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Marino Badiale – Sergio Rolando

Dipartimento di Matematica
Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
e-mail: marino.badiale@unito.it, sergio.rolando@unito.it

Abstract

We prove the existence of nonnegative symmetric solutions to the semilinear elliptic equation

$$-\triangle u+V\left(\left|y_{1}\right|,...,\left|y_{k}\right|\right)u=g\left(u\right)\quad \text{in } \mathbb{R}^{N}$$

where $x=(z,y_1,...,y_k)\in\mathbb{R}^{N_0}\times\mathbb{R}^{N_1}\times...\times\mathbb{R}^{N_k}=\mathbb{R}^N$ with $N\geq 3$, $k\geq 1$, $N_0\geq 0$ and $N_i\geq 2$ for $i>0$. The nonlinearity g and the potential V are, respectively, a continuous function, not necessarily superlinear at infinity, and a positive measurable function, not necessarily homogeneous but satisfying a subhomogeneity condition, which implies vanishing at infinity and singularity at least at the origin. This also yields the existence of nonrotating solitary waves and vortices with a critical frequency for nonlinear Schrödinger and Klein-Gordon equations with singular cylindrical potentials.

1. Introduction

Recent years have seen a growing interest in the study of nonlinear elliptic equations with decaying potentials (see e.g. [2, 8, 14, 18, 19, 23, 24, 25, 27, 28, 29, 41] and the references therein), mainly motivated by the investigation of stationary waves with a critical frequency for nonlinear Schrödinger equations (cf. [21, 22]).

In particular, different authors concerned themselves with semilinear equations of the form

$$-\triangle u+\mathcal{V}\left(x\right)u=g\left(u\right)\quad \text{in } \mathbb{R}^{N},\quad N\geq 3, \tag{1.1}$$

where $\mathcal{V}(x)$ is a positive potential, vanishing at infinity and exhibiting some symmetry and singularity. A first example is the case of the Hardy type potentials $\mathcal{V}(x)=V(x/|x|)\,|x|^{-2}$ considered in [42], where the author studies equation (1.1) with critical nonlinearity and completely solves the problem of radial solutions for constant V's (see also [9] for nonlinearities different from a pure power). Another example is given by the cylindrical potential

$$\mathcal{V}(x)=\frac{\lambda^2}{|y|^2},\qquad x=(z,y)\in\mathbb{R}^{N-K}\times\mathbb{R}^K,\quad N>K\geq 2,\tag{1.2}$$

which also arises in the search of solutions with nonvanishing angular momentum of evolution equations of Schrödinger or Klein-Gordon type (cf. [6, 7, 11]). In particular, equation (1.1) with potential (1.2) has been investigated in [8, 25, 36] and [6, 7], respectively with critical and double-power type nonlinearities. Also, in 1986, motivated by the search of nonradial symmetric solutions to complex valued elliptic equations arising in nonlinear field theories, P.L. Lions [35] studied problem (1.1) with

$$\mathcal{V}(x)=\sum_{i=1}^{k}\frac{\lambda_i^2}{|y_i|^2},\qquad x=(z,y_1,...,y_k)\in\mathbb{R}^{N_0}\times\mathbb{R}^{N_1}\times...\times\mathbb{R}^{N_k},\tag{1.3}$$

where $N_0=0,1$ and $N_i=2$ for $i>0$, finding solutions under quite general assumptions on g.

In all the aforementioned works, the potentials exhibit an inverse-square homogeneity, which gives relevant invariance properties to the equation. Besides, equation (1.1) with power type nonlinearities and radial nonquadratic potentials $\mathcal{V}(x)=V(|x|)$ satisfying $\liminf_{r\to 0} r^{\alpha_0}V(r)>0$ and $\liminf_{r\to+\infty} r^{\alpha_\infty}V(r)>0$ has been considered in [10, 26, 30, 40, 41], where it is shown that the existence of solutions relies on compatibility conditions between the growth rate of g and the singularity and decay rates of the potential. A similar phenomenon appears in [8], where the authors study the cylindrical problem

$$-\triangle u+\frac{\lambda^2}{|y|^\alpha}u=|u|^{p-1}u\quad\text{in }\mathbb{R}^N,\qquad x=(z,y)\in\mathbb{R}^{N-K}\times\mathbb{R}^K,\quad N>K\geq 2.$$

Finally, problem (1.1) with nonlinearities different from a pure power has been investigated in [9, 30] for both radial and cylindrical potentials of the form $\mathcal{V}(x)=V(|y|)$, and in [14, 15] for general potentials $\mathcal{V}\in L^{N/2}(\mathbb{R}^N)$.

Here we extend the results of some of the above mentioned papers by finding nonnegative solutions to the nonlinear equation

$$-\triangle u+V(|y_1|,...,|y_k|)\,u=g(u)\quad\text{in }\mathbb{R}^N\tag{1.4}$$

under some general hypotheses on V and g, where

(i) $N=\sum_{i=0}^k N_i\geq 3$ with $k\geq 1$, $N_0\geq 0$ and $N_i\geq 2$ for $i=1,...,k$, in such a way that $x\in\mathbb{R}^N$ splits as

$$x=(z,y_1,...,y_k)\in\mathbb{R}^{N_0}\times\mathbb{R}^{N_1}\times...\times\mathbb{R}^{N_k}=\mathbb{R}^N$$

(meaning $x=(y_1,...,y_k)$ if $N_0=0$)

(ii) $V:\mathbb{R}_+^k\to(0,+\infty)$ is a measurable function defined on

$$\mathbb{R}_+^k:=\left\{\mathbf{r}=(r_1,...,r_k)\in\mathbb{R}^k:r_i>0,\ \forall i=1,...,k\right\}$$

(iii) $g:\mathbb{R}\to\mathbb{R}$ is continuous and such that $g(0)=0$.

In particular, our existence result covers for example the case of equation (1.1) with potentials

$$\mathcal{V}(x)=\sum_{i=1}^{k}\frac{\lambda_i^2}{|y_i|^{\alpha_i}},\qquad \mathcal{V}(x)=\sum_{j=1}^{k}\frac{\lambda_j^2}{|y_i|^{\alpha_j}}\quad (i\text{ fixed})$$

with $\alpha_1,...,\alpha_k>0$ and $\boldsymbol{\lambda}=(\lambda_1,...,\lambda_k)\neq 0$, or

$$\mathcal{V}(x)=\min\left\{\frac{\lambda_1}{|y_i|^{\alpha_1}},\frac{\lambda_2}{|y_i|^{\alpha_2}}\right\},\qquad \mathcal{V}(x)=\max\left\{\frac{\lambda_1}{|y_i|^{\alpha_1}},\frac{\lambda_2}{|y_i|^{\alpha_2}}\right\}$$

with i fixed and $\lambda_1,\lambda_2,\alpha_1,\alpha_2>0$, or also

$$\mathcal{V}(x)=\frac{\lambda_0^2}{|y_1|^{\alpha_1}\cdots|y_k|^{\alpha_k}}+\frac{\lambda}{|y_i|^{\alpha_0}}\quad (i\text{ fixed}),\quad \mathcal{V}(x)=\frac{\lambda}{|y_1|^{\alpha_1}\cdots|y_k|^{\alpha_k}}\quad \text{if } N_0\neq 1$$

with $\sum_{j=1}^{k}\alpha_j>0$, $\alpha_0>0$ and $\lambda>0$ (see below for other more general examples). Observe that for $k=1$ and $N_0=0$, we also recover the case of radial potentials.

Going into detail, if $N_0\neq 1$ we will only require that the potential is a measurable function $V:\mathbb{R}_+^k\to(0,+\infty)$ satisfying

($\mathbf{V}_0$) $V\in L^\infty((a,b)^k)$ *for some* $b>a>0$

($\mathbf{V}_1$) $\exists\alpha>0$ *such that* $V(t\mathbf{r})\leq t^{-\alpha}V(\mathbf{r})$ *for all* $t>1$ *and almost every* $\mathbf{r}\in\mathbb{R}_+^k$

while, if $N_0=1$, we also need that

($\mathbf{V}_2$) $\exists i\in\{1,...,k\}$ *such that* $\operatorname*{essinf}_{\mathbf{r}\in\mathbb{R}_+^k,\,r_i<R}V(\mathbf{r})>0$ *for every* $R>0$.

Regarding the vanishing of V at infinity, it is not difficult to check that ($\mathbf{V}_0$) and ($\mathbf{V}_1$) imply

$$\operatorname*{essinf}_{|\mathbf{r}|>r}V(\mathbf{r})=0\quad\text{for every } r>0. \tag{1.5}$$

Similarly, ($\mathbf{V}_1$) implies

$$\operatorname*{esssup}_{|\mathbf{r}|<r}V(\mathbf{r})=+\infty\quad\text{for every } r>0 \tag{1.6}$$

provided that $\operatorname{essinf}_{a<|\mathbf{r}|<b}V(\mathbf{r})>0$ for some $b>a>0$.

As far as the nonlinearity is concerned, we will assume that $g:\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

($\mathbf{G}$) $G(s)>0$ *for some* $s>0$, where $G(s):=\int_0^s g(t)\,dt$, $\forall s\in\mathbb{R}$

($\mathbf{g}$) $\limsup_{s\to 0^+}\frac{|g(s)|}{s^{q-1}}<+\infty$ *for some* $q>2^*$, where $2^*:=\frac{2N}{N-2}$

together with one of the following conditions:

($\mathbf{g}_1$) $\limsup_{s\to+\infty} \frac{g(s)}{s^{p-1}} < +\infty$ *for some* $p < 2^*$

($\mathbf{g}_2$) $\exists s_0 > \inf\{s > 0 : G(s) > 0\}$ *such that* $g(s_0) = 0$.

Observe that these hypotheses do not fit the case of pure power nonlinearities and do not require that g is superlinear at infinity.

Our main result is Theorem 1.1 below, which concerns existence of solutions in the following weak sense: denoting $\mathbf{N} = (N_0, N_1, ..., N_K)$ and setting

$$H(\mathbf{N}, V) := \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(|y_1|, ..., |y_k|)\, u^2 dx < \infty \right\}, \tag{1.7}$$

we say that $u \in H(\mathbf{N}, V)$ is a *weak solution* to equation (1.4) if and only if

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla h \, dx + \int_{\mathbb{R}^N} V(|y_1|, ..., |y_k|)\, uh \, dx = \int_{\mathbb{R}^N} g(u)\, h \, dx, \quad \forall h \in H(\mathbf{N}, V). \tag{1.8}$$

Theorem 1.1. *Let* N, V, g *be as in* (i), (ii), (iii) *and assume that* ($\mathbf{V}_0$), ($\mathbf{V}_1$), ($\mathbf{g}$) *and* ($\mathbf{G}$) *hold together with* ($\mathbf{g}_1$) *or* ($\mathbf{g}_2$). *If* $N_0 = 1$ *also assume* ($\mathbf{V}_2$). *Then equation (1.4) has a nonzero nonnegative weak solution* $u \in H(\mathbf{N}, V)$ *satisfying*

- $u(x) = u(|z|, |y_1|, ..., |y_k|)$ *if* $N_0 \neq 1$
- $u(x) = u(z, |y_1|, ..., |y_k|)$ *if* $N_0 = 1$
- $\|u\|_{L^\infty(\mathbb{R}^N)} \leq s_0$ *if* ($\mathbf{g}_2$) *holds.*

With a slight abuse of notation, by $u(x) = u(|z|, |y_1|, ..., |y_k|)$ we naturally mean, here and in the rest of the paper, that $u(x) = u(S_0 z, S_1 y_1, ..., S_k y_k)$ for all isometries $S_i : \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$, $i = 0, 1, ..., k$, and almost every $x \in \mathbb{R}^N$. Similarly for $u(x) = u(z, |y_1|, ..., |y_k|)$, or other analogous writings.

Theorem 1.1 extends some of the results of [6, 7, 9, 30, 35], in the sense that it allows more general potentials, or nonlinearities, or both. In particular, comparing with [35], where solutions to equation (1.1) with potential (1.3) are found if ($\mathbf{G}$) holds together with

$$\limsup_{s\to 0^+} \frac{G(s)}{s^{2^*}} \leq 0 \quad \text{and} \quad \limsup_{s\to+\infty} \frac{G(s)}{s^{2^*}} \leq 0,$$

it seems that more restrictive hypotheses on the nonlinearity are the prize to pay for not requiring the inverse-square homogeneity of the potential. Similarly, comparing with [9], the subhomogeneity assumption ($\mathbf{V}_1$) is a prize to pay for allowing sublinear nonlinearities at infinity.

The main difficulties in proving Theorem 1.1 are the vanishing of the potential at infinity, which prevents the use of H^1 variational theory, and the lack of superlinearity assumptions on the nonlinearity, which does not allow to ensure the boundedness of the Palais-Smale sequences of the Euler functional associated to equation (1.4) by standard arguments of critical point theory. Such an obstacle will be handled in Section 3 by exploiting an abstract result from [30], which applies thanks to the subhomogeneity of the potential, while the former one will be overcome by setting the problem into the framework of the $L^p + L^q$ spaces (see Section 4), as the double-power behaviour of the nonlinearity allows us to do. Then the case $N_0 \neq 1$ turns out to be compact by the results of [4], while the noncompact case $N_0 = 1$ can be recovered by means of a concentration-compactness type result which we prove in Section 4.1 (cf. also Remark 1).

The following examples show some general potentials to which Theorem 1.1 applies.

Example 1.2. *Let $\boldsymbol{\lambda} \in \mathbb{R}^k \backslash \{0\}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{a}, \mathbf{b} \in \mathbb{R}_+^k$. For $i = 1, ..., k$, suppose $\alpha_i \leq \beta_i$ and $b_i \leq a_i$, and define*

$$U_i(s) := \begin{cases} a_i s^{-\alpha_i} & \text{if } 0 < s < 1 \\ b_i s^{-\beta_i} & \text{if } s \geq 1. \end{cases}$$

Then

$$V(\mathbf{r}) := \sum_{i=1}^{k} \lambda_i^2 U_i(r_i) \quad \text{for all } \mathbf{r} \in \mathbb{R}_+^k \tag{1.9}$$

satisfies $(\mathbf{V}_1)$ with $\alpha = \min_i \alpha_i$ and $(\mathbf{V}_2)$ for any $i \in \{1, ..., k\}$ such that $\lambda_i \neq 0$. Observe that U_i is not continuous if $b_i = a_i$. More generally, if $U_i : (0, +\infty) \to (0, +\infty)$ are any functions such that $s \mapsto s^{\alpha_i} U_i(s)$ is nonincreasing on $(0, +\infty)$ (not necessarily continuous), then one has

$$U_i(ts) \leq t^{-\alpha_i} U_i(s) \quad \text{for every } t > 1 \text{ and } s > 0, \tag{1.10}$$

so that the function $V : \mathbb{R}_+^k \to (0, +\infty)$ defined as in (1.9) satisfies

$$V(t\mathbf{r}) = \sum_{i=1}^{k} \lambda_i^2 U_i(tr_i) \leq \sum_{i=1}^{k} \lambda_i^2 t^{-\alpha_i} U_i(r_i) \leq t^{-\min_i \alpha_i} \sum_{i=1}^{k} \lambda_i^2 U_i(r_i) = t^{-\alpha} V(\mathbf{r}).$$

Moreover, for every $i \in \{1, ..., k\}$ and $R > 0$, it holds that

$$\forall \mathbf{r} \in \mathbb{R}_+^k, \quad r_i < R \;\Rightarrow\; V(\mathbf{r}) \geq \lambda_i^2 U_i(r_i) \geq \lambda_i^2 U_i(R),$$

which again ensures $(\mathbf{V}_2)$ for any $i \in \{1, ..., k\}$ such that $\lambda_i \neq 0$.

Example 1.3. *Let $\boldsymbol{\alpha} \in \mathbb{R}^k$. If $U_i : (0, +\infty) \to (0, +\infty)$, $i = 1, ..., k$, are any k functions such that $s \mapsto s^{\alpha_i} U_i(s)$ is nonincreasing on $(0, +\infty)$ (not necessarily continuous), then (1.10) holds and thus the mapping $V : \mathbb{R}_+^k \to (0, +\infty)$ defined by*

$$V(\mathbf{r}) := \prod_{i=1}^{k} U_i(r_i) \quad \text{for all } \mathbf{r} \in \mathbb{R}_+^k$$

satisfies

$$V(t\mathbf{r}) = \prod_{i=1}^{k} U_i(tr_i) \le \prod_{i=1}^{k} t^{-\alpha_i} U_i(r_i) = t^{-\sum_{i=1}^{k}\alpha_i} V(\mathbf{r}).$$

Hence $(\mathbf{V}_1)$ *holds provided that* $\sum_{i=1}^{k}\alpha_i > 0$.

Example 1.4. *More complex potentials satisfying* $(\mathbf{V}_1)$ *and* $(\mathbf{V}_2)$ *can be constructed starting from the ones of Examples 1.2 and 1.3. Indeed, if* $V_1, V_2 : \mathbb{R}_+^k \to (0,+\infty)$ *satisfy* $(\mathbf{V}_1)$ *for some* $\alpha_1, \alpha_2 > 0$, *then* $(\mathbf{V}_1)$ *holds for* $V_1 V_2$ *with* $\alpha = \alpha_1 + \alpha_2$. *Similarly,* $(\mathbf{V}_1)$ *holds with* $\alpha = \min\{\alpha_1, \alpha_2\}$ *for* $V_1 + V_2$, *which also satisfies* $(\mathbf{V}_2)$ *provided that it holds for just one of* V_1, V_2.

Theorem 1.1 bears remarkable applications to the theory of solitary waves and vortices (for which we refer to [5, 13] and the references therein). Roughly speaking, a *solitary wave* is a nonsingular solution of a field equation which travels as a localized packet in such a way that the physical quantities corresponding to the Noether invariances of the equation, such as the energy and the angular momentum, are finite and conserved in time; a *vortex* is a solitary wave with nonvanishing angular momentum. By the arguments of [6, 7], Theorem 1.1 yields new existence results for nonrotating solitary waves and vortices with critical frequency (in the sense of [21, 22]) in the nonlinear Schrödinger (NLS) and Klein-Gordon (NKG) equations with singular cylindrical potentials. Indeed, if $V : (0,+\infty) \to (0,+\infty)$ is a continuous function satisfying $(\mathbf{V}_1)$ (and thus $(\mathbf{V}_0)$ and $(\mathbf{V}_2)$) and $g \in C(\mathbb{R},\mathbb{R})$ satisfies $(\mathbf{g})$ and $(\mathbf{G})$ together with $(\mathbf{g}_1)$ or $(\mathbf{g}_2)$, Theorem 1.1 implies that $\forall k \in \mathbb{Z} \setminus \{0\}$ the equation

$$-\triangle u + \left(V(|y|) + \frac{k^2}{|y|^2} \right) u = g(u), \qquad x = (z,y) \in \mathbb{R} \times \mathbb{R}^2,$$

has a nonzero nonnegative solution $u = u(z,|y|)$ in the space (1.7), which is classical in $\mathbb{R} \times (\mathbb{R}^2 \setminus \{0\})$ by standard elliptic regularity theory and belongs to $L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ by the results of [31] (see also [39, Theorem 6.5]). Hence the standing wave

$$\psi(t,x) = u(z,|y|)\, e^{i(k\theta(y)-\omega t)}, \quad \theta(y) := \operatorname{Im}\log(y_1 + iy_2), \ \omega > 0,$$

is a solution to

$$i\frac{\partial \psi}{\partial t} = -\triangle\psi + (V(|y|) + \omega)\psi - g(|\psi|)\frac{\psi}{|\psi|} \tag{NLS}$$

and

$$\frac{\partial^2 \psi}{\partial t^2} - \triangle\psi + \left(V(|y|) + \omega^2\right)\psi = g(|\psi|)\frac{\psi}{|\psi|}, \tag{NKG}$$

(where $V(|y|)$ vanishes at infinity and is singular in $y = 0$, according to (1.5), (1.6)), which has finite energy

$$\mathcal{E}(\psi) = \frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + \left(V(|y|) + \frac{k^2}{|y|^2} \right) u^2 \right] dx - \int_{\mathbb{R}^3} G(u)\, dx + c_1 \int_{\mathbb{R}^3} u^2 dx$$

(where $c_1 = \omega/2$ for (NLS), $c_1 = \omega^2$ for (NKG)) and nonvanishing angular momentum

$$M(\psi) = c_2 \int_{\mathbb{R}^3} ku^2 dx$$

(where $c_2 = 1/2$ for (NLS), $c_2 = -\omega$ for (NKG)). Travelling vortices for (NLS) and (NKG) can be obtained by respectively applying Galileo and Lorentz transformations. The same argument also works for $k = 0$, yielding nonrotating solitary waves, provided that suitable decay conditions on V are satified in order that $u \in L^2(\mathbb{R}^3)$; for instance, it is sufficient that $\inf_{r>0} r^2 V(r) > 3/4$ (see [31, 39] again).

We conclude this introductory section by summarizing the notations of most frequent use throughout the paper.

Notations.

• $\mathbb{N}$ is the set of natural numbers, including 0.
• For any $r \in \mathbb{R}$ we set $r_+ := (|r| + r)/2$ and $r_- := (|r| - r)/2$, so that $r = r_+ - r_-$ with $r_+, r_- \geq 0$.
• The open ball $B_r(\xi_0) := \{\xi \in \mathbb{R}^d : |\xi - \xi_0| < r\}$ shall be simply denoted by B_r when $\xi_0 = 0$. The dimension d will be clear from the context.
• $|A|$ and χ_A respectively denote the d-dimensional Lebesgue measure and the characteristic function of any measurable set $A \subseteq \mathbb{R}^d$. We set $A^c := \mathbb{R}^d \setminus A$.
• By $\to$ and $\rightharpoonup$ we respectively mean *strong* and *weak* convergence in a Banach space X, whose dual space is denoted by X'.
• $\hookrightarrow$ denotes *continuous* embeddings.
• $C_c^\infty(\Omega)$ is the space of the infinitely differentiable real functions with compact support in the open set $\Omega \subseteq \mathbb{R}^d$.
• If $1 \leq r \leq \infty$ then $L^r(A)$ and $L^r_{\mathrm{loc}}(A)$ are the usual real Lebesgue spaces (for any measurable set $A \subseteq \mathbb{R}^d$).
• $r' = r/(r-1)$ is the Hölder-conjugate exponent of r, so that $L^{r'}$ is the dual of L^r.
• $2^* := 2N/(N-2)$, $N \geq 3$, is the critical exponent for the Sobolev embedding.
• $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ and $H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}$ are the usual Sobolev spaces, where $N \geq 3$ and $\Omega \subseteq \mathbb{R}^N$ is an open subset. $H^1_0(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

2. Variational approach

In this section we let N be as in (i), assume that $V : \mathbb{R}^k_+ \to (0, +\infty)$ is a measurable function satisfying $(\mathbf{V}_0)$ and describe the functional setting in which equation (1.4) can be cast into a variational form.

We equipp the linear space $H := H(\mathbf{N}, V)$ defined in (1.7) with the norm given by

$$\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(|y_1|, ..., |y_k|) u^2 dx, \tag{2.1}$$

so that it becomes a Hilbert space with the inner product

$$(u \mid v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} V(|y_1|, ..., |y_k|) uv \, dx.$$

Notice that $H \neq \varnothing$ thanks to $(\mathbf{V}_0)$, which implies $C_c^\infty(\Omega_0) \subseteq H$, where

$$\Omega_0 := \left\{ x \in \mathbb{R}^N : (|y_1|, ..., |y_k|) \in (a,b)^k \right\}. \tag{2.2}$$

Clearly $H \hookrightarrow D^{1,2}(\mathbb{R}^N)$ and, by well known embeddings of $D^{1,2}(\mathbb{R}^N)$, one has that $H \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $H \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $1 \leq p \leq 2^*$. In particular, the latter embedding is compact if $p < 2^*$ and thus it ensures that weak convergence in H implies (up to a subsequence) almost everywhere convergence in $\mathbb{R}^N$.

Due to the symmetries of the potential, we will mainly work in the symmetric subspace

$$\begin{aligned} H_{\mathrm{s}} := H_{\mathrm{s}}(\mathbf{N}, V) := \{ u \in H(\mathbf{N}, V) : u(x) = u(|z|, |y_1|, ..., |y_k|) \text{ if } N_0 \neq 1, \\ u(x) = u(z, |y_1|, ..., |y_k|) \text{ if } N_0 = 1 \} \end{aligned}$$

which, by pointwise convergence (up to a subsequence) of H-converging sequences, is closed in H and thus a Hilbert space with respect to the same norm (2.1) of H. Again, $(\mathbf{V}_0)$ implies that H_{s} is nonempty.

The following proposition clarifies the role of condition $(\mathbf{V}_2)$, which we will need in Section 4 when dealing with the case $N_0 = 1$.

Proposition 2.1. *Assume that $(\mathbf{V}_2)$ holds and denote*

$$\Omega_\rho^i := \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_{i-1}} \times B_\rho \times \mathbb{R}^{N_{i+1}} \times ... \times \mathbb{R}^{N_k} \subset \mathbb{R}^{N-N_0}.$$

Then H is continuously embedded into $H^1(\mathbb{R}^{N_0} \times \Omega_\rho^i)$ for every $\rho > 0$.

Proof. Let $\rho > 0$ and set

$$V_\rho := \operatorname*{essinf}_{(y_1,...,y_k) \in \Omega_\rho^i} V(|y_1|, ..., |y_k|) > 0$$

for brevity. Then for all $u \in H$ one has

$$\begin{aligned} \int_{\mathbb{R}^{N_0} \times \Omega_\rho^i} \left(|\nabla u|^2 + u^2 \right) dx &= \int_{\mathbb{R}^{N_0} \times \Omega_\rho^i} |\nabla u|^2 \, dx + \frac{1}{V_\rho} \int_{\mathbb{R}^{N_0} \times \Omega_\rho^i} V_\rho u^2 dx \\ &\leq \left(1 + \frac{1}{V_\rho} \right) \int_{\mathbb{R}^{N_0} \times \Omega_\rho^i} \left(|\nabla u|^2 + V(|y_1|, ..., |y_k|) \, u^2 \right) dx, \end{aligned}$$

which yields the result. ■

Now let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $(\mathbf{g})$ and $(\mathbf{G})$, together with $(\mathbf{g}_1)$ or $(\mathbf{g}_2)$. Define

$$f(s) := \chi(s) \, g(s) \quad \text{and} \quad F(s) := \int_0^s f(t) \, dt \qquad \text{for all } s \in \mathbb{R}$$

where $\chi := \chi_{(0,s_0)}$ if $(\mathbf{g}_2)$ holds, $\chi := \chi_{(0,+\infty)}$ otherwise. So, both if $(\mathbf{g}_1)$ or $(\mathbf{g}_2)$ holds, the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and F satisfies

(F) $\exists s_* > 0$ such that $F(s_*) > 0$.

Moreover, there exist $p, q \in \mathbb{R}$ such that $2 < p < 2^* < q$ and

(f$_\wedge$) $\exists M_1 > 0,\ \forall s \in \mathbb{R},\ |f(s)| \leq M_1 \min\{|s|^{p-1}, |s|^{q-1}\}$

(F$_\wedge$) $\exists M_2 > 0,\ \forall s \in \mathbb{R},\ |F(s)| \leq M_2 \min\{|s|^{p}, |s|^{q}\}$,

which yield in particular

(f$_*$) $|f(s)| \leq M_1 |s|^{2^*-1}$ *for all* $s \in \mathbb{R}$

(F$_*$) $|F(s)| \leq M_2 |s|^{2^*}$ *for all* $s \in \mathbb{R}$.

Thanks to (**f$_*$**), (**F$_*$**) and the continuous embedding $H \hookrightarrow L^{2^*}(\mathbb{R}^N)$, one checks (see for example [32]) that the functional $I : H \to \mathbb{R}$ defined by

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u)\, dx \qquad \text{for all } u \in H \tag{2.3}$$

is of class C^1 on H and has Fréchet derivative $I'(u) \in H'$ at any $u \in H$ given by

$$I'(u)h = (u \mid h) - \int_{\mathbb{R}^N} f(u)\, h\, dx \qquad \text{for all } h \in H. \tag{2.4}$$

The next propositions collect some quite standard facts about the restriction $I_{|H_s}$. In particular, the first one states a variational principle for finding weak solutions of equation (1.4) and the second one ensures that critical points for $I_{|H_s}$ are provided by the weak limits of its bounded criticizing sequences.

Proposition 2.2. *Let $g \in C(\mathbb{R}, \mathbb{R})$ satisfy* (**g**) *and* (**G**), *together with* (**g$_1$**) *or* (**g$_2$**). *Then every critical point of $I_{|H_s}$ is a nonnegative weak solution to equation (1.4) and, if* (**g$_2$**) *holds, it satisfies $u \leq s_0$ almost everywhere in $\mathbb{R}^N$.*

Proof. Let $u \in H_s$ be such that $I'(u)h = 0$ for all $h \in H_s$. Then, by virtue of the principle of symmetric criticality [37], u is a critical point of I, i.e., $I'(u)h = 0$ for all $h \in H$. Now, using $h = u_- \in H_s$ as test function in (2.4), one obtains $\|u_-\| = 0$, that is, $u \geq 0$. If $f = \chi_{(0,+\infty)}g$, this implies $f(u) = g(u)$ and thus (1.8) holds by (2.4). Otherwise, if $f = \chi_{(0,s_0)}g$, we compute (2.4) for $h = (u - s_0)_+ \in H_s$ and, since $f(u)(u - s_0)_+$ vanishes almost everywhere in $\mathbb{R}^N$, we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \nabla u \cdot \nabla (u - s_0)_+\, dx + \int_{\mathbb{R}^N} V(|y_1|, ..., |y_k|)\, u\, (u - s_0)_+\, dx \\ &\geq \int_{\mathbb{R}^N} \nabla u \cdot \nabla (u - s_0)_+\, dx = \int_{\mathbb{R}^N} \left|\nabla (u - s_0)_+\right|^2 dx\,. \end{aligned}$$

This implies $(u - s_0)_+ = 0$, i.e., $u \leq s_0$, which yields $f(u) = g(u)$ and thus proves (1.8) again. ■

Proposition 2.3. *Let $g \in C(\mathbb{R}, \mathbb{R})$ satisfy* $(\mathbf{g})$ *and* $(\mathbf{G})$*, together with* $(\mathbf{g}_1)$ *or* $(\mathbf{g}_2)$*. Then, for any $h \in H_{\mathrm{s}}$, the mapping $I'(\cdot)\, h : H_{\mathrm{s}} \to \mathbb{R}$ is weakly sequentially continuous.*

Proof. Of course one needs only consider the nonlinear term of $I'(\cdot)\, h$ and thus the claim follows from [6, Proposition 14], where the weak continuity of the mapping $u \mapsto \int_{\mathbb{R}^N} f(u)\, h\, dx$ has been proved on $D^{1,2}(\mathbb{R}^N)$ under condition $(\mathbf{f}_*)$. ∎

3. Existence of bounded Palais-Smale sequences

In this section we let N be as in (i) and assume that $V : \mathbb{R}_+^k \to (0, +\infty)$ is a measurable function satisfying $(\mathbf{V}_0)$ and $(\mathbf{V}_1)$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $(\mathbf{g})$ and $(\mathbf{G})$ together with $(\mathbf{g}_1)$ or $(\mathbf{g}_2)$. Our aim is to prove the existence of a bounded Palais-Smale sequence for the restriction $I_{|H_{\mathrm{s}}}$ of the functional defined in (2.3), by means of the following result.

Lemma 3.1. *Let $J : X \to \mathbb{R}$ be a C^1 functional on a Banach space $(X, \|\cdot\|)$, having the form*

$$J(u) = \frac{1}{\theta}\|u\|^{\theta} - B(u)$$

for some $\theta > 0$ and some continuous functional $B : X \to \mathbb{R}$. Assume that there exists a sequence of mappings $\{\psi_n\} \subset C(X, X)$ such that $\forall n$ there exist $\alpha_n > \beta_n > 0$ satisfying

$$\|u\|^{\theta} \geq \alpha_n \|\psi_n(u)\|^{\theta} \quad \text{and} \quad B(u) \leq \beta_n B(\psi_n(u)) \quad \text{for all } u \in X$$

and

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 1, \qquad \liminf_{n\to\infty} \frac{|1 - \beta_n|}{\alpha_n - \beta_n} < \infty.$$

If there exist $\rho > 0$ and $\bar{u} \in X$ with $\|\bar{u}\| > \rho$ such that

$$\inf_{\|u\|=\rho} J(u) > J(0) \geq J(\bar{u}) \quad \text{and} \quad \lim_{n\to\infty} \|\psi_n(0)\| = \lim_{n\to\infty} \|\psi_n(\bar{u}) - \bar{u}\| = 0$$

then J has a bounded Palais-Smale sequence $\{w_n\} \subset X$ at level

$$c_{J,\bar{u}} = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} J(u), \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = \bar{u}\}.$$

Proof. It is a particular case of [30, Theorem 1.1]. ∎

We begin by observing that, by $(\mathbf{F}_*)$ and the Sobolev inequality, there exists $C_0 > 0$ such that

$$I(u) \geq \frac{1}{2}\|u\|^2 - C_0\|u\|^{2^*} \quad \text{for all } u \in H_{\mathrm{s}},$$

so that we can fix $\rho > 0$ such that

$$\inf_{\|u\|=\rho} I(u) > 0 = I(0). \tag{3.1}$$

The next lemma exploits assumption (**G**) and (together with the subhomogeneity of V) ensures that $I_{|H_{\mathrm{s}}}$ has the mountain-pass geometry (see (3.2) of Lemma 3.3 below). Recall the definition (2.2) of Ω_0.

Lemma 3.2. *There exists $u_0 \in C_{\mathrm{c}}^{\infty}(\Omega_0) \cap H_{\mathrm{s}}$ such that $\int_{\mathbb{R}^N} F(u_0)\,dx > 0$.*

Proof. Let $a < a_n < a_0 < b_0 < b_n < b$ with $a_n \to a_0$ and $b_n \to b_0$ and take $\varphi_n \in C_{\mathrm{c}}^{\infty}((a_n, b_n))$ such that $\varphi_n(t) \equiv 1$ on (a_0, b_0) for all n and $0 \leq \varphi_n \leq 1$. Define

$$\phi_n(x) := s_* \varphi_n(|z|)\, \varphi_n(|y_1|) \cdots \varphi_n(|y_k|),$$

where $s_* > 0$ is given by condition (**F**), and set

$$\mathcal{K} := \left\{ x \in \mathbb{R}^N : (|z|, |y_1|, ..., |y_k|) \in [a_0, b_0]^{1+k} \right\}$$

(as usual, $\mathcal{K} = \{x \in \mathbb{R}^N : (|y_1|, ..., |y_k|) \in [a_0, b_0]^k\}$ if $N_0 = 0$). Then $\phi_n \in C_{\mathrm{c}}^{\infty}(\Omega_0) \cap H_{\mathrm{s}}$ and $\phi_n(x) \to s_* \chi_{\mathcal{K}}(x)$ for every $x \in \mathbb{R}^N$, whence $F(\phi_n(x)) \to F(s_*)\, \chi_{\mathcal{K}}(x)$ for every $x \in \mathbb{R}^N$, as F is continuous and $F(0) = 0$. Moreover

$$|F(\phi_n(x))| \leq M_2 |\phi_n(x)|^{2^*} \leq M_2 s_*^{2^*} \quad \text{for all } x \in \mathbb{R}^N$$

by ($\mathbf{F}_*$). Hence we get

$$\int_{\mathbb{R}^N} F(\phi_n)\,dx = \int_{\Omega_0} F(\phi_n)\,dx \to \int_{\Omega_0} F(s_*)\, \chi_{\mathcal{K}}(x)\,dx = F(s_*)\,|\mathcal{K}| > 0$$

by dominated convergence and we can take $u_0 = \phi_n$ with n large enough. ■

For every $t > 0$ and $u \in H_{\mathrm{s}}$, define $\psi^t(u) \in H_{\mathrm{s}}$ by setting

$$\psi^t(u)(x) := u(x/t).$$

Lemma 3.3. *For every $t > 1$ and $u \in H_{\mathrm{s}}$, one has $\psi^t \in C(H_{\mathrm{s}}, H_{\mathrm{s}})$ and*

$$\|u\|^2 \geq \frac{1}{t^{\max\{N-2, N-\alpha\}}} \left\|\psi^t(u)\right\|^2 \quad \text{and} \quad \int_{\mathbb{R}^N} F(u)\,dx = \frac{1}{t^N} \int_{\mathbb{R}^N} F\left(\psi^t(u)\right) dx.$$

Moreover, there exists $\bar{u} \in H_{\mathrm{s}}$ such that

$$\|\bar{u}\| > \rho, \quad I(\bar{u}) < 0 \quad \text{and} \quad \lim_{t \to 1^+} \left\|\psi^t(\bar{u}) - \bar{u}\right\| = 0. \tag{3.2}$$

Proof. Let $u \in H_{\mathrm{s}}$ and $t > 1$. Then $\nabla\left(\psi^t(u)\right)(x) = t^{-1} \nabla u(x/t)$ and from assumption ($\mathbf{V}_1$) it follows that

$$\begin{aligned}
\left\|\psi^t(u)\right\|^2 &= \frac{1}{t^2} \int_{\mathbb{R}^N} |\nabla u(x/t)|^2\,dx + \int_{\mathbb{R}^N} V(|y_1|, \ldots, |y_k|)\, u(x/t)^2\,dx \\
&= t^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + t^N \int_{\mathbb{R}^N} V(t|y_1|, \ldots, t|y_k|)\, u^2 dx \\
&\leq t^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + t^{N-\alpha} \int_{\mathbb{R}^N} V(|y_1|, \ldots, |y_k|)\, u^2 dx \\
&\leq t^{\max\{N-2, N-\alpha\}} \|u\|^2,
\end{aligned}$$

which also ensures that the linear mapping $\psi^t : H_s \to H_s$ is continuous. Now let $u_0 \in C_c^\infty(\Omega_0) \cap H_s$ be the mapping of Lemma 3.2. Since $\max\{N-2, N-\alpha\} < N$, as $t \to +\infty$ we have

$$I\left(\psi^t(u_0)\right) \leq \frac{1}{2} t^{\max\{N-2,N-\alpha\}} \|u_0\|^2 - t^N \int_{\mathbb{R}^N} F(u_0)\, dx \to -\infty$$

and

$$\left\|\psi^t(u_0)\right\|^2 \geq \int_{\mathbb{R}^N} \left|\nabla\left(\psi^t(u_0)\right)\right|^2 dx = t^{N-2} \int_{\mathbb{R}^N} |\nabla u_0|^2\, dx \to +\infty,$$

so that $\|\bar{u}\| > \rho$ and $I(\bar{u}) < 0$ for $\bar{u} = \psi^{\bar{t}}(u_0) \in C_c^\infty(\Omega_0)$ with $\bar{t} > 0$ large enough. Finally let $\{t_n\}$ be such that $t_n \to 1$ with $t_n > 1$ and fix $\Omega' \subset\subset \Omega_0$ such that both $\bar{u}$ and $\psi^{t_n}(\bar{u})$ belong to $C_c^\infty(\Omega')$ for n sufficiently large. Then by $(\mathbf{V}_0)$ one has

$$V_0 := \sup_{x \in \Omega'} V(|y_1|, ..., |y_k|) < +\infty$$

and we conclude

$$\begin{aligned} \int_{\mathbb{R}^N} V(|y_1|, ..., |y_k|) \left(\psi^{t_n}(\bar{u}) - \bar{u}\right)^2 dx &= \int_{\Omega'} V(|y_1|, ..., |y_k|) \left(\psi^{t_n}(\bar{u}) - \bar{u}\right)^2 dx \\ &\leq V_0 \int_{\Omega'} \left(\psi^{t_n}(\bar{u}) - \bar{u}\right)^2 dx \to 0 \end{aligned}$$

and

$$\int_{\mathbb{R}^N} \left|\nabla\left(\psi^{t_n}(\bar{u}) - \bar{u}\right)\right|^2 dx = \int_{\Omega'} \left|\nabla\left(\psi^{t_n}(\bar{u}) - \bar{u}\right)\right|^2 dx \to 0$$

as $n \to \infty$, since $\psi^{t_n}(\bar{u}) = \bar{u}\left(t_n^{-1}\cdot\right) \to \bar{u}$ e $\nabla\left(\psi^{t_n}(\bar{u})\right) = t_n^{-1} \nabla\bar{u}\left(t_n^{-1}\cdot\right) \to \nabla\bar{u}$ in $L^\infty(\mathbb{R}^N)$. ■

We can now easily deduce that the functional $I_{|H_s}$ has a bounded Palais-Smale sequence at the mountain-pass level

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], H_s) : \gamma(0) = 0,\ \gamma(1) = \bar{u}\}$ and $\bar{u}$ is given in Lemma 3.3.

Proposition 3.4. *There exists a bounded sequence* $\{w_n\} \subset H_s$ *such that* $I(w_n) \to c$ *and* $I'(w_n) \to 0$ *in* H_s'.

Proof. Let $\{t_n\}$ be any real sequence such that $t_n \to 1$ with $t_n > 1$ and define $\psi_n(u) := \psi^{t_n}(u)$, $\alpha_n := t_n^{-\max\{N-2,N-\alpha\}}$ and $\beta_n := t_n^{-N}$. Then $\alpha_n > \beta_n$ since $t_n > 1$ and $\max\{N-2, N-\alpha\} < N$, and one has

$$\lim_{n\to\infty} \frac{|1-\beta_n|}{\alpha_n - \beta_n} = \lim_{n\to\infty} \frac{1 - t_n^{-N}}{t_n^{-\max\{N-2,N-\alpha\}} - t_n^{-N}} = \frac{N}{N - \max\{N-2, N-\alpha\}} < \infty.$$

By (3.1) and Lemma 3.3, the conclusion follows from Lemma 3.1. ■

4. Compactness and proof of Theorem 1.1

In this section we give the proof of Theorem 1.1, which, by Propositions 2.2 and 2.3, is achieved if we show that at least one of the bounded Palais-Smale sequences which $I_{|H_s}$ admits according to Proposition 3.4 has a nonzero weak limit. We will see that this is actually true for any of such sequences if $N_0 \neq 1$, while some more work is needed if $N_0 = 1$. In any case, in order to bring the right amount of compactness, we need some preliminary results about the sum of Lebesgue spaces.

Let Ω be any measurable subset of $\mathbb{R}^N$, $N \geq 3$, and let $1 < p < q < \infty$. From the general theory of Banach spaces (see for example [17]), it is well known that

$$L^p + L^q(\Omega) := L^p(\Omega) + L^q(\Omega) = \{u_1 + u_2 : u_1 \in L^p(\Omega), u_2 \in L^q(\Omega)\}$$

is a Banach space equipped with the norm

$$\|u\|_{L^p+L^q(\Omega)} := \inf_{u=u_1+u_2} \Big(\|u_1\|_{L^p(\Omega)} + \|u_2\|_{L^q(\Omega)} \Big), \tag{4.1}$$

with respect to which it isometrically identifies with the dual space of $L^{p'}(\Omega) \cap L^{q'}(\Omega)$ endowed with the norm $\max\{\|\varphi\|_{L^{p'}(\Omega)}, \|\varphi\|_{L^{q'}(\Omega)}\}$. An equivalent norm is

$$\|u\|^*_{L^p+L^q(\Omega)} := \inf_{u=u_1+u_2} \max\{\|u_1\|_{L^p(\Omega)}, \|u_2\|_{L^q(\Omega)}\}$$

and one has (see [39, Theorem A.11])

$$\|u\|^*_{L^p+L^q(\Omega)} = \sup_{0\neq\varphi\in L^{p'}(\Omega)\cap L^{q'}(\Omega)} \frac{\int_\Omega u(x)\varphi(x)\,dx}{\|\varphi\|_{L^{p'}(\Omega)} + \|\varphi\|_{L^{q'}(\Omega)}}. \tag{4.2}$$

The set $L^p + L^q(\Omega)$ can be also characterized as follows (see [38, 39]): for any measurable function $u : \Omega \to \mathbb{R}$, the conditions

(1) $u \in L^p + L^q(\Omega)$

(2) $u \in L^p(\Omega') \cap L^q(\Omega \setminus \Omega')$ for some measurable set $\Omega' \subseteq \Omega$ (even depending on u)

(3) $u \in L^p(\Lambda_u) \cap L^q(\Omega \setminus \Lambda_u)$, where $\Lambda_u := \{x \in \Omega : |u(x)| > 1\}$

are equivalent. Moreover $u \in L^p + L^q(\Omega) \Leftrightarrow |u| \in L^p + L^q(\Omega)$, and $\|u\|_{L^p+L^q(\Omega)} = \||u|\|_{L^p+L^q(\Omega)}$ (see [39, Proposition A.6]).

The space $L^p + L^q(\Omega)$ has been studied and used in several recent works, such as [3, 4, 9, 12, 14, 15, 16, 30, 38, 39]. Here we recall from [4, 38, 39] some known results we will need in the following and prove a new concentration-compactness type result (Theorem 4.4 below), which yields Theorem 1.1 in the case $N_0 = 1$.

Proposition 4.1 ([37]). *If $f : \mathbb{R} \to \mathbb{R}$ is any continuous function satisfying $(\mathbf{f}_\wedge)$, then the Nemytskiĭ operator $u \mapsto f(u)$ is continuous from $L^p + L^q(\Omega)$ into $L^{p'}(\Omega) \cap L^{q'}(\Omega)$.*

If $p < 2^* < q$, the next proposition and the Sobolev inequality yield the continuous embedding:

$$D^{1,2}\left(\mathbb{R}^N\right) \hookrightarrow L^p + L^q\left(\mathbb{R}^N\right). \tag{4.3}$$

Proposition 4.2 ([37], [38, Proposition A.18]). *$L^r(\Omega)$ is continuously embedded into $L^p + L^q(\Omega)$ for every $r \in [p, q]$.*

For any $l \geq 1$ and $\mathbf{d} = (d_1, ..., d_l) \in (\mathbb{N} \setminus \{0\})^l$ such that $\sum_{i=1}^l d_i = N$, write $x = (x_1, ..., x_l) \in \mathbb{R}^{d_1} \times ... \times \mathbb{R}^{d_l} = \mathbb{R}^N$ and denote

$$D^{1,2}_{\mathbf{d}}\left(\mathbb{R}^N\right) := \left\{u \in D^{1,2}\left(\mathbb{R}^N\right) : u(x) = u(|x_1|, ..., |x_l|)\right\}. \tag{4.4}$$

Theorem 4.3 ([4, Theorem A.1]). *Let $l \geq 1$ and $\mathbf{d} \in (\mathbb{N} \setminus \{0, 1\})^l$ be such that $\sum_{i=1}^l d_i = N$. Then $D^{1,2}_{\mathbf{d}}(\mathbb{R}^N)$ is compactly embedded into every $L^p + L^q(\mathbb{R}^N)$ with $1 < p < 2^* < q$.*

The proof of Theorem 1.1 is now straightforward in the case with $N_0 \neq 1$ (and $H_{\mathrm{s}} = \{u \in H : u(x) = u(|z|, |y_1|, ..., |y_k|)\}$).

Proof of Theorem 1.1, case $N_0 \neq 1$. Assume that the hypotheses of the theorem hold with $N_0 \neq 1$ and let $\{w_n\} \subset H_{\mathrm{s}}$ be a bounded Palais-Smale sequence for the functional $I_{|H_{\mathrm{s}}}$ of Section 2 (which exists by Proposition 3.4). Since $H_{\mathrm{s}} \hookrightarrow D^{1,2}_{\mathbf{N}}(\mathbb{R}^N)$ (meaning as usual $\mathbf{N} = (N_1, ..., N_k)$ if $N_0 = 0$), Theorem 4.3 ensures that there exists $u \in H_{\mathrm{s}}$ such that (up to a subsequence) $w_n \to u$ in $L^p + L^q(\mathbb{R}^N)$. Hence, by Proposition 4.1 and the Riesz representation theorem, it is a standard exercise to conclude that $I'(w_n) \to 0$ in H'_{s} implies $w_n \to u$ in H_{s} and thus $I'(u) = 0$ in H'_{s}. Then $I(w_n) \to c > 0$ yields $u \neq 0$ and the assertion of the theorem finally follows from Proposition 2.2. ■

In the case with $N_0 = 1$ (and $H_{\mathrm{s}} = \{u \in H : u(x) = u(z, |y_1|, ..., |y_k|)\}$), Theorem 4.3 does not apply (cf. Remark 1 below) and we need the following concentration-compactness type result, which will be proved in the next subsection.

Let N be as in (i) with $N_0 \neq 0$, assume that $V : \mathbb{R}^k_+ \to (0, +\infty)$ is a measurable function satisfying $(\mathbf{V}_0)$ and $(\mathbf{V}_2)$, and define

$$H_{\mathrm{cyl}} := H_{\mathrm{cyl}}(\mathbf{N}, V) := \{u \in H(\mathbf{N}, V) : u(x) = u(z, |y_1|, ..., |y_k|)\}. \tag{4.5}$$

Notice that H_{cyl} is a nonempty closed subspace of H, which we equipp with the same norm (2.1).

Theorem 4.4. *If $\{u_n\} \subset H_{\mathrm{cyl}}$ is bounded, then, up to a subsequence, either $u_n \to 0$ in every $L^p + L^q(\mathbb{R}^N)$ with $2 < p < 2^* < q$, or there exist $\{z_n\} \subset \mathbb{R}$ and $u \in H_{\mathrm{cyl}} \setminus \{0\}$ such that $u_n^{z_n} \rightharpoonup u$ in H_{cyl}, where $u_n^{z_n}(x) := u_n(z + z_n, y_1, ..., y_k)$.*

By using Theorem 4.4, we can easily get the proof of Theorem 1.1 completed.

Proof of Theorem 1.1, case $N_0 = 1$. Assume that the hypotheses of the theorem hold with $N_0 = 1$ and let $\{w_n\}$ be any bounded Palais-Smale sequence for the restriction to H_{s} of the functional I defined in (2.3) of Section 2, which exists by Proposition 3.4. As $H_{\mathrm{s}} = H_{\mathrm{cyl}}$ for $N_0 = 1$, the sequence $\{w_n\}$ must satisfy one of the alternatives of Theorem 4.4, but the first one cannot occur: in fact, by $(\mathbf{f}_\wedge)$, (4.2) and Proposition 4.1, the operator $u \mapsto f(u)\,u$ turns out to be continuous from $L^p + L^q(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$ and thus $w_n \to 0$ in $L^p + L^q(\mathbb{R}^N)$ implies

$$\|w_n\|^2 = I'(w_n)\,w_n + \int_{\mathbb{R}^N} f(w_n)\,w_n dx \to 0,$$

which contradicts $I(w_n) \to c > 0$ and $I(0) = 0$. So there exist $\{z_n\} \subset \mathbb{R}$ and $u \in H_{\mathrm{s}} \setminus \{0\}$ such that $u_n := w_n^{z_n} \rightharpoonup u$ in H_{s}, where $\|I'(u_n)\|_{H_{\mathrm{s}}'} = \|I'(w_n)\|_{H_{\mathrm{s}}'} \to 0$ since one easily checks, by an obvious change of variables, that z-translations are isometries of H_{s}. The assertion of Theorem 1.1 then ensues by Propositions 2.3 and 2.2. ■

Remark 1. *Working in H_{cyl} instead of H_{s} since from Section 3, one can use Theorem 4.4 also for proving Theorem 1.1 with $N_0 > 1$, but the z-radiality of the solution is no more granted and assumption $(\mathbf{V}_2)$ is required. On the other hand, the above proof of the case $N_0 \neq 1$ does not work if $N_0 = 1$, since the embedding $D^{1,2}_{\mathbf{d}}(\mathbb{R}^N) \hookrightarrow L^p + L^q(\mathbb{R}^N)$ of Theorem 4.3 is not compact if $d_1 = 1$. Indeed, letting $\phi \in C_{\mathrm{c}}^\infty((0,1))$, $\phi \not\equiv 0$, $0 \leq \phi \leq 1$ and writing $x = (z,y) \in \mathbb{R} \times \mathbb{R}^{N-1}$, $N \geq 3$, it is easy to check that $\varphi_n(x) := \phi(|z| - n)\,\phi(|y|)$ defines a sequence which vanishes pointwise on $\mathbb{R}^N$, is bounded in $D^{1,2}_{\mathbf{d}}(\mathbb{R}^N)$, $\mathbf{d} = (1, N-1)$, and does not admit any subsequence converging to zero in some $L^p + L^q(\mathbb{R}^N)$ with $1 < p < q < \infty$.*

4.1. Proof of Theorem 4.4

We begin with some preliminaries: a characterization of the convergence in $L^p + L^q(\Omega)$ and a compactness result concerning the space (4.4) with $\mathbf{d} = (1, N-1)$, which derive from the arguments of [4] by the pointwise estimate of Lemma 4.6 below.

Let Ω be any measurable subset of $\mathbb{R}^N$, $N \geq 3$, and let $1 < p < q < \infty$. Denote

$$m(s) := m_{p,q}(s) := \min\{|s|^p, |s|^q\} \qquad \text{for all } s \in \mathbb{R}$$

and observe that $u \in L^p + L^q(\Omega)$ if and only if $m(u) \in L^1(\Omega)$, as follows from characterization (3) of page 13.

Proposition 4.5. *The operator $u \mapsto m(u)$ is continuous from $L^p + L^q(\Omega)$ into $L^1(\Omega)$. Moreover, if $\{u_n\}$ is any sequence of measurable functions $u_n : \Omega \to \mathbb{R}$, then $u_n \to 0$ in $L^p + L^q(\Omega)$ if and only if $m(u_n) \to 0$ in $L^1(\Omega)$.*

Proof. Setting $\bar{m}(s) := \min\{|s|^{p-1}, |s|^{q-1}\}$, by (4.2) and Proposition 4.1 it easy to see that the operator $u \mapsto m(u) = \bar{m}(u)\,|u|$ is continuous from $L^p + L^q(\Omega)$ into $L^1(\Omega)$. Moreover, if

$$\int_\Omega m(u_n) = \int_{\Lambda_{u_n}} |u_n|^p\, dx + \int_{\Lambda_{u_n}^c} |u_n|^q\, dx \to 0$$

(where, we recall, $\Lambda_{u_n} := \{x \in \Omega : |u_n(x)| > 1\}$), then by definition of the norm (4.1) one has

$$
\begin{aligned}
\|u_n\|_{L^p+L^q(\Omega)} &\leq \left\|u_n \chi_{\Lambda_{u_n}}\right\|_{L^p(\Omega)} + \left\|u_n \chi_{\Lambda_{u_n}^c}\right\|_{L^q(\Omega)} \\
&= \left(\int_{\Lambda_{u_n}} |u_n|^p \, dx\right)^{1/p} + \left(\int_{\Lambda_{u_n}^c} |u_n|^q \, dx\right)^{1/q} \to 0,
\end{aligned}
$$

since $u_n = u_n \chi_{\Lambda_{u_n}} + u_n \chi_{\Lambda_{u_n}^c}$ with $u_n \chi_{\Lambda_{u_n}} \in L^p(\Omega)$ and $u_n \chi_{\Lambda_{u_n}^c} \in L^q(\Omega)$. ■

Let $N \geq 3$ and recall the definition (4.4) of $D_{\mathbf{d}}^{1,2}(\mathbb{R}^N)$.

Lemma 4.6. *If $\mathbf{d} = (1, N-1)$, then there exists a constant $C_N > 0$ (only dependent on N) such that every $u \in D_{\mathbf{d}}^{1,2}(\mathbb{R}^N)$ nonincreasing with respect to $|x_1|$ satisfies*

$$
u(x) \leq C_N \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}^{\frac{N-2}{2N-2}} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{\frac{N}{2N-2}}}{|x_1|^{\frac{N-2}{2N-2}} |x_2|^{\frac{(N-2)^2}{2N-2}}} \quad \textit{for almost every } x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{N-1}.
$$

Proof. As it is easy to see by standard truncation and regularization arguments, the subset $\mathcal{S} \subset D_{\mathbf{d}}^{1,2}(\mathbb{R}^N)$ of the mappings which are nonincreasing with respect to $|x_1|$ is the closure of $C_c^\infty(\mathbb{R}^N) \cap \mathcal{S}$ in $D^{1,2}(\mathbb{R}^N)$, so that it is not restrictive to assume $u \in C_c^\infty(\mathbb{R}^N) \cap \mathcal{S}$ in proving the lemma. Then fix $x_1 \neq 0$ and define

$$
v(x_2) := \int_{|x_1|/2}^{|x_1|} u(t, x_2) \, dt \qquad \text{for all } x_2 \in \mathbb{R}^{N-1}.
$$

On the one hand, by symmetry and monotonicity properties of u, one has $v(x_2) = v(|x_2|)$ and

$$
v(x_2) \geq \frac{|x_1|}{2} u(|x_1|, x_2) = \frac{|x_1|}{2} u(x), \quad \forall x_2 \in \mathbb{R}^{N-1}. \tag{4.6}
$$

On the other hand, by Hölder inequality, we have

$$
|\nabla v(x_2)|^2 \leq \frac{|x_1|}{2} \int_{|x_1|/2}^{|x_1|} |\nabla_{x_2} u(t, x_2)|^2 \, dt \leq \frac{|x_1|}{2} \int_{|x_1|/2}^{|x_1|} |\nabla u(t, x_2)|^2 \, dt
$$

and

$$
v(x_2)^{2^*} \leq \left(\frac{|x_1|}{2}\right)^{2^*-1} \int_{|x_1|/2}^{|x_1|} u(t, x_2)^{2^*} \, dt
$$

for every $x_2 \in \mathbb{R}^{N-1}$, which imply

$$
\int_{\mathbb{R}^{N-1}} |\nabla v(x_2)|^2 \, dx_2 \leq \frac{|x_1|}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx
$$

and

$$
\int_{\mathbb{R}^{N-1}} v(x_2)^{2^*} \, dy \leq \left(\frac{|x_1|}{2}\right)^{2^*-1} \int_{\mathbb{R}^N} u^{2^*} dx.
$$

Hence we can apply Proposition II.1 of [33] and we get

$$\begin{aligned}
v\left(x_2\right) &\leq C\left\|\nabla v\right\|_{L^2(\mathbb{R}^{N-1})}^{\frac{N-2}{2N-2}}\left\|v\right\|_{L^{2^*}(\mathbb{R}^{N-1})}^{\frac{N}{2N-2}}\frac{1}{\left|x_2\right|^{(N-2)^2/(2N-2)}} \\
&\leq C\left(\frac{\left|x_1\right|}{2}\right)^{\frac{N-2}{4N-4}}\left\|\nabla u\right\|_{L^2(\mathbb{R}^N)}^{\frac{N-2}{2N-2}}\left(\frac{\left|x_1\right|}{2}\right)^{\frac{N+2}{4N-4}}\left\|u\right\|_{L^{2^*}(\mathbb{R}^N)}^{\frac{N}{2N-2}}\frac{1}{\left|x_2\right|^{(N-2)^2/(2N-2)}} \\
&= C\left(\frac{1}{2}\right)^{\frac{N}{2N-2}}\left\|\nabla u\right\|_{L^2(\mathbb{R}^N)}^{\frac{N-2}{2N-2}}\left\|u\right\|_{L^{2^*}(\mathbb{R}^N)}^{\frac{N}{2N-2}}\left|x_1\right|^{\frac{N}{2N-2}}\frac{1}{\left|x_2\right|^{(N-2)^2/(2N-2)}}
\end{aligned}$$

for almost every $x_2 \in \mathbb{R}^{N-1}$ and some constant $C > 0$ which only depends on N. Together with (4.6), this gives the estimate. ■

Proposition 4.7. *If* $\mathbf{d} = (1, N-1)$, *then every bounded sequence* $\{u_n\} \subset D_{\mathbf{d}}^{1,2}(\mathbb{R}^N)$ *such that* u_n *is nonincreasing with respect to* $|x_1|$ *is relatively compact in* $L^p + L^q(\mathbb{R}^N)$ *for every* $1 < p < 2^* < q$.

We observe that the requirement of nonincreasing in $|x_1|$ cannot be avoided, as the same counterexample of Remark 1 shows.

Proof. The proposition has been proved in [4, Theorem 3.2] for $N = 3$. In order to get the general case, one just has to adapt the same arguments, using Lemma 4.6 above instead of [33, Lemma II.1]. ■

We now turn to the proof of Theorem 4.4, which will be achieved through several lemmas. Accordingly, we hereafter let N be as in (i) with $N_0 \neq 0$ and assume that $V : \mathbb{R}_+^k \to (0, +\infty)$ is a measurable function satisfying $(\mathbf{V}_0)$ and such that condition $(\mathbf{V}_2)$ holds with $i = k$, which is not restrictive, up to a change of variable names.

Let $\{u_n\}$ be any bounded sequence in $H_{\text{cyl}} = H_{\text{cyl}}\left(\mathbf{N}, V\right)$ (recall definition (4.5)) and let $2 < p < 2^* < q$.

Fix $\varepsilon > 0$ arbitrarily.

Lemma 4.8. *There exist* n_ε *and* $R_\varepsilon > 1$ *such that*

$$\forall n \geq n_\varepsilon, \qquad \int_{\mathbb{R}^{N-N_k} \times B_{R_\varepsilon}^c} m\left(u_n\right) dx < \varepsilon \tag{4.7}$$

(where $B_{R_\varepsilon}^c \subset \mathbb{R}^{N_k}$*).*

Proof. It is convenient to denote

$$\tilde{z} = (z, y_1, ..., y_{k-1}) \in \mathbb{R} \times \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_{k-1}}$$

in such a way that $x \in \mathbb{R}^N$ splits as $x = (\tilde{z}, y_k) \in \mathbb{R}^{N-N_k} \times \mathbb{R}^{N_k}$. Recall that $N_i \geq 2$ for $i = 1, ..., k$ and observe that $k > 1$ implies $N - N_k \geq 3$, whereas $k = 1$ implies $N - N_k = 1$ (and $\tilde{z} = z$). Denote by u_n^* the spherically symmetric decreasing rearrangement of u_n with respect to $\tilde{z}$ (which is, more precisely, the $(N-N_k, N)$-Steiner symmetrization of u_n; see [20]). Then $\{u_n^*\}$ is bounded in $D_{\mathbf{d}}^{1,2}(\mathbb{R}^N)$ with $\mathbf{d} = (N-N_k, N_k)$ and we can apply Theorem 4.3 or Proposition 4.7, according as $k>1$ or $k=1$. As a consequence, there exists $u^* \in D_{\mathbf{d}}^{1,2}(\mathbb{R}^N)$ such that

$$u_n^* \to u^* \quad \text{in } L^p + L^q\left(\mathbb{R}^N\right)$$

(up to a subsequence) and

$$\int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}^c} m\left(u^*\right) dx < \frac{\varepsilon}{2} \qquad \text{for some suitable } R_\varepsilon > 1,$$

since $m(u^*) \in L^1(\mathbb{R}^N)$ by characterization (3) of page 13. So $m(u_n^*) \to m(u^*)$ in $L^1(\mathbb{R}^N)$ by Proposition 4.5, and this implies

$$\int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}^c} m\left(u_n^*\right) dx \to \int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}^c} m\left(u^*\right) dx.$$

As well known properties of symmetrizations ensure that

$$\int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}^c} m\left(u_n\right) dx = \int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}^c} m\left(u_n^*\right) dx$$

for all n, the conclusion readily follows. ■

Now fix n_ε and $R_\varepsilon > 1$ according to Lemma 4.8, take a V_ε such that

$$0 < V_\varepsilon < \min\left\{1, \operatorname*{essinf}_{(y_1,\dots,y_k)\in\mathbb{R}^{N-1},\, y_k\in B_{R_\varepsilon}} V\left(\left|y_1\right|,\dots,\left|y_k\right|\right)\right\}$$

according to $(\mathbf{V}_2)$ (with $i=k$) and set

$$\Lambda_{n,\varepsilon} := \left\{x \in \mathbb{R}^N : \left|u_n\left(x\right)\right| > V_\varepsilon^{\frac{p}{(p-2)(q-2^*)}}\varepsilon\right\}.$$

Lemma 4.9. *There exists a constant $\bar{C} > 0$, independent of ε, such that*

$$\int_{\Lambda_{n,\varepsilon}^c} m\left(u_n\right) dx \leq \bar{C}\varepsilon^{q-2^*} V_\varepsilon^{\frac{p}{p-2}} \qquad \textit{for all } n.$$

Proof. Since $q > 2^*$, for all n we have

$$\int_{\Lambda_{n,\varepsilon}^c} m\left(u_n\right) dx \leq \int_{\Lambda_{n,\varepsilon}^c} \left|u_n\right|^q dx \leq \left(V_\varepsilon^{\frac{p}{(p-2)(q-2^*)}}\varepsilon\right)^{q-2^*} \int_{\Lambda_{n,\varepsilon}^c} \left|u_n\right|^{2^*} dx$$

and the conclusion thus follows as $\{u_n\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$. ■

Lemma 4.10. *There exists a constant $\bar{C}_\varepsilon > 0$ such that for every measurable subset $\Omega \subseteq \mathbb{R}^N$ one has*

$$\int_{\Omega\cap\Lambda_{n,\varepsilon}} |u_n|^p \, dx \le \bar{C}_\varepsilon \, \|u_n\|^p_{L^p+L^q(\Omega)} \qquad \text{for all } n. \tag{4.8}$$

Proof. First notice that $u_n \in L^p(\Lambda_{n,\varepsilon})$ because $u_n \in L^p + L^q(\mathbb{R}^N)$ (recall the characterization (3) of page 13). Then assume $|\Omega \cap \Lambda_{n,\varepsilon}| \neq 0$ for all n, as (4.8) is obvious if $|\Omega \cap \Lambda_{n,\varepsilon}| = 0$. We use (4.2) and an argument from [12]: we get

$$\begin{aligned}
\|u_n\|^*_{L^p+L^q(\Omega)} &= \sup_{0\neq\varphi\in L^{p'}(\Omega)\cap L^{q'}(\Omega)} \frac{\left|\int_\Omega \varphi u_n dx\right|}{\|\varphi\|_{L^{p'}(\Omega)} + \|\varphi\|_{L^{q'}(\Omega)}} \\
&\ge \sup_{\substack{0\neq\varphi\in L^{p'}(\Omega)\cap L^{q'}(\Omega) \\ \varphi=0 \text{ in } \Omega\cap\Lambda^c_{n,\varepsilon}}} \frac{\left|\int_\Omega \varphi u_n dx\right|}{\|\varphi\|_{L^{p'}(\Omega)} + \|\varphi\|_{L^{q'}(\Omega)}} \\
&= \sup_{0\neq\varphi\in L^{p'}(\Omega\cap\Lambda_{n,\varepsilon})} \frac{\left|\int_{\Omega\cap\Lambda_{n,\varepsilon}} \varphi u_n dx\right|}{\|\varphi\|_{L^{p'}(\Omega\cap\Lambda_{n,\varepsilon})} + \|\varphi\|_{L^{q'}(\Omega\cap\Lambda_{n,\varepsilon})}} \\
&\ge \sup_{0\neq\varphi\in L^{p'}(\Omega\cap\Lambda_{n,\varepsilon})} \frac{\left|\int_{\Omega\cap\Lambda_{n,\varepsilon}} \varphi u_n dx\right|}{\left(1 + |\Omega\cap\Lambda_{n,\varepsilon}|^{1/p-1/q}\right)\|\varphi\|_{L^{p'}(\Omega\cap\Lambda_{n,\varepsilon})}} \\
&= \frac{1}{1 + |\Omega\cap\Lambda_{n,\varepsilon}|^{1/p-1/q}} \|u_n\|_{L^p(\Omega\cap\Lambda_{n,\varepsilon})},
\end{aligned}$$

where we have taken into account that $L^{p'}(\Omega\cap\Lambda_{n,\varepsilon}) \cap L^{q'}(\Omega\cap\Lambda_{n,\varepsilon}) = L^{p'}(\Omega\cap\Lambda_{n,\varepsilon})$ and $\|\varphi\|_{L^{q'}(\Omega\cap\Lambda_{n,\varepsilon})} \le |\Omega\cap\Lambda_{n,\varepsilon}|^{1/p-1/q} \|\varphi\|_{L^{p'}(\Omega\cap\Lambda_{n,\varepsilon})}$, since $p' > q'$ and

$$|\Omega\cap\Lambda_{n,\varepsilon}| \le |\Lambda_{n,\varepsilon}| \le V_\varepsilon^{-\frac{p^2}{(p-2)(q-2^*)}} \varepsilon^{-p} \int_{\Lambda_{n,\varepsilon}} |u_n(x)|^p \, dx < \infty.$$

Hence we have

$$\|u_n\|^*_{L^p+L^q(\Omega)} \ge \frac{\left(\int_{\Omega\cap\Lambda_{n,\varepsilon}} |u_n|^p \, dx\right)^{1/p}}{1 + |\Omega\cap\Lambda_{n,\varepsilon}|^{1/p-1/q}} \ge V_\varepsilon^{\frac{p}{(p-2)(q-2^*)}} \varepsilon \frac{|\Omega\cap\Lambda_{n,\varepsilon}|^{1/p}}{1 + |\Omega\cap\Lambda_{n,\varepsilon}|^{1/p-1/q}}$$

where the right hand side is unbounded if $\{|\Omega\cap\Lambda_{n,\varepsilon}|\}_n$ is unbounded. But $\{u_n\}$ is bounded in $L^p + L^q(\Omega)$ because it is bounded in $L^p + L^q(\mathbb{R}^N)$ (see (4.3)) and thus $\{|\Omega\cap\Lambda_{n,\varepsilon}|\}_n$ is bounded. For $\Omega = \mathbb{R}^N$ this yields in particular that $\{|\Lambda_{n,\varepsilon}|\}_n$ is bounded, and so we conclude

$$\begin{aligned}
\|u_n\|_{L^p(\Omega\cap\Lambda_{n,\varepsilon})} &\le \left(1 + |\Omega\cap\Lambda_{n,\varepsilon}|^{1/p-1/q}\right) \|u_n\|^*_{L^p+L^q(\Omega)} \\
&\le \left(1 + \sup_n |\Lambda_{n,\varepsilon}|^{1/p-1/q}\right) \|u_n\|_{L^p+L^q(\Omega)}.
\end{aligned}$$

This gives the result, since $1 + \sup_n |\Lambda_{n,\varepsilon}|^{1/p-1/q}$ is independent of Ω. ■

Lemma 4.11. *If*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}}\|u_n\|_{L^p+L^q(B_R(z)\times\mathbb{R}^{N-N_0})}=0 \quad \textit{for every } R>0, \tag{4.9}$$

then there exist $C>0$, independent of ε, and $C_\varepsilon>0$ such that for all n one has

$$\int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}} m(u_n)\,dx \le C\left(\varepsilon^{q-2^*}+C_\varepsilon o(1)_{n\to\infty}\right)^{1-2/p}$$

($o(1)_{n\to\infty}$ denotes any vanishing sequence, independent of ε).

Proof. Denote

$$\Omega_{R_\varepsilon}:=\mathbb{R}^{N_1}\times ...\times\mathbb{R}^{N_{k-1}}\times B_{R_\varepsilon}\subset\mathbb{R}^{N-N_0}$$

for brevity and decompose $\mathbb{R}^{N_0}$, up to null measure sets, as a disjoint countable union of open hypercubes Q_j of unitary edge. Let $\{z_j\}\subset\mathbb{R}^{N_0}$ and $R>0$ be such that $Q_j\subset B_R(z_j)$ for all j. Then for all n one has

$$\begin{aligned}\int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}} m(u_n)\,dx &= \sum_{j=1}^{\infty}\int_{Q_j\times\Omega_{R_\varepsilon}} m(u_n)\,dx \qquad (4.10)\\ &= \sum_{j=1}^{\infty}\left(\int_{Q_j\times\Omega_{R_\varepsilon}} m(u_n)\,dx\right)^{1-\frac{2}{p}}\left(\int_{Q_j\times\Omega_{R_\varepsilon}} m(u_n)\,dx\right)^{\frac{2}{p}},\end{aligned}$$

where, by Lemmas 4.9 and 4.10,

$$\begin{aligned}\int_{Q_j\times\Omega_{R_\varepsilon}} m(u_n)\,dx &\le \int_{\Lambda_{n,\varepsilon}^c} m(u_n)\,dx+\int_{(Q_j\times\Omega_{R_\varepsilon})\cap\Lambda_{n,\varepsilon}} m(u_n)\,dx\\ &\le \bar{C}V_\varepsilon^{\frac{p}{p-2}}\varepsilon^{q-2^*}+\int_{(Q_j\times\Omega_{R_\varepsilon})\cap\Lambda_{n,\varepsilon}}|u_n|^p\,dx\\ &\le \bar{C}V_\varepsilon^{\frac{p}{p-2}}\varepsilon^{q-2^*}+\bar{C}_\varepsilon\|u_n\|^p_{L^p+L^q(Q_j\times\Omega_{R_\varepsilon})}\\ &\le \bar{C}V_\varepsilon^{\frac{p}{p-2}}\varepsilon^{q-2^*}+\bar{C}_\varepsilon\sup_{z\in\mathbb{R}}\|u_n\|^p_{L^p+L^q\left(B_R(z)\times\mathbb{R}^{N-N_0}\right)}\\ &= \bar{C}V_\varepsilon^{\frac{p}{p-2}}\varepsilon^{q-2^*}+\bar{C}_\varepsilon o(1)_{n\to\infty}. \qquad (4.11)\end{aligned}$$

Now observe that, since $R_\varepsilon>1$, every domain $Q_j\times\Omega_{R_\varepsilon}$ satisfies the cone property by the same cone for which it holds for $Q_1\times\Omega_1$. Such a cone is independent of ε and therefore there exists a constant $C_*>0$, only dependent on $p\in(2,2^*)$ and the dimension $N\ge 3$, such that $\|w_n\|_{L^p(Q_j\times\Omega_{R_\varepsilon})}\le C_*\|w_n\|_{H^1(Q_j\times\Omega_{R_\varepsilon})}$ for all n (see [1, Lemma 5.12]). Hence, plugging (4.11) into (4.10) and denoting $\tilde{C}:=\bar{C}C_*^2$ for brevity, we get

$$\begin{aligned}&\int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}} m(u_n)\,dx\\ &\le \left(\bar{C}V_\varepsilon^{\frac{p}{p-2}}\varepsilon^{q-2^*}+\bar{C}_\varepsilon o(1)_{n\to\infty}\right)^{1-2/p}\sum_{j=1}^{\infty}\left(\int_{Q_j\times\Omega_{R_\varepsilon}}|u_n|^p\,dx\right)^{2/p}\end{aligned}$$

$$
\begin{aligned}
&\leq \bar{C}C_*^2\left(V_\varepsilon^{\frac{p}{p-2}}\varepsilon^{q-2^*}+\bar{C}_\varepsilon o\left(1\right)_{n\to\infty}\right)^{1-2/p}\cdot\\
&\quad\cdot\sum_{j=1}^{\infty}\left(\int_{Q_j\times\Omega_{R_\varepsilon}}\left|\nabla u_n\right|^2dx+\int_{Q_j\times\Omega_{R_\varepsilon}}\left|u_n\right|^2dx\right)\\
&\leq \tilde{C}\left(V_\varepsilon^{\frac{p}{p-2}}\varepsilon^{q-2^*}+\bar{C}_\varepsilon o\left(1\right)_{n\to\infty}\right)^{1-2/p}\cdot\\
&\quad\cdot\sum_{j=1}^{\infty}\left(\int_{Q_j\times\Omega_{R_\varepsilon}}\left|\nabla u_n\right|^2dx+\frac{1}{V_\varepsilon}\int_{Q_j\times\Omega_{R_\varepsilon}}V\left(\left|y_1\right|,...,\left|y_k\right|\right)\left|u_n\right|^2dx\right)\\
&\leq \tilde{C}\left(V_\varepsilon^{\frac{p}{p-2}}\varepsilon^{q-2^*}+\bar{C}_\varepsilon o\left(1\right)_{n\to\infty}\right)^{1-2/p}\frac{1}{V_\varepsilon}\left\|u_n\right\|^2\\
&= \tilde{C}\left\|u_n\right\|^2\left(\varepsilon^{q-2^*}+\bar{C}_\varepsilon V_\varepsilon^{-\frac{p}{p-2}}o\left(1\right)_{n\to\infty}\right)^{1-2/p}.
\end{aligned}
$$

The result then ensues, since $\{u_n\}$ is bounded in H_{cyl}. ■

Corollary 4.12. *If $u_n \nrightarrow 0$ in $L^p + L^q(\mathbb{R}^N)$, then, up to a subsequence, there exist $R > 0$ and $\{z_n\} \subset \mathbb{R}^{N_0}$ such that*

$$
\inf_n \|u_n\|_{L^p+L^q(B_R(z_n)\times\mathbb{R}^{N-N_0})} > 0. \tag{4.12}
$$

Proof. Assume on the contrary that (4.9) holds. Then, by (4.7) and Lemma 4.11, there exists $\bar{n}_\varepsilon \geq n_\varepsilon$ such that

$$
\begin{aligned}
\int_{\mathbb{R}^N} m\left(u_n\right)dx &= \int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}^c} m\left(u_n\right)dx+\int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}} m\left(u_n\right)dx\\
&\leq \varepsilon+C\left(\varepsilon^{q-2^*}+\varepsilon\right)^{1-2/p}
\end{aligned}
$$

for all $n \geq \bar{n}_\varepsilon$. As ε is arbitrary and C does not depend on ε, this means

$$
\int_{\mathbb{R}^N} m\left(u_n\right)dx \to 0 \quad \text{as } n\to\infty,
$$

i.e., $u_n \to 0$ in $L^p + L^q(\mathbb{R}^N)$ (see Proposition 4.5), which is a contradiction. ■

In order to conclude the proof of Theorem 4.4, we finally need the following compactness lemma, which relies on assumption $(\mathbf{V}_2)$ and follows from the results of [34].

Lemma 4.13. *H_{cyl} is compactly embedded into $L^p(B_R\times\mathbb{R}^{N-N_0-N_k}\times B_{R_\varepsilon})$ for every $R>0$.*

Proof. We show that if $v_n \rightharpoonup 0$ in H_{cyl} then $v_n \to 0$ in $L^p(B_R\times\mathbb{R}^{N-N_0-N_k}\times B_{R_\varepsilon})$. Let $\rho>0$ be such that $B_R\times B_{R_\varepsilon}\subset B_\rho\times B_\rho\subset\mathbb{R}^{N_0}\times\mathbb{R}^{N_k}$ and take $\phi\in C_c^\infty\left(B_\rho\times B_\rho\right)$ such that $\phi\left(z,y_k\right)\equiv 1$ on $B_R\times B_{R_\varepsilon}$ and $0\leq\phi\left(z,y_k\right)=\phi\left(z,\left|y_k\right|\right)\leq 1$. Then

$$
\int_{B_R\times\mathbb{R}^{N-N_0-N_k}\times B_{R_\varepsilon}}\left|v_n\right|^p dx \leq \int_{B_\rho\times\mathbb{R}^{N-N_0-N_k}\times B_\rho}\left|\phi v_n\right|^p dx
$$

and by Proposition 2.1 one easily checks that $\phi v_n \in H_0^1(B_\rho \times \mathbb{R}^{N-N_0-N_k} \times B_\rho) \cap H_{\mathrm{cyl}}$. The claim of the lemma thus follows from the compactness of the embedding

$$H_0^1\left(B_\rho \times \mathbb{R}^{N-N_0-N_k} \times B_\rho\right) \cap H_{\mathrm{cyl}} \hookrightarrow L^p\left(B_\rho \times \mathbb{R}^{N-N_0-N_k} \times B_\rho\right),$$

which is proved in [34, Theorem III.2], up to a change of variable names. ∎

Proof of Theorem 4.4. Assume that $u_n \nrightarrow 0$ in $L^p + L^q(\mathbb{R}^N)$ and let $R > 0$ and $\{z_n\} \subset \mathbb{R}^{N_0}$ be such that (4.12) holds (up to a subsequence), according to Corollary 4.12. Define $\{u_n^{z_n}\} \subset H_{\mathrm{cyl}}$ by setting

$$u_n^{z_n}(z, y_1, ..., y_k) := u_n(z + z_n, y_1, ..., y_k).$$

Then, by an obvious change of variables, it is easy to check that $\|u_n^{z_n}\| = \|u_n\|$ and $\|u_n^{z_n}\|_{L^p+L^q(B_R\times\mathbb{R}^{N-N_0})} = \|u_n\|_{L^p+L^q(B_R(z_n)\times\mathbb{R}^{N-N_0})}$, so that $\{u_n^{z_n}\}$ is bounded in H_{cyl} and, up to a subsequence, we can assume

$$u_n^{z_n} \rightharpoonup u \text{ in } H_{\mathrm{cyl}} \qquad \text{and} \qquad \inf_n \|u_n^{z_n}\|_{L^p+L^q(B_R\times\mathbb{R}^{N-N_0})} > 0. \tag{4.13}$$

The proof is complete if we show that $u \neq 0$, so assume $u = 0$ by contradiction. Then Lemma 4.13 gives

$$\int_{B_R\times\mathbb{R}^{N-N_0-N_k}\times B_{R_\varepsilon}} m(u_n^{z_n})\,dx \leq \int_{B_R\times\mathbb{R}^{N-N_0-N_k}\times B_{R_\varepsilon}} |u_n^{z_n}|^p\,dx \to 0$$

and thus, by (4.7), we obtain

$$\int_{B_R\times\mathbb{R}^{N-N_0}} m(u_n^{z_n})\,dx \leq o(1)_{n\to\infty} + \int_{\mathbb{R}^{N-N_k}\times B_{R_\varepsilon}^c} m(u_n^{z_n})\,dx \leq o(1)_{n\to\infty} + \varepsilon$$

for $n \geq n_\varepsilon$. As ε is arbitrary, this implies

$$\int_{B_R\times\mathbb{R}^{N-N_0}} m(u_n^{z_n})\,dx \to 0 \quad \text{as } n \to \infty,$$

which means $u_n^{z_n} \to 0$ in $L^p + L^q(B_R \times \mathbb{R}^{N-N_0})$ by Proposition 4.5 and therefore a contradiction ensues with (4.13). ∎

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