Then for $\lambda \notin \sigma(P)$ we obtain

$$(P - \lambda)(\partial_x^\alpha u) = \partial_x^\alpha f - \lambda\partial_x^\alpha u - [\partial_x^\alpha, P]u. \tag{3.3}$$

Fixed $\varepsilon > 0, \mu \geq \mu_{cr}$, we can now multiply both members of (3.3) by $\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}}$ and invert $P - \lambda$. We get

$$\begin{aligned}\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}}\partial_x^\alpha u &= \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}}(P-\lambda)^{-1}(\partial_x^\alpha f) - \lambda\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}}(P-\lambda)^{-1}(\partial_x^\alpha u)\\ &\quad -\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}}(P-\lambda)^{-1}[\partial_x^\alpha, P]u. \end{aligned}\tag{3.4}$$

Finally, taking H^s-norms and summing up for $|\alpha| \leq N$, we obtain

$$\begin{aligned}E_N^{s,\mu;\varepsilon}[u] &\leq \sum_{|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \left\|(P-\lambda)^{-1}(\partial_x^\alpha f)\right\|_s + |\lambda| \sum_{|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \left\|(P-\lambda)^{-1}(\partial_x^\alpha u)\right\|_s\\ &\quad + \sum_{|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \left\|(P-\lambda)^{-1}([\partial_x^\alpha, P]u)\right\|_s. \end{aligned}\tag{3.5}$$

We will prove the following result.

Theorem 3.1. *Let P in* (1.10) *satisfy* (1.11) *and assume that $\sigma(P) \neq \mathbb{C}$. Let moreover $f \in \mathcal{S}(\mathbb{R}^n)$ such that $|||f|||_{\{0,\mu;\varepsilon'\}} < +\infty$ for some $\mu \geq \mu_{cr}, \varepsilon' > 0$. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is a solution of the equation $Pu = f$, then $u \in \mathcal{S}(\mathbb{R}^n)$ and there exists $\varepsilon \in (0, \varepsilon']$ such that $|||u|||_{\{0,\mu;\varepsilon\}} < +\infty$. In particular, u satisfies* (3.1) *for some positive constant C.*

To prove the theorem we need to estimate the three terms in the right-hand side of (3.5) for $s = 0$ uniformly with respect to N. The most delicate term is the one containing commutators which must be written in a suitable form in order to get a sharp critical value for the regularity index μ. To treat it, we need some preliminary steps.

Lemma 3.2. *Let $\varrho \in]0,1[$, $r > 0$ and let b be a positive integer. Then*

$$t^{\varrho b} \leq r t^b + (1-\varrho)\left(\frac{\varrho}{r}\right)^{\varrho/(1-\varrho)}, \qquad t \geq 0. \tag{3.6}$$

Proof. Clearly we can assume $b = 1$, setting $t^b = z$. Define $g(z) = z^\varrho - rz$, $z \geq 0$. Since $g'(z) = \varrho z^{\varrho-1} - r = 0$ iff $z = z_{\varrho,r} = (\varrho/r)^{1/(1-\varrho)}$ we readily obtain that

$$\sup_{z\geq 0} g(z) = g(z_{\varrho,r}) = \left(\frac{\varrho}{r}\right)^{\varrho/(1-\varrho)} - r\left(\frac{\varrho}{r}\right)^{1/(1-\varrho)} = (1-\varrho)\left(\frac{\varrho}{r}\right)^{\varrho/(1-\varrho)}.$$

The proof is complete. □

Using Lemma 3.2 we can prove a crucial estimate.

Lemma 3.3. *Let $\mu > 0$, k, m be positive integers and let $\alpha, \gamma \in \mathbb{Z}_+^n$ such that $\alpha_j \geq 2\frac{\gamma_j(m+k)}{k} > 0$ for some $j \in \{1, \dots, n\}$ and let $\mu > 0$. Then for every $r > 0, \eta \geq 0$ we have*

$$\frac{\eta^{\alpha_j - \gamma_j \frac{m+k}{k}}}{|\alpha|^{\mu\left(\alpha_j - \gamma_j \frac{m+k}{k}\right)}} \leq r\frac{\eta^{\alpha_j}}{|\alpha|^{\mu\alpha_j}} + r^{1-\frac{\alpha_j k}{\gamma_j(m+k)}}. \tag{3.7}$$

Proof. We can write

$$\frac{\eta^{\alpha_j - \gamma_j \frac{m+k}{k}}}{|\alpha|^{\mu\left(\alpha_j - \gamma_j \frac{m+k}{k}\right)}} = \left(\frac{\eta}{|\alpha|^\mu}\right)^{\varrho\alpha_j}$$

where $\varrho = 1 - \frac{\gamma_j(m+k)}{\alpha_j k} \in (0,1)$. With this choice of ϱ we have $1 - \varrho = \frac{\gamma_j(m+k)}{\alpha_j k}$ and $\frac{\varrho}{1-\varrho} = \frac{\alpha_j k}{\gamma_j(m+k)} - 1$. Then applying Lemma 3.2 with $t = \frac{\eta}{|\alpha|^\mu}$ and $b = \alpha_j$, we obtain that for any $r > 0$:

$$\begin{aligned}
\frac{\eta^{\alpha_j - \gamma_j \frac{m+k}{k}}}{|\alpha|^{\mu\left(\alpha_j - \gamma_j \frac{m+k}{k}\right)}} &\leq r\frac{\eta^{\alpha_j}}{|\alpha|^{\mu\alpha_j}} + \frac{\gamma_j(m+k)}{\alpha_j k} r^{-\frac{\alpha_j k}{\gamma_j(m+k)}+1}\left(1 - \frac{\gamma_j(m+k)}{\alpha_j k}\right)^{\frac{\alpha_j k}{\gamma_j(m+k)}-1} \\
&= r\frac{\eta^{\alpha_j}}{|\alpha|^{\mu\alpha_j}} + \frac{\gamma_j(m+k)}{\alpha_j k}\frac{r^{-\frac{\alpha_j k}{\gamma_j(m+k)}+1}}{\left(1 - \frac{\gamma_j(m+k)}{\alpha_j k}\right)}\left(1 - \frac{\gamma_j(m+k)}{\alpha_j k}\right)^{\frac{\alpha_j k}{\gamma_j(m+k)}} \\
&\leq r\frac{\eta^{\alpha_j}}{|\alpha|^{\mu\alpha_j}} + \sup_{A\geq 2}\left[\frac{1}{A-1}\left(1 - \frac{1}{A}\right)^A\right] \cdot r^{1-\frac{\alpha_j k}{\gamma_j(m+k)}} \\
&\leq r\frac{\eta^{\alpha_j}}{|\alpha|^{\mu\alpha_j}} + r^{1-\frac{\alpha_j k}{\gamma_j(m+k)}}.
\end{aligned}$$

The lemma is proved. □

The following result is a straightforward consequence of Leibniz formula.

Lemma 3.4. *Let $\alpha, \rho, \sigma \in \mathbb{Z}_+^n$ and let k, m be positive integers. Then the following identity holds:*

$$\begin{aligned}
[\partial_x^\alpha, x^\sigma \partial_x^\rho]u &= \sum_{\substack{0\neq\gamma\leq\alpha \\ \gamma\leq\sigma}} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} x^{\sigma-\gamma}(\partial_x^{\alpha+\rho-\gamma}u) \\
&= \sum_{\substack{0\neq\gamma\leq\alpha \\ \gamma\leq\sigma}} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} x^{\sigma-\gamma}\partial_x^\rho \widetilde{\partial}^+(\partial_x^{\alpha-\gamma}\widetilde{\partial}^- u)
\end{aligned} \tag{3.8}$$

where $\widetilde{\partial}^{\pm} = \widetilde{\partial}^{\pm}_{\alpha,\gamma,k,m}$ *are the Fourier multipliers defined by the symbols*

$$\prod_{\substack{j=1 \\ \alpha_j > 2\gamma_j \frac{k+m}{k}}}^{n} |\xi_j|^{\pm \gamma_j m/k}. \tag{3.9}$$

To estimate the commutator, we use now the assumption $\mu \geq \mu_{cr}$.

Lemma 3.5. *Let* P *satisfy the assumptions of Theorem 3.1 and assume that* $\lambda \in \mathbb{C} \setminus \sigma(P)$*. Then for every* $u \in \mathcal{S}(\mathbb{R}^n)$ *and for every* $s \geq 0$ *there exist* $C_s > 0, \varepsilon > 0$ *such that*

$$\sum_{2n(k+m) \leq |\alpha| \leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \left\| (P-\lambda)^{-1} [\partial_x^\alpha, P] u \right\|_s \leq C_s \left(r E_N^{s,\mu;\varepsilon}[u] + \|u\|_{s+k+2m} \right). \tag{3.10}$$

for every integer $N \geq 2n(k+m)$*, for every* $r > 0$ *and for some* $\varepsilon > 0$ *independent of* N*.*

Proof. Let $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geq 2n(k+m)$. By Lemma 3.4 we can write

$$\begin{aligned} (P-\lambda)^{-1}[\partial_x^\alpha, P]u &= \sum_{\frac{|\rho|}{m} + \frac{|\sigma|}{k} \leq 1} c_{\rho\sigma} (P-\lambda)^{-1} \left([\partial_x^\alpha, x^\sigma] \partial_x^\rho u \right) \\ &= \sum_{\frac{|\rho|}{m} + \frac{|\sigma|}{k} \leq 1} c_{\rho\sigma} \sum_{\substack{0 \neq \gamma \leq \alpha \\ \gamma \leq \sigma}} \frac{\alpha!}{(\alpha-\gamma)!} \binom{\sigma}{\gamma} \times \\ &\qquad \times (P-\lambda)^{-1} \circ x^{\sigma-\gamma} \partial_x^\rho \widetilde{\partial}^+ \left(\partial_x^{\alpha-\gamma} \widetilde{\partial}^- u \right) \end{aligned} \tag{3.11}$$

with $\widetilde{\partial}^{\pm}$ defined as in (3.9). At this point, observe that the operator $(P-\lambda)^{-1} \circ x^{\sigma-\gamma} \partial_x^\rho \widetilde{\partial}^+$ is bounded from $H^s(\mathbb{R}^n)$ into itself for every $s \geq 0$ uniformly with respect to α, cf. [3]. Then, since $|\gamma| \leq |\sigma| \leq k$ in (3.11), we obtain

$$\frac{1}{|\alpha|^{\mu|\alpha|}} \|(P-\lambda)^{-1}[\partial_x^\alpha, P]u\|_s \leq C_s \frac{1}{|\alpha|^{\mu|\alpha|}} \sum_{\substack{0 \neq \gamma \leq \alpha \\ |\gamma| \leq k}} \prod_{i=1}^n \alpha_i^{\gamma_i} \cdot \|\partial_x^{\alpha-\gamma} \widetilde{\partial}^- u\|_s$$

for some positive constant C_s independent of α. Now, since $|\alpha| \geq 2n(k+m)$, we surely have $\alpha_j \geq 2\frac{k+m}{k}\gamma_j$ for some $j \in \{1, \dots, n\}$. Moreover we can write

$$\|\partial_x^{\alpha-\gamma} \widetilde{\partial}^- u\|_s = \left\| \langle \xi \rangle^s \prod_{\substack{j=1 \\ \alpha_j > 2\gamma_j \frac{k+m}{k}}}^{n} |\xi_j|^{\alpha_j - \gamma_j \frac{k+m}{k}} \cdot \prod_{\substack{h=1 \\ \alpha_h \leq 2\gamma_h \frac{k+m}{k}}}^{n} |\xi_h|^{\alpha_h - \gamma_h} \hat{u} \right\|, \tag{3.12}$$

where we denote by $\| \cdot \|$ the norm in $L^2(\mathbb{R}^n)$. On the other hand, for every $\mu \geq$

$\mu_{cr} = \frac{k}{k+m}$, we have

$$\begin{aligned}
\frac{\prod_{i=1}^n \alpha_i^{\gamma_i}}{|\alpha|^{\mu|\alpha|}} &\leq \left(\prod_{\substack{j=1 \\ \alpha_j > 2\gamma_j \frac{k+m}{k}}}^n \frac{|\alpha|^{\mu(\alpha_j - \gamma_j/\mu_{cr})}}{|\alpha|^{\mu\alpha_j - \gamma_j}} \cdot \frac{1}{|\alpha|^{\mu(\alpha_j - \gamma_j/\mu_{cr})}} \right) \cdot \prod_{\substack{h=1 \\ \alpha_h \leq 2\gamma_h \frac{k+m}{k}}}^n \frac{\alpha_h^{\gamma_h}}{|\alpha|^{\mu(\alpha_h - \gamma_h)}} \\
&\leq C \left(\prod_{\substack{j=1 \\ \alpha_j > 2\gamma_j \frac{k+m}{k}}}^n \frac{1}{|\alpha|^{\mu(\alpha_j - \gamma_j/\mu_{cr})}} \right) \cdot \prod_{\substack{h=1 \\ \alpha_h \leq 2\gamma_h \frac{k+m}{k}}}^n \frac{1}{|\alpha|^{\mu(\alpha_h - \gamma_h)}}. \qquad (3.13)
\end{aligned}$$

Now, for every $j \in \{1, \ldots, n\}$ such that $\alpha_j > 2\gamma_j \frac{k+m}{k}$ we can apply Lemma 3.3 with $\eta = |\xi_j|$ and we obtain that for every $r \in (0,1)$

$$\begin{aligned}
\frac{\prod_{i=1}^n \alpha_i^{\gamma_i}}{|\alpha|^{\mu|\alpha|}} \|\partial_x^{\alpha-\gamma} \widetilde{\partial}^- u\|_s &\leq \left\| \langle \xi \rangle^s \prod_{\substack{j=1 \\ \alpha_j > 2\gamma_j \frac{k+m}{k}}}^n r \frac{|\xi_j|^{\alpha_j}}{|\alpha|^{\mu\alpha_j}} \cdot \prod_{\substack{h=1 \\ \alpha_h \leq 2\gamma_h \frac{k+m}{k}}}^n \frac{|\xi_h|^{\alpha_h - \gamma_h}}{|\alpha|^{\mu(\alpha_h - \gamma_h)}} \hat{u} \right\| \\
&\quad + \left\| \langle \xi \rangle^s \prod_{\substack{h=1 \\ \alpha_h \leq 2\gamma_h \frac{k+m}{k}}}^n \frac{|\xi_h|^{\alpha_h - \gamma_h}}{|\alpha|^{\mu(\alpha_h - \gamma_h)}} \hat{u} \right\| \qquad (3.14)
\end{aligned}$$

Choosing $\varepsilon < 1$, summing over $|\alpha|$ and observing that

$$\begin{aligned}
&\sum_{2n(k+m) \leq |\alpha| \leq N} \varepsilon^{|\alpha|} \left\| \langle \xi \rangle^s \prod_{\substack{h=1 \\ \alpha_h \leq 2\gamma_h \frac{k+m}{k}}}^n \frac{|\xi_h|^{\alpha_h - \gamma_h}}{|\alpha|^{\mu(\alpha_h - \gamma_h)}} \hat{u} \right\| \\
&\leq C_s \|u\|_{s+k+2m} \sum_{2n(k+m) \leq |\alpha| \leq N} \varepsilon^{|\alpha|} \leq C'_s \|u\|_{s+k+2m}
\end{aligned}$$

for some constant $C'_s > 0$ independent of N, we finally deduce the estimate (3.10). $\square$

Proof of Theorem 3.1. By Corollary 8.1 in [3] we already know that $u \in \mathcal{S}(\mathbb{R}^n)$. To prove that $|||u|||_{\{0,\mu;\varepsilon\}} < +\infty$ we start from (3.5) for $s = 0$. Obviously, we have

$$\sum_{|\alpha| \leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \left\| (P - \lambda)^{-1} (\partial_x^\alpha f) \right\| \leq C |||f|||_{\{0,\mu;\varepsilon'\}} < +\infty \qquad (3.15)$$

for every $\varepsilon \leq \varepsilon'$. Concerning the second term, for every $\alpha \in \mathbb{Z}_+^n, \alpha \neq 0$ there exists $j = j_\alpha \in \{1, \ldots, n\}$ such that $\alpha_j > 0$. Writing $(P - \lambda)^{-1}(\partial_x^\alpha u) = (P - \lambda)^{-1} \circ \partial_{x_j} (\partial_x^{\alpha - e_j} u)$, by (3.2) the operator $(P - \lambda)^{-1} \circ \partial_{x_j}$ maps continuously $L^2(\mathbb{R}^n)$ into itself. Then we obtain

$$|\lambda| \sum_{|\alpha| \leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \left\| (P - \lambda)^{-1} (\partial_x^\alpha u) \right\| \leq C(\|u\| + \varepsilon E_{N-1}^{0,\mu;\varepsilon}[u]). \qquad (3.16)$$

The last term in (3.5) can be estimated applying Lemma 3.5. Then, choosing ε sufficiently small, there exists $C > 0$ such that for every $r \in (0,1)$ the following estimate holds:

$$E_N^{0,\mu;\varepsilon}[u] \le C \left(|\!|\!| f |\!|\!|_{\{0,\mu;\varepsilon'\}} + \varepsilon E_{N-1}^{0,\mu;\varepsilon}[u] + r E_N^{0,\mu;\varepsilon}[u] + \sum_{|\alpha|<2n(k+m)} \|\partial_x^\alpha u\| \right),$$

Taking now $r < C^{-1}$ we obtain

$$E_N^{0,\mu;\varepsilon}[u] \le \frac{C}{1-rC} \left(|\!|\!| f |\!|\!|_{\{0,\mu;\varepsilon'\}} + \varepsilon E_{N-1}^{0,\mu;\varepsilon}[u] + \sum_{|\alpha|<2n(k+m)} \|\partial_x^\alpha u\| \right).$$

Iterating this estimate and possibly shrinking ε, we conclude that $E_N^{0,\mu;\varepsilon}[u]$ is bounded with respect to N, hence $|\!|\!| u |\!|\!|_{\{0,\mu;\varepsilon\}} < +\infty$. □

Let us now study the decay of the solutions of $Pu = f$. Fixed $\nu \ge \nu_{cr} = \frac{m}{k+m}$, $\delta > 0, s \ge 0$ we define the norm

$$|\!|\!| u |\!|\!|_{s,\nu;\delta} = \sum_{\beta \in \mathbb{Z}_+^n} \frac{\delta^{|\beta|}}{|\beta|^{\nu|\beta|}} \|x^\beta u\|_s.$$

Using Sobolev embedding theorems it is easy to show that if $|\!|\!| u |\!|\!|_{s,\nu;\delta} < +\infty$ for some $\delta > 0, s > n/2$, then

$$\sup_{\beta \in \mathbb{Z}_+^n} C^{-|\beta|} (\beta!)^\nu \sup_{x \in \mathbb{R}^n} |x^\beta u(x)| < +\infty \tag{3.17}$$

for some positive constant C. Hence

$$\sup_{x \in \mathbb{R}^n} e^{A|x|^{1/\nu}} |u(x)| < +\infty \tag{3.18}$$

for some positive constant A.

We have the following result.

Theorem 3.6. *Let P in* (1.10) *satisfy the assumptions of Theorem 3.1 and let $f \in \mathcal{S}(\mathbb{R}^n)$ such that $|\!|\!| f |\!|\!|_{0,\nu;\delta'} < +\infty$ for some $\nu \ge \nu_{cr}, \delta' > 0$. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is a solution of the equation $Pu = f$, then $u \in \mathcal{S}(\mathbb{R}^n)$ and there exist $\delta \in (0, \delta']$ and $s > n/2$ such that $|\!|\!| u |\!|\!|_{s,\nu;\delta} < +\infty$. In particular, u satisfies* (3.18) *for some $A > 0$.*

Proof. As before, we know from [3] that $u \in \mathcal{S}(\mathbb{R}^n)$. Applying the Fourier transform to both members of $Pu = f$ and taking into account Proposition 2.3, we are reduced to study the equation $Q\hat{u} = \hat{f}$, where Q is (k,m)-globally elliptic and its symbol satisfies an estimate of the form (2.4). Moreover, if $\sigma(P) \neq \mathbb{C}$, then also $\sigma(Q) \neq \mathbb{C}$. Finally, the assumption on f and Parseval identity imply that $|\!|\!| \hat{f} |\!|\!|_{\{0,\nu;\delta'\}} < +\infty$. Then, interchanging k and m, we can apply Theorem 3.1 to the equation $Q\hat{u} = \hat{f}$ and we obtain that $|\!|\!| \hat{u} |\!|\!|_{\{0,\nu;\delta\}} < +\infty$ for some $\delta \in]0, \delta']$. Possibly shrinking δ this implies that $|\!|\!| \hat{u} |\!|\!|_{\{s,\nu;\delta\}} < +\infty$ for some integer $s > n/2$. But this is equivalent to say that $|\!|\!| u |\!|\!|_{s,\nu;\delta} < +\infty$. The theorem is then proved. □

Remark 1. *We observe that in the proof of Lemma 3.5 and Theorem 3.1 the parameter* $\varepsilon > 0$ *for which* $\|u\|_{0,\mu;\varepsilon} < +\infty$ *can be chosen independent of* u*. The same holds for the parameter* δ *in Theorem 3.6. As a consequence of this fact our result on pointwise decay estimates* (3.18) *can be reformulated more precisely as follows: there exists a constant* $A > 0$ *such that* (3.18) *holds for every solution* $u \in \mathcal{S}'(\mathbb{R}^n)$ *of the equation* $Pu = f$.

Using Theorems 3.1 and 3.6 we can easily prove Theorem 1.1.

Proof of Theorem 1.1. We can assume without loss of generality that P is self-adjoint; otherwise we can apply the L^2-adjoint $P^\star$ of P to both sides of $Pu = f$ and reduce to the equation $Su = g$ where $S = P^\star P$ is (m,k)-elliptic and self-adjoint and $g = P^\star f \in S^\mu_\nu(\mathbb{R}^n)$ if $f \in S^\mu_\nu(\mathbb{R}^n)$, cf. [12]. Hence in particular the condition $\sigma(P) \neq \mathbb{C}$ is fulfilled. Moreover, since $f \in S^\mu_\nu(\mathbb{R}^n)$ with $\mu \geq \mu_{cr}, \nu \geq \nu_{cr}$, then it satisfies the assumptions of both Theorems 3.1 and 3.6. Therefore if $u \in \mathcal{S}'(\mathbb{R}^n)$ is a solution of $Pu = f$, then u satisfies both (3.1) and (3.18). Hence by ii) of Proposition 2.4, we have $u \in S^\mu_\nu(\mathbb{R}^n)$. □

4 Nonlinear estimates

In this section we consider the general semilinear equation (1.15). Without loss of generality we can assume that $F(u) = u^\ell$ for some $\ell \in \mathbb{Z}, \ell \geq 2$, cf. (1.14). As in the previous section, we first discuss the Gevrey-analytic regularity of the solutions. We begin to prove that the solutions belong to $\mathcal{S}(\mathbb{R}^n)$ as in the linear case. However, as standard in the nonlinear case, we have now to require an initial regularity of u.

Lemma 4.1. *Let* P *in* (1.10) *satisfy* (1.11) *and assume that there exists* $\lambda \in \mathbb{C}\backslash\sigma(P)$*. Let* $u \in H^s(\mathbb{R}^n), s > n/2$ *be a solution of the equation* (1.15) *with* F *as in* (1.14) *and* $f \in \mathcal{S}(\mathbb{R}^n)$*. Then* $x^\sigma \partial_x^\rho u \in H^s(\mathbb{R}^n)$ *for any* $\rho, \sigma \in \mathbb{Z}^n_+$.

Proof. We argue by induction on $|\rho + \sigma|$. We first show that $\partial_{x_j} u \in H^s(\mathbb{R}^n)$ for any $j \in \{1, \dots, n\}$. From (1.15), introducing commutators, we get

$$(P-\lambda)(\partial_{x_j} u) = \partial_{x_j} f - \sum_{\substack{\frac{|\alpha|}{m}+\frac{|\beta|}{k}\leq 1 \\ \beta_j \neq 0}} c_{\alpha\beta}\beta_j x^{\beta-e_j}\partial_x^\alpha u - \lambda\partial_{x_j} u + \partial_{x_j}(u^\ell).$$

Inverting $P-\lambda$ and passing to Sobolev norms, it follows that

$$\begin{aligned} \|\partial_{x_j} u\|_s \leq\ & \|(P-\lambda)^{-1}(\partial_{x_j} f)\|_s + \sum_{\substack{\frac{|\alpha|}{m}+\frac{|\beta|}{k}\leq 1 \\ \beta_j \neq 0}} |c_{\alpha\beta}| \cdot \beta_j \cdot \|(P-\lambda)^{-1}(x^{\beta-e_j}\partial_x^\alpha u)\|_s \\ & + |\lambda| \cdot \|(P-\lambda)^{-1}(\partial_{x_j} u)\|_s + \|(P-\lambda)^{-1}(\partial_{x_j}(u^\ell))\|_s. \end{aligned} \tag{4.1}$$

Clearly $\|(P-\lambda)^{-1}(\partial_{x_j} f)\|_s < +\infty$ since $f \in \mathcal{S}(\mathbb{R}^n)$. Moreover, by (3.2) the operators $(P-\lambda)^{-1} \circ x^{\beta-e_j}\partial_x^\alpha$ and $(P-\lambda)^{-1} \circ \partial_{x_j}$ are bounded from $H^s(\mathbb{R}^n)$ into itself, then also the second and the third term in the right-hand side of (4.1) are finite. Concerning the nonlinear term, arguing as before and applying Schauder's lemma, we get

$$\|(P-\lambda)^{-1}(\partial_{x_j}(u^\ell))\|_s \leq C_s\|u^\ell\|_s \leq C'_s\|u\|_s^\ell < +\infty.$$

Hence $\partial_{x_j} u \in H^s(\mathbb{R}^n)$. Arguing similarly, it is easy to prove that $x_k u \in H^s(\mathbb{R}^n)$ for any $k \in \{1, \ldots, n\}$, then the lemma is true for $|\rho + \sigma| = 1$. For $|\rho + \sigma| > 1$, arguing as before we have

$$\begin{aligned} \|x^\sigma \partial_x^\rho u\|_s \ \le\ & \|(P-\lambda)^{-1}(x^\sigma \partial_x^\rho f)\|_s + |\lambda| \cdot \|(P-\lambda)^{-1}(x^\sigma \partial_x^\rho u)\|_s \\ & + \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \le 1} |c_{\alpha\beta}| \cdot \|(P-\lambda)^{-1}([x^\beta \partial_x^\alpha, x^\sigma \partial_x^\rho] u\|_s \\ & + \|(P-\lambda)^{-1}(x^\sigma \partial_x^\rho (u^\ell))\|_s. \end{aligned} \tag{4.2}$$

The first term in the right-hand side of (4.2) is clearly finite. Assume for example that $\sigma_j > 0$ for some $j \in \{1, \ldots, n\}$. Then by (3.2) we have

$$|\lambda| \cdot \|(P-\lambda)^{-1}(x^\sigma \partial_x^\rho u)\|_s = |\lambda| \cdot \|(P-\lambda)^{-1}(x_j x^{\sigma - e_j} \partial_x^\rho u)\|_s \le C_s \|x^{\sigma - e_j} \partial_x^\rho u\|_s < +\infty$$

by the inductive assumption. Concerning the third term, we can use the commutator identity

$$\begin{aligned} [x^\sigma \partial_x^\rho, x^\beta \partial_x^\alpha] u &= x^\sigma \partial_x^\rho (x^\beta \partial_x^\alpha u) - x^\beta \partial_x^\alpha (x^\sigma \partial_x^\rho u) \\ &= \sum_{\substack{\delta \le \rho \\ \delta \le \beta}}{}' \frac{\rho!}{(\rho - \delta)!} \binom{\beta}{\delta} \sum_{\substack{\gamma \le \sigma \\ \gamma \le \alpha}}{}' \frac{\sigma!}{(\sigma - \gamma)!} \binom{\alpha}{\gamma} (-1)^{|\gamma|} \times \\ & \quad \times x^{\beta - \delta} \partial_x^{\alpha - \gamma} (x^{\sigma - \gamma} \partial_x^{\rho - \delta} u), \end{aligned}$$

where $\sum' \sum'$ means that $|\gamma| + |\delta| > 0$. Then by (3.2) and by the inductive assumptions we get that

$$\sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \le 1} |c_{\alpha\beta}| \cdot \|(P-\lambda)^{-1}([x^\beta \partial_x^\alpha, x^\sigma \partial_x^\rho] u)\|_s < +\infty.$$

For the last term in (4.2) we can argue similarly. □

Theorem 4.2. *Let P in* (1.10) *be (m, k)-globally elliptic, i.e. it satisfies* (1.11)*, and assume that there exists $\lambda \in \mathbb{C} \setminus \sigma(P)$. Let $u \in H^s(\mathbb{R}^n), s > n/2$ be a solution of the equation* (1.15) *where F is of the form* (1.14) *and $f \in \mathcal{S}(\mathbb{R}^n)$ with $|||f|||_{\{s,\mu;\varepsilon'\}} < +\infty$ for some $\mu \ge \mu_{cr}, \varepsilon' > 0$. Then there exists $\varepsilon \in]0, \varepsilon']$ such that $|||u|||_{\{s,\tilde{\mu};\varepsilon\}} < +\infty$, where $\tilde{\mu} = \max\{\mu, 1\}$.*

Lemma 4.3. *Let P satisfy the assumptions of Theorem 4.2. Then for every $\mu \ge 1, s > n/2$, there exists a constant $C'_s > 0$ such that for any $\varepsilon > 0, N \in \mathbb{Z}_+$:*

$$\sum_{|\alpha| \le N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \left\| (P-\lambda)^{-1}(\partial_x^\alpha u^\ell) \right\|_s \le C'_s \left(\|u\|_s^\ell + \varepsilon (E_{N-1}^{s,\mu;\varepsilon}[u])^\ell \right). \tag{4.3}$$

Proof. For every $\alpha \in \mathbb{Z}_+^n, \alpha \neq 0$, there exists $j = j_\alpha \in \{1, \ldots, n\}$ such that $\alpha_j > 0$. Then, writing $(P-\lambda)^{-1} \circ \partial_x^\alpha = (P-\lambda)^{-1} \circ \partial_{x_j} \circ \partial_x^{\alpha - e_j}$ and applying (3.2) we get

$$\sum_{0 \neq |\alpha| \le N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \left\| (P-\lambda)^{-1}(\partial_x^\alpha (u^\ell)) \right\|_s \le C_s \varepsilon \sum_{0 \neq |\alpha| \le N} \frac{\varepsilon^{|\alpha| - 1}}{|\alpha|^{\mu|\alpha|}} \|\partial_x^{\alpha - e_j} (u^\ell)\|_s.$$

We now apply Leibniz rule which gives

$$\partial_x^{\alpha-e_j}(u^\ell) = \sum_{\alpha_1+\ldots+\alpha_\ell=\alpha-e_j} \frac{(\alpha-e_j)!}{\alpha_1!\ldots\alpha_\ell!}\partial_x^{\alpha_1}u\cdot\ldots\cdot\partial_x^{\alpha_\ell}u$$

and observe that for $\mu \geq 1$, we have the estimate

$$\frac{1}{|\alpha|^{\mu|\alpha|}}\frac{(\alpha-e_j)!}{\alpha_1!\cdot\ldots\cdot\alpha_\ell!} \leq \prod_{\rho=1}^{\ell}\frac{1}{|\alpha_\rho|^{\mu|\alpha_\rho|}}.$$

Hence, applying Schauder's lemma we easily obtain (4.3). □

Proof of Theorem 4.2. Arguing as for the linear case, we can write, for every $\varepsilon > 0, N \in \mathbb{Z}_+$:

$$\begin{aligned} E_N^{s,\tilde{\mu};\varepsilon}[u] \quad \leq \quad & \sum_{|\alpha|\leq N}\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\tilde{\mu}|\alpha|}}\left\|(P-\lambda)^{-1}(\partial_x^\alpha f)\right\|_s + |\lambda|\sum_{|\alpha|\leq N}\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\tilde{\mu}|\alpha|}}\left\|(P-\lambda)^{-1}(\partial_x^\alpha u)\right\|_s \\ & + \sum_{|\alpha|\leq N}\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\tilde{\mu}|\alpha|}}\left\|(P-\lambda)^{-1}([\partial_x^\alpha, P]u)\right\|_s \\ & + \sum_{|\alpha|\leq N}\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\tilde{\mu}|\alpha|}}\left\|(P-\lambda)^{-1}(\partial_x^\alpha(u^\ell))\right\|_s. \end{aligned} \tag{4.4}$$

The first two terms in the right-hand side of (4.4) can be estimated as in the proof of Theorem 3.1. The estimate of the term containing commutators is easier than in the linear case, because the case $\tilde{\mu} < 1$ is now excluded. In fact, we can write

$$(P-\lambda)^{-1}([\partial_x^\alpha, P]u) = \sum_{\frac{|\rho|}{m}+\frac{|\sigma|}{k}\leq 1} c_{\rho\sigma}\sum_{\substack{0\neq\gamma\leq\alpha\\ \gamma\leq\sigma}}\frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma}(P-\lambda)^{-1}\circ x^{\sigma-\gamma}\partial_x^\rho\left(\partial_x^{\alpha-\gamma}u\right).$$

Since $\tilde{\mu} \geq 1$ we have

$$\frac{1}{|\alpha|^{\tilde{\mu}|\alpha|}}\frac{\alpha!}{(\alpha-\gamma)!} \leq \frac{|\alpha|^{|\gamma|}}{|\alpha|^{\tilde{\mu}|\gamma|}}\cdot\frac{1}{|\alpha-\gamma|^{\tilde{\mu}|\alpha-\gamma|}} \leq \frac{1}{|\alpha-\gamma|^{\tilde{\mu}|\alpha-\gamma|}}.$$

Then we directly obtain

$$\sum_{|\alpha|\leq N}\frac{\varepsilon^{|\alpha|}}{|\alpha|^{\tilde{\mu}|\alpha|}}\left\|(P-\lambda)^{-1}([\partial_x^\alpha, P]u)\right\|_s \leq C_s\varepsilon E_{N-1}^{s,\tilde{\mu};\varepsilon}[u].$$

The last term can be estimated applying Lemma 4.3. Finally we get

$$E_N^{s,\tilde{\mu};\varepsilon}[u] \leq C_s\left(\|u\|_s + \|u\|_s^\ell + \|\!|f|\!\|_{\{s,\mu;\varepsilon'\}} + \varepsilon E_{N-1}^{s,\tilde{\mu};\varepsilon}[u] + \varepsilon\left(E_{N-1}^{s,\tilde{\mu};\varepsilon}[u]\right)^\ell\right),$$

and for ε sufficiently small we can iterate this estimate obtaining that $\sup_{N\in\mathbb{Z}_+} E_N^{s,\tilde{\mu};\varepsilon}[u] < +\infty$. This concludes the proof. □

To prove the decay properties for the solutions of (1.15), we can argue as in the previous section. Applying the Fourier transformation to (1.15) we obtain the new equation

$$Q\hat{u} = \hat{f} + \widehat{F(u)}, \tag{4.5}$$

where Q is (k,m)-globally elliptic.

Theorem 4.4. *Let P satisfy the assumptions of Theorem 4.2 and let $u \in H^s(\mathbb{R}^n)$, $s > n/2$ be a solution of (1.15), where F is of the form (1.14) and $f \in \mathcal{S}(\mathbb{R}^n)$ with $|||f|||_{s,\nu;\delta'} < +\infty$ for some $\nu \geq \nu_{cr}, \delta' > 0$. Then there exists $\delta \in (0, \delta']$ such that $|||u|||_{s,\nu;\delta} < +\infty$.*

Lemma 4.5. *Let Q be (k,m)-globally elliptic with $\sigma(Q) \neq \mathbb{C}$ and let $u \in \mathcal{S}(\mathbb{R}^n)$. Then, fixed $\lambda \in \mathbb{C} \setminus \sigma(Q)$, $s > n/2, \delta > 0, \nu \geq \nu_{cr}$ there exists $C > 0$ such that*

$$\sum_{|\alpha| \leq N} \frac{\delta^{|\alpha|}}{|\alpha|^{\nu|\alpha|}} \left\| (Q-\lambda)^{-1} (\partial_\xi^\alpha \widehat{u^\ell}) \right\|_s \leq C \left(\|\hat{u}\|_s^\ell + \delta \|u\|_s^{\ell-1} \cdot E_{N-1}^{s,\nu;\delta}[\hat{u}] \right). \tag{4.6}$$

for every $N \in \mathbb{Z}_+$.

Proof. If $\alpha_j \neq 0$ for some $j \in \{1, \ldots, n\}$, then we have that $(Q-\lambda)^{-1}(\partial_\xi^\alpha(\widehat{u^\ell})) = (Q-\lambda)^{-1}(\partial_{\xi_j} \partial_x^{\alpha - e_j} (\widehat{u^\ell}))$. Moreover, since the linear operator $(Q-\lambda)^{-1} \circ \partial_{\xi_j}$ is continuous from $H^s(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$, we obtain

$$\sum_{0 \neq |\alpha| \leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\nu|\alpha|}} \|(Q-\lambda)^{-1}(\partial_\xi^\alpha(u^\ell))\|_s \leq C_s \varepsilon \sum_{0 \neq |\alpha| \leq N} \frac{\varepsilon^{|\alpha|-1}}{|\alpha|^{\nu|\alpha|}} \|\partial_\xi^{\alpha - e_j} \widehat{u^\ell}\|_s.$$

Now, using standard properties of the Fourier transform and Sobolev embedding estimates we obtain

$$\begin{aligned}
\|\partial_\xi^{\alpha-e_j} \widehat{u^\ell}\|_s &= \|(\partial_\xi^{\alpha-e_j} \hat{u}) * \widehat{u^{\ell-1}}\|_s \\
&= \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{2s} \left| \mathcal{F}_{\xi \to \eta} \left(\partial_\xi^{\alpha-e_j} \hat{u} * \widehat{u^{\ell-1}} \right)(\eta) \right|^2 d\eta \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n} \langle \eta \rangle^{2s} \left| \widehat{\partial_\eta^{\alpha-e_j} \hat{u}}(\eta) \right|^2 \cdot \left| u^{\ell-1}(\eta) \right|^2 d\eta \right)^{1/2} \\
&\leq C_s \|u\|_s^{\ell-1} \cdot \|\partial_\xi^{\alpha-e_j} \hat{u}\|_s.
\end{aligned}$$

The lemma is then proved. □

Proof of Theorem 4.4. First of all, by Lemma 4.1, it follows that $u \in \mathcal{S}(\mathbb{R}^n)$. As in the proof of Theorem 3.6 it is sufficient to show that there exists $\delta > 0$ such that $|||\hat{u}|||_{\{s,\nu;\delta\}} < \infty$. Starting from (4.5) and taking $\lambda \in \mathbb{C} \setminus \sigma(Q)$, we get, for every $\alpha \in \mathbb{Z}_+^n$:

$$\partial_\xi^\alpha \hat{u} = (Q-\lambda)^{-1}(\partial_\xi^\alpha \hat{f}) - \lambda (Q-\lambda)^{-1}(\partial_\xi^\alpha \hat{u}) - (Q-\lambda)^{-1}[Q, \partial_\xi^\alpha]\hat{u} + (Q-\lambda)^{-1}(\partial_\xi^\alpha \widehat{u^\ell}).$$

We can now apply Lemma 3.5 and Lemma 4.5. We obtain that there exists $C > 0$ independent of N such that the following estimate holds true:

$$E_N^{s,\nu;\delta}[\hat{u}] \leq \frac{C}{1 - rC} \left(|||\hat{f}|||_{\{s,\nu;\delta'\}} + \delta E_{N-1}^{s,\nu;\delta}[\hat{u}] + \delta \|u\|_s^{\ell-1} \cdot E_{N-1}^{s,\nu;\delta}[\hat{u}] + \sum_{|\alpha| < 2n(k+m)} \|\partial_\xi^\alpha \hat{u}\|_s \right)$$

for every r with $0 < r < C^{-1}$ and for some $\delta \in]0, \delta']$. Iterating the last estimate and possibly shrinking δ we obtain that $|||\hat{u}|||_{\{s,\nu;\delta\}} < +\infty$. We leave the details to the reader. $\square$

Arguing as in the previous section, the proof of Theorem 1.3 is a direct consequence of Theorems 4.2 and 4.4 combined with Proposition 2.4.

We conclude this section giving the proof of Theorem 1.4. As for the equation (1.15), we prove separately regularity and decay estimates, but for the linear part of the equation the estimates are the same proved before. To conclude we only need to give estimates for the new nonlinear term coming from (1.18), (1.19). That is what we do in the next lemmas.

Lemma 4.6. *Let H be as in Theorem 1.4 and let $\lambda \in \mathbb{C} \setminus \sigma(H)$. Then, for every $\mu \geq 1, s > n/2, \varepsilon \in (0,1), \ell, N \in \mathbb{Z}_+, \ell \geq 2, q, \gamma \in \mathbb{Z}_+^n, |q| \leq h$, and for every $u \in H^{s+1}(\mathbb{R}^n)$ there exist positive constants C_s, C'_s such that the following estimates hold:*

$$\sum_{|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \|(H-\lambda)^{-1}(\partial_x^\alpha(x^q u^\ell))\|_s \leq C_s \left(\|u\|_s^\ell + \varepsilon (E_{N-1}^{s,\mu;\varepsilon}[u])^\ell \right); \tag{4.7}$$

$$\sum_{|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \|(H-\lambda)^{-1}(\partial_x^\alpha(u^\ell(\nabla u)^\gamma))\|_s \leq C'_s \left(\|u\|_{s+1}^{\ell+|\gamma|} + \varepsilon (E_{N-1}^{s,\mu;\varepsilon}[u])^{\ell+|\gamma|} \right). \tag{4.8}$$

Proof. We start by proving (4.7). Fixed $\alpha \neq 0$, let $j = j_\alpha \in \{1, \dots, n\}$ such that $\alpha_j > 0$. We have

$$\partial_x^\alpha(x^q u^\ell) = x^q \partial_{x_j} \partial_x^{\alpha - e_j}(u^\ell) + \sum_{\substack{\alpha' \leq \alpha \\ 0 \neq \alpha' \leq q}} \binom{\alpha}{\alpha'} \frac{q!}{(q-\alpha')!} x^{q-\alpha'} \partial_x^{\alpha-\alpha'}(u^\ell).$$

Observe that (3.2) with $m = 2, k = 2h$ implies that the operators $(H-\lambda)^{-1} \circ x^q \partial_{x_j}$ and $(H-\lambda)^{-1} \circ x^{q-\alpha'}$ are bounded from $H^s(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$ since $|q| \leq h$. Then, arguing as in the proof of Lemma 4.3, we easily obtain (4.7). Concerning (4.8), for $|\alpha| \geq 2$, we can write $(H-\lambda)^{-1} \circ \partial_x^\alpha = (H-\lambda)^{-1} \circ \partial_{x_i} \partial_{x_j} \circ \partial_x^{\alpha - e_i - e_j}$ for some $i, j \in \{1, \dots, n\}$ and apply (3.2), then

$$\begin{aligned} \sum_{2\leq|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \|(H-\lambda)^{-1}(\partial_x^\alpha(u^\ell(\nabla u)^\gamma))\|_s &\leq C'_s \sum_{2\leq|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\mu|\alpha|}} \|\partial_x^{\alpha-e_i-e_j}(u^\ell(\nabla u)^\gamma)\|_s \\ &\leq C'_s \varepsilon (E_{N-2}^{s,\mu;\varepsilon}[u])^\ell (E_{N-1}^{s,\mu;\varepsilon}[u])^{|\gamma|} \\ &\leq C'_s \varepsilon (E_{N-1}^{s,\mu;\varepsilon}[u])^{\ell+|\gamma|}. \end{aligned}$$

Then we get (4.8). $\square$

Repeating readily the steps of the proof of Theorem 4.2 with the aid of Lemma 4.6, we can easily prove that if $f \in \mathcal{S}(\mathbb{R}^n)$ with $|||f|||_{\{s,\mu;\varepsilon'\}} < +\infty$ for some $\mu \geq \mu_{cr}, s > n/2, \varepsilon' > 0$, and $u \in H^{s+1}(\mathbb{R}^n)$ is a solution of (1.20), then $|||u|||_{\{s,\tilde{\mu};\varepsilon\}} < +\infty$ for some $\varepsilon \in (0, \varepsilon']$, with $\tilde{\mu} = \max\{1, \mu\}$.
To prove decay estimates for (1.20) we apply the Fourier transform to both sides of (1.20). We obtain the new equation

$$\widehat{H}\hat{u} = \hat{f} + F(\widehat{x, u, \nabla} u), \tag{4.9}$$

where

$$\widehat{H} = Q(D) + |\xi|^2,$$

$Q(D)$ being an elliptic operator with constant coefficients of order $2h$. To prove regularity estimates for $\hat{u}$, we need the following lemma.

Lemma 4.7. *Let $\widehat{H}$ be the operator defined by* (4.9) *and let $\lambda \in \mathbb{C} \setminus \sigma(\widehat{H})$. Then, for every $\nu \geq \nu_{cr}, s > n/2, \varepsilon \in (0,1), \ell, N \in \mathbb{Z}_+, q, \gamma \in \mathbb{Z}^n_+$, with $|q| \leq h$ and $\ell + |\gamma| \geq 2$ and for every $u \in H^{s+1}(\mathbb{R}^n)$ there exist positive constants C_s, C'_s such that the following estimates hold:*

$$\sum_{|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\nu|\alpha|}} \|(\widehat{H}-\lambda)^{-1}(\partial_\xi^\alpha(\widehat{x^q u^\ell}))\|_s \leq C_s \left(\|\hat{u}\|_s^\ell + \varepsilon \|u\|_s^{\ell-1} \cdot E_{N-1}^{s,\nu;\varepsilon}[\hat{u}]\right); \tag{4.10}$$

$$\sum_{|\alpha|\leq N} \frac{\varepsilon^{|\alpha|}}{|\alpha|^{\nu|\alpha|}} \|(\widehat{H}-\lambda)^{-1}(\partial_\xi^\alpha(\widehat{u^\ell(\nabla u)^\gamma}))\|_s \leq C'_s \left(\|\hat{u}\|_{s+1}^{\ell+|\gamma|} + \varepsilon \|u\|_{s+1}^{\ell+|\gamma|-1} \cdot E_{N-1}^{s,\nu;\varepsilon}[\hat{u}]\right). \tag{4.11}$$

Proof. The proof of (4.10) is immediate. In fact, for every $\alpha \in \mathbb{Z}^n_+, \alpha \neq 0$ we have:

$$\|(\widehat{H}-\lambda)^{-1}(\partial_\xi^\alpha(\widehat{x^q u^\ell}))\|_s = \|(\widehat{H}-\lambda)^{-1}(\partial_\xi^{q+e_j}\partial_\xi^{\alpha-e_j}\widehat{u^\ell})\|_s \leq C_s \|\partial_\xi^{\alpha-e_j}\widehat{u^\ell}\|_s$$

and then we conclude as in the proof of Lemma 4.6. To prove (4.11), we observe that if $\ell \geq 1$, we have for every $\alpha \in \mathbb{Z}^n_+, \alpha \neq 0$:

$$\begin{aligned} \|(\widehat{H}-\lambda)^{-1}(\partial_\xi^\alpha(\widehat{u^\ell(\nabla u)^\gamma}))\|_s &\leq C_s \|\partial_\xi^{\alpha-e_j}(\hat{u} * (\widehat{u^{\ell-1}(\nabla u)^\gamma})\|_s \\ &\leq C_s \|\partial_\xi^{\alpha-e_j}\hat{u}\|_s \cdot \|u^{\ell-1}(\nabla u)^\gamma\|_s \\ &\leq C_s \|\partial_\xi^{\alpha-e_j}\hat{u}\|_s \cdot \|u\|_{s+1}^{\ell+|\gamma|-1}. \end{aligned}$$

For $\ell = 0, |\gamma| \geq 2$, we can argue similarly. We leave the details to the reader. □

With the aid of Lemma 4.7, arguing as in the proof of Theorem 4.4, we obtain that if $f \in \mathcal{S}(\mathbb{R}^n)$ is such that $|||f|||_{s,\nu;\delta'} < +\infty$ for some $s > n/2, \nu \geq \nu_{cr}, \delta' > 0$, and $u \in H^{s+1}(\mathbb{R}^n)$ is a solution of (1.20), then there exists $\delta \in (0, \delta']$ such that $|||u|||_{s,\nu;\delta} < +\infty$. We conclude observing that under the assumptions of Theorem 1.4, we have both $|||u|||_{\{s,\tilde{\mu};\varepsilon\}} < +\infty$ and $|||u|||_{s,\nu;\delta} < +\infty$ for some positive ε, δ and $s > n/2$. Combining these two estimates we easily obtain the proof of Theorem 1.4.

Remark 2. *We observe that our method can be easily adapted to a larger class of operators satisfying more general anisotropic estimates. Namely, fixed two multi-indices $k = (k_1, \ldots, k_n)$, $m = (m_1, \ldots, m_n)$, with $k_j > 0, m_j > 0$ for any $j = 1, \ldots, n$, we can consider an operator of the form*

$$P = \sum_{(\alpha,\beta)\in A} c_{\alpha\beta} x^\beta D_x^\alpha, \qquad c_{\alpha\beta} \in \mathbb{C}, \tag{4.12}$$

where

$$A = \{(\alpha,\beta) \in \mathbb{Z}^{2n}_+ : \frac{\alpha_1}{m_1} + \ldots + \frac{\alpha_n}{m_n} + \frac{\beta_1}{k_1} + \ldots + \frac{\beta_n}{k_n} \leq 1\}.$$

The principal symbol $p_{m,k}(x,\xi)$ of P is defined by

$$p_{m,k}(x,\xi) = \sum_{(\alpha,\beta)\in\widetilde{A}} c_{\alpha\beta}x^\beta\xi^\alpha,$$

with

$$\widetilde{A} = \{(\alpha,\beta) \in \mathbb{Z}_+^{2n} : \frac{\alpha_1}{m_1} + \ldots + \frac{\alpha_n}{m_n} + \frac{\beta_1}{k_1} + \ldots + \frac{\beta_n}{k_n} = 1\}.$$

The operator P in (4.12) is said to be (m,k)-globally elliptic if

$$\left|\sum_{(\alpha,\beta)\in A} c_{\alpha\beta}x^\beta\xi^\alpha\right| \geq \sum_{j=1}^{n}(|x_j|^{k_j} + |\xi_j|^{m_j}) \qquad \text{for} \quad |x| + |\xi| \geq R \tag{4.13}$$

for some positive constants C, R, or equivalently if

$$p_{m,k}(x,\xi) \neq 0 \qquad \text{for all} \quad (x,\xi) \neq (0,0).$$

For this class it is natural to prove estimates in general Gelfand-Shilov classes describing the regularity and decay properties with respect to each variable separately. We recall here the definition and refer the reader to [12] for a detailed presentation of these spaces.

Definition 4.8. *Let $\mu = (\mu_1, \ldots, \mu_n), \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$, with $\mu_j > 0, \nu_j > 0$ for all $j = 1, \ldots, n$. We denote by $S^\mu_\nu(\mathbb{R}^n)$ the space of all functions $u \in C^\infty(\mathbb{R}^n)$ such that*

$$\sup_{x\in\mathbb{R}^n} |x^\beta\partial_x^\alpha u(x)| \leq A^{|\alpha|+|\beta|+1}\alpha_1^{\alpha_1\mu_1}\cdot\ldots\cdot\alpha_n^{\alpha_n\mu_n}\beta_1^{\beta_1\nu_1}\cdot\ldots\cdot\beta_n^{\beta_n\nu_n}$$

for some constant $A > 0$.

We notice that Proposition 2.4 has an obvious extension to this class, cf. [9]. The assertion of Theorem 1.1 can be reformulated in this new framework as follows: if P is an operator of the form (4.12) satisfying (4.13) and $f \in S^\mu_\nu(\mathbb{R}^n)$ with $\mu_j \geq \frac{k_j}{k_j+m_j}, \nu_j \geq \frac{m_j}{k_j+m_j}$ for any $j = 1, \ldots, n$ then every solution $u \in \mathcal{S}'(\mathbb{R}^n)$ of the equation $Pu = f$ actually belongs to $S^\mu_\nu(\mathbb{R}^n)$. Similarly, for the semilinear equation $Pu = f + F(u)$ with $F(u)$ as in (1.14), starting from a solution $u \in H^s(\mathbb{R}^n)$, we can prove that $u \in S^{\tilde{\mu}}_\nu(\mathbb{R}^n)$, where $\tilde{\mu}_j = \max\{1, \mu_j\}$ for every $j = 1, \ldots, n$. We leave the details to the reader.

5 The one-dimensional case: examples

Fixing first attention on linear operators, we consider P as in (1.10) and $p_{m,k}(x,\xi)$ as in (2.1):

$$P = \sum_{\frac{\alpha}{m}+\frac{\beta}{k}\leq 1} c_{\alpha\beta}x^\beta D_x^\alpha, \tag{5.1}$$

$$p_{m,k}(x,\xi) = \sum_{\frac{\alpha}{m}+\frac{\beta}{k}=1} c_{\alpha\beta}x^\beta\xi^\alpha, \tag{5.2}$$

where now $x \in \mathbb{R}, \xi \in \mathbb{R}$; we recall that $D_x = -i\frac{d}{dx}$. Assume that P is (m,k)-globally elliptic, i.e. in view of Proposition 2.2

$$p_{m,k}(x,\xi) \neq 0 \qquad \textit{for all} \qquad (x,\xi) \neq (0,0). \tag{5.3}$$

Consider then the algebraic equations

$$p_{m,k}(\pm 1,\lambda) = \sum_{\frac{\alpha}{m}+\frac{\beta}{k}=1} c_{\alpha\beta}(\pm 1)^{\beta}\lambda^{\alpha}, \qquad \lambda \in \mathbb{C}.$$

In view of (5.3), the order of the equations is m and all the roots, counted with multiplicity, $\lambda_1^{\pm}, \dots, \lambda_m^{\pm}$ satisfy the condition $\text{Im}\lambda_j^{\pm} \neq 0$. We may apply to P the results of the asymptotic theory [14], [24], [28]; the following rough statements will be sufficient for our purposes in the following.

Proposition 5.1. *There exist two fundamental systems* $u_1^+, \dots, u_m^+$ *and* $u_1^-, \dots, u_m^-$ *of solutions of* $Pu = 0$, *of the form*

$$u_j^{\pm}(x) = \exp[i\lambda_j^{\pm}\nu|x|^{1/\nu}]v_j^{\pm}(x), \quad j = 1,\dots,m, \tag{5.4}$$

with $\nu = m/(k+m)$ *and*

$$|v_j^{\pm}(x)| \leq C\exp[\delta|x|^{\sigma}] \qquad \textit{for} \quad x \in \mathbb{R}_{\pm} \tag{5.5}$$

for some $\sigma < 1/\nu$ *and positive constants* C *and* δ *(in the case of a multiple root* $\lambda_j^{\pm}$, *any linear combination of the corresponding independent solutions* $u_j^{\pm}$ *also satisfies* (5.4), (5.5)*).*

We begin by giving a cheap proof of Theorem 1.1 in the case of a homogeneous ordinary differential equation.

Proposition 5.2. *Let* P *be defined as in* (5.1), (5.2), (5.3). *Assume* $Pu = 0, u \in \mathcal{S}'(\mathbb{R})$; *then* $u \in S_{\nu}^{\mu}(\mathbb{R})$, *with* $\mu = k/(k+m), \nu = m/(k+m)$.

Proof. Since (5.3) implies $\text{Im}\lambda_j^{\pm} \neq 0$ in (5.4) for all $j = 1,\dots,m$, all the solutions $u_j^{\pm}$ in Proposition (5.1) have exponential growth, if $\text{Im}\lambda_j^{\pm} < 0$, or exponential decay if $\text{Im}\lambda_j^{\pm} > 0$, in $\mathbb{R}_{\pm}$. On the other hand we know that a solution $u \in \mathcal{S}'(\mathbb{R})$ of $Pu = 0$ belongs to $\mathcal{S}(\mathbb{R})$, hence $u \in \mathcal{S}(\mathbb{R}_+)$ and $u \in \mathcal{S}(\mathbb{R}_-)$. This implies that u is a linear combination of the u_j^+ which have exponential decay in $\mathbb{R}_+$, and simultaneously linear combination of the u_j^- with exponential decay in $\mathbb{R}_-$. Note in particular that, if $\text{Im}\lambda_j^+ < 0$ for all $j = 1,\dots,m$, or $\text{Im}\lambda_j^- < 0$ for all $j = 1,\dots,m$, then non-trivial solutions $u \in \mathcal{S}'(\mathbb{R})$ cannot exist. Otherwise, from (5.4), (5.5) we have

$$|u(x)| \leq Ce^{-\delta|x|^{1/\nu}}, \qquad x \in \mathbb{R}, \tag{5.6}$$

for any constant δ satisfying

$$0 < \delta < \min\{\nu\text{Im}\lambda_j^{\pm} : \text{Im}\lambda_j^{\pm} > 0\}$$

and a suitable constant C depending on δ. We now use Proposition 2.3, namely for every solution $u \in \mathcal{S}'(\mathbb{R})$ of $Pu = 0$ we may write

$$\widehat{Pu} = Q\hat{u} = 0$$

where Q is now (k,m)-globally elliptic. We then apply to the ordinary differential equation $Q\hat{u} = 0$ the preceding arguments, exchanging the role of k and m. We deduce

$$|\hat{u}(\xi)| \leq C' e^{-\delta'|\xi|^{1/\mu}}, \qquad \xi \in \mathbb{R}, \tag{5.7}$$

for some $C' > 0, \delta' > 0$. According to [9], we may read (5.6) and (5.7) as

$$\sup_{x\in\mathbb{R}} |x^\beta u(x)| \leq C_1 A_1^{|\beta|}(\beta!)^\nu, \qquad \sup_{\xi\in\mathbb{R}} |\xi^\alpha \hat{u}(\xi)| \leq C_1 B_1^{|\alpha|}(\alpha!)^\mu$$

for all $\alpha, \beta \in \mathbb{Z}_+$ and suitable positive constants A_1, B_1, C_1 independent of α, β. In view of iii) of Proposition 2.4, these estimates give the conclusion $u \in S_\nu^\mu(\mathbb{R})$. □

As obvious byproduct of Proposition 5.2, we may recapture, in a particular case, the celebrated non-triviality theorem of Gelfand and Shilov, cf. [12].

Proposition 5.3. *Let $\mu > 0, \nu > 0, \mu + \nu = 1$. Assume that $\mu, \nu \in \mathbb{Q}$. Then $S_\nu^\mu(\mathbb{R}) \neq \{0\}$, i.e. there exists a non-trivial function $u \in S_\nu^\mu(\mathbb{R})$.*

Proof. Consider the basic example of $(2p, 2h)$-globally elliptic operator in $\mathbb{R}$

$$P = D_x^{2p} + x^{2h}. \tag{5.8}$$

The spectrum of P is discrete, with eigenvalues $\lambda_j \to +\infty$ and eigenfunctions $\varphi_j, j = 1, 2, \ldots$, forming a complete orthogonal system in $L^2(\mathbb{R})$, see [3]. Since also $P - \lambda_j$ is $(2p, 2h)$-globally elliptic, from Proposition 5.2 we have $\varphi_j \in S_{p/(h+p)}^{h/(h+p)}(\mathbb{R})$. It remains then to observe that, for any given $\mu \in \mathbb{Q}, 0 < \mu < 1$, we may write $\mu = h/(h+p)$ for two positive integers h and p, and consequently $\nu = 1 - \mu = p/(h+p)$. Hence we read $\varphi_j \in S_\nu^\mu(\mathbb{R})$. □

To see more explicit examples of functions in $S_\nu^\mu(\mathbb{R})$, we may address to similar ordinary differential operators with polynomial coefficients. In particular, we recall, cf. [17], [24] that the $(2, 2h)$-globally elliptic equation

$$(D^2 + x^{2h} - \rho x^{h-1})u = 0 \tag{5.9}$$

admits non-trivial solutions in $L^2(\mathbb{R})$, hence in $S_{1/(h+1)}^{h/(h+1)}(\mathbb{R})$, for special values of the parameter ρ, namely:

- When h is even, for $\rho = 2(h+1)N + h + 1, N \in \mathbb{Z}$, the solution is given in $\mathbb{R}_+$ by

$$u(x,\rho) = \exp[-x^{h+1}/(h+1)]\Psi\left(\frac{\rho+h}{2(h+1)}, \frac{h}{h+1}; \frac{2x^{h+1}}{h+1}\right) \tag{5.10}$$

whose analytic extension coincides in $\mathbb{R}_-$ with $u(-x, -\rho)$. To be definite, we recall the definition of the Tricomi function Ψ, cf. [27]:

$$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} \cdot \Phi(a - c + 1, 2 - c; x)$$

where the principal branch of x^{1-c} is chosen and Φ is the hypergeometric confluent function

$$\Phi(a, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \cdot \frac{x^n}{n!};$$

as standard, for $r \in \mathbb{R} : (r)_0 = 1, (r)_n = r(r+1) \cdot \ldots \cdot (r+n-1), n \geq 1$. We have

$$\Psi(a, c; x) \sim x^{-a} \sum_{n=0}^{\infty} \binom{c-a-1}{n} \frac{(a)_n}{x^n} \qquad \textit{for} \quad x \to +\infty, \tag{5.11}$$

which gives the expected exponential decay in (5.10).

• When h is odd, for $\rho = -2(h+1)N - h$ or else $\rho = -2(h+1)N - h - 2, N \in \mathbb{Z}$, the solution in $L^2(\mathbb{R})$ of (5.9) is of the form

$$u(x, \rho) = \exp[-x^{h+1}/(h+1)] P_\rho(x) \tag{5.12}$$

where $P_\rho(x)$ is a polynomial, cf. [26].
On the expression (5.12) we may directly recognize that $u \in S_{1/(h+1)}^{h/(h+1)}(\mathbb{R})$. It is natural to question whether solutions of the type exponential-polynomial occur for other (p, ph)-globally elliptic equations, when h is odd. For a detailed analysis of such solutions we address to [17], Section 7.4. As an example in the opposite direction: the $(3, 3h)$-globally elliptic equation

$$(D - ix^h)(D + ix^h)^2 u + \sigma x^{h-2} u = 0,$$

with h odd, $h \geq 3$, admits for some $\sigma \in \mathbb{C}$ solutions $u \in S_{1/(h+1)}^{h/(h+1)}(\mathbb{R})$ which are not of type (5.12), see [17], Section 7.3, for their explicit expression in terms of the Meijer's G-functions.

We pass now to consider nonlinear ordinary differential equations. We want to test the sharpness of Theorem 1.3 and Theorem 1.4 on a one-dimensional model. Generalizing the arguments in [7] we consider the equation

$$-u'' + x^{2h} u - h x^{h-1} u = x^h u^\ell - \ell u' u^{\ell-1}, \qquad x \in \mathbb{R}, \tag{5.13}$$

where $h, \ell \in \mathbb{Z}_+$, $\ell > 1, h > 1, h$ odd. We notice that (5.13) corresponds to the equation (1.20) for $n = 1, V(x) = x^{2h} - h x^{h-1}$ and $F(x, u, u') = x^h u^\ell - \ell u' u^{\ell-1}, f = 0$. First of all we observe that (5.13) can be re-written as follows:

$$\left(\frac{d}{dx} - x^h\right)(u' + x^h u) = \left(\frac{d}{dx} - x^h\right) u^\ell, \qquad x \in \mathbb{R}. \tag{5.14}$$

Then every solution $u \in H^2(\mathbb{R}^n)$ of the Bernoulli equation

$$u' + x^h u = u^\ell, \qquad x \in \mathbb{R}, \tag{5.15}$$

is also a solution of (5.14). We restrict our study to the solutions of (5.15). Fixing $u(0) = u_o > 0$, by standard arguments we obtain

$$u(x) = e^{-\frac{x^{h+1}}{h+1}} \left[u_o^{1-\ell} + (1-\ell) \int_0^x e^{-(\ell-1)\frac{t^{h+1}}{h+1}} dt \right]^{\frac{1}{1-\ell}}, \tag{5.16}$$

or equivalently,

$$u(x) = e^{-\frac{x^{h+1}}{h+1}} \left[\lambda + (\ell-1) \int_x^{+\infty} e^{-(\ell-1)\frac{t^{h+1}}{h+1}} dt \right]^{\frac{1}{1-\ell}}, \tag{5.17}$$

with $\lambda = u_o^{1-\ell} + (1-\ell)\int_0^{+\infty} e^{-(\ell-1)\frac{t^{h+1}}{h+1}}dt$. We notice from (5.16) that u is well defined for $x \leq 0$ and $u(x) \sim e^{-\frac{x^{h+1}}{h+1}}$ for $x \to -\infty$. To analyze the global behaviour of u on $\mathbb{R}$, it is convenient to express it in terms of special functions.
To be definite, write

$$\Gamma(\alpha) = \gamma(\alpha, x) + \Gamma(\alpha, x),$$

where

$$\gamma(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1}dt, \qquad \Gamma(\alpha, x) = \int_x^{+\infty} e^{-t}t^{\alpha-1}dt.$$

The function $\gamma(\alpha, x)$ is called the incomplete Gamma function, while $\Gamma(\alpha, x)$ is usually known as complementary incomplete Gamma function. We recall that $\Gamma(\alpha, x) = x^\alpha e^{-x}\Psi(1, \alpha+1, x)$; hence, in view of (5.11), for fixed $\alpha \in \mathbb{R}$ and for $x \to +\infty$, the function $\Gamma(\alpha, x)$ has the following asymptotic expansion

$$\Gamma(\alpha, x) \sim e^{-x}x^{\alpha-1}\sum_{n=0}^{+\infty}(-1)^n\frac{(1-\alpha)_n}{x^n}, \tag{5.18}$$

cf. [27]. By a change of variable it easily follows that

$$u(x) = e^{-\frac{x^{h+1}}{h+1}}\left[\lambda + \left(\frac{h+1}{\ell-1}\right)^{-\frac{h}{h+1}}\Gamma\left(\frac{1}{h+1}, \frac{\ell-1}{h+1}x^{h+1}\right)\right]^{\frac{1}{1-\ell}}, \tag{5.19}$$

with $\lambda = u_o^{1-\ell} - \left(\frac{h+1}{\ell-1}\right)^{-\frac{h}{h+1}}\Gamma\left(\frac{1}{h+1}\right)$. We can distinguish three cases:

a) $-\left(\frac{h+1}{\ell-1}\right)^{-\frac{h}{h+1}}\Gamma\left(\frac{1}{h+1}\right) < \lambda < 0$: In this case, the solution blows up at the point $x_o > 0$ defined by the equation

$$\lambda = (1-\ell)\int_{x_o}^{+\infty} e^{-(\ell-1)\frac{t^{h+1}}{h+1}}dt,$$

cf. (5.17).

b) $\lambda = 0$: The solution is well defined and analytic on $\mathbb{R}$. Moreover, by (5.18), (5.19), $u(x) \sim x^{\frac{h}{\ell-1}}$ for $x \to +\infty$. Therefore $u \in \mathcal{S}'(\mathbb{R}), u \notin \mathcal{S}(\mathbb{R})$. Notice that this does not contraddicts our results, since $u \notin H^s(\mathbb{R})$ for $s > 1/2$, hence the assumptions of Theorems 1.3, 1.4 are not fulfilled.

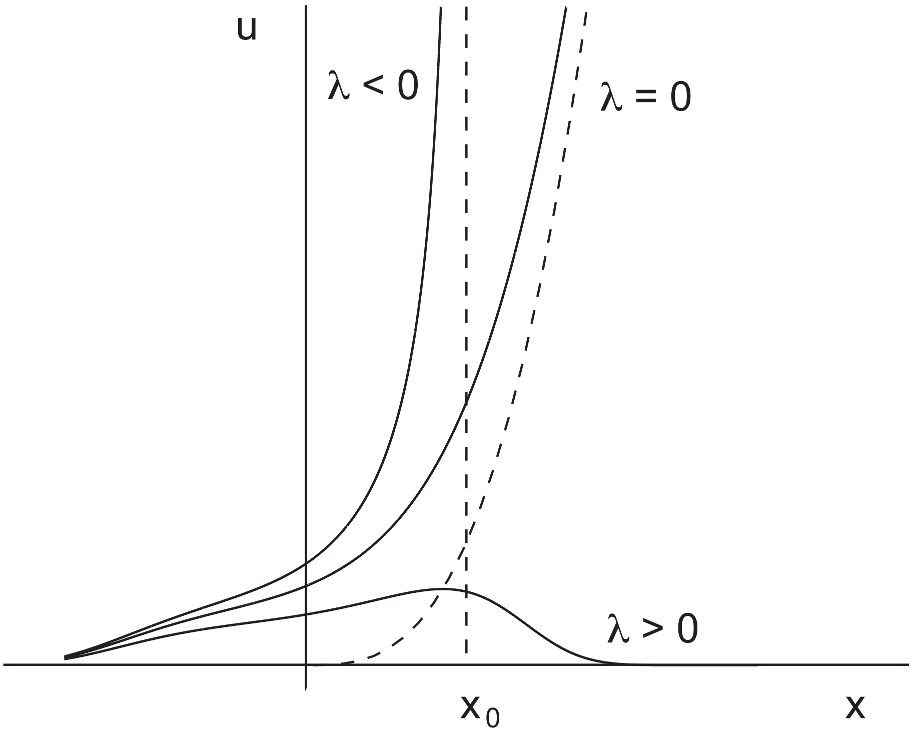
c) $\lambda > 0$: Also in this case, by (5.17), the solution u is analytic on $\mathbb{R}$. Moreover,

$$0 < u(x) < \lambda^{\frac{1}{1-\ell}}e^{-\frac{x^{h+1}}{h+1}}.$$

Now $u \in H^2(\mathbb{R})$ and Theorem 1.3 applies and gives the more precise information $u \in S^1_{\frac{1}{h+1}}(\mathbb{R})$. In particular, u admits a holomorphic extension $u(z)$ on a strip of the form $\{z \in \mathbb{C} : |\text{Im}z| < T\}$ for some $T > 0$. Nevertheless, the great Picard theorem of complex analysis implies that u cannot admit an entire extension on $\mathbb{C}$, since in (5.17) for any fixed $\lambda \in \mathbb{R}$, the equation

$$\lambda + (\ell-1)\int_z^{+\infty} e^{-(\ell-1)\frac{t^{h+1}}{h+1}}dt = 0$$

admits a solution z_o, cf. [27]. Hence we cannot expect to obtain $u \in S^{\mu}_{\frac{1}{h+1}}(\mathbb{R})$ for some $\mu < 1$. Thanks to the representation (5.19), we can illustrate the three cases a), b) and c) above on a MATLAB graphic for $h = 3, \ell = 2$. In the figure below the case $\lambda < 0$ corresponds to the choice $u_0 = 1$, while the case $\lambda > 0$ is obtained by choosing $u_0 = \frac{1}{2}$. Finally for $\lambda = 0$ we have $u(x) \sim x^3$ for $x \to +\infty$.



References

[1] S. Agmon, *Lectures on exponential decay of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, Math. Notes, Vol. 29 Princeton University Press, Princeton, 1982.

[2] H.A. Biagioni and T. Gramchev, *Fractional derivative estimates in Gevrey spaces, global regularity and decay for solutions to semilinear equations in* $\mathbb{R}^n$, J. Differential Equations **194** (2003), 140–165.

[3] P. Boggiatto, E. Buzano and L. Rodino, *Global hypoellipticity and spectral theory*, Math. Res. **92**, Akademie Verlag, Berlin, 1996.

[4] J. Bona and Y. Li, *Decay and analyticity of solitary waves* J. Math. Pures Appl. **76** (1997), 377–430.

[5] I. Bondareva and M. Shubin, *Equations of Korteweg–de Vries type in classes of increasing functions*, J. Soviet Math., **51** n. 3 (1990) 2323–2332.

[6] E. Buzano, *Super-exponential decay of solutions to differential equations in* $\mathbb{R}^d$. In "Modern Trends in Pseudo-Differential Operators", J. Toft, M.-W. Wong, H. Zhu Editors, Birkhäuser Verlag Basel/Switzerland (2006),117–133.

[7] M. Cappiello, T. Gramchev and L. Rodino, *Super-exponential decay and holomorphic extensions for semilinear equations with polynomial coefficients*, J. Functional Analysis **237** (2006), 634–654.

[8] M. Cappiello, T. Gramchev and L. Rodino, *Semilinear pseudo-differential equations and travelling waves*, In "Pseudo-differential operators: partial differential equations and time-frequency analysis", L. Rodino, B.W. Schulze, M.W. Wong Editors, Series: Fields Institute Communications **52** (2007), 213–238.

[9] J. Chung, S. Y. Chung and D. Kim, *Characterization of the Gelfand-Shilov spaces via Fourier transforms*, Proc. Am. Math. Soc. **124** (1996) n. 7, 2101–2108.

[10] E.B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics **92**, Cambridge University Press, Cambridge, 1989.

[11] E.B. Davies and B. Simon, *Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. **59** (1984) n.2, 335–395.

[12] I.M. Gel'fand and G.E. Shilov, *Generalized functions II*, Academic Press, New York, 1968.

[13] T. Gramchev, *Perturbative methods in scales of Banach spaces: applications for Gevrey regularity of solutions to semilinear partial differential equations*, Rend. Sem. Mat. Univ. Pol. Torino, **61** n. 2 (2003), 101–134.

[14] T. Gramchev and P. Popivanov, *Partial differential equations: approximate solutions in scales of functional spaces*, Math. Research, **108**, WILEY-VCH, Berlin, 2000.

[15] B. Helffer, *Théorie spectrale pour des opérateurs globalement elliptiques.* Astérisque, **112**. Société Mathématique de France, Paris, 1984.

[16] P.D. Hislop and I.M. Sigal, *Introduction to spectral theory*, Springer-Verlag, Berlin, 1996.

[17] M. Mascarello and L. Rodino, *Partial differential equations with multiple characteristics*, Akademie Verlag-Wiley, Berlin, 1997.

[18] B.S. Mityagin, *Nuclearity and other properties of spaces of type S*, Amer. Math. Soc. Transl., Ser. 2 **93** (1970), 45–59.

[19] S. Pilipovic, *Tempered ultradistributions*, Boll. Unione Mat. Ital., VII. Ser., B **2** (1988) n. 2, 235–251.

[20] P. J. Rabier, *Asymptotic behavior of the solutions of linear and quasilinear elliptic equations on* $\mathbb{R}^N$. Trans. Amer. Math. Soc. **356** (2004) n. 5, 1889–1907.

[21] P. J. Rabier and C. Stuart, *Exponential decay of the solutions of quasilinear second–order equations and Pohozaev identities*, J. Differential Equations, **165** (2000),199–234.

[22] V.S. Rabinovich, *Exponential estimates for eigenfunctions of Schrödinger operators with rapidly increasing and discontinuous potentials*, Contemporary Math. **364** (2004), 225–236.

[23] M. Shubin, *Pseudodifferential operators and spectral theory*, Springer Series in Soviet Mathematics, Springer Verlag, Berlin 1987.

[24] Y. Sibuya, *The Gevrey asymptotics in the case of singular perturbations*, J. Differential Equations **165** (2000) n. 2, 255–314.

[25] Y. Sibuya, *Formal power series solutions in a parameter*, J. Differential Equations **190** (2003), n. 2, 559–578.

[26] G. Szegö, *Orthogonal polynomials*, American Mathematical Society, 1959.

[27] F.G. Tricomi, *Funzioni speciali*, Ed. Tirrenia, Torino, 1965.

[28] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Wiley Interscience Publishers, New York, 1965.

[a] Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy
email: marco.cappiello@unito.it

[b] Dipartimento di Matematica e Informatica, Università di Cagliari, via Ospedale, 72, 09124 Cagliari, Italy
email: todor@unica.it

[c] Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy (Corresponding author)
email: luigi.rodino@unito.it